Sphere Packing in Lean

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Abstract

This blueprint consists of an adaptation of Maryna Viazovska's Fields Medal-winning paper proving that no packing of unit balls in Euclidean space $\mathbb{R}^8$ has density greater than that of the E_8-lattice packing. This blueprint is a work in progress, and will be frequently updated and restructured as the formalisation effort progresses. We recommend that you look at this webpage for the latest version.

Contents

1 Sphere packings

The Sphere Packing problem is a classic optimisation problem with widespread applications that go well beyond mathematics. The task is to determine the "densest" possible arrangement of spheres in a given space. It remains unsolved in all but finitely many dimensions.

It was famously determined, in [11], that the optimal arrangement in $\mathbb{R}^8$ is given by the E_8 lattice. The result is strongly dependent on the Cohn-Elkies linear programming bound (Theorem 3.1 in [3]), which, if a $\mathbb{R}^d \to \mathbb{R}$ function satisfying certain conditions exists, bounds the optimal density of sphere packings in $\mathbb{R}^d$ in terms of it. The proof in [11] uses the theory of modular forms to construct a function that can be used to bound the density of all sphere packings in $\mathbb{R}^8$ above by the density of the E_8 lattice packing. This then allows us to conclude that no packing in $\mathbb{R}^8$ can be denser than the E_8 lattice packing.

1.1 The Setup

This subsection gives an overview for the setup of the problem, both informally and in Lean. Throughout this blueprint, $\mathbb{R}^d$ will denote the Euclidean vector space equipped with distance $\|\cdot\|$ and Lebesgue measure $\mathrm{Vol}(\cdot)$. For any $x \in \mathbb{R}^d$ and $r \in \mathbb{R}_{>0}$, we denote by $B_d(x, r)$ the open ball in $\mathbb{R}^d$ with center x and radius r. While we will give a more formal definition of a sphere packing below (and in Lean), the underlying idea is that it is a union of balls of equal radius centred at points that are far enough from each other that the balls do not overlap.

Arguably the most important definition in this subsection is that of *packing density*, which measures which portion of d-dimensional Euclidean space is covered by a given sphere packing. Taking the supremum over all packings gives what we refer to as the *sphere packing constant*, which is the quantity we are interested in optimising.

Definition 1.1. *Given a set $X \subset \mathbb{R}^d$ and a real number $r > 0$ (known as the* separation radius*) such that $\|x - y\| \geq r$ for all distinct $x, y \in X$, we define the* sphere packing *$\mathcal{P}(X)$ with centres at X to be the union of all open balls of radius r centred at points in X:*

$$\mathcal{P}(X) := \bigcup_{x \in X} B_d(x, r)$$

Remark 1.2. *Note that a sphere packing is uniquely defined from a given set of centres (which, in order to be a valid set of centres, must admit a corresponding separation radius). Therefore, as a conscious choice during the formalisation process, we will define everything that depends on sphere packings in terms of* `SpherePacking`*, a* `structure` *that bundles all the identifying information of a packing, but not the actual balls themselves. For the purposes of this blueprint, however, we will refrain from making this distinction.*

We now define a notion of density for bounded regions of space by considering the density inside balls of finite radius.

Definition 1.3. *The* finite density *of a packing* $\mathcal{P}$ *is defined as*

$$\Delta_{\mathcal{P}}(R) := \frac{\mathrm{Vol}(\mathcal{P} \cap B_d(0, R))}{\mathrm{Vol}(B_d(0, R))}, \quad R > 0.$$

As intuitive as it seems to take the density of a packing to be the limit of the finite densities as the radius of the ball goes to infinity, it is not immediately clear that this limit exists. Therefore, we define the density of a sphere packing as a limit superior instead.

Definition 1.4. *We define the* density *of a packing* $\mathcal{P}$ *as the limit superior*

$$\Delta_{\mathcal{P}} := \limsup_{R \to \infty} \Delta_{\mathcal{P}}(R).$$

We may now define the sphere packing constant, the quantity that the sphere packing problem requires us to compute.

Definition 1.5. *The* sphere packing constant *is defined as supremum of packing densities over all possible packings:*

$$\Delta_d := \sup_{\substack{\mathcal{P} \subset \mathbb{R}^d \\ \text{sphere packing}}} \Delta_{\mathcal{P}}.$$

1.2 Scaling Sphere Packings

Given that the problem involves the *arrangement* of balls in space, it is intuitive not to worry about the radius of the balls (so long as they are all equal to each other). However, Definition 1.1 involves a choice of separation radius. In principle, we would want two sphere packing configurations that differ only in separation radii to 'encode the same information'. In this brief subsection, we will describe how to change the separation radius of a sphere packing by *scaling* the packing by a positive real number and prove that this does not affect its density. This will give us the freedom to choose any separation radius we like when attempting to define the optimal sphere packing in $\mathbb{R}^d$.

Definition 1.6. *Given a sphere packing* $\mathcal{P}(X)$ *with separation radius* r*, we defined the* scaled packing *with respect to a real number* $c > 0$ *to be the packing* $\mathcal{P}(cX)$*, where* $cX = \{cx \in V \mid x \in X\}$ *has separation radius* cr*.*

Lemma 1.7. *Let* $\mathcal{P}(X)$ *be a sphere packing and* c *a positive real number. Then, for all* $R > 0$*,*

$$\Delta_{\mathcal{P}(cX)}(cR) = \Delta_{\mathcal{P}(X)}(R).$$

Proof. The proof follows by direct computation:

$$\Delta_{\mathcal{P}(cX)}(cR) = \frac{\mathrm{Vol}(\mathcal{P}(cX) \cap B_d(0, cR))}{\mathrm{Vol}(B_d(0, cR))} = \frac{c^d \cdot \mathrm{Vol}(\mathcal{P}(X) \cap B_d(0, R))}{c^d \cdot \mathrm{Vol}(B_d(0, R))} = \Delta_{\mathcal{P}(X)}(R)$$

where the second equality follows from applying the fact that scaling a (measurable) set by a factor of c scales its volume by a factor of c^d to the fact that $\mathcal{P}(cX) \cap B_d(0, cR) = c \cdot (\mathcal{P}(X) \cap B_d(0, cR))$. ☐

Lemma 1.8. *Let $\mathcal{P}(X)$ be a sphere packing and c a positive real number. Then, the density of the scaled packing $\mathcal{P}(cX)$ is equal to the density of the original packing $\mathcal{P}(X)$.*

Proof. One can show, using relatively unsophisticated real analysis, that

$$\limsup_{R\to\infty} \Delta_{\mathcal{P}(cX)}(R) = \limsup_{cR\to\infty} \Delta_{\mathcal{P}(cX)}(cR)$$

Lemma 1.7 tells us that $\Delta_{\mathcal{P}(cX)}(cR) = \Delta_{\mathcal{P}(X)}(R)$ for every $R > 0$. Therefore,

$$\limsup_{cR\to\infty} \Delta_{\mathcal{P}(cX)}(cR) = \limsup_{cR\to\infty} \Delta_{\mathcal{P}(X)}(R) = \limsup_{R\to\infty} \Delta_{\mathcal{P}(X)}(R)$$

where the second equality is the result of a similar change of variables to the one done above. ☐

Therefore, as expected, we do not need to worry about the separation radius when constructing sphere packings. This will be useful when we attempt to construct the optimal sphere packing in $\mathbb{R}^8$—and even more so when attempting to *formalise* this construction—because the underlying structure of the packing is given by a set known as the E_8 lattice, which has separation radius $\sqrt{2}$.

We can also use Lemma 1.8 to simplify the computation of the sphere packing constant by taking the supremum not over *all* sphere packings but only over those with density 1.

Lemma 1.9.

$$\Delta_d = \sup_{\substack{\mathcal{P}\subset\mathbb{R}^d \\ \textit{sphere packing} \\ \textit{sep. rad.}=1}} \Delta_{\mathcal{P}}$$

Proof. That the supremum over packings of unit density is at most the sphere packing constant is obvious. For the reverse inequality, let $\mathcal{P}(X)$ be any sphere packing with separation radius r. We know, from Lemma 1.8, that the density of $\mathcal{P}(X)$ is equal to that of the scaled packing $\mathcal{P}\left(\frac{X}{r}\right)$. Since the scaled packing has separation radius 1, its density is naturally at most the supremum over all packings of unit density, meaning that the same is true of $\mathcal{P}(X)$. ☐

1.3 Lattices and Periodic packings

Definition 1.10. *We say that an additive subgroup $\Lambda \leq \mathbb{R}^d$ is a* lattice *if it is discrete and its $\mathbb{R}$-span contains all the elements of $\mathbb{R}^d$.*

Definition 1.11. *We say that a sphere packing $\mathcal{P}(X)$ is (Λ-)*periodic *if there exists a lattice $\Lambda \subset \mathbb{R}^d$ such that for all $x \in X$ and $y \in \Lambda$, $x + y \in X$ (ie, X is Λ-periodic).*

Lemma 1.12. *Every periodic sphere packing is a sphere packing.*

Proof. Mathematically, this lemma is hardly worth mentioning. We only do so to underscore the automatically constructed forgetful map `PeriodicSpherePacking.toSpherePacking` in Lean. ☐

Lemma 1.13. *If $\mathcal{P}(X)$ is a Λ-periodic sphere packing, then Λ acts on X by translation.*

Proof. This is immediate from the definition of a periodic sphere packing. ☐

Definition 1.14. *If Λ is a lattice, we call the Λ-periodic packing $\mathcal{P}(\Lambda)$ with centres at points in Λ a lattice packing.*

Remove this, since it's a duplicate of section 2.3 (Density of periodic packings.

Lemma 1.15. *If $X \subseteq \mathbb{R}^d$ is a set of sphere packing centres that is periodic with respect to some lattice Λ, then the density of the corresponding (periodic) sphere packing is given by*

$$\frac{|X/\Lambda|}{\mathrm{Vol}(\mathbb{R}^d/\Lambda)} \cdot \mathrm{Vol}(B_d(0,1))$$

where the quotients in the numerator and denominator correspond to the orbits of the action by translation of Λ on X and $\mathbb{R}^d$ respectively.

Remark 1.16. *This can be thought of as the "volume of spheres per fundamental domain": the number of spheres per fundamental domain is $|X/\Lambda|$, and the volume of each sphere is $\mathrm{Vol}(B_d(0,1))$.*

Definition 1.17. *The periodic sphere packing constant is defined to be*

$$\Delta_d^{periodic} := \sup_{\substack{P \subset \mathbb{R}^d \\ periodic\ packing}} \Delta_P$$

Theorem 1.18. *For all d, the periodic sphere packing constant in $\mathbb{R}^d$ is equal to the sphere packing constant in $\mathbb{R}^d$.*

Proof. **State this in Lean (ready). Fill in proof here (see [3, Appendix A]** ☐

In other words, it suffices to compute and optimise the periodic sphere packing constant.

1.4 Main Result

With the terminologies above, we can state the main theorem of this project.

Theorem 1.19. *All* periodic *packing $\mathcal{P} \subseteq \mathbb{R}^8$ has density satisfying $\Delta_{\mathcal{P}} \leq \Delta_{E_8}$, the density of the E_8 sphere packing (see Definition **??**).*

Proof. We will prove this theorem over the course of this blueprint. ☐

Corollary 1.20. *All packing $\mathcal{P} \subseteq \mathbb{R}^8$ has density satisfying $\Delta_{\mathcal{P}} \leq \Delta_{E_8}$.*

Proof. This is a direct consequence of Theorem 1.18 and Theorem **??**. ☐

Corollary 1.21. $\Delta_8 = \Delta_{E_8}$.

Proof. By definition, $\Delta_{E_8} \leq \Delta_8$, while Corollary **??** shows $\Delta_8 = \sup_{\text{packing } \mathcal{P}} \leq \Delta_{E_8}$, and the result follows. ☐

2 Density of packings

The definition of density given in Section 1 is inconvenient to work with, especially when our packing is a structured one, such as a periodic/lattice packing. This section fixes this problem.

2.1 Bounds on Finite Density of Packing

We first collect all the results we will prove here, then prove them separately below. We do this because some are proven already! Let $X \subseteq \mathbb{R}^d$ be a set of sphere packing centers with separation r.

Theorem 2.1. *We have the following theorem relating the finite density and the number of lattice points in a ball:*

$$\left| X \cap \mathcal{B}_d \left(R - \frac{r}{2} \right) \right| \cdot \frac{\mathrm{Vol}\left(\mathcal{B}_d \left(\frac{r}{2} \right) \right)}{\mathrm{Vol}(\mathcal{B}_d(R))} \leq \Delta_{\mathcal{P}}(R) \leq \left| X \cap \mathcal{B}_d \left(R + \frac{r}{2} \right) \right| \cdot \frac{\mathrm{Vol}\left(\mathcal{B}_d \left(\frac{r}{2} \right) \right)}{\mathrm{Vol}(\mathcal{B}_d(R))}$$

Proof. Proven by Gareth already. The high level idea is to prove that $\mathcal{P} \cap \mathcal{B}_d(R) = \left(\bigcup_{x \in X} \mathcal{B}_d \left(x, \frac{r}{2} \right) \right) \subseteq \bigcup_{x \in X \cap \mathcal{B}_d \left(R + \frac{r}{2} \right)} \mathcal{B}_d \left(x, \frac{r}{2} \right)$, and a similar inequality for the upper bound. The rest follows by rearranging and usign the fact that the balls are pairwise disjoint. $\square$

Suppose further that X is a periodic packing w.r.t. the lattice $\Lambda \subseteq \mathbb{R}^d$. Let $\mathcal{D}$ be a fundamental region of Λ, say the parallelopiped defined in the proof of Lemma 2.6, and let L be the bound on the norm of vectors in $\mathcal{D}$ (see Lemma 2.6).

Theorem 2.2. *For real numbers $R > L$, we have the following inequality relating the number of lattice points from Λ in a ball with the volume of the ball and the fundamental region $\mathcal{D}$:*

$$\frac{\mathrm{Vol}(\mathcal{B}_d(R - L))}{\mathrm{Vol}(\mathcal{D})} \leq |\Lambda \cap \mathcal{B}_d(R)| \leq \frac{\mathrm{Vol}(\mathcal{B}_d(R + L))}{\mathrm{Vol}(\mathcal{D})}$$

The proof can be found at Section 2.2.

Theorem 2.3. *For real numbers $R > L$, we have the following inequality relating the number of points from X (periodic w.r.t. Λ) in a ball with the number of points from Λ:*

$$|\Lambda \cap \mathcal{B}_d(R - L)| \, |X/\Lambda| \leq |X \cap \mathcal{B}_d(R)| \leq |\Lambda \cap \mathcal{B}_d(R + L)| \, |X/\Lambda|$$

The proof can be found at Section 2.2.

Finally, when we combine the inequalities above, we need one additional computational lemma.

Lemma 2.4. *For any constant $C > 0$, we have*

$$\lim_{R\to\infty} \frac{\mathrm{Vol}(\mathcal{B}_d(R))}{\mathrm{Vol}(\mathcal{B}_d(R+C))} = 1$$

Proof. Write out the formula for volume of a ball and simplify. More specifically, we have $\mathrm{Vol}(\mathcal{B}_d(R)) = R^d\pi^{d/2}/\Gamma\left(\frac{d}{2}+1\right)$, so $\mathrm{Vol}(\mathcal{B}_d(R))/\mathrm{Vol}(\mathcal{B}_d(R+C)) = R^d/(R+C)^d = \left(1-\frac{1}{R+C}\right)^d = 1$. $\square$

2.2 Bounds on Finite Density of Periodic Prcking

In this subsection, we build up results about the density of periodic packings. In particular, the density of a periodic packing, defined as the limit of the periodic packing intersected with a growing ball centered at the origin, is equal to the density within any fundamental region of the period lattice. The strategy is to prove lower and upper bounds for the number of lattice points in a ball in terms of the volume of the ball, correct up to the highest order term. Taking limit gives the correct density!

Below, let $X \subseteq \mathbb{R}^d$ be a set of periodic packing centers with respect to the lattice $\Lambda \subset \mathbb{R}^d$. We write $kX \coloneqq \{kv : v \in X\}$.

Definition 2.5. *Let $\Lambda \subset \mathbb{R}^d$ be a lattice. A set $\mathcal{D} \subseteq \mathbb{R}^d$ is a* fundamental domain *of Λ such that for all distinct $x, y \in \Lambda$, we have $(x+\mathcal{D}) \cap (y+\mathcal{D}) = \emptyset$ (disjointness) and $\bigcup_{x\in\Lambda} x+\mathcal{D} = \mathbb{R}^d$ (tiling).*

Lemma 2.6. *There always exists a* bounded *fundamental region $\mathcal{D}$ of Λ.*

Proof. Since lattices have $\mathbb{Z}$-bases, there exists a set of vectors $\mathcal{B} \subseteq \mathbb{R}^d$ such that $\Lambda = \mathrm{span}_{\mathbb{Z}}(\mathcal{B})$. We claim that $\mathcal{D}_\Lambda = \{\sum_i c_i\mathcal{B}_i \subseteq \mathbb{R}^n : c_i \in [0,1)^n\}$ is a fundamental domain. The rest exists in Mathlib already so I don't bother elaborating here :) From the definition, we see that for $v = \sum_i c_i\mathcal{B}_i \in \mathcal{D}_\Lambda$, we have $\|v\| \le \sum_i \|c_i\mathcal{B}_i\| \le \sum_i \|\mathcal{B}_i\|$, which is a constant. Hence, $\mathcal{D}_\Lambda$ is bounded. $\square$

We denote by L the bound of norm of vectors in the fundamental domain $\mathcal{D}$.

Lemma 2.7. *For all vectors $v \in \mathbb{R}^d$ there exists a unique lattice point $x \in \Lambda$ such that $v \in x+\mathcal{D}$.*

Proof. By the tiling property of the fundamental domain, we have $v \in \bigcup_{x\in\Lambda}(x+\mathcal{D})$. By definition, this means there exists a lattice point $x \in \Lambda$ such that $v \in x+\mathcal{D}$. To show that it is unique, suppose that $v \in (x+\mathcal{D})\cap(y+\mathcal{D})$ for distinct $x \neq y \in \Lambda$. By the disjointness property, $v \in \emptyset$, contradiction. $\square$

Proof of Theorem 2.2. For the first inequality, it suffices to prove that $\mathcal{B}_d(R-L) \subseteq \bigcup_{x\in\Lambda\cap\mathcal{B}_d(R)}(x+\mathcal{D})$, since the cosets on the right are almost disjoint. For a vector $v \in \mathcal{B}_d(R-L)$, we have $\|v\| < R-L$ by definition. By Lemma Lemma 2.7, there exists a lattice point $x \in \Lambda$ such that $v \in x+\mathcal{D}$. Rearranging gives $v - x \in \mathcal{D}$, which means $\|v-x\| \le L$. By the triangle inequality, $\|x\| < R$, i.e. $x \in \mathcal{B}_d(L)$, concluding the proof.

For the second inequality, we prove that $\bigcup_{x\in\Lambda\cap\mathcal{B}_d(R)}(x+\mathcal{D}) \subseteq \mathcal{B}_d(R+L)$. Consider a vector $x \in \Lambda\cap\mathcal{B}_d(R)$ and a vector $y \in x+\mathcal{D}$. From above, we know $\|x\| < R$ and $\|y-x\| \leq L$, so $\|y\| < R+L$, concluding the proof. □

Next, we build up to the proof for Theorem 2.3

Definition 2.8. *Suppose that Λ acts (additively) on X. We can associate an equivalence relation $\sim$ to the action generated by $x \sim y+x$ for all $x \in X$ and $y \in \Lambda$. We define the set $X/\Lambda := X/\sim$.* ***Give this a name? lol***

In Lean, X/Λ is defined as the type `Quotient S.addAction.orbitRel`.

Lemma 2.9. *For X a discrete set and Λ a $\mathbb{Z}$-lattice, X/Λ is finite.*

Proof. We first prove a bijection (as equivalence of types in Lean) between X/Λ and $X\cap\mathcal{D}$, where $\mathcal{D}$ is any bounded fundamental region of Λ.

Now to prove that $X\cap\mathcal{D}$ is finite, we argue by looking at the volume of $\mathcal{P}(X\cap\mathcal{D})$. Indeed, we have $\mathcal{P}(X\cap\mathcal{D}) = \bigcup_{x\in X\cap\mathcal{D}}\mathcal{B}_d\left(x,\frac{r}{2}\right) \subseteq \mathcal{B}_d\left(L+\frac{r}{2}\right)$, where L is the norm bound of $\mathcal{D}$ and r is the separation radius of X, by a straightforward application of the triangle inequality. Moreover, by definition of the separation radius, the balls in the LHS are also pairwise disjoint. Taking volumes on both sides, we have $\sum_{x\in X\cap\mathcal{D}}\mathrm{Vol}\left(\mathcal{B}_d\left(x,\frac{r}{2}\right)\right) = |X\cap\mathcal{D}|\mathrm{Vol}\left(\mathcal{B}_d\left(\frac{r}{2}\right)\right) \leq \mathrm{Vol}\left(\mathcal{B}_d\left(L+\frac{r}{2}\right)\right)$, concluding the proof.

Note that there is a minor technicality here, as the formula $\mathrm{Vol}\left(\bigcup_{x\in S}f(x)\right) = \sum_{x\in S}\mathrm{Vol}(f(x))$ used is only true when $S = X\cap\mathcal{D}$ is countable, which is not true here *a priori*. However, if $X\cap\mathcal{D}$ is uncountable, then we can simply take a countable subset of it and argue as above. Thanks to Etienne Marion on Zulip for the idea! □

Lemma 2.10. *For $R > L$,* ***Prove this shit lol***

$$\bigcup_{x\in\Lambda\cap\mathcal{B}_d(R-L)}(x+\mathcal{D}) \subseteq \mathcal{B}_d(R)$$

Proof. Trivial. **Fill this in** □

Proof of Theorem 2.3. Intersecting both sides of 2.10 with X and simplifying, we have

$$\left(\bigcup_{x\in\Lambda\cap\mathcal{B}_d(R-L)}(x+\mathcal{D})\right)\cap X = \bigcup_{x\in\Lambda\cap\mathcal{B}_d(R-L)}((x+\mathcal{D})\cap X) \subseteq \mathcal{B}_d(R)\cap X$$

Consider the (finite) cardinality on both sides and noting that $|(x+\mathcal{D})\cap X| = |X/\Lambda|$ for all x, we see that $|\Lambda \cap \mathcal{B}_d(R-L)||X/\Lambda| \leq |X \cap \mathcal{B}_d(R)|$, as desired.

Fill in the proof of the second inequality. $\square$

3 The E_8 lattice

3.1 Definitions of E_8 lattice

There are several equivalent definitions of the E_8 lattice. Below, we formalise two of them, and prove they are equivalent.

Definition 3.1. *E_8-lattice, Definition 1We define the E_8*-lattice *(as a subset of $\mathbb{R}^8$) to be*

$$\Lambda_8 = \{(x_i) \in \mathbb{Z}^8 \cup (\mathbb{Z} + \tfrac{1}{2})^8 | \sum_{i=1}^{8} x_i \equiv 0 \pmod 2\}.$$

Definition 3.2. *We define the* scaled E_8-lattice *(by a real number c) as*

$$c\Lambda_8 = \{c \cdot \vec{v} : \vec{v} \in \Lambda_8\}$$

.

Definition 3.3. *$E - 8$-lattice, Definition 2We define the E_8* basis vectors *to be the set of vectors*

$$\mathcal{B}_8 = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2 \\ -1/2 \\ -1/2 \\ -1/2 \\ -1/2 \\ -1/2 \\ -1/2 \\ -1/2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \right\}$$

Definition 3.4. *We define the* scaled E_8 basis vectors *(by a real number c) to be $c\mathcal{B}_8 = \{c \cdot \vec{v} : \vec{v} \in \mathcal{B}_8\}$.*

Theorem 3.5. *The two definitions above coincide, i.e. $c\Lambda_8 = \operatorname{span}_{\mathbb{Z}}(c\mathcal{B}_8)$.*

Proof. We prove each side contains the other side.

For a vector $\vec{v} \in \Lambda_8 \subseteq \mathbb{R}^8$, we have $\sum_i \vec{v}_i \equiv 0 \pmod 2$ and $\vec{v}_i$ being either all integers or all half-integers. After some modulo arithmetic, it can be seen that $\mathcal{B}_8^{-1}\vec{v}$ as integer coordinates (i.e. it is congruent to 0 modulo 1). Hence, $\vec{v} \in \operatorname{span}_{\mathbb{Z}}(\mathcal{B}_8)$.

For the opposite direction, we write the vector as $\vec{v} = \sum_i c_i \mathcal{B}_8^i \in \operatorname{span}_{\mathbb{Z}}(\mathcal{B}_8)$ with c_i being integers and $\mathcal{B}_8^i$ being the i-th basis vector. Expanding the definition then gives $\vec{v} = \left(c_1 - \frac{1}{2}c_7, -c_1 + c_2 - \frac{1}{2}c_7, \cdots, -\frac{1}{2}c_7\right)$. Again, after some modulo arithmetic, it can be seen that $\sum_i \vec{v}_i$ is indeed 0 modulo 2 and are all either integers and half-integers, concluding the proof.

(Note: this proof doesn't depend on that $\mathcal{B}_8$ is linearly independent.) $\square$

3.2 Basic Properties of E_8 lattice

In this section, we establish basic properties of the E_8 lattice and the $\mathcal{B}_8$ vectors.

Lemma 3.6. *For nonzero real numbers c, the set $c\mathcal{B}_8$ is a $\mathbb{R}$-basis of $\mathbb{R}^8$.*

Proof. It suffices to prove that $\mathcal{B}_8 \in \mathrm{GL}_8(\mathbb{R})$. We prove this by explicitly defining the inverse matrix $\mathcal{B}_8^{-1}$ and proving $\mathcal{B}_8\mathcal{B}_8^{-1} = I_8$, which implies that $\det(\mathcal{B}_8)$ is nonzero. $\square$

Lemma 3.7. *For real numbers c, $c\Lambda_8$ is an additive subgroup of $\mathbb{R}^8$.*

Proof. Trivially follows from that $\Lambda_8 \subseteq \mathbb{R}^8$ is the $\mathbb{Z}$-span of $\mathcal{B}_8$ and hence an additive group. $\square$

Lemma 3.8. *All vectors in Λ_8 have norm of the form $\sqrt{2n}$, where n is a nonnegative integer.*

Proof. Writing $\vec{v} = \sum_i c_i \mathcal{B}_8^i$, we have $\|v\|^2 = \sum_i \sum_j c_i c_j (\mathcal{B}_8^i \cdot \mathcal{B}_8^j)$. Computing all 64 pairs of dot products and simplifying, we get a massive term that is a quadratic form in c_i with even integer coefficients, concluding the proof. $\square$

Lemma 3.9. *For nonzero real numbers c, $c\Lambda_8$ is discrete, i.e. that the subspace topology induced by its inclusion into $\mathbb{R}^8$ is the discrete topology.*

Proof. We prove this for $c = 1$. Since Λ_8 is a topological group and $+$ is continuous, it suffices to prove that $\{0\}$ is open in Λ_8. This follows from the fact that there is an open ball $\mathcal{B}(\sqrt{2}) \subseteq \mathbb{R}^8$ around it containing no other lattice points, since the shortest nonzero vector has norm $\sqrt{2}$. $\square$

Lemma 3.10. *For nonzero real numbres c, $c\Lambda_8$ is a lattice, i.e. it is discrete and spans $\mathbb{R}^8$ over $\mathbb{R}$.*

Proof. The first part is the above lemma. The second part follows from that $\mathcal{B}_8$ is a basis and hence linearly independent over $\mathbb{R}$. $\square$

Prove E_8 is unimodular. Prove E_8 is positive-definite.

3.3 The E_8 sphere packing

Definition 3.11. *The* E_8 sphere packing *is the sphere packing with separation* 1*, whose set of centres is $\frac{1}{\sqrt{2}}\Lambda_8$.*

Theorem 3.12. *We have $\Delta_{E_8} = \frac{\pi^4}{384}$.*

Proof. **Finish proof. Preferably we want APIs about fundamental region of lattice, and use that to reduce this theorem to computation inside the fundamental region, and use formula for volume of ball.** □

4 Facts from Fourier analysis

In this section, we recall a few definitions from Fourier analysis.

Definition 4.1. *The Fourier transform of an L^1-function $f : \mathbb{R}^d \to \mathbb{C}$ is defined as*

$$\mathcal{F}(f)(y) = \widehat{f}(y) := \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle x, y \rangle} \, \mathrm{d}x, \quad y \in \mathbb{R}^d$$

where $\langle x, y \rangle = \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2 - \frac{1}{2}\|x - y\|^2$ is the standard scalar product in $\mathbb{R}^d$.

Definition 4.2. *A C^∞ function $f : \mathbb{R}^d \to \mathbb{C}$ is called a* Schwartz function *if it goes to zero as $\|x\| \to \infty$ faster then any inverse power of $\|x\|$, and the same holds for all partial derivatives of f.*

Definition 4.3. *The set of all Schwartz functions is called a* Schwartz space.

Lemma 4.4. *The Fourier transform is an automorphism of the space of Schwartz functions.*

Proof. **Fill in proof.** ☐

Lemma 4.5.

$$\mathcal{F}(e^{\pi i \|x\|^2 z})(y) = z^{-4} \, e^{\pi i \|y\|^2 \, (\frac{-1}{z})}.$$

Proof. **Fill in proof.** ☐

Theorem 4.6 ((Poisson summation formula))**.**

$$\sum_{\ell \in \Lambda} f(\ell) = \frac{1}{\mathrm{Vol}(\mathbb{R}^d / \Lambda)} \sum_{m \in \Lambda^*} \widehat{f}(m).$$

Proof. **Fill in proof.** ☐

5 Cohn-Elkies linear programming bounds

In 2003 Cohn and Elkies [3] developed linear programming bounds that apply directly to sphere packings. The goal of this section is to formalize the Cohn–Elkies linear programming bound.

The following theorem is the key result of [3]. (The original theorem is stated for a class of functions more general then Schwartz functions)

Theorem 5.1. *(Cohn, Elkies [3]) Suppose that $f : \mathbb{R}^d \to \mathbb{R}$ is a Schwartz function that is not identically zero and satisfies the following conditions:*

$$f(x) \leq 0 \text{ for } \|x\| \geq 1 \tag{1}$$

and

$$\widehat{f}(x) \geq 0 \text{ for all } x \in \mathbb{R}^d. \tag{2}$$

Then the density of d-dimensional sphere packings is bounded above by

$$\frac{f(0)}{\widehat{f}(0)} \cdot \mathrm{vol}(B_d(0, 1/2)).$$

Proof. Here we reproduce the proof given in [3]. We will first prove the theorem for periodic packings.

Let $X \subset \mathbb{R}^d$ be a discrete subset such that $\|x - y\| \geq 1$ for any distinct $x, y \in X$. Suppose that X is Λ-periodic with respect to some lattice $\Lambda \subset \mathbb{R}^d$.

The inequality

$$\sharp(X/\Lambda) \cdot f(0) \geq \sum_{x \in X} \sum_{y \in X/\Lambda} f(x - y) = \sum_{x \in X/\Lambda} \sum_{y \in X/\Lambda} \sum_{\ell \in \Lambda} f(x - y + l) \tag{3}$$

follows from the condition (1) of the theorem and the assumption on the distances between points in X. The equality

$$\sum_{x \in X/\Lambda} \sum_{y \in X/\Lambda} \sum_{\ell \in \Lambda} f(x - y + l) = \sum_{x \in X/\Lambda} \sum_{y \in X/\Lambda} \frac{1}{\mathrm{vol}(\mathbb{R}^d/\Lambda)} \sum_{m \in \Lambda^*} \widehat{f}(m)\, e^{2\pi i m(x-y)}.$$

follows from the Poisson summation formula. The right hand side of the above equation can be written as

$$\sum_{x \in X/\Lambda} \sum_{y \in X/\Lambda} \frac{1}{\mathrm{vol}(\mathbb{R}^d/\Lambda)} \sum_{m \in \Lambda^*} \widehat{f}(m)\, e^{2\pi i m(x-y)} = \frac{1}{\mathrm{vol}(\mathbb{R}^d/\Lambda)} \sum_{m \in \Lambda^*} \widehat{f}(m) \cdot \Big| \sum_{x \in X/\Lambda} e^{2\pi i m x} \Big|^2.$$

Note that $\big| \sum_{x \in X/\Lambda} e^{2\pi i m x} \big|^2 \geq 0$ for all $m \in \Lambda^*$. Moreover, the term corresponding to $m = 0$ satisfies

$\big|\sum_{x\in X/\Lambda} e^{2\pi i 0 x}\big|^2 = \sharp(X/\Lambda)^2$. Now we use the condition (2) and estimate

$$\frac{1}{\mathrm{vol}(\mathbb{R}^d/\Lambda)} \sum_{m\in\Lambda^*} \widehat{f}(m)\cdot\Big|\sum_{x\in X/\Lambda} e^{2\pi i m(x-y)}\Big|^2 \geq \frac{\sharp(X/\Lambda)^2}{\mathrm{vol}(\mathbb{R}^d/\Lambda)}\cdot\widehat{f}(0). \tag{4}$$

Comparing inequalities (3) and (4) we arrive at

$$\frac{\sharp(X/\Lambda)}{\mathrm{vol}(\mathbb{R}^d/\Lambda)} \leq \frac{f(0)}{\widehat{f}(0)}.$$

Now we see that the density of the periodic packing $\mathcal{P}_X$ with balls of radius $1/2$ is bounded by

$$\Delta(\mathcal{P}_X) = \frac{\sharp(X/\Lambda)}{\mathrm{vol}(\mathbb{R}^d/\Lambda)}\cdot\mathrm{vol}(B_d(0,1/2)) \leq \frac{f(0)}{\widehat{f}(0)}\cdot\mathrm{vol}(B_d(0,1/2)).$$

This finishes the proof of the theorem for periodic packings. Theorem 1.18 implies the desired result for arbitrary packings. □

The main step in our proof of Theorem **??** is the explicit construction of an optimal function. It will be convenient for us to scale this function by $\sqrt{2}$.

Theorem 5.2. *There exists a radial Schwartz function $g:\mathbb{R}^8\to\mathbb{R}$ which satisfies:*

$$g(x) \leq 0 \text{ for } \|x\| \geq \sqrt{2} \tag{5}$$

$$\widehat{g}(x) \geq 0 \text{ for all } x\in\mathbb{R}^8 \tag{6}$$

$$g(0) = \widehat{g}(0) = 1. \tag{7}$$

Theorem 5.1 applied to the optimal function $f(x) = g(x/\sqrt{2})$ immediately implies Theorem **??**.

6 Modular forms

In this section, we recall and develop some theory of (quasi)modular forms.

Let $\mathfrak{H}$ be the upper half-plane $\{z \in \mathbb{C} \mid \Im(z) > 0\}$.

Lemma 6.1. *The modular group* $\Gamma_1 := \mathrm{PSL}_2(\mathbb{Z})$ *acts on* $\mathfrak{H}$ *by linear fractional transformations*

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) z := \frac{az+b}{cz+d}.$$

Let N be a positive integer.

Definition 6.2. *The* level N principal congruence subgroup *of* Γ_1 *is*

$$\Gamma(N) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1 \middle| \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \bmod N \right\}.$$

Definition 6.3. *A subgroup* $\Gamma \subset \Gamma_1$ *is called a* congruence subgroup *if* $\Gamma(N) \subset \Gamma$ *for some* $N \in \mathbb{N}$.

An important example of a congruence subgroup is

$$\Gamma_0(N) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1 \middle| \; c \equiv 0 \bmod N \right\}.$$

Let $z \in \mathfrak{H}$, $k \in \mathbb{Z}$, and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$.

Definition 6.4. *The* automorphy factor *of weight* k *is defined as*

$$j_k(z, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)) := (cz+d)^{-k}.$$

Lemma 6.5. *The automorphy factor satisfies the* chain rule

$$j_k(z, \gamma_1\gamma_2) = j_k(z, \gamma_1)\, j_k(\gamma_2 z, \gamma_1).$$

Definition 6.6. *Let* F *be a function on* $\mathfrak{H}$ *and* $\gamma \in \mathrm{PSL}_2(\mathbb{Z})$. *Then the* slash operator *acts on* F *by*

$$(F|_k\gamma)(z) := j_k(z, \gamma)\, F(\gamma z).$$

Lemma 6.7. *The chain rule implies*

$$F|_k\gamma_1\gamma_2 = (F|_k\gamma_1)|_k\gamma_2.$$

Definition 6.8. *A* (holomorphic) modular form *of integer weight* k *and congruence subgroup* Γ *is a holomorphic function* $f : \mathfrak{H} \to \mathbb{C}$ *such that:*

1. $f|_k\gamma = f$ *for all* $\gamma \in \Gamma$

2. *for each* $\alpha \in \Gamma_1$ $f|_k\alpha$ *has the Fourier expansion* $f|_k\alpha(z) = \sum_{n=0}^{\infty} c_f(\alpha, \frac{n}{n_\alpha})\, e^{2\pi i \frac{n}{n_\alpha} z}$ *for some* $n_\alpha \in \mathbb{N}$ *and Fourier coefficients* $c_f(\alpha, m) \in \mathbb{C}$.

Definition 6.9. *Let* $M_k(\Gamma)$ *be the space of modular forms of weight* k *and congruence subgroup* Γ.

A key fact in the theory of modular forms is the following theorem:

Theorem 6.10. *The spaces* $M_k(\Gamma)$ *are finite dimensional.*

Let us consider several examples of modular forms.

Definition 6.11. *For an even integer* $k \geq 4$ *we define the* weight k Eisenstein series *as*

$$E_k(z) := \frac{1}{2\zeta(k)} \sum_{(c,d)\in\mathbb{Z}^2\setminus(0,0)} (cz+d)^{-k}. \tag{8}$$

Lemma 6.12. *For all* k, $E_k \in M_k(\Gamma_1)$

Proof. This follows from the fact that the sum converges absolutely. □

Lemma 6.13. *The Eisenstein series possesses the Fourier expansion*

$$E_k(z) = 1 + \frac{2}{\zeta(1-k)} \sum_{n=1}^{\infty} \sigma_{k-1}(n)\, e^{2\pi i z}, \tag{9}$$

where $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$. *In particular, we have*

$$\begin{aligned} E_4(z) &= 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)\, e^{2\pi i n z} \\ E_6(z) &= 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)\, e^{2\pi i n z}. \end{aligned}$$

The infinite sum (8) does not converge absolutely for $k = 2$. On the other hand, the expression (9) converges to a holomorphic function on the upper half-plane and therefore

Definition 6.14. *We set*

$$E_2(z) := 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n)\, e^{2\pi i n z}.$$

Lemma 6.15. *This function is not modular, however it satisfies*

$$z^{-2}\, E_2\Big(\frac{-1}{z}\Big) = E_2(z) - \frac{6i}{\pi}\,\frac{1}{z}.$$

The proof of this identity can be found in [12, Section 2.3]. The weight two Eisenstein series E_2 is an example of a *quasimodular form* [12, Section 5.1].

The discriminant form is a unique normalized cusp form of weight 12, which can be defined using E_4 and E_6.

Definition 6.16. *The* discriminant form $\Delta(z)$ *is given by*

$$\Delta(z) = \frac{E_4(z)^3 - E_6(z)^2}{1728} \tag{10}$$

Lemma 6.17. $\Delta(z) \in M_{12}(\Gamma_1)$ *and it vanishes at the unique cusp, i.e. it is a cusp form of level* Γ_1 *and weight* 12*.*

Proof. Being a modular form of desired weight and level directly follows from those of E_4 and E_6. It is a cusp form since the constant terms of Fourier expansions of E_4 and E_6 are both 1. ☐

It also admits a product formula, which allow us to prove positivity of $\Delta(it)$ for $t > 0$ later.

Lemma 6.18. *We have*

$$\Delta(z) = e^{2\pi i z} \prod_{n \geq 1} (1 - e^{2\pi i n z})^{24}. \tag{11}$$

Proof. There are several known proofs of this. One possible proof that we can formalize is from Kohnen [6], which prove

$$\frac{1}{2\pi i z} \frac{d}{dz} (\log \Delta) = 1 - 24 \sum_{n \geq 1} \frac{n e^{2\pi i n z}}{1 - e^{2\pi i n z}}. \tag{12}$$

by using a multiplicative analogue of the Hecke operator and the valence formula. ☐

Note that the RHS of (12) is equal to the $E_2(z)$. As a side note, we can also consider defining Δ as (11), and prove that it coincides with (10). Such an argument can be found in [2, Section 2.4].

Another example of modular forms we would like to consider are *theta functions* [12, Section 3.1].

Definition 6.19. *We define three different theta functions (so called "Thetanullwerte") as*

$$\begin{aligned}
\theta_{00}(z) &= \sum_{n \in \mathbb{Z}} e^{\pi i n^2 z} \\
\theta_{01}(z) &= \sum_{n \in \mathbb{Z}} (-1)^n \, e^{\pi i n^2 z} \\
\theta_{10}(z) &= \sum_{n \in \mathbb{Z}} e^{\pi i (n + \frac{1}{2})^2 z}.
\end{aligned}$$

The group Γ_1 is generated by the elements $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.

Lemma 6.20. *These elements act on the theta functions in the following way*

$$z^{-2} \, \theta_{00}^4 \left(\frac{-1}{z} \right) = - \, \theta_{00}^4(z) \tag{13}$$

$$
\begin{aligned}
z^{-2}\,\theta_{01}^4\left(\frac{-1}{z}\right) &= -\,\theta_{10}^4(z) && (14)\\
z^{-2}\,\theta_{10}^4\left(\frac{-1}{z}\right) &= -\,\theta_{01}^4(z) && (15)
\end{aligned}
$$

and

$$
\begin{aligned}
\theta_{00}^4(z+1) &= \theta_{01}^4(z) \\
\theta_{01}^4(z+1) &= \theta_{00}^4(z) && (16)\\
\theta_{10}^4(z+1) &= -\,\theta_{10}^4(z). && (17)
\end{aligned}
$$

Proof. The last three identities easily follow from the definition, and (15) and (14) are equivalent under $z \leftrightarrow -1/z$, so it is enough to show (13) and (15). These identities follow from the Poisson summation formula, which is already formalized by David Loeffler. More precisely, for the *two-variable* Jacobi theta function and its derivative (be careful for the variables, here we use τ instead of z)

$$\theta(z,\tau) = \sum_{n\in\mathbb{Z}} e^{2\pi i n z + \pi i n^2 \tau}$$

that specialize to $\theta_{00}(\tau) = \theta(0,\tau)$, $\theta_{01} = \theta(1/2,\tau)$, and $\theta_{10}(\tau) = e^{\pi i \tau/4}\theta(-\tau/2,\tau)$. Possion summation formula gives

$$\theta(z,\tau) = \frac{1}{\sqrt{-i\tau}} e^{-\frac{\pi i z^2}{\tau}} \theta\left(\frac{z}{\tau}, -\frac{1}{\tau}\right)$$

and applying the specializations above yield the identities. For example, (15) follows from

$$\theta_{01}(\tau) = \theta\left(\frac{1}{2},\tau\right) = \frac{1}{\sqrt{-i\tau}} e^{-\frac{\pi i}{4\tau}} \theta\left(\frac{1}{2\tau}, -\frac{1}{\tau}\right) = \frac{1}{\sqrt{-i\tau}} \theta_{10}\left(-\frac{1}{\tau}\right)$$

and taking 4th power. □

Using the above identities, we can prove that these are modular forms.

Lemma 6.21. *The theta functions $\theta_{00}^4, \theta_{01}^4$, and θ_{10}^4 belong to $M_2(\Gamma(2))$.*

Proof. Since the group $\Gamma(2)$ is generated by two elements

$$\alpha = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix},$$

it is enough to show that they are invariant under the slash actions with respect to α and β. For θ_{01}, (16) and (14)

$$(\theta_{01}^4|_2\alpha)(z) = \theta_{01}^4(z+2) = \theta_{00}^4(z+1) = \theta_{01}^4(z)$$

$$
\begin{aligned}
(\theta_{01}^4|_2\beta)(z) &= (2z+1)^{-2}\theta_{00}^4\left(\frac{z}{2z+1}\right) = (2z+1)^{-2}\cdot - \left(-\frac{2z+1}{z}\right)^2 \theta_{10}^4\left(-\frac{2z+1}{z}\right) \\
&= -z^{-2}\theta_{10}^4\left(-2-\frac{1}{z}\right) = -z^{-2}\theta_{10}^4\left(-\frac{1}{z}\right) = \theta_{01}^4(z).
\end{aligned}
$$

We can prove θ_{10}^4 and θ_{00}^4 similarly. ☐

We also have a nontrivial relation between these theta functions.

Lemma 6.22. *These three theta functions satisfy the* Jacobi identity

$$\theta_{01}^4 + \theta_{10}^4 = \theta_{00}^4.$$

Proof. One possible proof is to use dimesion formula of the space of modular forms of weight 2 and level $\Gamma(2)$. Especially, this space have dimension 2, with basis $\theta_{01}^4, \theta_{10}^4$. ☐

The *strict* positivity of Jacobi theta functions might needed later.

Lemma 6.23. *All three functions $t \mapsto \theta_{01}(it), \theta_{10}(it), \theta_{00}(it)$ are positive for $t > 0$.*

Proof. By Lemma 6.22 and the transformation law (14), it is enough to prove the positivity for $\theta_{10}(it)$, which is clear from its definition. ☐

Note that we only need the fourth powers of theta functions in Lemma 6.21 to define (7.13), not $\theta_{00}, \theta_{01}, \theta_{10}$ themselves.

Definition 6.24. *A* weakly-holomorphic modular form *of integer weight k and congruence subgroup Γ is a holomorphic function $f : \mathfrak{H} \to \mathbb{C}$ such that:*

1. *$f|_k\gamma = f$ for all $\gamma \in \Gamma$*
2. *for each $\alpha \in \Gamma_1$ $f|_k\alpha$ has the Fourier expansion $f|_k\alpha(z) = \sum_{n=n_0}^{\infty} c_f(\alpha, \frac{n}{n_\alpha})\, e^{2\pi i \frac{n}{n_\alpha} z}$ for some $n_0 \in \mathbb{Z}$ and $n_\alpha \in \mathbb{N}$.*

For an m-periodic holomorphic function f and $n \in \frac{1}{m}\mathbb{Z}$ we will denote the n-th Fourier coefficient of f by $c_f(n)$ so that

$$f(z) = \sum_{n \in \frac{1}{m}\mathbb{Z}} c_f(n)\, e^{2\pi i n z}.$$

We denote the space of weakly-holomorphic modular forms of weight k and group Γ by $M_k^!(\Gamma)$. The spaces $M_k^!(\Gamma)$ are infinite dimensional. Probably the most famous weakly-holomorphic modular form is the *elliptic j-invariant*

$$j := \frac{1728\, E_4^3}{E_4^3 - E_6^2}.$$

This function belongs to $M_0^!(\Gamma_1)$ and has the Fourier expansion

$$j(z) \;=\; q^{-1} + 744 + 196884\,q + 21493760\,q^2 + 864299970\,q^3 + 20245856256\,q^4 + O(q^5)$$

where $q = e^{2\pi i z}$. Using a simple computer algebra system such as PARI GP or Mathematica one can compute first hundred terms of this Fourier expansion within few seconds. An important question is to find an asymptotic formula for $c_j(n)$, the n-th Fourier coefficient of j. Using the Hardy-Ramanujan circle method [9] or the non-holomorphic Poincare series [8] one can show that

Lemma 6.25.

$$c_j(n) = \frac{2\pi}{n}\sum_{k=1}^{\infty}\frac{A_k(n)}{k}\, I_1\left(\frac{4\pi\sqrt{n}}{k}\right) \qquad n \in \mathbb{Z}_{>0}$$

where

$$A_k(n) = \sum_{\substack{h \bmod k \\ (h,k)=1}} e^{\frac{-2\pi i}{k}(nh+h')}, \quad hh' \equiv -1 (mod\, k),$$

and $I_\alpha(x)$ denotes the modified Bessel function of the first kind defined as in [1, Section 9.6].

A similar convergent asymptotic expansion holds for the Fourier coefficients of any weakly holomorphic modular form [5], [2, Propositions 1.10 and 1.12]. Such a convergent expansion implies effective estimates for the Fourier coefficients.

7 Fourier eigenfunctions with double zeroes at lattice points

In this section we construct two radial Schwartz functions $a, b : \mathbb{R}^8 \to i\mathbb{R}$ such that

$$\mathcal{F}(a) = a \tag{18}$$

$$\mathcal{F}(b) = -b \tag{19}$$

which double zeroes at all Λ_8-vectors of length greater than $\sqrt{2}$. Recall that each vector of Λ_8 has length $\sqrt{2n}$ for some $n \in \mathbb{N}_{\geq 0}$. We define a and b so that their values are purely imaginary because this simplifies some of our computations. We will show in Section 8 that an appropriate linear combination of functions a and b satisfies conditions (5)–(7).

First, we will define function a. To this end we consider the following functions:

Definition 7.1.

$$\begin{aligned} \phi_{-4} &:= -\, Dj\, E_6^{-1} \\ \phi_{-2} &:= \phi_{-4}\, E_2 + Dj\, E_4^{-1} \\ \phi_0 &:= \phi_{-4}\, E_2^2 + 2Dj\, E_4^{-1}\, E_2 + j - 1728. \end{aligned}$$

Here $Dj(z) = \frac{1}{2\pi i}\frac{d}{dz} j(z)$.

Lemma 7.2. *These functions have the Fourier expansions*

$$\phi_{-4}(z) = q^{-1} + 504 + 73764\, q + 2695040\, q^2 + 54755730\, q^3 + O(q^4) \tag{20}$$

$$\phi_{-2}(z) = 720 + 203040\, q + 9417600\, q^2 + 223473600\, q^3 + 3566782080\, q^4 + O(q^5)$$

$$\phi_0(z) = 518400\, q + 31104000\, q^2 + 870912000\, q^3 + 15697152000\, q^4 + O(q^5) \tag{21}$$

where $q = e^{2\pi i z}$.

The function $\phi_0(z)$ is not modular; however,

Lemma 7.3. *The identity 6.15 implies the following transformation rule:*

$$\phi_0\Big(\frac{-1}{z}\Big) = \phi_0(z) - \frac{12i}{\pi}\,\frac{1}{z}\,\phi_{-2}(z) - \frac{36}{\pi^2}\,\frac{1}{z^2}\,\phi_{-4}(z). \tag{22}$$

Definition 7.4. *For* $x \in \mathbb{R}^8$ *we define*

$$a(x) := \int_{-1}^{i} \phi_0\Big(\frac{-1}{z+1}\Big)\,(z+1)^2\, e^{\pi i \|x\|^2 z}\, dz + \int_{1}^{i} \phi_0\Big(\frac{-1}{z-1}\Big)\,(z-1)^2\, e^{\pi i \|x\|^2 z}\, dz \tag{23}$$

$$-2\int_0^i \phi_0\Big(\frac{-1}{z}\Big)\, z^2\, e^{\pi i\|x\|^2 z}\, dz + 2\int_i^{i\infty} \phi_0(z)\, e^{\pi i\|x\|^2 z}\, dz.$$

We observe that the contour integrals in (23) converge absolutely and uniformly for $x \in \mathbb{R}^8$. Indeed, $\phi_0(z) = O(e^{-2\pi i z})$ as $\Im(z) \to \infty$. Therefore, $a(x)$ is well defined. Now we prove that a satisfies condition (18).

Proposition 7.5.

The function a defined by (23) belongs to the Schwartz space and satisfies

$$\widehat{a}(x) = a(x).$$

Proof. First, we prove that a is a Schwartz function. From Lemma 6.13, Definition 6.14, and 6.25 we deduce that the Fourier coefficients of ϕ_0 satisfy

$$|c_{\phi_0}(n)| \leq 2\, e^{4\pi\sqrt{n}} \quad n \in \mathbb{Z}_{>0}.$$

Thus, there exists a positive constant C such that

$$|\phi_0(z)| \leq C\, e^{-2\pi \Im z} \qquad \text{for } \Im z > \frac{1}{2}.$$

We estimate the first summand in the right-hand side of (23). For $r \in \mathbb{R}_{\geq 0}$ we have

$$\Big|\int_{-1}^{i} \phi_0\Big(\frac{-1}{z+1}\Big)\, (z+1)^2\, e^{\pi i r^2 z}\, dz\Big| = \Big|\int_{i\infty}^{-1/(i+1)} \phi_0(z)\, z^{-4}\, e^{\pi i r^2(-1/z-1)}\, dz\Big| \leq$$
$$C_1 \int_{1/2}^{\infty} e^{-2\pi t}\, e^{-\pi r^2/t}\, dt \leq C_1 \int_0^{\infty} e^{-2\pi t}\, e^{-\pi r^2/t}\, dt = C_2\, r\, K_1(2\sqrt{2}\,\pi\, r)$$

where C_1 and C_2 are some positive constants and $K_\alpha(x)$ is the modified Bessel function of the second kind defined as in [1, Section 9.6]. This estimate also holds for the second and third summand in (23). For the last summand we have

$$\Big|\int_i^{i\infty} \phi_0(z)\, e^{\pi i r^2 z}\, dz\Big| \leq C \int_1^{\infty} e^{-2\pi t}\, e^{-\pi r^2 t}\, dt = C_3 \frac{e^{\pi(r^2+2)}}{r^2+2}.$$

Therefore, we arrive at

$$|a(r)| \leq 4C_2\, r\, K_1(2\sqrt{2}\pi r) + 2C_3 \frac{e^{-\pi(r^2+2)}}{r^2+2}.$$

It is easy to see that the left hand side of this inequality decays faster then any inverse power of r.

Analogous estimates can be obtained for all derivatives $\frac{d^k}{dr^k}a(r)$.

Now we show that a is an eigenfunction of the Fourier transform. We recall that the Fourier transform of a Gaussian function is

$$\mathcal{F}(e^{\pi i\|x\|^2 z})(y) = z^{-4}\, e^{\pi i\|y\|^2\,(\frac{-1}{z})}. \tag{24}$$

Next, we exchange the contour integration with respect to z variable and Fourier transform with respect to x variable in (23). This can be done, since the corresponding double integral converges absolutely. In this way we obtain

$$\widehat{a}(y) = \int\limits_{-1}^{i} \phi_0\Big(\frac{-1}{z+1}\Big)\,(z+1)^2\,z^{-4}\,e^{\pi i\|y\|^2\,(\frac{-1}{z})}\,dz + \int\limits_{1}^{i} \phi_0\Big(\frac{-1}{z-1}\Big)\,(z-1)^2\,z^{-4}\,e^{\pi i\|y\|^2\,(\frac{-1}{z})}\,dz$$
$$-2\int\limits_{0}^{i} \phi_0\Big(\frac{-1}{z}\Big)\,z^2\,z^{-4}\,e^{\pi i\|y\|^2\,(\frac{-1}{z})}\,dz + 2\int\limits_{i}^{i\infty} \phi_0(z)\,z^{-4}\,e^{\pi i\|y\|^2\,(\frac{-1}{z})}\,dz.$$

Now we make a change of variables $w = \frac{-1}{z}$. We obtain

$$\begin{aligned}\widehat{a}(y) = &\int\limits_{1}^{i} \phi_0\Big(1-\frac{1}{w-1}\Big)\,(\frac{-1}{w}+1)^2\,w^2\,e^{\pi i\|y\|^2\,w}\,dw\\ &+\int\limits_{-1}^{i} \phi_0\Big(1-\frac{1}{w+1}\Big)\,(\frac{-1}{w}-1)^2\,w^2\,e^{\pi i\|y\|^2\,w}\,dw\\ &-2\int\limits_{i\infty}^{i} \phi_0(w)\,e^{\pi i\|y\|^2\,w}\,dw + 2\int\limits_{i}^{0} \phi_0\Big(\frac{-1}{w}\Big)\,w^2\,e^{\pi i\|y\|^2\,w}\,dw.\end{aligned}$$

Since ϕ_0 is 1-periodic we have

$$\begin{aligned}\widehat{a}(y) \;=\; &\int\limits_{1}^{i} \phi_0\Big(\frac{-1}{z-1}\Big)\,(z-1)^2\,e^{\pi i\|y\|^2\,z}\,dz + \int\limits_{-1}^{i} \phi_0\Big(\frac{-1}{z+1}\Big)\,(z+1)^2\,e^{\pi i\|y\|^2\,z}\,dz\\ &+2\int\limits_{i}^{i\infty} \phi_0(z)\,e^{\pi i\|y\|^2\,z}\,dz - 2\int\limits_{0}^{i} \phi_0\Big(\frac{-1}{z}\Big)\,z^2\,e^{\pi i\|y\|^2\,z}\,dz\\ =\,& a(y).\end{aligned}$$

This finishes the proof of the proposition. $\square$

Next, we check that a has double zeroes at all Λ_8-lattice points of length greater then $\sqrt{2}$.

Proposition 7.6. *For $r > \sqrt{2}$ we can express $a(r)$ in the following form*

$$a(r) = -4\sin(\pi r^2/2)^2 \int\limits_0^{i\infty} \phi_0\Big(\frac{-1}{z}\Big)\, z^2\, e^{\pi i r^2\, z}\, dz. \tag{25}$$

Proof. We denote the right hand side of (25) by $d(r)$. It is easy to see that $d(r)$ is well-defined. Indeed, from the transformation formula (22) and the expansions (21)–(20) we obtain

$$\begin{aligned} \phi_0\Big(\frac{-1}{it}\Big) =& O(e^{-2\pi/t}) \quad \text{as } t\to 0 \\ \phi_0\Big(\frac{-1}{it}\Big) =& O(t^{-2}\, e^{2\pi t}) \quad \text{as } t\to \infty \end{aligned}$$

Hence, the integral (25) converges absolutely for $r>\sqrt{2}$. We can write

$$\begin{aligned} d(r) =& \int\limits_{-1}^{i\infty-1} \phi_0\Big(\frac{-1}{z+1}\Big)\,(z+1)^2\, e^{\pi i r^2\, z}\, dz - 2\int\limits_0^{i\infty} \phi_0\Big(\frac{-1}{z}\Big)\, z^2\, e^{\pi i r^2\, z}\, dz \\ &+ \int\limits_{1}^{i\infty+1} \phi_0\Big(\frac{-1}{z-1}\Big)\,(z-1)^2\, e^{\pi i r^2\, z}\, dz. \end{aligned}$$

From (22) we deduce that if $r>\sqrt{2}$ then $\phi_0\Big(\frac{-1}{z}\Big)\, z^2\, e^{\pi i r^2\, z} \to 0$ as $\Im(z)\to\infty$. Therefore, we can deform the paths of integration and rewrite

$$\begin{aligned} d(r) =& \int\limits_{-1}^{i} \phi_0\Big(\frac{-1}{z+1}\Big)\,(z+1)^2\, e^{\pi i r^2\, z}\, dz + \int\limits_{i}^{i\infty} \phi_0\Big(\frac{-1}{z+1}\Big)\,(z+1)^2\, e^{\pi i r^2\, z}\, dz \\ &-2\int\limits_{0}^{i} \phi_0\Big(\frac{-1}{z}\Big)\, z^2\, e^{\pi i r^2\, z}\, dz - 2\int\limits_{i}^{i\infty} \phi_0\Big(\frac{-1}{z}\Big)\, z^2\, e^{\pi i r^2\, z}\, dz \\ &+\int\limits_{1}^{i} \phi_0\Big(\frac{-1}{z-1}\Big)\,(z-1)^2\, e^{\pi i r^2\, z}\, dz + \int\limits_{i}^{i\infty} \phi_0\Big(\frac{-1}{z-1}\Big)\,(z-1)^2\, e^{\pi i r^2\, z}\, dz. \end{aligned}$$

Now from (22) we find

$$\begin{aligned} &\phi_0\Big(\frac{-1}{z+1}\Big)\,(z+1)^2 - 2\phi_0\Big(\frac{-1}{z}\Big)\, z^2 + \phi_0\Big(\frac{-1}{z-1}\Big)\,(z-1)^2 = \\ &\phi_0(z+1)\,(z+1)^2 - 2\phi_0(z)\, z^2 + \phi_0(z-1)\,(z-1)^2 \\ &-\frac{12i}{\pi}\Big(\phi_{-2}(z+1)\,(z+1) - 2\phi_{-2}(z)\, z + \phi_{-2}(z-1)\,(z-1)\Big) \\ &-\frac{36}{\pi^2}\Big(\phi_{-4}(z+1) - 2\phi_{-4}(z) + \phi_{-4}(z-1)\Big) = \\ &2\phi_0(z). \end{aligned}$$

Thus, we obtain

$$d(r) = \int_{-1}^{i} \phi_0\Big(\frac{-1}{z+1}\Big)\,(z+1)^2\,e^{\pi i r^2\,z}\,dz - 2\int_{0}^{i} \phi_0\Big(\frac{-1}{z}\Big)\,z^2\,e^{\pi i r^2\,z}\,dz$$
$$+ \int_{1}^{i} \phi_0\Big(\frac{-1}{z-1}\Big)\,(z-1)^2\,e^{\pi i r^2\,z}\,dz + 2\int_{i}^{i\infty} \phi_0(z)\,e^{\pi i r^2\,z}\,dz = a(r).$$

This finishes the proof. □

Finally, we find another convenient integral representation for a and compute values of $a(r)$ at $r = 0$ and $r = \sqrt{2}$.

Proposition 7.7. *For $r \geq 0$ we have*

$$\begin{aligned} a(r) =& 4i\,\sin(\pi r^2/2)^2\,\Bigg(\frac{36}{\pi^3\,(r^2-2)} - \frac{8640}{\pi^3\,r^4} + \frac{18144}{\pi^3\,r^2} \\ &+ \int_0^\infty \Big(t^2\,\phi_0\Big(\frac{i}{t}\Big) - \frac{36}{\pi^2}\,e^{2\pi t} + \frac{8640}{\pi}\,t - \frac{18144}{\pi^2}\Big)\,e^{-\pi r^2 t}\,dt\Bigg). \end{aligned} \tag{26}$$

The integral converges absolutely for all $r \in \mathbb{R}_{\geq 0}$.

Proof. Suppose that $r > \sqrt{2}$. Then by Proposition 7.6

$$a(r) = 4i\,\sin(\pi r^2/2)^2 \int_0^\infty \phi_0(i/t)\,t^2\,e^{-\pi r^2 t}\,dt.$$

From (21)–(22) we obtain

$$\phi_0(i/t)\,t^2 = \frac{36}{\pi^2}\,e^{2\pi t} - \frac{8640}{\pi}\,t + \frac{18144}{\pi^2} + O(t^2\,e^{-2\pi t}) \quad \text{as } t \to \infty. \tag{27}$$

For $r > \sqrt{2}$ we have

$$\int_0^\infty \Big(\frac{36}{\pi^2}\,e^{2\pi t} + \frac{8640}{\pi}\,t + \frac{18144}{\pi^2}\Big)\,e^{-\pi r^2 t}\,dt = \frac{36}{\pi^3\,(r^2-2)} - \frac{8640}{\pi^3\,r^4} + \frac{18144}{\pi^3\,r^2}.$$

Therefore, the identity (26) holds for $r > \sqrt{2}$.

On the other hand, from the definition (23) we see that $a(r)$ is analytic in some neighborhood of $[0, \infty)$. The asymptotic expansion (27) implies that the right hand side of (26) is also analytic in some neighborhood of $[0, \infty)$. Hence, the identity (26) holds on the whole interval $[0, \infty)$. This finishes the proof of the proposition. □

From the identity (26) we see that the values $a(r)$ are in $i\mathbb{R}$ for all $r \in \mathbb{R}_{\geq 0}$. In particular, we have

Proposition 7.8. *We have*

$$a(0) = \frac{-i\,8640}{\pi} \qquad a(\sqrt{2}) = 0 \qquad a'(\sqrt{2}) = \frac{i\,72\sqrt{2}}{\pi}.$$

Proof. These identities follow immediately from the previous proposition. □

Now we construct function b. To this end we consider the function

Definition 7.9.

$$h(z) \,:=\, 128\frac{\theta_{00}^4(z)+\theta_{01}^4(z)}{\theta_{10}^8(z)}. \tag{28}$$

It is easy to see that $h \in M^!_{-2}(\Gamma_0(2))$. Indeed, first we check that $h|_{-2}\gamma = h$ for all $\gamma \in \Gamma_0(2)$. Since the group $\Gamma_0(2)$ is generated by elements $\left(\begin{smallmatrix}1&0\\2&1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ it suffices to check that h is invariant under their action. This follows immediately from (13)–(17) and (28). Next we analyze the poles of h. It is known [7, Chapter I Lemma 4.1] that θ_{10} has no zeros in the upper-half plane and hence h has poles only at the cusps. At the cusp $i\infty$ this modular form has the Fourier expansion

$$h(z) \,=\, q^{-1} + 16 - 132q + 640q^2 - 2550q^3 + O(q^4).$$

Let $I = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$, $T = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$, and $S = \left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$ be elements of Γ_1.

Definition 7.10. *We define the followig three functions*

$$\begin{aligned}
\psi_I &:= h - h|_{-2}ST \qquad (29)\\
\psi_T &:= \psi_I|_{-2}T\\
\psi_S &:= \psi_I|_{-2}S. \qquad (30)
\end{aligned}$$

Lemma 7.11. *More explicitly, we have*

$$\begin{aligned}
\psi_I(z) &= 128\,\frac{\theta_{00}^4(z)+\theta_{01}^4(z)}{\theta_{10}^8(z)} \,+\, 128\frac{\theta_{01}^4(z)-\theta_{10}^4(z)}{\theta_{00}^8(z)}\\
\psi_T(z) &= 128\,\frac{\theta_{00}^4(z)+\theta_{01}^4(z)}{\theta_{10}^8(z)} \,+\, 128\,\frac{\theta_{00}^4(z)+\theta_{10}^4(z)}{\theta_{01}^8(z)}\\
\psi_S(z) &= -\,128\,\frac{\theta_{00}^4(z)+\theta_{10}^4(z)}{\theta_{01}^8(z)} - 128\,\frac{\theta_{10}^4(z)-\theta_{01}^4(z)}{\theta_{00}^8(z)}.
\end{aligned}$$

Lemma 7.12. *The Fourier expansions of these functions are*

$$\begin{aligned}
\psi_I(z) &= q^{-1} + 144 - 5120q^{1/2} + 70524q - 626688q^{3/2} + 4265600q^2 + O(q^{5/2}) \qquad (31)\\
\psi_T(z) &= q^{-1} + 144 + 5120q^{1/2} + 70524q + 626688q^{3/2} + 4265600q^2 + O(q^{5/2})\\
\psi_S(z) &= -\,10240q^{1/2} - 1253376q^{3/2} - 48328704q^{5/2} - 1059078144q^{7/2} + O(q^{9/2}). \qquad (32)
\end{aligned}$$

Definition 7.13. *For $x \in \mathbb{R}^8$ define*

$$b(x) := \int_{-1}^{i} \psi_T(z)\, e^{\pi i \|x\|^2 z}\, dz + \int_{1}^{i} \psi_T(z)\, e^{\pi i \|x\|^2 z}\, dz \tag{33}$$
$$-2 \int_{0}^{i} \psi_I(z)\, e^{\pi i \|x\|^2 z}\, dz - 2 \int_{i}^{i\infty} \psi_S(z)\, e^{\pi i \|x\|^2 z}\, dz.$$

Now we prove that b satisfies condition (19).

Proposition 7.14. *The function b defined by (33) belongs to the Schwartz space and satisfies*

$$\widehat{b}(x) = -b(x).$$

Proof. Here, we repeat the arguments used in the proof of Proposition 7.5. First we show that b is a Schwartz function. We have

$$\int_{-1}^{i} \psi_T(z)\, e^{\pi i r^2 z}\, dz = \int_{0}^{i+1} \psi_I(z)\, e^{\pi i r^2 (z-1)}\, dz =$$
$$\int_{i\infty}^{-1/(i+1)} \psi_I\Big(\frac{-1}{z}\Big)\, e^{\pi i r^2 (-1/z-1)}\, z^{-2}\, dz = \int_{i\infty}^{-1/(i+1)} \psi_S(z)\, z^{-4}\, e^{\pi i r^2 (-1/z-1)}\, dz.$$

There exists a positive constant C such that

$$|\psi_S(z)| \leq C\, e^{-\pi\, \Im z} \quad \text{for} \;\; \Im z > \frac{1}{2}.$$

Thus, as in the proof of Proposition 7.5 we estimate the first summand in the left-hand side of (33)

$$\Bigg|\int_{-1}^{i} \psi_T(z)\, e^{\pi i r^2 z}\, dz\Bigg| \leq C_1\, r\, K_1(2\pi r).$$

We combine this inequality with analogous estimates for the other three summands and obtain

$$|b(r)| \leq C_2\, r\, K_1(2\pi r) + C_3\, \frac{e^{-\pi(r^2+1)}}{r^2+1}.$$

Here C_1, C_2, and C_3 are some positive constants. Similar estimates hold for all derivatives $\frac{d^k}{d^k r} b(r)$.

Now we prove that b is an eigenfunction of the Fourier transform. We use identity (24) and change contour integration in z and Fourier transform in x. Thus we obtain

$$\mathcal{F}(b)(x) = \int\limits_{-1}^{i} \psi_T(z)\, z^{-4}\, e^{\pi i \|x\|^2 (\frac{-1}{z})}\, dz + \int\limits_{1}^{i} \psi_T(z)\, z^{-4}\, e^{\pi i \|x\|^2 (\frac{-1}{z})}\, dz$$
$$-2 \int\limits_{0}^{i} \psi_I(z)\, z^{-4}\, e^{\pi i \|x\|^2 (\frac{-1}{z})}\, dz - 2 \int\limits_{i}^{i\infty} \psi_S(z)\, z^{-4}\, e^{\pi i \|x\|^2 (\frac{-1}{z})}\, dz.$$

We make the change of variables $w = \frac{-1}{z}$ and arrive at

$$\mathcal{F}(b)(x) = \int\limits_{1}^{i} \psi_T\Big(\frac{-1}{w}\Big)\, w^2\, e^{\pi i \|x\|^2 w}\, dw + \int\limits_{-1}^{i} \psi_T\Big(\frac{-1}{w}\Big)\, w^2\, e^{\pi i \|x\|^2 w}\, dw$$
$$-2 \int\limits_{i\infty}^{i} \psi_I\Big(\frac{-1}{w}\Big)\, w^2\, e^{\pi i \|x\|^2 w}\, dw - 2 \int\limits_{i}^{0} \psi_S\Big(\frac{-1}{w}\Big)\, w^2\, e^{\pi i \|x\|^2 w}\, dw.$$

Now we observe that the definitions (29)–(30) imply

$$\begin{aligned} \psi_T|_{-2}S &= -\,\psi_T \\ \psi_I|_{-2}S &= \psi_S \\ \psi_S|_{-2}S &= \psi_I. \end{aligned}$$

Therefore, we arrive at

$$\mathcal{F}(b)(x) = \int\limits_{1}^{i} -\psi_T(z)\, e^{\pi i \|x\|^2 z}\, dz + \int\limits_{-1}^{i} -\psi_T(z)\, e^{\pi i \|x\|^2 z}\, dz$$
$$+2 \int\limits_{i}^{i\infty} \psi_S(z)\, e^{\pi i \|x\|^2 z}\, dz + 2 \int\limits_{0}^{i} \psi_I(z)\, e^{\pi i \|x\|^2 w}\, dw.$$

Now from (33) we see that

$$\mathcal{F}(b)(x) = -b(x).$$

□

Now we regard the radial function b as a function on $\mathbb{R}_{\geq 0}$. We check that b has double roots at Λ_8-points.

Proposition 7.15. *For $r > \sqrt{2}$ function $b(r)$ can be expressed as*

$$b(r) = -4 \sin(\pi r^2/2)^2 \int\limits_{0}^{i\infty} \psi_I(z)\, e^{\pi i r^2 z}\, dz. \qquad (34)$$

Proof. We denote the right hand side of (34) by $c(r)$. First, we check that $c(r)$ is well-defined. We have

$$\psi_I(it) = O(t^2\, e^{\pi/t}) \quad \text{as } t \to 0$$
$$\psi_I(it) = O(e^{2\pi t}) \quad \text{as } t \to \infty.$$

Therefore, the integral (34) converges for $r > \sqrt{2}$. Then we rewrite it in the following way:

$$c(r) = \int\limits_{-1}^{i\infty-1} \psi_I(z+1)\, e^{\pi i r^2 z}\, dz - 2\int\limits_{0}^{i\infty} \psi_I(z)\, e^{\pi i r^2 z}\, dz + \int\limits_{1}^{i\infty+1} \psi_I(z-1)\, e^{\pi i r^2 z}\, dz.$$

From the Fourier expansion (31) we know that $\psi_I(z) = e^{-2\pi i z} + O(1)$ as $\Im(z) \to \infty$. By assumption $r^2 > 2$, hence we can deform the path of integration and write

$$\int\limits_{-1}^{i\infty-1} \psi_I(z+1)\, e^{\pi i r^2 z}\, dz = \int\limits_{-1}^{i} \psi_T(z)\, e^{\pi i r^2 z}\, dz + \int\limits_{i}^{i\infty} \psi_T(z)\, e^{\pi i r^2 z}\, dz$$
$$\int\limits_{1}^{i\infty+1} \psi_I(z-1)\, e^{\pi i r^2 z}\, dz = \int\limits_{-1}^{i} \psi_T(z)\, e^{\pi i r^2 z}\, dz + \int\limits_{i}^{i\infty} \psi_T(z)\, e^{\pi i r^2 z}\, dz.$$

We have

$$\begin{aligned} c(r) = &\int\limits_{-1}^{i} \psi_T(z)\, e^{\pi i r^2 z}\, dz + \int\limits_{1}^{i} \psi_T(z)\, e^{\pi i r^2 z}\, dz - 2\int\limits_{0}^{i} \psi_I(z)\, e^{\pi i r^2 z}\, dz \\ &+ 2\int\limits_{i}^{i\infty} (\psi_T(z) - \psi_I(z))\, e^{\pi i r^2 z}\, dz. \end{aligned} \tag{35}$$

Next, we check that the functions ψ_I, ψ_T, and ψ_S satisfy the following identity:

$$\psi_T + \psi_S = \psi_I. \tag{36}$$

Indeed, from definitions (29)-(30) we get

$$\begin{aligned} \psi_T + \psi_S =& (h - h|_{-2}ST)|_{-2}T + (h - h|_{-2}ST)|_{-2}S \\ =& h|_{-2}T - h|_{-2}ST^2 + h|_{-2}S - h|_{-2}STS. \end{aligned}$$

Note that ST^2S belongs to $\Gamma_0(2)$. Thus, since $h \in M^!_{-2}\Gamma_0(2)$ we get

$$\psi_T + \psi_S = h|_{-2}T - h|_{-2}STS.$$

Now we observe that T and $STS(ST)^{-1}$ are also in $\Gamma_0(2)$. Therefore,

$$\psi_T + \psi_S = h|_{-2}T - h|_{-2}STS = h|_{-2} - h|ST = \psi_I.$$

Combining (35) and (36) we find

$$\begin{aligned} c(r) = & \int_{-1}^{i} \psi_T(z)\, e^{\pi i r^2 z}\, dz + \int_{1}^{i} \psi_T(z)\, e^{\pi i r^2 z}\, dz - 2\int_{0}^{i} \psi_I(z)\, e^{\pi i r^2 z}\, dz \\ & - 2\int_{i}^{i\infty} \psi_S(z)\, e^{\pi i r^2 z}\, dz \\ = & b(r). \end{aligned}$$

□

At the end of this section we find another integral representation of $b(r)$ for $r \in \mathbb{R}_{\geq 0}$ and compute special values of b.

Proposition 7.16. *For $r \geq 0$ we have*

$$b(r) = 4i\,\sin(\pi r^2/2)^2 \left(\frac{144}{\pi\, r^2} + \frac{1}{\pi\,(r^2-2)} + \int_0^\infty \left(\psi_I(it) - 144 - e^{2\pi t}\right)\, e^{-\pi r^2 t}\, dt \right). \tag{37}$$

The integral converges absolutely for all $r \in \mathbb{R}_{\geq 0}$.

Proof. The proof is analogous to the proof of Proposition 7.7. First, suppose that $r > \sqrt{2}$. Then by Proposition 7.15

$$b(r) = 4i\,\sin(\pi r^2/2)^2 \int_0^\infty \psi_I(it)\, e^{-\pi r^2 t}\, dt.$$

From (31) we obtain

$$\psi_I(it) = e^{2\pi t} + 144 + O(e^{-\pi t}) \quad \text{as } t \to \infty. \tag{38}$$

For $r > \sqrt{2}$ we have

$$\int_0^\infty \left(e^{2\pi t} + 144\right)\, e^{-\pi r^2 t}\, dt = \frac{1}{\pi\,(r^2-2)} + \frac{144}{\pi\, r^2}.$$

Therefore, the identity (37) holds for $r > \sqrt{2}$.

On the other hand, from the definition (33) we see that $b(r)$ is analytic in some neighborhood of $[0, \infty)$. The asymptotic expansion (38) implies that the right hand side of (37) is also analytic in some neighborhood of $[0, \infty)$. Hence, the identity (37) holds on the whole interval $[0, \infty)$. This finishes the proof of the proposition. $\square$

We see from (37) that $b(r) \in i\mathbb{R}$ far all $r \in \mathbb{R}_{\geq}0$. Another immediate corollary of this proposition is

Proposition 7.17. *We have*

$$b(0) = 0 \qquad b(\sqrt{2}) = 0 \qquad b'(\sqrt{2}) = \frac{i}{2\sqrt{2}\,\pi}.$$

8 Proof of Theorem 5.2

Our proof of the Theorem 5.2 relies on the following two inequalities for modular objects.

Proposition 8.1. *Consider the function $A : (0, \infty) \to \mathbb{C}$ defined as*

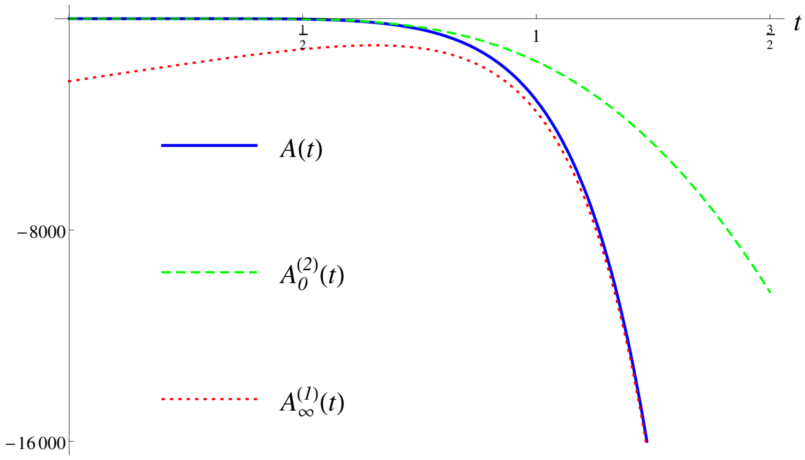
$$A(t) := -t^2 \phi_0(i/t) - \frac{36}{\pi^2}\,\psi_I(it).$$

Then $A(t) \in (-\infty, 0)$ for all $t \in (0, \infty)$.

Remark: *We might formalize the original proof from [11] or the proof of Dan Romik "On Viazovska's modular form inequalities" [10]. Below is the proof from [11].*

Proof. Function $A(t)$ is plotted in Figure 1.

Figure 1: Plot of the functions $A(t)$, $A_0^{(2)}(t) = -\frac{368640}{\pi^2}\,t^2\,e^{-\pi/t}$, and $A_\infty^{(1)}(t) = -\frac{72}{\pi^2}\,e^{2\pi t} + \frac{8640}{\pi}t - \frac{23328}{\pi^2}$.



We observe that we can compute the values of $A(t)$ for $t \in (0, \infty)$ with any given precision. Indeed, from identities (22) and (30) we obtain the following two presentations for $A(t)$

$$A(t) = -\,t^2 \phi_0(i/t) + \frac{36}{\pi^2}\,t^2\,\psi_S(i/t)$$

$$A(t) = -\,t^2 \phi_0(it) + \frac{12}{\pi}\,t\,\phi_{-2}(it) - \frac{36}{\pi^2}\,\phi_{-4}(it) - \frac{36}{\pi^2}\,\psi_I(i/t).$$

For an integer $n \geq 0$ let $A_0^{(n)}$ and $A_\infty^{(n)}$ be the functions such that

$$A(t) = A_0^{(n)}(t) + O(t^2\, e^{-\pi n/t}) \quad \text{as } t \to 0 \tag{39}$$

$$A(t) = A_\infty^{(n)}(t) + O(t^2\, e^{-\pi n t}) \quad \text{as } t \to \infty. \tag{40}$$

For each $n \geq 0$ we can compute these functions from the Fourier expansions (20)–(21), (31), and (32). For example, from (20)–(21) and (31) we compute

$$A_\infty^{(6)}(t) = -\frac{72}{\pi^2}\, e^{2\pi t} - \frac{23328}{\pi^2} + \frac{184320}{\pi^2}\, e^{-\pi t} - \frac{5194368}{\pi^2}\, e^{-2\pi t} + \frac{22560768}{\pi^2}\, e^{-3\pi t} - \frac{250583040}{\pi^2}\, e^{-4\pi t} + \frac{869916672}{\pi^2}\, e^{-5\pi t}$$
$$+ t\left(\frac{8640}{\pi} + \frac{2436480}{\pi}\, e^{-2\pi t} + \frac{113011200}{\pi}\, e^{-4\pi t}\right) - t^2(518400\, e^{-2\pi t} + 31104000\, e^{-4\pi t}).$$

From (20)–(21) and (32) we compute

$$A_0^{(6)}(t) = t^2\left(-\frac{368640}{\pi^2}\, e^{-\pi/t} - 518400\, e^{-2\pi/t} - \frac{45121536}{\pi^2}\, e^{-3\pi/t} - 31104000\, e^{-4\pi/t} - \frac{1739833344}{\pi^2}\, e^{-5\pi/t}\right).$$

Moreover, from the convergent asymptotic expansion for the Fourier coefficients of a weakly holomorphic modular form [2, Proposition 1.12] we find that the n-th Fourier coefficient $c_{\psi_I}(n)$ of ψ_I satisfies

$$|c_{\psi_I}(n)| \leq e^{4\pi\sqrt{n}} \qquad n \in \tfrac{1}{2}\mathbb{Z}_{>0}. \tag{41}$$

Similar inequalities hold for the Fourier coefficients of ψ_S, ϕ_0, ϕ_{-2}, and ϕ_{-4}:

$$\begin{aligned}
&|c_{\psi_S}(n)| \leq 2e^{4\pi\sqrt{n}} && n \in \tfrac{1}{2}\mathbb{Z}_{>0}\\
&|c_{\phi_0}(n)| \leq 2e^{4\pi\sqrt{n}} && n \in \mathbb{Z}_{>0}\\
&|c_{\phi_{-2}}(n)| \leq e^{4\pi\sqrt{n}} && n \in \mathbb{Z}_{>0}\\
&|c_{\phi_{-4}}(n)| \leq e^{4\pi\sqrt{n}} && n \in \mathbb{Z}_{>0}.
\end{aligned} \tag{42}$$

Therefore, we can estimate the error terms in the asymptotic expansions (39) and (40) of $A(t)$

$$\left|A(t) - A_0^{(m)}(t)\right| \leq \left(t^2 + \frac{36}{\pi^2}\right) \sum_{n=m}^{\infty} 2e^{2\sqrt{2}\pi\sqrt{n}}\, e^{-\pi n/t}$$
$$\left|A(t) - A_\infty^{(m)}(t)\right| \leq \left(t^2 + \frac{12}{\pi}\, t + \frac{36}{\pi^2}\right) \sum_{n=m}^{\infty} 2e^{2\sqrt{2}\pi\sqrt{n}}\, e^{-\pi n t}.$$

For an integer $m \geq 0$ we set

$$R_0^{(m)} := \left(t^2 + \frac{36}{\pi^2}\right) \sum_{n=m}^{\infty} 2e^{2\sqrt{2}\pi\sqrt{n}}\, e^{-\pi n/t}$$
$$R_\infty^{(m)} := \left(t^2 + \frac{12}{\pi}\, t + \frac{36}{\pi^2}\right) \sum_{n=m}^{\infty} 2e^{2\sqrt{2}\pi\sqrt{n}}\, e^{-\pi n t}.$$

Using interval arithmetic we check that

$$\begin{aligned}
\left|R_0^{(6)}(t)\right| &\leq \left|A_0^{(6)}(t)\right| \qquad \text{for} \quad t \in (0,1] \\
\left|R_\infty^{(6)}(t)\right| &\leq \left|A_\infty^{(6)}(t)\right| \qquad \text{for} \quad t \in [1,\infty) \\
A_0^{(6)}(t) &< 0 \qquad \text{for} \quad t \in (0,1] \\
A_\infty^{(6)}(t) &< 0 \qquad \text{for} \quad t \in [1,\infty).
\end{aligned}$$

Thus, we see that $A(t) < 0$ for $t \in (0,\infty)$. This finishes the proof of the Proposition. □

Proposition 8.2. *Consider the function $B : (0,\infty) \to \mathbb{C}$ defined as*

$$B(t) := -t^2\phi_0(i/t) + \frac{36}{\pi^2}\,\psi_I(it).$$

Then $B(t) \in (0,\infty)$ for all $t \in (0,\infty)$.

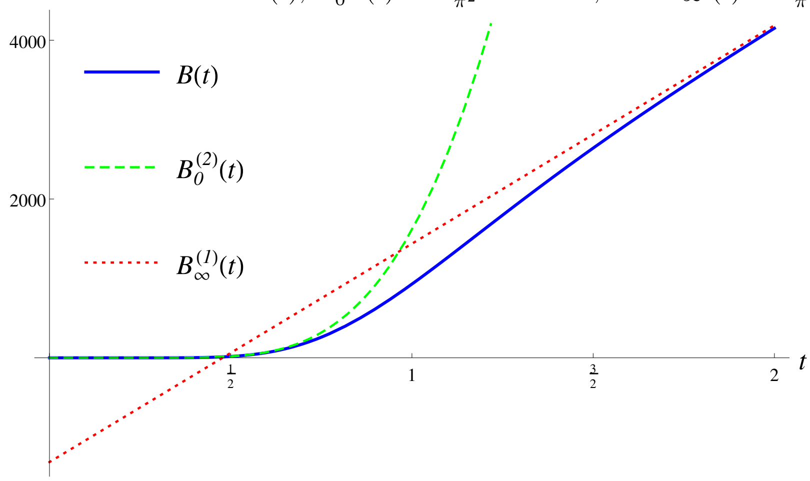
Remark *Below is the proof from [11]. Similarly to the previous proposition, another (hopefully easier for the formalization) proof of this inequality is given in [10].*

Proof. The function B can be also written as

$$\begin{aligned}
B(t) &= -\,t^2\phi_0(i/t) - \frac{36}{\pi^2}\,t^2\,\psi_S(i/t) \\
B(t) &= -\,t^2\phi_0(it) + \frac{12}{\pi}\,t\,\phi_{-2}(it) - \frac{36}{\pi^2}\,\phi_{-4}(it) + \frac{36}{\pi^2}\,\psi_I(i/t).
\end{aligned}$$

Our aim is to prove that $B(t) > 0$ for $t \in (0,\infty)$. A plot of $B(t)$ is given in Figure 2.

Figure 2: Plot of the functions $B(t)$, $B_0^{(2)}(t) = \frac{368640}{\pi^2}\,t^2\,e^{-\pi/t}$, and $B_\infty^{(1)}(t) = \frac{8640}{\pi}t - \frac{23328}{\pi^2}$.



For $n \geq 0$ let $B_0^{(n)}$ and $B_\infty^{(n)}$ be the functions such that

$$B(t) = B_0^{(n)}(t) + O(t^2\, e^{-\pi n/t}) \quad \text{as } t \to 0$$
$$B(t) = B_\infty^{(n)}(t) + O(t^2\, e^{-\pi n t}) \quad \text{as } t \to \infty.$$

We find

$$B_\infty^{(6)}(t) = -\tfrac{12960}{\pi^2} - \tfrac{184320}{\pi^2}\, e^{-\pi t} - \tfrac{116640}{\pi^2}\, e^{-2\pi t} - \tfrac{22560768}{\pi^2}\, e^{-3\pi t} + \tfrac{56540160}{\pi^2}\, e^{-4\pi t} - \tfrac{869916672}{\pi^2}\, e^{-5\pi t}$$
$$+ t\big(\tfrac{8640}{\pi} + \tfrac{2436480}{\pi}\, e^{-2\pi t} + \tfrac{113011200}{\pi}\, e^{-4\pi t}\big) - t^2\big(518400\, e^{-2\pi t} + 31104000\, e^{-4\pi t}\big)$$

and

$$B_0^{(6)}(t) = t^2\big(\tfrac{368640}{\pi^2}\, e^{-\pi/t} - 518400\, e^{-2\pi/t} + \tfrac{45121536}{\pi^2}\, e^{-3\pi/t} - 31104000\, e^{-4\pi/t} + \tfrac{1739833344}{\pi^2}\, e^{-5\pi/t}\big).$$

The estimates (41)–(42) imply that

$$\left|B(t) - B_0^{(6)}(t)\right| \leq R_0^{(6)}(t) \quad \text{for } t \in (0, 1]$$

and

$$\left|B(t) - B_\infty^{(6)}(t)\right| \leq R_\infty^{(6)}(t) \quad \text{for } t \in [1, \infty).$$

Using interval arithmetic we verify that

$$\left|R_0^{(6)}(t)\right| \leq \left|B_0^{(6)}(t)\right| \qquad \text{for } \ t \in (0, 1]$$
$$\left|R_\infty^{(6)}(t)\right| \leq \left|B_\infty^{(6)}(t)\right| \qquad \text{for } \ t \in [1, \infty)$$
$$B_0^{(6)}(t) > 0 \qquad \text{for } \ t \in (0, 1]$$
$$B_\infty^{(6)}(t) > 0 \qquad \text{for } \ t \in [1, \infty).$$

Now identity (44) implies (6). □

Finally, we are ready to prove Theorem 5.2.

Theorem 8.3. *The function*

$$g(x) := \frac{\pi\, i}{8640} a(x) + \frac{i}{240\pi}\, b(x)$$

satisfies conditions (5)–(7).

Proof. First, we prove that (5) holds. By Propositions 7.6 and 7.15 we know that for $r > \sqrt{2}$

$$g(r) = \frac{\pi}{2160} \sin(\pi r^2/2)^2 \int_0^\infty A(t)\, e^{-\pi r^2 t}\, dt \tag{43}$$

where

$$A(t) = -t^2 \phi_0(i/t) - \frac{36}{\pi^2}\, \psi_I(it).$$

from the Proposition 8.1 we know that $A(t) < 0 \quad$ for $t \in (0, \infty)$. Therefore identity (43) implies (5).

Next, we prove (6). By Propositions 7.7 and 7.16 we know that for $r > 0$

$$\widehat{g}(r) = \frac{\pi}{2160} \sin(\pi r^2/2)^2 \int_0^\infty B(t)\, e^{-\pi r^2 t}\, dt \tag{44}$$

where

$$B(t) = -t^2 \phi_0(i/t) + \frac{36}{\pi^2}\, \psi_I(it).$$

Finally, the property (7) readily follows from Proposition 7.8 and Proposition 7.17. This finishes the proof of Theorems 8.3 and 5.2. $\square$

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