Beauty of numbers

Beauty is the number of representations of this number as the sum of several consecutive positive integers. (A sum of one term is allowed.) For a given positive integer n, it is necessary to find the *smallest* beauty number n. For example, for $n = 4$ the smallest beauty number 4 is 15, because 15 can be represented as the sum of several consecutive numbers in exactly four ways: $15 = 7 + 8 = 4 + 5 + 6 = 1 + 2 + 3 + 4 + 5$.

We represent the desired integer s as the sum

$$s = (k+1) + (k+2) + \ldots + m = \frac{1}{2}m(m+1) - \frac{1}{2}k(k+1) = \frac{1}{2}(m-k)(m+k+1)$$

Subproblem 1. It follows from the equality that the beauty of the number s is equal to the number of integer positive solutions to the equation $2s = (m-k)(m+k+1)$. We select the values s of starting from 1, so that this equation has a given number of solutions n. Since the first factor is less than the second, it is enough to iterate over the values of $m - k$ from 1 to $\sqrt{2s}$. If for the current value of s the number of solutions to this equation does not coincide with n, increase s by one and repeat the search process.

For small values of n, this solution passes the tests of the first group and scores 20 points.

Subproblem 2. Note that the factors $m - k$ and $m + k + 1$ have different parity, since their sum is equal to an odd number $2m+1$. Therefore, each required representation of the number s necessarily corresponds to some odd divisor. The converse is also true — each odd divisor of s has its own representation of s as the sum of consecutive integers.

Therefore, *the beauty of the natural number s is equal to the number of its odd divisors*. For example, the numbers 15 and 30 have four odd divisors 1, 3, 5 and 15, so the beauty of each of them is four.

It also follows from the obtained statement that the required smallest number s does not contain *even* divisors, and therefore to solve the problem, you need to find the *smallest* natural number s with a given number of n odd divisors.

In subproblem 2, the number n is prime, so the number s is of the form p^{n-1}, p is prime. Indeed, if s contained some prime factor other than p, then the beauty of s would be a *composite* number. Among the numbers of the form p^{n-1}, the smallest is the number 3^{n-1}. Therefore, in subtask 2, you need to calculate the remainder of dividing 3^{n-1} by $10^9 + 9$.

This decision passes the tests of the second group and is gaining another 25 points.

Subproblems 3 and 4. We construct the desired number s as a product of prime factors. First of all, the expansion of the number s can include only 16 odd primes 3, 5, 7, 11, and so on up to 59. (The product of these numbers $3 \cdot 5 \cdot \ldots \cdot 59$ has 2^{16} odd divisors, which is less than 10^5. When adding the following simple of a factor of 61, the product will already be more than 10^5.) It remains to find the exponents with which these prime numbers enter the expansion of s:

$$s = 3^a \cdot 5^b \cdot 7^c \cdot \ldots .$$

Then the number of odd divisors of s is equal to $(a+1)(b+1)(c+1)\ldots = n$, and therefore the numbers $a+1$, $b+1$, $c+1, \ldots$ should be looked up among the divisors of the number n. It is easy to see that the required least number s has a, b, c, $\ldots$ are connected by the inequalities $a \geq b \geq c \geq \ldots$. Using simple recursion, we sort through *all* the possible factorizations of n in non-increasing order and among them we choose the smallest. For example, for $n = 20$ you need to make 4 numbers 3^{19}, $3^9 \cdot 5^1$, $3^4 \cdot 5^3$, $3^4 \cdot 5^1 \cdot 7^1$ and choose the smallest among them.

To compare numbers with each other, you can use long arithmetic. In this case, the solution passes the tests of the third subgroup and scores 70 points.

For a complete solution to the problem, it remains to note that all the constructed numbers do not need to be calculated — you can accurately compare the *logarithms* of these numbers among themselves. In this case, the solution passes all the tests and scores 100 points.