

Sphere Packing in Lean

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Abstract

This blueprint consists of an adaptation of Maryna Viazovska's Fields Medal-winning paper proving that no packing of unit balls in Euclidean space \mathbb{R}^8 has density greater than that of the E_8 -lattice packing. This blueprint is a work in progress, and will be frequently updated and restructured as the formalisation effort progresses. We recommend that you look at [this webpage](#) for the latest version.

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1 Sphere packings

1.1 Sphere packings

The sphere packing constant measures which portion of d -dimensional Euclidean space can be covered by non-overlapping unit balls. More precisely, let \mathbb{R}^d be the Euclidean vector space equipped with distance $\|\cdot\|$ and Lebesgue measure $\text{Vol}(\cdot)$. For $x \in \mathbb{R}^d$ and $r \in \mathbb{R}_{>0}$ we denote by $B_d(x, r)$ the ball in \mathbb{R}^d with center x and radius r . There are several types of sphere packings, which are determined by imposing certain conditions on the set of centres of the spheres involved in the packing. Therefore, we begin by defining those conditions on the centres of the spheres.

Below, c is any real number (the scaling factor), usually with constraints such as $c > 0$ or $c \neq 0$.

Definition 1. We say that $X \subseteq \mathbb{R}^d$ is a set of sphere packing centres with separation r if X is discrete and $\|x - y\| \geq r$ for all $x \neq y \in X$.

Definition 2. We say that $X \subseteq \mathbb{R}^d$ is a set of lattice packing centres with separation r if X is both a set of sphere packing centres with separation r and a lattice in \mathbb{R}^d .

Definition 3. Let $X \subset \mathbb{R}^d$ be a set of sphere packing centres of separation r . Then, the set

$$\mathcal{P} = \bigcup_{x \in X} B_d\left(x, \frac{r}{2}\right)$$

of unit balls centred at points in X is the corresponding sphere packing.

Remark 4. If X is a set of lattice/periodic packing centres (see Definitions 2 and 9), then the corresponding sphere packing \mathcal{P} is called a lattice/periodic packing.

Definition 5. The finite density of a packing \mathcal{P} is defined as

$$\Delta_{\mathcal{P}}(R) := \frac{\text{Vol}(\mathcal{P} \cap B_d(0, R))}{\text{Vol}(B_d(0, R))}, \quad R > 0.$$

Definition 6. We define the density of a packing \mathcal{P} as the limit supremum

$$\Delta_{\mathcal{P}} := \limsup_{R \rightarrow \infty} \Delta_{\mathcal{P}}(R).$$

Definition 7. The sphere packing problem is to compute the sphere packing constant, defined as supremum of packing densities over all possible packing

$$\Delta_d := \sup_{\substack{\mathcal{P} \subset \mathbb{R}^d \\ \text{sphere packing}}} \Delta_{\mathcal{P}}.$$

1.2 Lattices and Periodic packings

Definition 8. We say that an additive subgroup $\Lambda \leq \mathbb{R}^d$ is a lattice if it is discrete and its \mathbb{R} -span contains all the elements of \mathbb{R}^d .

Definition 9. We say that $X \subseteq \mathbb{R}^d$ is a set of periodic packing centres if X is a set of sphere packing centres and there exists a lattice $\Lambda \subset \mathbb{R}^d$ such that for any $x \in X$ and $y \in \Lambda$, their sum $x + y$ lies in X .

Lemma 10. If $X \subseteq \mathbb{R}^d$ is a set of sphere packing centres that is periodic with respect to some lattice Λ , then the density of the corresponding (periodic) sphere packing is given by

$$\frac{|X/\Lambda|}{\text{Vol}(\mathbb{R}^d/\Lambda)} \cdot \text{Vol}(B_d(0, 1))$$

where the quotients in the numerator and denominator correspond to the orbits of the action by translation of Λ on X and \mathbb{R}^d respectively.

Remark 11. This can be thought of as the “volume of spheres per fundamental domain”: the number of spheres per fundamental domain is $|X/\Lambda|$, and the volume of each sphere is $\text{Vol}(B_d(0, 1))$.

Definition 12. The periodic sphere packing constant is defined to be

$$\Delta_d^{\text{periodic}} := \sup_{\substack{P \subset \mathbb{R}^d \\ \text{periodic packing}}} \Delta_P$$

Theorem 13. For all d , the periodic sphere packing constant in \mathbb{R}^d is equal to the sphere packing constant in \mathbb{R}^d .

Proof. **State this in Lean (ready). Fill in proof here (see [3, Appendix A])** □

In other words, it suffices to compute and optimise the periodic sphere packing constant.

1.3 Density of periodic packings

In this subsection, we build up results about the density of periodic packings. In particular, the density of a periodic packing, defined as the limit of the periodic packing intersected with a growing ball centered at the origin, is equal to the density within any fundamental region of the period lattice.

1.4 Main Result

With the terminologies above, we can state the main theorem of this project.

Theorem 14. *All periodic packing $\mathcal{P} \subseteq \mathbb{R}^8$ has density satisfying $\Delta_{\mathcal{P}} \leq \Delta_{E_8}$, the density of the E_8 sphere packing (see Definition 28).*

Proof. We will prove this theorem over the course of this blueprint. □

Corollary 15. *All packing $\mathcal{P} \subseteq \mathbb{R}^8$ has density satisfying $\Delta_{\mathcal{P}} \leq \Delta_{E_8}$.*

Proof. This is a direct consequence of Theorem 13 and Theorem 14. □

Corollary 16. $\Delta_8 = \Delta_{E_8}$.

Proof. By definition, $\Delta_{E_8} \leq \Delta_8$, while Corollary 15 shows $\Delta_8 = \sup_{\text{packing } \mathcal{P}} \Delta_{\mathcal{P}} \leq \Delta_{E_8}$, and the result follows. □

2 The E_8 lattice

2.1 Definitions of E_8 lattice

There are several equivalent definitions of the E_8 lattice. Below, we formalise two of them, and prove they are equivalent.

Definition 17. *E_8 -lattice, Definition 1* We define the E_8 -lattice (as a subset of \mathbb{R}^8) to be

$$\Lambda_8 = \{(x_i) \in \mathbb{Z}^8 \cup (\mathbb{Z} + \frac{1}{2})^8 \mid \sum_{i=1}^8 x_i \equiv 0 \pmod{2}\}.$$

Definition 18. *We define the scaled E_8 -lattice (by a real number c) as*

$$c\Lambda_8 = \{c \cdot \vec{v} : \vec{v} \in \Lambda_8\}$$

.

Definition 19. *E_8 -lattice, Definition 2* We define the E_8 basis vectors to be the set of vectors

$$\mathcal{B}_8 = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2 \\ -1/2 \\ -1/2 \\ -1/2 \\ -1/2 \\ -1/2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Definition 20. *We define the scaled E_8 basis vectors (by a real number c) to be $c\mathcal{B}_8 = \{c \cdot \vec{v} : \vec{v} \in \mathcal{B}_8\}$.*

Theorem 21. *The two definitions above coincide, i.e. $c\Lambda_8 = \text{span}_{\mathbb{Z}}(c\mathcal{B}_8)$.*

Proof. We prove each side contains the other side.

For a vector $\vec{v} \in \Lambda_8 \subseteq \mathbb{R}^8$, we have $\sum_i \vec{v}_i \equiv 0 \pmod{2}$ and \vec{v}_i being either all integers or all half-integers. After some modulo arithmetic, it can be seen that $\mathcal{B}_8^{-1}\vec{v}$ as integer coordinates (i.e. it is congruent to 0 modulo 1). Hence, $\vec{v} \in \text{span}_{\mathbb{Z}}(\mathcal{B}_8)$.

For the opposite direction, we write the vector as $\vec{v} = \sum_i c_i \mathcal{B}_8^i \in \text{span}_{\mathbb{Z}}(\mathcal{B}_8)$ with c_i being integers and \mathcal{B}_8^i being the i -th basis vector. Expanding the definition then gives $\vec{v} = (c_1 - \frac{1}{2}c_7, -c_1 + c_2 - \frac{1}{2}c_7, \dots, -\frac{1}{2}c_7)$. Again, after some modulo arithmetic, it can be seen that $\sum_i \vec{v}_i$ is indeed 0 modulo 2 and are all either integers and half-integers, concluding the proof.

(Note: this proof doesn't depend on that \mathcal{B}_8 is linearly independent.) \square

2.2 Basic Properties of E_8 lattice

In this section, we establish basic properties of the E_8 lattice and the \mathcal{B}_8 vectors.

Lemma 22. *For nonzero real numbers c , the set $c\mathcal{B}_8$ is a \mathbb{R} -basis of \mathbb{R}^8 .*

Proof. It suffices to prove that $\mathcal{B}_8 \in \text{GL}_8(\mathbb{R})$. We prove this by explicitly defining the inverse matrix \mathcal{B}_8^{-1} and proving $\mathcal{B}_8 \mathcal{B}_8^{-1} = I_8$, which implies that $\det(\mathcal{B}_8)$ is nonzero. \square

Lemma 23. *For real numbers c , $c\Lambda_8$ is an additive subgroup of \mathbb{R}^8 .*

Proof. Trivially follows from that $\Lambda_8 \subseteq \mathbb{R}^8$ is the \mathbb{Z} -span of \mathcal{B}_8 and hence an additive group. \square

Lemma 24. *All vectors in Λ_8 have norm of the form $\sqrt{2n}$, where n is a nonnegative integer.*

Proof. Writing $\vec{v} = \sum_i c_i \mathcal{B}_8^i$, we have $\|\vec{v}\|^2 = \sum_i \sum_j c_i c_j (\mathcal{B}_8^i \cdot \mathcal{B}_8^j)$. Computing all 64 pairs of dot products and simplifying, we get a massive term that is a quadratic form in c_i with even integer coefficients, concluding the proof. \square

Lemma 25. *For nonzero real numbers c , $c\Lambda_8$ is discrete, i.e. that the subspace topology induced by its inclusion into \mathbb{R}^8 is the discrete topology.*

Proof. We prove this for $c = 1$. Since Λ_8 is a topological group and $+$ is continuous, it suffices to prove that $\{0\}$ is open in Λ_8 . This follows from the fact that there is an open ball $\mathcal{B}(\sqrt{2}) \subseteq \mathbb{R}^8$ around it containing no other lattice points, since the shortest nonzero vector has norm $\sqrt{2}$. \square

Lemma 26. *For nonzero real numbers c , $c\Lambda_8$ is a lattice, i.e. it is discrete and spans \mathbb{R}^8 over \mathbb{R} .*

Proof. The first part is the above lemma. The second part follows from that \mathcal{B}_8 is a basis and hence linearly independent over \mathbb{R} . \square

Prove E_8 is unimodular. Prove E_8 is positive-definite.

2.3 The E_8 sphere packing

Lemma 27. *For nonzero real numbers c , $c\Lambda_8$ is a valid set of sphere packing centres with separation $|c|\sqrt{2}$.*

Proof. This follows directly from Theorem 24. \square

Definition 28. *The E_8 sphere packing is the sphere packing with separation 1, whose set of centres is $\frac{1}{\sqrt{2}}\Lambda_8$.*

Theorem 29. *We have $\Delta_{E_8} = \frac{\pi^4}{384}$.*

Proof. **Finish proof. Preferably we want APIs about fundamental region of lattice, and use that to reduce this theorem to computation inside the fundamental region, and use formula for volume of ball.** \square

3 Facts from Fourier analysis

In this section, we recall a few definitions from Fourier analysis.

Definition 30. The Fourier transform of an L^1 -function $f : \mathbb{R}^d \rightarrow \mathbb{C}$ is defined as

$$\mathcal{F}(f)(y) = \widehat{f}(y) := \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle x, y \rangle} dx, \quad y \in \mathbb{R}^d$$

where $\langle x, y \rangle = \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2 - \frac{1}{2}\|x - y\|^2$ is the standard scalar product in \mathbb{R}^d .

Definition 31. A C^∞ function $f : \mathbb{R}^d \rightarrow \mathbb{C}$ is called a Schwartz function if it goes to zero as $\|x\| \rightarrow \infty$ faster than any inverse power of $\|x\|$, and the same holds for all partial derivatives of f .

Definition 32. The set of all Schwartz functions is called a Schwartz space.

Lemma 33. The Fourier transform is an automorphism of the space of Schwartz functions.

Proof. **Fill in proof.** □

Lemma 34.

$$\mathcal{F}(e^{\pi i \|x\|^2 z})(y) = z^{-4} e^{\pi i \|y\|^2 (\frac{-1}{z})}.$$

Proof. **Fill in proof.** □

Theorem 35 ((Poisson summation formula)).

$$\sum_{\ell \in \Lambda} f(\ell) = \frac{1}{\text{Vol}(\mathbb{R}^d/\Lambda)} \sum_{m \in \Lambda^*} \widehat{f}(m).$$

Proof. **Fill in proof.** □

4 Cohn-Elkies linear programming bounds

In 2003 Cohn and Elkies [3] developed linear programming bounds that apply directly to sphere packings. The goal of this section is to formalize the Cohn–Elkies linear programming bound.

The following theorem is the key result of [3]. (The original theorem is stated for a class of functions more general than Schwartz functions)

Theorem 36. (Cohn, Elkies [3]) Suppose that $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is a Schwartz function that is not identically zero and satisfies the following conditions:

$$f(x) \leq 0 \text{ for } \|x\| \geq 1 \tag{1}$$

and

$$\widehat{f}(x) \geq 0 \text{ for all } x \in \mathbb{R}^d. \tag{2}$$

Then the density of d -dimensional sphere packings is bounded above by

$$\frac{f(0)}{\widehat{f}(0)} \cdot \text{vol}(B_d(0, 1/2)).$$

Proof. Here we reproduce the proof given in [3]. We will first prove the theorem for periodic packings.

Let $X \subset \mathbb{R}^d$ be a discrete subset such that $\|x - y\| \geq 1$ for any distinct $x, y \in X$. Suppose that X is Λ -periodic with respect to some lattice $\Lambda \subset \mathbb{R}^d$.

The inequality

$$\#(X/\Lambda) \cdot f(0) \geq \sum_{x \in X} \sum_{y \in X/\Lambda} f(x - y) = \sum_{x \in X/\Lambda} \sum_{y \in X/\Lambda} \sum_{\ell \in \Lambda} f(x - y + \ell) \tag{3}$$

follows from the condition (1) of the theorem and the assumption on the distances between points in X . The equality

$$\sum_{x \in X/\Lambda} \sum_{y \in X/\Lambda} \sum_{\ell \in \Lambda} f(x - y + \ell) = \sum_{x \in X/\Lambda} \sum_{y \in X/\Lambda} \frac{1}{\text{vol}(\mathbb{R}^d/\Lambda)} \sum_{m \in \Lambda^*} \widehat{f}(m) e^{2\pi i m(x-y)}.$$

follows from the Poisson summation formula. The right hand side of the above equation can be written as

$$\sum_{x \in X/\Lambda} \sum_{y \in X/\Lambda} \frac{1}{\text{vol}(\mathbb{R}^d/\Lambda)} \sum_{m \in \Lambda^*} \widehat{f}(m) e^{2\pi i m(x-y)} = \frac{1}{\text{vol}(\mathbb{R}^d/\Lambda)} \sum_{m \in \Lambda^*} \widehat{f}(m) \cdot \left| \sum_{x \in X/\Lambda} e^{2\pi i m x} \right|^2.$$

Note that $\left| \sum_{x \in X/\Lambda} e^{2\pi i m x} \right|^2 \geq 0$ for all $m \in \Lambda^*$. Moreover, the term corresponding to $m = 0$ satisfies $\left| \sum_{x \in X/\Lambda} e^{2\pi i 0 x} \right|^2 = \#(X/\Lambda)^2$. Now we use the condition (2) and estimate

$$\frac{1}{\text{vol}(\mathbb{R}^d/\Lambda)} \sum_{m \in \Lambda^*} \widehat{f}(m) \cdot \left| \sum_{x \in X/\Lambda} e^{2\pi i m(x-y)} \right|^2 \geq \frac{\#(X/\Lambda)^2}{\text{vol}(\mathbb{R}^d/\Lambda)} \cdot \widehat{f}(0). \quad (4)$$

Comparing inequalities (3) and (4) we arrive at

$$\frac{\#(X/\Lambda)}{\text{vol}(\mathbb{R}^d/\Lambda)} \leq \frac{f(0)}{\widehat{f}(0)}.$$

Now we see that the density of the periodic packing \mathcal{P}_X with balls of radius $1/2$ is bounded by

$$\Delta(\mathcal{P}_X) = \frac{\#(X/\Lambda)}{\text{vol}(\mathbb{R}^d/\Lambda)} \cdot \text{vol}(B_d(0, 1/2)) \leq \frac{f(0)}{\widehat{f}(0)} \cdot \text{vol}(B_d(0, 1/2)).$$

This finishes the proof of the theorem for periodic packings. Theorem 13 implies the desired result for arbitrary packings. \square

The main step in our proof of Theorem 14 is the explicit construction of an optimal function. It will be convenient for us to scale this function by $\sqrt{2}$.

Theorem 37. *There exists a radial Schwartz function $g : \mathbb{R}^8 \rightarrow \mathbb{R}$ which satisfies:*

$$g(x) \leq 0 \text{ for } \|x\| \geq \sqrt{2} \quad (5)$$

$$\widehat{g}(x) \geq 0 \text{ for all } x \in \mathbb{R}^8 \quad (6)$$

$$g(0) = \widehat{g}(0) = 1. \quad (7)$$

Theorem 36 applied to the optimal function $f(x) = g(x/\sqrt{2})$ immediately implies Theorem 14.

5 Modular forms

In this section, we recall and develop some theory of (quasi)modular forms.

Let \mathfrak{H} be the upper half-plane $\{z \in \mathbb{C} \mid \Im(z) > 0\}$.

Lemma 38. *The modular group $\Gamma_1 := \text{PSL}_2(\mathbb{Z})$ acts on \mathfrak{H} by linear fractional transformations*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z := \frac{az + b}{cz + d}.$$

Let N be a positive integer.

Definition 39. *The level N principal congruence subgroup of Γ_1 is*

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1 \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}.$$

Definition 40. A subgroup $\Gamma \subset \Gamma_1$ is called a congruence subgroup if $\Gamma(N) \subset \Gamma$ for some $N \in \mathbb{N}$.

An important example of a congruence subgroup is

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1 \mid c \equiv 0 \pmod{N} \right\}.$$

Let $z \in \mathfrak{H}$, $k \in \mathbb{Z}$, and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.

Definition 41. The automorphy factor of weight k is defined as

$$j_k(z, \begin{pmatrix} a & b \\ c & d \end{pmatrix}) := (cz + d)^{-k}.$$

Lemma 42. The automorphy factor satisfies the chain rule

$$j_k(z, \gamma_1 \gamma_2) = j_k(z, \gamma_1) j_k(\gamma_2 z, \gamma_1).$$

Definition 43. Let F be a function on \mathfrak{H} and $\gamma \in \mathrm{PSL}_2(\mathbb{Z})$. Then the slash operator acts on F by

$$(F|_k \gamma)(z) := j_k(z, \gamma) F(\gamma z).$$

Lemma 44. The chain rule implies

$$F|_k \gamma_1 \gamma_2 = (F|_k \gamma_1)|_k \gamma_2.$$

Definition 45. A (holomorphic) modular form of integer weight k and congruence subgroup Γ is a holomorphic function $f : \mathfrak{H} \rightarrow \mathbb{C}$ such that:

1. $f|_k \gamma = f$ for all $\gamma \in \Gamma$
2. for each $\alpha \in \Gamma_1$ $f|_k \alpha$ has the Fourier expansion $f|_k \alpha(z) = \sum_{n=0}^{\infty} c_f(\alpha, \frac{n}{n_\alpha}) e^{2\pi i \frac{n}{n_\alpha} z}$ for some $n_\alpha \in \mathbb{N}$ and Fourier coefficients $c_f(\alpha, m) \in \mathbb{C}$.

Definition 46. Let $M_k(\Gamma)$ be the space of modular forms of weight k and congruence subgroup Γ .

A key fact in the theory of modular forms is the following theorem:

Theorem 47. The spaces $M_k(\Gamma)$ are finite dimensional.

Let us consider several examples of modular forms.

Definition 48. For an even integer $k \geq 4$ we define the weight k Eisenstein series as

$$E_k(z) := \frac{1}{2\zeta(k)} \sum_{(c,d) \in \mathbb{Z}^2 \setminus (0,0)} (cz + d)^{-k}. \quad (8)$$

Lemma 49. For all k , $E_k \in M_k(\Gamma_1)$

Proof. This follows from the fact that the sum converges absolutely. □

Lemma 50. The Eisenstein series possesses the Fourier expansion

$$E_k(z) = 1 + \frac{2}{\zeta(1-k)} \sum_{n=1}^{\infty} \sigma_{k-1}(n) e^{2\pi i n z}, \quad (9)$$

where $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$. In particular, we have

$$\begin{aligned} E_4(z) &= 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) e^{2\pi i n z} \\ E_6(z) &= 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) e^{2\pi i n z}. \end{aligned}$$

The infinite sum (8) does not converge absolutely for $k = 2$. On the other hand, the expression (9) converges to a holomorphic function on the upper half-plane and therefore

Definition 51. We set

$$E_2(z) := 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) e^{2\pi i n z}.$$

Lemma 52. This function is not modular, however it satisfies

$$z^{-2} E_2\left(\frac{-1}{z}\right) = E_2(z) - \frac{6i}{\pi} \frac{1}{z}.$$

The proof of this identity can be found in [12, Section 2.3]. The weight two Eisenstein series E_2 is an example of a *quasimodular form* [12, Section 5.1].

The discriminant form is a unique normalized cusp form of weight 12, which can be defined using E_4 and E_6 .

Definition 53. The discriminant form $\Delta(z)$ is given by

$$\Delta(z) = \frac{E_4(z)^3 - E_6(z)^2}{1728} \quad (10)$$

Lemma 54. $\Delta(z) \in M_{12}(\Gamma_1)$ and it vanishes at the unique cusp, i.e. it is a cusp form of level Γ_1 and weight 12.

Proof. Being a modular form of desired weight and level directly follows from those of E_4 and E_6 . It is a cusp form since the constant terms of Fourier expansions of E_4 and E_6 are both 1. \square

It also admits a product formula, which allow us to prove positivity of $\Delta(it)$ for $t > 0$ later.

Lemma 55. We have

$$\Delta(z) = e^{2\pi i z} \prod_{n \geq 1} (1 - e^{2\pi i n z})^{24}. \quad (11)$$

Proof. There are several known proofs of this. One possible proof that we can formalize is from Kohnen [6], which prove

$$\frac{1}{2\pi i z} \frac{d}{dz} (\log \Delta) = 1 - 24 \sum_{n \geq 1} \frac{n e^{2\pi i n z}}{1 - e^{2\pi i n z}}. \quad (12)$$

by using a multiplicative analogue of the Hecke operator and the valence formula. \square

Note that the RHS of (12) is equal to the $E_2(z)$. As a side note, we can also consider defining Δ as (11), and prove that it coincides with (10). Such an argument can be found in [2, Section 2.4].

Another example of modular forms we would like to consider are *theta functions* [12, Section 3.1].

Definition 56. We define three different theta functions (so called “Thetanullwerte”) as

$$\begin{aligned} \theta_{00}(z) &= \sum_{n \in \mathbb{Z}} e^{\pi i n^2 z} \\ \theta_{01}(z) &= \sum_{n \in \mathbb{Z}} (-1)^n e^{\pi i n^2 z} \\ \theta_{10}(z) &= \sum_{n \in \mathbb{Z}} e^{\pi i (n + \frac{1}{2})^2 z}. \end{aligned}$$

The group Γ_1 is generated by the elements $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Lemma 57. These elements act on the theta functions in the following way

$$z^{-2} \theta_{00}^4\left(\frac{-1}{z}\right) = -\theta_{00}^4(z) \quad (13)$$

$$z^{-2} \theta_{01}^4\left(\frac{-1}{z}\right) = -\theta_{10}^4(z) \quad (14)$$

$$z^{-2} \theta_{10}^4\left(\frac{-1}{z}\right) = -\theta_{01}^4(z) \quad (15)$$

and

$$\begin{aligned}\theta_{00}^4(z+1) &= \theta_{01}^4(z) \\ \theta_{01}^4(z+1) &= \theta_{00}^4(z)\end{aligned}\tag{16}$$

$$\theta_{10}^4(z+1) = -\theta_{10}^4(z).\tag{17}$$

Proof. The last three identities easily follow from the definition, and (15) and (14) are equivalent under $z \leftrightarrow -1/z$, so it is enough to show (13) and (15). These identities follow from the Poisson summation formula, which is already formalized by David Loeffler. More precisely, for the *two-variable* Jacobi theta function and its derivative (be careful for the variables, here we use τ instead of z)

$$\theta(z, \tau) = \sum_{n \in \mathbb{Z}} e^{2\pi i n z + \pi i n^2 \tau}$$

that specialize to $\theta_{00}(\tau) = \theta(0, \tau)$, $\theta_{01} = \theta(1/2, \tau)$, and $\theta_{10}(\tau) = e^{\pi i \tau/4} \theta(-\tau/2, \tau)$. Poisson summation formula gives

$$\theta(z, \tau) = \frac{1}{\sqrt{-i\tau}} e^{-\frac{\pi i z^2}{\tau}} \theta\left(\frac{z}{\tau}, -\frac{1}{\tau}\right)$$

and applying the specializations above yield the identities. For example, (15) follows from

$$\theta_{01}(\tau) = \theta\left(\frac{1}{2}, \tau\right) = \frac{1}{\sqrt{-i\tau}} e^{-\frac{\pi i}{4\tau}} \theta\left(\frac{1}{2\tau}, -\frac{1}{\tau}\right) = \frac{1}{\sqrt{-i\tau}} \theta_{10}\left(-\frac{1}{\tau}\right)$$

and taking 4th power. □

Using the above identities, we can prove that these are modular forms.

Lemma 58. *The theta functions θ_{00}^4 , θ_{01}^4 , and θ_{10}^4 belong to $M_2(\Gamma(2))$.*

Proof. Since the group $\Gamma(2)$ is generated by two elements

$$\alpha = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix},$$

it is enough to show that they are invariant under the slash actions with respect to α and β . For θ_{01} , (16) and (14)

$$\begin{aligned}(\theta_{01}^4|_2\alpha)(z) &= \theta_{01}^4(z+2) = \theta_{00}^4(z+1) = \theta_{01}^4(z) \\ (\theta_{01}^4|_2\beta)(z) &= (2z+1)^{-2} \theta_{00}^4\left(\frac{z}{2z+1}\right) = (2z+1)^{-2} \cdot -\left(-\frac{2z+1}{z}\right)^2 \theta_{10}^4\left(-\frac{2z+1}{z}\right) \\ &= -z^{-2} \theta_{10}^4\left(-2 - \frac{1}{z}\right) = -z^{-2} \theta_{10}^4\left(-\frac{1}{z}\right) = \theta_{01}^4(z).\end{aligned}$$

We can prove θ_{10}^4 and θ_{00}^4 similarly. □

We also have a nontrivial relation between these theta functions.

Lemma 59. *These three theta functions satisfy the Jacobi identity*

$$\theta_{01}^4 + \theta_{10}^4 = \theta_{00}^4.$$

Proof. One possible proof is to use dimension formula of the space of modular forms of weight 2 and level $\Gamma(2)$. Especially, this space have dimension 2, with basis $\theta_{01}^4, \theta_{10}^4$. □

The *strict* positivity of Jacobi theta functions might needed later.

Lemma 60. *All three functions $t \mapsto \theta_{01}(it), \theta_{10}(it), \theta_{00}(it)$ are positive for $t > 0$.*

Proof. By Lemma 59 and the transformation law (14), it is enough to prove the positivity for $\theta_{10}(it)$, which is clear from its definition. □

Note that we only need the fourth powers of theta functions in Lemma 58 to define (75), not $\theta_{00}, \theta_{01}, \theta_{10}$ themselves.

Definition 61. A weakly-holomorphic modular form of integer weight k and congruence subgroup Γ is a holomorphic function $f : \mathfrak{H} \rightarrow \mathbb{C}$ such that:

1. $f|_k \gamma = f$ for all $\gamma \in \Gamma$
2. for each $\alpha \in \Gamma_1$ $f|_k \alpha$ has the Fourier expansion $f|_k \alpha(z) = \sum_{n=n_0}^{\infty} c_f(\alpha, \frac{n}{n_\alpha}) e^{2\pi i \frac{n}{n_\alpha} z}$ for some $n_0 \in \mathbb{Z}$ and $n_\alpha \in \mathbb{N}$.

For an m -periodic holomorphic function f and $n \in \frac{1}{m}\mathbb{Z}$ we will denote the n -th Fourier coefficient of f by $c_f(n)$ so that

$$f(z) = \sum_{n \in \frac{1}{m}\mathbb{Z}} c_f(n) e^{2\pi i n z}.$$

We denote the space of weakly-holomorphic modular forms of weight k and group Γ by $M_k^!(\Gamma)$. The spaces $M_k^!(\Gamma)$ are infinite dimensional. Probably the most famous weakly-holomorphic modular form is the *elliptic j -invariant*

$$j := \frac{1728 E_4^3}{E_4^3 - E_6^2}.$$

This function belongs to $M_0^!(\Gamma_1)$ and has the Fourier expansion

$$j(z) = q^{-1} + 744 + 196884q + 21493760q^2 + 864299970q^3 + 20245856256q^4 + O(q^5)$$

where $q = e^{2\pi i z}$. Using a simple computer algebra system such as PARI GP or Mathematica one can compute first hundred terms of this Fourier expansion within few seconds. An important question is to find an asymptotic formula for $c_j(n)$, the n -th Fourier coefficient of j . Using the Hardy-Ramanujan circle method [9] or the non-holomorphic Poincare series [8] one can show that

Lemma 62.

$$c_j(n) = \frac{2\pi}{n} \sum_{k=1}^{\infty} \frac{A_k(n)}{k} I_1\left(\frac{4\pi\sqrt{n}}{k}\right) \quad n \in \mathbb{Z}_{>0}$$

where

$$A_k(n) = \sum_{\substack{h \bmod k \\ (h,k)=1}} e^{\frac{-2\pi i}{k}(nh+h')}, \quad hh' \equiv -1 \pmod{k},$$

and $I_\alpha(x)$ denotes the modified Bessel function of the first kind defined as in [1, Section 9.6].

A similar convergent asymptotic expansion holds for the Fourier coefficients of any weakly holomorphic modular form [5], [2, Propositions 1.10 and 1.12]. Such a convergent expansion implies effective estimates for the Fourier coefficients.

6 Fourier eigenfunctions with double zeroes at lattice points

In this section we construct two radial Schwartz functions $a, b : \mathbb{R}^8 \rightarrow i\mathbb{R}$ such that

$$\mathcal{F}(a) = a \tag{18}$$

$$\mathcal{F}(b) = -b \tag{19}$$

which double zeroes at all Λ_8 -vectors of length greater than $\sqrt{2}$. Recall that each vector of Λ_8 has length $\sqrt{2n}$ for some $n \in \mathbb{N}_{\geq 0}$. We define a and b so that their values are purely imaginary because this simplifies some of our computations. We will show in Section 7 that an appropriate linear combination of functions a and b satisfies conditions (5)–(7).

First, we will define function a . To this end we consider the following functions:

Definition 63.

$$\begin{aligned}\phi_{-4} &:= -Dj E_6^{-1} \\ \phi_{-2} &:= \phi_{-4} E_2 + Dj E_4^{-1} \\ \phi_0 &:= \phi_{-4} E_2^2 + 2Dj E_4^{-1} E_2 + j - 1728.\end{aligned}$$

Here $Dj(z) = \frac{1}{2\pi i} \frac{d}{dz} j(z)$.

Lemma 64. *These functions have the Fourier expansions*

$$\phi_{-4}(z) = q^{-1} + 504 + 73764q + 2695040q^2 + 54755730q^3 + O(q^4) \quad (20)$$

$$\begin{aligned}\phi_{-2}(z) &= 720 + 203040q + 9417600q^2 + 223473600q^3 + 3566782080q^4 + O(q^5) \\ \phi_0(z) &= 518400q + 31104000q^2 + 870912000q^3 + 15697152000q^4 + O(q^5)\end{aligned} \quad (21)$$

where $q = e^{2\pi iz}$.

The function $\phi_0(z)$ is not modular; however,

Lemma 65. *The identity 52 implies the following transformation rule:*

$$\phi_0\left(\frac{-1}{z}\right) = \phi_0(z) - \frac{12i}{\pi} \frac{1}{z} \phi_{-2}(z) - \frac{36}{\pi^2} \frac{1}{z^2} \phi_{-4}(z). \quad (22)$$

Definition 66. *For $x \in \mathbb{R}^8$ we define*

$$\begin{aligned}a(x) &:= \int_{-1}^i \phi_0\left(\frac{-1}{z+1}\right) (z+1)^2 e^{\pi i \|x\|^2 z} dz + \int_1^i \phi_0\left(\frac{-1}{z-1}\right) (z-1)^2 e^{\pi i \|x\|^2 z} dz \\ &\quad - 2 \int_0^i \phi_0\left(\frac{-1}{z}\right) z^2 e^{\pi i \|x\|^2 z} dz + 2 \int_i^\infty \phi_0(z) e^{\pi i \|x\|^2 z} dz.\end{aligned} \quad (23)$$

We observe that the contour integrals in (23) converge absolutely and uniformly for $x \in \mathbb{R}^8$. Indeed, $\phi_0(z) = O(e^{-2\pi i z})$ as $\Im(z) \rightarrow \infty$. Therefore, $a(x)$ is well defined. Now we prove that a satisfies condition (18).

Proposition 67.

The function a defined by (23) belongs to the Schwartz space and satisfies

$$\widehat{a}(x) = a(x).$$

Proof. First, we prove that a is a Schwartz function. From Lemma 50, Definition 51, and 62 we deduce that the Fourier coefficients of ϕ_0 satisfy

$$|c_{\phi_0}(n)| \leq 2 e^{4\pi\sqrt{n}} \quad n \in \mathbb{Z}_{>0}.$$

Thus, there exists a positive constant C such that

$$|\phi_0(z)| \leq C e^{-2\pi \Im z} \quad \text{for } \Im z > \frac{1}{2}.$$

We estimate the first summand in the right-hand side of (23). For $r \in \mathbb{R}_{\geq 0}$ we have

$$\begin{aligned}\left| \int_{-1}^i \phi_0\left(\frac{-1}{z+1}\right) (z+1)^2 e^{\pi i r^2 z} dz \right| &= \left| \int_{i\infty}^{-1/(i+1)} \phi_0(z) z^{-4} e^{\pi i r^2 (-1/z-1)} dz \right| \leq \\ C_1 \int_{1/2}^\infty e^{-2\pi t} e^{-\pi r^2/t} dt &\leq C_1 \int_0^\infty e^{-2\pi t} e^{-\pi r^2/t} dt = C_2 r K_1(2\sqrt{2}\pi r)\end{aligned}$$

where C_1 and C_2 are some positive constants and $K_\alpha(x)$ is the modified Bessel function of the second kind defined as in [1, Section 9.6]. This estimate also holds for the second and third summand in (23). For the last summand we have

$$\left| \int_i^{i\infty} \phi_0(z) e^{\pi i r^2 z} dz \right| \leq C \int_1^\infty e^{-2\pi t} e^{-\pi r^2 t} dt = C_3 \frac{e^{\pi(r^2+2)}}{r^2+2}.$$

Therefore, we arrive at

$$|a(r)| \leq 4C_2 r K_1(2\sqrt{2}\pi r) + 2C_3 \frac{e^{-\pi(r^2+2)}}{r^2+2}.$$

It is easy to see that the left hand side of this inequality decays faster than any inverse power of r . Analogous estimates can be obtained for all derivatives $\frac{d^k}{dr^k} a(r)$.

Now we show that a is an eigenfunction of the Fourier transform. We recall that the Fourier transform of a Gaussian function is

$$\mathcal{F}(e^{\pi i \|x\|^2 z})(y) = z^{-4} e^{\pi i \|y\|^2 (\frac{-1}{z})}. \quad (24)$$

Next, we exchange the contour integration with respect to z variable and Fourier transform with respect to x variable in (23). This can be done, since the corresponding double integral converges absolutely. In this way we obtain

$$\begin{aligned} \widehat{a}(y) &= \int_{-1}^i \phi_0\left(\frac{-1}{z+1}\right) (z+1)^2 z^{-4} e^{\pi i \|y\|^2 (\frac{-1}{z})} dz + \int_1^i \phi_0\left(\frac{-1}{z-1}\right) (z-1)^2 z^{-4} e^{\pi i \|y\|^2 (\frac{-1}{z})} dz \\ &\quad - 2 \int_0^i \phi_0\left(\frac{-1}{z}\right) z^2 z^{-4} e^{\pi i \|y\|^2 (\frac{-1}{z})} dz + 2 \int_i^{i\infty} \phi_0(z) z^{-4} e^{\pi i \|y\|^2 (\frac{-1}{z})} dz. \end{aligned}$$

Now we make a change of variables $w = \frac{-1}{z}$. We obtain

$$\begin{aligned} \widehat{a}(y) &= \int_1^i \phi_0\left(1 - \frac{1}{w-1}\right) \left(\frac{-1}{w} + 1\right)^2 w^2 e^{\pi i \|y\|^2 w} dw \\ &\quad + \int_{-1}^i \phi_0\left(1 - \frac{1}{w+1}\right) \left(\frac{-1}{w} - 1\right)^2 w^2 e^{\pi i \|y\|^2 w} dw \\ &\quad - 2 \int_{i\infty}^i \phi_0(w) e^{\pi i \|y\|^2 w} dw + 2 \int_i^0 \phi_0\left(\frac{-1}{w}\right) w^2 e^{\pi i \|y\|^2 w} dw. \end{aligned}$$

Since ϕ_0 is 1-periodic we have

$$\begin{aligned} \widehat{a}(y) &= \int_1^i \phi_0\left(\frac{-1}{z-1}\right) (z-1)^2 e^{\pi i \|y\|^2 z} dz + \int_{-1}^i \phi_0\left(\frac{-1}{z+1}\right) (z+1)^2 e^{\pi i \|y\|^2 z} dz \\ &\quad + 2 \int_i^{i\infty} \phi_0(z) e^{\pi i \|y\|^2 z} dz - 2 \int_0^i \phi_0\left(\frac{-1}{z}\right) z^2 e^{\pi i \|y\|^2 z} dz \\ &= a(y). \end{aligned}$$

This finishes the proof of the proposition. \square

Next, we check that a has double zeroes at all Λ_8 -lattice points of length greater than $\sqrt{2}$.

Proposition 68. For $r > \sqrt{2}$ we can express $a(r)$ in the following form

$$a(r) = -4 \sin(\pi r^2/2)^2 \int_0^{i\infty} \phi_0\left(\frac{-1}{z}\right) z^2 e^{\pi i r^2 z} dz. \quad (25)$$

Proof. We denote the right hand side of (25) by $d(r)$. It is easy to see that $d(r)$ is well-defined. Indeed, from the transformation formula (22) and the expansions (21)–(20) we obtain

$$\begin{aligned} \phi_0\left(\frac{-1}{it}\right) &= O(e^{-2\pi/t}) \quad \text{as } t \rightarrow 0 \\ \phi_0\left(\frac{-1}{it}\right) &= O(t^{-2} e^{2\pi t}) \quad \text{as } t \rightarrow \infty \end{aligned}$$

Hence, the integral (25) converges absolutely for $r > \sqrt{2}$. We can write

$$\begin{aligned} d(r) &= \int_{-1}^{i\infty-1} \phi_0\left(\frac{-1}{z+1}\right) (z+1)^2 e^{\pi i r^2 z} dz - 2 \int_0^{i\infty} \phi_0\left(\frac{-1}{z}\right) z^2 e^{\pi i r^2 z} dz \\ &\quad + \int_1^{i\infty+1} \phi_0\left(\frac{-1}{z-1}\right) (z-1)^2 e^{\pi i r^2 z} dz. \end{aligned}$$

From (22) we deduce that if $r > \sqrt{2}$ then $\phi_0\left(\frac{-1}{z}\right) z^2 e^{\pi i r^2 z} \rightarrow 0$ as $\Im(z) \rightarrow \infty$. Therefore, we can deform the paths of integration and rewrite

$$\begin{aligned} d(r) &= \int_{-1}^i \phi_0\left(\frac{-1}{z+1}\right) (z+1)^2 e^{\pi i r^2 z} dz + \int_i^{i\infty} \phi_0\left(\frac{-1}{z+1}\right) (z+1)^2 e^{\pi i r^2 z} dz \\ &\quad - 2 \int_0^i \phi_0\left(\frac{-1}{z}\right) z^2 e^{\pi i r^2 z} dz - 2 \int_i^{i\infty} \phi_0\left(\frac{-1}{z}\right) z^2 e^{\pi i r^2 z} dz \\ &\quad + \int_1^i \phi_0\left(\frac{-1}{z-1}\right) (z-1)^2 e^{\pi i r^2 z} dz + \int_i^{i\infty} \phi_0\left(\frac{-1}{z-1}\right) (z-1)^2 e^{\pi i r^2 z} dz. \end{aligned}$$

Now from (22) we find

$$\begin{aligned} &\phi_0\left(\frac{-1}{z+1}\right) (z+1)^2 - 2\phi_0\left(\frac{-1}{z}\right) z^2 + \phi_0\left(\frac{-1}{z-1}\right) (z-1)^2 = \\ &\phi_0(z+1) (z+1)^2 - 2\phi_0(z) z^2 + \phi_0(z-1) (z-1)^2 \\ &\quad - \frac{12i}{\pi} \left(\phi_{-2}(z+1) (z+1) - 2\phi_{-2}(z) z + \phi_{-2}(z-1) (z-1) \right) \\ &\quad - \frac{36}{\pi^2} \left(\phi_{-4}(z+1) - 2\phi_{-4}(z) + \phi_{-4}(z-1) \right) = \\ &2\phi_0(z). \end{aligned}$$

Thus, we obtain

$$\begin{aligned} d(r) &= \int_{-1}^i \phi_0\left(\frac{-1}{z+1}\right) (z+1)^2 e^{\pi i r^2 z} dz - 2 \int_0^i \phi_0\left(\frac{-1}{z}\right) z^2 e^{\pi i r^2 z} dz \\ &\quad + \int_1^i \phi_0\left(\frac{-1}{z-1}\right) (z-1)^2 e^{\pi i r^2 z} dz + 2 \int_i^{i\infty} \phi_0(z) e^{\pi i r^2 z} dz = a(r). \end{aligned}$$

This finishes the proof. \square

Finally, we find another convenient integral representation for a and compute values of $a(r)$ at $r = 0$ and $r = \sqrt{2}$.

Proposition 69. *For $r \geq 0$ we have*

$$a(r) = 4i \sin(\pi r^2/2)^2 \left(\frac{36}{\pi^3 (r^2 - 2)} - \frac{8640}{\pi^3 r^4} + \frac{18144}{\pi^3 r^2} \right. \\ \left. + \int_0^\infty \left(t^2 \phi_0\left(\frac{i}{t}\right) - \frac{36}{\pi^2} e^{2\pi t} + \frac{8640}{\pi} t - \frac{18144}{\pi^2} \right) e^{-\pi r^2 t} dt \right). \quad (26)$$

The integral converges absolutely for all $r \in \mathbb{R}_{\geq 0}$.

Proof. Suppose that $r > \sqrt{2}$. Then by Proposition 68

$$a(r) = 4i \sin(\pi r^2/2)^2 \int_0^\infty \phi_0(i/t) t^2 e^{-\pi r^2 t} dt.$$

From (21)–(22) we obtain

$$\phi_0(i/t) t^2 = \frac{36}{\pi^2} e^{2\pi t} - \frac{8640}{\pi} t + \frac{18144}{\pi^2} + O(t^2 e^{-2\pi t}) \quad \text{as } t \rightarrow \infty. \quad (27)$$

For $r > \sqrt{2}$ we have

$$\int_0^\infty \left(\frac{36}{\pi^2} e^{2\pi t} + \frac{8640}{\pi} t + \frac{18144}{\pi^2} \right) e^{-\pi r^2 t} dt = \frac{36}{\pi^3 (r^2 - 2)} - \frac{8640}{\pi^3 r^4} + \frac{18144}{\pi^3 r^2}.$$

Therefore, the identity (26) holds for $r > \sqrt{2}$.

On the other hand, from the definition (23) we see that $a(r)$ is analytic in some neighborhood of $[0, \infty)$. The asymptotic expansion (27) implies that the right hand side of (26) is also analytic in some neighborhood of $[0, \infty)$. Hence, the identity (26) holds on the whole interval $[0, \infty)$. This finishes the proof of the proposition. \square

From the identity (26) we see that the values $a(r)$ are in $i\mathbb{R}$ for all $r \in \mathbb{R}_{\geq 0}$. In particular, we have

Proposition 70. *We have*

$$a(0) = \frac{-i 8640}{\pi} \quad a(\sqrt{2}) = 0 \quad a'(\sqrt{2}) = \frac{i 72\sqrt{2}}{\pi}.$$

Proof. These identities follow immediately from the previous proposition. \square

Now we construct function b . To this end we consider the function

Definition 71.

$$h(z) := 128 \frac{\theta_{00}^4(z) + \theta_{01}^4(z)}{\theta_{10}^8(z)}. \quad (28)$$

It is easy to see that $h \in M_{-2}^1(\Gamma_0(2))$. Indeed, first we check that $h|_{-2}\gamma = h$ for all $\gamma \in \Gamma_0(2)$. Since the group $\Gamma_0(2)$ is generated by elements $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ it suffices to check that h is invariant under their action. This follows immediately from (13)–(17) and (28). Next we analyze the poles of h . It is known [7, Chapter I Lemma 4.1] that θ_{10} has no zeros in the upper-half plane and hence h has poles only at the cusps. At the cusp $i\infty$ this modular form has the Fourier expansion

$$h(z) = q^{-1} + 16 - 132q + 640q^2 - 2550q^3 + O(q^4).$$

Let $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ be elements of Γ_1 .

Definition 72. We define the followig three functions

$$\psi_I := h - h|_{-2}ST \quad (29)$$

$$\psi_T := \psi_I|_{-2}T$$

$$\psi_S := \psi_I|_{-2}S. \quad (30)$$

Lemma 73. More explicitly, we have

$$\begin{aligned} \psi_I(z) &= 128 \frac{\theta_{00}^4(z) + \theta_{01}^4(z)}{\theta_{10}^8(z)} + 128 \frac{\theta_{01}^4(z) - \theta_{10}^4(z)}{\theta_{00}^8(z)} \\ \psi_T(z) &= 128 \frac{\theta_{00}^4(z) + \theta_{01}^4(z)}{\theta_{10}^8(z)} + 128 \frac{\theta_{00}^4(z) + \theta_{10}^4(z)}{\theta_{01}^8(z)} \\ \psi_S(z) &= -128 \frac{\theta_{00}^4(z) + \theta_{10}^4(z)}{\theta_{01}^8(z)} - 128 \frac{\theta_{10}^4(z) - \theta_{01}^4(z)}{\theta_{00}^8(z)}. \end{aligned}$$

Lemma 74. The Fourier expansions of these functions are

$$\psi_I(z) = q^{-1} + 144 - 5120q^{1/2} + 70524q - 626688q^{3/2} + 4265600q^2 + O(q^{5/2}) \quad (31)$$

$$\psi_T(z) = q^{-1} + 144 + 5120q^{1/2} + 70524q + 626688q^{3/2} + 4265600q^2 + O(q^{5/2})$$

$$\psi_S(z) = -10240q^{1/2} - 1253376q^{3/2} - 48328704q^{5/2} - 1059078144q^{7/2} + O(q^{9/2}). \quad (32)$$

Definition 75. For $x \in \mathbb{R}^8$ define

$$\begin{aligned} b(x) &:= \int_{-1}^i \psi_T(z) e^{\pi i \|x\|^2 z} dz + \int_1^i \psi_T(z) e^{\pi i \|x\|^2 z} dz \\ &\quad - 2 \int_0^i \psi_I(z) e^{\pi i \|x\|^2 z} dz - 2 \int_i^{i\infty} \psi_S(z) e^{\pi i \|x\|^2 z} dz. \end{aligned} \quad (33)$$

Now we prove that b satisfies condition (19).

Proposition 76. The function b defined by (33) belongs to the Schwartz space and satisfies

$$\widehat{b}(x) = -b(x).$$

Proof. Here, we repeat the arguments used in the proof of Proposition 67. First we show that b is a Schwartz function. We have

$$\begin{aligned} \int_{-1}^i \psi_T(z) e^{\pi i r^2 z} dz &= \int_0^{i+1} \psi_I(z) e^{\pi i r^2 (z-1)} dz = \\ &= \int_{i\infty}^{-1/(i+1)} \psi_I\left(\frac{-1}{z}\right) e^{\pi i r^2 (-1/z-1)} z^{-2} dz = \int_{i\infty}^{-1/(i+1)} \psi_S(z) z^{-4} e^{\pi i r^2 (-1/z-1)} dz. \end{aligned}$$

There exists a positive constant C such that

$$|\psi_S(z)| \leq C e^{-\pi \Im z} \quad \text{for } \Im z > \frac{1}{2}.$$

Thus, as in the proof of Proposition 67 we estimate the first summand in the left-hand side of (33)

$$\left| \int_{-1}^i \psi_T(z) e^{\pi i r^2 z} dz \right| \leq C_1 r K_1(2\pi r).$$

We combine this inequality with analogous estimates for the other three summands and obtain

$$|b(r)| \leq C_2 r K_1(2\pi r) + C_3 \frac{e^{-\pi(r^2+1)}}{r^2+1}.$$

Here C_1 , C_2 , and C_3 are some positive constants. Similar estimates hold for all derivatives $\frac{d^k}{dr^k}b(r)$.

Now we prove that b is an eigenfunction of the Fourier transform. We use identity (24) and change contour integration in z and Fourier transform in x . Thus we obtain

$$\begin{aligned} \mathcal{F}(b)(x) &= \int_{-1}^i \psi_T(z) z^{-4} e^{\pi i \|x\|^2 (\frac{-1}{z})} dz + \int_1^i \psi_T(z) z^{-4} e^{\pi i \|x\|^2 (\frac{-1}{z})} dz \\ &\quad - 2 \int_0^i \psi_I(z) z^{-4} e^{\pi i \|x\|^2 (\frac{-1}{z})} dz - 2 \int_i^{i\infty} \psi_S(z) z^{-4} e^{\pi i \|x\|^2 (\frac{-1}{z})} dz. \end{aligned}$$

We make the change of variables $w = \frac{-1}{z}$ and arrive at

$$\begin{aligned} \mathcal{F}(b)(x) &= \int_1^i \psi_T\left(\frac{-1}{w}\right) w^2 e^{\pi i \|x\|^2 w} dw + \int_{-1}^i \psi_T\left(\frac{-1}{w}\right) w^2 e^{\pi i \|x\|^2 w} dw \\ &\quad - 2 \int_{i\infty}^i \psi_I\left(\frac{-1}{w}\right) w^2 e^{\pi i \|x\|^2 w} dw - 2 \int_i^0 \psi_S\left(\frac{-1}{w}\right) w^2 e^{\pi i \|x\|^2 w} dw. \end{aligned}$$

Now we observe that the definitions (29)–(30) imply

$$\begin{aligned} \psi_T|_{-2S} &= -\psi_T \\ \psi_I|_{-2S} &= \psi_S \\ \psi_S|_{-2S} &= \psi_I. \end{aligned}$$

Therefore, we arrive at

$$\begin{aligned} \mathcal{F}(b)(x) &= \int_1^i -\psi_T(z) e^{\pi i \|x\|^2 z} dz + \int_{-1}^i -\psi_T(z) e^{\pi i \|x\|^2 z} dz \\ &\quad + 2 \int_i^{i\infty} \psi_S(z) e^{\pi i \|x\|^2 z} dz + 2 \int_0^i \psi_I(z) e^{\pi i \|x\|^2 z} dz. \end{aligned}$$

Now from (33) we see that

$$\mathcal{F}(b)(x) = -b(x).$$

□

Now we regard the radial function b as a function on $\mathbb{R}_{\geq 0}$. We check that b has double roots at Λ_8 -points.

Proposition 77. *For $r > \sqrt{2}$ function $b(r)$ can be expressed as*

$$b(r) = -4 \sin(\pi r^2/2)^2 \int_0^{i\infty} \psi_I(z) e^{\pi i r^2 z} dz. \quad (34)$$

Proof. We denote the right hand side of (34) by $c(r)$. First, we check that $c(r)$ is well-defined. We have

$$\begin{aligned} \psi_I(it) &= O(t^2 e^{\pi/t}) \quad \text{as } t \rightarrow 0 \\ \psi_I(it) &= O(e^{2\pi t}) \quad \text{as } t \rightarrow \infty. \end{aligned}$$

Therefore, the integral (34) converges for $r > \sqrt{2}$. Then we rewrite it in the following way:

$$c(r) = \int_{-1}^{i\infty-1} \psi_I(z+1) e^{\pi i r^2 z} dz - 2 \int_0^{i\infty} \psi_I(z) e^{\pi i r^2 z} dz + \int_1^{i\infty+1} \psi_I(z-1) e^{\pi i r^2 z} dz.$$

From the Fourier expansion (31) we know that $\psi_I(z) = e^{-2\pi i z} + O(1)$ as $\Im(z) \rightarrow \infty$. By assumption $r^2 > 2$, hence we can deform the path of integration and write

$$\begin{aligned} \int_{-1}^{i\infty-1} \psi_I(z+1) e^{\pi i r^2 z} dz &= \int_{-1}^i \psi_T(z) e^{\pi i r^2 z} dz + \int_i^{i\infty} \psi_T(z) e^{\pi i r^2 z} dz \\ \int_1^{i\infty+1} \psi_I(z-1) e^{\pi i r^2 z} dz &= \int_{-1}^i \psi_T(z) e^{\pi i r^2 z} dz + \int_i^{i\infty} \psi_T(z) e^{\pi i r^2 z} dz. \end{aligned}$$

We have

$$\begin{aligned} c(r) &= \int_{-1}^i \psi_T(z) e^{\pi i r^2 z} dz + \int_1^i \psi_T(z) e^{\pi i r^2 z} dz - 2 \int_0^i \psi_I(z) e^{\pi i r^2 z} dz \\ &\quad + 2 \int_i^{i\infty} (\psi_T(z) - \psi_I(z)) e^{\pi i r^2 z} dz. \end{aligned} \tag{35}$$

Next, we check that the functions ψ_I, ψ_T , and ψ_S satisfy the following identity:

$$\psi_T + \psi_S = \psi_I. \tag{36}$$

Indeed, from definitions (29)-(30) we get

$$\begin{aligned} \psi_T + \psi_S &= (h - h|_{-2}ST)|_{-2}T + (h - h|_{-2}ST)|_{-2}S \\ &= h|_{-2}T - h|_{-2}ST^2 + h|_{-2}S - h|_{-2}STS. \end{aligned}$$

Note that ST^2S belongs to $\Gamma_0(2)$. Thus, since $h \in M_{-2}^1\Gamma_0(2)$ we get

$$\psi_T + \psi_S = h|_{-2}T - h|_{-2}STS.$$

Now we observe that T and $STS(ST)^{-1}$ are also in $\Gamma_0(2)$. Therefore,

$$\psi_T + \psi_S = h|_{-2}T - h|_{-2}STS = h|_{-2} - h|ST = \psi_I.$$

Combining (35) and (36) we find

$$\begin{aligned} c(r) &= \int_{-1}^i \psi_T(z) e^{\pi i r^2 z} dz + \int_1^i \psi_T(z) e^{\pi i r^2 z} dz - 2 \int_0^i \psi_I(z) e^{\pi i r^2 z} dz \\ &\quad - 2 \int_i^{i\infty} \psi_S(z) e^{\pi i r^2 z} dz \\ &= b(r). \end{aligned}$$

□

At the end of this section we find another integral representation of $b(r)$ for $r \in \mathbb{R}_{\geq 0}$ and compute special values of b .

Proposition 78. For $r \geq 0$ we have

$$b(r) = 4i \sin(\pi r^2/2)^2 \left(\frac{144}{\pi r^2} + \frac{1}{\pi(r^2-2)} + \int_0^\infty (\psi_I(it) - 144 - e^{2\pi t}) e^{-\pi r^2 t} dt \right). \quad (37)$$

The integral converges absolutely for all $r \in \mathbb{R}_{\geq 0}$.

Proof. The proof is analogous to the proof of Proposition 69. First, suppose that $r > \sqrt{2}$. Then by Proposition 77

$$b(r) = 4i \sin(\pi r^2/2)^2 \int_0^\infty \psi_I(it) e^{-\pi r^2 t} dt.$$

From (31) we obtain

$$\psi_I(it) = e^{2\pi t} + 144 + O(e^{-\pi t}) \quad \text{as } t \rightarrow \infty. \quad (38)$$

For $r > \sqrt{2}$ we have

$$\int_0^\infty (e^{2\pi t} + 144) e^{-\pi r^2 t} dt = \frac{1}{\pi(r^2-2)} + \frac{144}{\pi r^2}.$$

Therefore, the identity (37) holds for $r > \sqrt{2}$.

On the other hand, from the definition (33) we see that $b(r)$ is analytic in some neighborhood of $[0, \infty)$. The asymptotic expansion (38) implies that the right hand side of (37) is also analytic in some neighborhood of $[0, \infty)$. Hence, the identity (37) holds on the whole interval $[0, \infty)$. This finishes the proof of the proposition. \square

We see from (37) that $b(r) \in i\mathbb{R}$ for all $r \in \mathbb{R}_{\geq 0}$. Another immediate corollary of this proposition is

Proposition 79. We have

$$b(0) = 0 \quad b(\sqrt{2}) = 0 \quad b'(\sqrt{2}) = \frac{i}{2\sqrt{2}\pi}.$$

7 Proof of Theorem 37

Our proof of the Theorem 37 relies on the following two inequalities for modular objects.

Proposition 80. Consider the function $A : (0, \infty) \rightarrow \mathbb{C}$ defined as

$$A(t) := -t^2 \phi_0(i/t) - \frac{36}{\pi^2} \psi_I(it).$$

Then $A(t) \in (-\infty, 0)$ for all $t \in (0, \infty)$.

Remark: We might formalize the original proof from [11] or the proof of Dan Romik “On Viazovska’s modular form inequalities” [10]. Below is the proof from [11].

Proof. Function $A(t)$ is plotted in Figure 1.

We observe that we can compute the values of $A(t)$ for $t \in (0, \infty)$ with any given precision. Indeed, from identities (22) and (30) we obtain the following two presentations for $A(t)$

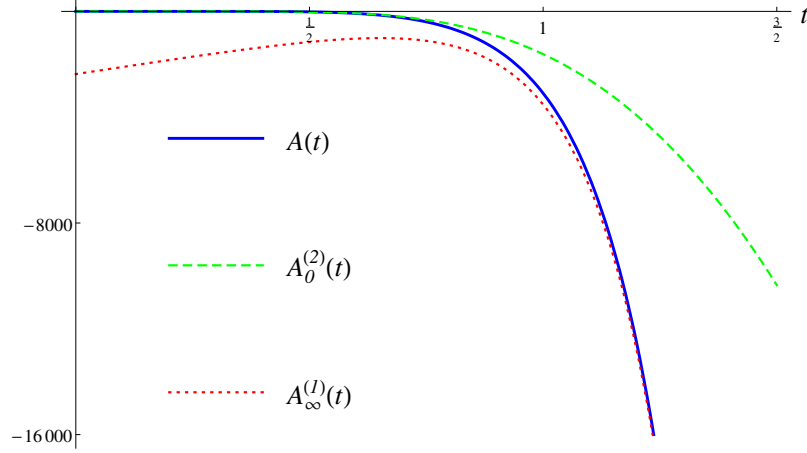
$$\begin{aligned} A(t) &= -t^2 \phi_0(i/t) + \frac{36}{\pi^2} t^2 \psi_S(i/t) \\ A(t) &= -t^2 \phi_0(it) + \frac{12}{\pi} t \phi_{-2}(it) - \frac{36}{\pi^2} \phi_{-4}(it) - \frac{36}{\pi^2} \psi_I(i/t). \end{aligned}$$

For an integer $n \geq 0$ let $A_0^{(n)}$ and $A_\infty^{(n)}$ be the functions such that

$$A(t) = A_0^{(n)}(t) + O(t^2 e^{-\pi n/t}) \quad \text{as } t \rightarrow 0 \quad (39)$$

$$A(t) = A_\infty^{(n)}(t) + O(t^2 e^{-\pi n t}) \quad \text{as } t \rightarrow \infty. \quad (40)$$

Figure 1: Plot of the functions $A(t)$, $A_0^{(2)}(t) = -\frac{368640}{\pi^2} t^2 e^{-\pi/t}$, and $A_\infty^{(1)}(t) = -\frac{72}{\pi^2} e^{2\pi t} + \frac{8640}{\pi} t - \frac{23328}{\pi^2}$.



For each $n \geq 0$ we can compute these functions from the Fourier expansions (20)–(21), (31), and (32). For example, from (20)–(21) and (31) we compute

$$A_\infty^{(6)}(t) = -\frac{72}{\pi^2} e^{2\pi t} - \frac{23328}{\pi^2} + \frac{184320}{\pi^2} e^{-\pi t} - \frac{5194368}{\pi^2} e^{-2\pi t} + \frac{22560768}{\pi^2} e^{-3\pi t} - \frac{250583040}{\pi^2} e^{-4\pi t} + \frac{869916672}{\pi^2} e^{-5\pi t} \\ + t\left(\frac{8640}{\pi} + \frac{2436480}{\pi} e^{-2\pi t} + \frac{113011200}{\pi} e^{-4\pi t}\right) - t^2(518400 e^{-2\pi t} + 31104000 e^{-4\pi t}).$$

From (20)–(21) and (32) we compute

$$A_0^{(6)}(t) = t^2\left(-\frac{368640}{\pi^2} e^{-\pi/t} - 518400 e^{-2\pi/t} - \frac{45121536}{\pi^2} e^{-3\pi/t} - 31104000 e^{-4\pi/t} - \frac{1739833344}{\pi^2} e^{-5\pi/t}\right).$$

Moreover, from the convergent asymptotic expansion for the Fourier coefficients of a weakly holomorphic modular form [2, Proposition 1.12] we find that the n -th Fourier coefficient $c_{\psi_I}(n)$ of ψ_I satisfies

$$|c_{\psi_I}(n)| \leq e^{4\pi\sqrt{n}} \quad n \in \frac{1}{2}\mathbb{Z}_{>0}. \quad (41)$$

Similar inequalities hold for the Fourier coefficients of ψ_S , ϕ_0 , ϕ_{-2} , and ϕ_{-4} :

$$\begin{aligned} |c_{\psi_S}(n)| &\leq 2e^{4\pi\sqrt{n}} & n \in \frac{1}{2}\mathbb{Z}_{>0} \\ |c_{\phi_0}(n)| &\leq 2e^{4\pi\sqrt{n}} & n \in \mathbb{Z}_{>0} \\ |c_{\phi_{-2}}(n)| &\leq e^{4\pi\sqrt{n}} & n \in \mathbb{Z}_{>0} \\ |c_{\phi_{-4}}(n)| &\leq e^{4\pi\sqrt{n}} & n \in \mathbb{Z}_{>0}. \end{aligned} \quad (42)$$

Therefore, we can estimate the error terms in the asymptotic expansions (39) and (40) of $A(t)$

$$\begin{aligned} |A(t) - A_0^{(m)}(t)| &\leq \left(t^2 + \frac{36}{\pi^2}\right) \sum_{n=m}^{\infty} 2e^{2\sqrt{2}\pi\sqrt{n}} e^{-\pi n/t} \\ |A(t) - A_\infty^{(m)}(t)| &\leq \left(t^2 + \frac{12}{\pi} t + \frac{36}{\pi^2}\right) \sum_{n=m}^{\infty} 2e^{2\sqrt{2}\pi\sqrt{n}} e^{-\pi n t}. \end{aligned}$$

For an integer $m \geq 0$ we set

$$\begin{aligned} R_0^{(m)} &:= \left(t^2 + \frac{36}{\pi^2}\right) \sum_{n=m}^{\infty} 2e^{2\sqrt{2}\pi\sqrt{n}} e^{-\pi n/t} \\ R_\infty^{(m)} &:= \left(t^2 + \frac{12}{\pi} t + \frac{36}{\pi^2}\right) \sum_{n=m}^{\infty} 2e^{2\sqrt{2}\pi\sqrt{n}} e^{-\pi n t}. \end{aligned}$$

Using interval arithmetic we check that

$$\begin{aligned} |R_0^{(6)}(t)| &\leq |A_0^{(6)}(t)| & \text{for } t \in (0, 1] \\ |R_\infty^{(6)}(t)| &\leq |A_\infty^{(6)}(t)| & \text{for } t \in [1, \infty) \\ A_0^{(6)}(t) &< 0 & \text{for } t \in (0, 1] \\ A_\infty^{(6)}(t) &< 0 & \text{for } t \in [1, \infty). \end{aligned}$$

Thus, we see that $A(t) < 0$ for $t \in (0, \infty)$. This finishes the proof of the Proposition. \square

Proposition 81. Consider the function $B : (0, \infty) \rightarrow \mathbb{C}$ defined as

$$B(t) := -t^2 \phi_0(i/t) + \frac{36}{\pi^2} \psi_I(it).$$

Then $B(t) \in (0, \infty)$ for all $t \in (0, \infty)$.

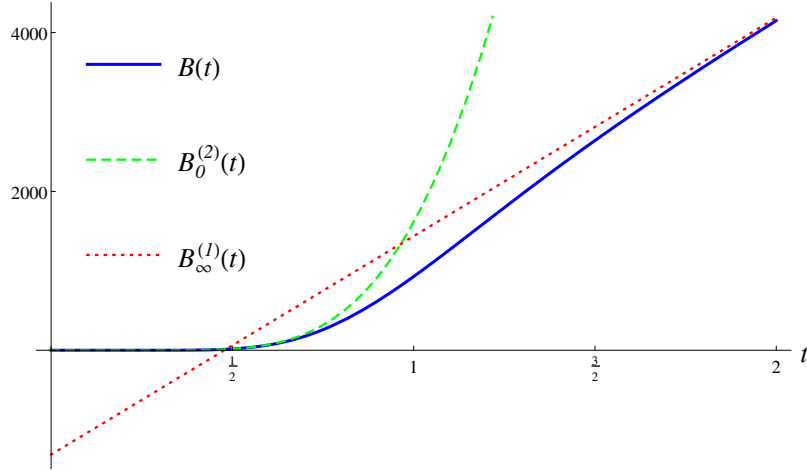
Remark Below is the proof from [11]. Similarly to the previous proposition, another (hopefully easier for the formalization) proof of this inequality is given in [10].

Proof. The function B can be also written as

$$\begin{aligned} B(t) &= -t^2 \phi_0(i/t) - \frac{36}{\pi^2} t^2 \psi_S(i/t) \\ B(t) &= -t^2 \phi_0(it) + \frac{12}{\pi} t \phi_{-2}(it) - \frac{36}{\pi^2} \phi_{-4}(it) + \frac{36}{\pi^2} \psi_I(i/t). \end{aligned}$$

Our aim is to prove that $B(t) > 0$ for $t \in (0, \infty)$. A plot of $B(t)$ is given in Figure 2.

Figure 2: Plot of the functions $B(t)$, $B_0^{(2)}(t) = \frac{368640}{\pi^2} t^2 e^{-\pi/t}$, and $B_\infty^{(1)}(t) = \frac{8640}{\pi} t - \frac{23328}{\pi^2}$.



For $n \geq 0$ let $B_0^{(n)}$ and $B_\infty^{(n)}$ be the functions such that

$$\begin{aligned} B(t) &= B_0^{(n)}(t) + O(t^2 e^{-\pi n/t}) & \text{as } t \rightarrow 0 \\ B(t) &= B_\infty^{(n)}(t) + O(t^2 e^{-\pi n t}) & \text{as } t \rightarrow \infty. \end{aligned}$$

We find

$$\begin{aligned} B_\infty^{(6)}(t) &= -\frac{12960}{\pi^2} - \frac{184320}{\pi^2} e^{-\pi t} - \frac{116640}{\pi^2} e^{-2\pi t} - \frac{22560768}{\pi^2} e^{-3\pi t} + \frac{56540160}{\pi^2} e^{-4\pi t} - \frac{869916672}{\pi^2} e^{-5\pi t} \\ &\quad + t\left(\frac{8640}{\pi} + \frac{2436480}{\pi} e^{-2\pi t} + \frac{113011200}{\pi} e^{-4\pi t}\right) - t^2(518400 e^{-2\pi t} + 31104000 e^{-4\pi t}) \end{aligned}$$

and

$$B_0^{(6)}(t) = t^2 \left(\frac{368640}{\pi^2} e^{-\pi/t} - 518400 e^{-2\pi/t} + \frac{45121536}{\pi^2} e^{-3\pi/t} - 31104000 e^{-4\pi/t} + \frac{1739833344}{\pi^2} e^{-5\pi/t} \right).$$

The estimates (41)–(42) imply that

$$\left| B(t) - B_0^{(6)}(t) \right| \leq R_0^{(6)}(t) \quad \text{for } t \in (0, 1]$$

and

$$\left| B(t) - B_\infty^{(6)}(t) \right| \leq R_\infty^{(6)}(t) \quad \text{for } t \in [1, \infty).$$

Using interval arithmetic we verify that

$$\begin{aligned} \left| R_0^{(6)}(t) \right| &\leq \left| B_0^{(6)}(t) \right| && \text{for } t \in (0, 1] \\ \left| R_\infty^{(6)}(t) \right| &\leq \left| B_\infty^{(6)}(t) \right| && \text{for } t \in [1, \infty) \\ B_0^{(6)}(t) &> 0 && \text{for } t \in (0, 1] \\ B_\infty^{(6)}(t) &> 0 && \text{for } t \in [1, \infty). \end{aligned}$$

Now identity (44) implies (6). □

Finally, we are ready to prove Theorem 37.

Theorem 82. *The function*

$$g(x) := \frac{\pi i}{8640} a(x) + \frac{i}{240\pi} b(x)$$

satisfies conditions (5)–(7).

Proof. First, we prove that (5) holds. By Propositions 68 and 77 we know that for $r > \sqrt{2}$

$$g(r) = \frac{\pi}{2160} \sin(\pi r^2/2)^2 \int_0^\infty A(t) e^{-\pi r^2 t} dt \tag{43}$$

where

$$A(t) = -t^2 \phi_0(i/t) - \frac{36}{\pi^2} \psi_I(it).$$

from the Proposition 80 we know that $A(t) < 0$ for $t \in (0, \infty)$. Therefore identity (43) implies (5).

Next, we prove (6). By Propositions 69 and 78 we know that for $r > 0$

$$\widehat{g}(r) = \frac{\pi}{2160} \sin(\pi r^2/2)^2 \int_0^\infty B(t) e^{-\pi r^2 t} dt \tag{44}$$

where

$$B(t) = -t^2 \phi_0(i/t) + \frac{36}{\pi^2} \psi_I(it).$$

Finally, the property (7) readily follows from Proposition 70 and Proposition 79. This finishes the proof of Theorems 82 and 37. □

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