

Sphere Packing in Lean

Maryna Viazovska

May 28, 2024

1 Basic definitions for sphere packings

1.1 Sphere packings

The sphere packing constant measures which portion of d -dimensional Euclidean space can be covered by non-overlapping unit balls. More precisely, let R^d be the Euclidean vector space equipped with distance $\|\cdot\|$ and Lebesgue measure $\text{Vol}(\cdot)$. For $x \in R^d$ and $r \in R_{>0}$ we denote by $B_d(x, r)$ the ball in R^d with center x and radius r .

Definition 1. Let $X \subset R^d$ be a discrete set of points such that $\|x - y\| \geq 2$ for any distinct $x, y \in X$. Then the union

$$P = \bigcup_{x \in X} B_d(x, 1)$$

is a sphere packing.

Definition 2. If X is a lattice in R^d then we say that P is a lattice sphere packing.

Definition 3. The finite density of a packing P is defined as

$$P(r) = \frac{\text{Vol}(P \cap B_d(0, r))}{\text{Vol}(B_d(0, r))}, \quad r > 0.$$

Definition 4. We define the density of a packing P as the limit superior

$$P = \limsup_r P(r).$$

Definition 5. The number we want to know is the supremum over all possible packing densities called the sphere packing constant.

The main result of this paper is the proof that

$$\delta_8 = \frac{4}{384} \approx 0.25367.$$

This is the density of the E_8 -lattice sphere packing.

Definition 6. Recall that the E_8 -lattice $\delta_8 \subset R^8$ is given by

$$\delta_8 = \{(x_i) \in Z^8 \mid (Z + \frac{1}{2})^8 \mid \sum_{i=1}^8 x_i \pmod{2}\}.$$

Lemma 7. δ_8 is a positive-definite, even, unimodular lattice of rank 8. The minimal distance between two points in δ_8 is 2.

Definition 8. The E_8 -lattice sphere packing is the packing of unit balls with centers at $\frac{1}{2}\delta_8$.

Our main result is

Theorem 9. No packing of unit balls in Euclidean space R^8 has density greater than that of the E_8 -lattice packing.

1.2 Lattices and Periodic packings

Definition 10. Let S be a discrete subgroup of \mathbb{R}^d . A set $X \subset \mathbb{R}^d$ is said to be S -periodic if for each $s \in S$ and $x \in X$ the vector $x + s$ belongs to X .

Definition 11. A lattice in the Euclidean space \mathbb{R}^d is a discrete, co-compact, abelian subgroup.