The universal number

The universal number $U(n)$ is the *smallest* positive integer, from which by removing some digits in its record you can get *any* a natural number from 1 to n. The number n can contain from 1 to 10^5 digits. It is required to find the *minimal* number $U(n)$ with this property.

The main topics of the task:

- dynamic programming on strings;
- recursion.

First, consider a few examples.

Sample 1 from the task: $U(10) = 1023456789$; we calculate $U(11) = 10123456789$. You can pay attention to the fact that from the found number $U(11)$ you can get any number from 1 to 19, so $U(11) = \ldots = U(19)$.

Sample 2. Find the universal number $U(222)$.

Answer: $U(222) = 12(0123456789)(0123456789)$.

Why is this and what other numbers can be obtained from $U(222)$?

Proof: From decimal notation $U(222)$ you can get, in particular, the numbers 111, 222, 100, 33, 44, ..., 99, therefore, the number contains at least *three* units, at least *three* twos, ..., at least *two* nines; total not less than $3 + 3 + 8 \cdot 2 = 22$ digits.

Consider a number $U(n) = 12(0123456789)(0123456789)$ in which each bracketed group contains all the digits from 0 to 9. It produces *all* two-digit and single-digit numbers, as well as all three-digit numbers with the first digit 1 or 2. It is easy to prove that the number found is the smallest.

It also follows from this example that for all $n \in [222; 299] : U(n) = U(222)$.

For all m-valued numbers from $\overline{nn\ldots nn}$ to $\overline{n9\ldots 99}$

$$U(\overline{nn\ldots nn}) = 12\ldots n(0123456789)^{m-1}.$$

A degree means that the set of numbers in parentheses is repeated a specified number of times.

An incomplete group $12\ldots n$ is called *prefix P* of the universal number U.

Sample 3. [Numbers in the range from $\overline{n0\ldots 00}$ to $\overline{nn\ldots n\,(n-1)}$.]

For numbers from $\overline{n0\ldots 00}$ to $\overline{nn\ldots n\,(n-1)}$ the prefix of universal number is $P = \overline{12\ldots(n-1)}$. In particular, for $n = 1$ there is no prefix.

Sample 4.

$U(10) = (1023456789)$

$U(100) = U(100) = \ldots = U(110) = (1023456789)(1023456789)$

$U(200) = U(201) = \ldots = U(221) = 1\,(1203456789)(1023456789)$.

The algorithm of solution. For arbitrary $n = \overline{a_1 a_2 \ldots a_m}$ we represent $U(n)$:

$$U(\overline{a_1 a_2 \ldots a_m}) = P + V(\overline{a_2 \ldots a_m}).$$

Here «+» — concatenation operation, $V(\overline{a_2 \ldots a_m})$ — a set of $(m-1)$ «full» groups (from numbers from 0 to 9 in some order).

To calculate $V(\overline{a_2 \ldots a_m})$, we use recursion.

$$V(a_2a_3\dots a_m) = (012\dots 9)_{a_3} + V(a_3\dots a_m), \quad \text{если } a_3 \neq 0.$$
$$V(a_20\dots a_m) = (012\dots 9)_1 + (0123456789)^{m-2}, \quad \text{если } a_3 = 0.$$

The group $(012\dots 9)_{a_3}$ is obtained from the «standard» set $(012\dots 9)$ by a cyclic left shift a_3 positions *first* (a_2+1) digits of this set, that is, a shift of the digits $01\dots a_2$.

Sample 5. For the number $U(210)$, the first group after the prefix is:

$$(012\dots 9)_1 = (012)_1(3456789) = (120)(3456789).$$

Sample 6.

$U(100) = (1023456789)(0123456789)$.

$U(300) = 12\,(1230456789)(0123456789)$.

$U(310) = 12 + V(310) = 12 + (1230456789) + V(10) = 12 + (1230456789) + (1023456789) =$
$= 12\,(1230456789)(1023456789)$.

$U(3310) = 12 + V(3310) = 12 + (3012456789) + V(310) = 12 + (3012456789) + (1230456789)(1023456789) =$
$= 12\,(3012456789)(1230456789)(1023456789)$.

The complexity of the algorithm $O(m)$, where m is the length of the number n.