GLASGOW COLLEGE UESTC

Exam paper

Elements of Information Theory (UESTC) (UESTC3021)

DATE: 9th Jan. 2019
Time: 09:30-11:30 am

Attempt all PARTS

The numbers in square brackets in the right-hand margin indicate the marks allotted to the part of the question against which the mark is shown.

An electronic calculator may be used provided that it does not have a facility for either textual storage or display, or for graphical display.

Continued overleaf

Q1. Given an arbitrary random variable X and its function $g(X)$, prove that the entropy $H[g(X)]$ of the function $g(X)$ is not higher than the entropy $H(X)$ of X by obeying the following steps. While proving (a) and (c), pleas describe the derivation process by using definition of entropy and related properties.

(a) $H(X, g(X)) = H(X) + H(g(X)|X)$

[5]

(b) $H(X, g(X)) = H(X)$

[5]

(c) $H(X, g(X)) = H\big(g(X)\big) + H(X|g(X))$

[5]

(d) $H(X, g(X)) \geq H(g(X))$

[5]

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Q2. Consider the random variable

$$X = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ \frac{1}{21} & \frac{2}{21} & \frac{3}{21} & \frac{4}{21} & \frac{5}{21} & \frac{6}{21} \end{pmatrix}$$

(a) Find a binary Huffman code for X. [6]

(b) Find the expected code length for this binary encoding and its coding efficiency. [4]

(c) Find a ternary Huffman code for X. [6]

(d) Find the expected code length for this ternary encoding and its coding efficiency. [4]

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Q3. Consider the discrete memoryless channel $Y = X + Z$ (mod 11), where

$$Z = \begin{bmatrix} 1 & 2 & 3 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

and $X \in \{0,1,\ldots,10\}$. Assume that Z is independent of X.

(a) Find the conditional entropy $H(Y|X)$. [6]

(b) Find the capacity of the channel. [8]

(c) What is the maximum probability distribution $\{p^*(X = i)|i = 0,1,\cdots,10\}$ of the channel input. [6]

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Q4. Consider a binary symmetric channel with crossover probability 0.1. A possible coding scheme for this channel with two codewords of length 3 is to encode message a_1 as 000 and a_2 as 111. With this coding scheme, we can consider the combination of encoder, channel and decoder as forming a new BSC, with two inputs a_1 and a_2 and two outputs b_1 and b_2 .

(a)Calculate the crossover probability of this channel. [5]

(b)What is the capacity of this channel in bits per transmission of the original channel? [5]

(c)What is the capacity of the original BSC with crossover probability 0.1? [5]

(d)Prove a general result that for any channel, considering the encoder, channel and decoder together as a new channel from messages to estimated messages will not increase the capacity in bits per transmission of the original channel. [5]

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Q5. Consider a discrete memoryless source uniformly distributed on the set $\{x_1, x_2, x_3\}$, i.e.,

$$\begin{bmatrix} X \\ p(x_i) \end{bmatrix} = \begin{Bmatrix} x_1 & x_2 & x_3 \\ 1/3 & 1/3 & 1/3 \end{Bmatrix},$$

and Hamming distortion is used as rate distortion function for this source, i.e.,

$$d(x, \hat{x}) = \begin{cases} 0 & \text{if } x = \hat{x} \\ & 1 \end{cases}$$

(a)Find $\mathrm{D_{min}}$, $\mathrm{R(D_{min})}$, and write the corresponding channel matrix for test. [6]

(b)Find $\mathrm{D_{max}}$, $\mathrm{R(D_{max})}$, and write the corresponding channel matrix for test. [6]

(c)If tolerated average distortion $\mathrm{D} = 1/3$, find the minimum number of binary digits needed to represents a source output in average. [8]

End of question paper