Sphere Packing in Lean

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1 Basic definitions for sphere packings

1.1 Sphere packings

The sphere packing constant measures which portion of d-dimensional Euclidean space can be covered by non-overlapping unit balls. More precisely, let R^d be the Euclidean vector space equipped with distance and Lebesgue measure Vol(). For $x\ \ R^d$ and $r\ \ R_{>0}$ we denote by $B_d(x, r)$ the ball in R^d with center x and radius r.

Definition 1. *Let $X\ \ R^d$ be a discrete set of points such that $x\ \ y\ \ 2$ for any distinct $x, y\ \ X$. Then the union*

$$P = \mathop{}_{xX} B_d(x, 1)$$

is a sphere packing.

Definition 2. *If X is a lattice in R^d then we say that P is a* lattice sphere packing*.*

Definition 3. *The* finite density *of a packing P is defined as*

$${}_P(r) = \frac{\mathrm{Vol}(P\ \ B_d(0, r))}{\mathrm{Vol}(B_d(0, r))}, \quad r > 0.$$

Definition 4. *We define the* density *of a packing P as the limit superior*

$${}_P = \limsup_{r} {}_P(r).$$

Definition 5. *The number be want to know is the supremum over all possible packing densities called the* sphere packing constant*.*

The main result of this paper is the proof that

$${}_8 = \frac{4}{384}\ \ 0.25367.$$

This is the density of the E_8-lattice sphere packing.

Definition 6. *Recall that the E_8-lattice ${}_8\ \ R^8$ is given by*

$${}_8 = \{(x_i)\ \ Z^8\ \ (Z + \tfrac{1}{2})^8 |\ \mathop{}_{i=1}^{8} x_i\ \ 0 \pmod 2\}.$$

Lemma 7. $_8$ *is a positive-definite, even, unimodular lattice of rank 8. The minimal distance between two points in* $_8$ *is* $\sqrt{2}$.

Definition 8. *The* E_8*-lattice sphere packing is the packing of unit balls with centers at* $\frac{1}{\sqrt{2}}{}_8$.

Our main result is

Theorem 9. *No packing of unit balls in Euclidean space* R^8 *has density greater than that of the* E_8*-lattice packing.*

1.2 Lattices and Periodic packings

Definition 10. *Let* S *be a discrete subgroup of* R^d. *A set* $X \quad R^d$ *is said to* S*-periodic if for each* $s \quad S$ *and* $x \quad X$ *the vector* $x + s$ *belongs to* X.

Definition 11. *A lattice in the Euclidean space* R^d *is a discrete, co-compact, abelian subgroup.*