HOLON INSTITUTE OF TECHNOLOGY, DEPARTMENT OF APPLIED MATHEMATICS

Sequential Models in Data Science
Recursive Bayesian Filtering

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INTRODUCTION

This exercise aims at giving an overall understanding of Bayesian Filtering and to implement it in Python.
The exercise is to be done by pairs of students. Each pair must present a pdf file with the theoretical answers and a Jupyter notebook with the Python functions. The notebook will be tested as this. No modification should be necessary for making it work.
All over the questions, x_k and y_k denotes respectively the state vector and the measurements.

1 KALMAN FILTER WITH BIASED NOISE

Derive the Kalman filter equations for the following linear-Gaussian filtering model with non-zero mean noises:

$$\begin{aligned} x_k &= A x_{k-1} + q_{k-1}, \\ y_k &= H x_k + r_k, \end{aligned}$$

where $q_{k-1} \sim \mathcal{N}(m_q, Q)$ and $r_k \sim \mathcal{N}(m_r, R)$.
It is recommended to use the two lemmas that were defined in task 2, section 3.

2 GAUSSIAN RANDOM WALK

Consider the Gaussian random walk:

$$\begin{aligned} x_k &= x_{k-1} + q_{k-1}, \\ y_k &= x_k + r_k, \end{aligned}$$

where $q_{k-1} \sim \mathcal{N}(0, Q)$ and $r_k \sim \mathcal{N}(0, R)$.

1. Write the Kalman equations in this case.
2. Implement a Gaussian random walk tracking in $\mathbb{R}^n$ and do a simulation in $\mathbb{R}^2$. Draw on the same graph in two different colors (with a clear caption) the trajectory of the system and the estimated state.

3 Extended Kalman Filter

Consider the following non-linear space model:

$$\begin{aligned} x_k &= x_{k-1} - 0.01\sin(x_{k-1}) + q_{k-1}, \\ y_k &= 0.5\sin(2x_k) + r_k, \end{aligned}$$

where $q_{k-1} \sim \mathcal{N}(0, 0.01^2)$ and $r_k \sim \mathcal{N}(0, 0.02)$.

1. Write the extended Kalman filter obtained by the Taylor linearization.
2. Implement in python this version of the EKF. Draw on the same graph in three different colors (with a clear caption) the trajectory of the system and the estimated states by the two different methods of EKF.

4 Grid Based Filter

Consider a situation where the state space is discrete and finite: $\forall n, x_n \in S$ and $|S| = N < \infty$. Let $S = \{x^1, \cdots, x^N\}$.
Define the coefficients $w^i_{k-1|k-1}$ by the following relation:

$$p(x_{k-1}|y_{1:k-1}) = \sum_{i=1}^{N} w^i_{k-1|k-1}\delta(x_{k-1} - x^i).$$

The prediction step leads to the following expression:

$$p(x_k|y_{1:k-1}) = \sum_{i=1}^{N} w^i_{k|k-1}\delta(x_{k-1} - x^i).$$

Then the update step leads to:

$$p(x_k|y_{1:k}) = \sum_{i=1}^{N} w^i_{k|k}\delta(x_{k-1} - x^i).$$

Prove the following relations:

1. $\forall i, w^i_{k|k-1} = \sum_{j=1}^{N} w^j_{k-1|k-1} p(x^i|x^j)$, where $p(x^i|x^j)$ is the transition probability from state x^j to state x^i,

2. $\forall i, w^i_{k|k} = \frac{w^i_{k|k-1} p(y_k|x^i)}{\sum_{j=1}^{N} w^j_{k|k-1} p(y_k|x^j)}$, where $p(y_k|x^l)$ is the probability distribution of the measurement y_k given the system is in state x^l.

3. Consider the following situation. Let x_k be a sequence of real numbers that satisfies:

$$\begin{aligned} x_k &= x_{k-1} + L v_{k-1} \\ y_k &= (\tfrac{1}{2} + r_k) x_k, \end{aligned}$$

where L is a positive real number, v_{k-1} is a Rademacher random variable the distribution of which is given by:

$$\mathbb{P}(v_{k-1} = i) = \begin{cases} \frac{1}{2} \text{ if } i \in \{-1, 1\} \\ 0 \text{ otherwise} \end{cases}$$

and r_k is a uniform random variable in $[0, 1/2]$. Let x_0 be a Gaussian random variable, such that $x_0 \sim \mathcal{N}(0, 1)$. Implement a grid based filter to track the state vector x_k over 100 time steps, within the state space $S = \{x_0, x_0 \pm L, \ldots, x_0 \pm NL\}$, where $N = 100$. Take $L = 2$. Draw a plot with the actual state and the estimated one with two different colors and an implicit legend, over the tracking interval.