一、填空题

1. $\dfrac{361}{400}$; 2. $\dfrac{1}{4}$; 3. $e^{-2}-e^{-3}$; 4. 2; 5. 1; 6. $\dfrac{1}{3}$

二、解答题

1. 设 $A=\{$考试及格$\}$，$B=\{$按时交作业$\}$

(1) 由全概率公式得

$$P(B)=P(A)P(B|A)+P(\bar{A})P(B|\bar{A})\ =0.9\times0.85+0.1\times0.35=0.8$$

(2) 由贝叶斯公式得

$$P(A\,|\,B)=\frac{P(A)P(B\,|\,A)}{P(B)}=\frac{0.9\times0.85}{0.8}=\frac{153}{160}=0.95625$$

2. (1) $P(X=k)=\dfrac{C_n^k p^k(1-p)^{n-k}}{1-(1-p)^n},\ \ k=1,2,...,n.$

(2) $E(X)=\displaystyle\sum_{k=1}^{n}k\,\frac{C_n^k p^k(1-p)^{n-k}}{1-(1-p)^n}=\frac{1}{1-(1-p)^n}\sum_{k=1}^{n}kC_n^k p^k(1-p)^{n-k}$

$$=\frac{1}{1-(1-p)^n}\sum_{k=0}^{n}kC_n^k p^k(1-p)^{n-k}=\frac{np}{1-(1-p)^n}$$

3. 设 X_i 为第 i 位经手人加入的游戏币数目, $i=1,2,...,200$. X 为最终箱子中游戏币增加的数目. 则：

$$P(X_i=k)=\frac{1}{5},\,k=-2,-1,0,1,2;\ \text{且}\ X_1,X_2,...,X_{200}\ \text{相互独立,}$$

$$E(X_i)=0,\,D(X_i)=E(X_i^2)=\frac{1}{5}\times[(-2)^2+(-1)^2+0+1^2+2^2]=2$$

$$X=\sum_{i=1}^{200}X_i,\,E(X)=0,\,D(X)=400$$

由中心极限定理, X 近似服从 $N(0,400)$, 于是

$$P(|X|\le10)=2\Phi\left(\frac{10}{\sqrt{400}}\right)-1=2\Phi(0.5)-1=2\times0.6915-1=0.383$$

4. (1) $f_X(x)=\displaystyle\int_{-\infty}^{+\infty}f(x,y)\mathrm{d}y=\begin{cases}\displaystyle\int_0^x 15xy^2\mathrm{d}y,&0<x<1\\0,&\text{其他}\end{cases}=\begin{cases}5x^4,&0<x<1\\0,&\text{其他}\end{cases}$;

(2) 当 $0<x<1$ 时

$$f_{Y|X}(y\,|\,x)=\frac{f(x,y)}{f_X(x)}=\begin{cases}\dfrac{15xy^2}{5x^4},&0<y<x\\0,&\text{其他}\end{cases}=\begin{cases}\dfrac{3y^2}{x^3},&0<y<x\\0,&\text{其他}\end{cases}$$

$$f_{Y|X}\left(y \left| \frac{1}{2}\right.\right) = \begin{cases} 24y^2, & 0 < y < \frac{1}{2}, \\ 0, & \text{其他} \end{cases} \quad P\left(Y < \frac{1}{4} \left| X = \frac{1}{2}\right.\right) = \int_0^{\frac{1}{4}} 24y^2 \mathrm{d}y = \frac{1}{8}$$

(3) Z 的取值范围为 $(0, 1)$. 当 $z < 0$ 时, $F_Z(z) = 0$; 当 $z \geq 1$ 时, $F_Z(z) = 1$.
当 $0 \leq z < 1$ 时,

$$F_Z(z) = P(Z \leq z) = P\left(\frac{Y}{X} \leq z\right) = \int_0^1 \mathrm{d}x \int_0^{zx} 15xy^2 \mathrm{d}y = \int_0^1 5z^3 x^4 \mathrm{d}x = z^3$$

于是

$$f_Z(z) = F_Z'(z) = \begin{cases} 3z^2, & 0 < z < 1 \\ 0, & \text{其他} \end{cases}$$

5. X 的密度函数为 $f(x) = \begin{cases} \mathrm{e}^{-x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$, $Y = \min\{X, 1\}$.

$$E(Y) = \int_0^1 x\mathrm{e}^{-x}\mathrm{d}x + \int_1^{+\infty} \mathrm{e}^{-x}\mathrm{d}x = 1 - \mathrm{e}^{-1};$$

$$E(Y^2) = \int_0^1 x^2\mathrm{e}^{-x}\mathrm{d}x + \int_1^{+\infty} \mathrm{e}^{-x}\mathrm{d}x = 2 - 4\mathrm{e}^{-1};$$

$$D(Y) = E(Y^2) - [E(Y)]^2 = 1 - 2\mathrm{e}^{-1} - \mathrm{e}^{-2}.$$

6. (1) $Z_1 = |X_1 - \mu|$ 的取值范围为 $[0, +\infty)$.

当 $z < 0$ 时, $F_Z(z) = 0$;

当 $z \geq 0$ 时, $F_Z(z) = P(Z_1 \leq z) = P(|X_1 - \mu| \leq z) = 2\Phi\left(\frac{z}{\sigma}\right) - 1$,

$$f_Z(z) = F_Z'(z) = \frac{2}{\sigma}\varphi\left(\frac{z}{\sigma}\right) = \frac{2}{\sqrt{2\pi}\sigma} e^{-\frac{z^2}{2\sigma^2}}$$

即 $$f_Z(z) = \begin{cases} \dfrac{2}{\sqrt{2\pi}\sigma} e^{-\frac{z^2}{2\sigma^2}}, & z \geq 0 \\ 0, & z < 0 \end{cases}$$

(2) $$L(\sigma) = \prod_{i=1}^n \frac{2}{\sqrt{2\pi}\sigma} e^{-\frac{z_i^2}{2\sigma^2}} = \frac{2^n}{(\sqrt{2\pi}\sigma)^n} e^{-\frac{1}{2\sigma^2}\sum\limits_{i=1}^n z_i^2}$$

$$\ln L(\sigma) = n\ln 2 - \frac{n}{2}\ln(2\pi) - n\ln\sigma - \frac{1}{2\sigma^2}\sum_{i=1}^n z_i^2$$

$$\frac{\mathrm{d}\ln L(\sigma)}{\mathrm{d}\sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^n z_i^2 = 0, \quad \text{解得} \; \hat{\sigma} = \sqrt{\frac{1}{n}\sum_{i=1}^n z_i^2}$$

故 σ 的极大似然估计量为 $\hat{\sigma} = \sqrt{\dfrac{1}{n}\sum\limits_{i=1}^n Z_i^2}$.

7. (1) $\dfrac{\overline{X}-\mu}{S/\sqrt{n}}=\dfrac{4(\overline{X}-\mu)}{S}\sim t(15)$，$P\left(-t_{0.975}(15)<\dfrac{4(\overline{X}-\mu)}{S}<t_{0.975}(15)\right)=0.95$

即 $P\left(\overline{X}-\dfrac{S}{4}t_{0.975}(15)<\mu<\overline{X}+\dfrac{S}{4}t_{0.975}(15)\right)=0.95$

故 μ 的置信度为 95% 的置信区间为

$$\left(\overline{x}-\dfrac{s}{4}\times t_{0.975}(15),\overline{x}-\dfrac{s}{4}\times t_{0.975}(15)\right)=\left(5.2-\dfrac{1.6}{4}\times 2.1315,5.2+\dfrac{1.6}{4}\times 2.1315\right)$$

$$=(4.3437,\,6.0526)$$

(2) 检验统计量为 $\chi^2=\dfrac{15S^2}{\sigma_0^2}=\dfrac{15S^2}{2.5}=6S^2$，

检验拒绝域为 $W=\{\chi^2=6S^2>\chi^2_{0.95}(15)\}=\{\chi^2=6S^2>24.996\}$

带入 $s^2=2.56$，计算得 $6s^2=6\times 2.56=15.36<24.996$

故不能拒绝原假设，不能认为 σ^2 显著大于 2.5.