Sphere Packing in Lean

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July 31, 2024

Abstract

This blueprint consists of an adaptation of Maryna Viazovska's Fields Medal-winning paper proving that no packing of unit balls in Euclidean space $\mathbb{R}^8$ has density greater than that of the E_8-lattice packing. This blueprint is a work in progress, and will be frequently updated and restructured as the formalisation effort progresses. We recommend that you look at this webpage for the latest version.

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1 Sphere packings

1.1 Sphere packings

The sphere packing constant measures which portion of d-dimensional Euclidean space can be covered by non-overlapping unit balls. More precisely, let $\mathbb{R}^d$ be the Euclidean vector space equipped with distance $\|\cdot\|$ and Lebesgue measure $\mathrm{Vol}(\cdot)$. For $x \in \mathbb{R}^d$ and $r \in \mathbb{R}_{>0}$ we denote by $B_d(x, r)$ the ball in $\mathbb{R}^d$ with center x and radius r. There are several types of sphere packings, which are determined by imposing certain conditions on the set of centres of the spheres involved in the packing. Therefore, we begin by defining those conditions on the centres of the spheres.

Below, c is any real number (the scaling factor), usually with constraints such as $c > 0$ or $c \neq 0$.

Definition 1. *We say that $X \subseteq \mathbb{R}^d$ is a set of* sphere packing centres with separation r *if X is discrete and $\|x - y\| \geq r$ for all $x \neq y \in X$.*

Definition 2. *We say that $X \subseteq \mathbb{R}^d$ is a set of* lattice packing centres with separation r *if X is both a set of sphere packing centres with separation r and a lattice in $\mathbb{R}^d$.*

Definition 3. *Let $X \subset \mathbb{R}^d$ be a set of sphere packing centres of separation r. Then, the set*

$$\mathcal{P} = \bigcup_{x \in X} B_d\left(x, \frac{r}{2}\right)$$

of unit balls centred at points in X is the corresponding sphere packing.

Remark 4. *If X is a set of lattice/periodic packing centres (see Definitions 2 and 9), then the corresponding sphere packing $\mathcal{P}$ is called a* lattice/periodic packing.

Definition 5. *The* finite density *of a packing $\mathcal{P}$ is defined as*

$$\Delta_{\mathcal{P}}(R) := \frac{\mathrm{Vol}(\mathcal{P} \cap B_d(0, R))}{\mathrm{Vol}(B_d(0, R))}, \quad R > 0.$$

Definition 6. *We define the* density *of a packing $\mathcal{P}$ as the limit supremum*

$$\Delta_{\mathcal{P}} := \limsup_{R \to \infty} \Delta_{\mathcal{P}}(R).$$

Definition 7. *The sphere packing problem is to compute the* sphere packing constant*, defined as supremum of packing densities over all possible packing*

$$\Delta_d := \sup_{\substack{\mathcal{P} \subset \mathbb{R}^d \\ \text{sphere packing}}} \Delta_{\mathcal{P}}.$$

1.2 Lattices and Periodic packings

Definition 8. *We say that an additive subgroup $\Lambda \leq \mathbb{R}^d$ is a* lattice *if it is discrete and its $\mathbb{R}$-span contains all the elements of $\mathbb{R}^d$.*

Definition 9. *We say that $X \subseteq \mathbb{R}^d$ is a set of* periodic packing centres *if X is a set of sphere packing centres and there exists a lattice $\Lambda \subset \mathbb{R}^d$ such that for any $x \in X$ and $y \in \Lambda$, their sum $x + y$ lies in X.*

Remove this, since it's a duplicate of section 2.3 (Density of periodic packings.

Lemma 10. *If $X \subseteq \mathbb{R}^d$ is a set of sphere packing centres that is periodic with respect to some lattice Λ, then the density of the corresponding (periodic) sphere packing is given by*

$$\frac{|X/\Lambda|}{\mathrm{Vol}(\mathbb{R}^d/\Lambda)} \cdot \mathrm{Vol}(B_d(0, 1))$$

where the quotients in the numerator and denominator correspond to the orbits of the action by translation of Λ on X and $\mathbb{R}^d$ respectively.

Remark 11. *This can be thought of as the "volume of spheres per fundamental domain": the number of spheres per fundamental domain is* $|X/\Lambda|$, *and the volume of each sphere is* $\mathrm{Vol}(B_d(0,1))$.

Definition 12. *The periodic sphere packing constant is defined to be*

$$\Delta_d^{periodic} := \sup_{\substack{P \subset \mathbb{R}^d \\ periodic\ packing}} \Delta_P$$

Theorem 13. *For all* d, *the periodic sphere packing constant in* $\mathbb{R}^d$ *is equal to the sphere packing constant in* $\mathbb{R}^d$.

Proof. **State this in Lean (ready). Fill in proof here (see [3, Appendix A]** ☐

In other words, it suffices to compute and optimise the periodic sphere packing constant.

1.3 Main Result

With the terminologies above, we can state the main theorem of this project.

Theorem 14. *All* periodic *packing* $\mathcal{P} \subseteq \mathbb{R}^8$ *has density satisfying* $\Delta_{\mathcal{P}} \leq \Delta_{E_8}$, *the density of the* E_8 *sphere packing (see Definition 36).*

Proof. We will prove this theorem over the course of this blueprint. ☐

Corollary 15. *All packing* $\mathcal{P} \subseteq \mathbb{R}^8$ *has density satisfying* $\Delta_{\mathcal{P}} \leq \Delta_{E_8}$.

Proof. This is a direct consequence of Theorem 13 and Theorem 14. ☐

Corollary 16. $\Delta_8 = \Delta_{E_8}$.

Proof. By definition, $\Delta_{E_8} \leq \Delta_8$, while Corollary 15 shows $\Delta_8 = \sup_{\text{packing } \mathcal{P}} \leq \Delta_{E_8}$, and the result follows. ☐

2 Density of packings

The definition of density given in Section 1 is inconvenient to work with, especially when our packing is a structured one, such as a periodic/lattice packing. This section fixes this problem.

2.1 Bounds on Finite Density of Packing

We first collect all the results we will prove here, then prove them separately below. We do this because some are proven already! Let $X \subseteq \mathbb{R}^d$ be a set of sphere packing centers with separation r.

Theorem 17. *We have the following theorem relating the finite density and the number of lattice points in a ball:*

$$\left|X \cap \mathcal{B}_d\left(R - \frac{r}{2}\right)\right| \cdot \frac{\mathrm{Vol}\left(\mathcal{B}_d\left(\frac{r}{2}\right)\right)}{\mathrm{Vol}(\mathcal{B}_d(R))} \leq \Delta_{\mathcal{P}}(R) \leq \left|X \cap \mathcal{B}_d\left(R + \frac{r}{2}\right)\right| \cdot \frac{\mathrm{Vol}\left(\mathcal{B}_d\left(\frac{r}{2}\right)\right)}{\mathrm{Vol}(\mathcal{B}_d(R))}$$

Proof. Proven by Gareth already. The high level idea is to prove that $\mathcal{P} \cap \mathcal{B}_d(R) = \left(\bigcup_{x \in X} \mathcal{B}_d\left(x, \frac{r}{2}\right)\right) \subseteq \bigcup_{x \in X \cap \mathcal{B}_d\left(R + \frac{r}{2}\right)} \mathcal{B}_d\left(x, \frac{r}{2}\right)$, and a similar inequality for the upper bound. The rest follows by rearranging and usign the fact that the balls are pairwise disjoint. $\square$

Suppose further that X is a periodic packing w.r.t. the lattice $\Lambda \subseteq \mathbb{R}^d$. Let $\mathcal{D}$ be a fundamental region of Λ, say the parallelopiped defined in the proof of Lemma 22, and let L be the bound on the norm of vectors in $\mathcal{D}$ (see Lemma 22).

Theorem 18. *For real numbers $R > L$, we have the following inequality relating the number of lattice points from Λ in a ball with the volume of the ball and the fundamental region $\mathcal{D}$:*

$$\frac{\mathrm{Vol}(\mathcal{B}_d(R - L))}{\mathrm{Vol}(\mathcal{D})} \leq |\Lambda \cap \mathcal{B}_d(R)| \leq \frac{\mathrm{Vol}(\mathcal{B}_d(R + L))}{\mathrm{Vol}(\mathcal{D})}$$

The proof can be found at Section 2.2.

Theorem 19. *For real numbers $R > L$, we have the following inequality relating the number of points from X (periodic w.r.t. Λ) in a ball with the number of points from Λ:*

$$|\Lambda \cap \mathcal{B}_d(R - L)|\,|X/\Lambda| \leq |X \cap \mathcal{B}_d(R)| \leq |\Lambda \cap \mathcal{B}_d(R + L)|\,|X/\Lambda|$$

Link the proof

Finally, when we combine the inequalities above, we need one additional computational lemma.

Lemma 20. *For any constant $C > 0$, we have*

$$\lim_{R \to \infty} \frac{\mathrm{Vol}(\mathcal{B}_d(R))}{\mathrm{Vol}(\mathcal{B}_d(R + C))} = 1$$

Proof. Write out the formula for volume of a ball and simplify. More specifically, we have $\mathrm{Vol}(\mathcal{B}_d(R)) = R^d \pi^{d/2} / \Gamma\left(\frac{d}{2} + 1\right)$, so $\mathrm{Vol}(\mathcal{B}_d(R))/\mathrm{Vol}(\mathcal{B}_d(R + C)) = R^d/(R + C)^d = \left(1 - \frac{1}{R+C}\right)^d = 1$. $\square$

2.2 Bounds on Finite Density of Periodic Prcking

In this subsection, we build up results about the density of periodic packings. In particular, the density of a periodic packing, defined as the limit of the periodic packing intersected with a growing ball centered at the origin, is equal to the density within any fundamental region of the period lattice. The strategy is to prove lower and upper bounds for the number of lattice points in a ball in terms of the volume of the ball, correct up to the highest order term. Taking limit gives the correct density!

Below, let $X \subseteq \mathbb{R}^d$ be a set of periodic packing centers with respect to the lattice $\Lambda \subset \mathbb{R}^d$. We write $kX := \{kv : v \in X\}$.

Definition 21. *Let $\Lambda \subset \mathbb{R}^d$ be a lattice. A set $\mathcal{D} \subseteq \mathbb{R}^d$ is a* fundamental domain *of Λ such that for all distinct $x, y \in \Lambda$, we have $(x + \mathcal{D}) \cap (y + \mathcal{D}) = \emptyset$ (disjointness) and $\bigcup_{x \in \Lambda} x + \mathcal{D} = \mathbb{R}^d$ (tiling).*

Lemma 22. *There always exists a* bounded *fundamental region $\mathcal{D}$ of Λ.*

Proof. Since lattices have $\mathbb{Z}$-bases, there exists a set of vectors $\mathcal{B} \subseteq \mathbb{R}^d$ such that $\Lambda = \operatorname{span}_{\mathbb{Z}}(\mathcal{B})$. We claim that $\mathcal{D}_\Lambda = \{\sum_i c_i \mathcal{B}_i \subseteq \mathbb{R}^n : c_i \in [0,1)^n\}$ is a fundamental domain. The rest exists in Mathlib already so I don't bother elaborating here :) From the definition, we see that for $v = \sum_i c_i \mathcal{B}_i \in \mathcal{D}_\Lambda$, we have $\|v\| \leq \sum_i \|c_i \mathcal{B}_i\| \leq \sum_i \|\mathcal{B}_i\|$, which is a constant. Hence, $\mathcal{D}_\Lambda$ is bounded. $\square$

We denote by L the bound of norm of vectors in the fundamental domain $\mathcal{D}$.

Lemma 23. *For all vectors $v \in \mathbb{R}^d$ there exists a unique lattice point $x \in \Lambda$ such that $v \in x + \mathcal{D}$.*

Proof. By the tiling property of the fundamental domain, we have $v \in \bigcup_{x \in \Lambda}(x + \mathcal{D})$. By definition, this means there exists a lattice point $x \in \Lambda$ such that $v \in x + \mathcal{D}$. To show that it is unique, suppose that $v \in (x+\mathcal{D}) \cap (y+\mathcal{D})$ for distinct $x \neq y \in \Lambda$. By the disjointness property, $v \in \emptyset$, contradiction. $\square$

Proof of Theorem 18. For the first inequality, it suffices to prove that $\mathcal{B}_d(R-L) \subseteq \bigcup_{x \in \Lambda \cap \mathcal{B}_d(R)}(x+\mathcal{D})$, since the cosets on the right are almost disjoint. For a vector $v \in \mathcal{B}_d(R-L)$, we have $\|v\| < R - L$ by definition. By Lemma Lemma 23, there exists a lattice point $x \in \Lambda$ such that $v \in x + \mathcal{D}$. Rearranging gives $v - x \in \mathcal{D}$, which means $\|v - x\| \leq L$. By the triangle inequality, $\|x\| < R$, i.e. $x \in \mathcal{B}_d(L)$, concluding the proof.

«««< HEAD For the second inequality, we prove that $\bigcup_{x \in \Lambda \cap \mathcal{B}_d(R)}(x + \mathcal{D}) \subseteq \mathcal{B}_d(R + L)$. Consider a vector $x \in \Lambda \cap \mathcal{B}_d(R)$ and a vector $y \in x + \mathcal{D}$. From above, we know $\|x\| < R$ and $\|y - x\| \leq L$, so $\|y\| < R + L$, concluding the proof. $\square$

Next, we build up to the proof for **??**

Definition 24. ***Here we define X/Λ.***

Proof of **??**. **Fill in proof.** $\square$

3 The E_8 lattice

3.1 Definitions of E_8 lattice

There are several equivalent definitions of the E_8 lattice. Below, we formalise two of them, and prove they are equivalent.

Definition 25. *E_8-lattice, Definition 1We define the E_8-*lattice *(as a subset of $\mathbb{R}^8$) to be*

$$\Lambda_8 = \{(x_i) \in \mathbb{Z}^8 \cup (\mathbb{Z} + \tfrac{1}{2})^8 \mid \sum_{i=1}^{8} x_i \equiv 0 \pmod 2\}.$$

Definition 26. *We define the* scaled E_8-lattice *(by a real number c) as*

$$c\Lambda_8 = \{c \cdot \vec{v} : \vec{v} \in \Lambda_8\}$$

.

Definition 27. *E_8-lattice, Definition 2We define the E_8* basis vectors *to be the set of vectors*

$$\mathcal{B}_8 = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2 \\ -1/2 \\ -1/2 \\ -1/2 \\ -1/2 \\ -1/2 \\ -1/2 \\ -1/2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \right\}$$

Definition 28. *We define the* scaled E_8 basis vectors *(by a real number c) to be $c\mathcal{B}_8 = \{c\cdot\vec{v} : \vec{v} \in \mathcal{B}_8\}$.*

Theorem 29. *The two definitions above coincide, i.e. $c\Lambda_8 = \operatorname{span}_{\mathbb{Z}}(c\mathcal{B}_8)$.*

Proof. We prove each side contains the other side.

For a vector $\vec{v} \in \Lambda_8 \subseteq \mathbb{R}^8$, we have $\sum_i \vec{v}_i \equiv 0 \pmod 2$ and $\vec{v}_i$ being either all integers or all half-integers. After some modulo arithmetic, it can be seen that $\mathcal{B}_8^{-1}\vec{v}$ as integer coordinates (i.e. it is congruent to 0 modulo 1). Hence, $\vec{v} \in \operatorname{span}_{\mathbb{Z}}(\mathcal{B}_8)$.

For the opposite direction, we write the vector as $\vec{v} = \sum_i c_i \mathcal{B}_8^i \in \operatorname{span}_{\mathbb{Z}}(\mathcal{B}_8)$ with c_i being integers and $\mathcal{B}_8^i$ being the i-th basis vector. Expanding the definition then gives $\vec{v} = \left(c_1 - \frac{1}{2}c_7, -c_1 + c_2 - \frac{1}{2}c_7, \cdots, -\frac{1}{2}c_7\right)$. Again, after some modulo arithmetic, it can be seen that $\sum_i \vec{v}_i$ is indeed 0 modulo 2 and are all either integers and half-integers, concluding the proof.

(Note: this proof doesn't depend on that $\mathcal{B}_8$ is linearly independent.) $\square$

3.2 Basic Properties of E_8 lattice

In this section, we establish basic properties of the E_8 lattice and the $\mathcal{B}_8$ vectors.

Lemma 30. *For nonzero real numbers c, the set $c\mathcal{B}_8$ is a $\mathbb{R}$-basis of $\mathbb{R}^8$.*

Proof. It suffices to prove that $\mathcal{B}_8 \in \mathrm{GL}_8(\mathbb{R})$. We prove this by explicitly defining the inverse matrix $\mathcal{B}_8^{-1}$ and proving $\mathcal{B}_8\mathcal{B}_8^{-1} = I_8$, which implies that $\det(\mathcal{B}_8)$ is nonzero. $\square$

Lemma 31. *For real numbers c, $c\Lambda_8$ is an additive subgroup of $\mathbb{R}^8$.*

Proof. Trivially follows from that $\Lambda_8 \subseteq \mathbb{R}^8$ is the $\mathbb{Z}$-span of $\mathcal{B}_8$ and hence an additive group. $\square$

Lemma 32. *All vectors in Λ_8 have norm of the form $\sqrt{2n}$, where n is a nonnegative inteeger.*

Proof. Writing $\vec{v} = \sum_i c_i \mathcal{B}_8^i$, we have $\|v\|^2 = \sum_i \sum_j c_i c_j (\mathcal{B}_8^i \cdot \mathcal{B}_8^j)$. Computing all 64 pairs of dot products and simplifying, we get a massive term that is a quadratic form in c_i with even integer coefficients, concluding the proof. $\square$

Lemma 33. *For nonzero real numbers c, $c\Lambda_8$ is discrete, i.e. that the subspace topology induced by its inclusion into $\mathbb{R}^8$ is the discrete topology.*

Proof. We prove this for $c = 1$. Since Λ_8 is a topological group and $+$ is continuous, it suffices to prove that $\{0\}$ is open in Λ_8. This follows from the fact that there is an open ball $\mathcal{B}(\sqrt{2}) \subseteq \mathbb{R}^8$ around it containing no other lattice points, since the shortest nonzero vector has norm $\sqrt{2}$. $\square$

Lemma 34. *For nonzero real numbres c, $c\Lambda_8$ is a lattice, i.e. it is discrete and spans $\mathbb{R}^8$ over $\mathbb{R}$.*

Proof. The first part is the above lemma. The second part follows from that $\mathcal{B}_8$ is a basis and hence linearly independent over $\mathbb{R}$. $\square$

Prove E_8 is unimodular. Prove E_8 is positive-definite.

3.3 The E_8 sphere packing

Lemma 35. *For nonzero real numbers c, $c\Lambda_8$ is a valid set of sphere packing centres with separation $|c|\sqrt{2}$.*

Proof. This follows directly from Theorem 32. $\square$

Definition 36. *The E_8* sphere packing *is the sphere packing with separation 1, whose set of centres is $\frac{1}{\sqrt{2}}\Lambda_8$.*

Theorem 37. *We have $\Delta_{E_8} = \frac{\pi^4}{384}$.*

Proof. **Finish proof. Preferably we want APIs about fundamental region of lattice, and use that to reduce this theorem to computation inside the fundamental region, and use formula for volume of ball.** $\square$

4 Facts from Fourier analysis

In this section, we recall a few definitions from Fourier analysis.

Definition 38. *The Fourier transform of an L^1-function $f : \mathbb{R}^d \to \mathbb{C}$ is defined as*

$$\mathcal{F}(f)(y) = \widehat{f}(y) := \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle x,y \rangle} \, \mathrm{d}x, \quad y \in \mathbb{R}^d$$

where $\langle x, y \rangle = \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2 - \frac{1}{2}\|x-y\|^2$ is the standard scalar product in $\mathbb{R}^d$.

Definition 39. *A C^∞ function $f : \mathbb{R}^d \to \mathbb{C}$ is called a* Schwartz function *if it goes to zero as $\|x\| \to \infty$ faster then any inverse power of $\|x\|$, and the same holds for all partial derivatives of f.*

Definition 40. *The set of all Schwartz functions is called a* Schwartz space.

Lemma 41. *The Fourier transform is an automorphism of the space of Schwartz functions.*

Proof. **Fill in proof.** □

Lemma 42.

$$\mathcal{F}(e^{\pi i \|x\|^2 z})(y) = z^{-4} \, e^{\pi i \|y\|^2 \, (\frac{-1}{z})}.$$

Proof. **Fill in proof.** □

Theorem 43 ((Poisson summation formula))**.**

$$\sum_{\ell \in \Lambda} f(\ell) = \frac{1}{\mathrm{Vol}(\mathbb{R}^d/\Lambda)} \sum_{m \in \Lambda^*} \widehat{f}(m).$$

Proof. **Fill in proof.** □

5 Cohn-Elkies linear programming bounds

In 2003 Cohn and Elkies [3] developed linear programming bounds that apply directly to sphere packings. The goal of this section is to formalize the Cohn–Elkies linear programming bound.

The following theorem is the key result of [3]. (The original theorem is stated for a class of functions more general then Schwartz functions)

Theorem 44. *(Cohn, Elkies [3]) Suppose that $f : \mathbb{R}^d \to \mathbb{R}$ is a Schwartz function that is not identically zero and satisfies the following conditions:*

$$f(x) \leq 0 \text{ for } \|x\| \geq 1 \tag{1}$$

and

$$\widehat{f}(x) \geq 0 \text{ for all } x \in \mathbb{R}^d. \tag{2}$$

Then the density of d-dimensional sphere packings is bounded above by

$$\frac{f(0)}{\widehat{f}(0)} \cdot \mathrm{vol}(B_d(0, 1/2)).$$

Proof. Here we reproduce the proof given in [3]. We will first prove the theorem for periodic packings.

Let $X \subset \mathbb{R}^d$ be a discrete subset such that $\|x - y\| \geq 1$ for any distinct $x, y \in X$. Suppose that X is Λ-periodic with respect to some lattice $\Lambda \subset \mathbb{R}^d$.

The inequality

$$\sharp(X/\Lambda) \cdot f(0) \geq \sum_{x \in X} \sum_{y \in X/\Lambda} f(x - y) = \sum_{x \in X/\Lambda} \sum_{y \in X/\Lambda} \sum_{\ell \in \Lambda} f(x - y + l) \tag{3}$$

follows from the condition (1) of the theorem and the assumption on the distances between points in X. The equality

$$\sum_{x \in X/\Lambda} \sum_{y \in X/\Lambda} \sum_{\ell \in \Lambda} f(x - y + l) = \sum_{x \in X/\Lambda} \sum_{y \in X/\Lambda} \frac{1}{\mathrm{vol}(\mathbb{R}^d/\Lambda)} \sum_{m \in \Lambda^*} \widehat{f}(m)\, e^{2\pi i m(x-y)}.$$

follows from the Poisson summation formula. The right hand side of the above equation can be written as

$$\sum_{x \in X/\Lambda} \sum_{y \in X/\Lambda} \frac{1}{\mathrm{vol}(\mathbb{R}^d/\Lambda)} \sum_{m \in \Lambda^*} \widehat{f}(m)\, e^{2\pi i m(x-y)} = \frac{1}{\mathrm{vol}(\mathbb{R}^d/\Lambda)} \sum_{m \in \Lambda^*} \widehat{f}(m) \cdot \Big| \sum_{x \in X/\Lambda} e^{2\pi i m x} \Big|^2.$$

Note that $\big|\sum_{x \in X/\Lambda} e^{2\pi i m x}\big|^2 \geq 0$ for all $m \in \Lambda^*$. Moreover, the term corresponding to $m = 0$ satisfies $\big|\sum_{x \in X/\Lambda} e^{2\pi i 0 x}\big|^2 = \sharp(X/\Lambda)^2$. Now we use the condition (2) and estimate

$$\frac{1}{\mathrm{vol}(\mathbb{R}^d/\Lambda)} \sum_{m \in \Lambda^*} \widehat{f}(m) \cdot \Big| \sum_{x \in X/\Lambda} e^{2\pi i m(x-y)} \Big|^2 \geq \frac{\sharp(X/\Lambda)^2}{\mathrm{vol}(\mathbb{R}^d/\Lambda)} \cdot \widehat{f}(0). \tag{4}$$

Comparing inequalities (3) and (4) we arrive at

$$\frac{\sharp(X/\Lambda)}{\mathrm{vol}(\mathbb{R}^d/\Lambda)} \leq \frac{f(0)}{\widehat{f}(0)}.$$

Now we see that the density of the periodic packing $\mathcal{P}_X$ with balls of radius $1/2$ is bounded by

$$\Delta(\mathcal{P}_X) = \frac{\sharp(X/\Lambda)}{\mathrm{vol}(\mathbb{R}^d/\Lambda)} \cdot \mathrm{vol}(B_d(0, 1/2)) \leq \frac{f(0)}{\widehat{f}(0)} \cdot \mathrm{vol}(B_d(0, 1/2)).$$

This finishes the proof of the theorem for periodic packings. Theorem 13 implies the desired result for arbitrary packings. □

The main step in our proof of Theorem 14 is the explicit construction of an optimal function. It will be convenient for us to scale this function by $\sqrt{2}$.

Theorem 45. *There exists a radial Schwartz function $g : \mathbb{R}^8 \to \mathbb{R}$ which satisfies:*

$$g(x) \leq 0 \text{ for } \|x\| \geq \sqrt{2} \tag{5}$$

$$\widehat{g}(x) \geq 0 \text{ for all } x \in \mathbb{R}^8 \tag{6}$$

$$g(0) = \widehat{g}(0) = 1. \tag{7}$$

Theorem 44 applied to the optimal function $f(x) = g(x/\sqrt{2})$ immediately implies Theorem 14.

6 Modular forms

In this section, we recall and develop some theory of (quasi)modular forms.

Let $\mathfrak{H}$ be the upper half-plane $\{z \in \mathbb{C} \mid \Im(z) > 0\}$.

Lemma 46. *The modular group* $\Gamma_1 := \mathrm{PSL}_2(\mathbb{Z})$ *acts on* $\mathfrak{H}$ *by linear fractional transformations*

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) z := \frac{az+b}{cz+d}.$$

Let N be a positive integer.

Definition 47. *The* level N principal congruence subgroup *of* Γ_1 *is*

$$\Gamma(N) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1 \middle| \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \bmod N \right\}.$$

Definition 48. *A subgroup* $\Gamma \subset \Gamma_1$ *is called a* congruence subgroup *if* $\Gamma(N) \subset \Gamma$ *for some* $N \in \mathbb{N}$.

An important example of a congruence subgroup is

$$\Gamma_0(N) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1 \middle| \ c \equiv 0 \bmod N \right\}.$$

Let $z \in \mathfrak{H}$, $k \in \mathbb{Z}$, and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$.

Definition 49. *The* automorphy factor *of weight* k *is defined as*

$$j_k(z, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)) := (cz+d)^{-k}.$$

Lemma 50. *The automorphy factor satisfies the* chain rule

$$j_k(z, \gamma_1\gamma_2) = j_k(z, \gamma_1)\, j_k(\gamma_2 z, \gamma_1).$$

Definition 51. *Let* F *be a function on* $\mathfrak{H}$ *and* $\gamma \in \mathrm{PSL}_2(\mathbb{Z})$. *Then the* slash operator *acts on* F *by*

$$(F|_k\gamma)(z) := j_k(z, \gamma)\, F(\gamma z).$$

Lemma 52. *The chain rule implies*

$$F|_k\gamma_1\gamma_2 = (F|_k\gamma_1)|_k\gamma_2.$$

Definition 53. *A* (holomorphic) modular form *of integer weight* k *and congruence subgroup* Γ *is a holomorphic function* $f : \mathfrak{H} \to \mathbb{C}$ *such that:*

1. $f|_k\gamma = f$ *for all* $\gamma \in \Gamma$
2. *for each* $\alpha \in \Gamma_1$ $f|_k\alpha$ *has the Fourier expansion* $f|_k\alpha(z) = \sum_{n=0}^{\infty} c_f(\alpha, \frac{n}{n_\alpha})\, e^{2\pi i \frac{n}{n_\alpha} z}$ *for some* $n_\alpha \in \mathbb{N}$ *and Fourier coefficients* $c_f(\alpha, m) \in \mathbb{C}$.

Definition 54. *Let* $M_k(\Gamma)$ *be the space of modular forms of weight* k *and congruence subgroup* Γ.

A key fact in the theory of modular forms is the following theorem:

Theorem 55. *The spaces* $M_k(\Gamma)$ *are finite dimensional.*

Let us consider several examples of modular forms.

Definition 56. *For an even integer* $k \geq 4$ *we define the* weight k Eisenstein series *as*

$$E_k(z) := \frac{1}{2\zeta(k)} \sum_{(c,d) \in \mathbb{Z}^2 \setminus (0,0)} (cz+d)^{-k}. \tag{8}$$

Lemma 57. *For all* k, $E_k \in M_k(\Gamma_1)$

Proof. This follows from the fact that the sum converges absolutely. □

Lemma 58. *The Eisenstein series possesses the Fourier expansion*

$$E_k(z) = 1 + \frac{2}{\zeta(1-k)} \sum_{n=1}^{\infty} \sigma_{k-1}(n)\, e^{2\pi i z}, \tag{9}$$

where $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$*. In particular, we have*

$$\begin{aligned} E_4(z) &= 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)\, e^{2\pi i n z} \\ E_6(z) &= 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)\, e^{2\pi i n z}. \end{aligned}$$

The infinite sum (8) does not converge absolutely for $k = 2$. On the other hand, the expression (9) converges to a holomorphic function on the upper half-plane and therefore

Definition 59. *We set*

$$E_2(z) := 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n)\, e^{2\pi i n z}.$$

Lemma 60. *This function is not modular, however it satisfies*

$$z^{-2}\, E_2\left(\frac{-1}{z}\right) = E_2(z) - \frac{6i}{\pi}\,\frac{1}{z}.$$

The proof of this identity can be found in [12, Section 2.3]. The weight two Eisenstein series E_2 is an example of a *quasimodular form* [12, Section 5.1].

The discriminant form is a unique normalized cusp form of weight 12, which can be defined using E_4 and E_6.

Definition 61. *The* discriminant form $\Delta(z)$ *is given by*

$$\Delta(z) = \frac{E_4(z)^3 - E_6(z)^2}{1728} \tag{10}$$

Lemma 62. $\Delta(z) \in M_{12}(\Gamma_1)$ *and it vanishes at the unique cusp, i.e. it is a cusp form of level* Γ_1 *and weight* 12*.*

Proof. Being a modular form of desired weight and level directly follows from those of E_4 and E_6. It is a cusp form since the constant terms of Fourier expansions of E_4 and E_6 are both 1. ☐

It also admits a product formula, which allow us to prove positivity of $\Delta(it)$ for $t > 0$ later.

Lemma 63. *We have*

$$\Delta(z) = e^{2\pi i z} \prod_{n \geq 1} (1 - e^{2\pi i n z})^{24}. \tag{11}$$

Proof. There are several known proofs of this. One possible proof that we can formalize is from Kohnen [6], which prove

$$\frac{1}{2\pi i z}\frac{d}{dz}(\log \Delta) = 1 - 24 \sum_{n \geq 1} \frac{n e^{2\pi i n z}}{1 - e^{2\pi i n z}}. \tag{12}$$

by using a multiplicative analogue of the Hecke operator and the valence formula. ☐

Note that the RHS of (12) is equal to the $E_2(z)$. As a side note, we can also consider defining Δ as (11), and prove that it coincides with (10). Such an argument can be found in [2, Section 2.4].

Another example of modular forms we would like to consider are *theta functions* [12, Section 3.1].

Definition 64. *We define three different theta functions (so called "Thetanullwerte") as*

$$\begin{aligned}\theta_{00}(z) &= \sum_{n\in\mathbb{Z}} e^{\pi i n^2 z}\\ \theta_{01}(z) &= \sum_{n\in\mathbb{Z}} (-1)^n\, e^{\pi i n^2 z}\\ \theta_{10}(z) &= \sum_{n\in\mathbb{Z}} e^{\pi i (n+\frac{1}{2})^2 z}.\end{aligned}$$

The group Γ_1 is generated by the elements $T = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$.

Lemma 65. *These elements act on the theta functions in the following way*

$$z^{-2}\,\theta_{00}^4\left(\frac{-1}{z}\right) = -\,\theta_{00}^4(z) \tag{13}$$

$$z^{-2}\,\theta_{01}^4\left(\frac{-1}{z}\right) = -\,\theta_{10}^4(z) \tag{14}$$

$$z^{-2}\,\theta_{10}^4\left(\frac{-1}{z}\right) = -\,\theta_{01}^4(z) \tag{15}$$

and

$$\theta_{00}^4(z+1) = \theta_{01}^4(z)$$

$$\theta_{01}^4(z+1) = \theta_{00}^4(z) \tag{16}$$

$$\theta_{10}^4(z+1) = -\,\theta_{10}^4(z). \tag{17}$$

Proof. The last three identities easily follow from the definition, and (15) and (14) are equivalent under $z \leftrightarrow -1/z$, so it is enough to show (13) and (15). These identities follow from the Poisson summation formula, which is already formalized by David Loeffler. More precisely, for the *two-variable* Jacobi theta function and its derivative (be careful for the variables, here we use τ instead of z)

$$\theta(z,\tau) = \sum_{n\in\mathbb{Z}} e^{2\pi i n z + \pi i n^2 \tau}$$

that specialize to $\theta_{00}(\tau) = \theta(0,\tau)$, $\theta_{01} = \theta(1/2,\tau)$, and $\theta_{10}(\tau) = e^{\pi i\tau/4}\theta(-\tau/2,\tau)$. Possion summation formula gives

$$\theta(z,\tau) = \frac{1}{\sqrt{-i\tau}} e^{-\frac{\pi i z^2}{\tau}}\,\theta\left(\frac{z}{\tau}, -\frac{1}{\tau}\right)$$

and applying the specializations above yield the identities. For example, (15) follows from

$$\theta_{01}(\tau) = \theta\left(\frac{1}{2},\tau\right) = \frac{1}{\sqrt{-i\tau}} e^{-\frac{\pi i}{4\tau}}\,\theta\left(\frac{1}{2\tau}, -\frac{1}{\tau}\right) = \frac{1}{\sqrt{-i\tau}}\theta_{10}\left(-\frac{1}{\tau}\right)$$

and taking 4th power. □

Using the above identities, we can prove that these are modular forms.

Lemma 66. *The theta functions $\theta_{00}^4, \theta_{01}^4$, and θ_{10}^4 belong to $M_2(\Gamma(2))$.*

Proof. Since the group $\Gamma(2)$ is generated by two elements

$$\alpha = \begin{pmatrix}1&2\\0&1\end{pmatrix}, \quad \beta = \begin{pmatrix}1&0\\2&1\end{pmatrix},$$

it is enough to show that they are invariant under the slash actions with respect to α and β. For θ_{01}, (16) and (14)

$$\begin{aligned}(\theta_{01}^4|_2\alpha)(z) &= \theta_{01}^4(z+2) = \theta_{00}^4(z+1) = \theta_{01}^4(z)\\ (\theta_{01}^4|_2\beta)(z) &= (2z+1)^{-2}\theta_{00}^4\left(\frac{z}{2z+1}\right) = (2z+1)^{-2}\cdot - \left(-\frac{2z+1}{z}\right)^2 \theta_{10}^4\left(-\frac{2z+1}{z}\right)\\ &= -z^{-2}\theta_{10}^4\left(-2-\frac{1}{z}\right) = -z^{-2}\theta_{10}^4\left(-\frac{1}{z}\right) = \theta_{01}^4(z).\end{aligned}$$

We can prove θ_{10}^4 and θ_{00}^4 similarly. □

We also have a nontrivial relation between these theta functions.

Lemma 67. *These three theta functions satisfy the* Jacobi identity

$$\theta_{01}^4 + \theta_{10}^4 = \theta_{00}^4.$$

Proof. One possible proof is to use dimesion formula of the space of modular forms of weight 2 and level $\Gamma(2)$. Especially, this space have dimension 2, with basis $\theta_{01}^4, \theta_{10}^4$. $\square$

The *strict* positivity of Jacobi theta functions might needed later.

Lemma 68. *All three functions $t \mapsto \theta_{01}(it), \theta_{10}(it), \theta_{00}(it)$ are positive for $t > 0$.*

Proof. By Lemma 67 and the transformation law (14), it is enough to prove the positivity for $\theta_{10}(it)$, which is clear from its definition. $\square$

Note that we only need the fourth powers of theta functions in Lemma 66 to define (83), not $\theta_{00}, \theta_{01}, \theta_{10}$ themselves.

Definition 69. *A* weakly-holomorphic modular form *of integer weight k and congruence subgroup Γ is a holomorphic function $f : \mathfrak{H} \to \mathbb{C}$ such that:*

1. *$f|_k\gamma = f$ for all $\gamma \in \Gamma$*
2. *for each $\alpha \in \Gamma_1$ $f|_k\alpha$ has the Fourier expansion $f|_k\alpha(z) = \sum_{n=n_0}^{\infty} c_f(\alpha, \frac{n}{n_\alpha})\, e^{2\pi i \frac{n}{n_\alpha} z}$ for some $n_0 \in \mathbb{Z}$ and $n_\alpha \in \mathbb{N}$.*

For an m-periodic holomorphic function f and $n \in \frac{1}{m}\mathbb{Z}$ we will denote the n-th Fourier coefficient of f by $c_f(n)$ so that

$$f(z) = \sum_{n \in \frac{1}{m}\mathbb{Z}} c_f(n)\, e^{2\pi i n z}.$$

We denote the space of weakly-holomorphic modular forms of weight k and group Γ by $M_k^!(\Gamma)$. The spaces $M_k^!(\Gamma)$ are infinite dimensional. Probably the most famous weakly-holomorphic modular form is the *elliptic j-invariant*

$$j := \frac{1728\, E_4^3}{E_4^3 - E_6^2}.$$

This function belongs to $M_0^!(\Gamma_1)$ and has the Fourier expansion

$$j(z) = q^{-1} + 744 + 196884\, q + 21493760\, q^2 + 864299970\, q^3 + 20245856256\, q^4 + O(q^5)$$

where $q = e^{2\pi i z}$. Using a simple computer algebra system such as PARI GP or Mathematica one can compute first hundred terms of this Fourier expansion within few seconds. An important question is to find an asymptotic formula for $c_j(n)$, the n-th Fourier coefficient of j. Using the Hardy-Ramanujan circle method [9] or the non-holomorphic Poincare series [8] one can show that

Lemma 70.

$$c_j(n) = \frac{2\pi}{n} \sum_{k=1}^{\infty} \frac{A_k(n)}{k}\, I_1\left(\frac{4\pi\sqrt{n}}{k}\right) \qquad n \in \mathbb{Z}_{>0}$$

where

$$A_k(n) = \sum_{\substack{h \bmod k \\ (h,k)=1}} e^{\frac{-2\pi i}{k}(nh+h')}, \quad hh' \equiv -1 (mod\, k),$$

and $I_\alpha(x)$ denotes the modified Bessel function of the first kind defined as in [1, Section 9.6].

A similar convergent asymptotic expansion holds for the Fourier coefficients of any weakly holomorphic modular form [5], [2, Propositions 1.10 and 1.12]. Such a convergent expansion implies effective estimates for the Fourier coefficients.

7 Fourier eigenfunctions with double zeroes at lattice points

In this section we construct two radial Schwartz functions $a, b : \mathbb{R}^8 \to i\mathbb{R}$ such that

$$\mathcal{F}(a) = a \tag{18}$$
$$\mathcal{F}(b) = -b \tag{19}$$

which double zeroes at all Λ_8-vectors of length greater than $\sqrt{2}$. Recall that each vector of Λ_8 has length $\sqrt{2n}$ for some $n \in \mathbb{N}_{\geq 0}$. We define a and b so that their values are purely imaginary because this simplifies some of our computations. We will show in Section 8 that an appropriate linear combination of functions a and b satisfies conditions (5)–(7).

First, we will define function a. To this end we consider the following functions:

Definition 71.

$$\begin{aligned}\phi_{-4} &:= -\,Dj\,E_6^{-1}\\ \phi_{-2} &:= \phi_{-4}\,E_2 + Dj\,E_4^{-1}\\ \phi_0 &:= \phi_{-4}\,E_2^2 + 2Dj\,E_4^{-1}\,E_2 + j - 1728.\end{aligned}$$

Here $Dj(z) = \frac{1}{2\pi i}\frac{d}{dz}j(z)$.

Lemma 72. *These functions have the Fourier expansions*

$$\phi_{-4}(z) = q^{-1} + 504 + 73764\,q + 2695040\,q^2 + 54755730\,q^3 + O(q^4) \tag{20}$$
$$\phi_{-2}(z) = 720 + 203040\,q + 9417600\,q^2 + 223473600\,q^3 + 3566782080\,q^4 + O(q^5)$$
$$\phi_0(z) = 518400\,q + 31104000\,q^2 + 870912000\,q^3 + 15697152000\,q^4 + O(q^5) \tag{21}$$

where $q = e^{2\pi i z}$.

The function $\phi_0(z)$ is not modular; however,

Lemma 73. *The identity 60 implies the following transformation rule:*

$$\phi_0\Big(\frac{-1}{z}\Big) = \phi_0(z) - \frac{12i}{\pi}\,\frac{1}{z}\,\phi_{-2}(z) - \frac{36}{\pi^2}\,\frac{1}{z^2}\,\phi_{-4}(z). \tag{22}$$

Definition 74. *For $x \in \mathbb{R}^8$ we define*

$$\begin{aligned}a(x) :=& \int_{-1}^{i} \phi_0\Big(\frac{-1}{z+1}\Big)\,(z+1)^2\,e^{\pi i \|x\|^2 z}\,dz + \int_{1}^{i} \phi_0\Big(\frac{-1}{z-1}\Big)\,(z-1)^2\,e^{\pi i \|x\|^2 z}\,dz\\ &-2\int_{0}^{i} \phi_0\Big(\frac{-1}{z}\Big)\,z^2\,e^{\pi i \|x\|^2 z}\,dz + 2\int_{i}^{i\infty} \phi_0(z)\,e^{\pi i \|x\|^2 z}\,dz.\end{aligned} \tag{23}$$

We observe that the contour integrals in (23) converge absolutely and uniformly for $x \in \mathbb{R}^8$. Indeed, $\phi_0(z) = O(e^{-2\pi i z})$ as $\Im(z) \to \infty$. Therefore, $a(x)$ is well defined. Now we prove that a satisfies condition (18).

Proposition 75.
The function a defined by (23) belongs to the Schwartz space and satisfies

$$\widehat{a}(x) = a(x).$$

Proof. First, we prove that a is a Schwartz function. From Lemma 58, Definition 59, and 70 we deduce that the Fourier coefficients of ϕ_0 satisfy

$$|c_{\phi_0}(n)| \leq 2\,e^{4\pi\sqrt{n}} \quad n \in \mathbb{Z}_{>0}.$$

Thus, there exists a positive constant C such that

$$|\phi_0(z)| \le C\, e^{-2\pi \Im z} \qquad \text{for } \Im z > \frac{1}{2}.$$

We estimate the first summand in the right-hand side of (23). For $r \in \mathbb{R}_{\ge 0}$ we have

$$\Big|\int\limits_{-1}^{i} \phi_0\Big(\frac{-1}{z+1}\Big)\,(z+1)^2\, e^{\pi i r^2 z}\,dz\Big| = \Big|\int\limits_{i\infty}^{-1/(i+1)} \phi_0(z)\, z^{-4}\, e^{\pi i r^2(-1/z-1)}\,dz\Big| \le$$
$$C_1 \int\limits_{1/2}^{\infty} e^{-2\pi t}\, e^{-\pi r^2/t}\,dt \le C_1 \int\limits_{0}^{\infty} e^{-2\pi t}\, e^{-\pi r^2/t}\,dt = C_2\, r\, K_1(2\sqrt{2}\,\pi\, r)$$

where C_1 and C_2 are some positive constants and $K_\alpha(x)$ is the modified Bessel function of the second kind defined as in [1, Section 9.6]. This estimate also holds for the second and third summand in (23). For the last summand we have

$$\Big|\int\limits_{i}^{i\infty} \phi_0(z)\, e^{\pi i r^2 z}\,dz\Big| \le C \int\limits_{1}^{\infty} e^{-2\pi t}\, e^{-\pi r^2 t}\,dt = C_3 \frac{e^{\pi(r^2+2)}}{r^2+2}.$$

Therefore, we arrive at

$$|a(r)| \le 4C_2\, r\, K_1(2\sqrt{2}\pi r) + 2C_3 \frac{e^{-\pi(r^2+2)}}{r^2+2}.$$

It is easy to see that the left hand side of this inequality decays faster then any inverse power of r. Analogous estimates can be obtained for all derivatives $\frac{d^k}{dr^k}a(r)$.

Now we show that a is an eigenfunction of the Fourier transform. We recall that the Fourier transform of a Gaussian function is

$$\mathcal{F}(e^{\pi i \|x\|^2 z})(y) = z^{-4}\, e^{\pi i \|y\|^2\,(\frac{-1}{z})}. \tag{24}$$

Next, we exchange the contour integration with respect to z variable and Fourier transform with respect to x variable in (23). This can be done, since the corresponding double integral converges absolutely. In this way we obtain

$$\widehat{a}(y) = \int\limits_{-1}^{i} \phi_0\Big(\frac{-1}{z+1}\Big)\,(z+1)^2\, z^{-4}\, e^{\pi i \|y\|^2\,(\frac{-1}{z})}\,dz + \int\limits_{1}^{i} \phi_0\Big(\frac{-1}{z-1}\Big)\,(z-1)^2\, z^{-4}\, e^{\pi i \|y\|^2\,(\frac{-1}{z})}\,dz$$
$$-2\int\limits_{0}^{i} \phi_0\Big(\frac{-1}{z}\Big)\, z^2\, z^{-4}\, e^{\pi i \|y\|^2\,(\frac{-1}{z})}\,dz + 2\int\limits_{i}^{i\infty} \phi_0(z)\, z^{-4}\, e^{\pi i \|y\|^2\,(\frac{-1}{z})}\,dz.$$

Now we make a change of variables $w = \frac{-1}{z}$. We obtain

$$\widehat{a}(y) = \int\limits_{1}^{i} \phi_0\Big(1 - \frac{1}{w-1}\Big)\,(\frac{-1}{w}+1)^2\, w^2\, e^{\pi i \|y\|^2\, w}\,dw$$
$$+ \int\limits_{-1}^{i} \phi_0\Big(1 - \frac{1}{w+1}\Big)\,(\frac{-1}{w}-1)^2\, w^2\, e^{\pi i \|y\|^2\, w}\,dw$$
$$-2\int\limits_{i\infty}^{i} \phi_0(w)\, e^{\pi i \|y\|^2\, w}\,dw + 2\int\limits_{i}^{0} \phi_0\Big(\frac{-1}{w}\Big)\, w^2\, e^{\pi i \|y\|^2\, w}\,dw.$$

Since ϕ_0 is 1-periodic we have

$$\begin{aligned}\widehat{a}(y) \;=& \int_{1}^{i} \phi_0\Big(\frac{-1}{z-1}\Big)\,(z-1)^2\,e^{\pi i \|y\|^2\,z}\,dz + \int_{-1}^{i} \phi_0\Big(\frac{-1}{z+1}\Big)\,(z+1)^2\,e^{\pi i \|y\|^2\,z}\,dz\\ &+2\int_{i}^{i\infty} \phi_0(z)\,e^{\pi i\|y\|^2\,z}\,dz - 2\int_{0}^{i}\phi_0\Big(\frac{-1}{z}\Big)\,z^2\,e^{\pi i\|y\|^2\,z}\,dz\\ &= a(y).\end{aligned}$$

This finishes the proof of the proposition. $\square$

Next, we check that a has double zeroes at all Λ_8-lattice points of length greater then $\sqrt{2}$.

Proposition 76. *For $r>\sqrt{2}$ we can express $a(r)$ in the following form*

$$a(r) = -4\sin(\pi r^2/2)^2 \int_{0}^{i\infty} \phi_0\Big(\frac{-1}{z}\Big)\,z^2\,e^{\pi i r^2\,z}\,dz. \tag{25}$$

Proof. We denote the right hand side of (25) by $d(r)$. It is easy to see that $d(r)$ is well-defined. Indeed, from the transformation formula (22) and the expansions (21)–(20) we obtain

$$\begin{aligned}\phi_0\Big(\frac{-1}{it}\Big) =& O(e^{-2\pi/t}) \quad \text{as } t\to 0\\ \phi_0\Big(\frac{-1}{it}\Big) =& O(t^{-2}\,e^{2\pi t}) \quad \text{as } t\to\infty\end{aligned}$$

Hence, the integral (25) converges absolutely for $r>\sqrt{2}$. We can write

$$\begin{aligned}d(r) =& \int_{-1}^{i\infty-1} \phi_0\Big(\frac{-1}{z+1}\Big)\,(z+1)^2\,e^{\pi i r^2\,z}\,dz - 2\int_{0}^{i\infty}\phi_0\Big(\frac{-1}{z}\Big)\,z^2\,e^{\pi i r^2\,z}\,dz\\ &+\int_{1}^{i\infty+1} \phi_0\Big(\frac{-1}{z-1}\Big)\,(z-1)^2\,e^{\pi i r^2\,z}\,dz.\end{aligned}$$

From (22) we deduce that if $r>\sqrt{2}$ then $\phi_0\Big(\frac{-1}{z}\Big)\,z^2\,e^{\pi i r^2\,z}\to 0$ as $\Im(z)\to\infty$. Therefore, we can deform the paths of integration and rewrite

$$\begin{aligned}d(r) =& \int_{-1}^{i} \phi_0\Big(\frac{-1}{z+1}\Big)\,(z+1)^2\,e^{\pi i r^2\,z}\,dz + \int_{i}^{i\infty} \phi_0\Big(\frac{-1}{z+1}\Big)\,(z+1)^2\,e^{\pi i r^2\,z}\,dz\\ &-2\int_{0}^{i}\phi_0\Big(\frac{-1}{z}\Big)\,z^2\,e^{\pi i r^2\,z}\,dz - 2\int_{i}^{i\infty}\phi_0\Big(\frac{-1}{z}\Big)\,z^2\,e^{\pi i r^2\,z}\,dz\\ &+\int_{1}^{i} \phi_0\Big(\frac{-1}{z-1}\Big)\,(z-1)^2\,e^{\pi i r^2\,z}\,dz + \int_{i}^{i\infty} \phi_0\Big(\frac{-1}{z-1}\Big)\,(z-1)^2\,e^{\pi i r^2\,z}\,dz.\end{aligned}$$

Now from (22) we find

$$\begin{aligned}&\phi_0\Big(\frac{-1}{z+1}\Big)\,(z+1)^2 - 2\phi_0\Big(\frac{-1}{z}\Big)\,z^2 + \phi_0\Big(\frac{-1}{z-1}\Big)\,(z-1)^2 =\\ &\phi_0(z+1)\,(z+1)^2 - 2\phi_0(z)\,z^2 + \phi_0(z-1)\,(z-1)^2\\ &-\frac{12i}{\pi}\Big(\phi_{-2}(z+1)\,(z+1) - 2\phi_{-2}(z)\,z + \phi_{-2}(z-1)\,(z-1)\Big)\\ &-\frac{36}{\pi^2}\Big(\phi_{-4}(z+1) - 2\phi_{-4}(z) + \phi_{-4}(z-1)\Big) =\\ &2\phi_0(z).\end{aligned}$$

Thus, we obtain

$$d(r) = \int_{-1}^{i} \phi_0\Big(\frac{-1}{z+1}\Big)\,(z+1)^2\,e^{\pi i r^2\,z}\,dz - 2\int_{0}^{i} \phi_0\Big(\frac{-1}{z}\Big)\,z^2\,e^{\pi i r^2\,z}\,dz$$
$$+ \int_{1}^{i} \phi_0\Big(\frac{-1}{z-1}\Big)\,(z-1)^2\,e^{\pi i r^2\,z}\,dz + 2\int_{i}^{i\infty} \phi_0(z)\,e^{\pi i r^2\,z}\,dz = a(r).$$

This finishes the proof. $\square$

Finally, we find another convenient integral representation for a and compute values of $a(r)$ at $r=0$ and $r=\sqrt{2}$.

Proposition 77. *For $r \geq 0$ we have*

$$a(r) = 4i\,\sin(\pi r^2/2)^2 \Bigg(\frac{36}{\pi^3\,(r^2-2)} - \frac{8640}{\pi^3\,r^4} + \frac{18144}{\pi^3\,r^2}$$
$$+ \int_{0}^{\infty} \Big(t^2\,\phi_0\Big(\frac{i}{t}\Big) - \frac{36}{\pi^2}\,e^{2\pi t} + \frac{8640}{\pi}\,t - \frac{18144}{\pi^2}\Big)\,e^{-\pi r^2 t}\,dt\Bigg). \tag{26}$$

The integral converges absolutely for all $r \in \mathbb{R}_{\geq 0}$.

Proof. Suppose that $r > \sqrt{2}$. Then by Proposition 76

$$a(r) = 4i\,\sin(\pi r^2/2)^2 \int_{0}^{\infty} \phi_0(i/t)\,t^2\,e^{-\pi r^2 t}\,dt.$$

From (21)–(22) we obtain

$$\phi_0(i/t)\,t^2 = \frac{36}{\pi^2}\,e^{2\pi t} - \frac{8640}{\pi}\,t + \frac{18144}{\pi^2} + O(t^2\,e^{-2\pi t}) \quad \text{as } t \to \infty. \tag{27}$$

For $r > \sqrt{2}$ we have

$$\int_{0}^{\infty} \Big(\frac{36}{\pi^2}\,e^{2\pi t} + \frac{8640}{\pi}\,t + \frac{18144}{\pi^2}\Big)\,e^{-\pi r^2 t}\,dt = \frac{36}{\pi^3\,(r^2-2)} - \frac{8640}{\pi^3\,r^4} + \frac{18144}{\pi^3\,r^2}.$$

Therefore, the identity (26) holds for $r > \sqrt{2}$.

On the other hand, from the definition (23) we see that $a(r)$ is analytic in some neighborhood of $[0,\infty)$. The asymptotic expansion (27) implies that the right hand side of (26) is also analytic in some neighborhood of $[0,\infty)$. Hence, the identity (26) holds on the whole interval $[0,\infty)$. This finishes the proof of the proposition. $\square$

From the identity (26) we see that the values $a(r)$ are in $i\mathbb{R}$ for all $r \in \mathbb{R}_{\geq 0}$. In particular, we have

Proposition 78. *We have*

$$a(0) = \frac{-i\,8640}{\pi} \qquad a(\sqrt{2}) = 0 \qquad a'(\sqrt{2}) = \frac{i\,72\sqrt{2}}{\pi}.$$

Proof. These identities follow immediately from the previous proposition. $\square$

Now we construct function b. To this end we consider the function

Definition 79.

$$h(z) := 128\frac{\theta_{00}^4(z) + \theta_{01}^4(z)}{\theta_{10}^8(z)}. \tag{28}$$

It is easy to see that $h \in M^!_{-2}(\Gamma_0(2))$. Indeed, first we check that $h|_{-2}\gamma = h$ for all $\gamma \in \Gamma_0(2)$. Since the group $\Gamma_0(2)$ is generated by elements $\left(\begin{smallmatrix}1&0\\2&1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ it suffices to check that h is invariant under their action. This follows immediately from (13)–(17) and (28). Next we analyze the poles of h. It is known [7, Chapter I Lemma 4.1] that θ_{10} has no zeros in the upper-half plane and hence h has poles only at the cusps. At the cusp $i\infty$ this modular form has the Fourier expansion

$$h(z) \;=\; q^{-1} + 16 - 132q + 640q^2 - 2550q^3 + O(q^4).$$

Let $I = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$, $T = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$, and $S = \left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$ be elements of Γ_1.

Definition 80. *We define the followig three functions*

$$\begin{aligned} \psi_I &:= h - h|_{-2}ST && (29)\\ \psi_T &:= \psi_I|_{-2}T \\ \psi_S &:= \psi_I|_{-2}S. && (30) \end{aligned}$$

Lemma 81. *More explicitly, we have*

$$\begin{aligned} \psi_I(z) &= 128\,\frac{\theta_{00}^4(z)+\theta_{01}^4(z)}{\theta_{10}^8(z)} \;+\; 128\,\frac{\theta_{01}^4(z)-\theta_{10}^4(z)}{\theta_{00}^8(z)}\\ \psi_T(z) &= 128\,\frac{\theta_{00}^4(z)+\theta_{01}^4(z)}{\theta_{10}^8(z)} \;+\; 128\,\frac{\theta_{00}^4(z)+\theta_{10}^4(z)}{\theta_{01}^8(z)}\\ \psi_S(z) &= -\,128\,\frac{\theta_{00}^4(z)+\theta_{10}^4(z)}{\theta_{01}^8(z)} - 128\,\frac{\theta_{10}^4(z)-\theta_{01}^4(z)}{\theta_{00}^8(z)}. \end{aligned}$$

Lemma 82. *The Fourier expansions of these functions are*

$$\begin{aligned} \psi_I(z) &= q^{-1} + 144 - 5120q^{1/2} + 70524q - 626688q^{3/2} + 4265600q^2 + O(q^{5/2}) && (31)\\ \psi_T(z) &= q^{-1} + 144 + 5120q^{1/2} + 70524q + 626688q^{3/2} + 4265600q^2 + O(q^{5/2})\\ \psi_S(z) &= -\,10240q^{1/2} - 1253376q^{3/2} - 48328704q^{5/2} - 1059078144q^{7/2} + O(q^{9/2}). && (32) \end{aligned}$$

Definition 83. *For $x \in \mathbb{R}^8$ define*

$$\begin{aligned} b(x) := \int_{-1}^{i} \psi_T(z)\, e^{\pi i \|x\|^2 z}\, dz + \int_{1}^{i} \psi_T(z)\, e^{\pi i \|x\|^2 z}\, dz && (33)\\ -2\int_{0}^{i} \psi_I(z)\, e^{\pi i \|x\|^2 z}\, dz - 2\int_{i}^{i\infty} \psi_S(z)\, e^{\pi i \|x\|^2 z}\, dz. \end{aligned}$$

Now we prove that b satisfies condition (19).

Proposition 84. *The function b defined by (33) belongs to the Schwartz space and satisfies*

$$\widehat{b}(x) = -b(x).$$

Proof. Here, we repeat the arguments used in the proof of Proposition 75. First we show that b is a Schwartz function. We have

$$\int_{-1}^{i} \psi_T(z)\, e^{\pi i r^2 z}\, dz = \int_{0}^{i+1} \psi_I(z)\, e^{\pi i r^2 (z-1)}\, dz =$$
$$\int_{i\infty}^{-1/(i+1)} \psi_I\Big(\frac{-1}{z}\Big)\, e^{\pi i r^2(-1/z-1)}\, z^{-2}\, dz = \int_{i\infty}^{-1/(i+1)} \psi_S(z)\, z^{-4}\, e^{\pi i r^2(-1/z-1)}\, dz.$$

There exists a positive constant C such that

$$|\psi_S(z)| \le C\, e^{-\pi\,\Im z} \quad \text{for } \Im z > \frac{1}{2}.$$

Thus, as in the proof of Proposition 75 we estimate the first summand in the left-hand side of (33)

$$\left|\int\limits_{-1}^{i} \psi_T(z)\, e^{\pi i r^2 z}\, dz\right| \le C_1\, r\, K_1(2\pi r).$$

We combine this inequality with analogous estimates for the other three summands and obtain

$$|b(r)| \le C_2\, r\, K_1(2\pi r) + C_3\, \frac{e^{-\pi(r^2+1)}}{r^2+1}.$$

Here C_1, C_2, and C_3 are some positive constants. Similar estimates hold for all derivatives $\frac{d^k}{d^k r} b(r)$.

Now we prove that b is an eigenfunction of the Fourier transform. We use identity (24) and change contour integration in z and Fourier transform in x. Thus we obtain

$$\begin{aligned}\mathcal{F}(b)(x) = &\int\limits_{-1}^{i} \psi_T(z)\, z^{-4}\, e^{\pi i \|x\|^2 (\frac{-1}{z})}\, dz + \int\limits_{1}^{i} \psi_T(z)\, z^{-4}\, e^{\pi i \|x\|^2 (\frac{-1}{z})}\, dz \\ &-2\int\limits_{0}^{i} \psi_I(z)\, z^{-4}\, e^{\pi i \|x\|^2 (\frac{-1}{z})}\, dz - 2\int\limits_{i}^{i\infty} \psi_S(z)\, z^{-4}\, e^{\pi i \|x\|^2 (\frac{-1}{z})}\, dz.\end{aligned}$$

We make the change of variables $w = \frac{-1}{z}$ and arrive at

$$\begin{aligned}\mathcal{F}(b)(x) = &\int\limits_{1}^{i} \psi_T\Big(\frac{-1}{w}\Big)\, w^2\, e^{\pi i \|x\|^2 w}\, dw + \int\limits_{-1}^{i} \psi_T\Big(\frac{-1}{w}\Big)\, w^2\, e^{\pi i \|x\|^2 w}\, dw \\ &-2\int\limits_{i\infty}^{i} \psi_I\Big(\frac{-1}{w}\Big)\, w^2\, e^{\pi i \|x\|^2 w}\, dw - 2\int\limits_{i}^{0} \psi_S\Big(\frac{-1}{w}\Big)\, w^2\, e^{\pi i \|x\|^2 w}\, dw.\end{aligned}$$

Now we observe that the definitions (29)–(30) imply

$$\begin{aligned}\psi_T|_{-2}S &= -\,\psi_T \\ \psi_I|_{-2}S &= \psi_S \\ \psi_S|_{-2}S &= \psi_I.\end{aligned}$$

Therefore, we arrive at

$$\begin{aligned}\mathcal{F}(b)(x) = &\int\limits_{1}^{i} -\psi_T(z)\, e^{\pi i \|x\|^2 z}\, dz + \int\limits_{-1}^{i} -\psi_T(z)\, e^{\pi i \|x\|^2 z}\, dz \\ &+2\int\limits_{i}^{i\infty} \psi_S(z)\, e^{\pi i \|x\|^2 z}\, dz + 2\int\limits_{0}^{i} \psi_I(z)\, e^{\pi i \|x\|^2 w}\, dw.\end{aligned}$$

Now from (33) we see that

$$\mathcal{F}(b)(x) = -b(x).$$

□

Now we regard the radial function b as a function on $\mathbb{R}_{\ge 0}$. We check that b has double roots at Λ_8-points.

Proposition 85. *For $r > \sqrt{2}$ function $b(r)$ can be expressed as*

$$b(r) = -4\sin(\pi r^2/2)^2 \int_0^{i\infty} \psi_I(z)\, e^{\pi i r^2 z}\, dz. \tag{34}$$

Proof. We denote the right hand side of (34) by $c(r)$. First, we check that $c(r)$ is well-defined. We have

$$\begin{aligned}\psi_I(it) &= O(t^2\, e^{\pi/t}) \quad \text{as } t \to 0\\ \psi_I(it) &= O(e^{2\pi t}) \quad \text{as } t \to \infty.\end{aligned}$$

Therefore, the integral (34) converges for $r > \sqrt{2}$. Then we rewrite it in the following way:

$$c(r) = \int_{-1}^{i\infty-1} \psi_I(z+1)\, e^{\pi i r^2 z}\, dz - 2\int_0^{i\infty} \psi_I(z)\, e^{\pi i r^2 z}\, dz + \int_1^{i\infty+1} \psi_I(z-1)\, e^{\pi i r^2 z}\, dz.$$

From the Fourier expansion (31) we know that $\psi_I(z) = e^{-2\pi i z} + O(1)$ as $\Im(z) \to \infty$. By assumption $r^2 > 2$, hence we can deform the path of integration and write

$$\begin{aligned}\int_{-1}^{i\infty-1} \psi_I(z+1)\, e^{\pi i r^2 z}\, dz &= \int_{-1}^{i} \psi_T(z)\, e^{\pi i r^2 z}\, dz + \int_i^{i\infty} \psi_T(z)\, e^{\pi i r^2 z}\, dz\\ \int_1^{i\infty+1} \psi_I(z-1)\, e^{\pi i r^2 z}\, dz &= \int_{-1}^{i} \psi_T(z)\, e^{\pi i r^2 z}\, dz + \int_i^{i\infty} \psi_T(z)\, e^{\pi i r^2 z}\, dz.\end{aligned}$$

We have

$$\begin{aligned}c(r) = &\int_{-1}^{i} \psi_T(z)\, e^{\pi i r^2 z}\, dz + \int_1^{i} \psi_T(z)\, e^{\pi i r^2 z}\, dz - 2\int_0^{i} \psi_I(z)\, e^{\pi i r^2 z}\, dz \\ &+ 2\int_i^{i\infty} (\psi_T(z) - \psi_I(z))\, e^{\pi i r^2 z}\, dz.\end{aligned} \tag{35}$$

Next, we check that the functions ψ_I, ψ_T, and ψ_S satisfy the following identity:

$$\psi_T + \psi_S = \psi_I. \tag{36}$$

Indeed, from definitions (29)-(30) we get

$$\begin{aligned}\psi_T + \psi_S =&(h - h|_{-2}ST)|_{-2}T + (h - h|_{-2}ST)|_{-2}S\\ =&h|_{-2}T - h|_{-2}ST^2 + h|_{-2}S - h|_{-2}STS.\end{aligned}$$

Note that ST^2S belongs to $\Gamma_0(2)$. Thus, since $h \in M^!_{-2}\Gamma_0(2)$ we get

$$\psi_T + \psi_S = h|_{-2}T - h|_{-2}STS.$$

Now we observe that T and $STS(ST)^{-1}$ are also in $\Gamma_0(2)$. Therefore,

$$\psi_T + \psi_S = h|_{-2}T - h|_{-2}STS = h|_{-2} - h|ST = \psi_I.$$

Combining (35) and (36) we find

$$\begin{aligned}c(r) = &\int_{-1}^{i} \psi_T(z)\, e^{\pi i r^2 z}\, dz + \int_1^{i} \psi_T(z)\, e^{\pi i r^2 z}\, dz - 2\int_0^{i} \psi_I(z)\, e^{\pi i r^2 z}\, dz\\ &- 2\int_i^{i\infty} \psi_S(z)\, e^{\pi i r^2 z}\, dz\\ =&b(r).\end{aligned}$$

$\square$

At the end of this section we find another integral representation of $b(r)$ for $r \in \mathbb{R}_{\geq 0}$ and compute special values of b.

Proposition 86. *For $r \geq 0$ we have*

$$b(r) = 4i\, \sin(\pi r^2/2)^2 \left(\frac{144}{\pi\, r^2} + \frac{1}{\pi\, (r^2-2)} + \int_0^\infty \left(\psi_I(it) - 144 - e^{2\pi t}\right) e^{-\pi r^2 t}\, dt \right). \tag{37}$$

The integral converges absolutely for all $r \in \mathbb{R}_{\geq 0}$.

Proof. The proof is analogous to the proof of Proposition 77. First, suppose that $r > \sqrt{2}$. Then by Proposition 85

$$b(r) = 4i\, \sin(\pi r^2/2)^2 \int_0^\infty \psi_I(it)\, e^{-\pi r^2 t}\, dt.$$

From (31) we obtain

$$\psi_I(it) = e^{2\pi t} + 144 + O(e^{-\pi t}) \quad \text{as } t \to \infty. \tag{38}$$

For $r > \sqrt{2}$ we have

$$\int_0^\infty \left(e^{2\pi t} + 144\right) e^{-\pi r^2 t}\, dt = \frac{1}{\pi\, (r^2-2)} + \frac{144}{\pi\, r^2}.$$

Therefore, the identity (37) holds for $r > \sqrt{2}$.

On the other hand, from the definition (33) we see that $b(r)$ is analytic in some neighborhood of $[0, \infty)$. The asymptotic expansion (38) implies that the right hand side of (37) is also analytic in some neighborhood of $[0, \infty)$. Hence, the identity (37) holds on the whole interval $[0, \infty)$. This finishes the proof of the proposition. $\square$

We see from (37) that $b(r) \in i\mathbb{R}$ far all $r \in \mathbb{R}_{\geq} 0$. Another immediate corollary of this proposition is

Proposition 87. *We have*

$$b(0) = 0 \qquad b(\sqrt{2}) = 0 \qquad b'(\sqrt{2}) = \frac{i}{2\sqrt{2}\,\pi}.$$

8 Proof of Theorem 45

Our proof of the Theorem 45 relies on the following two inequalities for modular objects.

Proposition 88. *Consider the function $A : (0, \infty) \to \mathbb{C}$ defined as*

$$A(t) := -t^2 \phi_0(i/t) - \frac{36}{\pi^2}\, \psi_I(it).$$

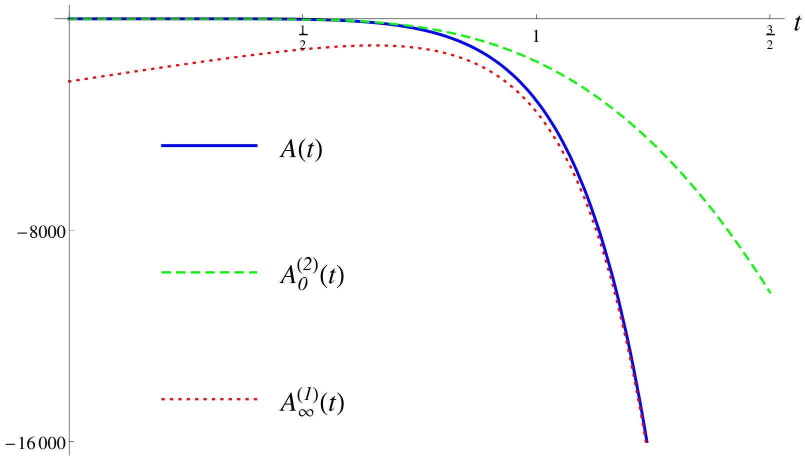
Then $A(t) \in (-\infty, 0)$ for all $t \in (0, \infty)$.

Remark: *We might formalize the original proof from [11] or the proof of Dan Romik "On Viazovska's modular form inequalities" [10]. Below is the proof from [11].*

Proof. Function $A(t)$ is plotted in Figure 1.
We observe that we can compute the values of $A(t)$ for $t \in (0, \infty)$ with any given precision. Indeed, from identities (22) and (30) we obtain the following two presentations for $A(t)$

$$\begin{aligned} A(t) &= -\, t^2 \phi_0(i/t) + \frac{36}{\pi^2}\, t^2\, \psi_S(i/t) \\ A(t) &= -\, t^2 \phi_0(it) + \frac{12}{\pi}\, t\, \phi_{-2}(it) - \frac{36}{\pi^2}\, \phi_{-4}(it) - \frac{36}{\pi^2}\, \psi_I(i/t). \end{aligned}$$

Figure 1: Plot of the functions $A(t)$, $A_0^{(2)}(t) = -\frac{368640}{\pi^2}\, t^2\, e^{-\pi/t}$, and $A_\infty^{(1)}(t) = -\frac{72}{\pi^2}\, e^{2\pi t} + \frac{8640}{\pi} t - \frac{23328}{\pi^2}$.



For an integer $n \geq 0$ let $A_0^{(n)}$ and $A_\infty^{(n)}$ be the functions such that

$$A(t) = A_0^{(n)}(t) + O(t^2\, e^{-\pi n/t}) \quad \text{as } t \to 0 \tag{39}$$

$$A(t) = A_\infty^{(n)}(t) + O(t^2\, e^{-\pi n t}) \quad \text{as } t \to \infty. \tag{40}$$

For each $n \geq 0$ we can compute these functions from the Fourier expansions (20)–(21), (31), and (32). For example, from (20)–(21) and (31) we compute

$$A_\infty^{(6)}(t) = -\frac{72}{\pi^2}\, e^{2\pi t} - \frac{23328}{\pi^2} + \frac{184320}{\pi^2}\, e^{-\pi t} - \frac{5194368}{\pi^2}\, e^{-2\pi t} + \frac{22560768}{\pi^2}\, e^{-3\pi t} - \frac{250583040}{\pi^2}\, e^{-4\pi t} + \frac{869916672}{\pi^2}\, e^{-5\pi t}$$
$$+ t\left(\frac{8640}{\pi} + \frac{2436480}{\pi}\, e^{-2\pi t} + \frac{113011200}{\pi}\, e^{-4\pi t}\right) - t^2 (518400\, e^{-2\pi t} + 31104000\, e^{-4\pi t}).$$

From (20)–(21) and (32) we compute

$$A_0^{(6)}(t) = t^2 \left(-\frac{368640}{\pi^2}\, e^{-\pi/t} - 518400\, e^{-2\pi/t} - \frac{45121536}{\pi^2}\, e^{-3\pi/t} - 31104000\, e^{-4\pi/t} - \frac{1739833344}{\pi^2}\, e^{-5\pi/t}\right).$$

Moreover, from the convergent asymptotic expansion for the Fourier coefficients of a weakly holomorphic modular form [2, Proposition 1.12] we find that the n-th Fourier coefficient $c_{\psi_I}(n)$ of ψ_I satisfies

$$|c_{\psi_I}(n)| \leq e^{4\pi\sqrt{n}} \qquad n \in \tfrac{1}{2}\mathbb{Z}_{>0}. \tag{41}$$

Similar inequalities hold for the Fourier coefficients of ψ_S, ϕ_0, ϕ_{-2}, and ϕ_{-4}:

$$\begin{aligned}
|c_{\psi_S}(n)| &\leq 2e^{4\pi\sqrt{n}} && n \in \tfrac{1}{2}\mathbb{Z}_{>0} \\
|c_{\phi_0}(n)| &\leq 2e^{4\pi\sqrt{n}} && n \in \mathbb{Z}_{>0} \\
|c_{\phi_{-2}}(n)| &\leq e^{4\pi\sqrt{n}} && n \in \mathbb{Z}_{>0} \\
|c_{\phi_{-4}}(n)| &\leq e^{4\pi\sqrt{n}} && n \in \mathbb{Z}_{>0}.
\end{aligned} \tag{42}$$

Therefore, we can estimate the error terms in the asymptotic expansions (39) and (40) of $A(t)$

$$\left|A(t) - A_0^{(m)}(t)\right| \leq \left(t^2 + \frac{36}{\pi^2}\right) \sum_{n=m}^{\infty} 2e^{2\sqrt{2}\pi\sqrt{n}}\, e^{-\pi n/t}$$

$$\left|A(t) - A_\infty^{(m)}(t)\right| \leq \left(t^2 + \frac{12}{\pi}\, t + \frac{36}{\pi^2}\right) \sum_{n=m}^{\infty} 2e^{2\sqrt{2}\pi\sqrt{n}}\, e^{-\pi n t}.$$

For an integer $m \geq 0$ we set

$$R_0^{(m)} := \left(t^2 + \frac{36}{\pi^2}\right) \sum_{n=m}^{\infty} 2e^{2\sqrt{2}\pi\sqrt{n}}\, e^{-\pi n/t}$$

$$R_\infty^{(m)} := \left(t^2 + \frac{12}{\pi}\, t + \frac{36}{\pi^2}\right) \sum_{n=m}^{\infty} 2e^{2\sqrt{2}\pi\sqrt{n}}\, e^{-\pi n t}.$$

Using interval arithmetic we check that

$$
\begin{aligned}
\left|R_0^{(6)}(t)\right| &\le \left|A_0^{(6)}(t)\right| \qquad \text{for } \ t\in(0,1]\\
\left|R_\infty^{(6)}(t)\right| &\le \left|A_\infty^{(6)}(t)\right| \qquad \text{for } \ t\in[1,\infty)\\
A_0^{(6)}(t) &< 0 \qquad \text{for } \ t\in(0,1]\\
A_\infty^{(6)}(t) &< 0 \qquad \text{for } \ t\in[1,\infty).
\end{aligned}
$$

Thus, we see that $A(t)<0$ for $t\in(0,\infty)$. This finishes the proof of the Proposition. □

Proposition 89. *Consider the function $B:(0,\infty)\to\mathbb{C}$ defined as*

$$B(t) := -t^2\phi_0(i/t) + \frac{36}{\pi^2}\,\psi_I(it).$$

Then $B(t)\in(0,\infty)$ for all $t\in(0,\infty)$.

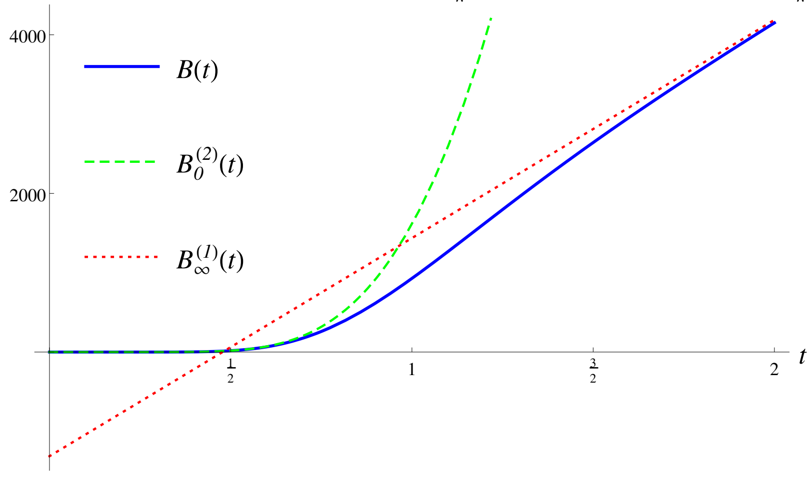
Remark *Below is the proof from [11]. Similarly to the previous proposition, another (hopefully easier for the formalization) proof of this inequality is given in [10].*

Proof. The function B can be also written as

$$
\begin{aligned}
B(t) &= -\,t^2\phi_0(i/t) - \frac{36}{\pi^2}\,t^2\,\psi_S(i/t)\\
B(t) &= -\,t^2\phi_0(it) + \frac{12}{\pi}\,t\,\phi_{-2}(it) - \frac{36}{\pi^2}\,\phi_{-4}(it) + \frac{36}{\pi^2}\,\psi_I(i/t).
\end{aligned}
$$

Our aim is to prove that $B(t)>0$ for $t\in(0,\infty)$. A plot of $B(t)$ is given in Figure 2.

Figure 2: Plot of the functions $B(t)$, $B_0^{(2)}(t)=\frac{368640}{\pi^2}\,t^2\,e^{-\pi/t}$, and $B_\infty^{(1)}(t)=\frac{8640}{\pi}t-\frac{23328}{\pi^2}$.



For $n\ge 0$ let $B_0^{(n)}$ and $B_\infty^{(n)}$ be the functions such that

$$
\begin{aligned}
B(t) =& B_0^{(n)}(t) + O(t^2\,e^{-\pi n/t}) \quad \text{ as } t\to 0\\
B(t) =& B_\infty^{(n)}(t) + O(t^2\,e^{-\pi n t}) \quad \text{ as } t\to \infty.
\end{aligned}
$$

We find

$$
\begin{aligned}
B_\infty^{(6)}(t) = &-\frac{12960}{\pi^2} - \frac{184320}{\pi^2}\,e^{-\pi t} - \frac{116640}{\pi^2}\,e^{-2\pi t} - \frac{22560768}{\pi^2}\,e^{-3\pi t} + \frac{56540160}{\pi^2}\,e^{-4\pi t} - \frac{869916672}{\pi^2}\,e^{-5\pi t}\\
&+ t\Big(\frac{8640}{\pi} + \frac{2436480}{\pi}\,e^{-2\pi t} + \frac{113011200}{\pi}\,e^{-4\pi t}\Big) - t^2\big(518400\,e^{-2\pi t} + 31104000\,e^{-4\pi t}\big)
\end{aligned}
$$

and

$$B_0^{(6)}(t) = t^2\big(\tfrac{368640}{\pi^2}\,e^{-\pi/t} - 518400\,e^{-2\pi/t} + \tfrac{45121536}{\pi^2}\,e^{-3\pi/t} - 31104000\,e^{-4\pi/t} + \tfrac{1739833344}{\pi^2}\,e^{-5\pi/t}\big).$$

The estimates (41)–(42) imply that

$$\left|B(t) - B_0^{(6)}(t)\right| \le R_0^{(6)}(t) \quad \text{for } t \in (0,1]$$

and

$$\left|B(t) - B_\infty^{(6)}(t)\right| \le R_\infty^{(6)}(t) \quad \text{for } t \in [1,\infty).$$

Using interval arithmetic we verify that

$$\begin{aligned}
\left|R_0^{(6)}(t)\right| &\le \left|B_0^{(6)}(t)\right| \quad &&\text{for } \ t \in (0,1]\\
\left|R_\infty^{(6)}(t)\right| &\le \left|B_\infty^{(6)}(t)\right| \quad &&\text{for } \ t \in [1,\infty)\\
B_0^{(6)}(t) &> 0 \quad &&\text{for } \ t \in (0,1]\\
B_\infty^{(6)}(t) &> 0 \quad &&\text{for } \ t \in [1,\infty).
\end{aligned}$$

Now identity (44) implies (6). □

Finally, we are ready to prove Theorem 45.

Theorem 90. *The function*

$$g(x) := \frac{\pi\, i}{8640} a(x) + \frac{i}{240\pi}\, b(x)$$

satisfies conditions (5)*–*(7)*.*

Proof. First, we prove that (5) holds. By Propositions 76 and 85 we know that for $r > \sqrt{2}$

$$g(r) = \frac{\pi}{2160}\,\sin(\pi r^2/2)^2 \int_0^\infty A(t)\, e^{-\pi r^2 t}\, dt \tag{43}$$

where

$$A(t) = -t^2\phi_0(i/t) - \frac{36}{\pi^2}\,\psi_I(it).$$

from the Proposition 88 we know that $A(t) < 0 \quad$ for $t \in (0,\infty)$. Therefore identity (43) implies (5).

Next, we prove (6). By Propositions 77 and 86 we know that for $r > 0$

$$\widehat{g}(r) = \frac{\pi}{2160}\,\sin(\pi r^2/2)^2 \int_0^\infty B(t)\, e^{-\pi r^2 t}\, dt \tag{44}$$

where

$$B(t) = -t^2\phi_0(i/t) + \frac{36}{\pi^2}\,\psi_I(it).$$

Finally, the property (7) readily follows from Proposition 78 and Proposition 87. This finishes the proof of Theorems 90 and 45. □

References

[1] M. Abramowitz, I. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Applied Mathematics Series 55 (10 ed.), New York, USA: United States Department of Commerce, National Bureau of Standards; Dover Publications, 1964.

[2] J. Bruinier, *Borcherds products on O(2,l) and Chern classes of Heegner divisors*, Springer Lecture Notes in Mathematics 1780 (2002)

[3] H. Cohn, N. Elkies, *New upper bounds on sphere packings I*, Annals of Math. 157 (2003) pp. 689–714.

[4] F. Diamond, J. Shurman, *A First Course in Modular Forms*, Springer New York, 2005.

[5] D. Hejhal, *The Selberg trace formula for* $\mathrm{PSL}(2,\mathbb{R})$, Springer Lecture Notes in Mathematics 1001 (1983)

[6] W. Kohnen, *A Very Simple Proof of the q-Product Expansion of the* Δ*-Function*, Ramanujan Journal

[7] D. Mumford, *Tata Lectures on Theta I*, Birkhäuser, 1983.

[8] H. Petersson, *Ueber die Entwicklungskoeffizienten der automorphen Formen*, Acta Mathematica, Bd. 58 (1932), pp. 169–215.

[9] H. Rademacher and H. S. Zuckerman, *On the Fourier coefficients of certain modular forms of positive dimension*, Annals of Math. (2) 39 (1938), pp. 433–462.

[10] Dan Romik *On Viazovska's modular form inequalities* Proceedings of the National Academy of Sciences (USA) 120 (2023), article e2304891120

[11] Maryna S. Viazovska, *The sphere packing problem in dimension 8* , Pages 991–1015 from Volume 185 (2017), Issue 3.

[12] D. Zagier, *Elliptic Modular Forms and Their Applications*, In: The 1-2-3 of Modular Forms, (K. Ranestad, ed.) Norway, Springer Universitext, 2008.

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