Fundamentals of Image Processing

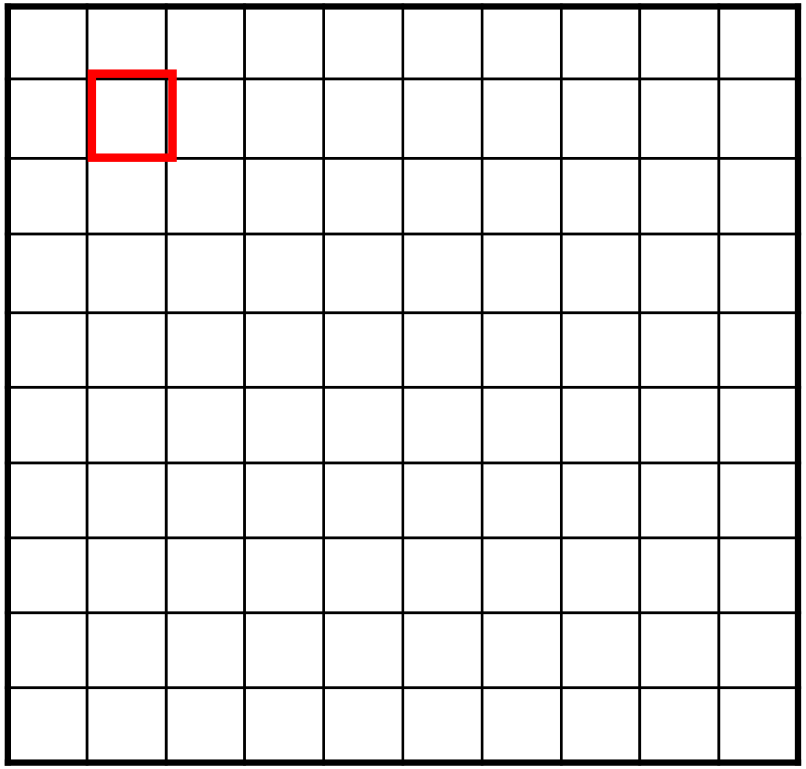
Review – Spatial Filtering

$$g[\cdot,\cdot]\ \frac{1}{9}\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$$

$f[.,.]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$h[.,.]$



$$h[m,n]=\sum_{k,l} g[k,l]\, f[m+k,n+l]$$

Slide credit: S. Seitz

Review – Spatial Filtering

$g[\cdot\,,\cdot\,]\ \frac{1}{9}$

1	1	1
1	1	1
1	1	1

$f[.,.]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$h[.,.]$

	0	10							

$$h[m,n]=\sum_{k,l} g[k,l]\, f[m+k,n+l]$$

Slide credit: S. Seitz

Review – Spatial Filtering

$$g[\cdot,\cdot]\ \frac{1}{9}\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}$$

$f[.,.]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$h[.,.]$

	0	10	20						

$$h[m,n]=\sum_{k,l} g[k,l]\, f[m+k,n+l]$$

Slide credit: S. Seitz

Review – Spatial Filtering

$$g[\cdot\,,\cdot\,]\ \frac{1}{9}\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}$$

$f[.,.]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$h[.,.]$

	0	10	20	30					

$$h[m,n]=\sum_{k,l} g[k,l]\, f[m+k,n+l]$$

Slide credit: S. Seitz

Review – Spatial Filtering

$$g[\cdot\,,\cdot\,]\ \frac{1}{9}\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$$

$$f[.,.]$$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$$h[.,.]$$

	0	10	20	30	30				

$$h[m,n]=\sum_{k,l} g[k,l]\, f[m+k,n+l]$$

Slide credit: S. Seitz

Review – Spatial Filtering

$g[\cdot,\cdot]\ \frac{1}{9}$

1	1	1
1	1	1
1	1	1

$f[.,.]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$h[.,.]$

	0	10	20	30	30	30	20	10	
	0	20	40	60	60	60	40	20	
	0	30	60	90	90	90	60	30	
	0	30	50	80	80	90	60	30	
	0	30	50	80	80	90	60	30	
	0	20	30	50	50	60	40	20	
	10	20	30	30	30	30	20	10	
	10	10	10	0	0	0	0	0	

$$h[m,n] = \sum_{k,l} g[k,l]\, f[m+k, n+l]$$

Slide credit: S. Seitz

Why does a lower resolution image still make sense to us? What do we lose?

Image: http://www.flickr.com/photos/igorms/136916757/

Slide credit: D. Hoiem

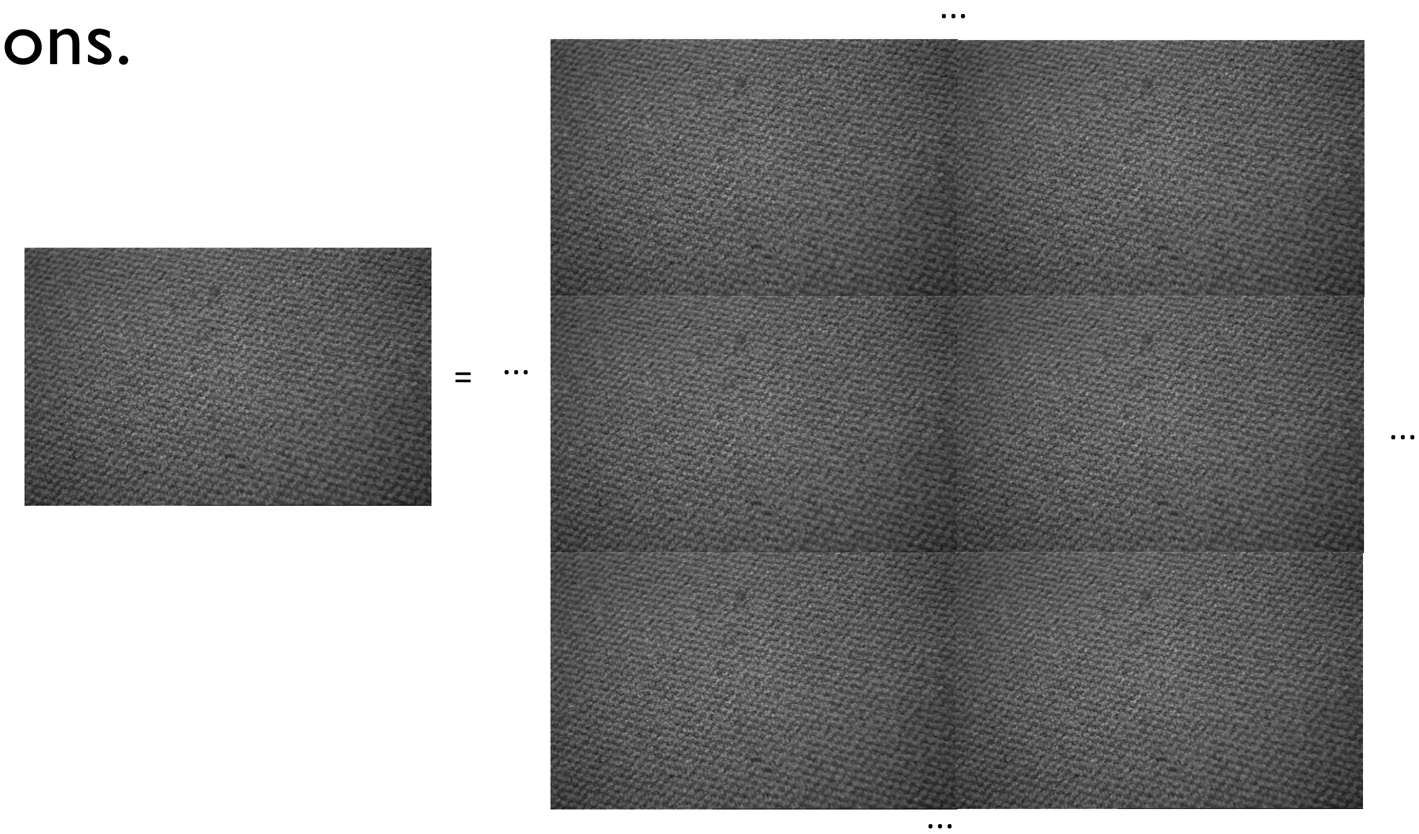
How is it that a 4MP image can be compressed to a few hundred KB without a noticeable change?

Slide credit: J. Hays

Answer to these questions?

- Thinking images in terms of frequency.

- Treat images as infinite-size, continuous periodic functions.



Jean Baptiste Joseph Fourier (1768-1830)

had idea (1807):

Any *univariate function can be rewritten as a weighted sum of sines and cosines of different frequencies.*

Slide credit: A. Efros

Jean Baptiste Joseph Fourier (1768-1830)

had an idea (1807):

Any *univariate function can be rewritten as a weighted sum of sines and cosines of different frequencies.*

- Don't believe it?
  - Neither did Lagrange, Laplace, Poisson and other big wigs
  - Not translated into English until 1878!
- But it's (mostly) true!
  - called Fourier Series
  - there are some subtle restrictions

Slide credit: A. Efros

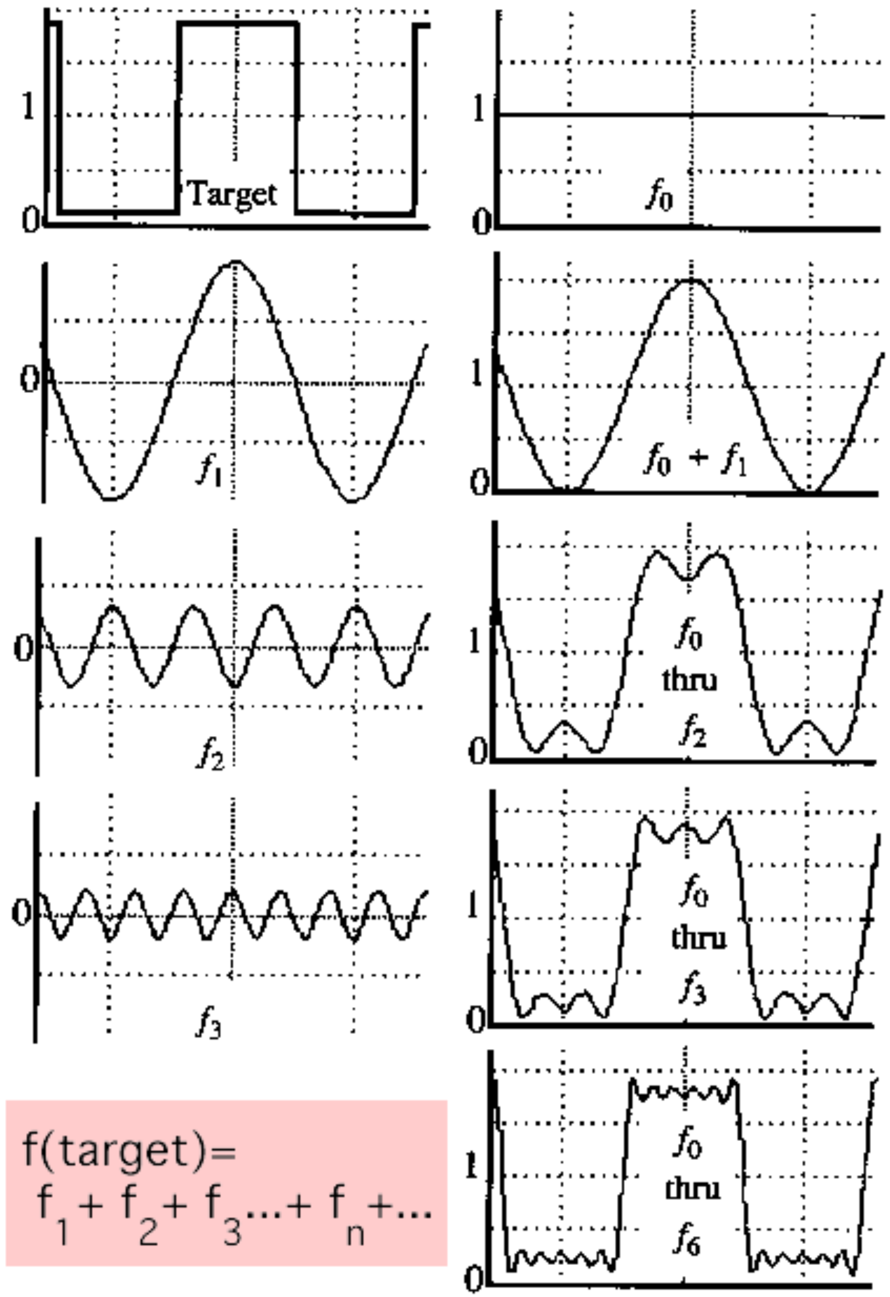
A sum of sines

Our building block:

$$A\sin(\omega x+\phi)$$

The Fourier theory shows how most real function can be represented in terms of a basis of sinusoids.

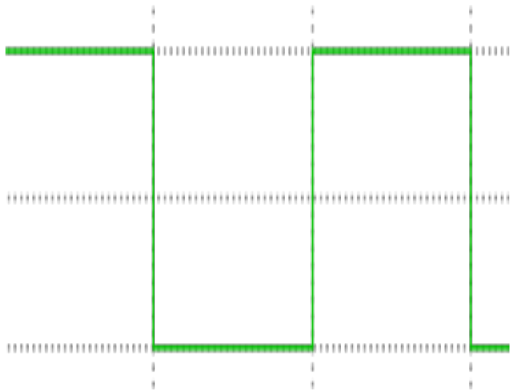
Add enough of them to get any signal *f(x)* you want.



f(target)=
$f_1 + f_2 + f_3 ... + f_n + ...$

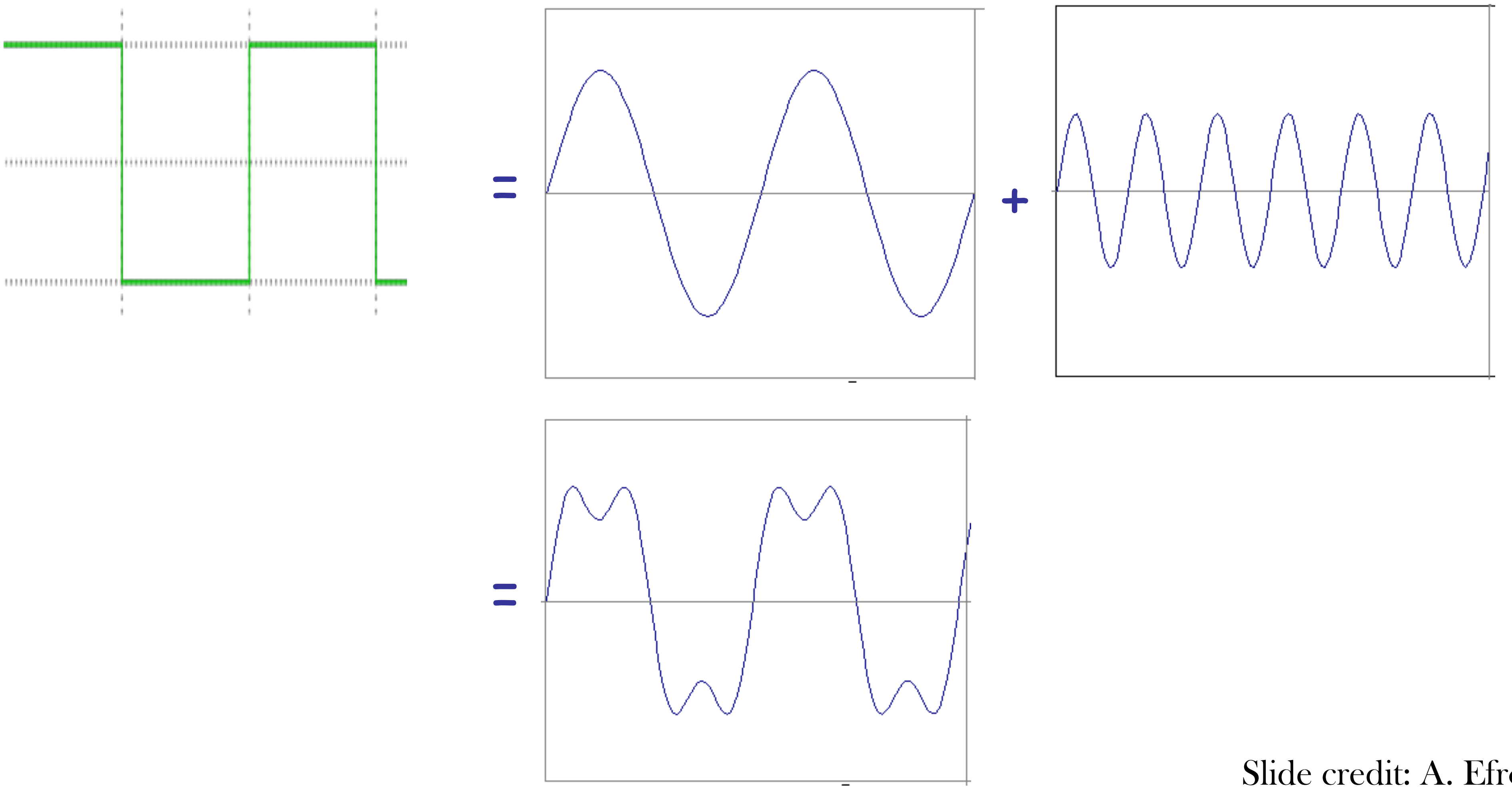
Slide credit: A. Efros

Frequency Spectra



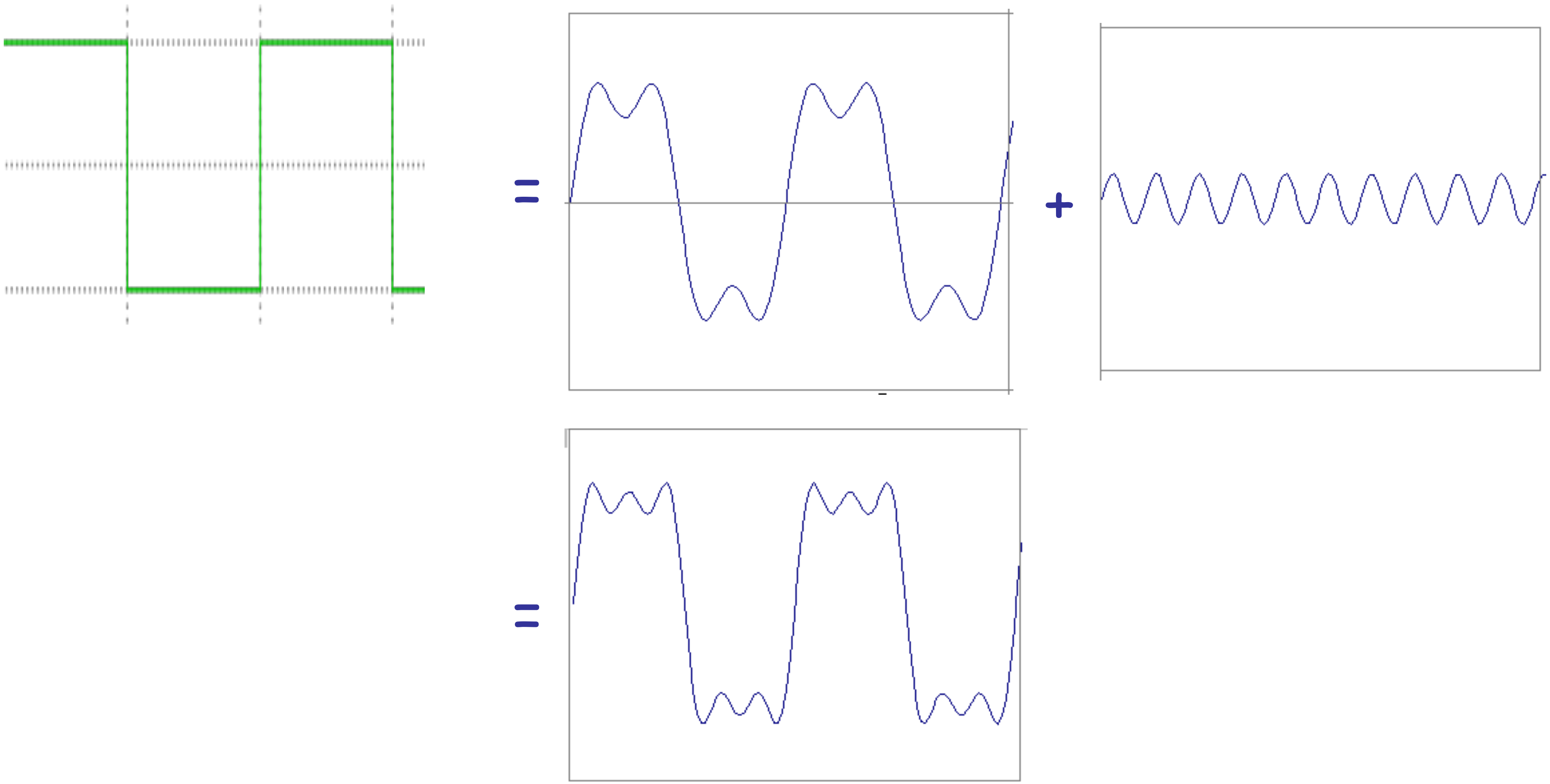
Slide credit: A. Efros

Frequency Spectra



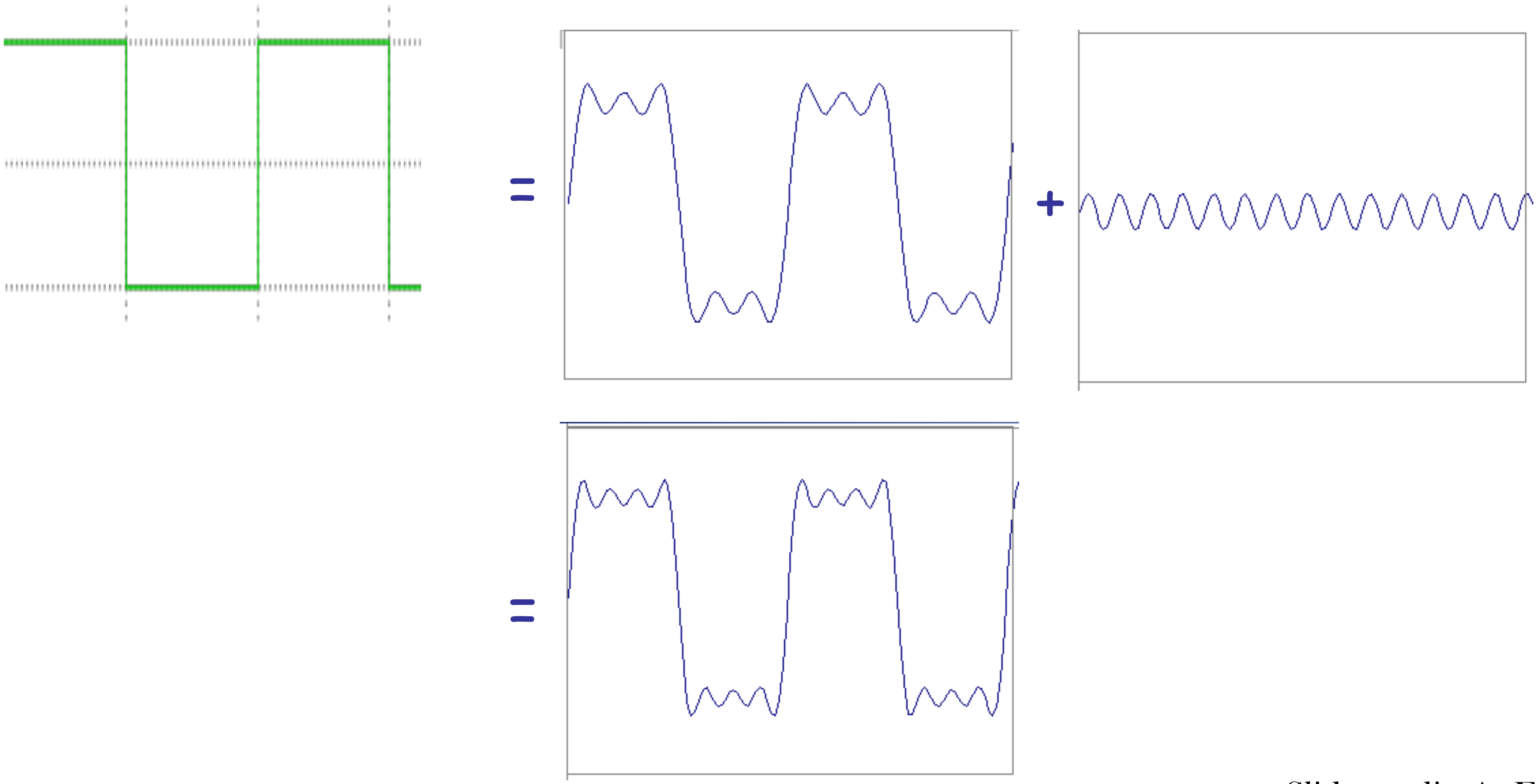
Slide credit: A. Efros

Frequency Spectra



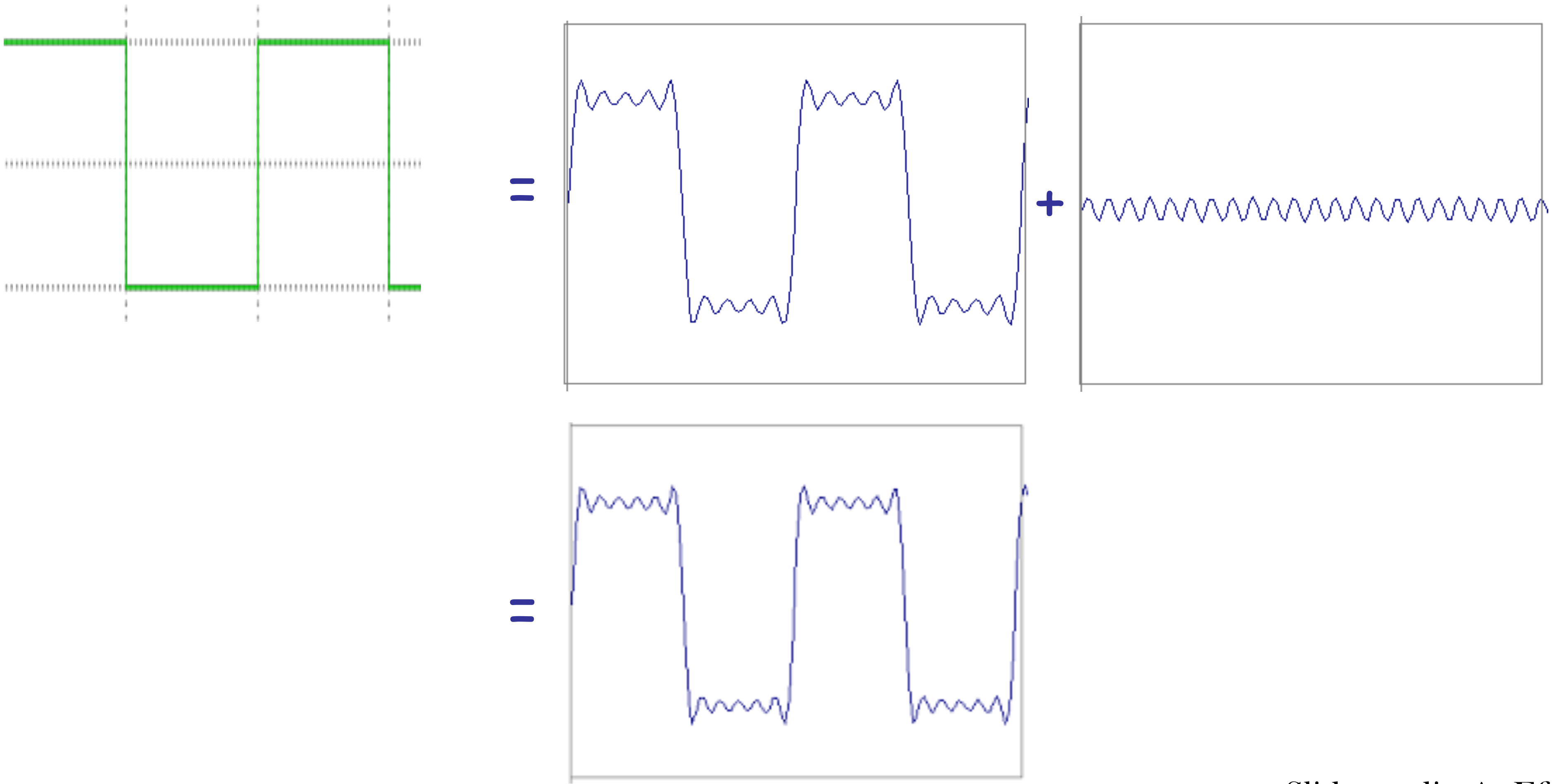
Slide credit: A. Efros

Frequency Spectra



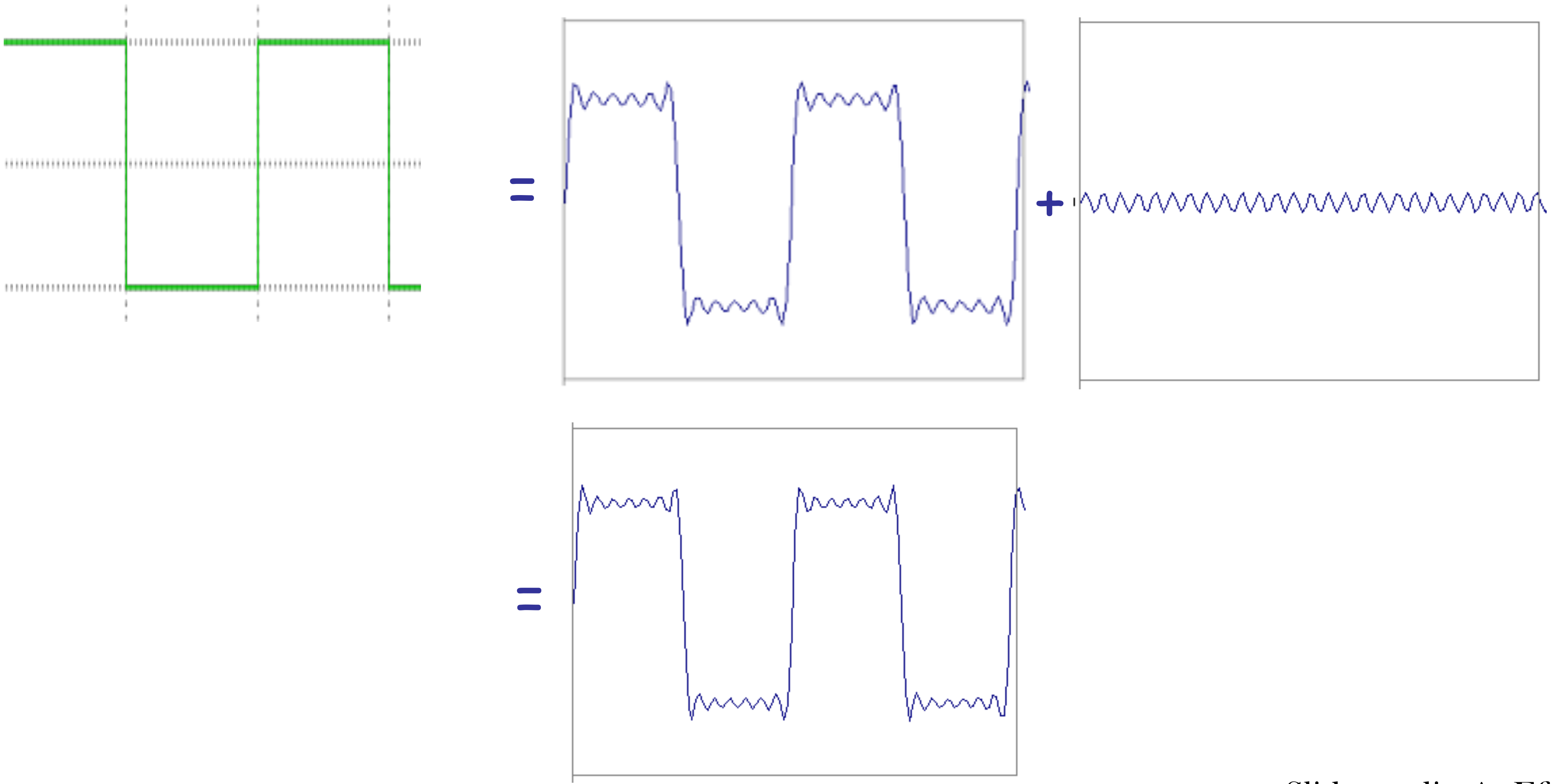
Slide credit: A. Efros

Frequency Spectra



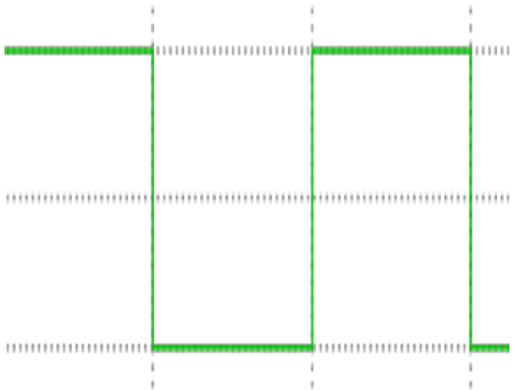
Slide credit: A. Efros

Frequency Spectra

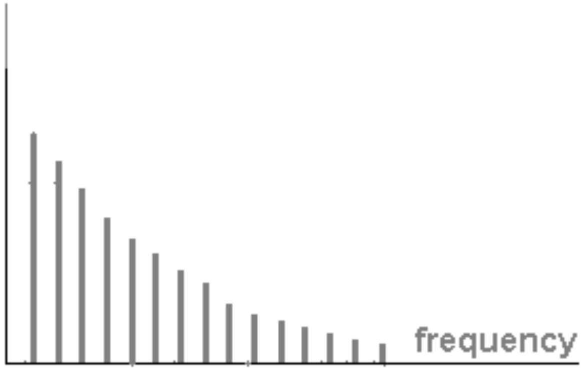


Slide credit: A. Efros

Frequency Spectra



$$= \boxed{A\sum_{k=1}^{\infty}\frac{1}{k}\sin(2\pi kt)}$$



Slide credit: A. Efros

Frequency Spectra

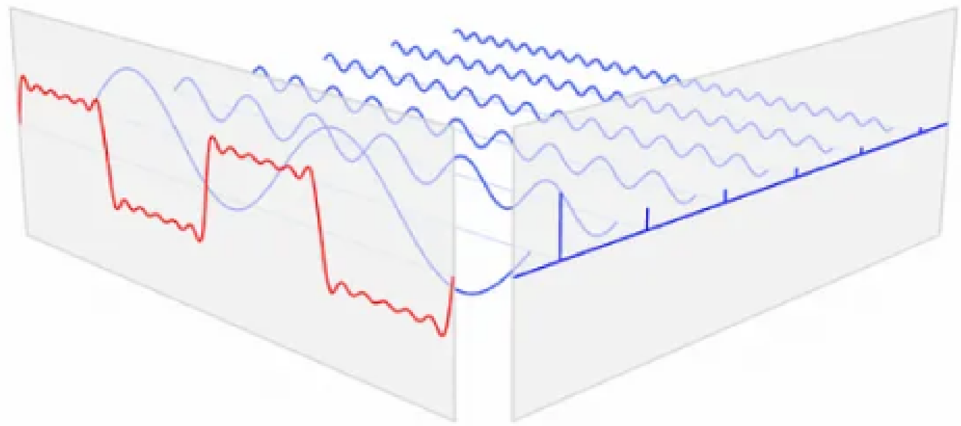
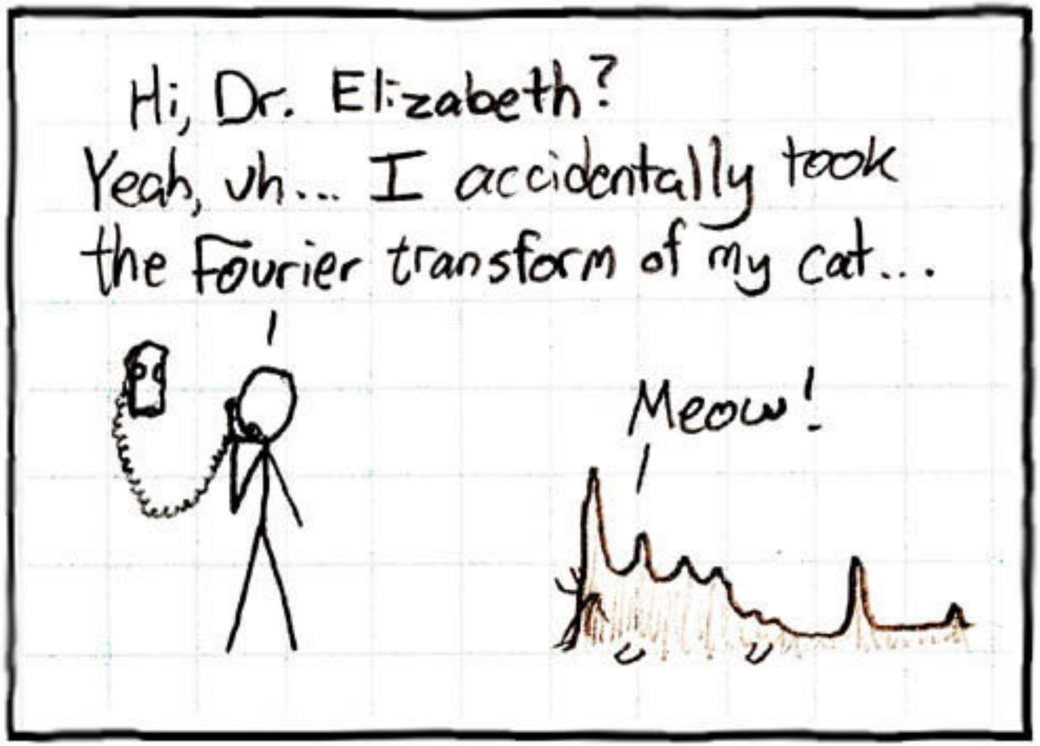


Image credit: Lucas V. Barbosa

Other signals

- We can also think of all kinds of other signals the same way



xkcd.com

Slide credit: J. Hays

Fourier Transform

We want to understand the frequency *w* of our signal. So, let's reparametrize the signal by *w* instead of *x*:

f(x) → Fourier Transform → ***F(w)***

For every *w* from 0 to inf, ***F(w)*** holds the amplitude *A* and phase *f* of the corresponding sine $A\sin(\omega x+\phi)$

- How can *F* hold both? Complex number trick!

$$F(\omega)=R(\omega)+iI(\omega)$$

$$A=\pm\sqrt{R(\omega)^2+I(\omega)^2} \qquad \phi=\tan^{-1}\frac{I(\omega)}{R(\omega)}$$

We can always go back:

F(w) → Inverse Fourier Transform → ***f(x)***

Slide credit: A. Efros

Fourier Transform

- Fourier transform stores the magnitude and phase at each frequency
  - Magnitude encodes how much signal there is at a particular frequency
  - Phase encodes spatial information (indirectly)
  - For mathematical convenience, this is often notated in terms of real and complex numbers

Amplitude: $A = \pm\sqrt{R(\omega)^2 + I(\omega)^2}$

Phase: $\phi = \tan^{-1} \frac{I(\omega)}{R(\omega)}$

Discrete Fourier transform

- Forward transform

$$F(u,v) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-j2\pi(ux/M + vy/N)}$$

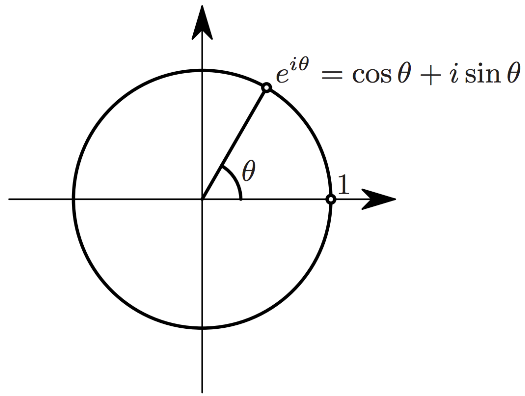
for $u = 0,1,2,\ldots,M-1, v = 0,1,2,\ldots,N-1$

- Inverse transform

$$f(x,y) = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,v) e^{j2\pi(ux/M + vy/N)}$$

for $x = 0,1,2,\ldots,M-1, y = 0,1,2,\ldots,N-1$

u, *v* : the transform or frequency variables
x, *y* : the spatial or image variables



Euler's definition of $e^{i\theta}$

Slide credit: B. Freeman and A. Torralba

The Fourier Transform

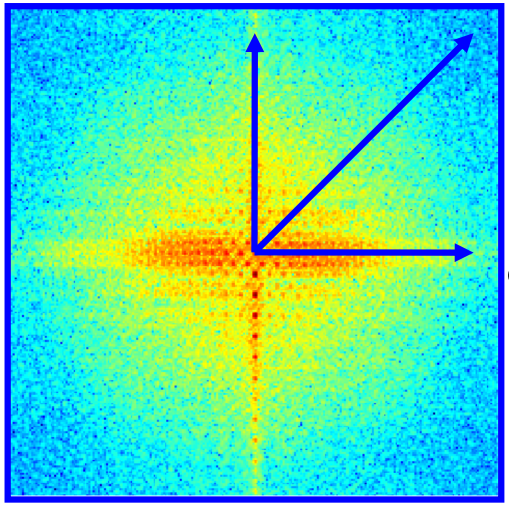
- Represent function on a new basis
  - Think of functions as vectors, with many components
  - We now apply a linear transformation to transform the basis
    - dot product with each basis element
- In the expression, u and v select the basis element, so a function of x and y becomes a function of u and v
- basis elements have the form $e^{-i2\pi(ux+vy)}$

Slide credit: S. Thrun

How to interpret 2D Fourier Spectrum

Vertical orientation

0



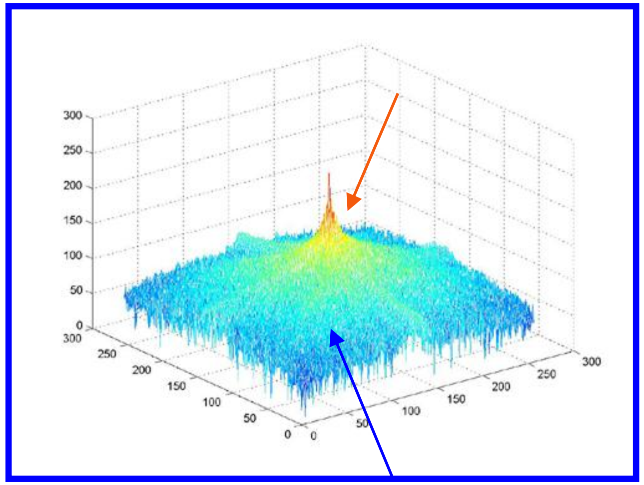
45 deg.

Horizontal orientation

0 f_{max}

fx in cycles/image

Low spatial frequencies



High spatial frequencies

Log power spectrum

Slide credit: B. Freeman and A. Torralba

Fourier basis element

$$e^{-i2\pi(ux+vy)}$$

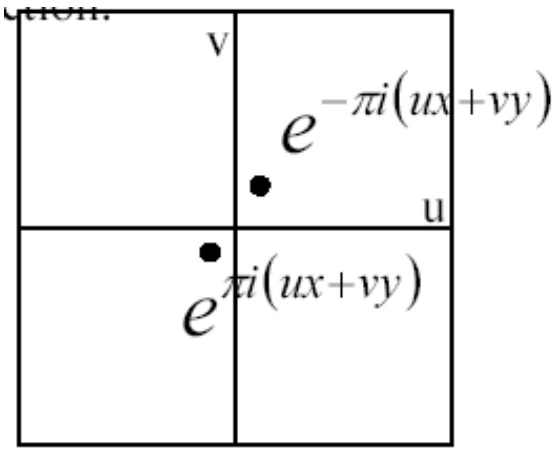
example, real part

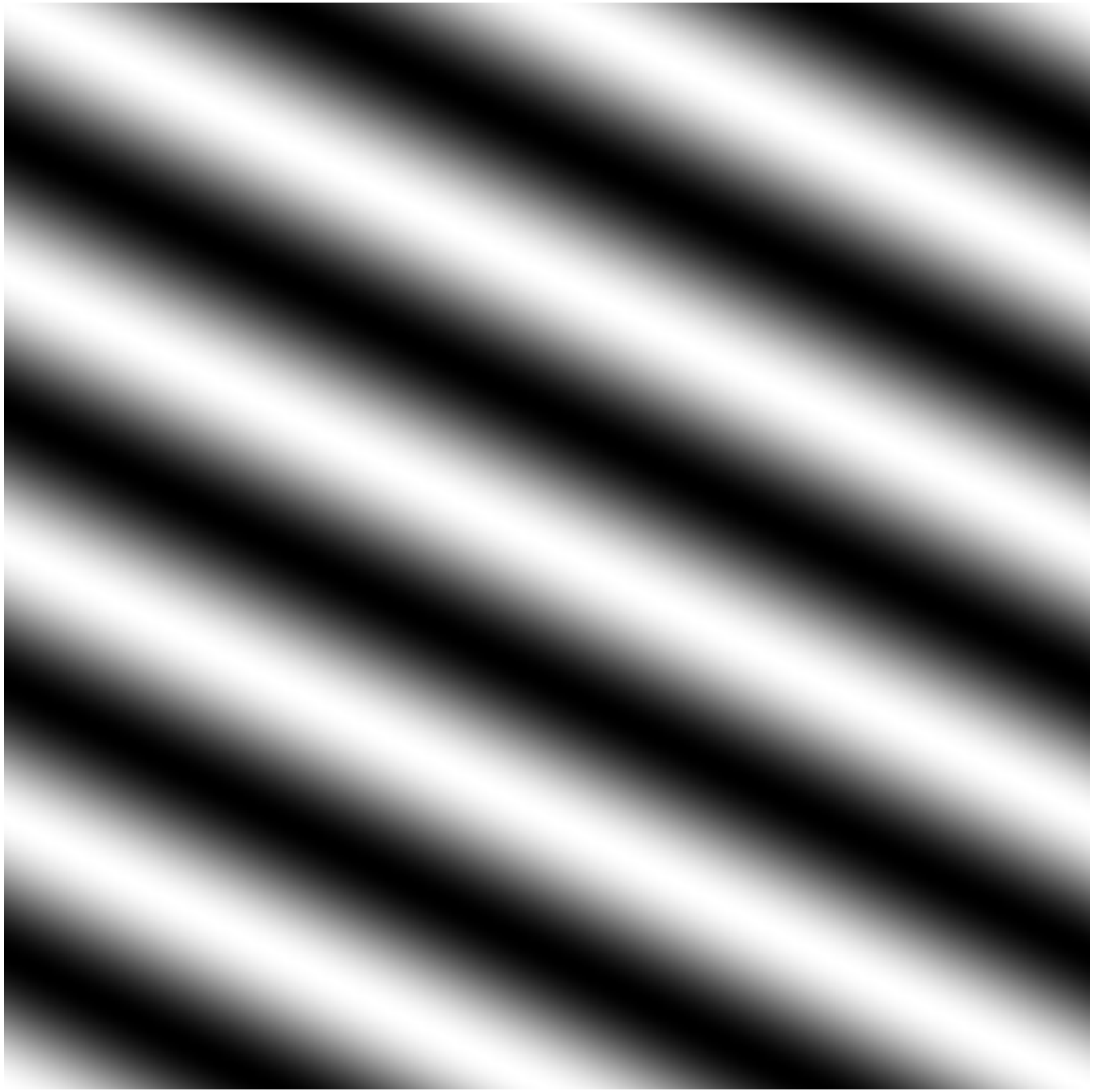
$F^{u,v}(x,y)$

$F^{u,v}(x,y)$=const. for $(ux+vy)$=const.

Vector (u,v)

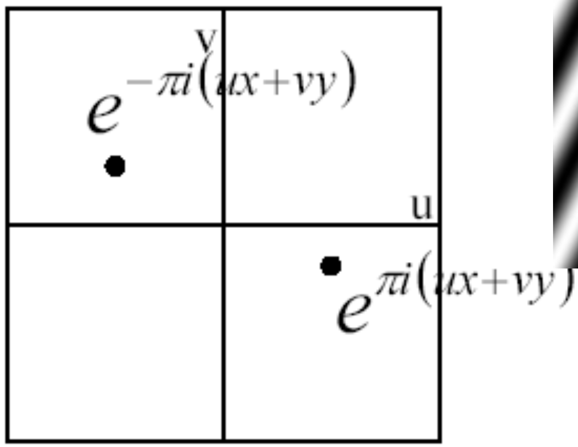
- Magnitude gives frequency
- Direction gives orientation.





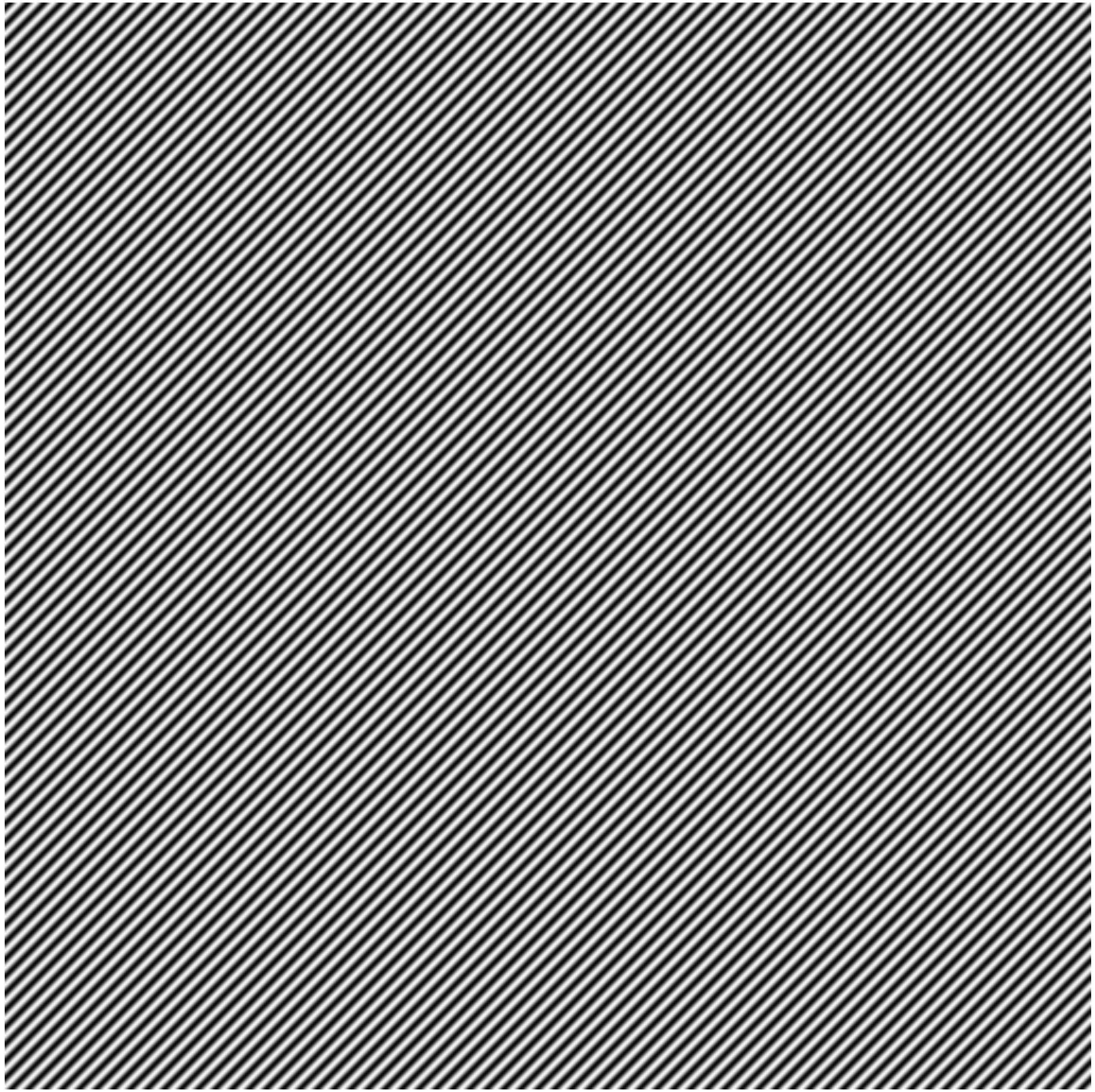
Slide credit: S. Thrun

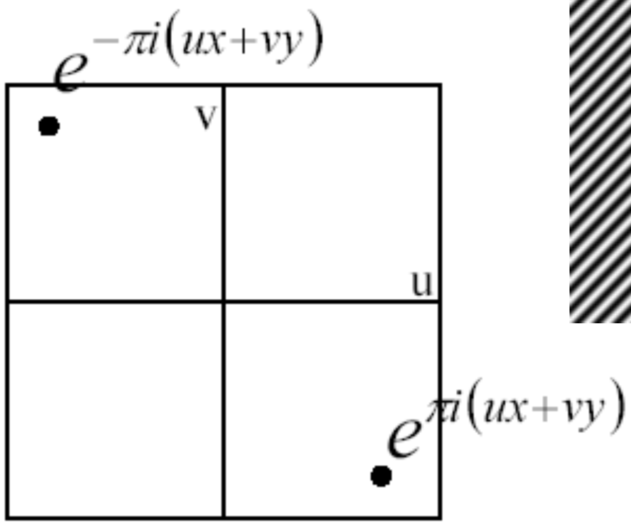
Here u and v are larger than in the previous slide.



Slide credit: S. Thrun

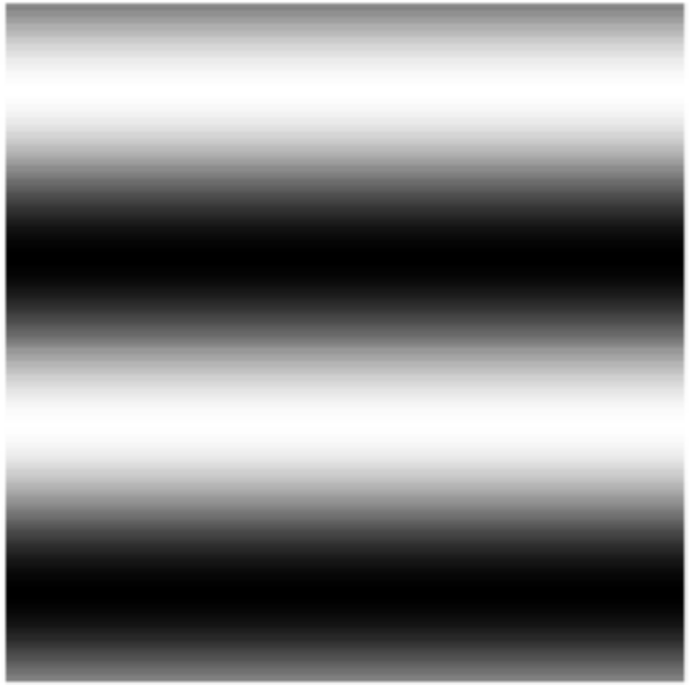
And larger still...

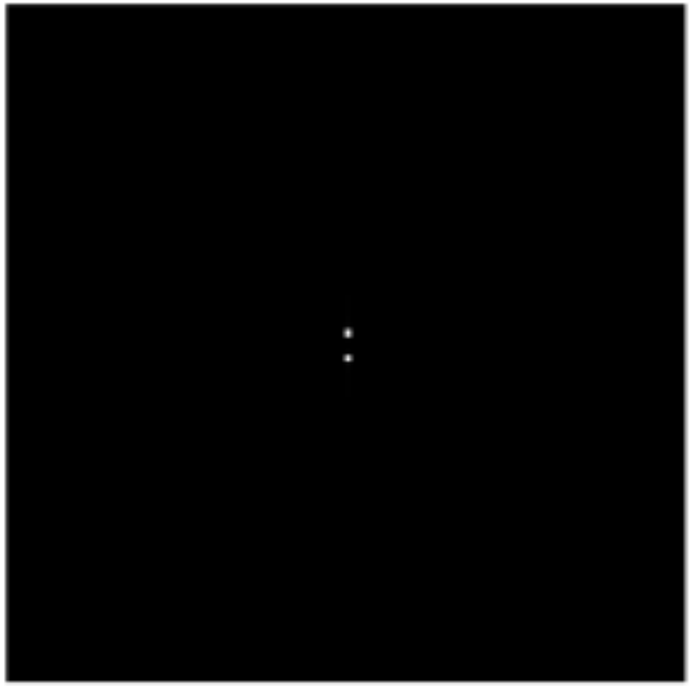




Slide credit: S. Thrun

2D FFT

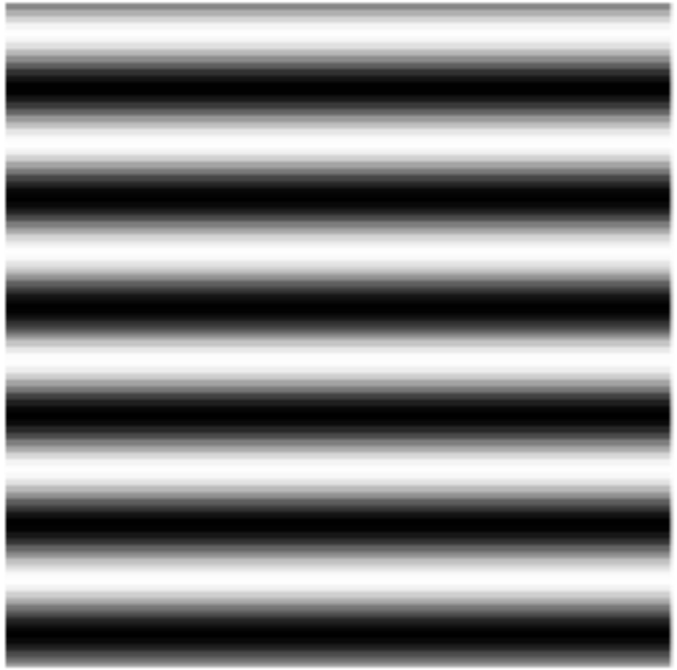


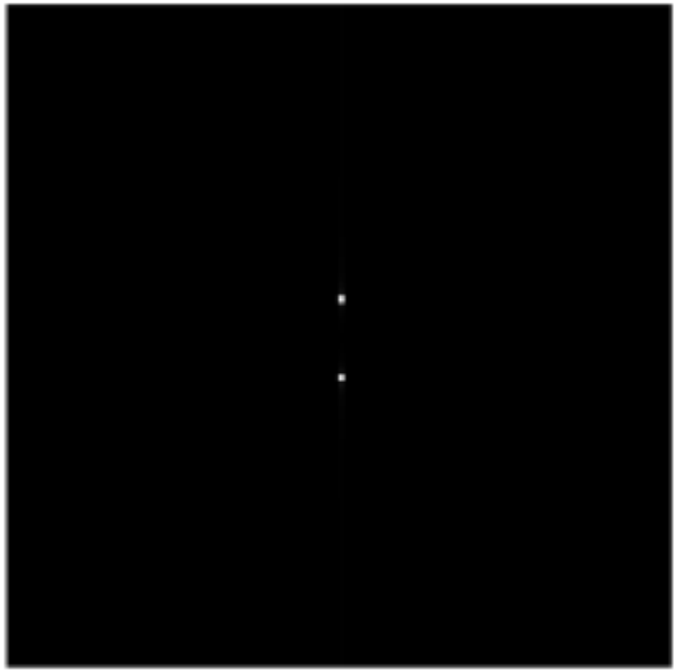


Sinusoid with frequency = 1 and its FFT

Slide credit: M. H. Yang

2D FFT

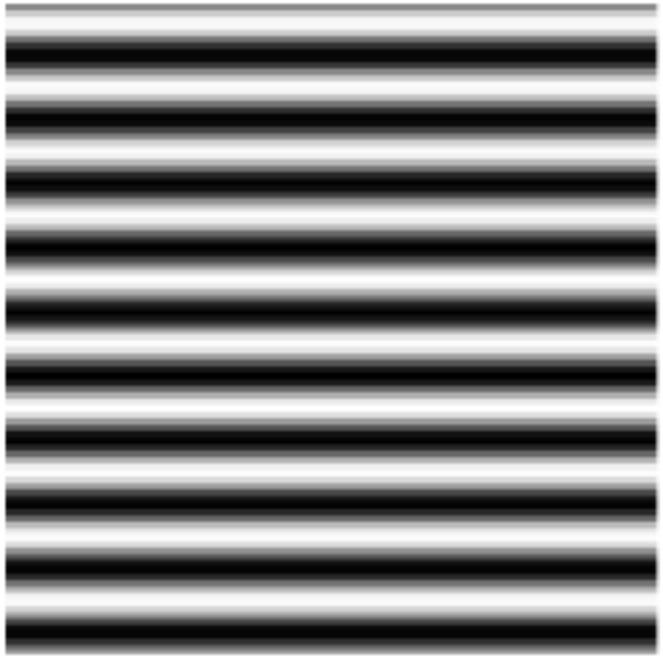


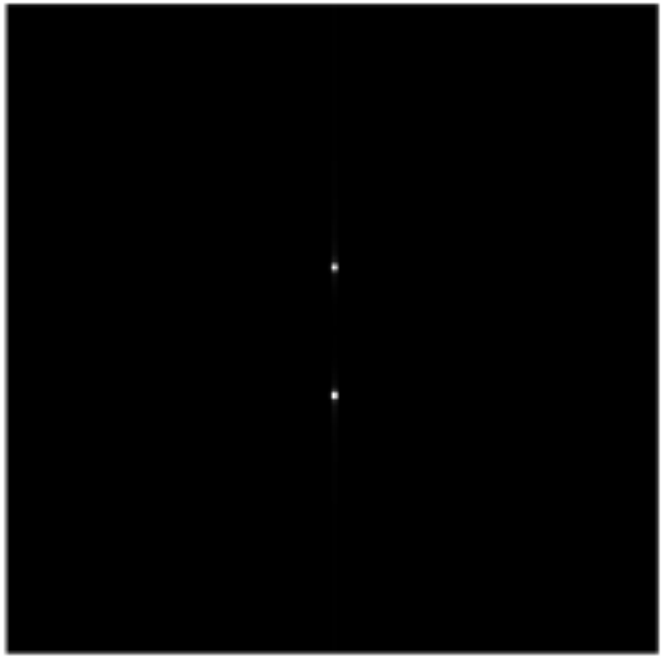


Sinusoid with frequency = 3 and its FFT

Slide credit: M. H. Yang

2D FFT

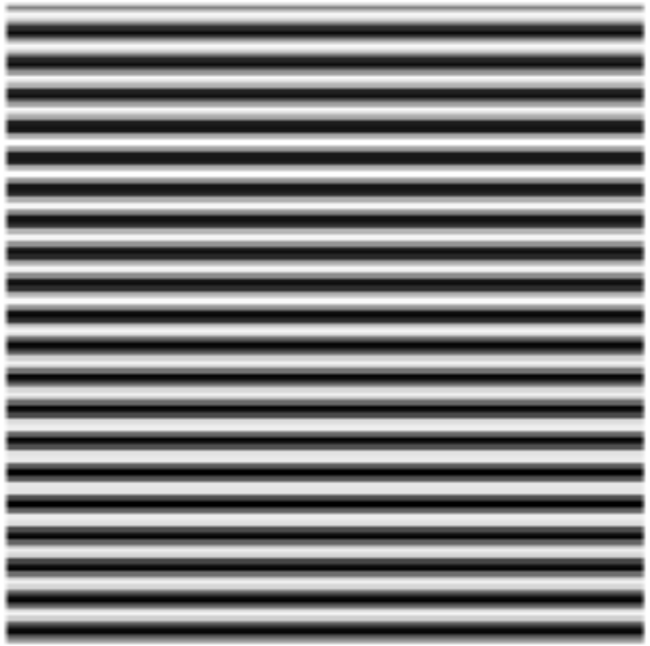


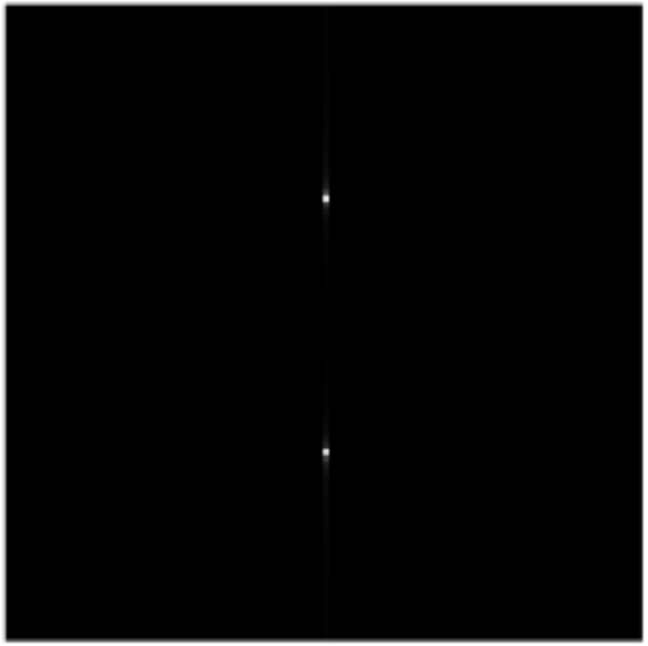


Sinusoid with frequency = 5 and its FFT

Slide credit: M. H. Yang

2D FFT

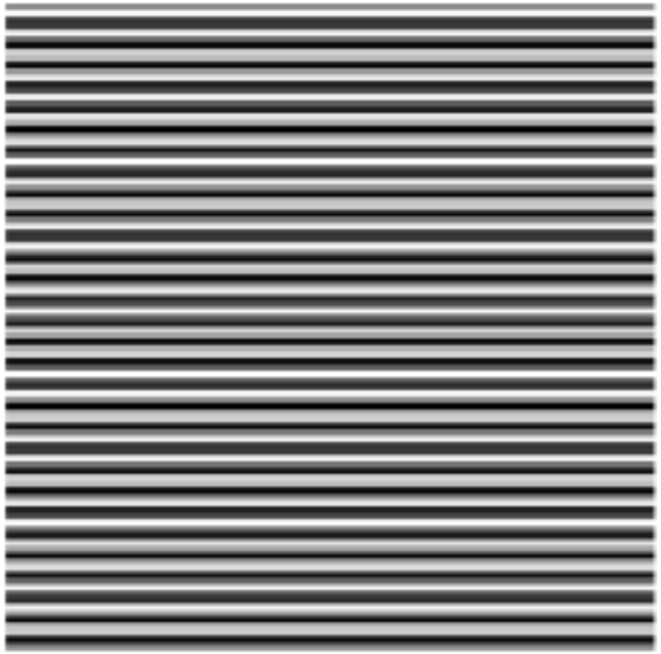


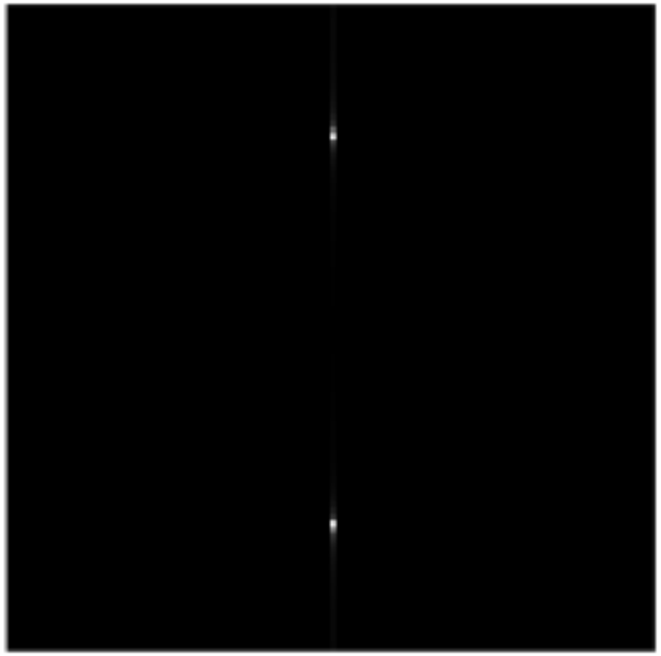


Sinusoid with frequency = 10 and its FFT

Slide credit: M. H. Yang

2D FFT

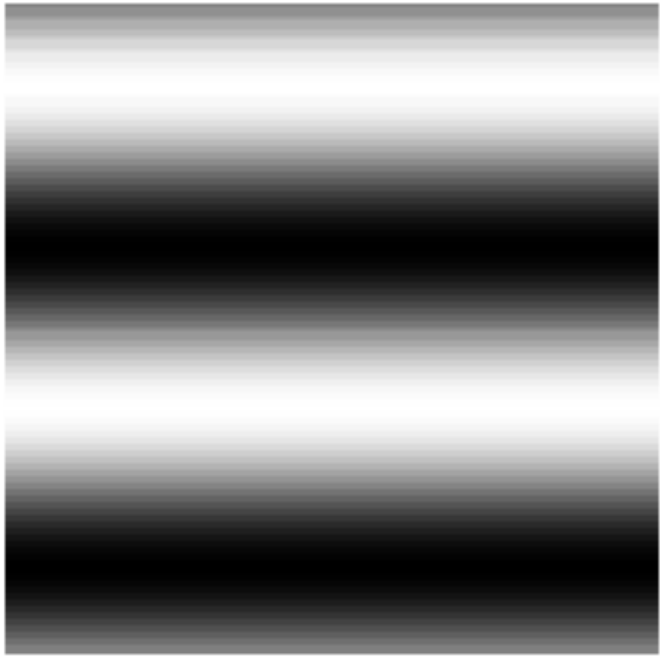


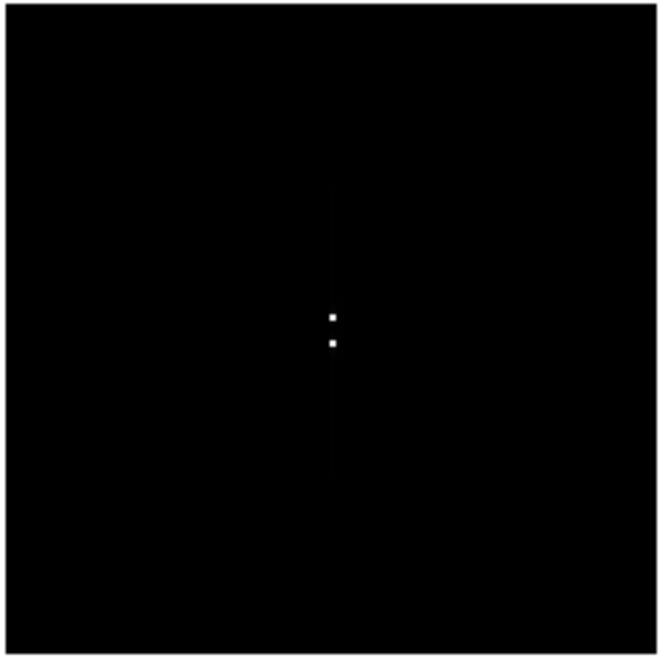


Sinusoid with frequency = 15 and its FFT

Slide credit: M. H. Yang

2D FFT





Sinusoid with varying frequency and their FFT

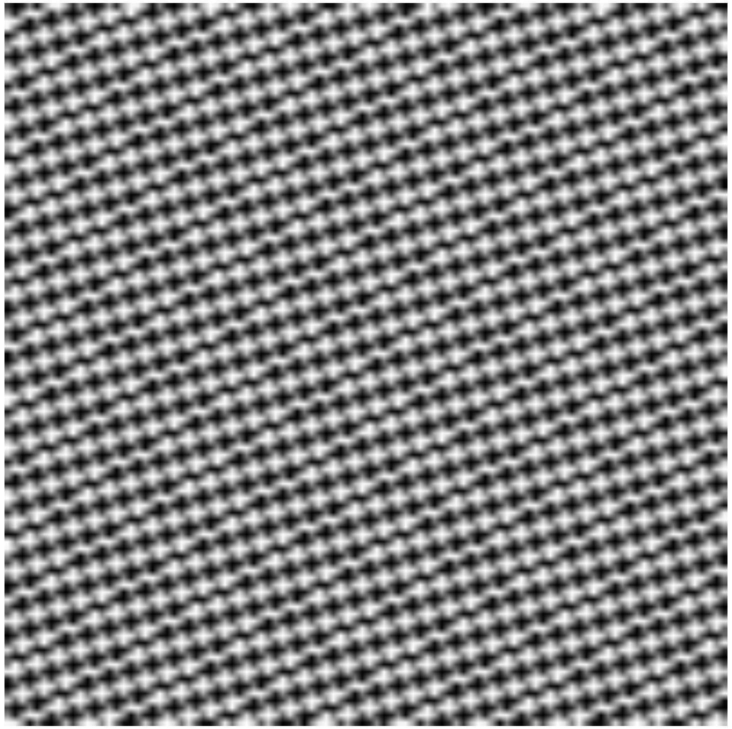
Slide credit: M. H. Yang

Rotation

Sinusoid rotated at 30 degrees and its FFT

Slide credit: M. H. Yang

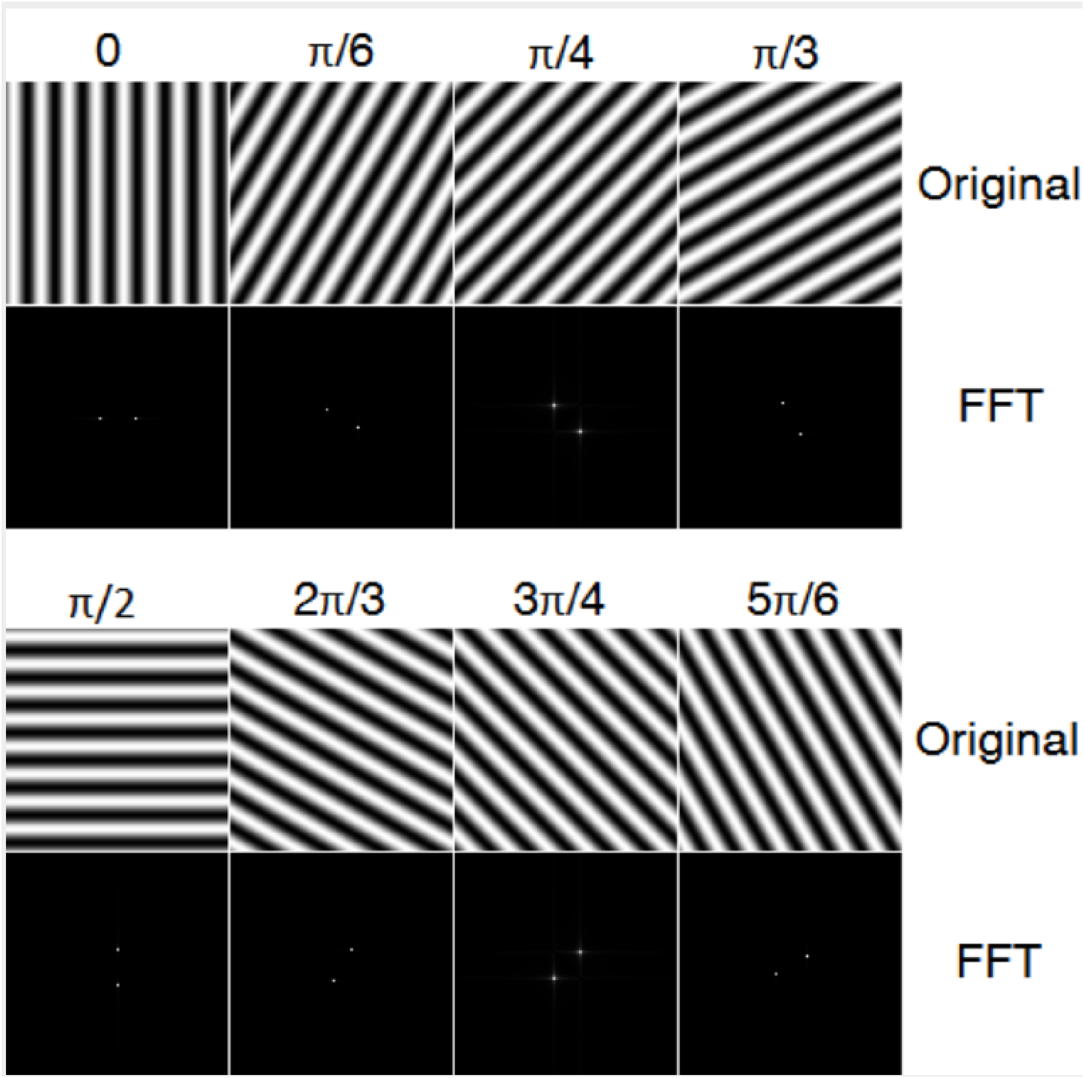
2D FFT



Sinusoid rotated at 60 degrees and its FFT

Slide credit: M. H. Yang

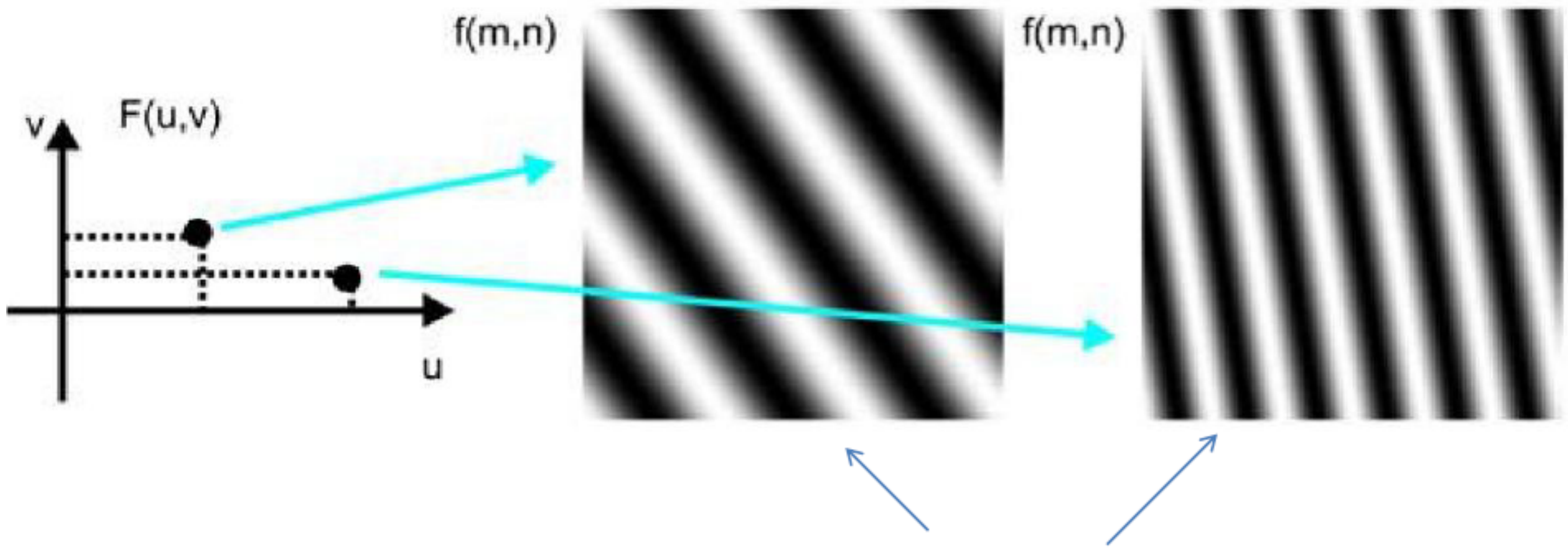
2D FFT



Rotation of sinusoid from 0 to π and their FFTs.

2D FFT

$$F(u,v) = \frac{1}{MN} \cdot \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-i2\pi(xu/M + yv/N)}$$



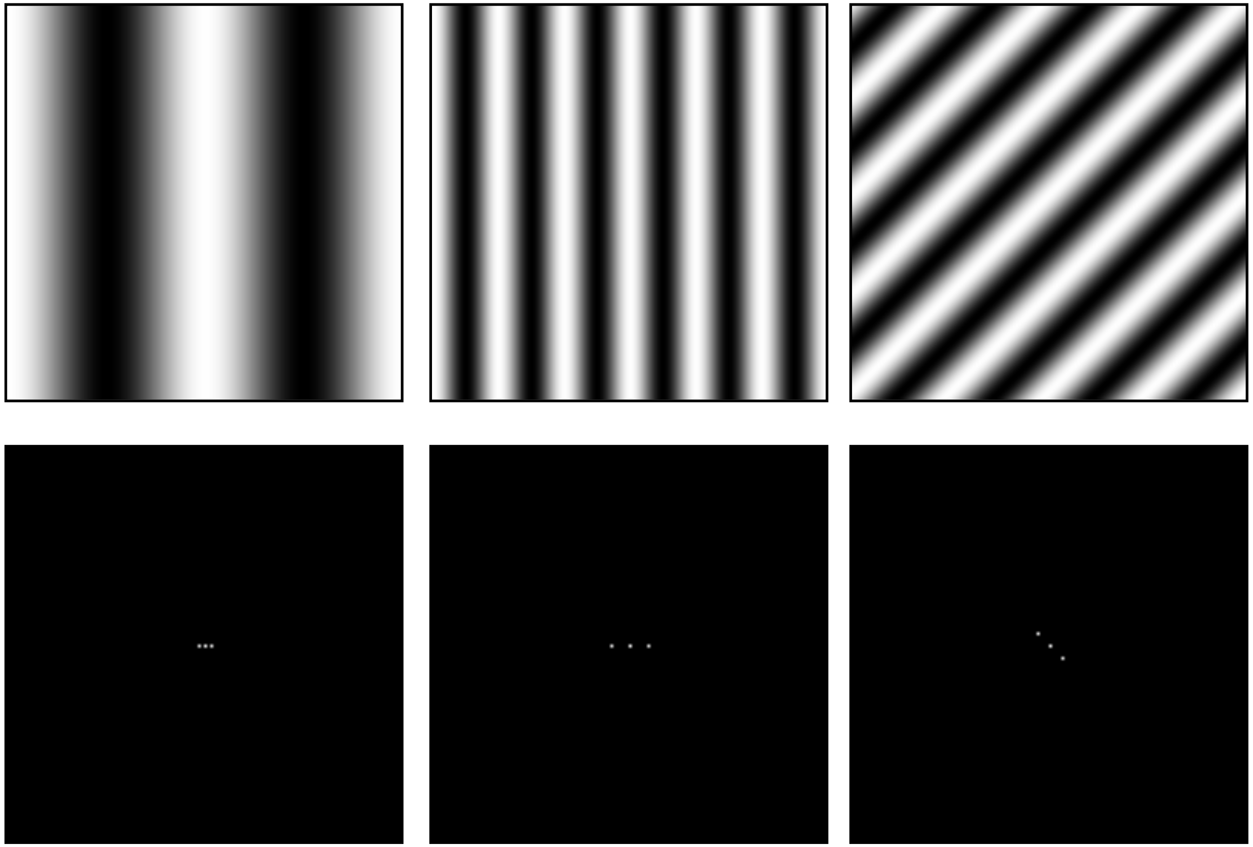
Convolution masks for different frequencies

Slide credit: M. H. Yang

Fourier analysis in images

Intensity Image

Fourier Image

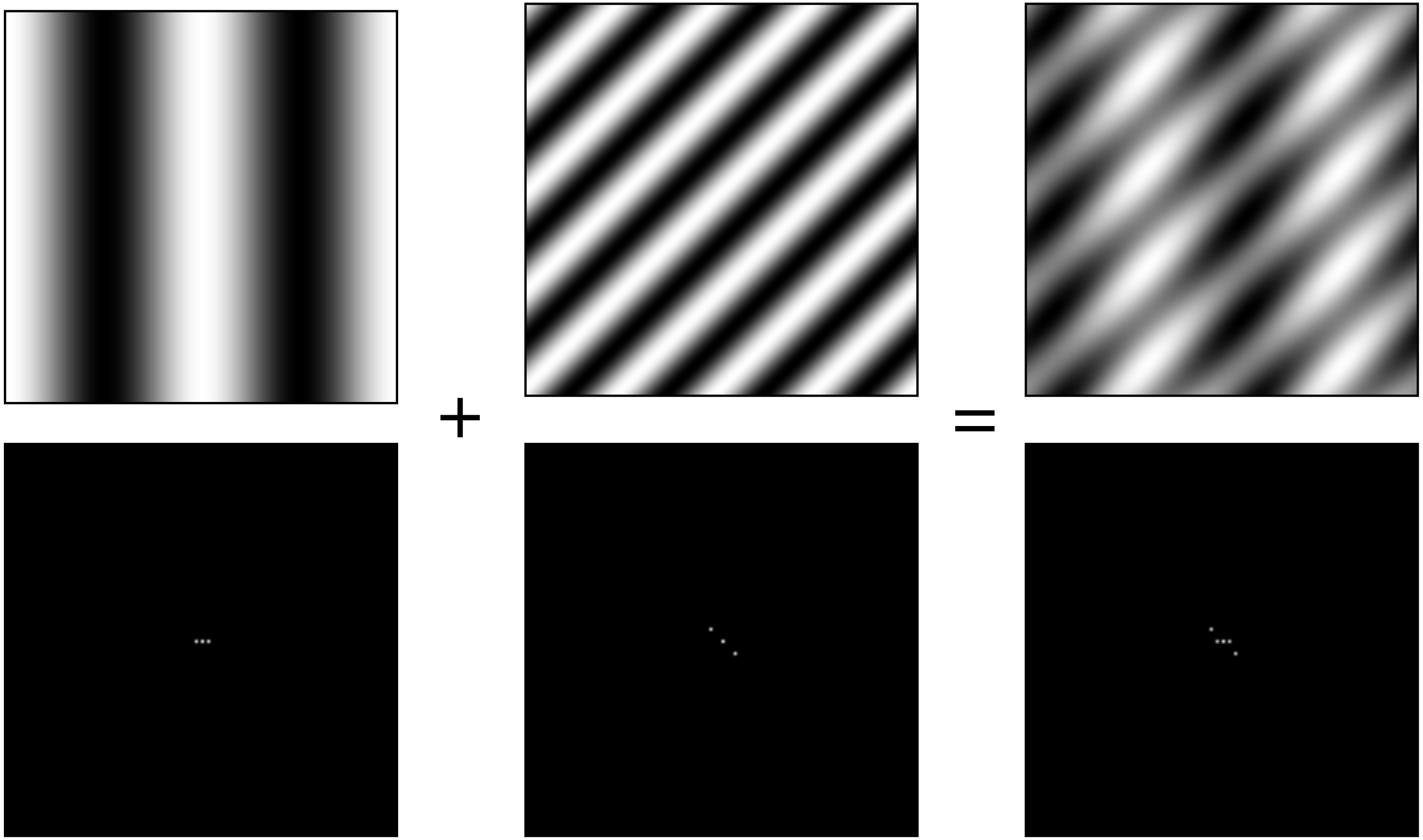


http://sharp.bu.edu/~slehar/fourier/fourier.html#filtering

More: http://www.cs.unm.edu/~brayer/vision/fourier.html

Slide credit: A. Efros

Signals can be composed

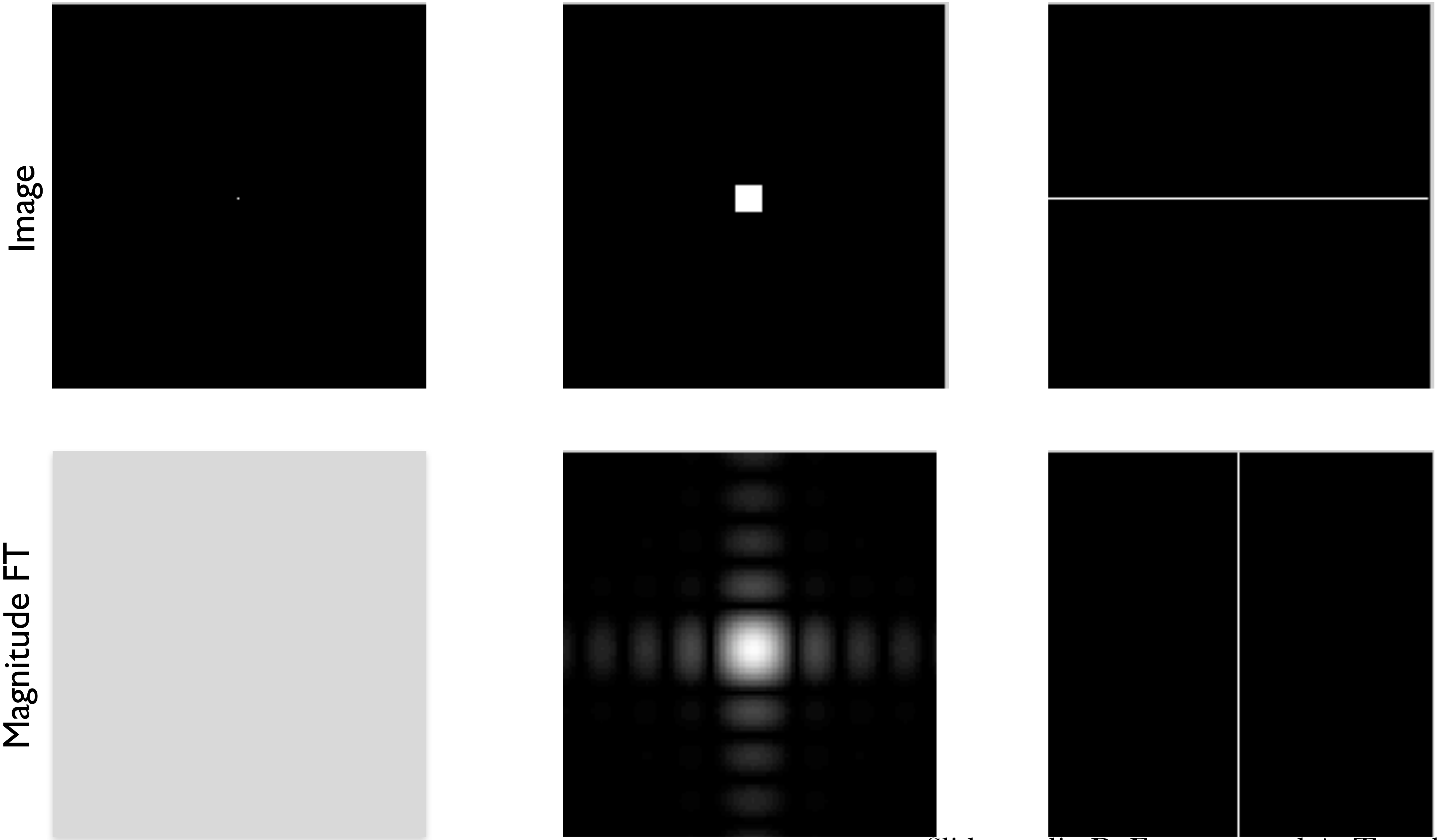


http://sharp.bu.edu/~slehar/fourier/fourier.html#filtering

More: http://www.cs.unm.edu/~brayer/vision/fourier.html

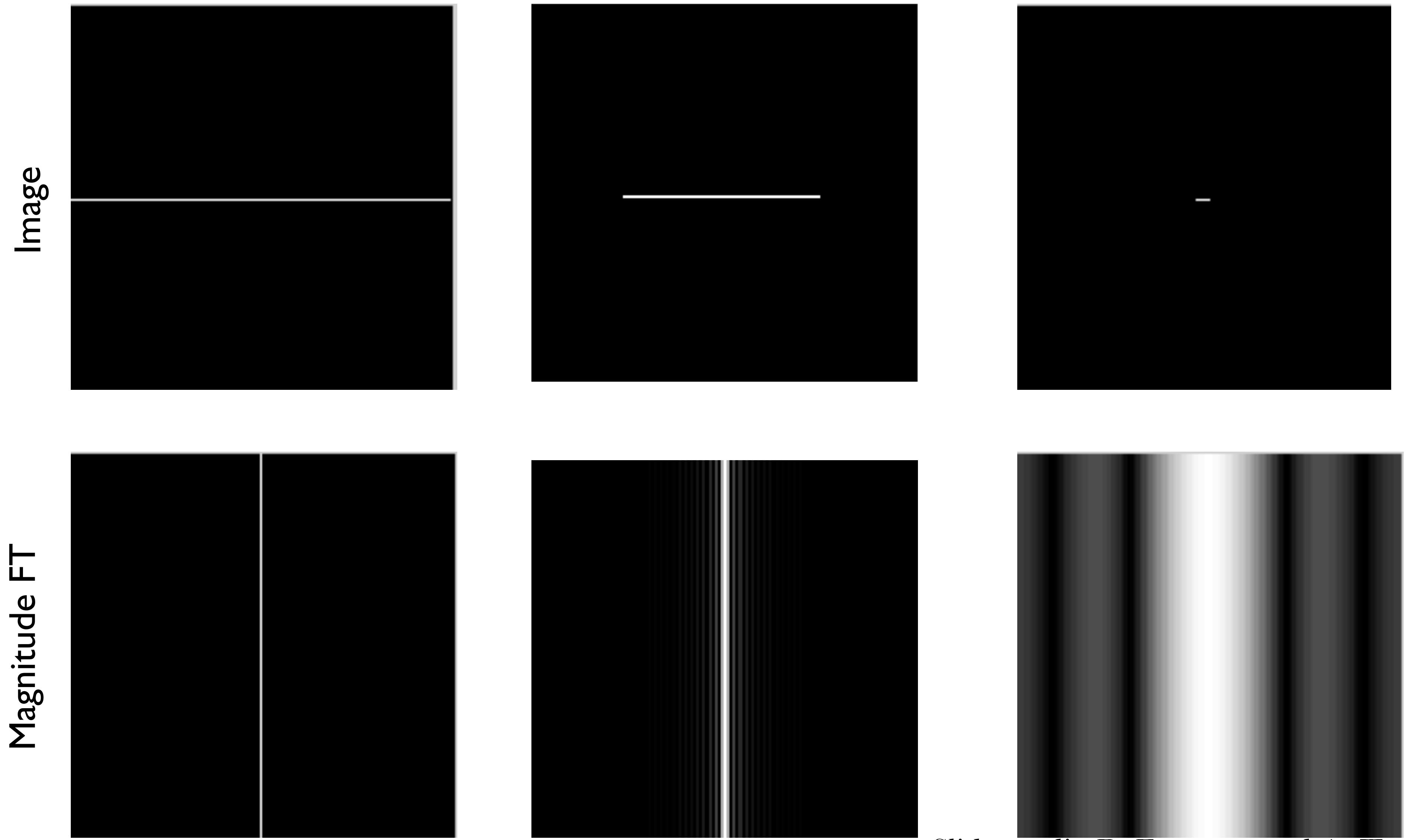
Slide credit: A. Efros

Some important Fourier Transforms



Slide credit: B. Freeman and A. Torralba

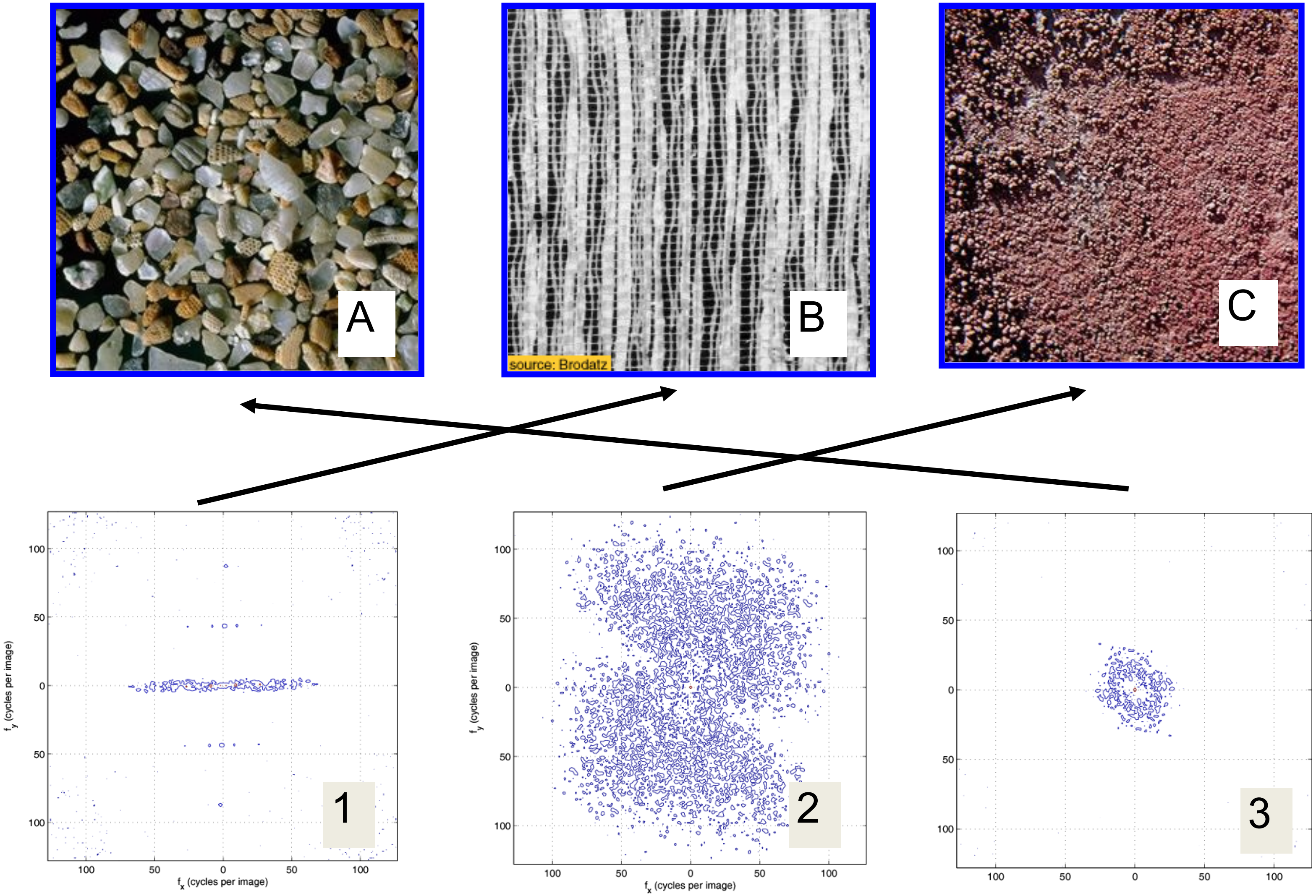
Some important Fourier Transforms



Slide credit: B. Freeman and A. Torralba

Fourier Amplitude Spectrum

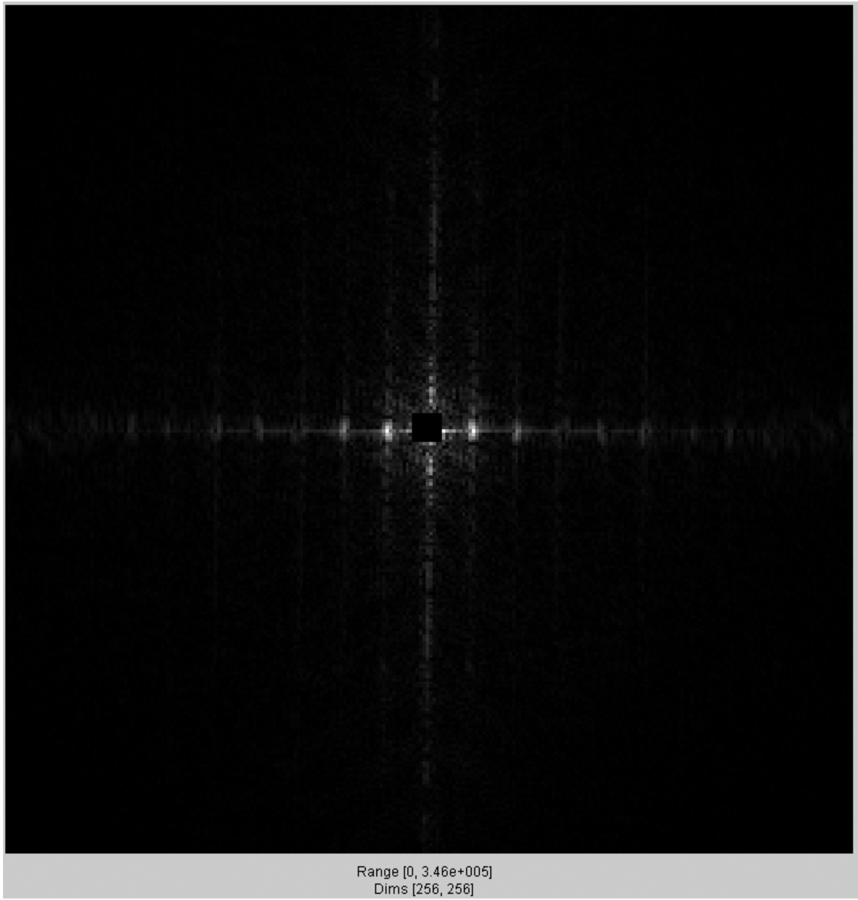
Images



fx(cycles/image pixel size) fx(cycles/image pixel size) fx(cycles/image pixel size)

Slide credit: B. Freeman and A. Torralba

Fourier transform magnitude



Slide credit: B. Freeman and A. Torralba

The Convolution Theorem

- The Fourier transform of the convolution of two functions is the product of their Fourier transforms

$$\mathrm{F}[g * h] = \mathrm{F}[g]\,\mathrm{F}[h]$$

- The inverse Fourier transform of the product of two Fourier transforms is the convolution of the two inverse Fourier transforms

$$\mathrm{F}^{-1}[gh] = \mathrm{F}^{-1}[g] * \mathrm{F}^{-1}[h]$$

- **Convolution** in spatial domain is equivalent to **multiplication** in frequency domain!

Slide credit: A. Efros

Properties of Fourier Transforms

- Linearity $\mathcal{F}[ax(t) + by(t)] = a\mathcal{F}[x(t)] + b\mathcal{F}[y(t)]$

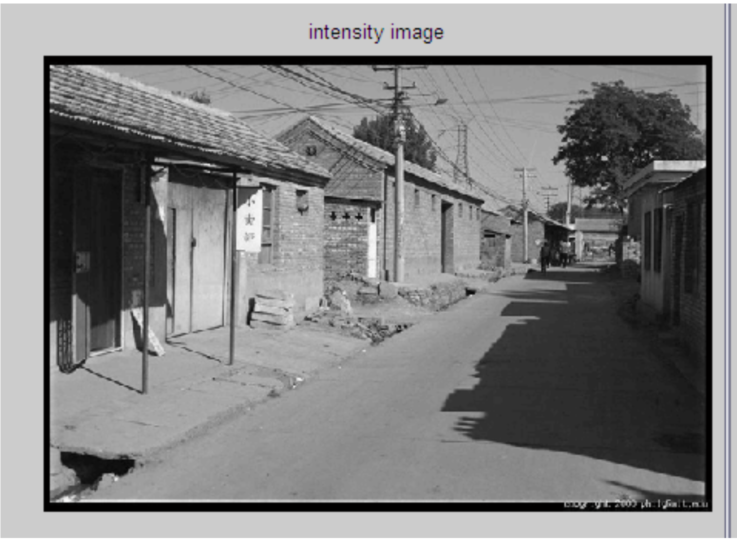
- Fourier transform of a real signal is symmetric about the origin

- The energy of the signal is the same as the energy of its Fourier transform

Slide credit: J. Hays

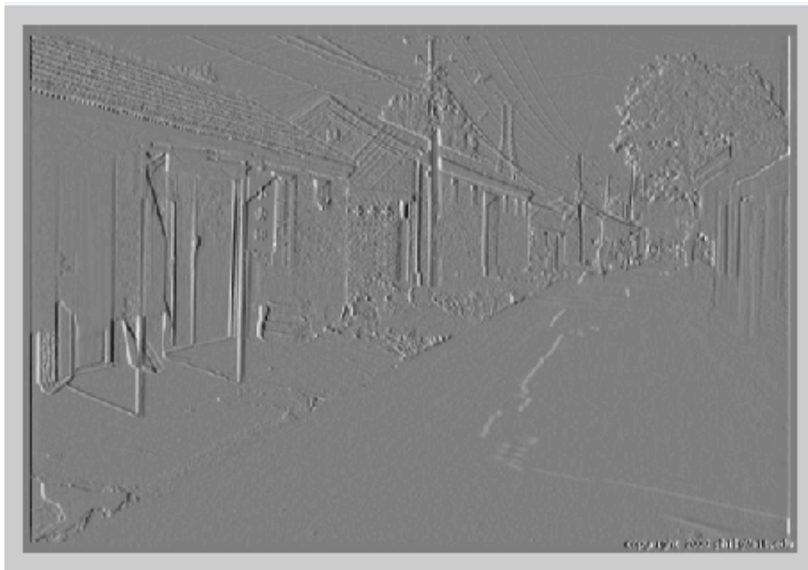
Filtering in spatial domain

1	0	-1
2	0	-2
1	0	-1



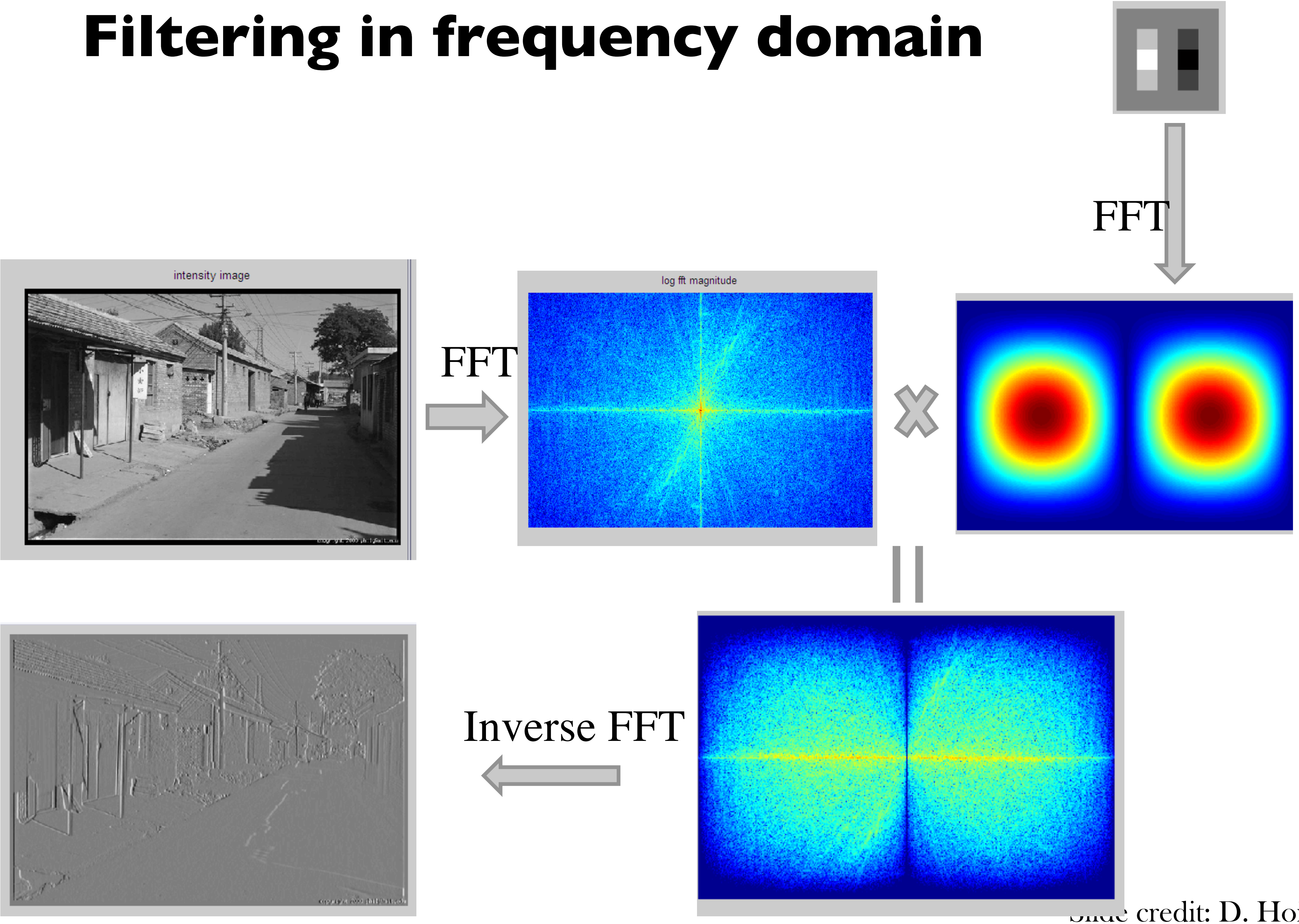
*

=



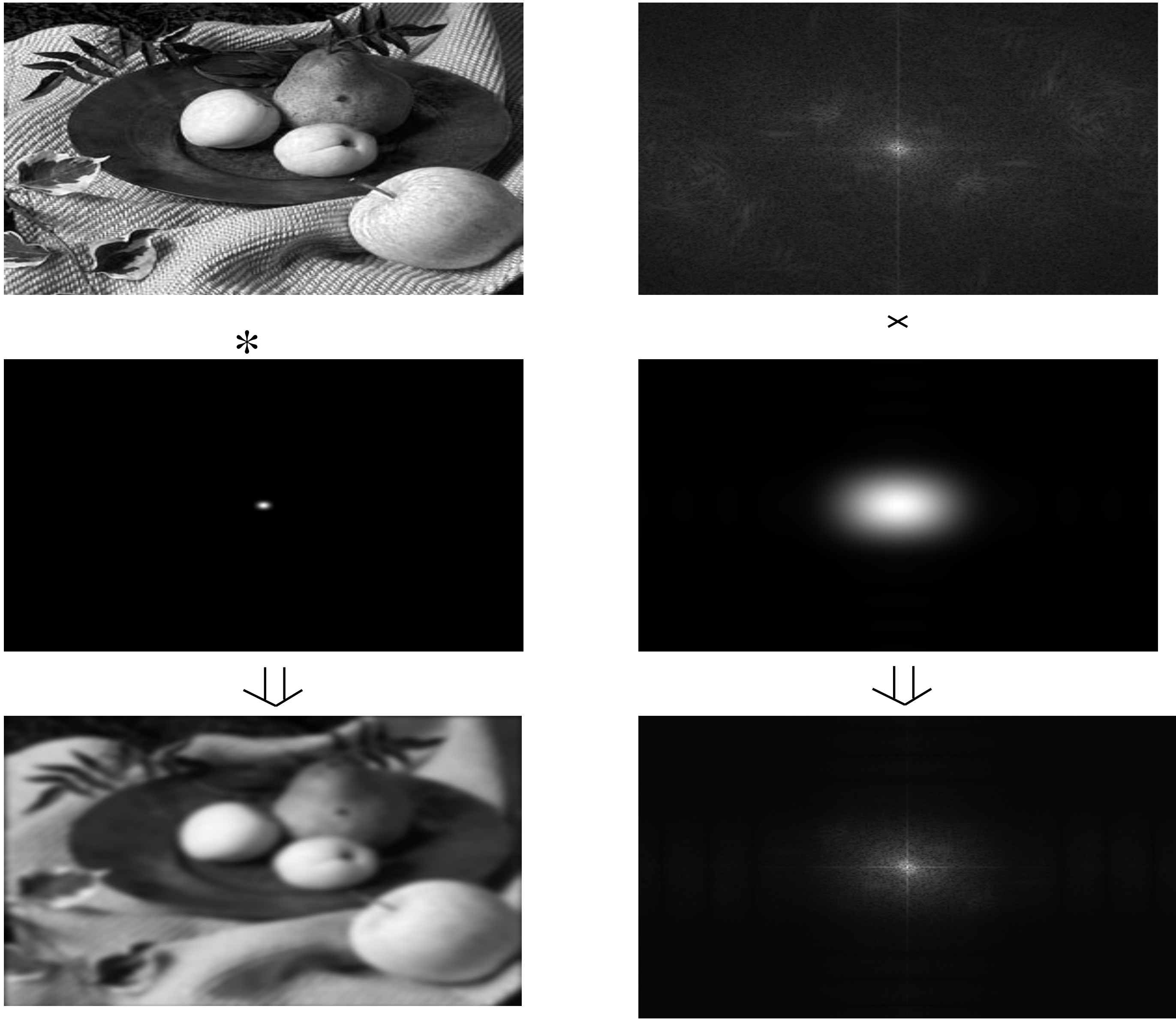
Slide credit: D. Hoiem

Filtering in frequency domain



Slide credit: D. Hoiem

2D convolution theorem example



$f(x,y)$ $|F(s_x,s_y)|$

$*$ $\times$

$h(x,y)$ $|H(s_x,s_y)|$

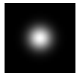
$\Downarrow$ $\Downarrow$

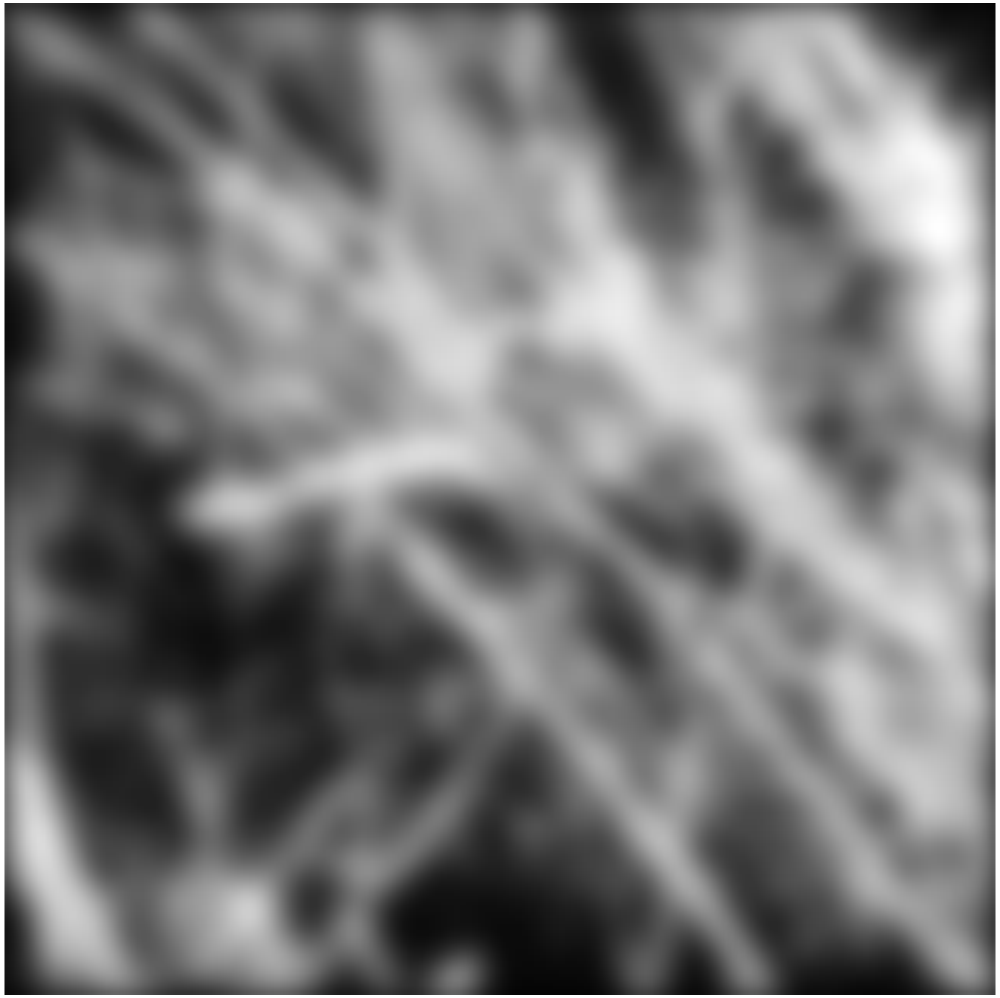
$g(x,y)$ $|G(s_x,s_y)|$

Slide credit: A. Efros

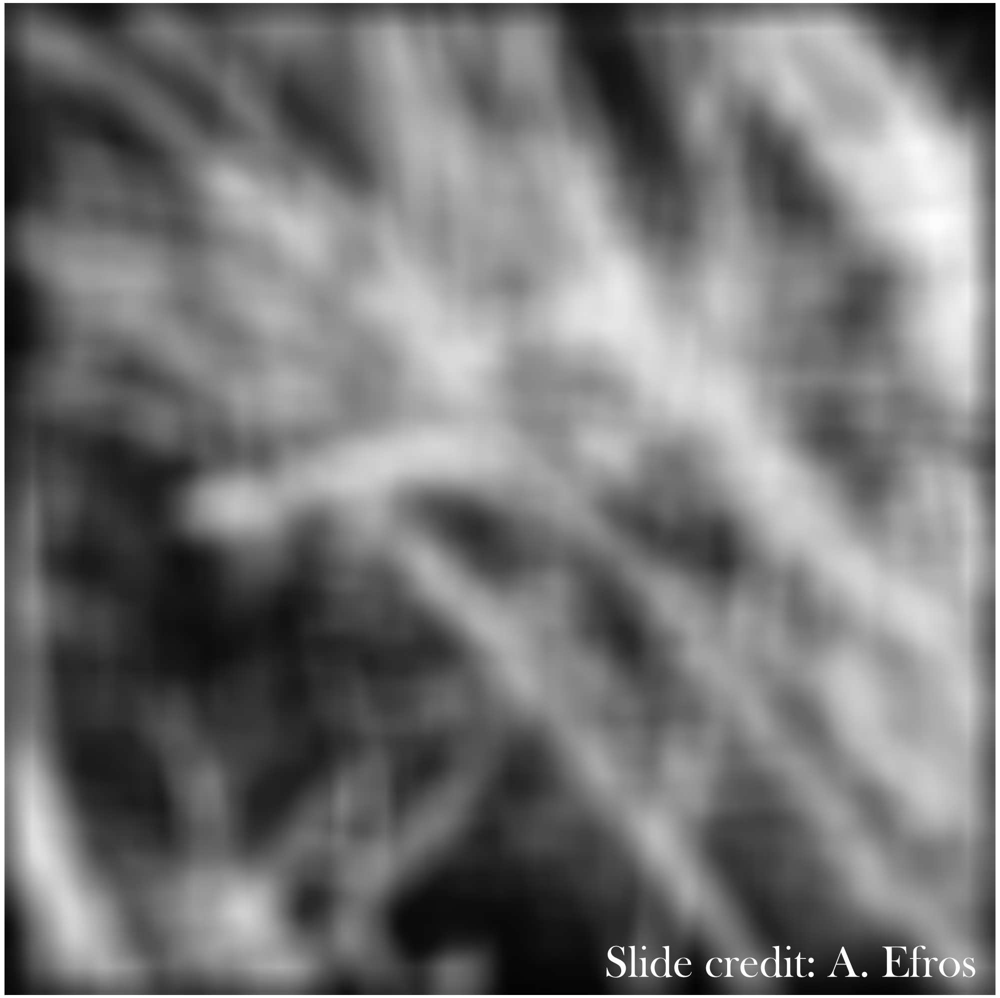
Filtering

Why does the Gaussian give a nice smooth image, but the square filter give edgy artifacts?

Gaussian 



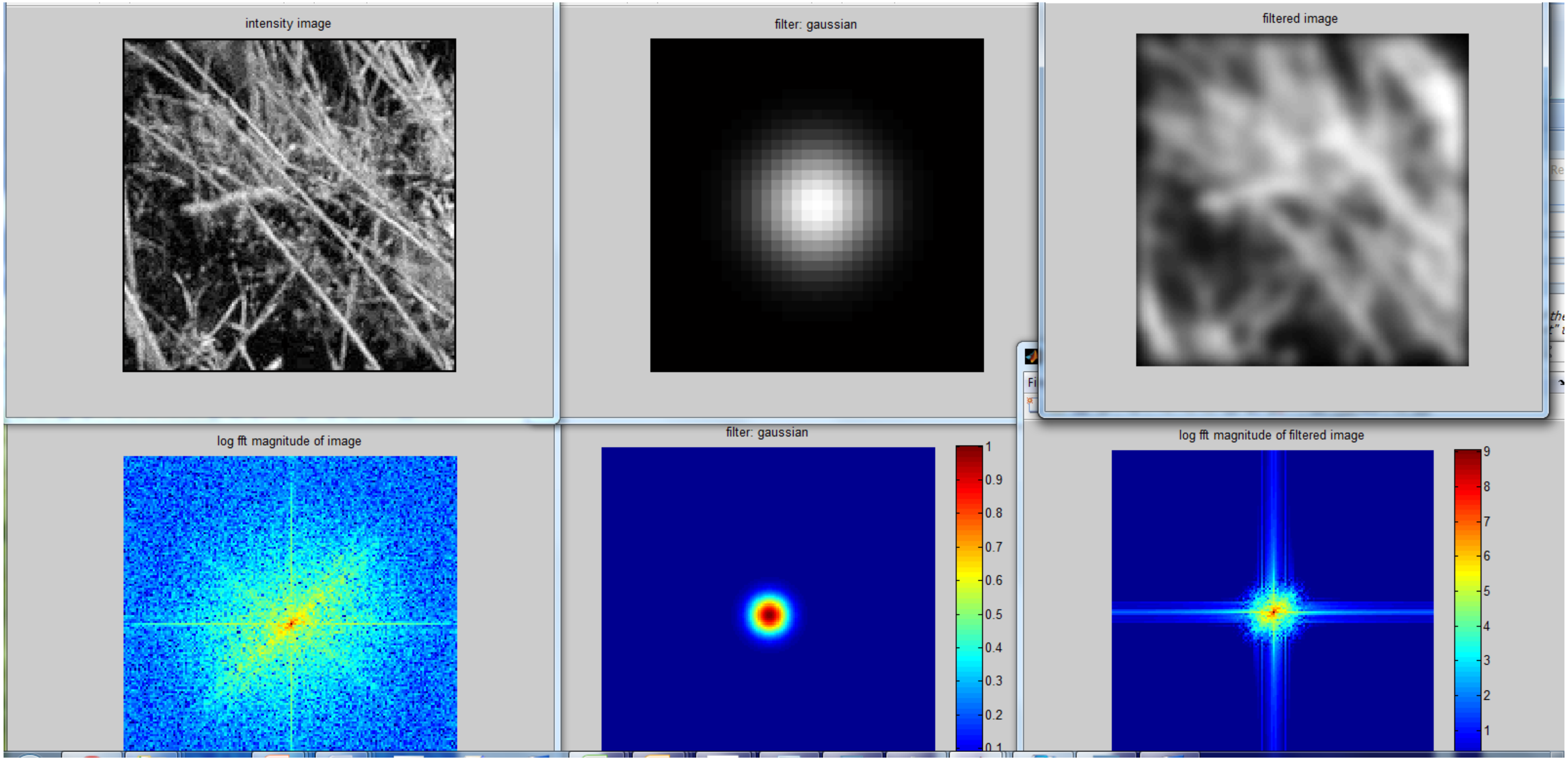
Box filter 



Slide credit: A. Efros

Filtering

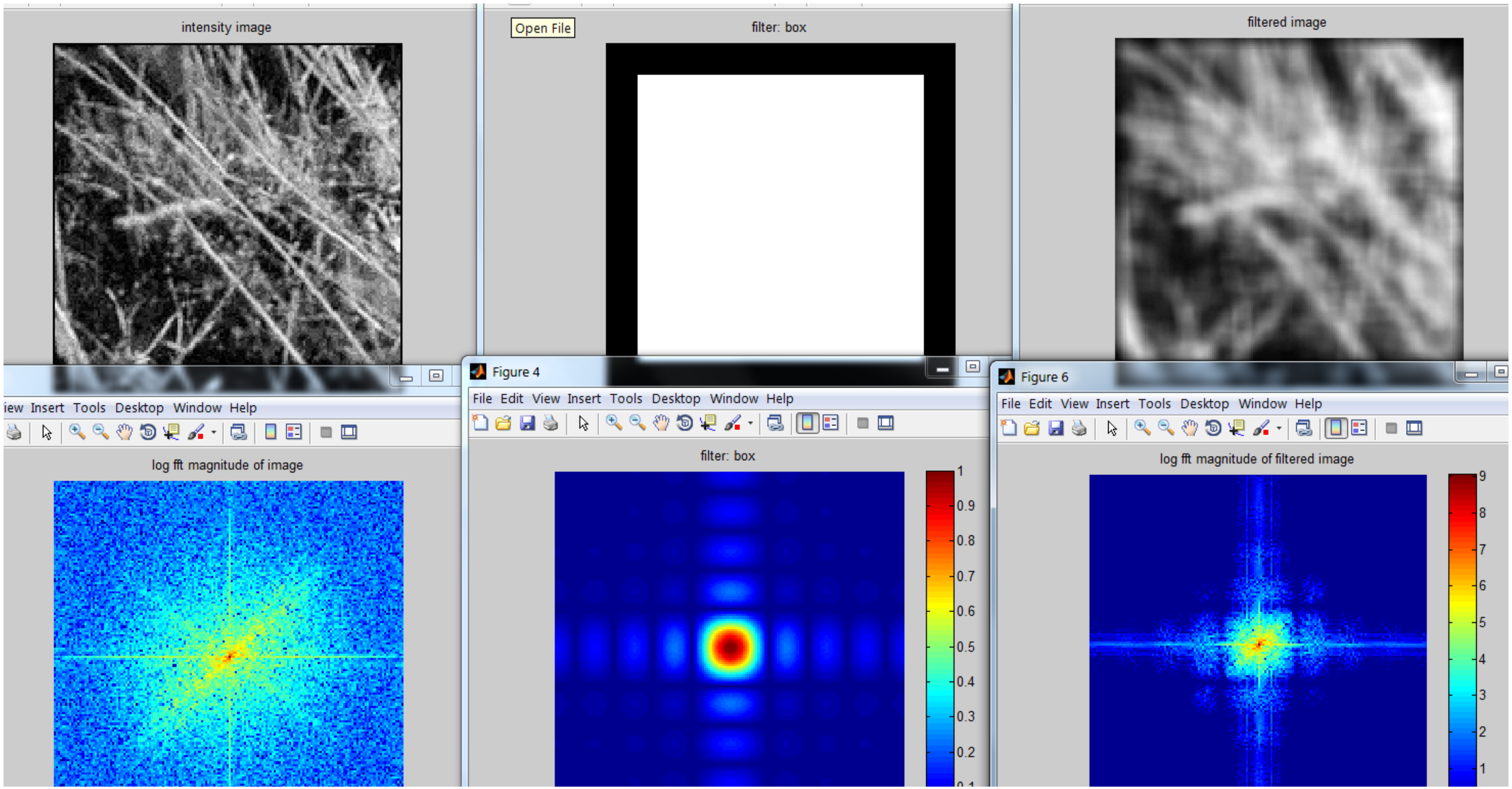
Gaussian



Slide credit: A. Efros

Filtering

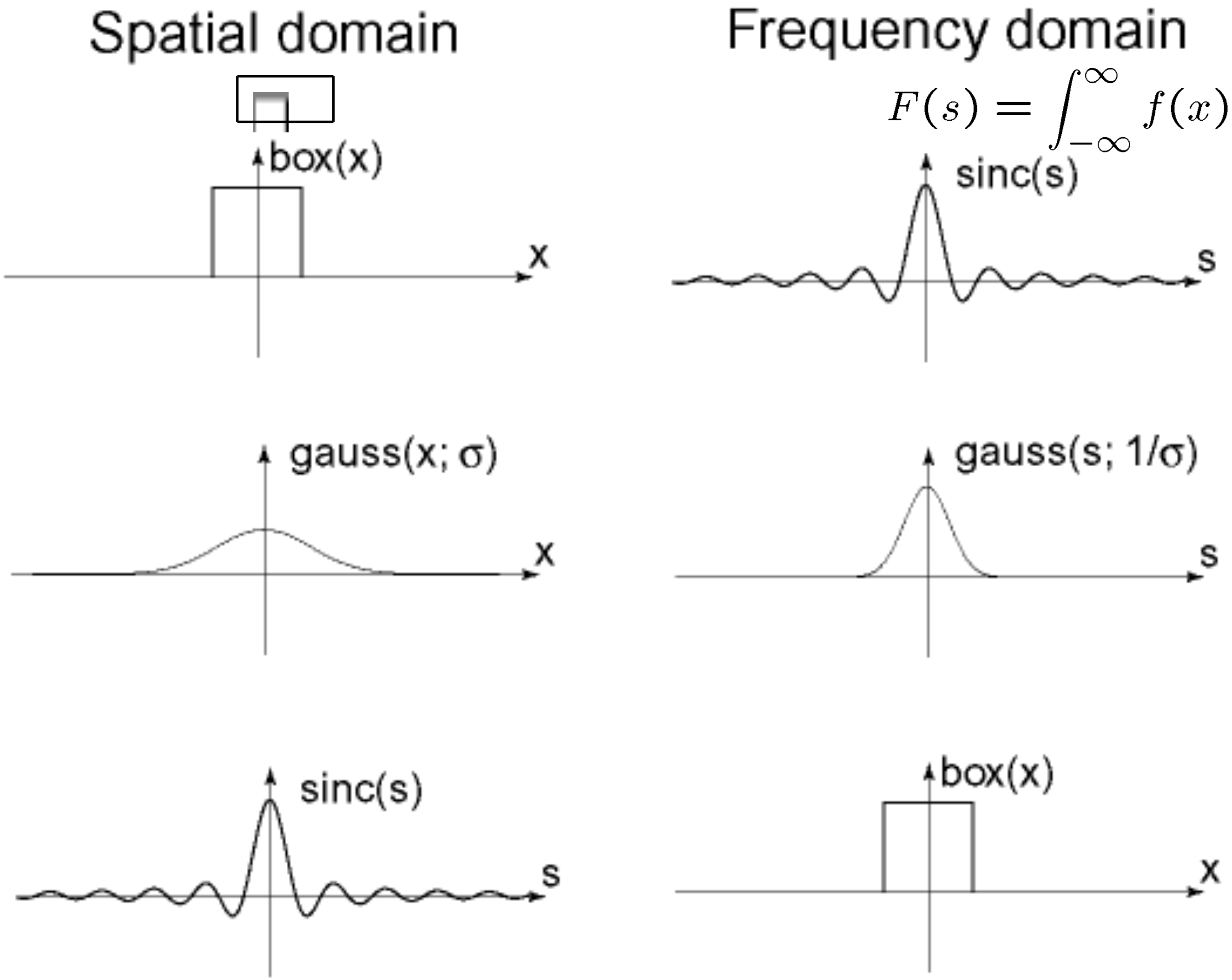
Box Filter



Slide credit: A. Efros

Fourier Transform pairs

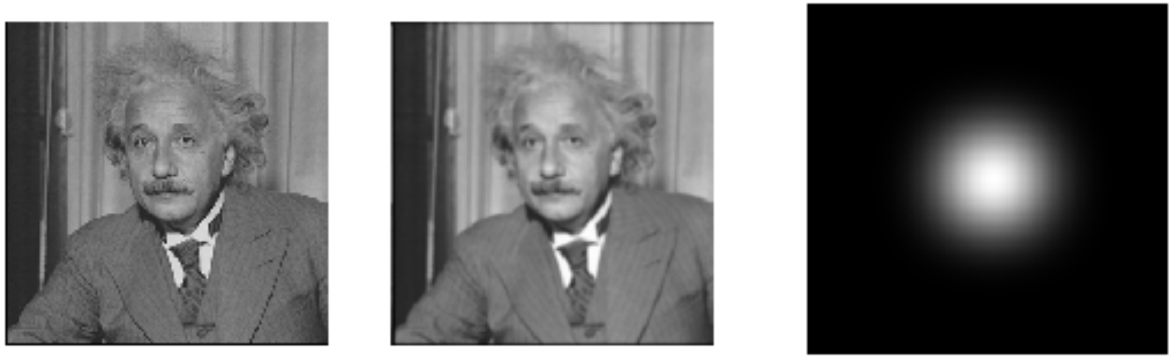
$$F(s) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi sx} dx$$



Slide credit: A. Efros

Low-pass, Band-pass, High-pass filters

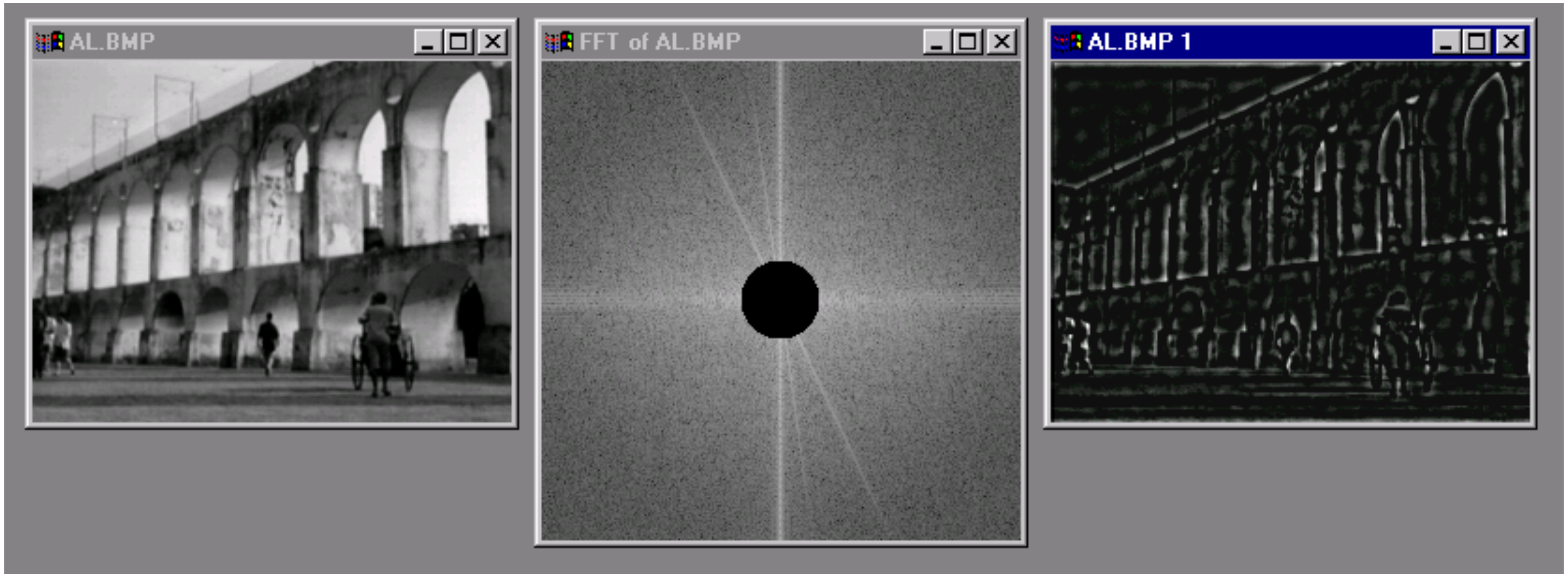
low-pass:



High-pass / band-pass:

Slide credit: A. Efros

Edges in images



Slide credit: A. Efros

Phase and Magnitude

- Curious fact
  - all natural images have about the same magnitude transform
  - hence, phase seems to matter, but magnitude largely doesn't
- Demonstration
  - Take two pictures, swap the phase transforms, compute the inverse - what does the result look like?

Image with cheetah phase (and zebra magnitude)

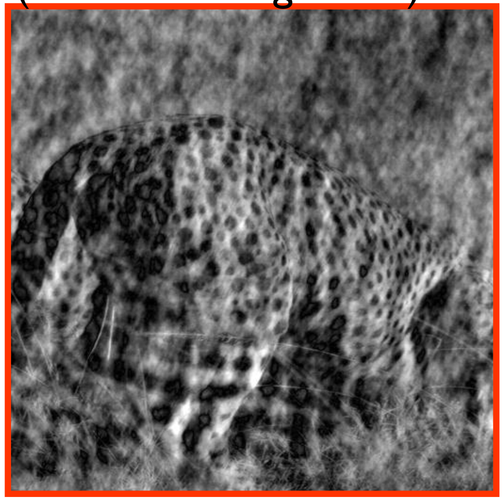
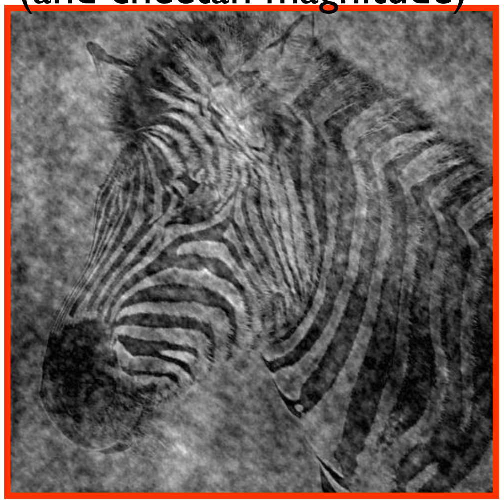


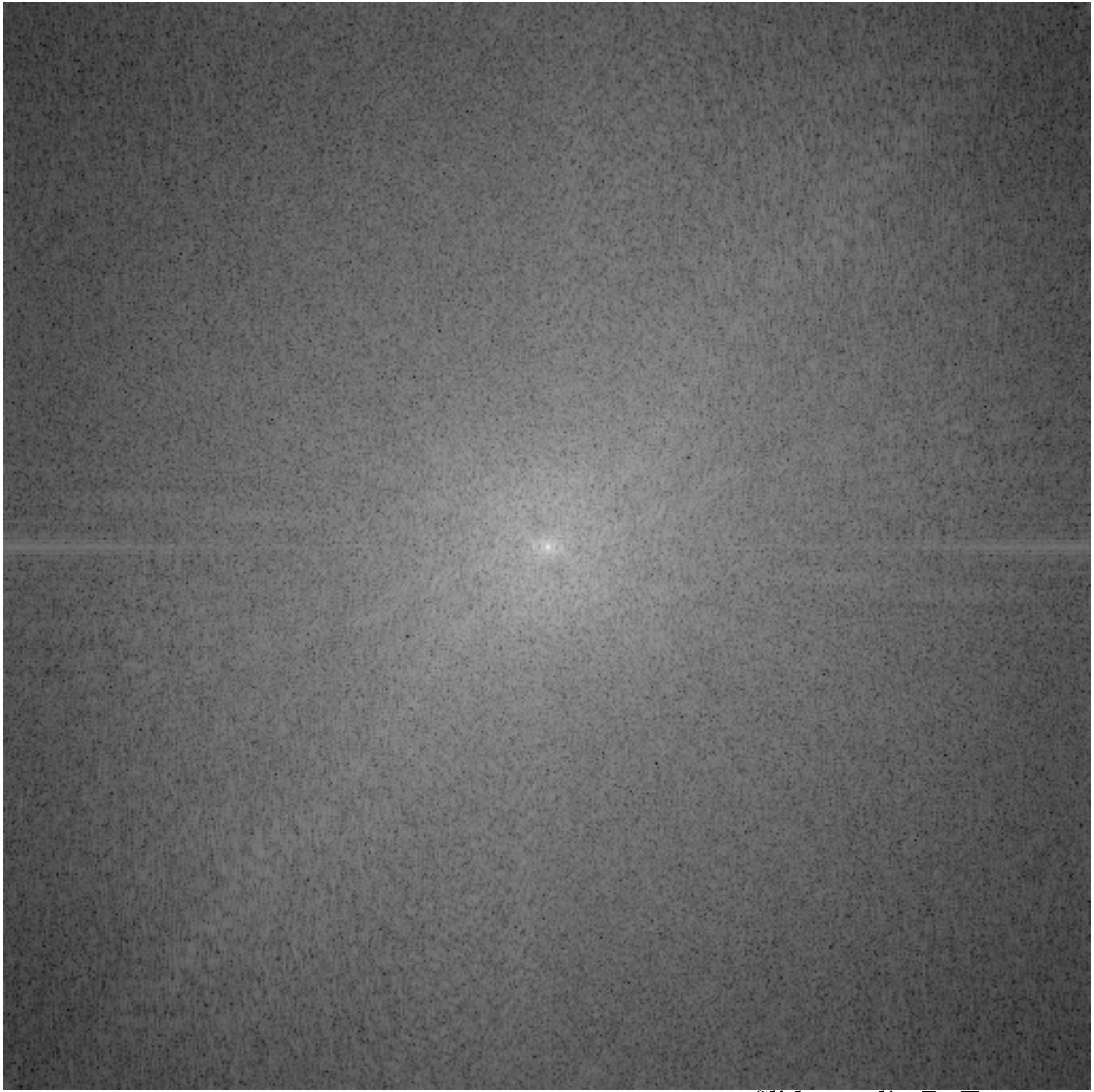
Image with zebra phase (and cheetah magnitude)



Slide credit: B. Freeman and A. Torralba

Slide credit: B. Freeman and A. Torralba

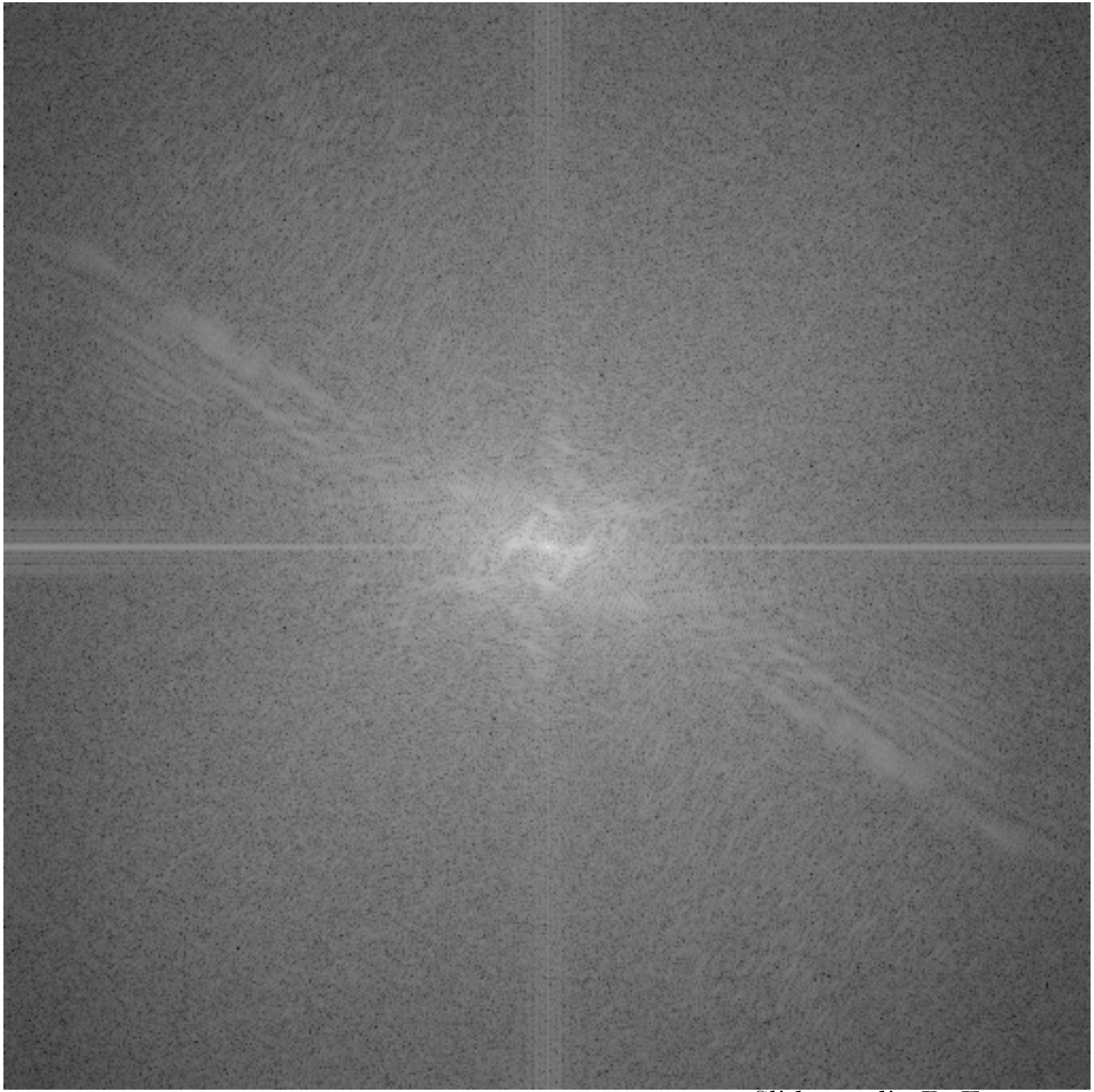
This is the magnitude transform of the cheetah picture



Slide credit: B. Freeman and A. Torralba

Slide credit: B. Freeman and A. Torralba

This is the magnitude transform of the zebra picture

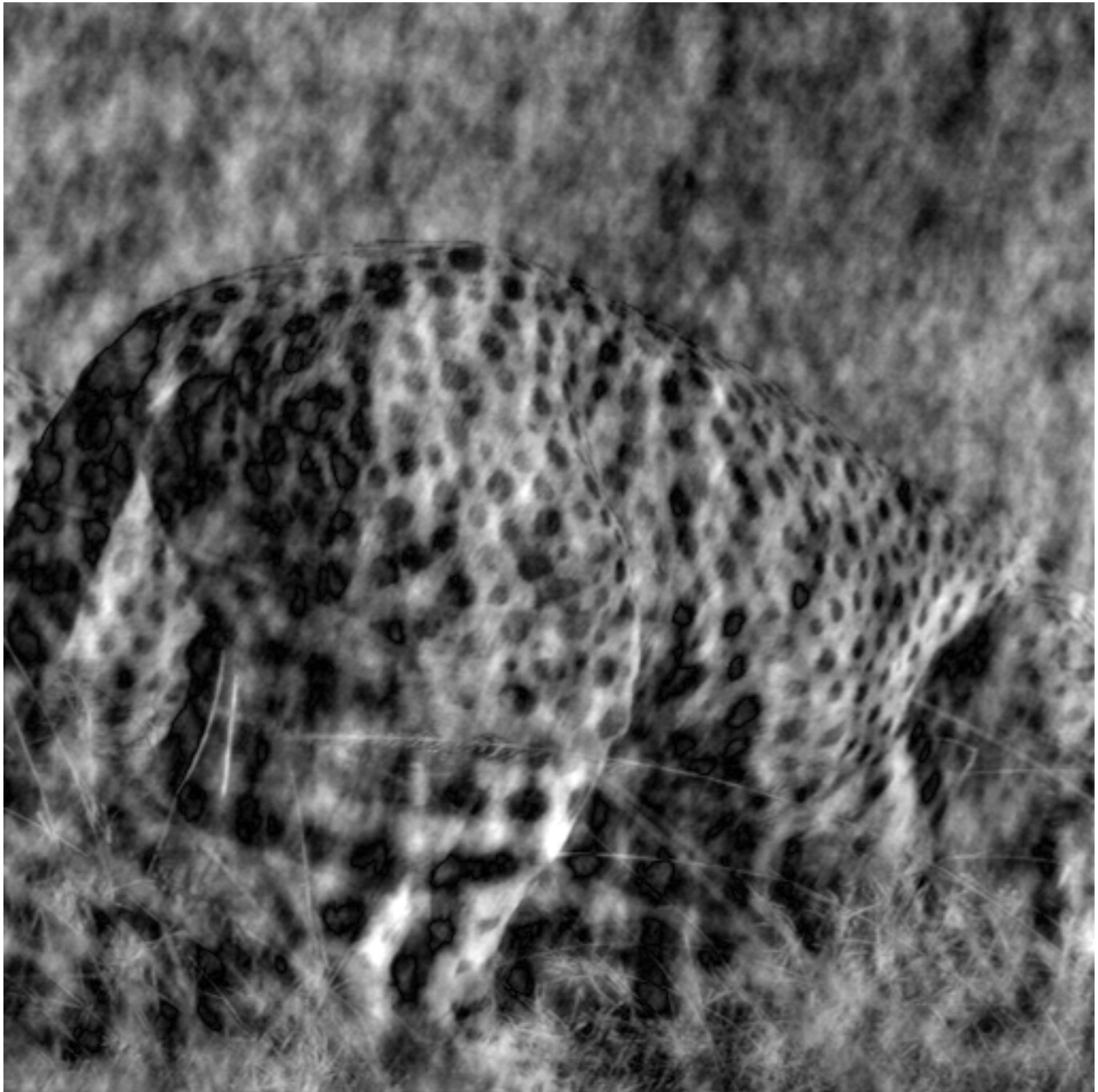


Slide credit: B. Freeman and A. Torralba

Reconstruction with zebra phase, cheetah magnitude

Slide credit: B. Freeman and A. Torralba

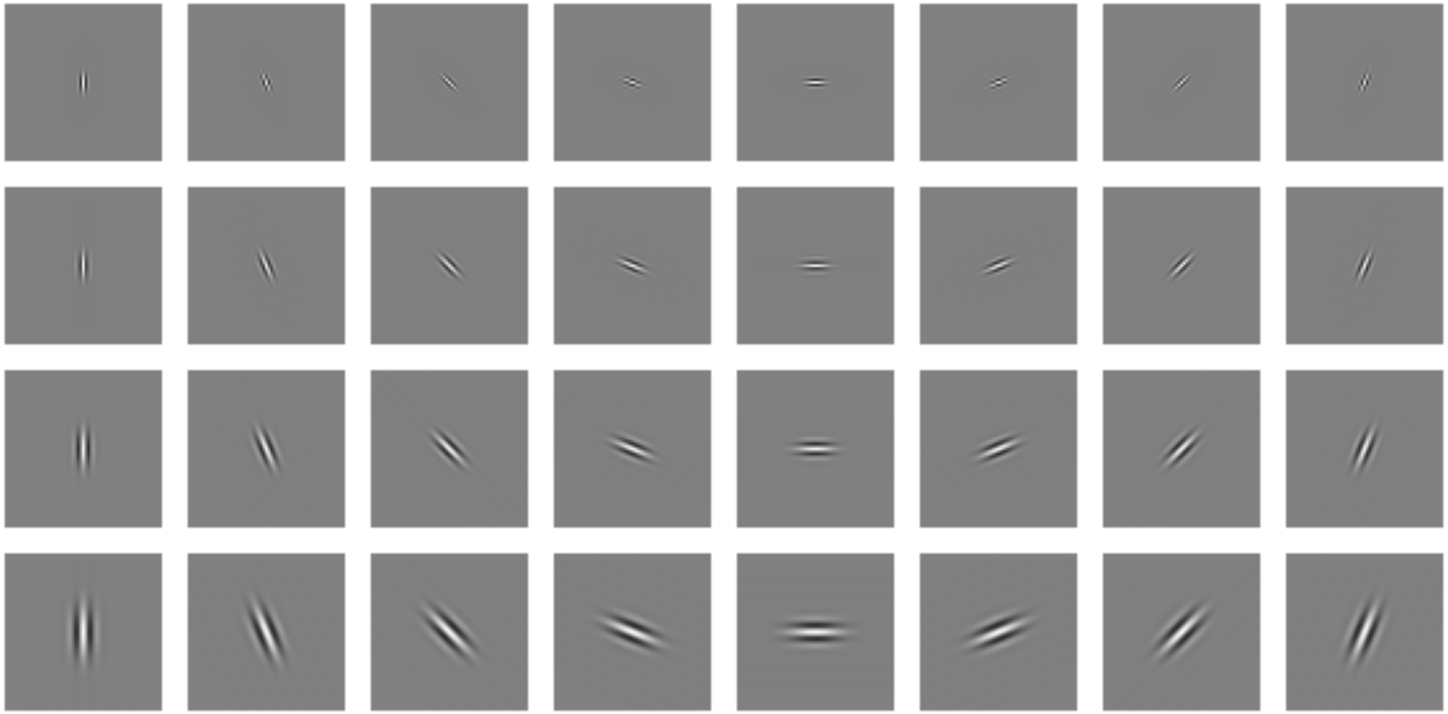
Reconstruction with cheetah phase, zebra magnitude



Slide credit: B. Freeman and A. Torralba

Clues from Human Perception

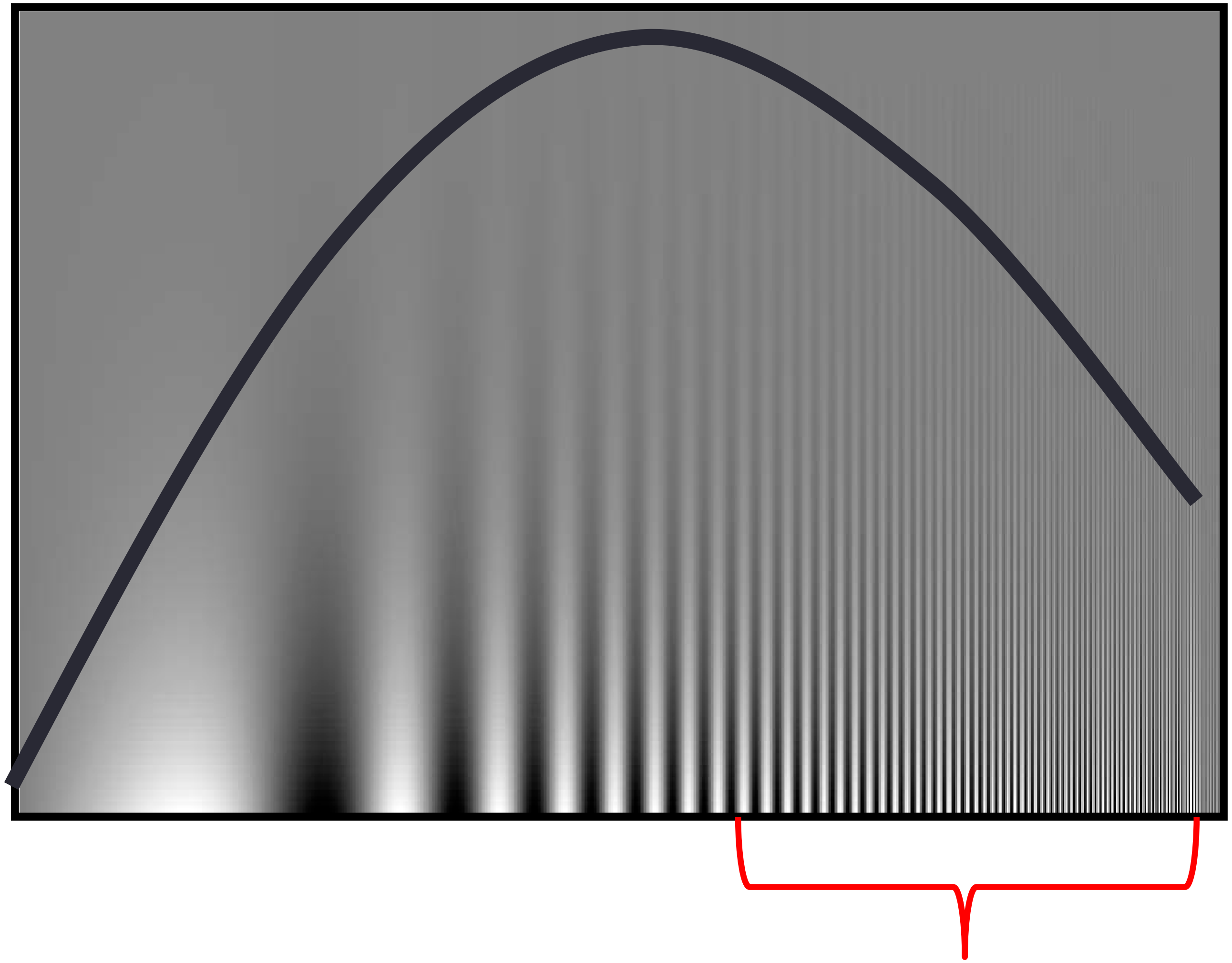
- Early processing in humans filters for various orientations and scales of frequency
- Perceptual cues in the mid-high frequencies dominate perception
- When we see an image from far away, we are effectively subsampling it



Early Visual Processing: Multi-scale edge and blob filters

Slide credit: J. Hays

Campbell-Robson contrast sensitivity curve

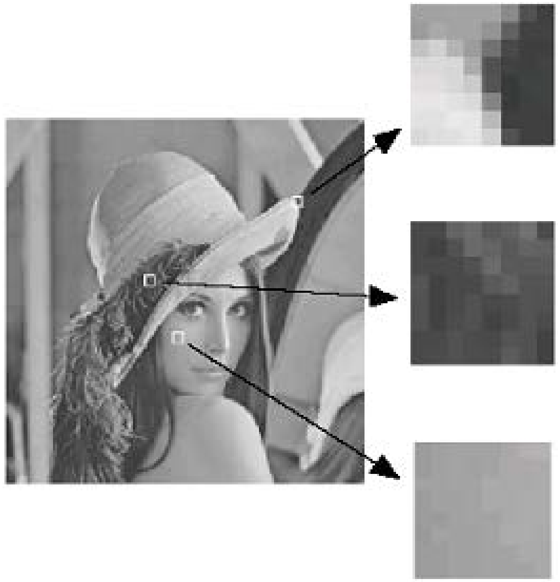


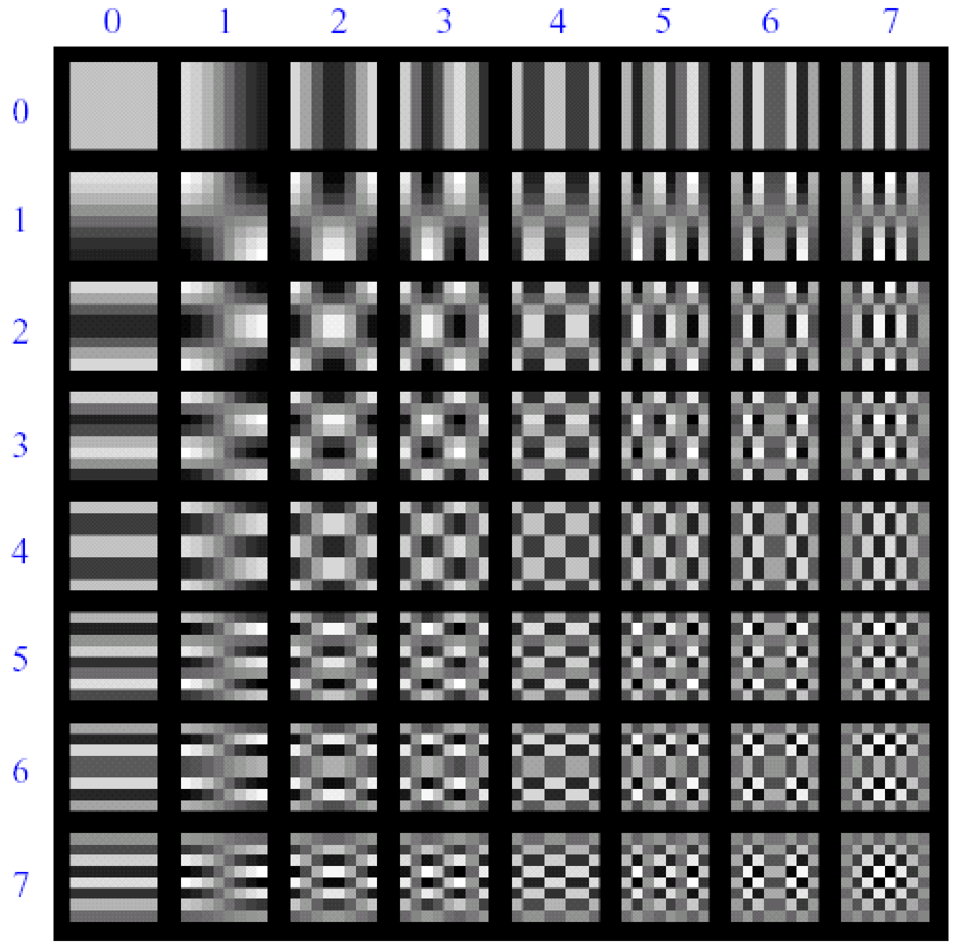
The higher the frequency the less sensitive human visual system is…

Slide credit: J. Hays

Lossy Image Compression (JPEG)

$$X_{k_1,k_2} = \sum_{n1=0}^{N_1-1} \sum_{n2=0}^{N_2-1} x_{n_1,n_2} \cos\left[\frac{\pi}{N_1}\left(n_1+\frac{1}{2}\right)k_1\right] \cos\left[\frac{\pi}{N_2}\left(n_2+\frac{1}{2}\right)k_2\right].$$

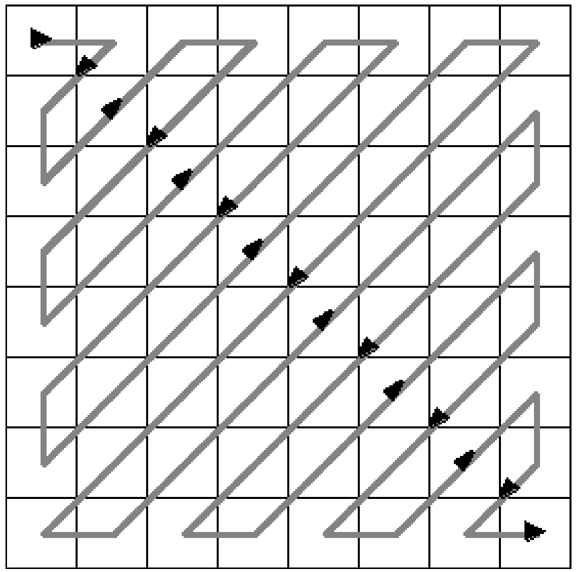




Block-based Discrete Cosine Transform (DCT) on 8x8 Slide credit: A. Bobick

Using DCT in JPEG

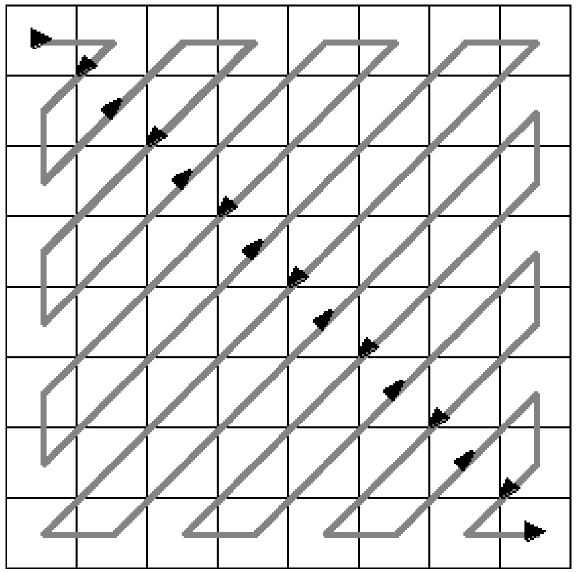
- The first coefficient B(0,0) is the DC component, the average intensity
- The top-left coeffs represent low frequencies, the bottom right – high frequencies



Slide credit: A. Bobick

Image compression using DCT

- DCT enables image compression by concentrating most image information in the low frequencies
- Loose unimportant image info (high frequencies) by cutting B(*u*,*v*) at bottom right
- The decoder computes the inverse DCT – IDCT



Slide credit: A. Bobick

JPEG compression comparison

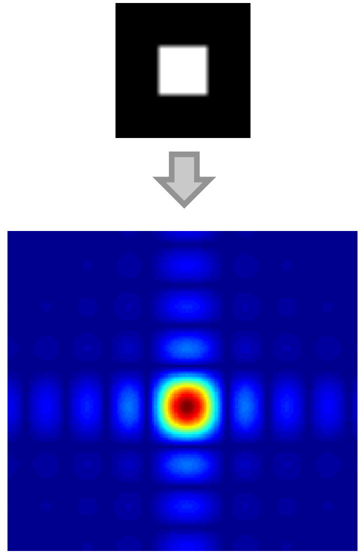
89k

12k

Slide credit: A. Bobick

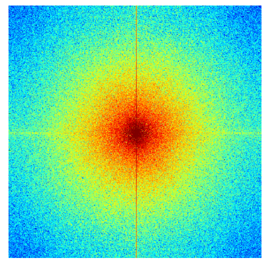
Things to Remember

- Sometimes it makes sense to think of images and filtering in the frequency domain
  - Fourier analysis



- Can be faster to filter using FFT for large images (N logN vs. N^2 for auto-correlation)

- Images are mostly smooth
  - Basis for compression

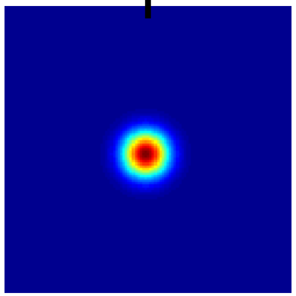


Slide credit: J. Hays

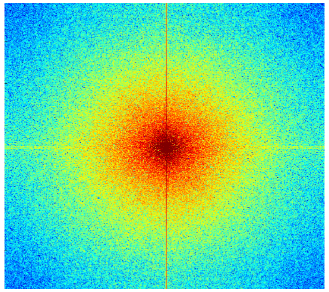
Practice question

1. Match the spatial domain image to the Fourier magnitude image

1



2



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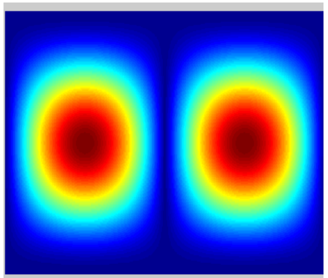

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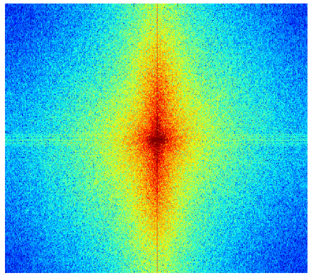

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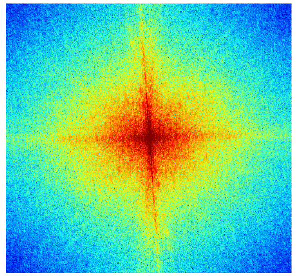
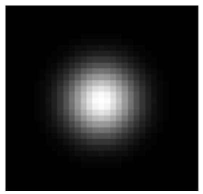


A

B

C

D



E

Slide credit: J. Hays