Q1. Maximum Likelihood Estimation of λ

1.

$$\log P(D|\lambda) = \log \frac{\lambda^{\sum x_i} e^{-n\lambda}}{\prod_{i=1}^{n} x_i!} = \sum_{i=1}^{N} x_i \log\lambda - n\lambda - \log \prod_{i=1}^{n} x_i!$$

2. $$\frac{d}{d\lambda}\left[\log P(D|\lambda)\right] = \frac{\sum x_i}{\lambda} - n - 0 = \sum_{i=1}^{N} x_i \frac{1}{\lambda} - N$$

3. $$\frac{\sum x_i}{\lambda} - n = 0$$

$$\frac{\sum x_i}{\lambda} = n$$

$$\hat{\lambda} = \frac{\sum x_i}{n} = \hat{x}$$

Thus, we know λ is the average number of occurrences.

Q2. Maximum A Posteriori Estimate of λ with a Gamma Prior

1. Let $f(x_i|\lambda) = \frac{e^{\lambda}\lambda^{x_i}}{x_i!}$; $x_i = 0, 1, 2, 3, \ldots$ $f(x_i) = \frac{\beta^{\alpha}\lambda^{x_i}e^{-\lambda\beta}}{\Gamma(\alpha)}$

chere λ is greater than zero.

$$\Rightarrow f(x_i,\lambda) = \prod_{i=1}^{b} f(x_i|\lambda)f(\lambda) = \frac{e^{-N\lambda}\lambda^{\sum_{i=1}^{N}x_i}}{\prod_{i=1}^{N}x_i!} * \frac{e^{-\lambda\beta}\lambda^{\alpha-1}\beta^{\alpha}}{\Gamma(\alpha)}$$

$$f(x) = \int_{\lambda} f(x_i,\lambda)\,d\lambda = \int_0^{\infty} \frac{\beta^{\alpha}}{\prod_{i=1}^{N} x_i\Gamma(\alpha)} e^{-N\lambda} e^{-\lambda\beta} \lambda^{\sum_{i=1}^{N}x_i} \lambda^{\alpha-1}\,d\lambda$$

$$= \frac{\beta^{\alpha}}{\prod_{i=1}^{N} x_i\Gamma(\alpha)} \int_0^{\infty} e^{-\lambda(N+\beta)} \lambda^{\sum_{i=1}^{N}x_i+\alpha-1}\,d\lambda$$

$$= \frac{\dfrac{\beta^{\alpha}}{\prod_{i=1}^{N} x_i\Gamma(\alpha)} \displaystyle\int_0^{\infty} e^{-\lambda(N+\beta)} \lambda^{\sum_{i=1}^{N}x_i+\alpha-1}\,d\lambda}{\dfrac{\beta^{\alpha}}{\prod_{i=1}^{N} x_i\Gamma(\alpha)} * \dfrac{\Gamma\sum_{i=1}^{N}x_i+\alpha}{(N+B)^{\sum_{i=1}^{N}x_i+\alpha}}}$$

$$= \frac{1}{\Gamma\sum_{i=1}^{N}x_i+\alpha} e^{-\lambda(N+\beta)}\lambda^{\sum_{i=1}^{N}x_i+\alpha-1}(N+B)^{\sum_{i=1}^{N}x_i+\alpha}$$

$$\ln P(\lambda|D) = -\ln\left[\Gamma\left(\sum_{i=1}^{N}x_i+\alpha\right)\right] - \lambda(N+B) + \ln\lambda\left(\sum_{i=1}^{N}x_i+\alpha-1\right)$$

$$+ \ln(N+B)\left(\sum_{i=1}^{N}x_i+\alpha\right) = C - \lambda(N+B) + \ln\lambda\left(\sum_{i=1}^{N}x_i+\alpha-1\right)$$

2. $\frac{\partial}{\partial\lambda} P(\lambda|D) = -(N+B) + \frac{\sum_{i=1}^{N} x_i + \alpha - 1}{\lambda}$

3. $-(N+B) + \frac{\sum_{i=1}^{N} x_i + \alpha - 1}{\lambda} = 0$

$\sum_{i=1}^{N} x_i + \alpha - 1 = \lambda(N+B)$

$\lambda = \frac{\sum_{i=1}^{N} x_i + \alpha - 1}{N+B} = \hat{\lambda} \Rightarrow$ be the Mapestimator of λ

Q3 Deriving the Posterior of a Poisson-Gamma Model

$\log P(\lambda|D) \propto \log(D|\lambda) + \log(\lambda)$

$P(\lambda|D) = P(D|\lambda) \cdot P(\lambda)$

$= \prod_{i=1}^{n} \frac{\beta^{\alpha} \lambda^{\alpha-1} e^{-\beta\lambda}}{F(\alpha)} \cdot \frac{\beta^{\alpha} \lambda^{\alpha-1} e^{-\beta\lambda}}{F(\alpha)}$

$\propto e^{-N\lambda} \cdot \lambda^{\sum_{i=1} x_i} \cdot \lambda^{\alpha-1} e^{-\beta\lambda}$

$\propto e^{-\lambda(N+B)} \cdot \lambda^{(\sum_{i=1} x_i + \alpha - 1)}$

There, $P(\lambda|D) \propto \text{Gamma}\left(\sum_i x_i + \alpha - 1, \beta + n\right)$