Optimization

CMSC 150 Numerical and Symbolic Computation

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Lecture Outline

I. Introduction to Scientific Computing
II. Approximation and Error Analysis
III. Linear Systems
IV. Interpolation and Curve Fitting
V. Optimization
    A. Introduction
    B. Unconstrained (One & Multidimensional)
    C. **Constrained (Graphical and Simplex Method)**
VI. Roots of Equations
VII. Numerical Differentiation and Integration

Introduction to Linear Programming

- Linear Programming
  - an optimization approach that deals with meeting a desired objective such as maximizing profit or minimizing cost in the presence of constraints such as limited resources
  - Linear: mathematical functions representing the objective and constraints
  - Programming: scheduling or setting an agenda

Introduction to Linear Programming

- Linear Programming
  - For maximization problem:
  - Objective Function
  - Maximize $Z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$

  where c_j = payoff of each unit of the jth activity

  x_j = magnitude of the jth activity

- THUS, the value of the objective function z is the total payoff due to the total number of activities n.

Introduction to Linear Programming

- Linear Programming
  - Constraints

$$a_{i1}\,x1 + a_{i2}\,x_2 + \ldots + a_{\mathrm{in}}\,x_n \le b_i$$

where a_{ij} = amount of the ith resource needed for each jth activity

b_i = amount of the available ith resource

- Nonnegativity constraints X_i >= 0

Linear Programming

- Setting up Linear Programming

The engineer operating this plant must decide how much of each gas to produce to maximize profits. If the amount of regular and premium produced weekly are designated as x1 and x2, respectively, the total weekly profit can be calculated as $Total profit = 150\,x_1 + 175\,x_2$

	Products		
Resource	Regular	Premium	Resource Availability
Raw Gas	7 m^3/ton	11 m^3/ton	77 m^3/week
Production time	10 hr/ton	8 hr/ton	80 hr/week
Storage	9 tons	6 tons	
Profit	150/ton	175/ton	

Linear Programming

- Setting up Linear Programming

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$$\boxed{Maximize\ Z = 150x_1 + 175x_2}$$

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Linear Programming

- Constraints

$$\boxed{Maximize\ Z = 150\,x_1 + 175\,x}$$

material constraints $7\,x_1 + 11\,x_2 \leq 77$

time constraints $10\,x_1 + 8\,x_2 \leq 80$

regular storage constraints $x_1 \leq 9$

premium storage constraints $x_2 \leq 6$

positivity constraints $x_1\,,\,x_2 \geq 0$

Linear Programming

- Constraints

$$Maximize\ Z = 150x_1 + 175x_2$$

$$7x_1 + 11x_2 + S1 = 77 \qquad x_1 + S3 = 9$$

$$10x_1 + 8x_2 + S2 = 80 \qquad x_2 + S4 = 6$$

material constraints $7x_1 + 11x_2 \leq 77$

time constraints $10x_1 + 8x_2 \leq 80$

regular storage constraints $x_1 \leq 9$

premium storage constraints $x_2 \leq 6$

positivity constraints $x_1, x_2 \geq 0$

Simplex Method

- Step 1: Using slack variables, convert LP problem to a system of linear equations.
- Step 2: Set up initial tableau. Derive basic solution.
- Step 3: Select pivot column.
- Step 4: Select pivot element.
- Step 5: Use the pivot to clear pivot column using Gauss-Jordan.
- Step 6: Repeat steps 3-5 until there are no more negative numbers in the bottom row.
- The solution for the LP problem is the basic solution from the final tableau.

Simplex Method

- Step 1: Using slack variables, convert LP problem to a system of linear equations.
- Step 2: Set up initial tableau. Derive basic solution.
  - Look for the columns that are **cleared** (all zeros except for one entry). We assign to the corresponding variable the test ratio. We call these variables the **active** variables. All variables whose columns are not cleared are assigned zero and are called **inactive**.

Simplex Method

$$Maximize\ Z = 150x_1 + 175x_2$$

– Step 1 & 2

x1	x2	S1	S2	S3	S4	Z	Answer
7	11	1	0	0	0	0	77
10	8	0	1	0	0	0	80
1	0	0	0	1	0	0	9
0	1	0	0	0	1	0	6
-150	-175	0	0	0	0	1	0

Basic Solution			
S1=77	S2=80	S3=9	S4=6
x1=0	x2=0	Z=0	

Simplex Method

- Step 3: Select pivot column.
  - Look at all the numbers in the bottom row, excluding the Answer column. From these, choose the **negative number with the largest magnitude**. Its column is the pivot column. (If there are two candidates, choose either one.) If all the numbers in the bottom row are zero or positive, then computation is done. The basic solution is the optimal solution.

Simplex Method

- Step 4: Select pivot element.
  - 1) The pivot must always be a positive number. (This rules out zeros and negative numbers.)
  - 2) For each positive entry b in the pivot column, compute the ratio **a/b**, where a is the number in the rightmost column in that row. This is called the **test ratio**.
  - 3) Among these ratios, choose the **smallest** one. The corresponding number b is the pivot element.

Simplex Method

$$Maximize\, Z = 150x_1 + 175x_2$$

– Step 3 & 4

$$Pivot\, Col = X2$$

$$Pivot\, Element = 1$$

x1	x2	S1	S2	S3	S4	Z	Answer
7	11	1	0	0	0	0	77
10	8	0	1	0	0	0	80
1	0	0	0	1	0	0	9
0	$\boxed{1}$	0	0	0	1	0	6
-150	-175	0	0	0	0	1	0

Basic Solution			
S1=77	S2=80	S3=9	S4=6
x1=0	x2=0	Z=0	

Simplex Method

$$Maximize\, Z = 150\,x_1 + 175\,x_2$$

– Step 5 & 2
(basic solution)

x1	x2	S1	S2	S3	S4	Z	Answer
7	0	1	0	0	-11	0	11
10	0	0	1	0	-8	0	32
1	0	0	0	1	0	0	9
0	1	0	0	0	1	0	6
-150	0	0	0	0	175	1	1050

Basic Solution			
S1=11	S2=32	S3=9	S4=0
x1=0	x2=6	Z=1050	

Simplex Method

$$Maximize\ Z = 150x_1 + 175x_2$$

– Step 3 & 4

$Pivot\ Col = X1$

$Pivot\ Element = 7$

x1	x2	S1	S2	S3	S4	Z	Answer
7	0	1	0	0	-11	0	11
10	0	0	1	0	-8	0	32
1	0	0	0	1	0	0	9
0	1	0	0	0	1	0	6
-150	0	0	0	0	175	1	1050

Basic Solution			
S1=11	S2=32	S3=9	S4=0
x1=0	x2=6	Z=1050	

Simplex Method

$$Maximize\; Z = 150\,x_1 + 175\,x_2$$

– Step 5 & 2
(basic solution)

x1	x2	S1	S2	S3	S4	Z	Answer
1	0	0.14285	0	0	-1.57142	0	1.57142
0	0	-1.4285	1	0	7.7142	0	16.2858
0	0	-0.14285	0	1	1.57142	0	7.42858
0	1	0	0	0	1	0	6
0	0	21.4275	0	0	-60.713	1	1285.713

Basic Solution			
S1=0	S2=16.2858	S3=7.42858	S4=0
x1=1.57142	x2=6	Z=1285.713	

Simplex Method

$$Maximize\ Z = 150x_1 + 175x_2$$

– Step 3 & 4

$$Pivot\ Col = X1$$
$$Pivot\ Element = 7$$

x1	x2	S1	S2	S3	S4	Z	Answer
1	0	0.14285	0	0	-1.57142	0	1.57142
0	0	-1.4285	1	0	7.7142	0	16.2858
0	0	-0.14285	0	1	1.57142	0	7.42858
0	1	0	0	0	1	0	6
0	0	21.4275	0	0	-60.713	1	1285.713

Basic Solution			
S1=0	S2=16.2858	S3=7.42858	S4=0
x1=1.57142	x2=6	Z=1285.713	

Simplex Method

x1	x2	S1	S2	S3	S4	Z	Answer
1	0	-0.14786	0.2027	0	0	0	4.888
0	0	-0.185	0.129	0	1	0	2.111
0	0	0.1478	-0.2027	1	0	0	4.111
0	1	0.185	-0.129	0	0	0	3.889
0	0	10.1955	7.8319	0	0	1	1413.878

Basic Solution			
S1=0	S2=0	S3=4.111	S4=2.111
x1=4.888	x2=3.889	Z=1413.878	

Since all the elements in the objective function are positive, we now have our optimal solution.

Linear Programming

- Constraints

$$\boxed{Maximize\ Z = 150\,x_1 + 175\,x_2}$$

material constraints $7\,x_1 + 11\,x_2 \leq 77$

time constraints $10\,x_1 + 8\,x_2 \leq 80$

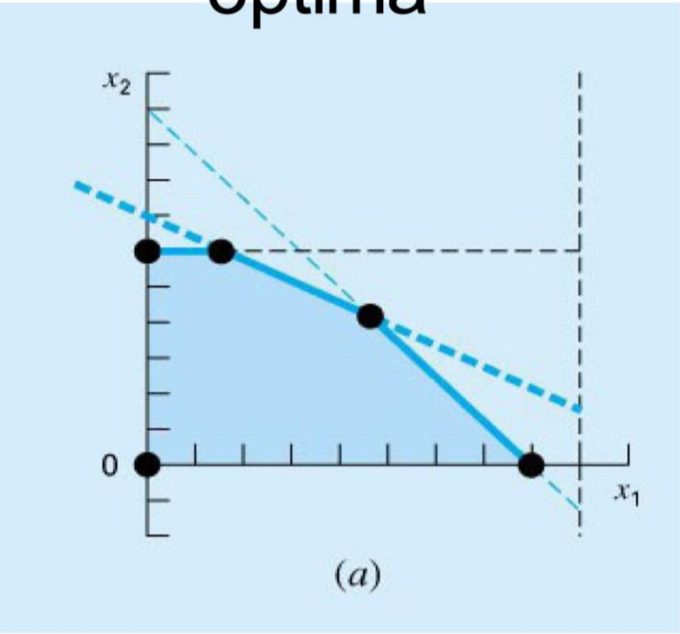
regular storage constraints $x_1 \leq 9$

premium storage constraints $x_2 \leq 6$

positivity constraints $x_1, x_2 \geq 0$

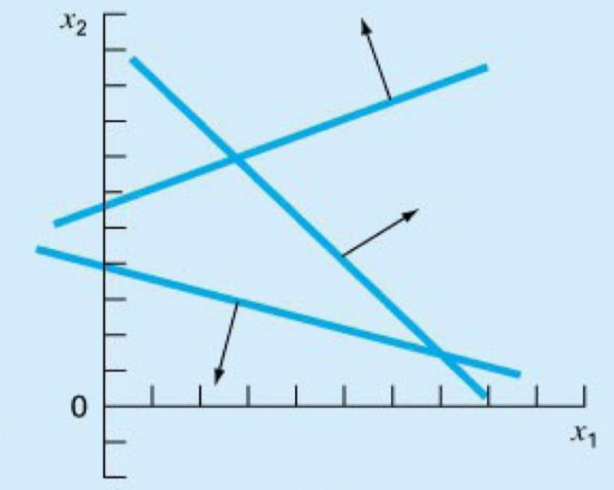
Linear Programming

- Possible Outcomes
  - **Unique Solution** - the maximum objective function intersects a single point
  - Alternate Solution - obtain an infinite number of optima



Linear Programming

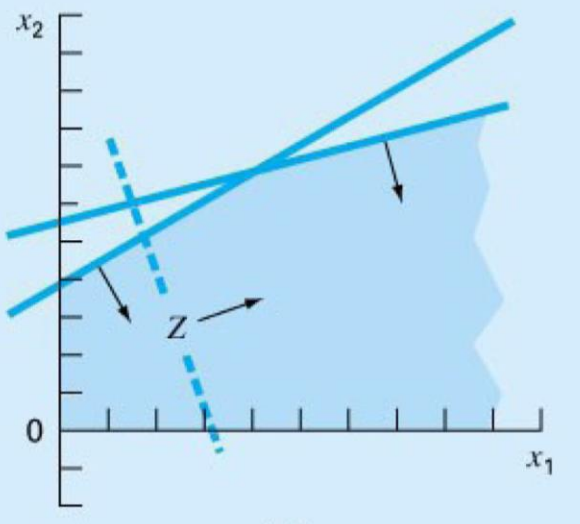
- Possible Outcomes
  - **No Feasible Solution** – can be caused by error in setting up the problem. This may also be caused if is over-constrained where no solution can satisfy all the constraints.



(b)

Linear Programming

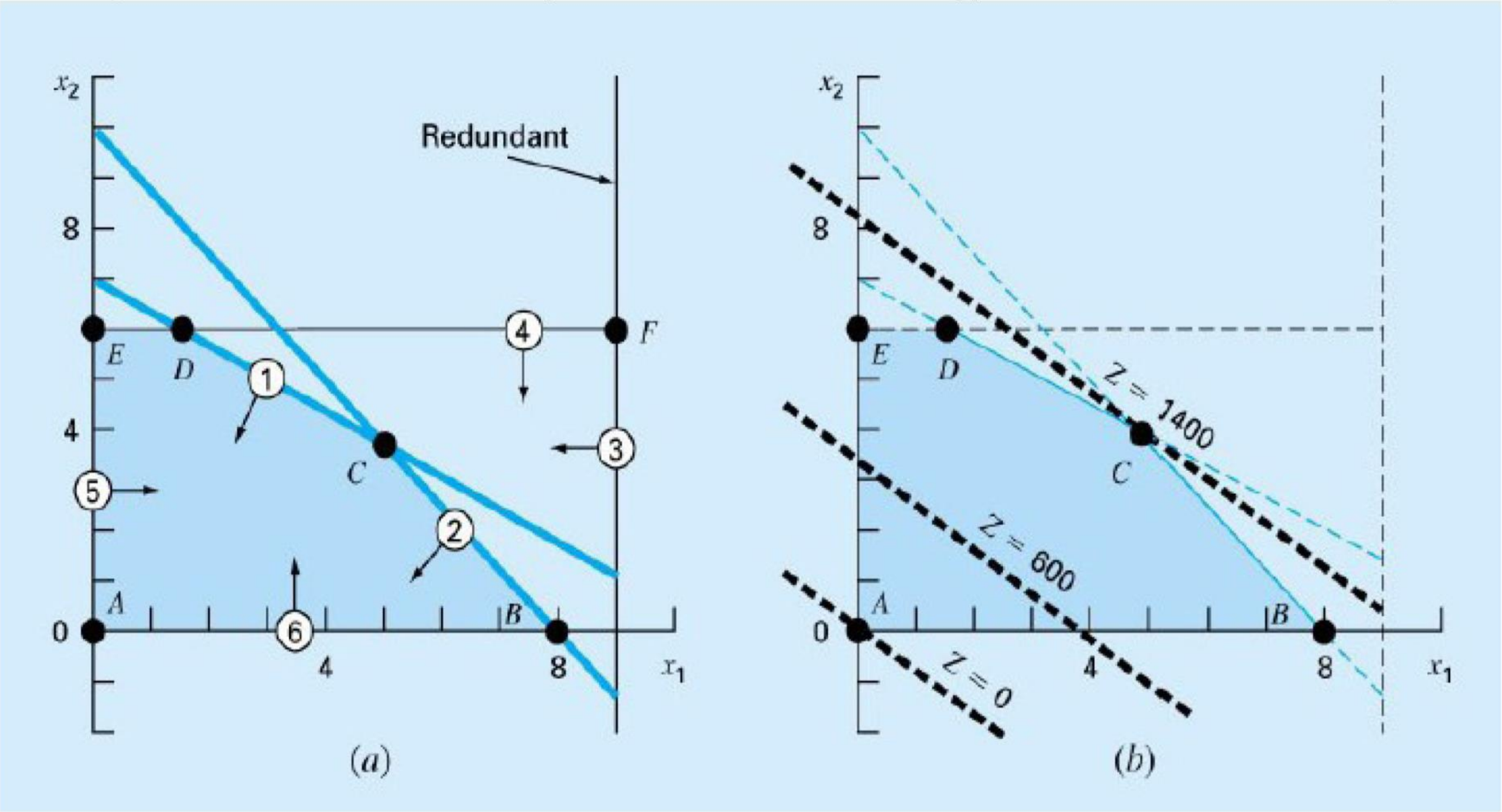
- Possible Outcomes
  - **Unbounded Problems** – the problem is under-constrained and therefore, open-ended



(c)

Linear Programming

- Graphical Solution (Iso-Profit Line || Corner-Points)



Linear Programming

- A little history
  - Linear Programming was conceptually developed before World War II, by an outstanding Soviet mathematician, A.N. Kolmogorov.
  - An application was develop for the well known diet problem by Stigler in 1945.
  - As LP advance over the year, George Dantzig developed a solution procedure known as simplex algorithm.