Curve Fitting and Interpolation

CMSC 150 Numerical and Symbolic Computation

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Lei Kristoffer R. Lactuan

Institute of Computer Science
University of the Philippines Los Baños

Lecture Outline

I. Introduction to Scientific Computing

II. Approximation and Error Analysis

III. Linear Systems

IV. Interpolation and Curve Fitting

   A. Interpolating Polynomials

   B. Divided Difference

   C. Lagrarian Polynomials and Neville’s Method

   D. Linear, Quadratic and Cubic Spline Interpolation

V. Optimization

VI. Roots of Equations

VII. Numerical Differentiation and Integration

Lagrange interpolating Polynomial

General Form:

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(xi) \qquad L_i(x)$$

$$= \prod_{j=0, j\neq i}^{n} \frac{x - xj}{x_i - xj}$$

First-order:

$$f_1(x) = f(x_0)\frac{x - x_1}{x_0 - x_1} + f(x_1)\frac{x - x_0}{x_1 - x_0}$$

Curve Fitting and Interpolation

Lagrange interpolating Polynomial

Second-order:

$$f_2(x) = f(x_0)\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + f(x_1)\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$$

$$+f(x_2)\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

Lagrange interpolating Polynomial

Example:

- Estimate **ln 2**
- Given: $X_0 = 1$, $X_1 = 3$, $X_2 = 4$, and $X_3 = 5$

  ln 1 = 0 *ln 3 = 1.0986*

  ln 4 = 1.3863 *ln 5 = 1.6094*
- Solution: Let's try different orders of polynomial!

$f_1(2) = 0.5493$ $e_T = 20.75\%$

$f_2(2) = 0.6365$ $e_T = 8.17\%$

$f_3(2) = 0.6640$ $e_T = 4.2\%$

Neville's Method

General Form:

$$P_{i,k} = \frac{(x - x_i) \times \; P_{i+1,k-1} + (x_{i+k} - x) \times \; P_{i,k-1}}{x_{i+k} - x_i}$$

In Neville's, data have to be arranged according to their closeness to x

Given:

Estimate x=2

x	f(x)	$(x-x_i)$
4.25	1.4469	2.25
1	0	1
3.5	1.2528	1.5
5	1.6904	3

Curve Fitting and Interpolation

Neville's Method

i	x_i	$f(x_i) = P_{i0}$	P_{i1}	P_{i2}	P_{i3}
0	1	0	0.5011		
1	3.5	1.2528	0.8646		
2	4.25	1.4469	0.9594		
3	5	1.6094			

$$P_{0,1} = \frac{(2-1)\times 1.2528+(3.5-2)\times 0}{3.5-1} = 0.5011$$

$$P_{1,1} = \frac{(2-3.5)\times 1.4469+(4.25-2)\times 1.2528}{4.25-3.5} = 0.8646$$

$$P_{2,1} = \frac{(2-4.25)\times 1.6094+(5-2)\times 1.4469}{35-4.25} = 0.9594$$

Curve Fitting and Interpolation

Neville's Method

i	x_i	$f(x_i) = P_{i0}$	P_{i1}	P_{i2}	P_{i3}
0	1	0	0.5011	0.6129	0.6521
1	3.5	1.2528	0.8646	0.7698	
2	4.25	1.4469	0.9594		
3	5	1.6094			

$$P_{0,2} = \frac{(2-1)\times 0.8646+(4.25-2)\times 0.5011}{4.25-1} = 0.6129$$

$$P_{1,2} = \frac{(2-3.5)\times 0.9594+(5-2)\times 0.8646}{5-3.5} = 0.7698$$

$$P_{0,3} = \frac{(2-1)\times 0.7698+(5-2)\times 0.6129}{5-1} = 0.9594$$

Curve Fitting and Interpolation

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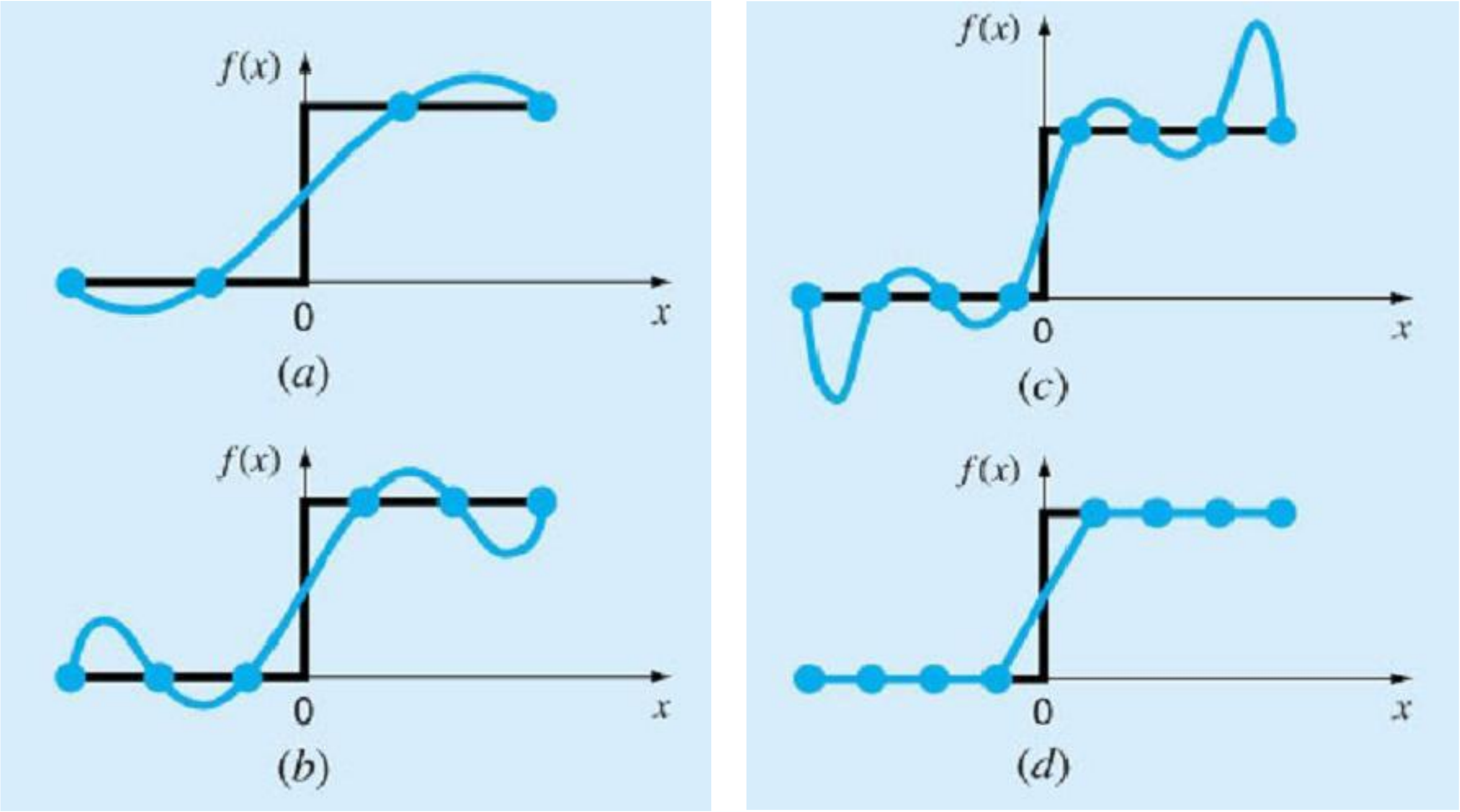
Spline Interpolation

- Review: nth order polynomial -> n+1 data points
- Prone to error: round-off errors and overshoots
- Alternative approach: apply lower-order polynomials to subsets of data points called spline interpolation

Spline Interpolation



Curve Fitting and Interpolation

Linear Spline Interpolation

- The simplest connection between two points is a ***line***.
- First-order splines for a group of ordered data points can be defined as a set of linear functions:

$$f(x) = f(x_0) + m_0 (x-x_0) \quad x_0 \leq x \leq x_1$$
$$f(x) = f(x_1) + m_1 (x-x_1) \quad x_1 \leq x \leq x_2$$
$$...$$
$$f(x) = f(x_{n-1}) + m_{n-1} (x-x_{n-1}) \quad x_{n-1} \leq x \leq x_n$$

- Where $\boldsymbol{m_i}$ is the slope of the straight line connecting the points. $\boldsymbol{m_i} = \frac{f(xi_{+1)} - f(xi)}{x_{i+1} - x_i}$

Curve Fitting and Interpolation

Linear Spline Interpolation

- We can now evaluate at any point between x_0 and x_n using these equations by locating which interval the point lies in.

$$
\boxed{
\begin{aligned}
f(x) &= f(x_0) + m_0\,(x-x_0) \\
f(x) &= f(x_1) + m_1\,(x-x_1) \\
&\dots \\
f(x) &= f(x_{n-1}) + m_{n-1}\,(x-x_{n-1})
\end{aligned}
}
\qquad
\begin{aligned}
& x_0 \leq x \leq x_1 \\
& x_1 \leq x \leq x_2 \\
& \\
& x_{n-1} \leq x \leq x_n
\end{aligned}
$$

- Then, the appropriate equations is used

Linear Spline Interpolation

- This is similar to the linear interpolation we discussed earlier.

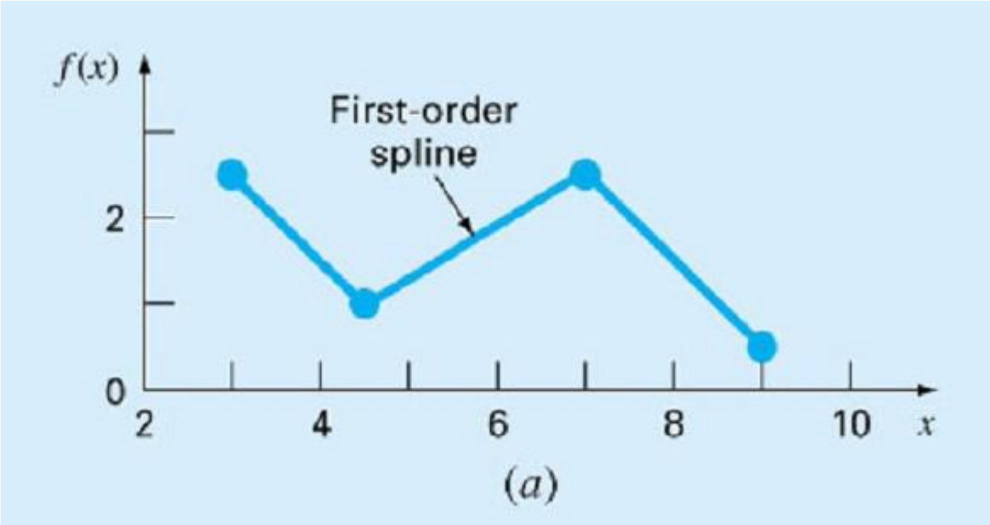
- Disadvantage: Slope abruptly changes on knots.

- Knots: data points where two splines meet.

Linear Spline Interpolation

- Fit the data using first-order splines. Evaluate the function at x=5.

x	f(x)
3.0	2.5
4.5	1.0
7.0	2.5
9.0	0.5



Curve Fitting and Interpolation

Linear Spline Interpolation

- Solution: x=5. Within the interval 4.5 to 7

x	f(x)
3.0	2.5
4.5	1.0
7.0	2.5
9.0	0.5

$$m_1 = \frac{2.5-1}{7-4.5} = 0.60$$

$$f(x) = f(x_1) + m_1\,(x - x_1)$$

$$f(x) = f(4.5) + 0.60\,(5 - 4.5)$$

$$= 1.3$$

Linear Spline Interpolation

- In linear spline, slope abruptly changes on knots.

- Mathematically speaking, the first derivative of the fucntion (slope) is discontinuous at these points (slopes differ).

- To ensure that the nth order derivatives are continuous at the knots, a spline of at least n+1 order must be used.

Quadratic Spline Interpolation

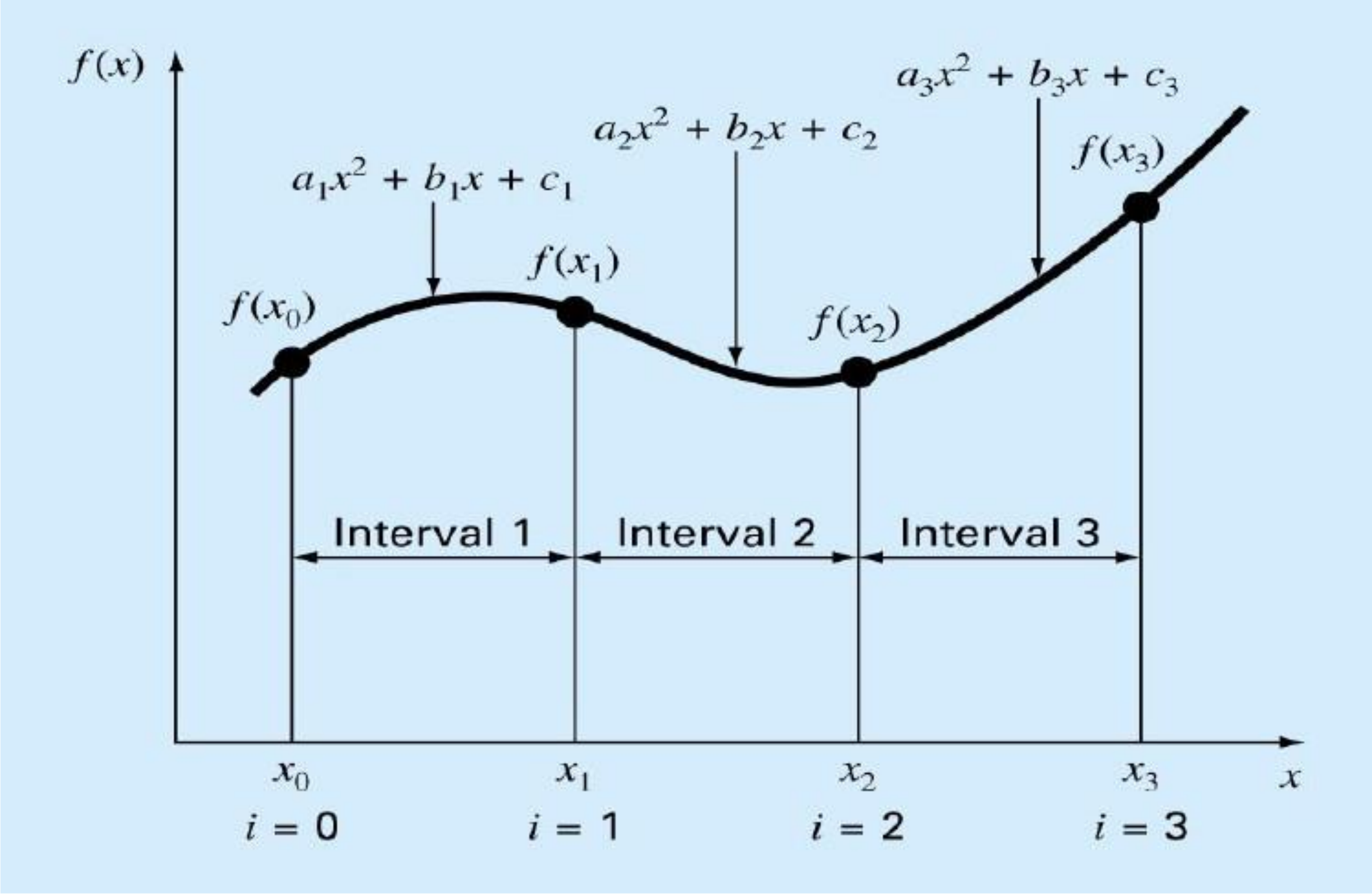
- Quadratic splines: Continuous first derivative at the knots.

- Objective: To derive a second order polynomial for each interval between data points.

- Polynomial for each interval:

$$f_i(x) = a_i x^2 + b_i x + c_i$$

Quadratic Spline Interpolation



Curve Fitting and Interpolation

Quadratic Spline Interpolation

- n+1 data points, n intervals.
- 3 unknowns for every interval. Therefore, we have 3n unknowns for the system.
- We need to generate 3n equations.

Quadratic Spline Interpolation

3n equations (condition 1)

For i=2 to n:

$$\boxed{\begin{array}{l} a_{i-1}(x_{i-1})^2 + b_{i-1}x_{i-1} + c_{i-1} = f(x_{i-1}) \\ a_i(x_{i-1})^2 + b_i x_{i-1} + c_i = f(x_{i-1}) \end{array}}$$

- The function values of adjacent polynomials must be equal to the interior knots.
- Because only interior knots are used, the two equations above each generate n-1 conditions for a total of 2n-2 equations.

Quadratic Spline Interpolation

3n equations (condition 2)

$$a_1(x_0)^2 + b_1 x_0 + c_1 = f(x_0)$$
$$a_n(x_n)^2 + b_n x_n + c_n = f(x_n)$$

- The first and last functions must pass through the end points
- Additional two equations: 2n-2+2 = 2n equations.

Quadratic Spline Interpolation

3n equations (condition 3)

For i=2 to n:

$$\boxed{2a_{i-1}(x_{i-1}) + b_{i-1} = 2a_i(x_{i-1}) + b_i}$$

- The first derivative at the interior knots must be equal.

$f_i(x) = a_i x^2 + b_i x + c_i \Rightarrow f'(x) = 2ax + bx + c$

Quadratic Spline Interpolation

3n equations (condition 4)

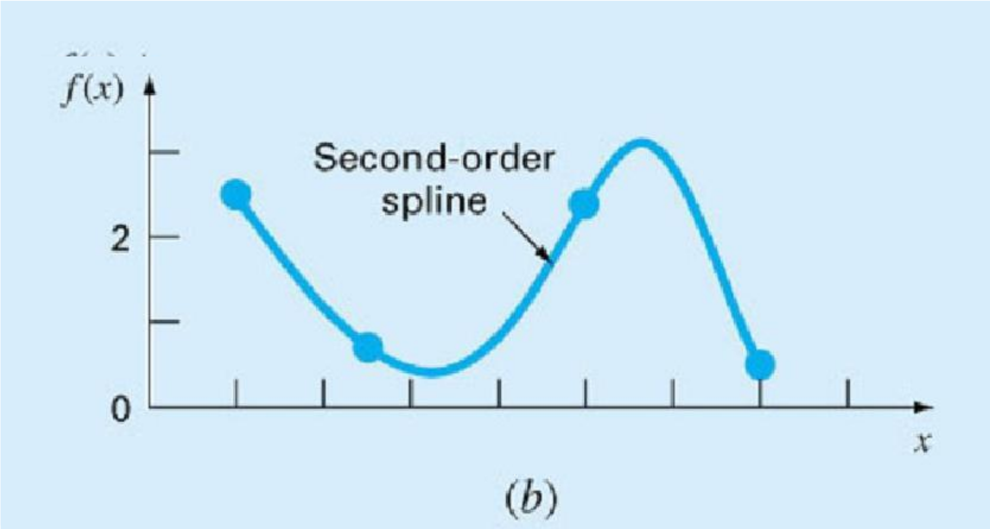
$$\boxed{a_1 = 0}$$

- Assume that the second derivative is zero at the first point.

Quadratic Spline Interpolation

- Fit the data using second-order splines. Evaluate the function at x=5.

x	f(x)
3.0	2.5
4.5	1.0
7.0	2.5
9.0	0.5



Solution: 4 data points. n=3 intervals. 3(3) equations. Generate 9 equations.

Curve Fitting and Interpolation

Quadratic Spline Interpolation

Solution (condition 1)

For i=2 to n:

$$\boxed{\begin{aligned} &a_{i-1}(x_{i-1})^2 + b_{i-1}x_{i-1} + c_{i-1} = f(x_{i-1}) \\ &a_i(x_{i-1})^2 + b_i x_{i-1} + c_i = f(x_{i-1}) \end{aligned}}$$

$$20.25a_1 + 4.5b_1 + c_1 = 1.0$$
$$20.25a_2 + 4.5b_2 + c_2 = 1.0$$
$$49a_2 + 7b_2 + c_2 = 2.5$$
$$49a_3 + 7b_3 + c_3 = 2.5$$

x	f(x)
3.0	2.5
4.5	1.0
7.0	2.5
9.0	0.5

Curve Fitting and Interpolation

Quadratic Spline Interpolation

Solution (condition 2)

$$a_1(x_0)^2 + b_1 x_0 + c_1 = f(x_0)$$
$$a_n(x_n)^2 + b_n x_{in} + c_n = f(x_n)$$

$$9a_1 + 3b_1 + c_1 = 2.5$$
$$81a_3 + 9b_3 + c_3 = 0.5$$

x	f(x)
3.0	2.5
4.5	1.0
7.0	2.5
9.0	0.5

Quadratic Spline Interpolation

Solution (condition 3)

For i=2 to n:

$$\boxed{2a_{i-1}(x_{i-1}) + b_{i-1} = 2a_i(x_{i-1}) + b_i}$$

$$9a_1 + b_1 = 9a_2 + b_2$$
$$14a_2 + b_2 = 14a_3 + b_3$$

x	f(x)
3.0	2.5
4.5	1.0
7.0	2.5
9.0	0.5

Quadratic Spline Interpolation

8 Equations:

$$20.25a_1 + 4.5b_1 + c_1 = 1.0$$
$$20.25a_2 + 4.5b_2 + c_2 = 1.0$$
$$49a_2 + 7b_2 + c_2 = 2.5$$
$$49a_3 + 7b_3 + c_3 = 2.5$$
$$9a_1 + 3b_1 + c_1 = 2.5$$
$$81a_3 + 9b_3 + c_3 = 0.5$$
$$9a_1 + b_1 = 9a_2 + b_2$$
$$14a_2 + b_2 = 14a_3 + b_3$$

Quadratic Spline Interpolation

Matrix form:

$$
\begin{bmatrix}
4.5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 20.25 & 4.5 & 1 & 0 & 0 & 0 \\
0 & 0 & 49 & 7 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 49 & 7 & 1 \\
3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 81 & 9 & 1 \\
1 & 0 & -9 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 14 & 1 & 0 & -14 & -1 & 0
\end{bmatrix}
\begin{bmatrix}
b_1 \\ c_1 \\ a_2 \\ b_2 \\ c_2 \\ a_3 \\ b_3 \\ c_3
\end{bmatrix}
=
\begin{bmatrix}
1 \\ 1 \\ 2.5 \\ 2.5 \\ 2.5 \\ 0.5 \\ 0 \\ 0
\end{bmatrix}
$$

Quadratic Spline Interpolation

Solution

- The matrix can be solved using one of the techniques we studied earlier and it will yield the following results:

$a_1 = 0$	$b_1 = -1$	$c_1 = 5.5$
$a_2 = 0.64$	$b_2 = -6.76$	$c_2 = 18.46$
$a_3 = -1.6$	$b_3 = 24.6$	$c_3 = -91.3$

Quadratic Spline Interpolation

Solution

- We can now substitute these values into the original quadratic equation:

$$f_1(x) = -x + 5.5 \qquad 3.0 \le x \le 4.5$$
$$f_2(x) = 0.64x^2 – 6.76x + 18.46 \qquad 4.5 \le x \le 7.0$$
$$f_3(x) = -1.6x^2 + 24.6x – 91.3 \qquad 7.0 \le x \le 9.0$$

$$f_2(5) = 0.64(5)^2 – 6.76(5) + 18.46$$
$$= 0.66$$