It might seem silly to write it in this way, but when we get to larger systems of linear equations it is helpful to keep this very simple example in mind.

Why do we always use the symbol x for the unknown? Why ex?

Apparently, it comes from the Arabic word for *thing*: شيء It is pronounced something like *she'en* in Arabic.* Because the Spanish who were invaded partially by the Arabs starting in 711 did not have a sound for "sh", they instead used "xei" and related it to the way the Greek symbol "chi" is pronounced. Chi is usually spelled as χ. It was then translated into the usual x once the rest of Europe caught up with this cool stuff from a faraway land. Algebra comes from the Arabic word *Al-Jabra*, which I have been told loosely translates into a "reunion of broken parts": الجبر. Poetry meets mathematics or mathematics meets poetry. To me it is the same, both present a beautiful realm in a different way. Both have constraints on what one can create in that realm.

Al-Jabra is the name of a mathematical treatise written in the year 820 in Baghdad, now part of Iraq.

To find an unknown thing you have to inverse something known and multiply it with the known. That is how you get the unknown from the known. That sounds a bit like interpolation, doesn't it? But it is different. It is a little bit more complicated but not too complicated.

As we have seen earlier, 1D linear equations are easily solved for a single unknown thing.

Now let's turn to a larger system like†:

$$4x + 6y = 0$$

$$3x + y = -10.$$

In this case, we have to solve for two unknown things that we labeled x and y that have to satisfy the two equations simultaneously.

By the way, I hated this stuff in high school. I am not a number cruncher. We have dumb computers for doing that. But let's go through the exercise of solving this puzzle.

---

* I do not speak or read Arabic. I heard the pronunciation through Google translate. There is where I also got the beautiful scriptures for *thing* and *Al-Jabra*. To me the fact that the sign for *thing* is as complicated as the *reunion of broken parts* is a complete mystery.

† We omit the "×" multiplication symbol. It is silently implied. $3 \times x = 3x$. It saves space and there is less confusion between the two exes.

One strategy to solve these equations is to first solve for y from the second equation in terms of x and then substitute that result in the first equation to find x. Then we go back to our second equation and solve for y.

Let's go and do it.

From the second equation we get that

$$y = -10 - 3x$$

Therefore, by substituting this expression for y in the first equation we have that

$$4x + 6(-10 - 3x) = 0$$

$$4x - 60 - 18x = 0$$

$$-14x - 60 = 0$$

$$x = -\frac{60}{14} = -\frac{30}{7}$$

That is just straightforward high school math. Hence, we now know the value for x and consequently we can find the value for y by substitution:

$$y = -10 + 3 \times \frac{30}{7} = \frac{-70 + 90}{7} = \frac{20}{7}$$

So, after all these boring computations we conclude that the solution is $x = \frac{-30}{7}$ and $y = \frac{20}{7}$.

Our two unknowns are now two knowns.

Since I am bad at this stuff let's make sure this is the right solution as I do not want to embarrass myself in my own book. All we have to do is just plug in the solution back into the equations and see if they are satisfied.

$$4\left(\frac{-30}{7}\right) + 6\left(\frac{20}{7}\right) = \frac{-120 + 120}{7} = 0$$

$$3\left(\frac{-30}{7}\right) + \left(\frac{20}{7}\right) = \frac{-90 + 20}{7} = \frac{-70}{7} = -10.$$

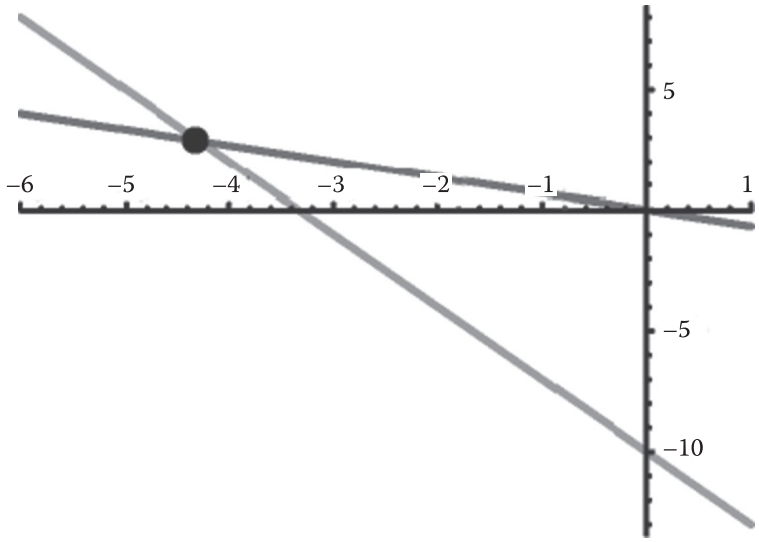


FIGURE 7.1 The solution of a 2D linear system can be computed geometrically.

Yes they are satisfied! High fives! Every smart reader of this book is now shaking their head.

Now let's make this a bit more fun, at least to me.

Let's first look at these linear equations from a geometrical point of view.

The two equations aforementioned define two lines in the plane and their intersection is the solution to the equations. This is illustrated in Figure 7.1. You can verify that the coordinates of the point of intersection matches the result we just painfully derived. Isn't it more fun to draw two lines rather than going through a series of calculations?

I think so.

By the way this geometrical approach can be generalized to any dimension not just two. In three dimensions, three 2D planes intersect in one point. Well okay not always, like in the case when some of them are parallel. That is when it gets tricky. Mathematicians call these cases *degenerate dimensions*. For four dimensional linear systems, we intersect four 3D *hyperplanes*. For N dimensional systems we intersect N $(N–1)$-dimensional hyperplanes.

Do not ask me how to visualize them. That is someone else's job.

That is why we introduce *matrices*. Matrices are like apartment buildings that contain numbers, not bugs. They are useful to solve linear problems. The 2D linear system earlier can be written using a matrix as:

$$\begin{pmatrix} 4 & 6 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ -10 \end{pmatrix}$$

In plain speak: a known thing called a matrix multiplies an unknown thing called a vector, which has to be equal to a known vector. It generalizes the simple 1D problem mentioned earlier. The solution is now

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 3 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ -10 \end{pmatrix}$$

This seems more posh than the usual high school derivation. The caveat of course is: what is the inverse of a matrix? For a number which is a special case of a matrix it is just one over the number as long as that number is not zero. Degenerates cases like that are a pain, but there are known ways to deal with them without anyone getting hurt.

Computing inverses of matrices is crucial to animating fluids. We will see that in practice you do not have to *explicitly* compute the inverse.

We have just replaced common numbers with matrices and vectors. Mathematicians take the liberty of treating them as another type of *number*. They are not the familiar numbers we are used to, but you can multiply them and invert them just like our familiar numbers.

Why stick to only two unknowns and two by two apartment buildings? This approach also works for 3D equations: in this case our unknowns are x, y, and z. What about bigger systems? We seem to be running out of characters. We could use the other roman characters but we are left with only 23 of them. Dr. Seuss in another genius book called *On Beyond Zebra* introduced other symbols coming out of his imagination to extend the roman characters. But even Dr. Seuss was running out of symbols. Mathematicians borrow heavily from Greek symbols and Hebrew symbols. But that is still not enough.

The way out of this conundrum is to *label* a single character. Let's use the Arabic *thing* character x, which is the most popular character for an unknown that we are trying to figure out from known things. A label is just a way to distinguish between identical looking things. Imagine being in a former communist country where every shampoo bottle has the same shape. Which one would you buy? No advertisements back then. Well you would check out the labels. That is the same for our unknowns. Each x now has a label attached to it. Usually, it works as follows: "thing one," "thing two," thing three," "thing four," "thing five," "all the intermediate things," "the size of my linear system."

Actually, when we discussed diffusion if you remember we already introduced the label notation.

But let me repeat it. Repetition is a good way to understand stuff. At least that works for me.

Instead of (x, y) we write $\mathbf{x} = (x_1, x_2)$. We can generalize this and now we can deal with big systems of equations. Let's say the size is $N = 1{,}000{,}234{,}000{,}120$ then our vector of unknowns can be written as follows:

$$\mathbf{x} = (x_1, x_2, x_3, x_4, x_5, \ldots, x_N).$$

Mathematicians are lazy but they are smart and that is why they can be lazy. I did not write out all 1,000,234,000,120 labeled things.* The "…" does the trick. It means that you are smart enough to figure it out, you can see the pattern. If you are bored you can always complete the entire sequence. But why bother? In fact I could have listed only two things separated by the "…": the first one has the label "1" and the last one has the label "the size of my linear system."

Now that we know how to label vectors of any size how do we label matrices? If you think of a matrix as an apartment building it is pretty obvious that you need the floor number and the apartment number on that floor. That is two numbers. *Bingo!* We can write a matrix using two labels. Also our apartment buildings are square. No hipsters are allowed in. Each floor has as many apartments as the number of floors.

Therefore our matrix of size N can be written as follows:

$$\mathbf{A} = \begin{pmatrix} a_{11} & \cdots & a_{1N} \\ \vdots & \ddots & \vdots \\ a_{N1} & \cdots & a_{NN} \end{pmatrix}.$$

Similarly, the known vector on the right-hand side can be written as (it should be a column vector really):

$$\mathbf{b} = (b_1, \ldots, b_N).$$

* My editor would be happy since I would then meet the quota of pages that I promised to write.

And here comes the expression for a general linear system:

$$\mathbf{A}\,\mathbf{x} = \mathbf{b}.$$

How do you multiply a vector by a matrix? It should be obvious from the simple 2D example.

But here is the rule just in case:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11}\ x_1 + \ a_{12}\ x_2 \\ a_{21}\ x_1 + \ a_{22}\ x_2 \end{pmatrix}.$$

The matrix multiplies a vector in order to produce another vector.

That is how simple linear systems are.

And it works for any size.

And the solution is readily available:

$$\mathbf{x} = \mathbf{A}^{-1}\,\mathbf{b}.$$

That is it.

Of course that is not it. We still have to compute the inverse of the matrix, we need another *Intermezzo*.

To summarize: Matrices and vectors are useful to solve linear problems. Matrices are kind of like numbers as you can add, subtract, multiply, and sometimes divide them. There is one crucial difference, however; that is, that matrices do not always **commute**: **AB**–**BA** is not always equal to zero.*

7.2 INTERMEZZO DUE: THE GENERAL SOLUTION OF A LINEAR SYSTEM

> [I was advised] to read Jordan's *Cours d'Analyse*; and I shall never forget the astonishment with which I read that remarkable work, the first inspiration for so many mathematicians of my generation, and learnt for the first time as I read it what mathematics really meant.
>
> GODFREY HAROLD HARDY (FAMOUS ENGLISH MATHEMATICIAN)

* Heisenberg used this result to derive his uncertainty principle. The funny fact is that Heisenberg did not know that his operators were in fact infinite-dimensional matrices. He did not know about matrices! Fellow mathematicians, when he showed his results to them, they were like: "Dude those are matrices."

If a Jordan canonical form of a matrix is familiar to you than you can skip this *Intermezzo* and go to the washroom.

This *Intermezzo* starts with a bombastic statement.

> There is a fundamental result in mathematics that says that every matrix can be put in a *Jordan canonical form*. This result is named after the French mathematician, Camille Jordan (1838–1922), not after a country in the Middle East.

This result basically says that

> For each matrix there is a linear transform that will put it into a much simpler form.

It is vague but it captures the gist. We will make it more concrete.

Let me explain this concept with a simple example of an ellipse living in the plane. Ellipses can have various shapes and orientations as shown in Figure 7.2a. The circle is a special case. In general, an ellipse is defined by the lengths of its principal directions and also by a rotation angle.

In the bottom of Figure 7.2, we show the same ellipse represented using different reference frames illustrated by the two arrows in the center of the ellipse. If the arrows are aligned with the principal axes of the ellipses (Figure 7.2b, right) then the description of the ellipse is much simpler.

The two descriptions are equally valid. Just one of them is simpler. Simple is good because it involves less boring math and more fun math.

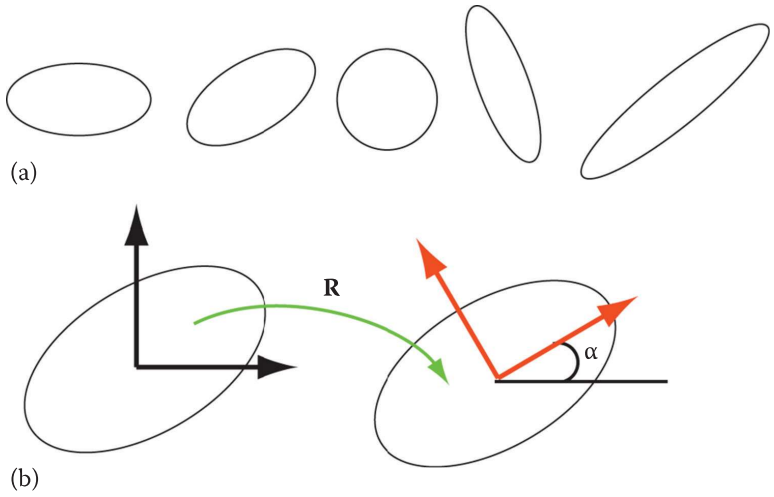


FIGURE 7.2 Examples of ellipses in the plane (a) and two descriptions of the same ellipse (b).

How do you describe an ellipse? There are really many ways. But let's take the example of a circle of unit length first which is a special case of an ellipse. In this case the two principal axes have identical lengths: equal to one in this case. The unit circle can be defined as the set of all points that are at a distance of one from the center of the circle. If x and y are the coordinates of a point in the plane, then according to Pythagoras the following identity has to hold:

$$x^2 + y^2 = 1$$

What does this have to do with matrices?

Well we can alternatively write this relationship as follows:

$$(x \quad y)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 1$$

There you go: here is a lucky boring matrix squeezed in between two unknown vectors. They are identical twins actually: one is lying down and one is standing up straight.

In fancy vector notations this becomes

$$\mathbf{x}^T\ \mathbf{A}\ \mathbf{x} = 1.^*$$

In the case of the circle, the matrix is called the *identity matrix*. It is the equivalent of the number "1" for matrices. Hence the name, you multiply a vector by the identity matrix and you get the same vector back. For numbers, for example, we have the obvious fact that $1 \times 2345 = 2345$. This is the reason that 1 is not considered a prime number by the way. It would contradict the "unique prime decomposition theorem."

Any number multiplied by one gives us back the same number. I think most people are taught that in grade school.

---

* What about that "T" superscript? It is used to denote the transpose of a vector. This is a matter of convention. I like my vectors to stand upright by default: a column of numbers. To get a row of numbers you just rotate your tall and thin vector by −90°. And *voila!* You get the transpose of a vector: a vector lying on its back. Just to make it clear what this equation says

$$(x \quad y)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (x \quad y)\begin{pmatrix} x \\ y \end{pmatrix} = x^2 + y^2$$

For noncircular ellipses the corresponding matrix is not as simple. However, it can be made simple through a particular linear transformation: a rotation in the plane.*

We will spend a little bit more time on this problem.

This exercise shows the beauty of abstraction in mathematics.

It is the art of replacing concrete things by nonsensical abstract symbols that have to satisfy certain rules. You apply these rules to these nonsensical symbols and you get results. Then these results can be applied to a variety of concrete problems. That is the power of abstraction. Recall the Arabic origin of the word *Algebra*: it is a "reunion of broken parts."

The general equation of an ellipse in the plane is

$$A x^2 + 2B x y + C y^2 = 1$$

In matrix form this can be written as

$$(x \quad y)\begin{pmatrix} A & B \\ B & C \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 1$$

Our matrix is now not as simple as in the case of the unit circle. The equation stands for any ellipse after all. But the matrix is special, it is *symmetrical*. What that means is that if you take the mirror image of the matrix along its left to right diagonal you get the same matrix. More concretely

$$\begin{pmatrix} A & B \\ B & C \end{pmatrix} = \begin{pmatrix} A & B \\ B & C \end{pmatrix}^T.$$

This is not always the case. Consider the following nonsymmetrical matrix:

$$\begin{pmatrix} 3 & 6 \\ 10 & 1 \end{pmatrix} \neq \begin{pmatrix} 3 & 10 \\ 6 & 1 \end{pmatrix}.$$

---

* Here is the exact expression if you need to know: $\mathbf{R}(\alpha) = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{pmatrix}$.

In Figure 7.2b, we show how to put our matrix in a simpler form. All we have to do is to find the rotation matrix that transforms the black frame on the left into the red frame on the right. The right frame is nicer since it is aligned with the principal axes of the ellipse. All we are doing is just changing our frame of reference. The ellipse has not changed. We are just looking at it in a different way.

We just changed the description of something without changing it.

The new coordinates are obtained from the old ones through the rotation transformation:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \mathbf{R}(\alpha) \begin{pmatrix} x \\ y \end{pmatrix}$$

In these new coordinates the equation of the ellipse becomes

$$\begin{pmatrix} x' & y' \end{pmatrix} \begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = 1$$

And guess what? The numbers a and b are the lengths of the principal axes of the ellipse. So, what we have done through a rotation is to simplify the ellipsoidal matrix to a *diagonal* matrix. This is an apartment building that is empty except for the rooms that have the same number as the floor it is on. Lots of empty rooms, everyone will sleep well.

Hence, for our ellipsoidal matrix we have that

$$\begin{pmatrix} A & B \\ B & C \end{pmatrix} = \mathbf{R}^T \begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{pmatrix} \mathbf{R}$$

This is where the beauty of abstraction kicks in. We can write this result for any symmetrical matrix **A**.

There is a transformation **R** and diagonal matrix **D** such that

$$\mathbf{A} = \mathbf{R}^T \mathbf{D} \mathbf{R}$$

Well we just showed that our ellipse matrix is equivalent to a diagonal matrix. We just changed how we view things and *voilà!*

We used ellipses just to visualize the transformation and the subsequent simplification.

We now present a higher-dimensional example. Like this matrix of size 10×10:

$$\begin{bmatrix} 2 & -1 & 4 & -3 & 6 & 7 & 8 & 2 & 5 & -2 \\ -1 & 3 & -7 & 8 & 2 & -3 & -5 & 1 & 0 & 4 \\ 4 & -7 & 1 & 5 & 9 & 4 & -4 & -3 & 8 & -1 \\ -3 & 8 & 5 & 7 & -4 & 1 & 5 & 3 & 7 & 3 \\ 6 & 2 & 9 & -4 & 5 & -6 & 7 & -8 & 1 & -4 \\ 7 & -3 & 4 & 1 & -6 & -1 & 4 & 6 & 0 & 2 \\ 8 & -5 & -4 & 5 & 7 & 4 & 6 & 1 & 4 & 0 \\ 2 & 1 & -3 & 3 & -8 & 6 & 1 & -4 & 2 & 6 \\ 5 & 0 & 8 & 7 & 1 & 0 & 4 & 2 & 1 & -3 \\ -2 & 4 & -1 & 3 & -4 & 2 & 0 & 6 & -3 & -8 \end{bmatrix}$$

There exists a linear transformation that will turn this matrix into the following simpler diagonal form:

$$\begin{bmatrix} 1.6362 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8.9481 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 13.930 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 19.872 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 25.620 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3.0797 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -7.8420 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -12.452 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -13.986 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -20.646 \end{bmatrix}$$

To compute this matrix I used the symbolic math package called *Maple*. I love these symbolic packages as they take care of the boring math. If a computer can do the math than you know it is boring math. My contribution was to make sure that the 10×10 matrix is indeed symmetrical. That's it. Then I just told Maple to compute the *eigenvalues*

of the matrix, and Bingo, it returned the values on the diagonal of the matrix. I truncated them too. The numbers on the diagonal are not the exact values.

Eigenvalues? *Eigen* is German for *self* or *own*. Not that that is a very helpful fact to know. These values are specific to the matrix and are owned by it. Not really. Different matrices can have the same eigenvalues. But they are basically the same as in the case of our ellipse. However, they have different eigenvectors.

Why bother with this math involving German-sounding names and transformations? Well it can help you to compute the inverse of a matrix. Inverting a diagonal matrix is just like inverting numbers. All you have to do is invert every number on the diagonal. For example:

$$\begin{pmatrix} 14 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -6 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{14} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{6} \end{pmatrix}.$$

The beauty is that solving for inverses is now as simple as the case of a single number. Of course we have not mentioned how to compute this magical transform that takes our dense matrix to the nicer diagonal matrix. The Maple software took care of that for our 10D linear system. To compute the transform usually involves computing *eigenvectors*. We are not going to dwell too much on this subject. I just wanted to mention that through clever mathematics any linear system can be brought into a simpler linear system. This all works out nicely if the magical transform can easily be computed. That is the key. We will see that for fluids living on donuts the transform is well known and it is called the *Fourier transform*.

Everything I said is valid for symmetrical matrices of any size, not only 3×3 or 10×10.

What about arbitrary matrices that are not symmetrical?

Camille Jordan, the French mathematician I mentioned at the beginning of this section, took care of that. He proved that for any arbitrary matrix you can find a linear transformation that decomposes the matrix into a bunch of *Jordan blocks*.

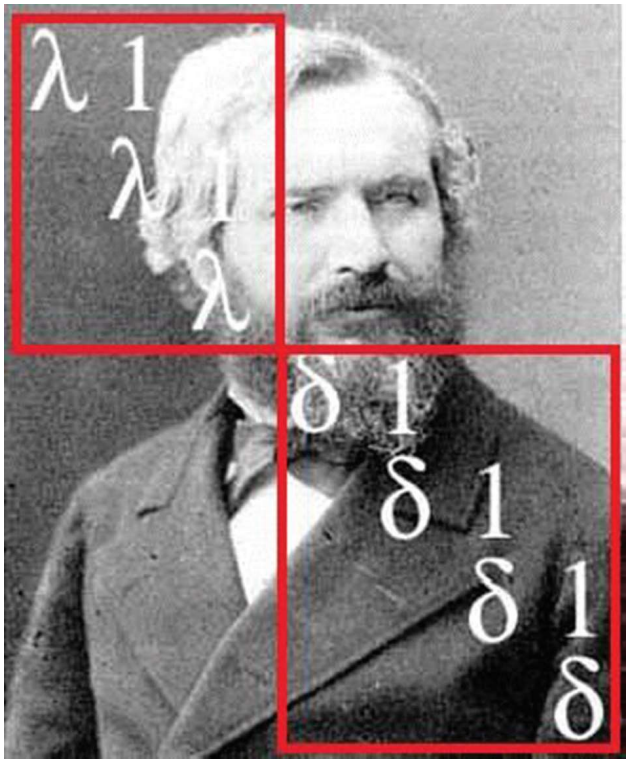


FIGURE 7.3 Jordan (in background) with two Jordan blocks (foreground). In this case, there are two blocks and they are bounded by red rectangles. One has a size of 3×3 and another has a size of 4×4. (Modified from Public Domain.)

Figure 7.3 shows a picture of Camille Jordan with two blocks superimposed on his face representing some 7×7 matrix. In this case there are two blocks. The first block is of size 3×3 and the second one is of size 4×4. Notice that the matrix is in general almost diagonal except for the pesky number ones on the heads of some of the diagonal numbers.

Jordan's achievement is monumental.

It is amazing that all matrices can be reduced to this simple form. Symmetrical matrices are a special case where all Jordan blocks have a size of 1×1. No one has a pesky one on their head in this case.

That is it for linear systems then, right?

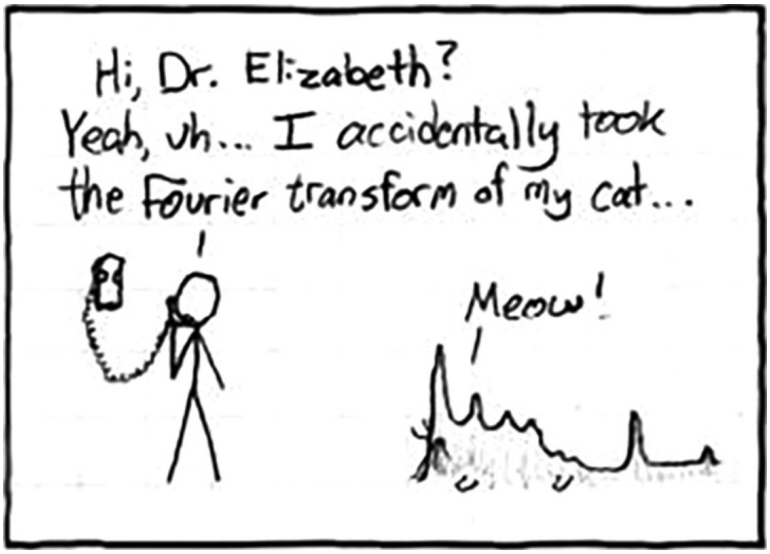
No it is still not it.

In theory, yes, but in practice we do not normally compute the Jordan canonical form for linear systems. Not for fluid animation at any rate. But it is very cool to know that this result is rock solid and has a mathematically rigorous proof.*

* I have used it in some of my other research on *subdivision surfaces*. I used the fact that taking powers of matrices is the same as taking powers of the Jordan blocks, which are much simpler to compute.

To summarize: Any matrix can be put into a simpler form. A diagonal matrix for symmetrical matrices and for general matrices is the Jordan canonical form: blocks with ones on their heads. This can help to invert and understand matrices.

7.3 INTERMEZZO TRE: CIRCULANT MATRICES AND THE FOURIER TRANSFORM



CREATED BY RANDALL MUNROE CHECK OUT HIS STUFF AT HTTP://XKCD.COM

Before we move on let me describe a beautiful result that will help us solve the motion of fluids on donuts. Remember those pesky bugs crawling all over your yummy morning donut.

This involves the all mighty *Fourier transform*. If you are already familiar with this most well-known transform of all transforms you can skip this *Intermezzo* and go stretch your legs.

It is not usually explained in the following manner, at least not to engineers I believe. This is how it was roughly explained in one of my Abstract Algebra classes 27 years ago. This is from fuzzy memory.

The Fourier transform on the other hand is heavily used by engineers.

We have to know about things called *roots of unity* first. This can be explained with algebraic things or using geometrical figures.

Let's start with the algebraic approach involving things. In this case the problem becomes: "which things, when raised to some given power are

equal to one." Warning: there are going to be a lot of *ones* in the following paragraphs.

The simplest case is

$$z = 1.$$

Wow, that was easy. Only one solution: only one is equal to one. It seems almost absurd to have an equation for such a simple fact, but to some people it looks impressive.

Next we can try

$$z^2 = 1.$$

Now we have two solutions: $z = -1$ and $z = 1$. Minus one times minus one is equal to one and one times one is equal to one. That was kind of easy too.

Alright, what about

$$z^3 = 1.$$

Now it gets more interesting. We should have three solutions right? The number one is of course one of them since one times one times one times one equals one. What about the two other things? It is not going to be minus one this time because minus one times minus one times minus one is equal to minus one, not one. Got that?

We are stuck only if we consider the numbers that we are used to. The way out of this trap is to just add another dimension. We extend our numbers stuck on a 1D line into 2D numbers. This is just like the vectors in the plane we described earlier. Two-dimensional numbers are really the sweet spot for these types of problems. Three-dimensional numbers are useless. But 4D numbers called *quaternions* are useful to describe rotations and other stuff. Game developers use them, for example. We do not need them in this book. The basic relations of quaternions were engraved on Broome Bridge in Dublin, Ireland, by Hamilton when he discovered them on a stroll through the city. Hamilton was an Irish mathematician. Check it out when you happen to be in Dublin. Not your usual tourist attraction. Most people go to Temple Bar or the Guinness Brewery. But check it out nevertheless if you get a chance.

Generally, the 2D numbers we need are called *complex numbers* or *imaginary numbers*. In my opinion, these are badly chosen names for these numbers because they actually make everything simpler, not more complex. There is nothing imaginary about them either. They should be called *awesome numbers* instead.

I am not going to present the basic theory of awesome numbers (I mean *cough!* complex numbers) as there are thousands of excellent books that do a wonderful job explaining why they are so awesome. I encourage readers to look it up. This is a book about fluid animation after all not about awesome numbers. But awesome numbers can help to animate awesome-looking fluids on donuts.

I am just going to stick to awesome numbers (vectors) that are of length one: the vectors centered at the origin whose tip lies on a unit circle. In the previous examples, we had our unknown thing z raised to the power one, power two, and power three and we wanted to find the unknown things when raised to these powers to be equal to the awesome number one. The awesome number one in the awesome system of numbers is the vector of length one that lies on the horizontal line and points to the right.

Geometrically for a given power n the solutions are all the vectors that are a multiple of the angle of 360° divided by n. So for $n=1$ it is just the vector of length one that points in the positive direction because a rotation of 360° will bring it back to itself. For $n=2$ we of course have the usual vector of length one but also the vector in the opposite direction because it is a rotation of 180 which is half of 360.* So far, our solutions live in the familiar 1D numbers domain. But they are embedded in the 2D space of awesome numbers. They are a subset. Most awesome numbers eschew the lowly 1D numbers stuck on their infinite line.

For $n=3$ it gets more interesting. Now our rotation angle is 360°/3° = 120°. Again we have our vector of unit length but also the rotated ones. They are depicted on the left side of Figure 7.4. Next to the $n=3$ (cubic) case, we also depict the solutions for $n=4$, 5, 10. The pattern is pretty obvious. The solutions are evenly spaced like the spokes on a bicycle wheel. In each case, the red vectors can be brought to align with the black one by rotating them counterclockwise a number of times with an angle equal to 360/n. The "number of times" is a number between zero and n–1.

* Those of you who paid attention so far will recall that this is why Euler's beautiful formula is true.

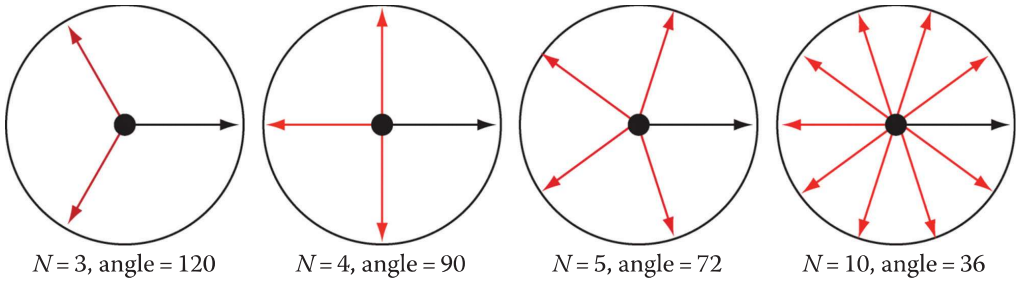


FIGURE 7.4 Roots of unity for degrees 3, 4, 5, and 10.

What is the connection between raising some number to a given power and the roots of unity?*

To take the power of a unit awesome number you just multiply its angle with respect to the awesome one vector times the power. Remember an awesome unit vector is defined by a rotation angle with respect to the awesome one number.

Let us do it for the power three. Remember I am a *freak* and I like to explain things with simple examples.

$$3 \times 0 = 0$$

$$3 \times 120 = 360 = 0$$

$$3 \times 240 = 720 = 0.$$

This seems like crazy math but really that is not the case. Remember these angles are like bicycle spokes or the hands on your wristwatch.

They rotate and circulate.

They eventually end up at the same spot. Not like usual numbers that run off to infinity. *Hilbert's infinite hotel* trick doesn't apply in this case. "Sorry we are full we cannot move guests around because we are fully booked because this is a finite hotel." Down the street you will find Hilbert's infinite hotel which has cheaper rates. Hilbert's hotel always has a vacant room. All you have to do is move the guest from room 1 to room 2, move the guest from room 2 to room 3, move the guest from room 3 to room 4, and so on. This is a bit of a pain for the guests but hey the rates are cheap. Pack lightly when you intend to check in at "Hilbert's infinite hotel."

* For those of you who know this kind of math it is because: $(e^{i2\pi j/n})^n = e^{i2\pi j} = 1$. With $j = 0, \ldots n-1$.

The wristwatch is a good analogy. When you adjust your watch for 12 hours you end up with the same time. When I travel to China, I do not have to adjust my watch since there is a 12-hour time difference between China and Toronto. I mention China, not a specific city or region because they are all on the same time zone.

It does not help to get over your jet lag however. The Air Canada flight from Beijing to Toronto leaves at 6 p.m. and gets you back at 6 p.m. the same day.

In Figure 7.4 (left), the black vector needs to be rotated zero times by 120°, and the first red vector needs to be rotated counterclockwise two times by 120° to align it with the black one. The second red vector has to be rotated only counterclockwise once by 120° to align it with the black one.

Just from visually inspecting Figure 7.4, one can notice some interesting properties of the roots of unity. Every root has a partner root, which is its reflection with respect to the horizontal axis. In this ballroom you are also allowed to be your own partner. Awesome one is allowed to dance with itself. If the degree n is an even number like 4 and 10 then every root has a partner pointing in the opposite direction. All these facts can be rigorously proved. But they are pretty self-evident from Figure 7.4.

Also their total sum is always equal to zero. For an even number of roots this is pretty obvious: every root cancels out his/her partner root. For odd number roots it works out as well.

Keep these roots of awesome one in mind.

We will now turn to circulant matrices and later bring back our waltzing roots of unity to join the party.

There is a good reason why these matrices are called circulant and they are intimately related to our roots of unity that live on a circle.

To construct circulant matrices we start with a vector of any size. To clear any confusion I have to state that awesome numbers can be represented by 2D vectors. The vectors that are multiplied by the circulant matrices on the other hand can be vectors of arbitrary size that sometimes contain awesome numbers.

Just to make it more concrete we will start with a vector of size three:

$$(a,b,c).$$

The numbers a, b, and c can have arbitrary values.

One example could be

$$(1, -40, 1024).$$

From this vector we can create what is usually called a *circulant matrix*. The way you construct such a matrix is to fill the rows with a right-shifted version of the vector to create a matrix. In our example, you get the following 3×3 matrix:

$$\mathbf{A} = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}.$$

Figure 7.5 shows how to construct the circulant matrix from a vector. The figure uses the same pattern of unitary roots as shown in Figure 7.4 in the case of three roots. This time we label each root with the numbers stored in the vector.

This is how it works.

We fill in each row of the matrix counterclockwise from each circle starting from the awesome one. Subsequently, we rotate the labels counterclockwise and fill out the remaining four rows similarly. Hopefully, this process is clear from comparing Figure 7.5 and the circulant matrix earlier.

It is pretty easy to spot the pattern in the matrix as well: identical numbers flowing diagonally from left to right.

Let us go back to the roots of unity for a power of three. These roots can be labeled from the second root denoted by ω. This root raised to any power

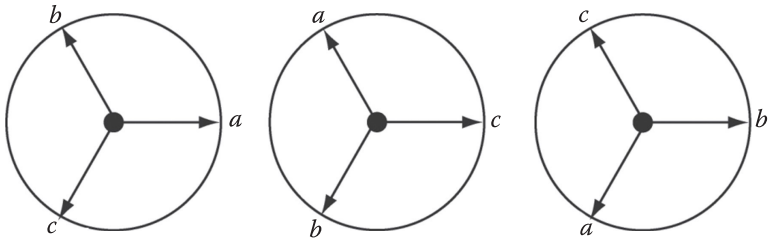


FIGURE 7.5 The rows of a 3×3 circulant matrix are obtained by counterclockwise rotations.

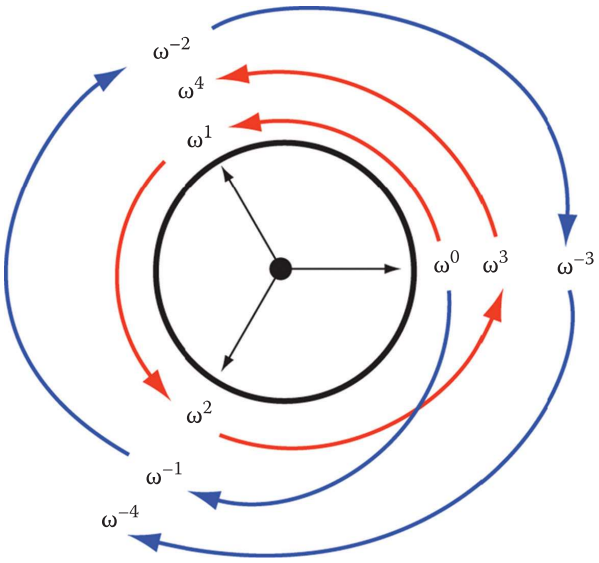


FIGURE 7.6 All the roots of unity for the case of a power 3 can be obtained by taking powers of the second root denoted by ω^1.

in the awesome system of numbers gives us all the roots. In fact $\omega^0 = 1$: returns the awesome number one which is the first root. In Figure 7.6, we show that all the roots of unity can be obtained by taking the powers from the second root of unity.

This root is usually denoted by the symbol ω. It comes from the Greek alphabet and is pronounced *omega*. It also looks pretty. Taking powers of this pretty symbol gives us all the roots of unity as shown in Figure 7.6. If we follow the red arrows we get all the roots counterclock wise. On the other hand, if we follow the blue arrows we get all the roots clockwise. The only difference is that in one case we take positive powers (red) while in the other case (blue) we take negative powers. This can go on forever.

Notice that $\omega^3 = 1$. This is the case for all powers that are multiples of three.

Therefore, all the following numbers are all equal to the awesome number one.

$$1 = \ldots, \omega^{-6}, \omega^{-3}, \omega^0, \omega^3, \omega^6, \ldots$$

Another important property is that the sum of the roots of unity is equal to zero. Geometrically, if you connect all the roots of unity you get a regular polygon. In this case it is a triangle. But it works for all other cases as well.

Looking at Figure 7.4 we can easily construct regular polygons that connect the roots of unity: we get a triangle (3), a square (4), a pentagon (5), and a decagon (10). For the general case of n roots you get what mathematicians call an *n-gon*. No fancy Greek this time.

The center of mass of an arbitrary regular polygon is clearly zero

Mathematically, for our degree three example we have that

$$1+\omega+\omega^2=0.$$

Now here enters the magic.

For circulant matrices the transform, which by the way is also a matrix, is constructed as follows for the 3×3 case:

$$\mathbf{F}=\frac{1}{\sqrt{3}}\begin{pmatrix}1 & 1 & 1\\ 1 & \omega & \omega^2\\ 1 & \omega^2 & \omega^4\end{pmatrix}.$$

Using the properties of the roots shown in Figure 7.6 and by following the red arrows we can rewrite this matrix as follows:

$$\mathbf{F}=\frac{1}{\sqrt{3}}\begin{pmatrix}1 & 1 & 1\\ 1 & \omega & \omega^2\\ 1 & \omega^2 & \omega\end{pmatrix}.$$

For circulant matrices this magical transform will make it diagonal. The transform takes us into the nice clean space where the matrix is diagonal. But how do we get back to the messy hood? We replace every power of the root ω by its inverse. We now follow the blue arrows in Figure 7.6. We get that

$$\mathbf{F}^{-1}=\frac{1}{\sqrt{3}}\begin{pmatrix}1 & 1 & 1\\ 1 & \omega^2 & \omega\\ 1 & \omega & \omega^2\end{pmatrix}.$$

Sanity check: Let's verify that the transform times the inverse gives us the identity matrix. Remember the identity matrix? That is the matrix that has only 1s on the diagonal.

Here is the math:

$$\mathbf{F}^{-1}\,\mathbf{F} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}\frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 3 & 1+\omega+\omega^2 & 1+\omega^2+\omega \\ 1+\omega^2+\omega & 3 & 1+\omega+\omega^2 \\ 1+\omega+\omega^2 & 1+\omega^2+\omega & 3 \end{pmatrix} = \begin{pmatrix} \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} \end{pmatrix}.$$

All those off-diagonal identical terms are equal to zero because the roots of unity add up to zero. Recall the regular polygons and the corresponding algebraic relation:

$$1+\omega+\omega^2 = 0.$$

All is good. We now have a recipe to compute the corresponding diagonal matrix **D** from the circulant matrix **A**.

To simplify the math we use the result that every 3×3 circulant matrix can be written as

$$\mathbf{A} = a\,\mathbf{I} + b\,\mathbf{S} + c\,\mathbf{S}^2.$$

This is just a way of using algebra to rewrite the circulant matrix. The **S** matrix just models a shift. In fact it is equal to

$$\mathbf{S} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \text{ and therefore } \mathbf{S}^2 = \mathbf{S}\times\mathbf{S} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Go ahead and verify this fact at home.

The important point here is that this result is true for any vector (a,b,c). From some simple algebra, we conclude that the corresponding diagonal matrix is equal to

$$\mathbf{D} = \mathbf{F}^{-1}\,\mathbf{A}\,\mathbf{F} = a\,\mathbf{F}^{-1}\mathbf{F} + b\,\mathbf{F}^{-1}\,\mathbf{S}\,\mathbf{F} + c\,\mathbf{F}^{-1}\,\mathbf{S}^2\,\mathbf{F}.$$

I made it simple by considering only small 3×3 matrices. For larger systems you have to take more powers of the matrix **S**.

For the 3×3 case when working out the math we get that

$$\mathbf{D} = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} + \begin{pmatrix} b & 0 & 0 \\ 0 & b\omega & 0 \\ 0 & 0 & b\omega^2 \end{pmatrix} + \begin{pmatrix} c & 0 & 0 \\ 0 & c\omega^2 & 0 \\ 0 & 0 & c\omega \end{pmatrix}$$

$$= \begin{pmatrix} a+b+c & 0 & 0 \\ 0 & a+b\omega+c\omega^2 & 0 \\ 0 & 0 & a+c\omega+b\omega^2 \end{pmatrix}.$$

That is it.

There are our magical transformations and the resulting diagonal matrix.

I know this has been a bit heavy on the cool math. However, the only numbers we used were 1, 2, 3, and $\sqrt{3}$. All the rest was algebra that applies to many more numbers, even awesome ones. This is the kind of math I love. And perhaps I have spent a little too much space on explaining the discrete Fourier transform. The theory is beautiful and very useful at the same time: a double whammy.

To summarize: The Fourier transform provides the magical transformation from circulant matrices to diagonal matrices. The Fourier transform and its inverse can be written explicitly and elegantly in closed form. We presented it for the special case of 3×3 matrices. But the theory can be extended in a straightforward manner for larger systems. In fact it works for any circulant matrix of any size. How cool is that?

7.4 INTERMEZZO QUATTRO: NUMERICAL SOLUTION OF LINEAR SYSTEMS

Mathematics is the science of what is clear by itself.

CARL GUSTAV JACOB JACOBI (GREAT GERMAN MATHEMATICIAN)

When I have clarified and exhausted a subject, then I turn away from it, in order to go into darkness again.

CARL FRIEDRICH GAUSS (FAMOUS GERMAN MATHEMATICIAN)

I will try to make this a short *Intermezzo* as there are thousands of excellent books on this topic that explain this material really well. My hope is to give enough information in order for the reader to understand some of the code as follows.

To reiterate: if you are familiar with this material skip this *Intermezzo* and get a drink. Take a break. I recommend a Chinotto or a Campari on the rocks.

Thus far, we have seen that any matrix can be transformed in a simpler form involving Jordan blocks. For circulant matrices this transformation is explicitly given by the discrete Fourier transform.

Now welcome to the messy world of numerical matrices. Welcome to downtown and make sure you carry a computer that is password protected.

Numerical matrices contain computer numbers. Computer numbers are not like *ideal* mathematical numbers. They are finite. They have finite precision. They have to because the computer can only handle a finite number of bits. Remember? Numerical numbers are tricky. But I will not dwell on this topic in this book. Someone ought to write a book on this subject and enlighten us all once and for all on this tricky stuff. In my experience, you deal with them like you deal with strangers whose language you barely speak. You get used to them, learn from their behavior, and you try not to get in trouble.

There are many packages commercial and free for all that manipulate numerical matrices. These packages can compute fun things like the German eigenvectors and eigenvalues, but also the inverses, transposes, pseudo-inverses, Russian Choleski decompositions, and what not for numerical matrices. Their output usually consists of numerical vectors, matrices, or even a single number as in the case of the determinant of a matrix. For example, if the determinant of your matrix is zero then you know that it is degenerate, it has a nontrivial null space. Or said differently, the matrix's kernel is nonzero. This is crazy talk. But that is how mathematicians talk and it makes complete sense if you know the lingo.

These numerical packages compute numerical unknowns from numerical knowns. Everything is done in the computer. No blackboards or whiteboards are involved and no black coffee.

Computer coders are great at naming these numerical packages. Here is a sample: BCSLIB-EXT, MA57, MUMPS, Oblio, PARDISO, SPOOLES, SPRSBLKLLT, TAUCS, UMFPACK, WSMP, and so on. This is for real.

Welcome to geek central.

I do not use any of these packages.* Not because I am lazy but because I am lucky. I like to write code instead of mashing various downloaded codes together and have them work with the latest Ubuntu Linux release, even having to rebuild the kernel in some cases. I have nothing against Linux by the way. It is just another hood to code in. I have written code there without getting hurt. But if you download code that was not written in your hood it can be painful.

Why am I lucky? This is why:

Linear systems in fluid animation are sparse.

We already observed in *Intermezzo duo* that any matrix can be transformed into a simple form, thanks to the French mathematician Camille Jordan. These matrices comprised of simple Jordan blocks are sparse. These matrices have very few elements that are not equal to zero. In fact they are almost vectors: quasi-vectors. It seems like a waste of space to write sparse matrices in matrix form. So much space wasted on lowly zeroes.

The linear systems in fluid animation are simple to start with. Their matrices contain mostly zeroes. They are sparse. They can be made sparser with Jordan's help. But this is not necessary. Computing Jordan blocks is a messy business unless you are lucky and you want to solve a fluid on a donut. In the latter case you can use the almighty Fourier transform and all your Jordan blocks are of size one.

Here is an example of a sparse matrix of size 10×10:

$$\begin{pmatrix} * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & * & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & * & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & * & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * \end{pmatrix}.$$

* I am such a liar. I used FFTW from MIT because it so cool. More about that in the following when we discuss bugs crawling on donuts.

The "*" stand for any number that is not zero, they visualize the sparseness. They do not stand for unknown things that we have to solve for. They are not exes you dumped or who dumped you. They are just a visual aid to figure out the structure of a matrix. The actual numbers are irrelevant as long as they are nonzero.

How do you measure sparseness? One measure of sparseness could be the number of "*" divided by the number of 0's? Consequently, this means that the following 20×20 matrix is sparser than the 10×10 matrix aforementioned.

$$\begin{pmatrix}
* & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \\
* & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * \\
* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & *
\end{pmatrix}.$$

Bottom line: sparseness just means you do not have to store your entire matrix in memory because most of the entries are zero. So, really in this case you are dealing with vectors not matrices. No one wants to waste memory on the zero minions.

Apart from sparseness, there is another reason why I am lucky. You can tell this from the two matrices just shown. The reason is that:

Matrices in fluid animation have very regular patterns.

Sparseness and regularity make it much easier to solve the linear systems encountered in fluid animation.

All sparse matrices are not necessarily regular. And all regular matrices are not necessarily sparse.

Here are two examples.

$$\begin{bmatrix} \text{sparse} \\ \textit{but} \\ \text{not regular} \end{bmatrix} = \begin{pmatrix} * & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 \\ * & * & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 \\ * & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & * \end{pmatrix}$$

$$\text{Vs} \begin{bmatrix} \text{regular} \\ \textit{but} \\ \text{not sparse} \end{bmatrix} = \begin{pmatrix} * & 0 & * & * & * & * & * & * & * & * \\ * & * & 0 & * & * & * & * & * & * & * \\ * & * & * & 0 & * & * & * & * & * & * \\ * & * & * & * & 0 & * & * & * & * & * \\ * & * & * & * & * & 0 & * & * & * & * \\ * & * & * & * & * & * & 0 & * & * & * \\ * & * & * & * & * & * & * & 0 & * & * \\ * & * & * & * & * & * & * & * & 0 & * \\ * & * & * & * & * & * & * & * & * & 0 \\ 0 & * & * & * & * & * & * & * & * & * \end{pmatrix}.$$

The bottom line is that when you have to wrestle with a sparse but nonregular matrix, you probably will be better off hiring a wrestler from a sparse matrix library. On the other hand, if you have to wrestle a nonsparse but regular matrix get a wrestler from your local mathematics department. For circulant matrices, you would have to go to Grenoble in France and ask to see *Monsieur* Fourier (never mind that he is dead). If on the other hand, no clever math dude can figure it out you will have to go to your local Walmart and buy a numerical dense matrix solver software package. The greeter at the entrance will point you to the right aisle. "It is right next to the gun and ammo on aisle 15."

However, in fluid animation we are lucky. Our matrices are both sparse and regular. Therefore, we can write our own code and stay away from Walmart. I have nothing against Walmart by the way; they have cheap black T-shirts that fit me.

Let us consider a simple example as usual to see how a sparse regular linear system can be solved.

The following is the depicted heat transfer in a 1D bar. We have two boundaries. The left-hand side is heated at a constant temperature of 1 (pick your unit) and the right-hand side is held at a constant temperature of 0 (using the same units as 1). Units do not matter in this example.

The setting is pretty straightforward. We have a bar that is heated at one end and is cooled at the other end and we want to know how the temperature evolves over time. Figure 7.7 depicts the situation. We know that eventually the bar will have a steady distribution of temperature. Not constant in this case because both ends are fixed with a fixed temperature.

This is a diffusion process. Remember those crawling bugs? In this case they are stuck on a line.

Let us assume that we discretize the problem as depicted in Figure 7.7 with 8 nodes. There is nothing special about the number eight. This example of course works for any number of nodes. Nodes 1 and 8 are fixed. The temperature of node 1 is fixed to be 1 and the temperature of node 8 is fixed to be 0. That leaves us with six unknowns: node 2, node 3, node 4, node 5, node 6, and node 7.

We are interested only in the final steady state of the temperature not the evolution of temperature over time.

The temperature is higher on the left-hand side of the bar than on the right-hand side of the bar. These values are fixed once and for all. We are interested in computing the six unknown values in between.

FIGURE 7.7 Heat flow in a 1D bar.

This is the corresponding linear system:

$$\begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The matrix is sparse and regular. Not that sparse in this case. But larger systems will become sparser and sparser since there will be more zeroes as was shown in the examples earlier. It is definitely regular. I think that is pretty clear.

The solution is actually known precisely in this case:

$$\begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} = \begin{pmatrix} \frac{6}{7} \\ \frac{5}{7} \\ \frac{4}{7} \\ \frac{3}{7} \\ \frac{2}{7} \\ \frac{1}{7} \end{pmatrix}.$$

The steady state is a linear interpolation between the left-end temperature 1 and the right-hand temperature of 0. Try to verify this at home by multiplying the known regular-sparse matrix with the previously unknown vector that is now known in order to get the known vector on the right.

Why bother trying to solve this problem numerically since we already know the answer?

In my experience, this is a good strategy when implementing computer code for solvers. Think of these as sanity checks. If your code does not

reproduce the sequence of six numbers for this problem, then there is something wrong with the implementation of your linear solver. Sorry, end of argument: your code is wrong. That is why mathematics is useful in writing halal code.

On the other hand, sometimes faulty code can result in amazing unexpected visual effects. In our business of computer animation we call this: "a bug has just turned into a feature."

Customer: "This is so cool!"

Coder: "Thanks. But it is wrong."

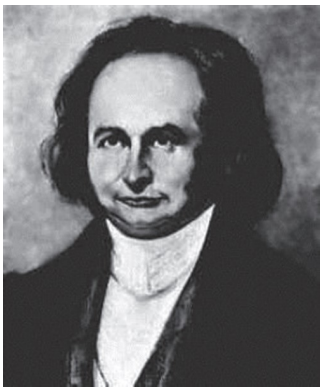
If the coolness of the bug does not break anything else in the software, we can always introduce a switch between the *cool behavior* and the *right behavior*. Turning the switch on or off is up to the user also known as the customer. Yes, a customer. Some people still pay for software, especially if it has cool bugs.

Geez and now you also have to find a name for your newly introduced switch. Like "Psychedelic Effect Number 4320" or "Psychedelic Chickens Sliding down Mount Everest." The guys writing the documentation for the software are like: what? A more boring name would be "New Incompressibility."

I am not a big fan of this approach. But if the faulty code turns out to reveal something new about how we think about a problem, then I am a big fan of this approach. Remember the motto: *if it looks good it is good*. On the other hand, I think we have to understand why it looks good. That is what my job in research is all about. If not you are just throwing dirt code on a wall and scraping off whatever sticks.

When working with computer numbers we solve these linear equations *iteratively*. We start with a guess and then refine it using many steps until we are satisfied that we are near the solution. Depending on the application this nearness condition can vary.

We will consider two techniques in this *Intermezzo*: the *Jacobi iteration* and *Gauss–Seidel iteration*. They are named after three German mathematicians: Carl Gustav Jacob Jacobi (1804–1851), Carl Friedrich Gauss (1777–1855), and Philipp Ludwig von Seidel (1821–1896). They are all depicted in Figure 7.8.



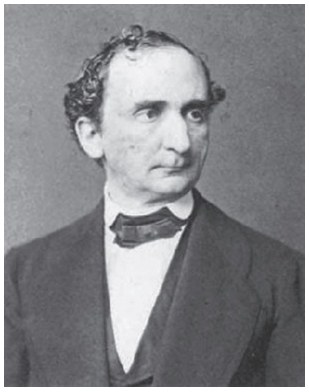


FIGURE 7.8 Jacobi, Gauss, and Seidel.

Just to clarify: *iteration* is the process of computing a sequence of vectors. Each vector is obtained from the previous one, starting with an initial vector $\mathbf{x}^{(0)}$. From that a sequence of vectors is obtained:

$$\mathbf{x}^{(0)} \to \mathbf{x}^{(1)} \to \mathbf{x}^{(2)} \to \mathbf{x}^{(3)} \to \mathbf{x}^{(4)} \to \mathbf{x}^{(5)} \to \mathbf{x}^{(6)} \to \mathbf{x}^{(7)} \to \mathbf{x}^{(8)} \to \mathbf{x}^{(9)} \to \cdots \to \mathbf{x}^{(\infty)}.$$

If all goes according to the game plan then we have in the end that

$$\mathbf{A}\,\mathbf{x}^{(\infty)} = \mathbf{b}.$$

How to obtain one iterate from the other is the art of solving linear systems iteratively. In fact, we only have to explain one step in this chain since the same rule applies to all steps separated by an arrow.*

Therefore all we have to explain is a single step:

$$\mathbf{x}^{(known)} \to \mathbf{x}^{(unknown)}.$$

Again: we are trying to get the unknown from the known. Jacobi and Gauss–Seidel have two different ways of doing this.

Jacobi updates the unknowns only using the knowns. Gauss–Seidel updates the unknowns from the knowns and the unknowns that have already been turned into knowns.

* This is not always the case for more sophisticated linear solvers.

Their behavior and implementation differs: Jacobi is easy to do in parallel and Gauss–Seidel eats up less memory and converges faster.

There are many ways of explaining these methods. Let us use an algebraic approach, first in cartoon matrix math and then making it explicit for our heated bar. The idea is to separate a matrix into three regions: "the diagonal," "the lower left," and the "upper right." Visually, it is something like this:

$$\begin{pmatrix} * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & * & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & * & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & * & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * \end{pmatrix} = \begin{pmatrix} * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 \end{pmatrix} + \begin{pmatrix} 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We can condense this into a more compact notation as follows:

$$\mathbf{A} = \mathbf{D} + \mathbf{L} + \mathbf{R}.$$

As shown earlier, the inverse of the diagonal matrix D is easy to compute. Just take the inverse of all the diagonal elements. This allows us to write

$$\mathbf{A}\,\mathbf{x} = \mathbf{D}\,\mathbf{x} + \mathbf{L}\,\mathbf{x} + \mathbf{R}\,\mathbf{x}.$$

To cut to the chase we will show how both Jacobi and Gauss–Seidel determine unknowns from knowns using this decomposition:

Jacobi:

$$\mathbf{x}^{(unknown)} = \mathbf{D}^{-1}(\mathbf{b} - \mathbf{L}\,\mathbf{x}^{(known)} - \mathbf{R}\,\mathbf{x}^{(known)}).$$

Gauss–Seidel:

$$\mathbf{x}^{(unknown)} = \mathbf{D}^{-1}(\mathbf{b} - \mathbf{L}\,\mathbf{x}^{(unknown)} - \mathbf{R}\,\mathbf{x}^{(known)}).$$

A subtle difference for sure, but it makes a big difference in practice. I wrote a simple piece of code that solves the heat transfer in a bar with eight nodes using the C++ language. This code tries to solve the linear heat equation up to three decimal places. In this case, the Jacobi approach requires 54 iterations to achieve that precision while Gauss–Seidel requires 29 iterations to achieve the same precision. The values of the nodal values are shown in Figure 7.9: on the left for Jacobi and on the right for Gauss–Seidel.

Clearly, Gauss–Seidel outperforms Jacobi.

However, Jacobi is easier to implement on parallel architectures like the GPU. Gauss–Seidel also has the problem that there is an order bias. If you reshuffle the order of the nodes you will get a different sequence on the right-hand side of Figure 7.9. This is usually not a problem when we seek a steady-state solution.

Here is the code that created the numbers in Figure 7.9 in its entirety.

```
#include <stdio.h>
const int max_iterations=10000000;
const double tolerance=1e-3;
const doublerhs[8]={0.0,-1.0,0.0,0.0,0.0,0.0,0.0,0.0};

static double abs (double a){
 return (a >= 0.0 ? a: -a);
}
static void initialize (double * a) {
 for (int i=0; i<8; i++) {a[i]=0.0;}
}
static void copy (double * a, double * b) {
 for (int i=0; i<8; i++) {a[i]=b[i];}
}
```

```
static double difference (double * a, double * b) {
 double d=0.0;
 for (int i=0; i<8; i++) {d += abs (a[i] - b[i]);}
 return (d);
}
static void iterate (double*a,double*b,double * c) {
 initialize (a); initialize (c);
 for (int k=0; k<max_iterations; k++) {
 printf ("%3d: ", k);
 for (int i=1; i<=6; i++) {
 a[i]=(rhs[i] - b[i-1] - c[i+1]) * (-0.5);
 printf ("%5.3f ", abs(a[i]));
} printf ("\n");
 double err=difference (a, c); copy (c, a);
 if (err < tolerance) {break;}
}
}
int main (int argc, char ** argv)
{
 double unknown[8], known[8];
 printf("\nJacobi iterations:\n\n");iterate(unknown,
known,known);
 printf ("\nGauss-Seidel iterations:\n\n"); iterate
(unknown, unknown, known);
 return (1);
}
```

We had to include the *stdio* header file so that our Slave can communicate the results to the **Human** or **Pet** through a **printf**. The header file is hood and device agnostic and will take care of all the messy details. It helps the **Compiler** (remember him?) to figure out how to generate ones and zeroes that instruct the Slave how to read and print stuff. The text will be printed pretty much in any hood and pretty much on any device. To nonexperts *stdio* sounds like a cheesy 1970s nightclub in Manhattan or a European hair gel. To the *cognoscenti* it stands for *standard input/output*.

Notice that we did not use a matrix type because of the *regularity* and the *sparseness* of our matrix. There are only vectors for (1) the right-hand side, (2) the known, and (3) the unknown. We padded the right-hand side of vector `rhs` left and right with a zero to make the inner loops simpler. So the vector is of size 8 not 6. The two added values correspond to the ghost values we mentioned earlier.

```
 0 : 0.500 0.000 0.000 0.000 0.000 0.000
 1 : 0.500 0.250 0.000 0.000 0.000 0.000
 2 : 0.625 0.250 0.125 0.000 0.000 0.000
 3 : 0.625 0.375 0.125 0.063 0.000 0.000
 4 : 0.688 0.375 0.219 0.063 0.031 0.000
 5 : 0.688 0.453 0.219 0.125 0.031 0.016
 6 : 0.727 0.453 0.289 0.125 0.070 0.016
 7 : 0.727 0.508 0.289 0.180 0.070 0.035
 8 : 0.754 0.508 0.344 0.180 0.107 0.035
 9 : 0.754 0.549 0.344 0.226 0.107 0.054
10 : 0.774 0.549 0.387 0.226 0.140 0.054
11 : 0.774 0.581 0.387 0.263 0.140 0.070
12 : 0.790 0.581 0.422 0.263 0.167 0.070
13 : 0.790 0.606 0.422 0.294 0.167 0.083
14 : 0.803 0.606 0.450 0.294 0.189 0.083
15 : 0.803 0.627 0.450 0.320 0.189 0.094
16 : 0.813 0.627 0.473 0.320 0.207 0.094
17 : 0.813 0.643 0.473 0.340 0.207 0.104
18 : 0.822 0.643 0.492 0.340 0.222 0.104
19 : 0.822 0.657 0.492 0.357 0.222 0.111
20 : 0.828 0.657 0.507 0.357 0.234 0.111
21 : 0.828 0.668 0.507 0.370 0.234 0.117
22 : 0.834 0.668 0.519 0.370 0.244 0.117
23 : 0.834 0.676 0.519 0.381 0.244 0.122
24 : 0.838 0.676 0.529 0.381 0.252 0.122
25 : 0.838 0.683 0.529 0.390 0.252 0.126
26 : 0.842 0.683 0.537 0.390 0.258 0.126
27 : 0.842 0.689 0.537 0.397 0.258 0.129
28 : 0.845 0.689 0.543 0.397 0.263 0.129
29 : 0.845 0.694 0.543 0.403 0.263 0.132
30 : 0.847 0.694 0.549 0.403 0.267 0.132
31 : 0.847 0.698 0.549 0.408 0.267 0.134
32 : 0.849 0.698 0.553 0.408 0.271 0.134
33 : 0.849 0.701 0.553 0.412 0.271 0.135
34 : 0.850 0.701 0.556 0.412 0.274 0.135
35 : 0.850 0.703 0.556 0.415 0.274 0.137
36 : 0.852 0.703 0.559 0.415 0.276 0.137
37 : 0.852 0.705 0.559 0.418 0.276 0.138
38 : 0.853 0.705 0.562 0.418 0.278 0.138
39 : 0.853 0.707 0.562 0.420 0.278 0.139
40 : 0.854 0.707 0.563 0.420 0.279 0.139
41 : 0.854 0.708 0.563 0.421 0.279 0.140
42 : 0.854 0.708 0.565 0.421 0.280 0.140
43 : 0.854 0.710 0.565 0.423 0.280 0.140
44 : 0.855 0.710 0.566 0.423 0.281 0.140
45 : 0.855 0.710 0.566 0.424 0.281 0.141
46 : 0.855 0.710 0.567 0.424 0.282 0.141
47 : 0.855 0.711 0.567 0.425 0.282 0.141
48 : 0.856 0.711 0.568 0.425 0.283 0.141
49 : 0.856 0.712 0.568 0.425 0.283 0.141
50 : 0.856 0.712 0.569 0.425 0.283 0.141
51 : 0.856 0.712 0.569 0.426 0.283 0.142
52 : 0.856 0.712 0.569 0.426 0.284 0.142
53 : 0.856 0.713 0.569 0.426 0.284 0.142
54 : 0.856 0.713 0.570 0.426 0.284 0.142
```

```
 0 : 0.500 0.250 0.125 0.063 0.031 0.016
 1 : 0.625 0.375 0.219 0.125 0.070 0.035
 2 : 0.688 0.453 0.289 0.180 0.107 0.054
 3 : 0.727 0.508 0.344 0.226 0.140 0.070
 4 : 0.754 0.549 0.387 0.263 0.167 0.083
 5 : 0.774 0.581 0.422 0.294 0.189 0.094
 6 : 0.790 0.606 0.450 0.320 0.207 0.104
 7 : 0.803 0.627 0.473 0.340 0.222 0.111
 8 : 0.813 0.643 0.492 0.357 0.234 0.117
 9 : 0.822 0.657 0.507 0.370 0.244 0.122
10 : 0.828 0.668 0.519 0.381 0.252 0.126
11 : 0.834 0.676 0.529 0.390 0.258 0.129
12 : 0.838 0.683 0.537 0.397 0.263 0.132
13 : 0.842 0.689 0.543 0.403 0.267 0.134
14 : 0.845 0.694 0.549 0.408 0.271 0.135
15 : 0.847 0.698 0.553 0.412 0.274 0.137
16 : 0.849 0.701 0.556 0.415 0.276 0.138
17 : 0.850 0.703 0.559 0.418 0.278 0.139
18 : 0.852 0.705 0.562 0.420 0.279 0.140
19 : 0.853 0.707 0.563 0.421 0.280 0.140
20 : 0.854 0.708 0.565 0.423 0.281 0.141
21 : 0.854 0.710 0.566 0.424 0.282 0.141
22 : 0.855 0.710 0.567 0.425 0.283 0.141
23 : 0.855 0.711 0.568 0.425 0.283 0.142
24 : 0.856 0.712 0.569 0.426 0.284 0.142
25 : 0.856 0.712 0.569 0.426 0.284 0.142
26 : 0.856 0.713 0.570 0.427 0.284 0.142
27 : 0.856 0.713 0.570 0.427 0.285 0.142
28 : 0.856 0.713 0.570 0.427 0.285 0.142
29 : 0.857 0.713 0.570 0.428 0.285 0.143
```

FIGURE 7.9 Jacobi output (left) and Gauss–Seidel output (right).

This code is easy to generalize to higher-dimensional systems. Try this at home. If you cut and paste this code and know how to compile C++ code in your hood you are all set. It is also easily turned into *pure* C.

To summarize: Very simple code can help you to solve a linear system. These systems are very common in science and it is amazing that such simple code can solve them, thanks to Jacobi, Gauss, and Seidel. Note you do not have to store the matrix explicitly. No space is wasted on useless zeroes. The code shown earlier demonstrates that.

End of story for the solution of linear systems.

Again sorry to disappoint: this is not the end of the story. These methods are simple but they do not converge to the solution very rapidly. It takes many steps to get close to the solution. Geeks like things that are rapid. No *slow food* for programmers. Actually, they have liquid protein mixes now for programmers so they don't waste time and chill out at a decent restaurant for lunch or dinner.

Sure Gauss–Seidel is better than Jacobi. But there is better stuff out there.

For example:

Conjugate Gradient, Preconditioned Conjugate Gradient, GMRES, Multi-Grid, Preconditioned Multi-Grid, and so on. These are all very cool techniques that I have used and they are covered in thousands of books.

Look this stuff up.

CHAPTER 8

A Simple Fluid Solver

> Simplicity is the ultimate sophistication.
>
> LEONARDO DA VINCI (THE ULTIMATE RENAISSANCE MAN)

It is now prime time to present a simple fluid solver that I wrote about 15 years ago. It is based on the techniques that I first crammed into a paper presented in 1999 at the annual SIGGRAPH conference. The story I like to tell, which is true by the way, is that it was the paper with the lowest scores that got accepted that year at the conference.

Allow me to reiterate.

In academia you submit a paper which then gets assigned to expert reviewers who in turn judge your paper, write a review, and give it a score. Two of these reviewers are part of a committee who make the final decision whether your paper is accepted or rejected at the conference. Since SIGGRAPH is the premier conference in computer graphics, it is very prestigious to get your paper accepted there. The acceptance rate is usually about 20% or 30% of all submitted papers. For academics, having papers accepted at SIGGRAPH is important in order to get a job or even better, a job for life at a university. This is called tenure. For real, a well-paid job for life. This is a kind of cynical however. Mostly researchers want to get a paper at SIGGRAPH so they can share and present their work to a large audience and also share their contributions. It is all about sharing something you have worked really hard on. At least that is what I want to believe.

I created a simple solver.

But what do we mean by saying something is simple? Remember Kolmogorov and his turbulence model mentioned earlier? According to Kolmogorov, a problem is simple (not complex) when the computer code that solves the problem is small. Here is the catch though. What programming language should we choose? I can create a language, of course called *JosStamBogusFluidLanguage*, where there is only one instruction called `Solve _ Fluid`. We are all done. Yay! But it is one line of code written in a bogus language invented by yours truly. No one is impressed.

That of course is a cheat. No matter what language we use it all gets translated down into zeroes and ones. But only our Slave likes zeroes and ones. The Compiler turns our program written in our favorite programming language into these pesky 0s and 1s.

The solver I will describe was first presented at *GDC* 2003 in San Jose, California. *GDC* stands for *G*ame *D*eveloper *C*onference, and it is usually held yearly somewhere in the San Francisco Bay area. I will present the solver in *pure C* language. It uses all the concepts introduced in the previous sections and intermezzi.* Pure C is my favorite language. This language is the closest thing to programming languages called *assembly languages*.

Assembly languages give you a lot of control to create 0s and 1s. No human I know can type in 0s and 1s and create a program that performs a certain task. I often stray away from those types who are able to perform such a task unless they are fun to hang out with and also know about French philosophers or Seattle rock bands.

Coding in Assembly is cool because you are closer to the Slave. Like a secret language that less geeky and more sophisticated programmers will not understand. Let them wrestle with their C++ templates. If the last sentence does not make any sense to you, consider yourself lucky. But there will be one template class in the following text.

The problem with Assembly is that it is mostly Slave dependent. It depends on the hardware. So you spend weeks coding for the iPad in Assembly (good luck!) and then someone wants your App to be ported to the Surface, and you have to code it all over again in another Assembly (good luck again!).

---

* The original paper was called "Real-Time Fluid Dynamics for Games." You can find it on the web by Googling the title.

Pure C is cool because you are close to the Slave but not too close. You are not dealing with just one Slave but potentially with many Slaves at the same time living in different hoods.

In Figure 8.1 we show a simple C program (left) translated into binary code and assembly language.* Figure 8.1 is interesting as it shows that the C code on the left is pretty readable, while the Assembly version on the right is not (to most mere mortals!). The binary version of the code is also shown. They are the numbers on the left-hand side or the right-hand side. They are hexadecimal numbers. Look it up. Hint "E" equals "1110" in binary and is equal to 14 in the decimal system. I was taught this in grade one in Geneva by the way as part of the *new math* movement pioneered by the Swiss philosopher and linguist Jean Piaget. We used to spot him riding his bike in our neighborhood in Geneva in the 1970s.

This program computes the so-called *Fibonacci numbers*. This is how they are computed. You start with 1 and 1. Then repeatedly the numbers are computed by adding together the two previous numbers. The sequence goes like this:

1 1 2 3 5 8 13 21 34 55 89 144 233 377 610 987 1597 2584 4181 6765 10946 17711 28657 46368 75025 121393 196418 317811 514229 …

These numbers grow really fast and have many interesting properties. The ratio of two consecutive Fibonacci numbers *converges* to the golden ratio that is equal to $1+\sqrt{5}/2 \cong 1.61803\ldots$ You can use those German eigenvectors and eigenvalues to prove this. Look it up. The golden ratio also shows up in ancient Greek architecture to construct esthetically pleasing buildings. The golden ratio shows up in surprising ways all over mathematics.

I think everyone will agree that the pure C code is easier to decipher. However, note that for this particular example the pure C code is much smaller than this particular compiled Assembly version. Also the C code is more readable. That is the whole point. We do not want get too close to the Slave.

I can sense some people yawning reading this. Yes okay tell me something I don't know. However, clarifying things is always helpful, I think.

To summarize: We made it clear what simple code is. In the end, it is equal to the number of ones and zeroes that are served to the Slave for breakfast. No one codes in Assembly language anymore. No one types in

* This was created with the following wonderful tool available on the web: http://assembly.ynh.io/

```
#include <stdio.h>
static void Fibonacci ( int N ) {
  int i, F0=1, F1=1, F=0, Ftemp;
  printf ( "1 1 ");
  for ( i=2 ; i<N ; i++ ) {
    F = F0 + F1;
    Ftemp = F; F0 = F1; F1 = Ftemp;
    printf ( "%d ", F );
  }
  printf ( "\n" );
}
int main ( int argc, char ** argv ) {
  Fibonacci ( 30 );
  return 0;
}
```

```
                        .Ltext0:
                .section    .rodata
            .LC0:
0000 31203120               .string     "1 1 "
     00
            .LC1:
0005 25642000               .string     "%d "
                .text
            Fibonacci:
            .LFB0:
                .cfi_startproc
0000 55                     pushq       %rbp
                .cfi_def_cfa_offset 16
                .cfi_offset 6, -16
0001 4889E5                 movq        %rsp, %rbp
                .cfi_def_cfa_register 6
0004 4883EC30               subq        $48, %rsp
0008 897DDC                 movl        %edi, -36(%rbp)
000b C745F001               movl        $1, -16(%rbp)
     000000
0012 C745F401               movl        $1, -12(%rbp)
     000000
0019 C745F800               movl        $0, -8(%rbp)
     000000
0020 BF000000               movl        $.LC0, %edi
     00
0025 B8000000               movl        $0, %eax
     00
002a E8000000               call        printf
     00
002f C745EC02               movl        $2, -20(%rbp)
     000000
0036 EB35                   jmp         .L2
            .L3:
0069 8345EC01               addl        $1, -20(%rbp)
0038 8B45F4                 movl        -12(%rbp), %eax
003b 8B55F0                 movl        -16(%rbp), %edx
003e 01D0                   addl        %edx, %eax
0040 8945F8                 movl        %eax, -8(%rbp)
0043 8B45F8                 movl        -8(%rbp), %eax
0046 8945FC                 movl        %eax, -4(%rbp)
0049 8B45F4                 movl        -12(%rbp), %eax
004c 8945F0                 movl        %eax, -16(%rbp)
004f 8B45FC                 movl        -4(%rbp), %eax
0052 8945F4                 movl        %eax, -12(%rbp)
0055 8B45F8                 movl        -8(%rbp), %eax
0058 89C6                   movl        %eax, %esi
005a BF000000               movl        $.LC1, %edi
     00
005f B8000000               movl        $0, %eax
     00
0064 E8000000               call        printf
     00
            .L2:
006d 8B45EC                 movl        -20(%rbp), %eax
0070 3B45DC                 cmpl        -36(%rbp), %eax
0073 7CC3                   jl          .L3
0075 BF0A0000               movl        $10, %edi
     00
007a E8000000               call        putchar
     00
007f C9                     leave
                .cfi_def_cfa 7, 8
0080 C3                     ret
                .cfi_endproc
            .LFE0:
```

(a)

FIGURE 8.1 C code (a) and the corresponding Assembly code (b). (*Continued*)

```
                        .globl  main
            main:
            .LFB1:
                        .cfi_startproc
0081 55                 pushq   %rbp
                        .cfi_def_cfa_offset 16
                        .cfi_offset 6, -16
0082 4889E5             movq    %rsp, %rbp
                        .cfi_def_cfa_register 6
0085 4883EC10           subq    $16, %rsp
0089 897DFC             movl    %edi, -4(%rbp)
008c 488975F0           movq    %rsi, -16(%rbp)
0090 BF1E0000           movl    $30, %edi
   00
0095 E866FFFF           call    Fibonacci
   FF
009a B8000000           movl    $0, %eax
   00
009f C9                 leave
                        .cfi_def_cfa 7, 8
00a0 C3                 ret
                        .cfi_endproc
            .LFE1:
            .Letext0:
```

(b)

FIGURE 8.1 *(Continued)* C code (a) and the corresponding Assembly code (b).

zeroes and ones anymore. The bottom line is the size of the executable that Slave has to digest.

Now let's all watch a math horror flick.

8.1 A MATH HORROR FLICK: OPERATOR SPLITTING

> I can't really make fun of zombies. They're not liars. They're not cheats.
>
> GEORGE ROMERO (AMERICAN FILM MAKER KNOWN FOR HIS EPIC ZOMBIE MOVIES)

The title probably sounds like nonsense to most people. Operator splitting sounds like a nasty thing to do to operators. But really what it means is to take a complicated problem and split it up into simpler parts. No one gets hurt in the process. Not that I know of.

It is sort of the opposite of the meaning of the Arabic word *algebra*. Now we solve for each broken part and assemble it in a reunion. We do not solve for the reunion and then resolve the broken parts. But some of the broken parts use algebra. Math is multifaceted.

What have we learned about fluids so far? One can crudely summarize it as follows. Things are moved around, things are diffused, things are stirred around by forces, and things have to conserve mass.

Okay.

The strategy in *operator splitting* methods is to solve each effect separately and then iterate this process over discrete intervals of time to get an animation of a fluid.

In my experience, this is a good strategy to solve problems. Understand the parts and solve them separately and consequently; interesting emergent behaviors will result from their interactions.

8.2 CODE PLEASE?

> The best programs are written so that computing machines can perform them quickly and so that human beings can understand them clearly. A programmer is ideally an essayist who works with traditional aesthetic and literary forms as well as mathematical concepts, to communicate the way that an algorithm works and to convince a reader that the results will be correct.
>
> DONALD ERVIN KNUTH (FAMOUS COMPUTER SCIENTIST AND PROFESSOR AT STANFORD)

Here we go. I assume that you are familiar with the C language. If not you might get the gist of how to write a fluid solver in about 100 lines of readable C code. And hey maybe you will get hooked on some pure grade C. That would be cool. Figure 8.2 shows me giving a keynote talk

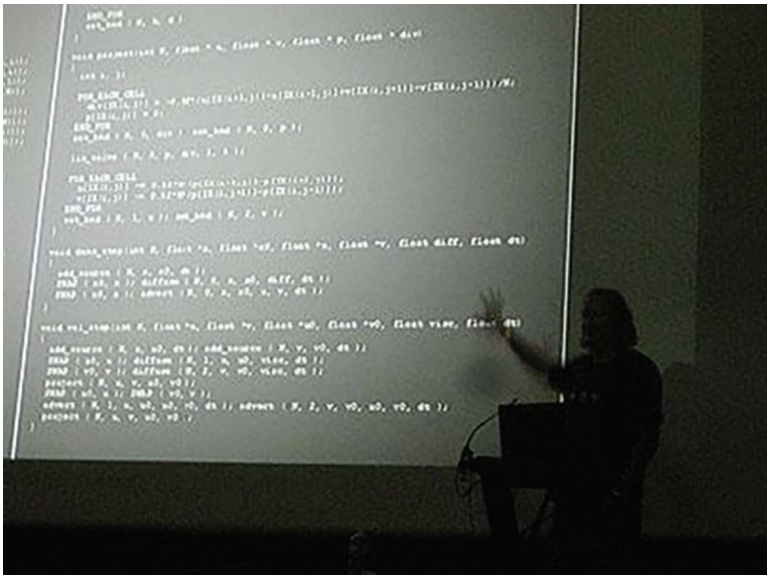


FIGURE 8.2 Future Game On, Paris, September 10, 2010 (photo by Jason de la Roca). Yeah, that is me on the right with my fuzzy claw.

and showing the code that fits on a single PowerPoint slide at a gamer conference in Paris on September 10, 2010.

Let us start with the data structures. Those are the parts of the computer memory that will store our densities and our velocity fields. To simplify things, we assume that we are only solving for densities being moved around in a fluid field with a constant density. Whoa?

There is a common misunderstanding about fluid densities that I have encountered when presenting this material in my talks that I have to clear right up front.

When I say density, I mean the density of something like perfume or dog farts being infused, transported, and diffused in a fluidlike air or water. The density of the fluid is fixed once and for all in our simple example code. It is equal to the number one, yes "1," in our solver. In standard units, this is close to the density of air. But as we saw earlier, it does not matter since we can always rescale our numbers in a manner consistent to another system of units.

One more time, when I say density, it is the density of something immersed in the fluid—not the density of the fluid. I hope that at this point you will shake your head and think: "Yes, alright man I got it. Can we move on a little?"

We are going to describe a fluid solver that lives on a grid. Please refer back to Figure 6.5. Remember the poor bugs stuck in their hideous apartment building surrounded by ghosts? Well, let us turn that into C code. The apartments will be stored in a chunk of memory. Remember we needed two coordinates to label each apartment. But we prefer one coordinate for efficiency reasons. It is simpler and faster.

Here is the solution: we simply chop off each floor and put them side by side to create a motel. This would not work in Paris, there's no space, so we will have to imagine a place in New Mexico perhaps. This transformation is illustrated in Figure 8.3. The relationship between the labeling of the hotel rooms and the motel rooms is pretty straightforward:

$$\text{Index}_{Motel} = \text{room}_{Hotel} + N \times \text{floor}_{Hotel}.$$

Here, N is the number of rooms on one floor of the hotel, and it is also the number of floors of the hotel. In Figure 8.3, $N = 4$. As an example, room (3,2) in the hotel corresponds to room 11 in the motel. This is what mathematicians call a *one-to-one mapping*. It should be pretty obvious how to

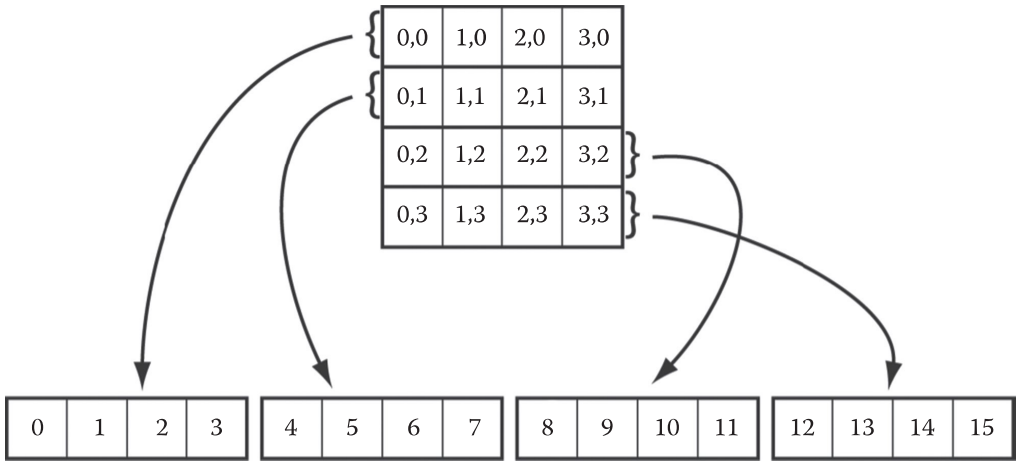


FIGURE 8.3 Mapping from a 2D grid to a 1D grid.

get back to the hotel given the motel room number. Can you figure it out? I am feeling generous so I'll give you the solution in pure C code:

```
int room_number=motel_number%N;
int floor_number=motel_number/N;
```

This is how you get two numbers out of one number. What is "%"? The creators of pure C decided that this symbol stands for the reminder of a division: high school stuff.

All this mapping business is there just to explain that we will only have 1D arrays of type *float*. But they are proxies (substitutes) for our higher dimensional grids. They make the code more readable. That's it.

Let us get back to our data structures. Data and memory are especially important on game consoles since memory is limited. You do not want to waste it.

We use floats as our numerical substitutes for ideal mathematical numbers. We need six arrays: two for the density, two for the horizontal component, and two for the vertical component of the velocity: $2+2+2=6$. Here is how it is done in C:

```
static float u[size], v[size], u_prev[size],
  v_prev[size];
static float dens[size], dens_prev[size];
```

In the case of the fluid solver the `size` is equal to `(N+2)*(N+2)`. Why the plus two? That is because we have to include the ghost cells so that our

boundary conditions are met. What about `static`? That is just so that Slave is not confused if another piece of code included in the software uses the same name. The name will only exist in this file. This avoids arguments of the following type:

Slave. "Back off Master two. That data is owned by Master one."

Master Two. "Well just ask your Master one to use `static` on his data. Wow he must be a rookie. Where did you guys meet anyway?"

Slave. "How arrogant you are Master two. Maybe you should use `static` on *your* data."

We can model our mapping between hotels and motels using something called a *macro* in the C language. Actually, macro is shorthand for *macroinstruction*. A macro turns small stuff into bigger stuff. Programmers are like mathematicians and are lazy so they create things that automatically create code for them. The macro is the least sophisticated of these techniques. And I will stick to it. No templates right now in this book. Templates can be very useful however but do not abuse them.

Here is how you can define a macro to model the relation between hotels and motels:

```
#define IX(i,j) ((i)+(N+2)*(j))
```

For example, if `N=34565` then

```
IX(31119,23) = (31119) + (34567)*(23) = 826,160.
```

But it can also work for `N=3` or any other number.

Let us now introduce another macro that will make it easier to go through our grids and update them. Here it is:

```
#define FOR_EACH_CELL for ( j=1 ; j<=N ; j++ ) {\
                      for ( i=1 ; i<=N ; i++ ) {
#define END_FOR }}
```

We also need a macro to swap array pointers. A pointer is just the address of the start of an array in memory. Here is the macro:

```
#define SWAP(x0,x) {float *tmp=x0;x0=x;x=tmp;}
```

In the code that follows, we have simplified things somewhat. Some people might wonder what happened to the grid spacing. This is usually denoted by the symbol "h." In our solver we are assuming that we are dealing with a unit square domain: each side is of size one. Therefore:

```
h=1/N, h^2=1/N^2, 1/h=N and 1/h^2=N^2.
```

Keep this in mind in trying to understand the code. This has confused many people but taking these shortcuts makes the code more compact.

8.2.1 Moving Densities

We first describe code that moves and diffuses densities in a static velocity field. The beauty is that we can reuse the same code to animate the velocity field as well. Historically, I first wrote the density animation code from a given static velocity field and then realized it could be applied to the velocity field as well.

That was another epiphany.

The procedure is shown in Figure 8.4. We start with an initial array of densities specified by the Human or the Pet. This is called an initial value problem in mathematics. Then the density evolves through three steps in our solver: we add densities, diffuse the densities, and then move the density through the fixed (for now) velocity field. That is operator splitting for you.

Let us go through each step and show the corresponding code.

Step 1: Add Sources to the density! This is simple: we just add densities to the grid. For example, human or pet could be using a mouse, a stylus, or

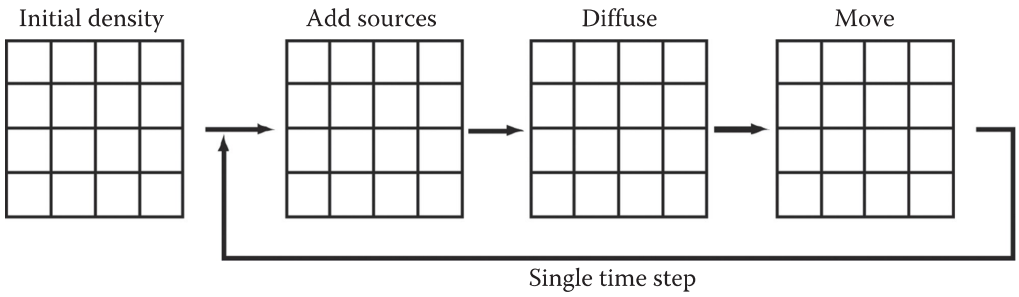


FIGURE 8.4 Schematic depiction of how the density grid is updated over fixed time steps.

their fingers or paws to add densities to some locations in the grid. Or it could be generated using some procedure. Like: "every 5 frames put some density in the middle cell of the grid." In code:

```
void add_source ( int N, float * x, float * s,
  float dt )
{
     int i, size=(N+2)*(N+2);
     for ( i=0 ; i<size ; i++ ) x[i] += dt*s[i];
}
```

Step 2: Diffuse the density! We encountered this before and we will use an implicit technique to solve the resulting linear system. If you do not know what I am talking about go back to Intermezzo Quattro.

The process of diffusion is depicted in Figure 8.5 where a single cell of density is diffused over a time step "`dt`" throughout the grid.

We will use a linear Gauss–Seidel solver because it converges faster than a Jacobi solver. If you are converting this to parallel code you might want to use Jacobi or *red-black* Gauss–Seidel.

In Intermezzo Quattro, we already gave an implementation for the diffusion of temperature in a rod. But here is an implementation that is more general in pure C. It depends on two variables: "`a`" and "`c`." These two variables will allow us to *solve* all the linear systems in our fluid solver. The variable "`b`" models how the ghosts should behave, more about that in the following text.

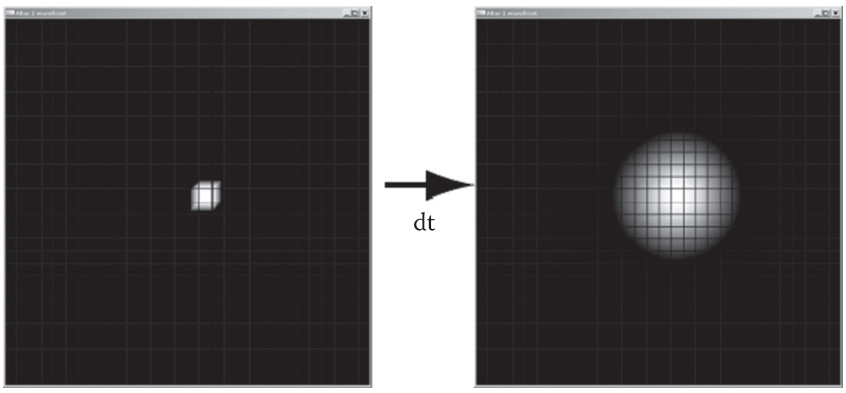


FIGURE 8.5 A single diffusion for the density step of the solver.

Here is the code for the linear solver using Gauss–Seidel iterations:

```
void lin_solve(int N, int b, float *x, float *x0,
  float a, float c) float c)
{
  int i, j, n;
  for ( n=0 ; n<20 ; n++ ) {
    FOR_EACH_CELL
      x[IX(i,j)] = (x0[IX(i,j)]+a*(x[IX(i-1,j)]+
       x[IX(i+1,j)]+x[IX(i,j-1)]+x[IX(i,j+1)]))/c;
    END_FOR
    set_bnd ( N, b, x );
  }
}
```

In this code initially the array "x0" is the known and the array "x" is initially the unknown, which becomes a known array through iteration.

Why 20 iterations? Well you are free to pick any number you want. The larger the number of iterations, the slower but the more precise your solution will be. The smaller the number of iterations, the faster the code will be and the less precise it will be. For the grid sizes, I was using in the late 1990s when I wrote this code it was sort of a compromise between these two behaviors. Feel free to experiment.

Of course there are more principled ways to choose the number of iterations. For example, it could be based on how close the known is to the unknown. But with Gauss–Seidel you will have to wait a long time. If you want some good speed, check out the stuff that I mentioned in Intermezzo Quattro. In fact the code in that Intermezzo the iterations were based on an error criterion. Of course it was set pretty low. But it was just to make a point.

What about "set_bnd" and "b"? In hindsight I could have called them "set_ghost_cells" and "ghost_type." But I wrote this before I discovered the existence of ghosts. Just kidding.

Here is "set_bnd." It just fills the boundary ghost cells from the interior cells.

There are three options for the ghost values. Here they are

(0) *Ghost*: do exactly what your neighbor does.

(1) *Ghost*: do exactly the opposite that your horizontal neighbor does.

(2) *Ghost*: do exactly the opposite that your vertical neighbor does.

That is what the following code implements. Only case (0) is used for the density code. Conditions (1) and (2) are used in the following velocity solver to ensure that no flow escapes our apartments.

```
void set_bnd(int N, int b, float *x)
{
  int i;
  for ( i=1 ; i<=N ; i++ ) {
    x[IX(0  ,i)]=b==1 ? -x[IX(1,i)] : x[IX(1,i)];
    x[IX(N+1,i)]=b==1 ? -x[IX(N,i)] : x[IX(N,i)];
    x[IX(i,0  )]=b==2 ? -x[IX(i,1)] : x[IX(i,1)];
    x[IX(i,N+1)]=b==2 ? -x[IX(i,N)] : x[IX(i,N)];
  }
  x[IX(0  ,0  )]=0.5f*(x[IX(1,0  )]+x[IX(0  ,1)]);
  x[IX(0  ,N+1)]=0.5f*(x[IX(1,N+1)]+x[IX(0  ,N)]);
  x[IX(N+1,0  )]=0.5f*(x[IX(N,0  )]+x[IX(N+1,1)]);
  x[IX(N+1,N+1)]=0.5f*(x[IX(N,N+1)]+x[IX(N+1,N)]);
}
```

Notice how we first fill the four boundaries. And then the ghosts in their fancy corner apartments are just an average of the values of their neighbor ghosts. Ghosts do not have any free will. They are ghosts after all. I will explain the meaning of "b" when we get to flow fields using this routine.

Using this code, we can write a routine that diffuses density in a stable manner.

```
void diffuse(int N, int b, float *x, float *x0, float
diff, float dt) float dt)
{
  float a=dt*diff*N*N;
  lin_solve ( N, b, x, x0, a, 1+4*a );
}
```

We will use this routine again to handle viscosity for the fluid's velocity. That is the beauty of this.

Step 3: Move the density! Here is the most fun part. How do you move a density through a fixed velocity field? This is like me and a whole other horde of tourists wandering through Venice as mentioned previously. The setting is depicted in Figure 8.6. The velocity field in red is fixed. The figure shows the evolution of the density over a single time step. Two things

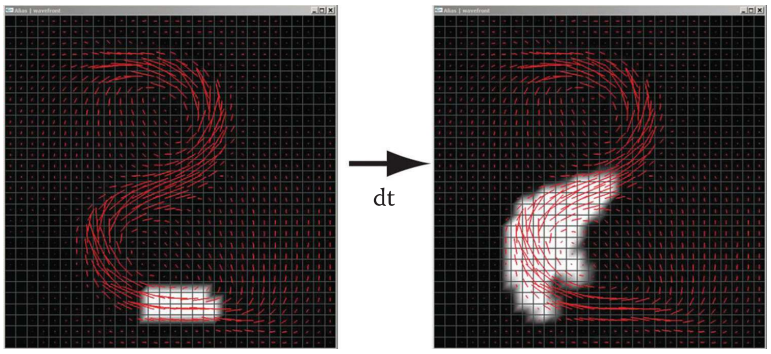


FIGURE 8.6 One step of a density (in white) being moved by a static fluid field (in red).

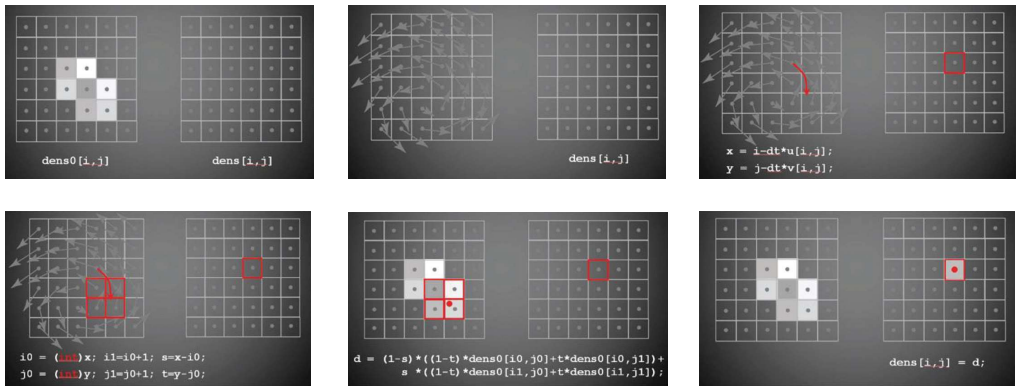


FIGURE 8.7 Transport of densities in a grid using the semi-Lagrangian technique.

to notice: (1) the density flows along the velocity field and (2) the density seems to spread out. The first property is exactly what we want. The second one is an artefact of the semi-Lagrangian technique that we use in this code. No free lunch. This technique is stable but it adds artificial diffusion. The simple code uses the semi-Lagrangian method.

We already described this technique earlier using bugs and psychopathic landlords. Now let us translate that into C code.

Figure 8.7 shows the six steps involved and the corresponding C code. Think of the shades of grey as now being a density of bugs.

Like we mentioned earlier, we need two grids: the initial density stored in `dens0` and the moved density in the array `dens`. We break this moving business down into six steps.

1. Current density on the left and the new densities on the right which we are going to update.

2. We show the velocity field that is used to transport the density field.
3. This is how we transport the center of a cell using the simplest possible technique to other grid cells.
4. We determine where the cell ends up and find the four neighbors.
5. Once we have the four neighbors we can interpolate the density values.
6. Yay! We have the new interpolated density value and we transport it to the same location in the new grid.

That's it.

This is really the coolest part of the solver. The other parts are just text book linear algebra mathematics dressed in tight code. At least to me and most hackers*: writing tight and fast code is cool.

Okay, here is the code that moves densities around.

```
void advect(int N, int b, float *d, float *d0, float
 *u, float *v, float dt)
{
  int i, j, i0, j0, i1, j1;
  float x, y, s0, t0, s1, t1, dt0;
  dt0=dt*N;
  FOR_EACH_CELL
    x=i-dt0*u[IX(i,j)]; y=j-dt0*v[IX(i,j)];
    if (x<0.5f) x=0.5f; if (x>N+0.5f) x=N+0.5f;
     i0=(int)x; i1=i0+1;
    if (y<0.5f) y=0.5f; if (y>N+0.5f) y=N+0.5f;
     j0=(int)y; j1=j0+1;
    s1=x-i0; s0=1-s1; t1=y-j0; t0=1-t1;
    d[IX(i,j)]=s0*(t0*d0[IX(i0,j0)]+t1*d0[IX
     (i0,j1)])+s1*(t0*d0[IX(i1,j0)]+t1*d0[IX(i1,j1)]);
  END_FOR
  set_bnd ( N, b, d );
}
```

* I am not talking about hackers that break into someone else's computer and steal their data, plant viruses, and commit other acts of shameless vandalism. These guys are just losers and give coders a bad name. True hackers only care about crafting the most elegant and rapid code that solves a problem.

Step 4: Let's put it all together! This is the code to solve for the density given a fixed nonevolving vector field

```
void dens_step ( int N, float * x, float * x0,
 float * u, float * v, float diff, float dt )
{
    add_source ( N, x, x0, dt );
    SWAP ( x0, x ); diffuse ( N, 0, x, x0, diff, dt );
    SWAP ( x0, x ); advect ( N, 0, x, x0, u, v, dt );
}
```

If you have a vector field available on a grid, this is one way to evolve densities over time. You can create a lot of nifty swirly effects over time.

Go wild. Please.

You do not have to stick to a single density by the way, of course not. You can evolve different densities using the same velocity field. I wrote programs that evolve the densities for the red, green, and blue components of the density and then mixed them together to create psychedelic effects. Also you can have different densities *react* between each other, like in chemical reactions.

This density solver is really a building block to create many cool effects.

The code for a density solver moving in a fluid flow can be used to compute the motion of the fluid itself. Our density solver will still work but now the fluid velocity is just assumed to be static only for the current frame.

To summarize: We provided simple code to describe the evolution of a density immersed in a static velocity field. The beauty is that this code can be reused to solve the motion of fluids.

Next, we describe how we reuse the code given previously to solve for the fluid's velocity.

8.2.2 Moving Fluids That Change Themselves: Nonlinearity

Recall the result first found by Euler: fluids move themselves. That is why they are complicated and nonlinear and hard to solve. We just covered the case of fluids frozen in time moving densities immersed in them.

We provided the code.

Actually, the situation shown in Figure 8.4 applies to the evolution of the velocity of the fluid as well. This is depicted in Figure 8.8.

It perfectly mirrors the density solver. That was a key insight.

This was another epiphany.

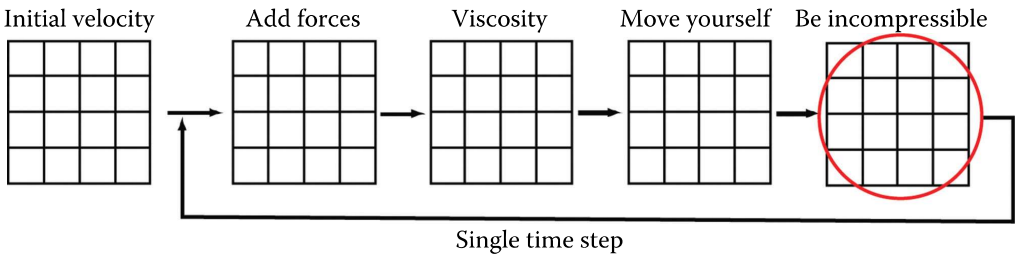


FIGURE 8.8 A single step of the solver for the velocity of a fluid is very similar to the solver for the density shown in Figure 8.4. Except for that big red circle.

And that is why I presented it in this manner.

Forces are added by human fingers or pet paws. Then, depending on the property of the fluid, the velocities will be diffused through the effect of viscosity. Finally the velocity will move itself. Forces of course can also be a function of temperature (buoyancy), mass densities (gravity) or chemical reactions and other effects cooked up by a coder or an animator.

The density solver code can be reused to animate the velocity of the fluids.

Are we done then?

Not quite. There is another step not present in the density solver that is circled in red in Figure 8.8. The velocity flow has to be incompressible. Remember Euler's first result: what flows in has to flow out. We also mentioned the fundamental result of Helmholtz and Hodge: every vector field can be written as the sum of an incompressible vector (which we want) and a gradient field (which we do not want).

The key is to compute the gradient part of the velocity field and then subtract it from the field itself.

This involves three substeps.

1. Compute the amount of nonincompressibility in each cell.
2. Compute the pressure for each cell that will fix that.
3. Subtract the gradient of this pressure field from the original nonincompressible fluid.

Substep 1 just computes the difference between inflow and outflow. If it is zero: cool we are all done. But in general, this is not the case because the previous four steps shown in Figure 8.8 do not guarantee this property.

Substep 2 is the tricky part. We have to solve a linear equation to get the unknown pressure field from the known nonincompressibility. This is the so-called *Poisson equation*. But no worries. We can solve it using the linear solver that we used to handle the diffusion of the density. I am not going to write down the Poisson equation, because I promised you that I was not going to use a single partial differential equation in this book. But the equation basically says: "What is the pressure whose gradient will make our flow incompressible?" I cheated again. I mentioned the word gradient. But recall that it is just the direction of steepest negative descent or positive ascent when you are skiing.

You can also view it as a *projection*. Think of incompressible vector fields as special fields living in the whole infinite space of all imaginable vector fields. Think of Plato's cave mentioned earlier or the process of creating an image from a 3D representation. Plato's shadows are projections. Projection is a process where you reveal one aspect of something. For a specific fluid we reveal its incompressible side.

Substep 3 is easy. We just subtract the gradient of the pressure from the vector field. The gradient is just the difference in pressure between adjacent cells.

Here is the code.

```
void project(int N, float * u, float * v, float * p,
  float * div)
{
  int i, j;
  FOR_EACH_CELL
    div[IX(i,j)] = -0.5f*(u[IX(i+1,j)]-u[IX(i-1,j)]+
                  v[IX(i,j+1)]-v[IX(i,j-1)])/N;
    p[IX(i,j)] = 0;
  END_FOR

  set_bnd ( N, 0, div ); set_bnd ( N, 0, p );
  lin_solve ( N, 0, p, div, 1, 4 );

  FOR_EACH_CELL
    u[IX(i,j)] -= 0.5f*N*(p[IX(i+1,j)]-p[IX(i-1,j)]);
    v[IX(i,j)] -= 0.5f*N*(p[IX(i,j+1)]-p[IX(i,j-1)]);
  END_FOR

  set_bnd ( N, 1, u ); set_bnd ( N, 2, v );
}
```

In this code we also make sure to populate the ghost apartments, I mean boundary grid cells.

The solver code for the evolution of the fluid's velocity is

```
void vel_step(int N, float *u, float *v, float *u0,
float *v0, float visc, float dt)
{
  add_source ( N, u, u0, dt ); add_source ( N, v,
    v0, dt );
  SWAP ( u0, u ); diffuse ( N, 1, u, u0, visc, dt );
  SWAP ( v0, v ); diffuse ( N, 2, v, v0, visc, dt );
  project ( N, u, v, u0, v0 );
  SWAP ( u0, u ); SWAP ( v0, v );
  advect ( N, 1, u, u0, u0, v0, dt );
  advect ( N, 2, v, v0, u0, v0, dt );
  project ( N, u, v, u0, v0 );
}
```

Pretty cool no?

Notice that the projection step is called twice: before and after advection. Semi-Lagrangian advection works better if the velocity field is close to incompressible. Advection on the other hand messes up incompressibility so it does not hurt to apply an incompressibility step afterwards. Sounds like a recipe doesn't it?

It is.

Please feel free to play around with these basic pieces in different orders. I won't provide fancy mathematical proofs that this all converges to a solution. Every piece is stable so you will get something interesting that looks fluidlike. This is unless computer errors are introduced in the code of course. But even then, as we have seen earlier, some customers might like the effects created by these defects.

Alright, here is the full version of the solver without my fuzzy scary claw:

```
#define IX(i,j) ((i)+(N+2)*(j))
#define SWAP(x0,x) {float * tmp=x0;x0=x;x=tmp;}
#define FOR_EACH_CELL for ( j=1 ; j<=N ; j++ ) {\
                      for ( i=1 ; i<=N ; i++ ) {
#define END_FOR }}
void add_source(int N, float *x, float *s, float dt)
{
  int i, size=(N+2)*(N+2);
```

```
  for ( i=0 ; i<size ; i++ ) x[i] += dt*s[i];
}
void set_bnd(int N, int b, float *x)
{
  int i;
  for ( i=1 ; i<=N ; i++ ) {
    x[IX(0 ,i)]=b==1 ? -x[IX(1,i)] : x[IX(1,i)];
    x[IX(N+1,i)]=b==1 ? -x[IX(N,i)] : x[IX(N,i)];
    x[IX(i,0 )]=b==2 ? -x[IX(i,1)] : x[IX(i,1)];
    x[IX(i,N+1)]=b==2 ? -x[IX(i,N)] : x[IX(i,N)];
  }
  x[IX(0  ,0  )]=0.5f*(x[IX(1,0  )]+x[IX(0  ,1)]);
  x[IX(0  ,N+1)]=0.5f*(x[IX(1,N+1)]+x[IX(0  ,N)]);
  x[IX(N+1,0  )]=0.5f*(x[IX(N,0  )]+x[IX(N+1,1)]);
  x[IX(N+1,N+1)]=0.5f*(x[IX(N,N+1)]+x[IX(N+1,N)]);
}
void lin_solve(int N, int b, float *x, float *x0,
float a, float c)
{
  int i, j, n;
  for ( n=0 ; n<20 ; n++ ) {
    FOR_EACH_CELL
      x[IX(i,j)]=(x0[IX(i,j)]+a*(x[IX(i-1,j)]+
        x[IX(i+1,j)]+x[IX(i,j-1)]+x[IX(i,j+1)]))/c;
    END_FOR
    set_bnd ( N, b, x );
  }
}
void diffuse(int N, int b, float *x, float *x0,
float diff, float dt)
{
  float a=dt*diff*N*N;
  lin_solve ( N, b, x, x0, a, 1+4*a );
}
void advect(int N, int b, float *d, float *d0, float *u,
  float *v, float dt)
{
  int i, j, i0, j0, i1, j1;
  float x, y, s0, t0, s1, t1, dt0;
  dt0=dt*N;
  FOR_EACH_CELL
    x=i-dt0*u[IX(i,j)]; y=j-dt0*v[IX(i,j)];
    if (x<0.5f) x=0.5f; if (x>N+0.5f) x=N+0.5f; i0=(int)
      x; i1=i0+1;
    if (y<0.5f) y=0.5f; if (y>N+0.5f) y=N+0.5f; j0=(int)
      y; j1=j0+1;
    s1=x-i0; s0=1-s1; t1=y-j0; t0=1-t1;
    d[IX(i,j)]=s0*(t0*d0[IX(i0,j0)]+t1*d0[IX(i0,j1)])+
               s1*(t0*d0[IX(i1,j0)]+t1*d0[IX(i1,j1)]);
```

```
  END_FOR
  set_bnd ( N, b, d );
}
void project(int N, float * u, float * v, float * p,
  float * div)
{
  int i, j;
  FOR_EACH_CELL
    div[IX(i,j)] = -0.5f*(u[IX(i+1,j)]-u[IX(i-1,j)]+
                  v[IX(i,j+1)]-v[IX(i,j-1)])/N;
    p[IX(i,j)] = 0;
  END_FOR
  set_bnd ( N, 0, div ); set_bnd ( N, 0, p );
  lin_solve ( N, 0, p, div, 1, 4 );
  FOR_EACH_CELL
    u[IX(i,j)] -= 0.5f*N*(p[IX(i+1,j)]-p[IX(i-1,j)]);
    v[IX(i,j)] -= 0.5f*N*(p[IX(i,j+1)]-p[IX(i,j-1)]);
  END_FOR
  set_bnd ( N, 1, u ); set_bnd ( N, 2, v );
}
void dens_step(int N, float *x, float *x0, float *u,
  float *v, float diff, float dt)
{
  add_source ( N, x, x0, dt );
  SWAP ( x0, x ); diffuse ( N, 0, x, x0, diff, dt );
  SWAP ( x0, x ); advect ( N, 0, x, x0, u, v, dt );
}
void vel_step(int N, float *u, float *v, float *u0, float
  *v0, float visc, float dt)
{
  add_source ( N, u, u0, dt ); add_source ( N, v,
    v0, dt );
  SWAP ( u0, u ); diffuse ( N, 1, u, u0, visc, dt );
  SWAP ( v0, v ); diffuse ( N, 2, v, v0, visc, dt );
  project ( N, u, v, u0, v0 );
  SWAP ( u0, u ); SWAP ( v0, v );
  advect ( N, 1, u, u0, u0, v0, dt ); advect ( N, 2, v,
    v0, u0, v0, dt );
  project ( N, u, v, u0, v0 );
}
```

I know that to some of you this code looks even scarier than the Holy Scriptures. But you can actually find English words in the program. If you know English you can get the gist of this program. To programmers these characters are clear to them if they know how to program in C or some other dialect of it. To the Slave it is crystal clear after we invite Mister Compiler to join the party and translate it all into zeroes and ones.

This code compiles *as is* by the way. Your geeky friend in your hood can help you if needed.

This figure is just slightly bigger than the fancy equations shown in Figures 3.23 and 3.24. The code is entirely self-contained. It does not rely on any external libraries. Some people think that makes it *elegant*. I do.*

To summarize: We provided code for the evolution of the velocity field by reusing code from the simple density solver. We however needed to provide additional code to make the velocity incompressible. But luckily we were able to reuse the linear solver code with different parameters.

8.3 BUGS CRAWLING ON DONUTS, THE FFT, AND ~60 LINES OF C CODE

> Two things are infinite: the universe and human stupidity; and I am not sure about the universe.
>
> ALBERT EINSTEIN

In this section, I will present a custom-made solver for a flow that lives on a Donut. It is based mostly on the theory of Monsieur Fourier. Refer to Intermezzo Tre aforementioned. Also it is useful to go back to Figure 3.28 and see how a 2D square can be wrapped around a donut.

This is not just an academic exercise or something out of a mathematical cabinet of curiosity.

Because our squares are actually donuts, they can be used as wallpaper. We solve a fluid on a small square and then we can replicate it over an entire wall because the fluids seamlessly fit across opposite boundaries. This tiling is shown in Figure 8.9 where a single snapshot of a donut fluid is replicated four times. The repetition is pretty obvious in this case. But this is just a single snapshot of an animated sequence. Remember visually, fluids like air are nowhere to be seen. Only their effects are visible on things being stirred around by them. So, even though the four tiles on the right-hand side of Figure 8.9 look repetitive, their influence on the motion of things is not necessarily visible. Also keep in mind that we are only showing a single snapshot in time of the fluid.

---

* It is also readable. If you are as old as I am, you might remember the one liners published in the *Amiga World* magazine that could only handle 256 characters. One of them was a one liner that ray traced a sphere. Search for *obfuscated code* on the Internet and you will know what I mean by *readable code*.

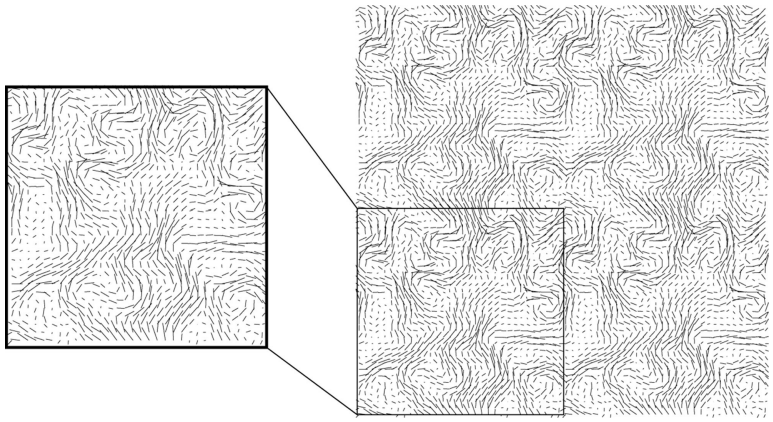


FIGURE 8.9 Fluids on donuts can be tiled infinitely all over the entire plane.

The fluid's velocity is sort of a background for many effects. Usually the flow is not shown; only its effect on things is obvious. I have said this before but it is an important fact to keep in mind when dealing with fluids using velocity fields.

We can create more out of less.

In fact it is possible to create an infinite fluid out of a finite fluid. It is sort of like how a fractal works but in a different way. Why? We cannot zoom into this fluid infinitely many times. Actually, that is not true. We can make our original fluid as small as we want. For the sake of argument, let us limit the size of the smallest grid to the so-called *Planck length* which is roughly 10^{-35} m. It is the smallest scale handled by physics. Based on size you can scale the velocity field: remember Kolmogorov. Continuous space is a mathematical fiction but mathematics is a good way to abstract things. Fiction is cool and so is the math involved. In practice, we use larger domains for starting grids. Usually of length one by one.

In Intermezzo Tre, we dealt with the discrete Fourier transform. We worked out all the details for a 3×3 matrix. The continuous Fourier transform is what is normally taught at universities, especially to engineers. But it is also a fiction. The discrete Fourier transform is immensely useful in diagonalizing matrices. But in practice the Fourier transform is used because of something called the *FFT*: *F*ast *F*ourier *T*ransform. This ingenious technique was discovered by Gauss, the guy depicted in the center of Figure 7.8, in 1805. The FFT was later rediscovered by Cooley and Tukey at IBM in 1965. They were not aware of Gauss's work. That happens all the time in research. Gauss did not publish his result, however. It was found only after his death and published posthumously.

We will not explain the FFT in this work because that would take another Intermezzo and there are thousands of books that explain it in great detail. Look it up if you are interested.

However, I will briefly mention the *FFTW*: the *F*astest *F*ourier *T*ransform in the *W*est.* I already apologized in a footnote (how cowardly is that?) that I sometimes use external libraries. FFTW is one of the cool ones. It is written in C and it works for any size, not only powers of two like the original FFT. Hey and it is from MIT.† It is a good stuff, trust me.

In Intermezzo Tre, we saw that Fourier transforms were useful to analytically put circulant matrices (structured but not sparse) into a simple form. But what is a Fourier transform of a fluid? The crucial link is that fluids on donuts wrap around like the circulant matrices. They are related even when we are talking about very different objects. Again, this is the beauty of mathematics.

That is what we will discuss next.

The Fourier transform is used heavily in image processing. This transform gives an alternative picture of a picture that is equivalent to the picture. Got that? A picture, after being Fourier transformed, reveals another aspect of itself. This other view presents the *spectral content* of the picture. This is when the wave numbers come into play that we mentioned earlier. In Figure 8.10, we show four waves and their corresponding wave numbers. The arrow is like the one mentioned in Figure 5.2 in reference with Kolmogorov turbulence. The bigger the vector the more detail in the wave: the smaller the scale. The vector also shows the direction of the wave like that of a wind direction.

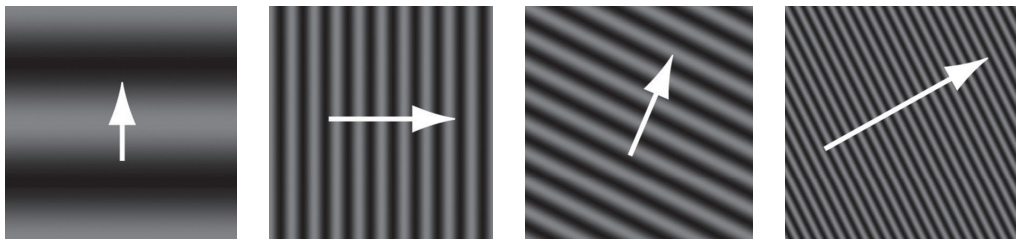


FIGURE 8.10 Four wave vectors and their corresponding waves.

* http://www.fftw.org.

† MIT stands for the Massachusetts Institute of Technology. That is where the top brainy kids go to study and do cool geek stuff. I never applied there.

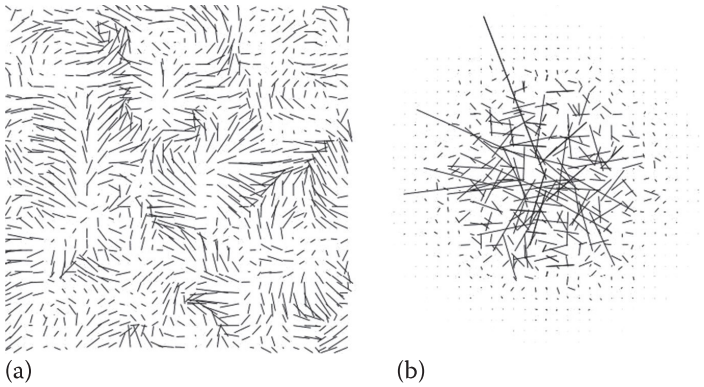


FIGURE 8.11 A spatial vector field (a) and its Fourier transform (b).

Figure 8.11 shows a vector field (a) and its Fourier transform (b). The field in the spatial domain is clearly periodic. You can decorate arbitrary sized wallpaper with it. The Fourier transformed field on the right is very different. Most of the action is concentrated in the center. The vectors in the center are actually the largest and then they drop off. Each vector in the Fourier (Figure 8.11b) field says how much the wave vector (one of those in Figure 8.11) contributes to the spatial field. There is a drop off because the field is of finite size, or in the continuous case, the effect of viscosity limits the amount of scales.

It is instructive to look at the similar Fourier transform for pictures. Figure 8.12 shows two pictures and their respective Fourier transforms. Notice that the two pictures are random. They are noisy. But they are not completely random or noisy because their respective Fourier transforms are bounded. The spread is inversely related to the *size* of the structures in the spatial images. Sound familiar? It is similar to a dish of turbulence served *à la* Kolmogorov.

Actually, fields that are fractals never go to zero, but they still drop off with scales. Kolmogorov's turbulence is like that. In practice, we just cut off the Fourier representation when the values decrease with increasing scale.

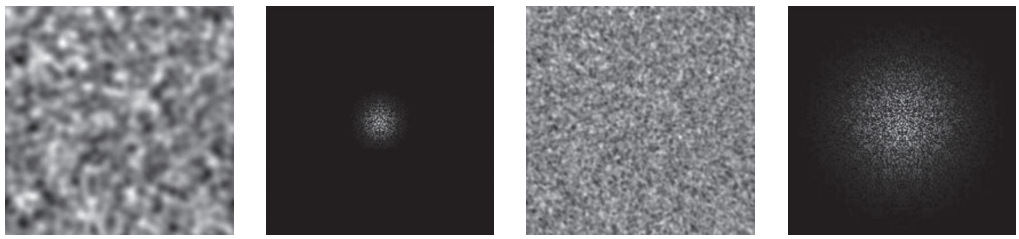


FIGURE 8.12 Two random images and their respective Fourier transforms.

When these values are below a certain threshold dictated by the Master, they are by *fiat* set to zero. Remember Slave can only handle that many bits. Let's not waste time and memory on tiny numbers.

In the Fourier domain, our flows are not as visually appealing as in our usual space.

Usually, the Fourier transform is used to solve for incompressibility for periodic flows living on donuts. Recall the Helmholtz–Hodge decomposition shown in Figure 3.13. In the Fourier domain, this becomes the decomposition depicted in Figure 8.13.

I think this is the best geometrical explanation of the Helmholtz–Hodge decomposition. This was another epiphany for me. Any vector field in the Fourier domain is the sum of the following:

1. An incompressible field whose vectors are all tangent to circles.
2. A gradient field whose vectors are all normal to circles.

Figure 8.14 illustrates this situation. The orange vector is the wave vector $\mathbf{k}$ defined by a direction and a magnitude. It corresponds to a planar wave as shown in Figure 8.10. At the tip of this wave vector a particular Fourier velocity $\mathbf{v}$ is shown in black. This corresponds to one of the vectors of the field on the left-hand side of Figure 8.13. This vector field can be decomposed into a tangential component $\mathbf{v}_T$ (in green) and a perpendicular component $\mathbf{v}_N$ (in blue). I know there are a lot of arrows in the figure. But focus on $\mathbf{k}$ the wave vector and $\mathbf{v}$ the vector in Fourier space. The components of the Fourier vector (green/blue) are just the coordinates with respect to the circle.

This is similar to our ellipse example described in Intermezzo Due earlier. In this case, we have that $\mathbf{v}=\mathbf{v}_T+\mathbf{v}_N$, therefore $\mathbf{v}_T=\mathbf{v}-\mathbf{v}_N$. This is

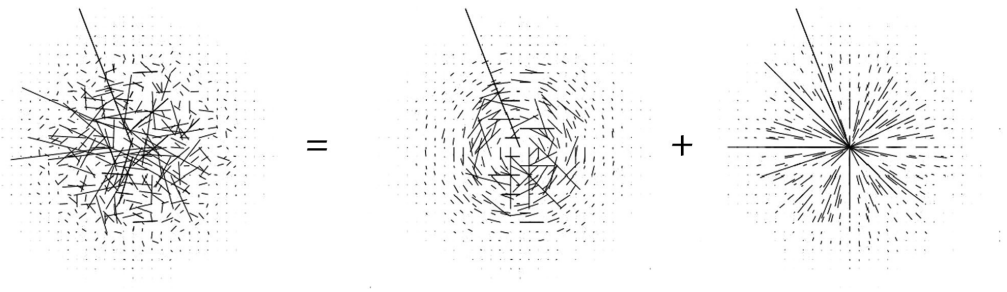


FIGURE 8.13 The Helmholtz–Hodge decomposition in the Fourier domain.

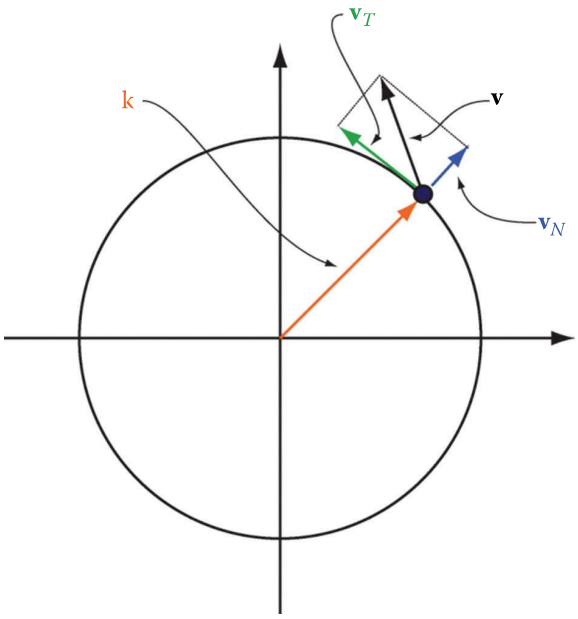


FIGURE 8.14 Every vector in the Fourier domain is separated into a tangential component and a perpendicular component.

essentially the Helmholtz–Hodge decomposition. In plain speak: we get an incompressible vector field by projecting the original field onto the line tangential to the circle defined by the wave vector. Just so you can understand the following code let's go over the mathematical steps of a projection for a single vector **v**. We are given a tangential vector **T** and we want to project our vector on this tangential vector. Every tangential vector has an evil twin which is perpendicular to itself. We call it **N** because it is *normal* to the tangential vector. The vector $\mathbf{v}_N$ is obtained by projecting the vector **v** onto the normal vector **N**:

$$\mathbf{v}_N = \mathbf{v} * \mathbf{N}.$$

The star "*" this time stands for a dot product between vectors. The dot product is pretty simple; it takes two vectors and turns them into a number. For example:

$$(a,b) * (c,d) = ac + bd.$$

But also:

$$(a,b,c,d,e,f) * (g,h,i,j,k,m) = ag + bh + ci + dj + ek + fm.$$

Hopefully these two examples clarify what a dot product between vectors is. It works for any dimension.

To get the tangential part of the vector we just have to subtract the normal component of the original vector from itself:

$$\mathbf{v}_T = \mathbf{v} - \mathbf{v}_N = \mathbf{v} - \mathbf{v}^* \mathbf{N}.$$

This is another, simpler way to describe the Helmholtz–Hodge result. It is simpler because we looked at this problem from the Fourier point of view.

If you have digested the code described earlier for a simple solver, then the code in Figure 8.15 should be pretty easy to swallow. I have even added some code showing how to talk to the FFTW library. First you have to initialize some of their data structures. Also I introduced a FFT macro which takes care of direct and inverse transforms using the FFTW library. The size of the grid is hard coded to 32×32. But it does not have to be. Any rookie programmer can change the code to allow any sizes. Even better, you can turn it into an input specified by a Human or a Pet. By the way, you do not have to use the FFTW. It should be pretty clear how to modify the code with your favorite fast Fourier transform. I decided to include the low level stuff anyway for the FFTW. Why? Because FFTW is a cool solver and can handle any grid size. I restricted the size to an even number as you can tell from the code. It makes some of the math easier. If you are going to use FFTW, it is up to you to get it working in your hood. Installing libraries, like adding up numbers, bores me to death. It's the kind of stuff you keep for sleepy Friday mornings.

This looks complicated and simple at the same time. Well, at least to me.

Anyone who has written code will agree that it is simple for what it is able to achieve. A noncoder will just shrug and view this code as some other abstract nonsense like the Lagrangian of the standard model. The difference is that you can make it work on your laptop with the help of your geeky girlfriend/boyfriend. She or He will get it to work in your hood.

The most difficult part is to go back and forth between the FFTW data structures and our velocity arrays. It took me awhile to get it right. But I give these relations here for free for this particular piece of code. Use it in any other application that requires simulating something on a torus. The basic idea is:

```
#include <srfftw.h> /* Heresy! */

#define SIZE 32
#define SIZE2 (SIZE*SIZE)
#define SIZE2P ((SIZE+2)*(SIZE))

static float u[SIZE2], v[SIZE2], u0[SIZE2P], v0[SIZEP2};
static int rfftwnd_plan plan_rc, plan_cr;

#define FFT(s,n,u) \
if (s==1) rfftwnd_one_real_to_complex ( plan_rc, (fftw_real    *)u, (fftw_complex *)u );\
else      rfftwnd_one_complex_to_real ( plan_cr, (fftw_complex *)u, (fftw_real    *)u )

int init_solve ( int n )
{
   int i;

   if ( n%2 ) return 0;

   plan_rc = rfftw2d_create_plan ( n, n,  FFTW_REAL_TO_COMPLEX, FFTW_IN_PLACE );
   plan_cr = rfftw2d_create_plan ( n, n,  FFTW_COMPLEX_TO_REAL, FFTW_IN_PLACE );

   for ( i=0 ; i<n*n ; i++ ) u[i] = v[i] = 0;
   for ( i=0 ; i<(n+2)*n ; i++ ) u0[i] = v0[i] = 0;

   return 1;
}

#define floor(x) ((x)>=0.0?((int)(x)):(-((int)(1-(x)))))

void fourier_solve ( int n, float * u, float * v, float * u0, float * v0, float visc, float dt )
{
   float x, y, x0, y0, f, r, U[2], V[2];
   int i, j, i0, j0, i1, j1, s, t;

   for ( i=0 ; i<n*n ; i++ ){
      u[i] += dt*u0[i]; u0[i] = u[i];
      v[i] += dt*v0[i]; v0[i] = v[i];
   }

   for ( x=0.5/n,i=0 ; i<n ; i++,x+=1.0/n ){
      for ( y=0.5/n,j=0 ; j<n ; j++,y+=1.0/n ){
         x0 = n*(x-dt*u0[i+n*j])-0.5; y0 = n*(y-dt*v0[i+n*j])-0.5;
         i0 = floor(x0); s = x0-i0; i0 = (n+(i0%n))%n; i1 = (i0+1)%n;
         j0 = floor(y0); t = y0-j0; j0 = (n+(j0%n))%n; j1 = (j0+1)%n;
         u[i+n*j] = (1-s)*((1-t)*u0[i0+n*j0]+t*u0[i0+n*j1])+
                        s *((1-t)*u0[i1+n*j0]+t*u0[i1+n*j1]);
         v[i+n*j] = (1-s)*((1-t)*v0[i0+n*j0]+t*v0[i0+n*j1])+
                        s *((1-t)*v0[i1+n*j0]+t*v0[i1+n*j1]);
      }
   }

   for ( i=0 ; i<n ; i++ )
      for ( j=0 ; j<n ; j++ )
         { u0[i+(n+2)*j] = u[i+n*j]; v0[i+(n+2)*j] = v[i+n*j]; }

   FFT(1,n,u0); FFT(1,n,v0);

   for ( i=0 ; i<=n ; i+=2 ){
      x = 0.5*i;
      for ( j=0 ; j<n ; j++ ){
         y = j<=n/2 ? j : j-n;
         r = x*x+y*y;
         if ( r==0 ) continue;
         f = 1 / ( 1 + r*dt*visc);
         U[0] = u0[i  +(n+2)*j]; V[0] = v0[i  +(n+2)*j];
         U[1] = u0[i+1+(n+2)*j]; V[1] = v0[i+1+(n+2)*j];
         u0[i  +(n+2)*j] = f*( (1-x*x/r)*U[0]     -x*y/r *V[0] );
         u0[i+1+(n+2)*j] = f*( (1-x*x/r)*U[1]     -x*y/r *V[1] );
         v0[i+  (n+2)*j] = f*(   -y*x/r *U[0] + (1-y*y/r)*V[0] );
         v0[i+1+(n+2)*j] = f*(   -y*x/r *U[1] + (1-y*y/r)*V[1] );
      }
   }

   FFT(-1,n,u0); FFT(-1,n,v0);

   f = 1.0/(n*n);
   for ( i=0 ; i<n ; i++ )
      for ( j=0 ; j<n ; j++ )
         { u[i+n*j] = f*u0[i+(n+2)*j]; v[i+n*j] = f*v0[i+(n+2)*j]; }
}
```

FIGURE 8.15 Entire code to simulate a fluid on a torus using the FFT.

1. Transform your data from the spatial domain to the Fourier domain.
2. Solve your problem in the Fourier domain.
3. Transform your data back from the Fourier domain to the spatial domain.

In our donut fluid solver, we use the Fourier transform to turn the Helmholtz–Hodge decomposition—which is *global* in the spatial domain—into a *local* projection in the Fourier domain. A global thing has been turned into a local thing. The cost we have to pay for this is that we have to transform our data to the Fourier domain. That transformation however is global. So superficially nothing has been gained. But thanks to the fast Fourier transform this is not true. The global Fourier transform can be made faster computationally. Think again of our ellipse example aforementioned. By changing the point of view—the coordinate system—we made the thing simpler. But we did not alter the thing. It is all about how you look at it. But you have to do it in a rapid way in practice. If not nothing is gained.

This in a nutshell is why the FFT is so cool. Too bad its usage is restricted to donuts. Well not quite. The FFT also works for higher-dimensional donuts, for all spaces that have the property that when you cross a boundary you end up on the opposite boundary of the domain. Some scientists even speculate that our universe is finite in that sense. You go one way and you end up at the same spot. Is that crazy? Maybe. But remember people used to believe the earth was flat. Enough said.

You can use the code in Figure 8.15 *as is*. Other people have done this before. You have to install the FFTW package first however. Talk to your favorite geek in your hood if you cannot do it yourself. Trust me, cool geeks love doing this sort of stuff and will help you out, as long as you buy them a six-pack or a fancy latté.

Notice that we only use the FFTW to enforce incompressibility. The transport/advection step is still done in physical space.

Because I am being generous, the 3D version of the code is shown in Figure 8.16. Like the 2D version it is given *as is*. The reason I included this code is that it is not too hard to convert it to higher dimensions. It would work in 510,384 dimensions. This would involve a lot of typing unless you design some clever macros or templates.

```
#include <srfftw.h> /* Heresy again!*/

#define SIZE 32
#define SIZE3 (SIZE*SIZE*SIZE)
#define SIZE3P ((SIZE+2)*(SIZE)*(SIZE))

static float u[SIZE3], v[SIZE3], w[SIZE3], u0[SIZE3P], v0[SIZ3P], w0[SIZE3P];

static rfftwnd_plan plan_rc, plan_cr;

#define FFT(s,u) \
if (s==1) rfftwnd_one_real_to_complex ( plan_rc, (fftw_real    *)u,(fftw_complex *)u );\
else      rfftwnd_one_complex_to_real ( plan_cr, (fftw_complex *)u,(fftw_real    *)u )

int init_solve3 ( int n )
{
   int i;

   if ( n%2 ) return 0;

   plan_rc = rfftw3d_create_plan ( n, n, n, FFTW_REAL_TO_COMPLEX, FFTW_IN_PLACE );
   plan_cr = rfftw3d_create_plan ( n, n, n, FFTW_COMPLEX_TO_REAL, FFTW_IN_PLACE );

   for ( i=0 ; i<n*n*n ; i++ ) u[i] = v[i] = w[i] = 0.0f;
   for ( i=0 ; i<(n+2)*n*n ; i++ ) u0[i] = v0[i] = w0[i] = 0.0f;

   return 1;
}

#define floor(x) ((x)>=0.0?((int)(x)):(-((int)(1-(x)))))

void fourier_solve3 ( int n, float * u, float * v, float * w, float * u0, float * v0, float * w0, float visc, float dt )
{
   float x, y, z, f, r, s, t, U[2], V[2], W[2], dtn;
   int idx, idx0, idx1, idx000, idx001, idx010, idx011, idx100, idx101, idx110, idx111;
   int i, j, k, i0, j0, k0, i1, j1, k1;

   for ( i=0 ; i<n*n*n ; i++ ){
      u[i] += dt*u0[i]; u0[i] = u[i];
      v[i] += dt*v0[i]; v0[i] = v[i];
      w[i] += dt*w0[i]; w0[i] = w[i];
   }

   dtn = n*dt;

   for ( i=0 ; i<n ; i++ ){
      for ( j=0 ; j<n ; j++ ){
         for ( k=0 ; k<n ; k++ ){
            idx = i+n*(j+n*k);
            x = i-dtn*u0[idx]; y = j-dtn*v0[idx]; z = k-dtn*w0[idx];
            i0 = floor(x); r = x-i0; i0 = (n+(i0%n))%n; i1 = (i0+1)%n;
            j0 = floor(y); s = y-j0; j0 = (n+(j0%n))%n; j1 = (j0+1)%n;
            k0 = floor(z); t = z-k0; k0 = (n+(k0%n))%n; k1 = (k0+1)%n;
            idx000 = i0+n*(j0+n*k0); idx001 = i0+n*(j0+n*k1); idx010 = i0+n*(j1+n*k0); idx011 = i0+n*(j1+n*k1);
            idx100 = i1+n*(j0+n*k0); idx101 = i1+n*(j0+n*k1); idx110 = i1+n*(j1+n*k0); idx111 = i1+n*(j1+n*k1);
            u[idx] = (1-r) * ( (1-s) * ( (1-t)*u0[idx000] + t*u0[idx001] ) + s * ( (1-t)*u0[idx010] + t*u0[idx011] ) ) +
                        r  * ( (1-s) * ( (1-t)*u0[idx100] + t*u0[idx101] ) + s * ( (1-t)*u0[idx110] + t*u0[idx111] ) );
            v[idx] = (1-r) * ( (1-s) * ( (1-t)*v0[idx000] + t*v0[idx001] ) + s * ( (1-t)*v0[idx010] + t*v0[idx011] ) ) +
                        r  * ( (1-s) * ( (1-t)*v0[idx100] + t*v0[idx101] ) + s * ( (1-t)*v0[idx110] + t*v0[idx111] ) );
            w[idx] = (1-r) * ( (1-s) * ( (1-t)*w0[idx000] + t*w0[idx001] ) + s * ( (1-t)*w0[idx010] + t*w0[idx011] ) ) +
                        r  * ( (1-s) * ( (1-t)*w0[idx100] + t*w0[idx101] ) + s * ( (1-t)*w0[idx110] + t*w0[idx111] ) );
         }
      }
   }

   for ( i=0 ; i<n ; i++ ){
      for ( j=0 ; j<n ; j++ ){
         for ( k=0 ; k<n ; k++ ){
            idx1 = i+n*(j+n*k); idx0 = i+(n+2)*(j+n*k);
            u0[idx0] = u[idx1]; v0[idx0] = v[idx1]; w0[idx0] = w[idx1];
         }
      }
   }

   FFT(1,u0); FFT(1,v0); FFT(1,w0);

   for ( i=0 ; i<=n ; i+=2 )
   {
      x = 0.5f*i;
      for ( j=0 ; j<n ; j++ ){
         y = (float)(j<=n/2 ? j : j-n);
         for ( k=0 ; k<n ; k++ ){
            z = (float)(k<=n/2 ? k : k-n);
            r = x*x+y*y+z*z;
            if ( r==0.0f ) continue;
            f = 1 / ( 1 + r*dt*visc);
            idx0 = i  +(n+2)*(j+n*k);
            idx1 = i+1+(n+2)*(j+n*k);
            U[0] = u0[idx0]; V[0] = v0[idx0]; W[0] = w0[idx0];
            U[1] = u0[idx1]; V[1] = v0[idx1]; W[1] = w0[idx1];
            u0[idx0] = f*( (1-x*x/r)*U[0]     -x*y/r *V[0]     -x*z/r *W[0] );
            u0[idx1] = f*( (1-x*x/r)*U[1]     -x*y/r *V[1]     -x*z/r *W[1] );
            v0[idx0] = f*(   -y*x/r *U[0] + (1-y*y/r)*V[0]     -y*z/r *W[0] );
            v0[idx1] = f*(   -y*x/r *U[1] + (1-y*y/r)*V[1]     -y*z/r *W[1] );
            w0[idx0] = f*(   -z*x/r *U[0]     -z*y/r *V[0] + (1-z*z/r)*W[0] );
```

FIGURE 8.16 Three-dimensional version of the code. (*Continued*)

```
            w0[idx1] = f*(   -z*x/r *U[1]     -z*y/r *V[1] + (1-z*z/r)*W[1] );
         }
      }
   }

   FFT(-1,u0); FFT(-1,v0); FFT(-1,w0);

   f = 1.0f/(n*n*n);
   for ( i=0 ; i<n ; i++ ){
      for ( j=0 ; j<n ; j++ ){
         for ( k=0 ; k<n ; k++ ){
            idx1 = i+n*(j+n*k); idx0 = i+(n+2)*(j+n*k);
            u[idx1] = f*u0[idx0]; v[idx1] = f*v0[idx0]; w[idx1] = f*w0[idx0];
         }
      }
   }
}
```

FIGURE 8.16 *(Continued)* Three-dimensional version of the code.

What would the advection step be in the Fourier space? This is an interesting question that is rarely addressed in fluid dynamics books. Instead of a local spatial phenomenon it becomes a global Fourier phenomenon. Therefore advection is best done in the spatial domain. Why do something in the Fourier domain that is not Fourier friendly. Fourier space is good for some things but not for others. It takes some practice and flair to use this transform adequately. However, a uniform translational advection in the spatial domain corresponds to a rotation in the Fourier domain.

To summarize: We used the Fourier transform to come up with a simple fluid solver on donuts. We cheated a bit as we used a library from MIT called FFTW. The Fourier transform came in very handy to solve for the incompressibility condition for these fluids. Another cool aspect of these fluids is that donuts can be unfolded into an infinite tiling of the plane. You basically get an infinite fluid from a finite one. How cool is that.

8.4 FOUR-DIMENSIONAL TURBULENT VECTOR FIELDS AND TURBULENCE

> There are really four dimensions, three which we call the three planes of Space, and a fourth, Time.
>
> H. G. WELLS (AMERICAN AUTHOR AS QUOTED FROM HIS NOVEL: *THE TIME MACHINE*)

We can reuse this Fourier code to create turbulent textures. Everything shown earlier was illustrated in a 2D space and a corresponding 2D Fourier space.

Remember turbulence?

That stuff that no one understands. We will use Kolmogorov's simple model to create 4D turbulence fields. It is like wallpaper but it wraps in space and in time: four dimensions. They are like a fatter cousin of the

fields we have considered so far. These cousins wrap around not only in space but also in time. That is the wickedest part. These fields run forever in time and tile space endlessly as well. This is sort of trippy. It is useful, as you can use these turbulent fields to *decorate* existing velocity fields to generate more swirling detail on top of them.

This is very common in computer graphics. That is how we add detail to the surfaces. It is called *texture mapping*. Instead of modeling all the detail, a surface is modeled at a coarser level and detail is added later on. So, rather than modeling every pore and zit on a virtual character, these are added using a pore and zit map applied to the surface. More from less.

This is the same with a turbulent vector field: it adds detail and liveliness to a coarser fluid velocity field.

This is how it works.

Take a 4D noise sampled on a grid. This just means assign a random vector to each cell in a 4D grid. Then take a 4D Fourier transform. We make the spatial part incompressible using the fast projection. We then multiply this field by a Kolmogorov-like spectrum. Of course, anyone can use their own spectrum invented in their basement. Then we Fourier transform this field back to the spatial domain. *Voilà!* You now have a 4D grid of turbulent vectors. You only have to compute it once and then you can reuse it in your favorite app to add cool details to your flow. This stuff got me my first SIGGRAPH paper and fueled part of the particle system in Alias's Power Animator software and then our MAYA software. This all happened in the early 1990s in Toronto.

```
gen_turb ( v ) {
  for ( it=0 ; it<=Nt/2 ; it++ ) {
    l=it/Nt;
    for ( ix,iy,iz=0 ; ix,iy,iz<Nx,Ny,Nz ;
      ix,iy,iz++ ) {
       kx, ky, kz=ix/Nx, iy/Ny, iz/Nz;
       if ( ix > Nx/2 ) kx=kx - 1.0;
       if ( iy > Nx/2 ) ky=ky - 1.0;
       if ( iz > Nx/2 ) kz=kz - 1.0;
       k2=kx*kx+ky*ky+kz*kz;
       /* generate random complex vector */
       tx, ty, tz=unif(0.0,2*PI);
       RE(Wx,Wy,Wz)=S(sqrt(k2),l)*cos(tx,ty,tz);
       IM(Wx,Wy,Wz)=S(sqrt(k2),l)*sin(tx,ty,tz);1 3
       /* project onto plane normal to {kx,ky,kz} */
```

```
        Vx=(1-kx*kx/k2)*Wx - kx*ky/k2 *Wy - kx*kz/k2
          *Wz;
        Vy=-ky*kx/k2 *Wx+(1-ky*ky/k2)*Wy - ky*kz/k2
          *Wz;
        Vz=-kz*kx/k2 *Wx - kz*ky/k2 *Wy+(1-kz*kz/
          k2)*Wz;
        /* store and ensure "mirror" symmetries */
        w[ix][iy][iz][it]={Vx,Vy,Vz};
        w[(Nx-ix)%Nx][(Ny-iy)%Ny][(Nz-iz)%Nz]
          [(Nt-it)%Nt]={CC(Vx),CC(Vy),CC(Vz)};
    }
  }
  /* "mirror" symmetries at the axis of the
    symmetry */
  for ( it,ix,iy,iz=0,0,0,0 and Nt/2,Nx/2,Ny/2,Nz/2 )
    {
    IM(vel[ix][iy][iz][it])=0.0;
  }
  X(v)=invFFT4(X(w));
  Y(v)=invFFT4(Y(w));
  Z(v)=invFFT4(Z(w));
  /* normalize each component of v in the range
    [-1,+1] */
}
/* In the code we used the following macros:
RE(c)=real part of complex variable
IM(c)=complex part of complex variable
CC(c)=complex conjugate of the variable
X(v)=x component of a vector
Y(v)=y component of a vector
Z(v)=z component of a vector
S(k,t)=Kolmogorov's spectrum
*/
```

Creating flow turbulence is mostly a purely kinematic approach. We only model the thing that affects the thing; not the thing that affects the thing that then affects the thing. That would be dynamics. We model a first-order effect, not a second-order effect. All the babble mentioned above involved mostly dynamics. Although the dynamics are cooler, kinematics can be cool too. Kinematics is the nice neighbor that is more predictable than the eccentric dynamics neighbor two blocks down the road. But dynamics may be more fun to hang out with.

To summarize: Using a 4D Fourier transform, you can create interesting flow texture maps that wrap both in time and in space. You only have to compute this field once and then store the result. These precomputed flows can then be used in many applications.

8.5 DECORATING FLUIDS

> Please do not stand too close to my paintings, the smell will make you sick.
>
> REMBRANDT VAN RIJN (LEGENDARY DUTCH MASTER PAINTER AND ETCHER, 1606–1669)

In the previous section, we described a technique to decorate the fluid's velocity. We described how to modulate the thing that affects the things. We will describe a technique to directly modify the appearance of a fluid's density using evolving texture maps. The idea is to simulate the velocity at some level and then add detail.

First, we have to explicitly define what a texture map is. Let us do it in two dimensions so that we can clearly illustrate this concept.

A texture map is usually a 2D picture of something. A photograph perhaps: a picture of your dog or a picture of some interesting texture you happened to photograph on Jeju Island in South Korea, for example. This picture can then be mapped onto a digital surface. This is illustrated in Figure 8.17. On the left there is a picture of our family poodle "Luke" and on the right is a surface. Both the photograph and the surface are 2D surfaces. The one on the left corresponding to the photograph is flat and undeformed while the surface on the right has been deformed. You can tell from the shading. On the right-hand side of Figure 8.17, we depict

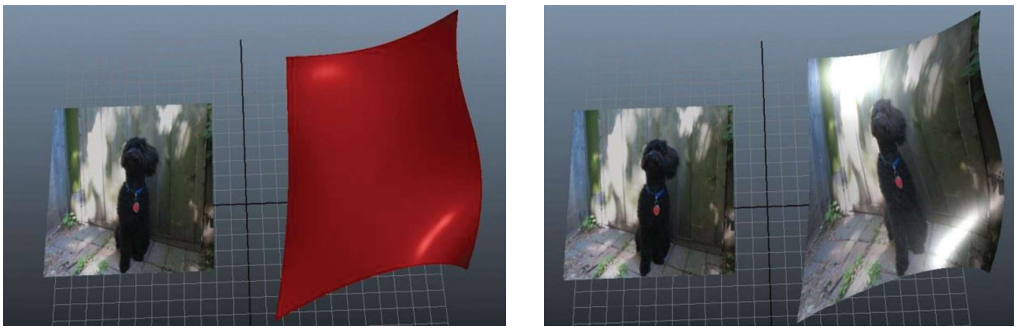


FIGURE 8.17 Left: The texture map and the surface. Right: The texture map applied to the surface.

what happens if our picture is *texture mapped* onto the deformed surface. Generally for each pixel element, a pixel of the texture is mapped to a unique point on the surface. This is really oversimplifying the situation. What if the texture has only a few pixels and the surface is large in comparison? What if the texture map is huge and the surface is tiny in the background? Welcome to the world of optimal sampling. Look it up. This book is not about optimal texture mapping. But it is a fascinating subject.

The point that I am making here is that you can decorate—add detail—to a surface through a mechanism known as texture mapping.

The texture map is often rectangular. Computer graphics people refer to each point in the texture map by two coordinates usually denoted by u and v. These coordinates are often bounded between zero and one. The left bottom corner is (0,0) and gets mapped to the left bottom corner of the surface. The top right corner is (1,1) and gets mapped to the top right corner of the surface. In between: well you just get a mapping in between these two extremes. Notice that the texture map looks distorted on the surface. This is usually an undesirable artefact. But in fluid animation we like this. We want stuff to swirl around and evolve over time and look cool.

So, how do we decorate a fluid with a texture map?*

The key insight is to treat the "u" and the "v" coordinates as density fields that are going to move, diffuse, and wiggle in a vector field. The initial state of the coordinates is depicted in Figure 8.18. Initially, they are just linear ramps. This means there is a one-to-one correspondence between the texture and the fluid when it is in a rest state. Texture coordinates are treated just like the density of perfume, dog farts, or Beijing smog being stirred and diffused in the air, which remember is a fluid. That is how we can dress up fluids in pretty skirts: *Fluides en Jupons*.

I hope that Figure 8.19 illustrates the basic concept. As the u and v coordinates are being moved around, the mapping to the original texture is being affected. Figure 8.19 shows a point at the center of the fluid. Initially its u–v coordinates are (0.5,0.5): it refers to the exact same pixel located at the center of the texture map. However, over time the u–v *densities* evolve because they are immersed in a fluid. The point in the middle now refers to some other u–v coordinate in the original texture map. In this case,

* I did not invent this technique from scratch. Nelson Max and Barry Becker in their 1992 paper entitled "Flow Visualization Using Moving Textures" inspired this work. Research is never done in isolation.

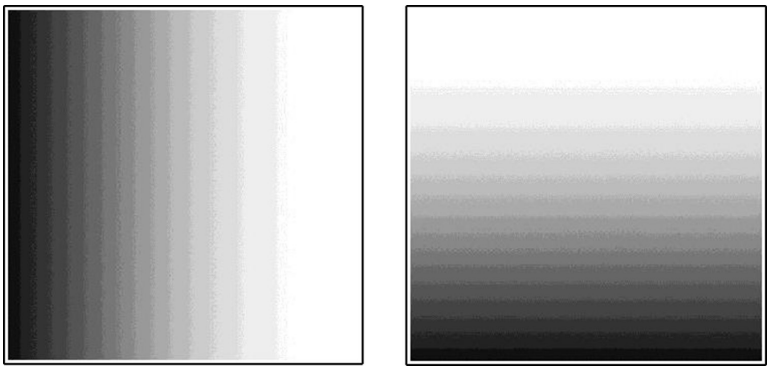


FIGURE 8.18 Initial u and v maps treated as density fields.

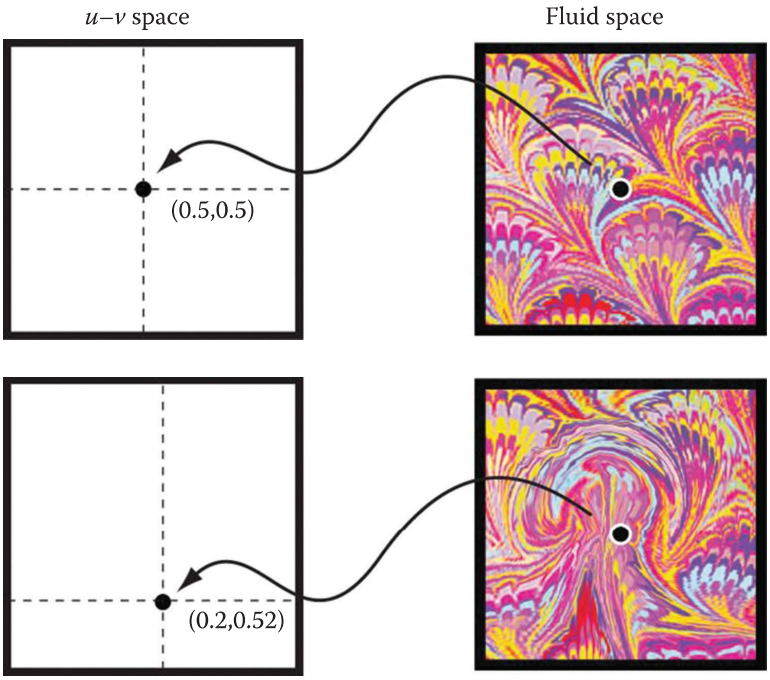


FIGURE 8.19 Texture advection. The u–v coordinates are modified by the fluid. Each cell in the fluid grid has a unique u–v coordinate that corresponds to a point in the texture map.

it's a point in the lower part of the texture. *Et Voila!* This results in the visual illusion of an upward motion.

In fact there is more to this technique that I have to explain.

One problem is that because of numerical diffusion when you run this method for some time everything gets blurred out. In this context, it means that your entire fluid will be equal to a single pixel of the texture

map. In the case of the texture map in Figure 8.19, the fluid might end up being all pink just like in the world of Dr. Seuss' *The Cat in the Hat Comes Back*.

To solve this problem, we use three sets of u–v coordinates. Ugh! This makes everything much slower. But thanks to Moore's law it will be fast at some time in the future. Besides, it creates much cooler animations. And it runs for reasonable resolutions in real-time on an iPhone.

This is how it works.

The idea is to blend three sets of u–v densities multiplied by a weight at each time frame. The weights add up to one and they evolve over time in a staggered manner.

Figure 8.20 shows one choice of blend functions: 2D teepees. I always go for the simplest stuff because I am lazy. And if that doesn't work I will have to look for something fancier.

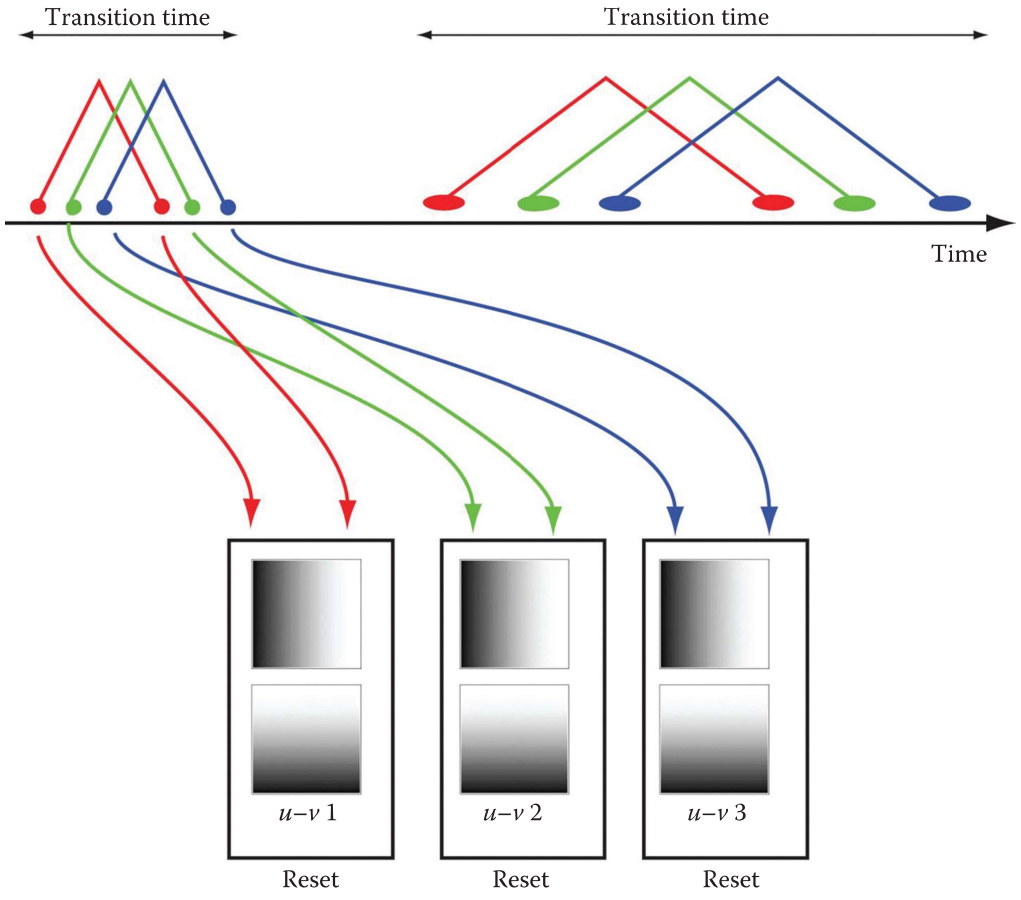


FIGURE 8.20 Weighting different texture maps over time to reduce loss of detail. In this example three texture map coordinates are used.

Figure 8.20 is kind of funky and looks like some crazy triptych or a sanitized version of a Basquiat painting.* I will attempt to explain it in plain English. So why bother with a figure? Well, sometimes it's clearer in words and images. They complement each other. At least that usually works for me.

Three texture coordinates are being evolved over time through a fluid. When the tent function gets to zero, the fluid coordinates are reset as shown in Figure 8.20. No worries; their weight is zero so no one will notice. The other two weights are still alive and well. But then from then on, the defunct texture comes immediately alive after it has been reset and after its weight is set to zero. No worries again; it will rise to power again and then disappear again only to rise again. It is a cyclical process. It could go on forever until your program crashes because of a memory leak, for example.

You do not need three texture maps and a simple tent *teepee* function. I encourage readers to experiment with other combinations at home. I am just saying that this worked out for me. The transition time between texture maps can be easily modeled through a scaling of time. This is depicted on the right-hand side of Figure 8.20. It is just a knob that an animator can use. If it is small, texture coordinates will blend more rapidly and if it is large, texture coordinates will blend at a slower pace. There is no scientific justification for this parameter. Well not that I know of right now. But who cares? This book is about fluid animation. And this technique is way cool. People use it to create nifty effects.

To summarize: Fluids can be dressed up with fancy flowery dresses. Sometimes you get more from less, when it involves mapping a cool texture map with coordinates evolving through the fluid. In addition, these animations can be made livelier by having three sets of u–v coordinates and blending between them over time.

* Jean-Michel Basquiat (1960–1988) was from Haitian descent and was born and raised in New York. He is a true 1980s legend. Check out his art.

CHAPTER 9

The Little Computers Who Can Handle Fluids

> Intellectual property has the shelf life of a banana.
>
> BILL GATES (COFOUNDER OF MICROSOFT)

Imagine 1999.

Yeah, back when people like me were worried about the Y2K bug because we did not trust the COBOL code that was driving bank transactions and Wall Street trading. Hopefully, a lot of my readers won't even remember that time. I am talking about the time that the small computers were crawling out of their rabbit holes. Thanks to Steve Jobs and Bill Gates. Who doesn't have a *smart phone* today?

Of course nowadays smart phones are everywhere and are taken for granted.

I wrote a fluid simulator for the Pocket PC back in 2000 and showed it at our annual computer graphics conference at SIGGRAPH in 2001 during a speech in front of a large audience and of course elsewhere: in pubs and production houses in swinging Soho in London, in the Metro in Paris, in hotel lobbies and lounges in Los Angeles, good thing I didn't drop my iPAQ in the pool at the Figueroa Hotel, also at family gatherings in Geneva, and so on. The list just goes on.

It is cool to share creations in such an effortless manner with other people. Don't get me wrong. A lot of effort went into creating the App of course. I wrote a lot of code and had to download SDKs and emulators to

test and run the code. That takes time and effort. But then you can just show it off in a laid back manner to a wide-ranging audience.

"Check this out man."

Reactions ranged from "awesome stuff dude!" to "what is this rubbish mate?" Good times. To reiterate a point I made earlier: high praise or dumb criticism is better than no reaction at all. Some people get it and some do not.

This section is about how I taught little computers that fit in your pocket to do fluid simulations.

The Palm and the Pocket PC did not have hardware support for floating points. They did not even have fancy graphics interfaces like OpenGL. Still at the time they were cool devices. Even though these devices came out in the late 1990s, it felt like I was back in the 1980s again, yay!

I am going to keep this short.

I felt amused in 2008 when a kid told me after an invited talk: "Why are you showing me a demo on a Palm device that my dad used to figure out his finances?" My response: "Because I can."

I am not going to dwell on how I implemented the basic rendering pipeline of graphics like I did in the late 1990s and in the early 1980s as a teenager. Because now we have *OpenGL ES* that runs on any iOS device. I wasted so much time on re-implementing low-level rendering algorithms. It was fun and I do now understand these algorithms better. Because to code is to really understand something. That is if it runs and works at all. I understand the rendering pipeline. I have coded it many times.

I am now going to open a nasty can of worms that contains **`floats, doubles, doubledoubles,`** and what not. These are *pretend real numbers*. They are stuck in the discrete realm that the Slave understands and lives in. How do you represent continuous real numbers that the Slave understands?

Bill Kahan is a Toronto-born computer scientist who now lives in Berkeley, California. He won a Turing Award in 1989 partly for coming up with a standard for floating numbers. It is called the *IEEE 754 Standard*. Before that the floating point world was the equivalent of an urban jungle. You would get your code to work with your most favorite *pretend reals* in your hood before finding out that the results would be completely different in another hood that was dealing with another version of *pretend reals*. You were stuck in your hood back then. Professor Kahan took us out of this urban jungle. All computer processors now use the same patois for *real numbers*. Phew! Now consistency, and consistent craziness, can

happen in all hoods. That is progress. Respect to Prof. Kahan. His work was a monumental achievement. Chapeau!

Floating points are tricky. Trust me.

I want to introduce *fixed-point* reals instead. They are of course not my invention. I just used them and played with them and had fun with them. They are less tricky and more honest than their IEEE cousins. But they are still tricky. Just in a more predictable way.

Small computers in the late 1990s and PCs in the early 1980s did not support floating point arithmetic on their processors. If you wanted your code to run efficiently you had to use integers. Yes, that's right: a finite number of bits that our Slave can understand, dressed up like in a float dress.

I already spent too much time on this topic. It is *passé*. Let me just show you in Figure 9.1, out of nostalgia, the header file that allows one to implement fixed-point arithmetic instead of floats. To me, it is an achievement that I want to share: albeit a relic from the past. It is a useless piece of code since float operations are actually faster than integers in most cases nowadays. But it is tight and elegant more so than a float dress.

It is very instructive to play with fixed numbers. It shows the kind of traps you can be caught in with float numbers. Try this at home. No one will get hurt.

Using this header file a usual float statement like "`x = a*(b/c)`" becomes in fixed point "`x = XM(a,XD(b,c))`." Yeah it is uglier. Besides, you ask, why am I using 16 bits? That was because it was way back in 1999. Observe how I am using the "`(long)`" type (these babies had 32 bits back then) for multiplication and division. That is why I am using 16 bits. You can make it prettier in C++ or some other object-oriented language by using operator overloading. This has been extremely geeky. If you are into this stuff look it up on the Internet.

Figure 9.2 shows two snapshots of fluids running on a Palm (a) and on an iPAQ Pocket PC (b).

```
/* Each freal in bits is xxxxxxxx.xxxxxxxx That is 16 bits */
#define freal short
#define X1      (1<<8)
#define I2X(i)  ((i)<<8)
#define X2I(x)  ((x)>>8)
#define F2X(f)  ((f)*X1)
#define X2F(x)  ((float)(x)/(float)X1)
#define XM(x,y) ((freal)(((long)(x)*(long)(y))>>8))
#define XD(x,y) ((freal)(((long)(x))<<8)/(long)(y)))
```

FIGURE 9.1 Fixed-point arithmetic implemented using a header file.

FIGURE 9.2 Two snapshots of fluid animations on a small computer. The Palm (a) and the iPAQ (b).

Actually, the Palm version had an accelerometer SD Card with an API. Remember this was the early naught years. I got this SD card from a Swiss-Italian engineer called Paolo Bernasconi, who worked for a startup company in Silicon Valley called *Motion Sense*. He gave me the API and the SD card. We met at a pub in San Jose, California, for a beer during GDC, the annual gamer conference. I gave a talk there on fluid dynamics.

To summarize: I wrote the first fluid interactive animation app for handheld devices. I had to use fixed-point arithmetic and my own renderer to have these simulations run on such a small device. I created it basically from scratch: fun times. This was in 2000.

CHAPTER 10

The Smart Phones That Can Handle Fluids

> I have conquered an empire but I have not been able to conquer myself.
>
> PETER THE GREAT (CZAR OF RUSSIA, 1672–1725)

> Innovation distinguishes between a leader and a follower.
>
> STEVE JOBS (COFOUNDER OF APPLE, 1955–2011)

Now imagine 2007.

Apple comes out with the iPhone: accelerometer built in, support of OpenGL ES, fast processor, Internet access built in, multitouch interface, and so on. Shortly thereafter, the iTouch came out. It was pretty close in coolness to the iPhone. What is not to love? Wow, why did I not buy it in 2007? Because you could not natively write code on the device: only through web apps.

Come on Apple.

All that cool hardware fits in our hand and that we cannot directly access. What a tease. Also I did not want to go the *Jail Break* way. I did not want to get in trouble, and I thought it was the wrong approach.

Then everything changed in early 2008.

Steve Jobs announced that they were releasing a native SDK with an API. I immediately started to download their beta release, bought an iTouch,

and started to code a fluid solver on the device. Not only a fluid solver but also other apps like a simulation of hard spheres interacting. Like those poor hamsters stuck in their plastic balls banging into each other.

Writing an app back then was a pain. This is because I had never wandered into the Mac hood before. This was a new neighborhood. Actually, not that different: just a new patois and a lot of glassy condos. But geez these dudes did not make it easy on me. But in the end I was able to write code on one of the most beautiful designed smart phones at the time.

Here is the breakdown of what I had to go through in the new hood:

1. Get an iDevice, in my case an iTouch
2. Get a MacBook Pro or some other Apple computer
3. Learn Xcode
4. Learn Objective C
5. Learn Cocoa (Touch)
6. Learn how to set up an Apple Dev Account
7. Learn how to set up Keychain, Provisioning, etc.
8. Write the app
9. Add Autodesk Marketing Stuff
10. Get the app approved by Apple

Ironically, one of the easiest parts was writing the app (step 8). But in the end I got an Apple app up and running. The app came out officially in early November 2008 on the App Store and was called *Autodesk Fluid*. This was the first app that Autodesk ever released on an iDevice, and I was the *Agent* for all of Autodesk for 2 years. As a bonus I now know how to write code for iOS devices and the Mac. Figure 10.1 shows three different snapshots of the app in action: another triptych of some sorts. The cheesy branding is what I had to put with in order to get my app out there.

This app was free and it got around 300,000 downloads worldwide. For some crazy reason there were 175,000 downloads from the United Kingdom alone in a couple of days. I have no idea what triggered this frenzy. The number of downloads for a certain app is usually an exponential process that decays very rapidly. There are of course exceptions.

FIGURE 10.1 Three snapshots of the *Autodesk Fluid* app. This was the first app that Autodesk ever released.

At that time, it was the number one free app on the UK App Store. I kept a snapshot of the iTunes UK main page as shown in Figure 10.2. Having that many people download your app is cool. I did not make a single penny out of this app by the way. I did not do it for the money. I just did it to show something cool I created to a wider audience. On the other

FIGURE 10.2 Autodesk Fluid was the number one free app in the United Kingdom.

hand, you have to be ready to face a lot of *trolling*. Just to give you an idea: "the best part of this app is when I deleted it from my iPhone," "pixelated rubbish," "I wish I could give it a minus one star," and so on. Yes it is hard to be popular in the United Kingdom. Of course there was a lot of praise as well: "this is so sick man!," "cool!," "awesome!" and so on. No bad or hurt feelings by the way. In the end, I created something and I shared it. The trolls are like the little boys who trash your sand castle on the beach. Destroying something makes them feel better.

My experience with the Pocket PC was much easier back in 2000. No approval by Microsoft was necessary. Anyone could just beam their apps to each other just like with the Palm. I always compare iOS to a totalitarian regime that creates beautiful cities. Think of Paris, France, rebuilt by Haussmann under the supervision of Emperor Napoleon III or St. Petersburg built from scratch under the orders of Czar Peter the Great. Both cities have a very coherent look because they were built under the supervision of pretty much a single person. I think of Microsoft Windows more like Toronto, Seattle, or Los Angeles: sort of ugly in appearance but more flexible. Neighborhoods come and go. There is contemporary stuff next to old buildings. There is very little consistency in appearance in these cities.

I prefer to use Apple devices, but I prefer to code in the Windows world. I like to visit Paris, but I prefer to live and work in Toronto.

To summarize: I wrote a fluid app for the iPhone in 2008. The first official app released by the company that I work for: Autodesk. The process was somewhat painful but well worth it. There were 175,000 downloads in the United Kingdom alone in a couple of days.

CHAPTER 11

Fluid FX

Version 2.0 of Autodesk Fluid

Making Fluids fun for the masses.

In 2010, our consumer division at Autodesk decided to create a more sophisticated version of the *Autodesk Fluid app*. Better user interface (UI), the ability to warp texture maps like we described earlier and set things on fire! My original *Autodesk Fluid app* was really a mash up of OpenGL ES and Objective C code that I found on the apple.com website. My fluid code however was still my own. I spent some time optimizing it for the iOS operating system. The beauty of Objective C is that you can include plain good old C in your project. The same is true of C++.

But really under the hood my app was ugly: except of course for the fluid code part.

For *Fluid FX* we actually had expert Mac coders, meetings, product managers, and so on. Not me hacking away alone in my fifth floor cave anymore.

This resulted in *Fluid FX* and other apps for the Mac. You can find them at http://usa.autodesk.com/fx-apps/.

Check it out. This is not a shameless promotion to try to sell stuff and make me rich. I don't get a penny. Like I said before, if you create something cool you want to share it with as many people as possible. Then your work was worth all the time and the effort.

One of the lesser known apps is *Motion FX*. It only runs on the Mac under Lion and upgrades. If you have a Mac and don't want to spend a

FIGURE 11.1 Here we go again. *Fluid FX* was the number one *App gratuite* in France.

dime, download it. It is free. It uses the camera on your Mac to track your body movements, including eyes, to create a lot of fun animations. You can get it from the link provided earlier. Try this at home and spread the link to your friends. Spread the fun.

This time, *Fluid FX* became the biggest download on the French App Store. This is shown in Figure 11.1.

To summarize: We did it again but better. *Fluid FX*, *Motion FX*, and *Time FX* are all based on the basic code given in this book. Of course a lot of effort has been put into the design of the UI and the proper writing of iOS code. This was a team effort. I thank the entire team for this to happen.

CHAPTER 12

Show Time! MAYA Fluid Effects

माया

MAYA (ILLUSION) IN SANSKRIT

Let us go back in time. As mentioned in the Prolegomenon (Preface) at the beginning of this book, the fluid solver has been put into our flagship animation software called MAYA.

A bit of history is in order.

Once upon a time there was a company called Alias. It was founded in 1984 in Toronto, Canada. Their first products were geared toward the design of modeling shapes.* Legend has it that General Motors bought our software when some of our engineers showed the GM execs a view from inside a virtual car before it was built. Hard to do that with a clay model, isn't it? The software was very *rock "n" roll* back then. Menus would pop up from the bottom, and there was no *undo* feature.

I joined Alias as a part-time employee in 1994 to put some particle stuff into their Power Animator software when I was doing my PhD at the University of Toronto. In 1995, Alias from Toronto and *Wavefront* from sunny Santa Barbara were both acquired by Silicon Graphics from less

* Back then in the 1980s our coders could say they were into modeling in nightclubs. Yeah right.

sunny Mountain View and merged to create Alias *wavefront*.* At the same time, people were working on a new revolutionary piece of animation software called MAYA. It was finally released in 1998. This software is used in many production houses and was awarded an Oscar in 2003. And yes, the Oscar is in the lobby of our office in Toronto. Well, a good replica in any case, the real one is stored in a secure vault somewhere. We do not want some visitor to run away with the real thing.

At any rate, in 2000 the company decided to put my fluid solver prototype into MAYA. After an epic meeting in Santa Barbara (remember Duncan?), I went back to Seattle and my colleagues went back to Toronto. After a lot of coding and back and forth meetings, MAYA Fluid Effects came out in 2002 with the MAYA 4.5 release.

One of the biggest hurdles that we faced, as I recall it, was to add *volume rendering* to MAYA in a somewhat efficient way. Volumes are like clouds, smoke, fire, steam, and so on. Not like surfaces. Volumes are three dimensional and they scatter, absorb, and sometimes emit light (think of a flame). This makes modeling their effects much harder than shading a surface. There were also software architectural reasons why it was difficult to add volumes to the rendering engine of MAYA. I had to partly deal with these problems, thank you very much. So I won't dwell further on how it was solved.

But still, Figure 12.1 depicts the situation in a simplified manner. Our volume of fluid, the density, is actually sliced into discrete squares from the point of view of the camera. Then all the illuminations from each slice are blended together from front to back (following the arrow). The situation is a little more complicated in practice, of course. But that is the basic idea.

How to render volumes was a topic part of my PhD thesis. So it was not too much of a challenge to write this code. There is an interplay between dynamics and rendering. Yeah sure you figured out the dynamics (fluid animation), but in the end one still has to render these volumes. I am not going to talk about how to render—create an image—of a volume. There are many good books written on that topic. Look it up if you are interested.

We also added 3D textures to decorate our 3D fluids. Or in some cases just to create a static texture, like in creating background clouds. This is just a pretty straightforward 3D extension of the procedure we explained

* At the time there was a contest within the company to find a new name for the merger. At the end, we just added the two names together. This reminds me of the naming of the 1998 soccer stadium in Paris for the World Cup Final. The best that the committee could come up with was *Stade de France*. Come on!

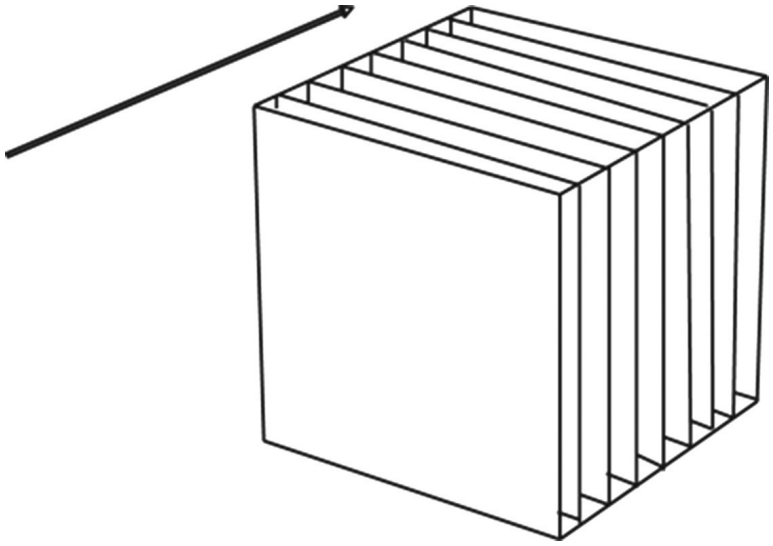


FIGURE 12.1 The rendering of a 3D volume.

above for 2D textures. Just the rendering part is trickier. This is shown in Figure 12.2, which is a still from an animation. The camera is able to *fly through the clouds*. Notice all the subtle effects due to scattering and self-shadowing. There is no emission in this case. The animator through various knobs can control the appearance of the clouds. In Figure 12.2, the animator decided to make them nice and fluffy like cotton balls.

FIGURE 12.2 A volumetric cloud texture map in MAYA.

It is hard to demonstrate the power of MAYA Fluid Effects without showing actual animations. But if you have gone to the movies you have seen their application. Just to name a few: *The Lord of the Rings, Ghost Rider, The Day After Tomorrow, 300, Avengers, Tintin*, and so on. In fact MAYA Fluid Effects got a Technical Achievement Award from the Academy of Movie Picture Arts and Sciences in 2008. I had to wear a tuxedo again.

Following is an important point I want to make. Just to make it crystal clear.

> The artists who create the visuals you see in movies really deserve all the credit for the final shots. We provide the brushes, the tools. When I used to paint with the airbrush, it wasn't Jens Andreas Paasche* who made the painting but he—the inventor—definitely helped me by creating paintings using such a beautiful, elegant and easy to use tool: his airbrush. That was his creation. My paintings using his tools were my creations.

The following figures show many examples using MAYA Fluid Effects created by my friend Duncan Brinsmead who is also my colleague at Autodesk.

Figure 12.3 shows four frames of ink being dropped in a fluid. The forces exerted on the density of ink are to counter gravity. They descend and create an external force on the fluid. The incompressibility of the fluid makes the density curl around and create cool patterns. These four frames do not do justice to the actual animation. But the stills are still cool.

The opposite effect is shown in Figure 12.4. In this case, a nuclear explosion is simulated using a density field and a temperature field. In this case, the motion is upward. We show four frames of the animation. Of course these animations use some spicy magic sauces concocted by Duncan that are not covered by the basic fluid solver.

MAYA Fluid Effects can also be used to create fire-like effects, this is shown in Figure 12.5. It uses a temperature field and a density field that influence the air flow. The color can be controlled by the animator through a ramp.

There are a lot more of effects, of course. We can simulate snow avalanches (Figure 12.6) and liquids (Figure 12.7). The liquid simulations were really hard to implement because of the gain and loss of the mass of

* Paasche founded the company in 1906 in Chicago.

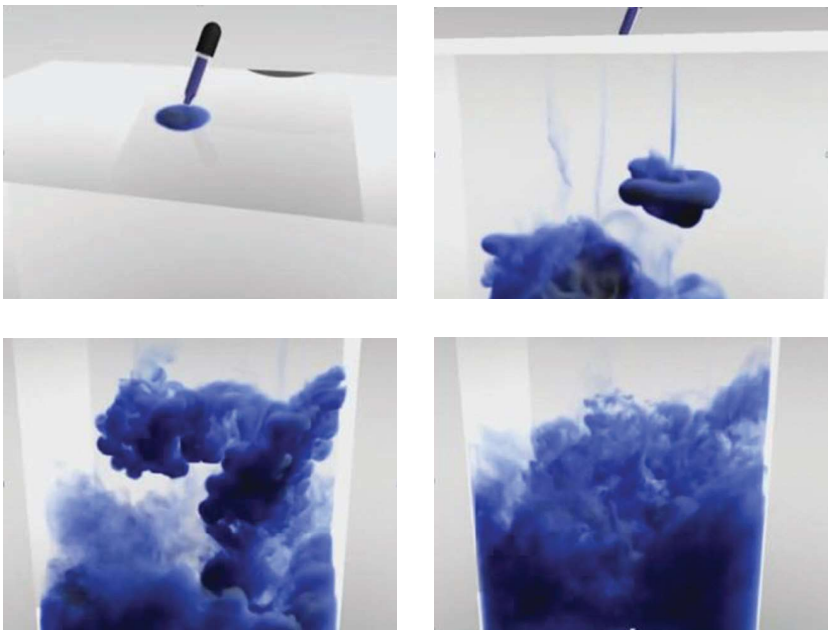


FIGURE 12.3 Four frames of an animation created using MAYA Fluid Effects.

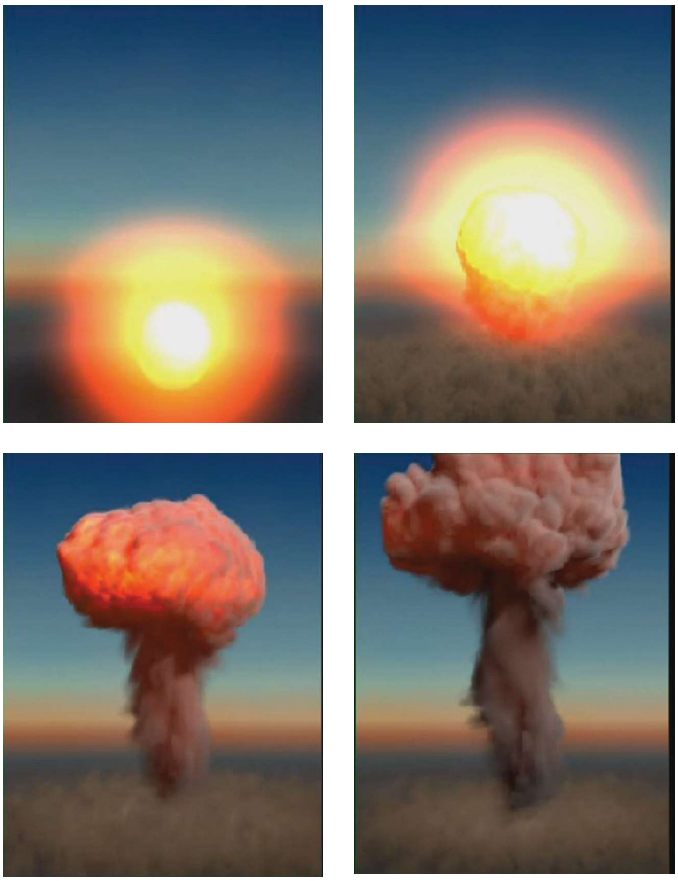


FIGURE 12.4 Four frames of a nuclear explosion created with MAYA Fluid Effects.

FIGURE 12.5 Four frames of a simple flame created with MAYA Fluid Effects.

FIGURE 12.6 Simulation of a snow avalanche using MAYA Fluid Effects.

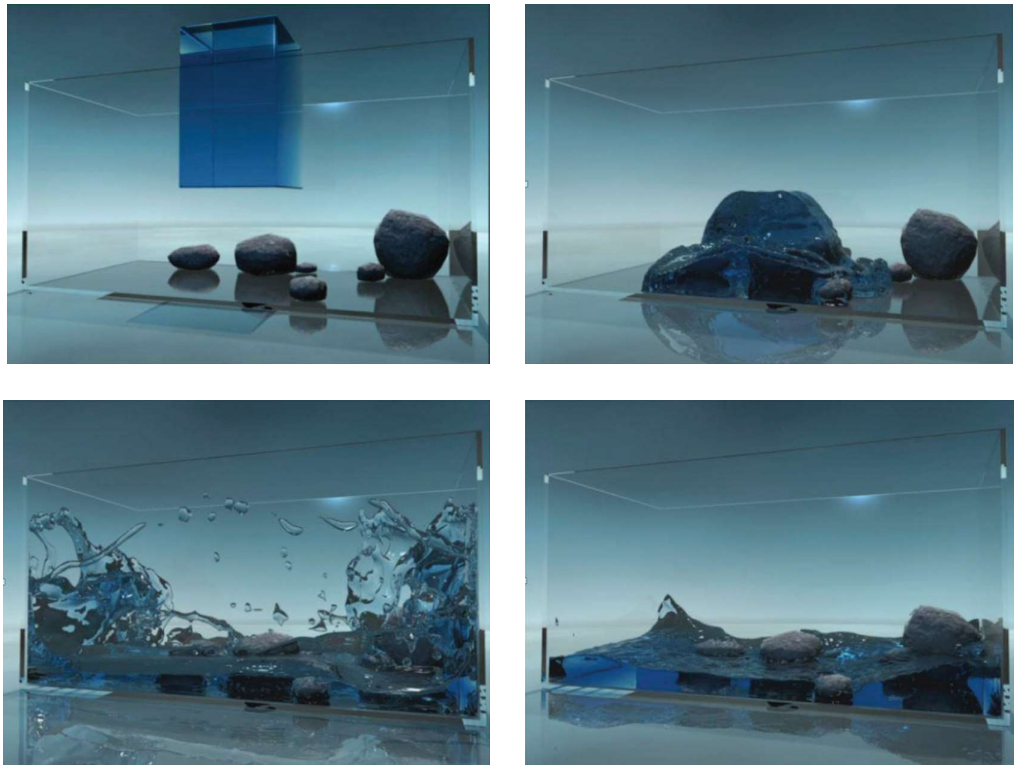


FIGURE 12.7 Four frames of a liquid simulation using MAYA Fluid Effects.

the liquid in the advection process. The way we solved these artefacts is somewhat inelegant. We used a reaction-diffusion solver to *straighten* the boundary between the air and the liquid. We also modified the Fish equation to compensate for mass gain and mass loss. We basically make the fluid compressible and divergent where we detect mass gain and mass loss.

Unfortunately, this book will not deal with all the details of liquid simulation in our MAYA software. Why not? It was really sort of a *whack a mole* approach. I hope there is something more elegant out there.

I hope this gives the reader a broad overview of what MAYA Fluid Effects can achieve.

When I was visiting Weta Digital in Wellington, New Zealand, I was talking in French with one of the animators called Mathieu Chardonnet at a local bar. It turned out that he grew up in *La Haute Savoie* next door to Geneva. I used to ski there. After an hour he asked me what I was doing. It turns out that he is a heavy user of MAYA Fluids: so cool.* I love whimsical experiences like that.

Get our MAYA software and try this at home. You can download a free trial of our MAYA software for 30 days. Just search for *free MAYA software* in your favorite search engine.

To summarize: Putting research code into a complex commercial software is not easy. It is a team effort. Volume rendering ended up being a challenge. Putting a basic solver in a powerful software package like MAYA enables people to create a lot of cool visual effects. It has been used in feature movies. We received an Academy Award for it. I had to wear a tuxedo for the second time in my life.

* You can find his art at: http://www.krop.com/mathieuchardonnet/

CHAPTER 13

Fluids on Arbitrary Surfaces

The true sign of intelligence is not knowledge but imagination.

ALBERT EINSTEIN (WELL YOU KNOW WHO HE IS)

Previously, we talked about fluids on donuts, also known as tori, which we harnessed using the almighty Fourier transform. Now I will briefly describe a technique to animate fluids on other surfaces: surfaces with more holes or less than donuts. A good example is our good old earth, which is topologically equivalent to a sphere: a big blue round ball with some fluffy clouds floating on it. No holes. But there are surfaces with more holes. Some of those creatures are depicted in Figure 13.1. The creature on the far right is not even a closed surface, but it is has *one hole*.

First I have to describe how these Subdivision Surfaces are created.

How did I create these creatures? That was back in 2002 using our MAYA software in the loft in Seattle. I used subdivision surfaces. This is a technology I worked on. And for some results I created I shared an Academy Award with Ed Catmull and Tony de Rose from Pixar in 2006: these surfaces are used in many movies.

The concept of subdivision surfaces is best explained as a process. First you start with a discrete surface made of polygons. Then like a sadist you chop off corners; in the end after an infinite number of sadistic acts you end up with a nice smooth version of the discrete surface that you

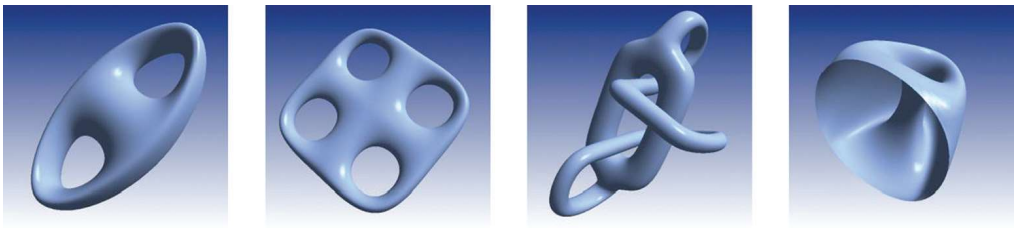


FIGURE 13.1 Four examples of surfaces on which we can model fluids.

started with.* In 1997, I showed that you could save the sadist a lot of time. You don't have to chop off corners. Just compute eigenvalues and eigenvectors. Bring in the German Eigen's to the rescue. From the initial surface you can get directly to the same surface that would have been reached by chopping off corners. Computer graphics researchers had done this before for the initial set of points. I showed that you could do it for all points: even an infinite number of them, directly from the initial set of polygons.

This works not only just in this case. There is a class of subdivision surfaces called "Catmull–Clark subdivision surfaces." They are named after the inventors Ed Catmull and Jim Clark: two computer graphics legends. Catmull started Pixar and Clark founded Silicon Graphics. Their work generalized some smooth surface stuff that only worked for regular quad meshes to arbitrary shapes. This is how it works. It is a recursive process. At each step new points are introduced for each quadrilateral, *quad*, and then all the points are smoothed: the old ones and the new ones.

This is to be repeated *ad infinitum*. In the end, you get a smooth looking surface.

This is illustrated for a cube shrinking into a *sphere* in Figure 13.2. Actually, the limit surface is not a perfect sphere: just an approximation of a sphere. This process is gentler than the sadistic corner cutting procedure described earlier. But in the end you get a nice looking surface, which can have any topology. The gentle approach results in smoother surfaces than the sadist's approach.

I always wanted to combine my subdivision work and my fluid work.

This dream resulted in a 2003 SIGGRAPH publication. Academics were puzzled about why I was wasting my time and not doing something useful for my company. Just minutes before my talk, I was asked to justify why

* I am oversimplifying of course. But that is the gist of the subdivision process. This book is not about subdivision surfaces. Look it up if you want more information.

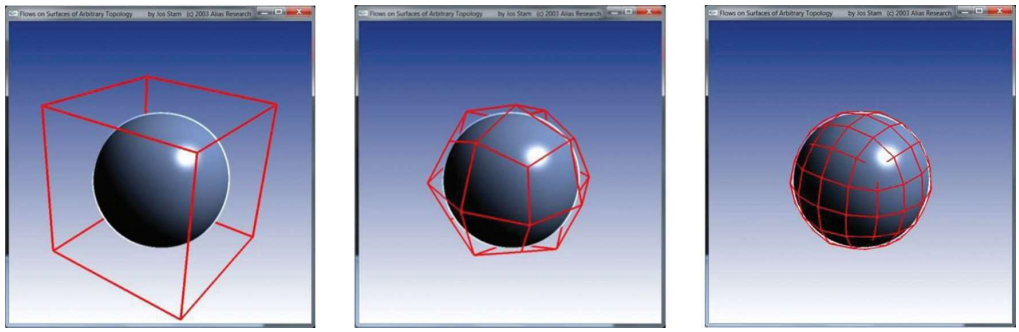


FIGURE 13.2 Three steps of Catmull–Clark subdivision (in red) and the limit surface in blue.

I created this. In my talk I just said: "because it is a cool thing to do"*: Yeah whatever. But really, you never know where research will lead you, and the benefits that will come of it.

This is how I proceeded. I assigned a 2D fluid solver to each initial quad that defines a Catmull–Clark subdivision surface. Using my limit surface technique, I was able to map the fluid solve to the surface. The mapping for three quads of a cube is shown in Figure 13.3.

One tricky aspect of this technique is how to handle exchanges of information between adjacent patches. In order to handle this, we use our ghosts located at the boundary of each quad domain. Remember Figure 6.5 described earlier.

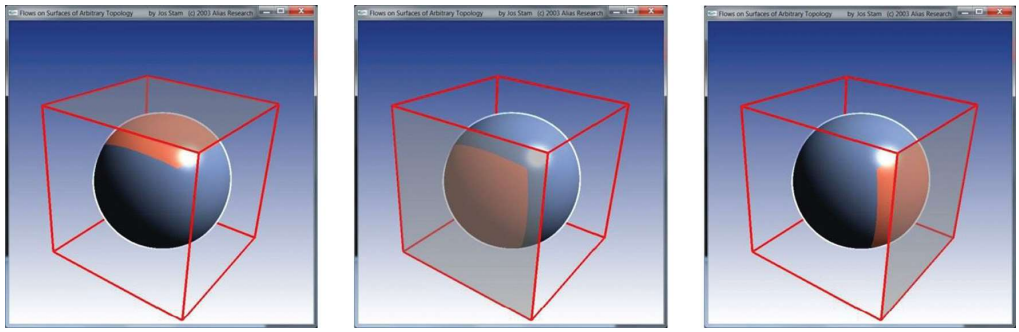


FIGURE 13.3 Using an exact evaluation, each quad from the initial mesh can be mapped to a smooth patch on the surface.

* Just as an anecdote. My paper got accepted because of a snowstorm in Colorado. Getting papers accepted at SIGGRAPH is tricky. But who cares, it got accepted. I think it is a beautiful paper. In 2011, I was invited specifically to present these results at IMPA in Rio de Janeiro. I include these results in all my invited presentations on fluid animations.

"Okay neighbor quad I will tell your ghosts what to do at our common border if you tell my ghosts what to do, deal?"

This sounds easy but it is not. It is a lot trickier. I am not going to provide any code here. For that, check out my SIGGRAPH 2003 paper and the follow-up work. I am just going to show some of the results that can be achieved with this technique. Some are depicted in Figure 13.4. These are snapshots from a demo program I wrote, which runs in real time on a decent PC. I showed it in San Diego, California, at SIGGRAPH 2003. These pictures are all rendered using Open GL using transparency and texture maps. Back in 2003 this was considered pretty cool.

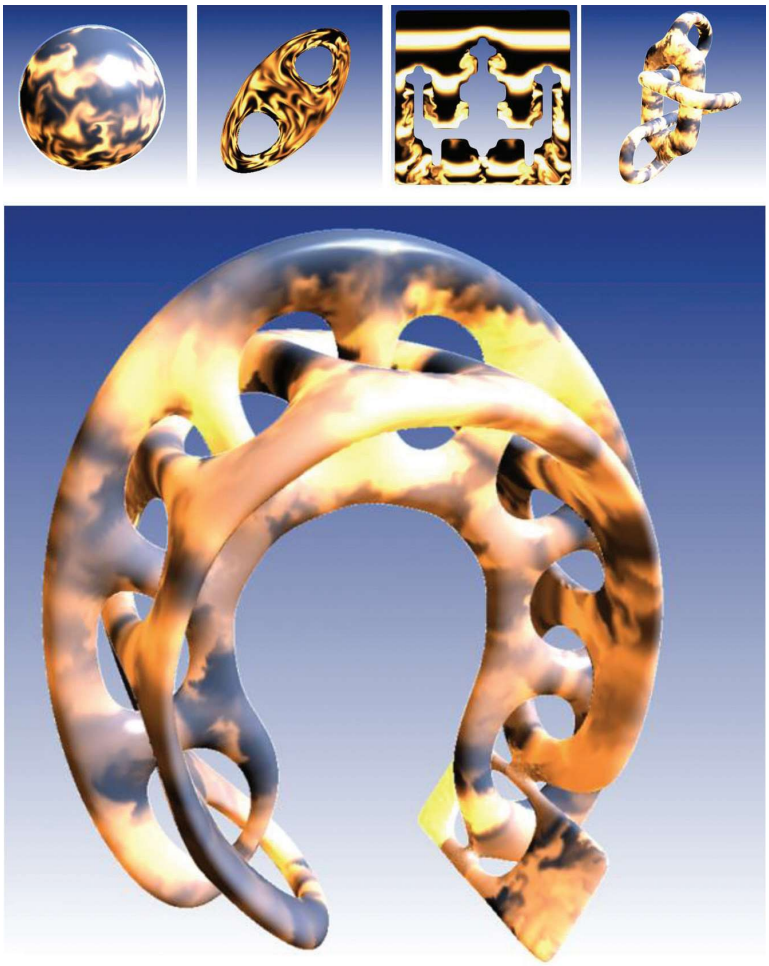


FIGURE 13.4 Images created using fluid simulations on arbitrary surfaces. They are all rendered using OpenGL.

To summarize: I combined some of my subdivision surface work and some of my fluids work. In this manner, I was able to simulate fluids on arbitrary surfaces. I left out many details, but I hope the reader gets the gist of the method. If you want to know more, check out my SIGGRAPH 2003 paper.

CHAPTER 14

Control Freaks! How to Make Fluids Do What We Want

Only you can control your future.

DR. SEUSS (AMERICAN GENIUS WRITER AND ILLUSTRATOR, 1904–1991)

Go To Statement Considered Harmful.

TITLE OF A FAMOUS ARTICLE BY EDSGAR DIJKSTRA (LEGENDARY DUTCH-AMERICAN COMPUTER SCIENTIST)

Usually, we stir up fluids with a swine's hair and see what happens. That is not acceptable for animators who want to create a particular special effect involving a fluid animation. Let's say you want your fluid over time to move some density into a particular shape. In the interim, you want the fluid to look natural and nice and smooth while possibly following the laws of physics.

Let's control fluids!

Professor Zoran Popovic from the University of Washington in Seattle proposed a solution to this problem. I was hanging out at "You Dub" at

the time in rainy Seattle when I was not in the Pioneer Square loft. At first I was very skeptical that this approach would work at all. Two graduate students we worked with were known as A&A: Adrien Treuille and Antoine McNamara. They implemented a solution first and made it work. They deserve all the kudos for this work.

They proved me wrong. Anytime please. I like to be proven wrong. That is progress to me. Being proven wrong means you have to work harder to be confident in your solution next time. Being wrong some of the time is the name of the game in research. Learn from it. I hope I do.

I did write my own version in that loft in Seattle once I knew it was possible. And I did not use FORTRAN like A&A. I used *f2c* a tool that translates FORTRAN code to C code. The translator called f2c creates one of the craziest and ugliest C codes on the planet. But hey it will talk to your C code without you having to buy and install a FORTRAN compiler. FORTRAN compilers are free if you live in the Linux hood of town. To use f2c, you have to include a header file called f2c.h, which starts with the following two comments:

```
/* f2c.h  --  Standard Fortran to C header file */

/**  barf  [ba:rf]  2.  "He suggested using FORTRAN,
     and everybody barfed."

      - From The Shogakukan DICTIONARY OF NEW ENGLISH
        (Second edition) */
```

That is geek humor folks. Here is a snippet of code of what f2c translates some code from FORTRAN to C:

```
/* Subroutine */ int poistg_(nperod, n, mperod, m, a,
 b, c__, idimy, y, ierror, w)
integer *nperod, *n, *mperod, *m;
real *a, *b, *c__;
integer *idimy;
real *y;
integer *ierror;
real *w;
{
    /* System generated locals */
    integer y_dim1, y_offset, i__1, i__2;
```

```
    /* Local variables */
    static integer iwba, iwbb, iwbc, modd, mhmi, mhpi,
    irev, i__, j, k;
    static real a1;
    static integer mskip;
    extern /* Subroutine */ int postg2_();
    static integer mh, mp, np, iwtcos, ipstor, iwd,
      iwp, mhm1, iwb2, iwb3, nby2, iww1, iww2, iww3;

/* more ugly nonsense and then */

/* … */

/* Parameter adjustments */
    --a;
    --b;
    --c__;
    y_dim1 = *idimy;
    y_offset = 1 + y_dim1 * 1;
    y -= y_offset;
    --w;

/* … */

L107:
    switch ((int)*nperod) {
      case 1:  goto L108;
      case 2:  goto L108;
      case 3:  goto L108;
      case 4:  goto L119;
    }


/* etc etc etc and more automatically generated ugly
  crazy C code*/
```

What is up with the double underscores by the way?

In the C language, it is considered a big *no no* to use the *goto* statement. But sometimes you have to use it to get out of a rabbit hole. But it is considered heresy. In FORTRAN, gotos are all over the place. For those of you who are not familiar with gotos or jumps, they tell your Slave to take

a hike and go somewhere else, sometimes far away. Kind of like a *chutes* and *ladders* game.

I have to tell this story. I sent some of this automatically translated code to a programmer in Toronto from my loft in Seattle. He thought I was crazy to write code like that. No human C coder writes code like that. Only computers automatically write code like that. I have since then written my own Fish Solver in clean C. See earlier text.

14.1 SHOOTING CANNONBALLS IN TWO DIMENSIONS

> I hate small towns because once you've seen the cannon in the park there's nothing else to do.
>
> LENNY BRUCE (AMERICAN COMEDIAN, 1925–1966)

Here is the basic idea.

In the spirit of keeping things simple in this book, I will illustrate this idea with a simple example in two dimensions first.

Figure 14.1 shows the situation. The goal is to shoot a 2D cannonball (blue) to hit the target shown in red on the right. The curve in red illustrates the actual trajectory that the red *cannonball* takes given the initial velocity: angle plus speed. In the first two snapshots on the left of Figure 14.1, the blue cannonball misses the red target. But, yes the blue guy hits the red guy in the third snapshot.

Question: How did we get there?

Answer: Through the yellow curve depicted at the top of the four frames in Figure 14.1. This curve is the distance squared between where the blue ball ends up and the red target. The solution is where the corresponding yellow dot is at a minimum of the yellow curve (actually zero in this case). One solution is shown in the third snapshot of Figure 14.1. But there is also another solution shown in the fourth snapshot of Figure 14.1. In this case, the trajectory of the red cannonball takes less time. This is probably

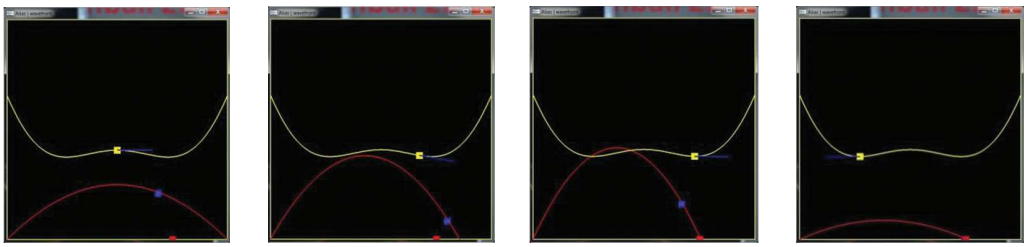


FIGURE 14.1 Optimizing the trajectory of a cannonball in two dimensions.

the preferred one: this is Occam's razor principle (mentioned earlier) all over again. From multiple solutions one usually chooses the simplest one.

There are two solutions because the distance function in yellow has two *valleys* of equal depth.

This is an important point.

We already discussed nonlinear problems earlier. For this particular nonlinear problem, the two valleys are equally valid solutions. If we add another term that is equal to the time that the blue cannonball takes, then the first valley is the right solution.

The gradient of the yellow curve is used to eventually find the solution. Remember the gradient is the direction of steepest ascent and its inverse is the direction of biggest descent. The gradient of the yellow slope is illustrated as a blue vector in Figure 14.1.

Let us analyze this situation. What we have here is

1. The *controls*: initial velocity of the blue cannonball. Two parameters: in this case angle and speed.
2. The *simulation* that creates a result from the controls.
3. The *goal* we want to achieve. In this case it is the distance squared between the red dot and the blue cannonball when it hits the floor.
4. The *optimizer*: a mechanism that updates the initial velocity (controls) given the goal function.

The key ingredient is Step (4). In our simple cannonball example, we were just interactively updating the controls by following the gradient of the goal function. Just like those reckless skiers going for the direction of steepest descent until they hit the bottom of the hill.

14.2 COMPUTER OPTIMIZERS

> Premature optimization is the root of all evil.
>
> DONALD ERWIN KNUTH
> (AMERICAN COMPUTER SCIENCE LEGEND)

We can abstract the simple situation described earlier with the following nomenclature. Mathematicians like to do these kinds of things.

1. A set of controls
2. A simulator
3. A cost function (*energy function*)
4. An optimizer

This is a very general framework. You can add additional constraints to this list as required. The bottom line is that this list is an outline of a *nonlinear* optimization problem.

To find a solution, keep on refining the controls based on the result of the objective function and on using an optimizer. This is shown in Figure 14.2.

This figure illustrates what happens when you run a simulation of something that is dependent on some *controls*. The *simulated output* is entirely determined by the controls that drive the simulation. Once that output is generated by the simulation, it is compared to the *desired output* proposed by the animator. The mismatch between the output of the simulation and the desired state is then passed onto the optimizer. The optimizer takes these results and updates the controls accordingly.

The magic happens in the optimizer.

Then we go through this again and again until our fatigued *optimizer* tells us: "Enough, guys. We got a solution and I have to take a nap." Or alternatively: "Your input is garbage, get your gradients right."

The optimizer is the key to this process.

What Zoran and A&A proposed was to use an optimizer called BFGS: another geek name. It is an acronym for the four researchers who came up with the technique in the early 1970s. Their last names are **B**royden, **F**letcher, **G**oldfarb and **S**hanno: hence the name **BFGS**.

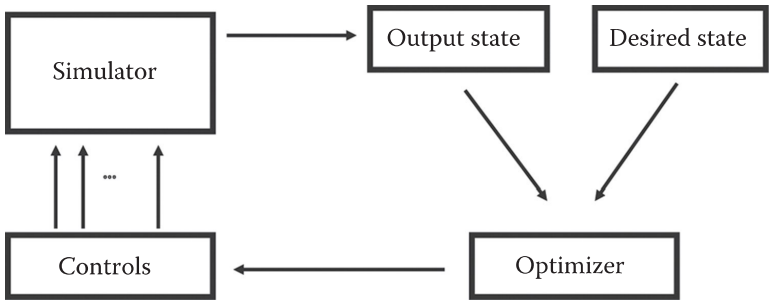


FIGURE 14.2 The basic framework of optimization.

Back then I had never heard of this optimization technique. That is why it is good idea to work with smart people. You learn from them while having fun. Geez and as a bonus, A&A also speak French.

Let us treat BFGS as a black box.* By *black box* I mean we are not going into the internals of the optimizer. This has nothing to do with orange boxes in airplanes. What it needs is the current controls, the value of the goal function and the *gradient* of the goal function and *voilà* you get a new set of controls.

The controls are explicit. The goal function is just computed by running the simulation and comparing the final state to the desired state: easy.

However, what about computing the gradient?

To explain this I will guide you through a simple example. It illustrates how you can compute the gradient of some code with respect to some controls *automatically*. For real.

Ready?

The simple code is shown in Figure 14.3.

It is a toy, not a serious example, of course. The goal is to minimize the goal function `f`. But what about the value of the gradient of `f` with respect to the controls `u` and `v`? To this end, we add a set of new variables. We actually differentiate each line of code. For each variable, we associate a cousin who is bigger. The cousin's size is relative to the number of controls: two in this case. Not that big really, but hold on.

Figure 14.4 shows the original code with their bigger cousins. Each operation is also applied to the cousins. They are different but related through calculus.

Shame on me, I promised not to mention calculus in this book. Readers who know calculus will figure out the recipe. Others please go along

```
float x, y, z;                      /* variables */
float u, v;                         /* controls */
float f;                            /* cost variable */

x = y + z + u*u;    /* statement #1 */
y = z*y + x*v;      /* statement #2 */

f = x*x + y*y + u*u + v*v; /* cost function */
```

FIGURE 14.3 Some simple code.

* Okay, if you know the lingo: it is a Quasi-Newton method, which maintains a sparse inverse of the Hessian updated using a user-provided gradient. It also involves a local line search in each step.

```
float x, y, z, dx[2], dy[2], dz[2]; /* variables */
float u, v, du[2], dv[2];           /* two controls */
float f, df[2];                     /* cost variable */

dx[0]=dx[1]=dy[0]=dy[1]=dz[0]=dz[1]=df[0]=df[1]=0;
du[0]=1; du[1]=0; dv[0]=0; dv[1]=1;

x = y + z + u*u;    /* statement #1 */
   dx[0] = dy[0] + dz[0] + 2*u*du[0];
   dx[1] = dy[1] + dz[1] + 2*u*du[1];

y = z*y + x*v;      /* statement #2 */
   dy[0] = z*dy[0] + y*dz[0] + v*dx[0] + x*dv[0];
   dy[1] = z*dy[1] + y*dz[1] + v*dx[1] + x*dv[1];

f = x*x + y*y + u*u + v*v; /* cost function */
   df[0] = 2*x*dx[0] + 2*y*dy[0] + 2*u*du[0] + 2*v*dv[0];
   df[1] = 2*x*dx[1] + 2*y*dy[1] + 2*u*du[1] + 2*v*dv[1];
```

FIGURE 14.4 The code from Figure 14.3 augmented with their big cousins.

the flow. Or skip the math and the code and move on. Maybe it is time for an Intermezzo, perhaps?

For those who are still with me, here is the method to this madness.

I am not going to dwell on this further but in C++ you can overload operators. That to me is one of the coolest features of C++. The computer language called *Java*, for example, does not have this feature that makes for uglier code but according to their *aficionados* makes it more explicit and more readable and less bug prone. I am not religious about these matters. Use whatever computer language that works for you. Some people dress like hipsters, squares, mods, punks, and so on. Who cares?

So, welcome to the world of *Automatic Differentiation*, also known in vernacular geek speak as *AD*.

By overloading the usual operators like "+," "–," "*," "/," "sin," "cos," and so on, with including your big cousins, you can just write, for example:

```
dfloat<666+1> x, y, z;
dfloat<666+1> f = x*x + y*z;
```

What is a **dfloat**? It is a float forced to hang out with his bigger fellow cousin at all times. The big cousins are told exactly what to do. In Figure 14.5, we provide a C++ class that implements automatic differentiation for N controls. This class can be used for pretty much any optimization problem that uses a gradient like BFGS. I have to mention that for some problems that are not smooth it might fail miserably. There might not even be a gradient defined at some places. But BFGS works surprisingly well even in these particular nonsmooth cases.

```
#include <math.h>

template <int N> class dfloat
{
public:
       float v[N+1]; // change this to your favorite "pretend real."
// and all references to floats. Use a typedef or a define or whatever.

       dfloat () { for ( int i=0 ; i<=N ; i++ ) v[i] = 0.0f; }
       dfloat ( float s ) { v[0] = s; for ( int i=1 ; i<=N ; i++ ) v[i] = 0.0f; }
           float val ( void ) { return ( v[0] ); }
       float val ( int i ) { return ( v[i] ); }
       void val ( float s ) { v[0] = s; }
       void val ( int i, float s ) { v[i] = s; }

       dfloat & operator = ( dfloat & a ) {
                       for ( int i=0 ; i<=N ; i++ ) v[i] = a.v[i];
                       return ( *this ); }

       dfloat & operator = ( float s ) {
              v[0] = s;
              for ( int i=1 ; i<=N ; i++ ) v[i] = 0.0f;
              return ( *this ); }

       dfloat & operator += ( dfloat & a ) {
              for ( int i=0 ; i<=N ; i++ ) v[i] += a.v[i];
              return ( *this ); }

       dfloat & operator -= ( dfloat & a ) {
              for ( int i=0 ; i<=N ; i++ ) v[i] -= a.v[i];
              return ( *this ); }

       dfloat & operator *= ( dfloat & a ) {
              for ( int i=1 ; i<=N ; i++ ) v[i] = v[i]*a.v[0] + v[0]*a.v[i];
              v[0] *= a.v[0];
              return ( *this ); }

       dfloat & operator /= ( dfloat & a ) {
              float g = a.v[0]*a.v[0];
              for ( int i=1 ; i<=N ; i++ ) v[i] = (v[i]*a.v[0]-v[0]*a.v[i])/g;
              v[0] /= a.v[0];
              return ( *this ); }

       dfloat & operator += ( float s ) {
              v[0] += s;
              return ( *this ); }

       dfloat & operator -= ( float s ) {
              v[0] -= s;
              return ( *this ); }

       dfloat & operator *= ( float s ) {
              for ( int i=0 ; i<=N ; i++ ) v[i] *= s;
              return ( *this ); }

       dfloat & operator /= ( float s ) {
              for ( int i=0 ; i<=N ; i++ ) v[i] /= s;
              return ( *this ); }

       dfloat operator - ( void ) {
              dfloat c;
              for ( int i=0 ; i<=N ; i++ ) c.v[i] = -v[i];
              return ( c ); }

       dfloat operator + ( dfloat & a, dfloat & b ) {
              dfloat c;
              for ( int i=0 ; i<=N ; i++ ) c.v[i] = a.v[i] + b.v[i];
              return ( c ); }

       dfloat operator - ( dfloat & a, dfloat & b ) {
              dfloat c;
```

FIGURE 14.5 Automatic differentiation implementation in C++. Operator overloading rules! Templates in this case too. (*Continued*)

```
        for ( int i=0 ; i<=N ; i++ ) c.v[i] = a.v[i] - b.v[i];
        return ( c ); }

dfloat operator * ( dfloat & a, dfloat & b ) {
        dfloat c;
        c.v[0] = a.v[0] * b.v[0];
        for ( int i=1 ; i<=N ; i++ ) c.v[i] = a.v[i]*b.v[0] + a.v[0]*b.v[i];
        return ( c ); }

dfloat operator / ( dfloat & a, dfloat & b ) {
        dfloat c;
        c.v[0] = a.v[0] / b.v[0];
        float g = b.v[0]*b.v[0];
        for ( int i=1 ; i<=N ; i++ ) c.v[i] = (a.v[i]*b.v[0] - a.v[0]*b.v[i])/g;
        return ( c ); }

dfloat operator + ( float s, dfloat & a ) {
        dfloat c;
        c.v[0] = s + a.v[0];
        for ( int i=1 ; i<=N ; i++ ) c.v[i] = a.v[i];
        return ( c ); }

dfloat operator + ( dfloat & a, float s ) {
        dfloat c;
        c.v[0] = a.v[0] + s;
        for ( int i=1 ; i<=N ; i++ ) c.v[i] = a.v[i];
        return ( c ); }

dfloat operator - ( float s, dfloat & a ) {
        dfloat c;
        c.v[0] = s - a.v[0];
        for ( int i=1 ; i<=N ; i++ ) c.v[i] = -a.v[i];
        return ( c ); }

dfloat operator - ( dfloat & a, float s ) {
        dfloat c;
        c.v[0] = a.v[0] - s;
        for ( int i=1 ; i<=N ; i++ ) c.v[i] = a.v[i];
        return ( c ); }

dfloat operator * ( float s, dfloat & a ) {
        dfloat c;
        for ( int i=0 ; i<=N ; i++ ) c.v[i] = s*a.v[i];
        return ( c ); }

dfloat operator * ( dfloat & a, float s ) {
        dfloat c;
        for ( int i=0 ; i<=N ; i++ ) c.v[i] = a.v[i]*s;
        return ( c ); }

dfloat operator / ( float s, dfloat & a ) {
        dfloat c;
        c.v[0] = s/a.v[0];
        float g = a.v[0]*a.v[0];
        for ( int i=1 ; i<=N ; i++ ) c.v[i] = -s*a.v[i]/g;
        return ( c ); }

dfloat operator / ( dfloat & a, float s ) {
        dfloat c;
        for ( int i=0 ; i<=N ; i++ ) c.v[i] = a.v[i]/s;
        return ( c ); }

dfloat dsqrt ( dfloat & a ) {
        dfloat c;
        c.v[0] = sqrtf(a.v[0]);
        for ( int i=1 ; i<=N ; i++ ) c.v[i] = 0.5f * a.v[i] / c.v[0];
        return ( c ); }

dfloat dacos ( dfloat & a ) {
        dfloat c;
        c.v[0] = (float) acos(a.v[0]);
```

FIGURE 14.5 (*Continued*) Automatic differentiation implementation in C++. Operator overloading rules! Templates in this case too. (*Continued*)

```
                float g = -1.0f/sqrtf(1-a.v[0]*a.v[0]);
                for ( int i=1 ; i<=N ; i++ ) c.v[i] = a.v[i] * g;
                return ( c ); }

};
```

FIGURE 14.5 (*Continued*) Automatic differentiation implementation in C++. Operator overloading rules! Templates in this case too.

You can use this code to optimize or *tame* any existing code you have. The AD of *pretend real numbers* do not have to be **`dfloats`** by the way: they could be **`ddouble`** or **`ddoubledouble`**, and so on. It is pretty easy to change the code provided in Figure 14.5 to accommodate other worms. Just change the one line at the top as mentioned in the red comment and replace floats by whatever worm you are using.

For example, the code shown in Figure 14.4 can be written more succinctly using the code from Figure 14.5. Let us call that file "ad.h" so we can cleanly include it in the code below. Make sure it lives in your project directory. Or change your IDE to include the directory where it lives. The latter is probably better as you can use the code for different projects that need it.

With this header file, our useless code becomes "AD&C++-sanitized" as shown in Figure 14.6.

The code is much simpler. Even better you can take any of your favorite code and simply replace your **`floats`** with **`dfloats`**. Make sure to include the "ad.h" file and use C++ not C. Your C code will not compile with "ad.h." There is no need to include fancy libraries either. All the automatic differential meat is in the header file.

For fun I applied this methodology to some *vanilla* rigid body code that was gathering dust in my drawer in that Seattle loft. The experiment

```
#include "ad.h"

dfloat<2> x, y; // variables
dfloat<2> u, v; // controls
dfloat<2> f;    // cost variable

u.val(0) = 1.0f; v.val(1) = 1.0f;

x = y + z + u*u;    // statement #1
y = z*y + x*v;      // statement #2
f = x*x + y*y + u*u + v*v; // cost function
```

FIGURE 14.6 The code of Figure 14.4 rewritten using the Automatic differentiating code.

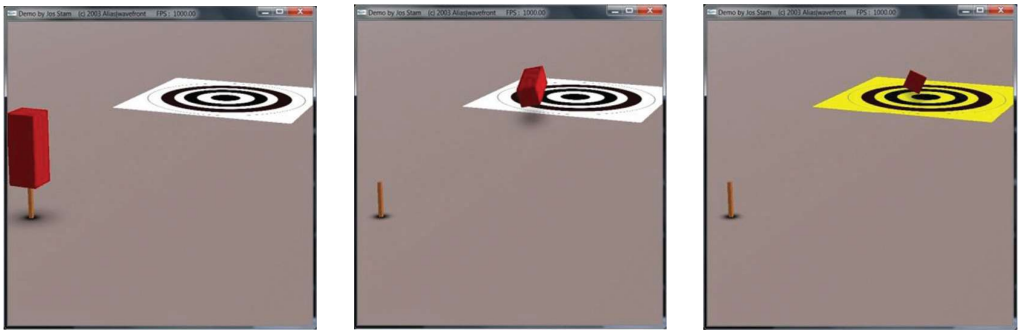


FIGURE 14.7 Three snapshots of a control of a rigid body trying to hit the target when it turns yellow.

the code implements is the simple cannonball simulation illustrated in Figure 14.1 earlier, but it runs in 3D space. However, in this case, there are three controls modeling the initial impulse exerted on the rigid body at the start of the simulation. Just like in the 2D cannonball example, the rigid body has to hit a target when it turns yellow at a specified time. This is illustrated in Figure 14.7.

The demo is cool. The three frames shown in Figure 14.7 do not capture the optimization process. The figure only shows one simulation attempt at hitting the target when it turns yellow. In this case it failed. But lo and behold, the almighty BFGS optimizer will modify the controls and try to improve the next attempt at reaching the bullets' eye when it turns yellow.

For fluids we control the motion using an array of forces that guide an initial blob of density into a particular shape. The controls in this case are an array of 5×5 forces that are applied at different times. So if there are N frames, then the number of controls is $N \times 25$. The basic setup is depicted in Figure 14.8.

Let me explain this figure again.

The figure shows all the controls that are the force vectors for each frame arranged in an area. The fluid simulator uses these force vectors to move densities around. At frame N, the end frame, the resulting state of the density is compared to the desired state. The difference (squared) between the simulated density and the desired state is then computed and fed to the optimizer beast along with the current state of the $N \times 25$ force controls. The optimizer then updates the force controls. This is similar to the earlier simpler examples. Depending on the value of frames N and the number of forces, you can see that we get many controls.

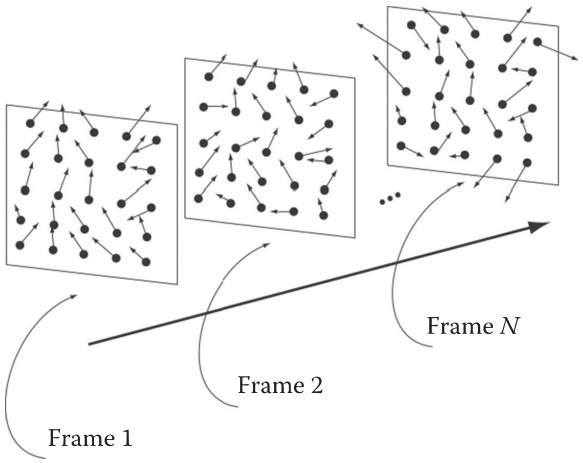


FIGURE 14.8 The force controls for a fluid simulation. In this case an array of 5 × 5 force grids separated over *N* intervals.

Figure 14.9 shows a sequence of simulations of a piece of density initially localized at the bottom: just a rectangular blob. The objective of this blob is to morph in the letter "C" using the array of control forces distributed over time. Notice how the forces shown as yellow arrows change over time. The red arrows show the drop off vector field created by the force fields. The red vectors are the ones that actually affect the evolution of the density. Of course, we also enforce incompressibility. That is what makes the fluids look cool and swirly.

Figure 14.10 shows an animation where an initial blob of density morphs into our old MAYA software logo using this technique. The cool aspect is that we go from a coherent blob of density to another coherent state. It is of course easy to go from a coherent state to a chaotic state. Just drop some milk in your morning coffee. It will swirl around and diffuse and mix your brew into a light brown state. But I bet it won't eventually turn into our MAYA logo in your coffee cup. That would be cool though but you would have to be lucky I guess. Like winning the lotto:

FIGURE 14.9 An initial blob of density is morphed in a fluidlike manner into a "C" character. The yellow force arrows drive the density to its target over time.

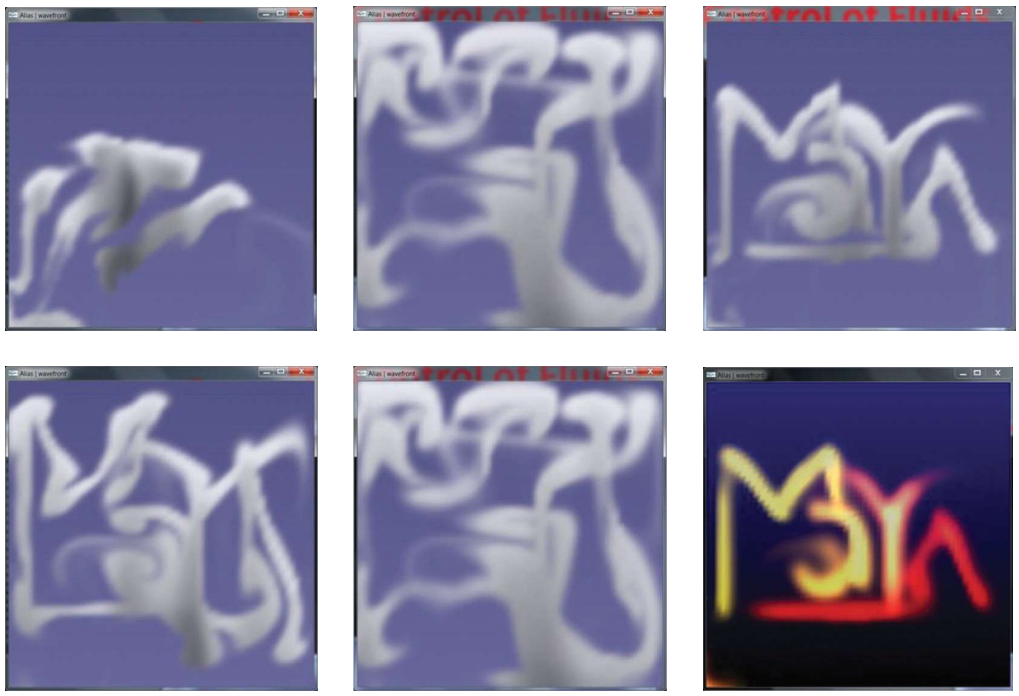


FIGURE 14.10 Five frames from an animation of a blob of density turning into our now defunct MAYA logo. The sixth one is more psychedelic just for fun.

very improbable. By the way, this is also related to the Second Law of Thermodynamics: entropy increases over time. If it wasn't for the sun, the entire world would just be covered with a homogeneous glob of stuff: probably ice.

Sometimes the optimizer beast goes crazy on a subsequent attempt but then miraculously recovers eventually. It is very entertaining to watch how the BFGS optimizer finds a solution to this problem.

What happens when there are a lot of controls? What if you have 1,002,345,402,401 controls? In this case you get differentials like:

```
dfloat<1002345402401+1>one_skinny_
    dude_with_a_huge_cousin;
```

Each differential cousin of a variable dude is getting bigger and bigger. That is lots of extra weight to carry around. "Cousin Dude you are taking up too much space and slowing me down."

What to do?

What if we replace the big *differential* cousin dude with their skinnier *adjoint* cousin dude?

Adjoint cousins are faster and light weight and they are weird. Cool!

To summarize: Most optimizers are hungry for differentials. One way to feed them big cousins from good old code is to use automatic differentiation. Take any good old code and use C++ and the header file provided earlier to feed the beast. In exchange, the beast will tell you how to update your controls to achieve whatever goal you want to achieve: Etta my boy.

14.3 THE AD, THE JOINT, AND THE PATH BACK TO THE OPTIMIZER

> And, uh, lotta strands to keep in my head, man. Lotta strands in old Duder's head. Luckily I'm adhering to a pretty strict, uh, drug regimen to keep my mind, you know, limber.
>
> THE *DUDE* IN THE BIG LEBOWSKI MOVIE PLAYED BY AMERICAN ACTOR JEFF BRIDGES

The title of this section sounds like a stoner trip but it is not.

The *adjoint method* is a somewhat obscure technique. At least it was obscure to me back in 2003. Its explanation is usually obfuscated with fancy mathematics. I will try to explain the adjoint method through the process of *generalization*. Sometimes it is easier to explain something in a general abstract framework. That is simple mathematics without complicated mathematics.

You get rid of all the messy details like triple integrals, partial derivatives, and other complicated mathematical exotica. I love mathematical exotica by the way. But it gets in the way.

Through generalization you get to the core of the problem and strip it bare. The benefit is that you understand the core technique, and this technique can potentially be applied to a variety of other problems as well. I hope to share with you the power of this methodology in the explanation that follows.

This is an outline of my explanation.

> Start with concrete code and show that it can be turned into linear algebra. Then explain the adjoint method in one figure using simple algebra. After that I will go to my basement and get the contents of my safe upstairs without having to move the safe upstairs.

If you want to follow the entire argument and are not too familiar with vectors and matrices, please consult the material in the first Intermezzo

```
y = z*y + x*v;
   dy[0] = z*dy[0] + y*dz[0] + v*dx[0] + x*dv[0];
   dy[1] = z*dy[1] + y*dz[1] + v*dx[1] + x*dv[1];
```

FIGURE 14.11 Concrete code and their differential cousins in red.

provided earlier. If that is too much work, just skip all the algebra and think of an analogy about a safe full of cash in my basement and an angry landlord knocking on my door. I will explain what I mean in a minute.

Figure 14.11 shows some concrete C code that we have encountered before. The original code and their cousins below can be cast into a matrix vector equation. This works for any code by the way.

The code shown in Figure 14.12 earlier can be turned into matrix algebra for the red-faced cousins. Here is how.

The cousins stored in a vector are continually being multiplied by matrices that are derived from each line of the code. So really we have the following sequence for the code in Figure 14.6, in math speak:

$$X_3 = A_3 A_2 A_1 X_0$$

This sequence starts with the initial 6×2 "vector" X_0 equal to

$$X_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

In the final step, we care only about the two last elements of the vector, namely **`df[0]`** and **`df[1]`**. This seems like a phenomenal waste of space

$$\begin{pmatrix} dx[0] & dx[1] \\ dy[0] & dy[1] \\ dz[0] & dz[1] \\ du[0] & du[1] \\ dv[0] & dv[1] \\ df[0] & df[1] \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ v & z & y & 0 & x & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} dx[0] & dx[1] \\ dy[0] & dy[1] \\ dz[0] & dz[1] \\ du[0] & du[1] \\ dv[0] & dv[1] \\ df[0] & df[1] \end{pmatrix}$$

$$X_2 = A_2 X_1$$

FIGURE 14.12 The differential of one line of code is turned into a matrix equation.

and effort. All the matrices are large and sparse in general and in the end we care only about the value of the last two cousins at the bottom. Another thing to notice is that these two cousins have been carried all along in their comfy limousines only to be put to work all the way at the end. This does not seem fair. None of the previous matrices woke them up and slapped them in the face. Their other cousins did all the hard work.

We can concatenate all the matrices into the following single equation:

$$X_{final} = A\, X_{start}\,.$$

This is the beauty of abstraction and generalization. We just condensed the differential of any code in a simple equation with three symbols. This not only works for our silly example. It is a generalization, remember.

The differential we have to feed to the optimizer beast can be obtained by stuffing the bottom cousins in a vector through a projection. Mathematically

$$df_{big\ cousins} = p^T X_{final} \quad \text{where } p = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

This equation just says: "use a projection to grab the value of the cousins at the bottom. Those are the only dudes the beast wants to eat."

This is how the adjoint method works.

We run the simulation forward in time to get the value of the objective function. This is typically the difference between the final result and the desired state squared. From that we compute the projection vector. And then we go backward in time and update skinny cousins using the adjoint of the matrices that correspond to the adjoint of the differentials of the code. Mathematically

$$p_{initial} = A^T p_{final} \quad \text{and} \quad df_{skinny\ cousins} = p_{initial}{}^T X_{initial}\,.$$

The crucial question is why are $df_{big\ cousins} = df_{skinny\ cousins}$ using a different technique?

The proof of this fact is so simple that I will provide it here in a one-liner proof:

$$df_{big\ cousins} = p_{final}{}^{T} X_{final} = p_{final}{}^{T} A\, X_{initial} = (A^{T} p_{final})^{T} X_{initial}$$
$$= p_{initial}{}^{T} X_{initial} = df_{skinny\ cousins}$$

That is it!

This was another epiphany.

This proof was so cool to me. The simple proof shows that the adjoint method can be explained in such a simple manner with just standard linear algebra.

Figure 14.13 is a visual depiction of the sizes involved in the forward differential (top) versus the adjoint method (bottom). This figure shows that the skinny cousins do the same job as their bigger twins. That is the magic of the adjoint method in a nutshell.

It is hard to explain the adjoint method honestly without mathematics.

But let me try with an analogy.

Mathematics is basically making stories more precise and making them sometimes more amenable to computer implementations. But they are still stories in the end.

Analogies are very helpful and make it more fun to memorize math.

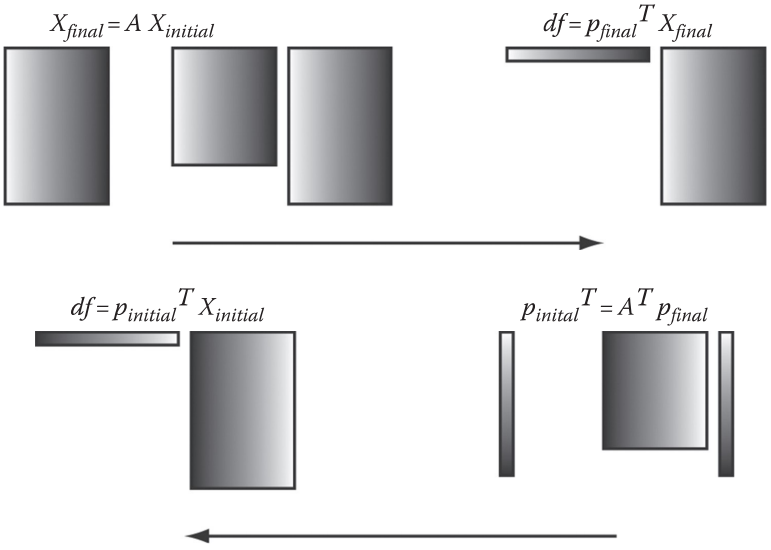


FIGURE 14.13 Pictorial comparison of big cousins being moved around versus skinny cousins.

Let's say you have a big heavy safe in your basement that contains some important stuff like loads of cash. You want to retrieve some of that cash from the safe because your landlord is knocking on your door.

Your first option is the differential method. You carry your safe upstairs and then use your key to open the safe and collect the cash. You probably also have to move the safe downstairs after that. The idle big cousins can come in handy at this point. By the way the differential technique does not require you to do that. But who wants a safe in their living room? I don't. They are usually ugly looking and are an easy target for hoodlums breaking into your house.

Your second option is the adjoint method. You take your key downstairs in your basement, open the safe, and grab the amount of cash that you owe to the landlord. Then lock the safe, if you still have any money left. The adjoint method requires you to take the key upstairs by the way. But hey a key fits in your pocket: no sweat. A safe doesn't fit in your pocket on the other hand. Not in my world at least. Besides, you do not want the key near the safe in the case those hoodlums break into your house and wander into your basement.

Figure 14.14 illustrates two strategies of unlocking a safe in the basement to pay the rent to the landlord who is knocking on your door. This is an example of how mathematics can be explained without mathematics at all. Alright, I know the figures are decorated with math symbols. But the basic argument can be explained without mathematics. At least I tried my best.

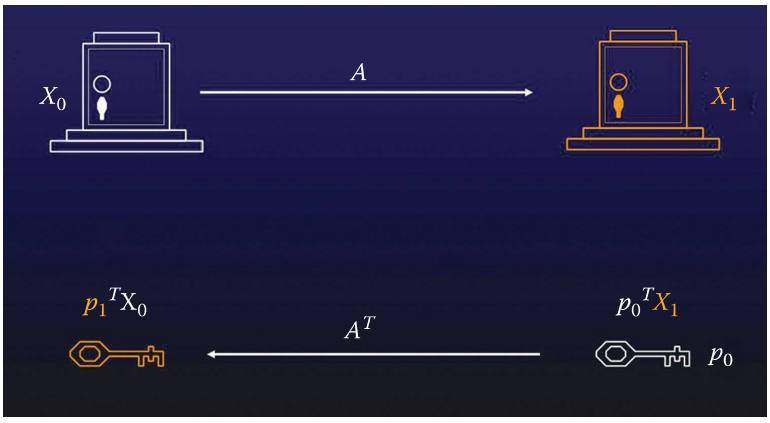


FIGURE 14.14 Figure 14.13 explained with a safe in the basement and greedy landlords.

I hope you got the gist of the adjoint technique through this analogy. You can find more information on the web of course. I also wrote a more extensive version with adjoint code in a paper I wrote for a SIGGRAPH 2004 course. It also includes an adjoint version of my simple fluid code.*

To summarize: The adjoint technique is a more efficient method to feed the optimizer beast with a gradient. However, it is less straightforward than automatic differentiation. The adjoint method is more complicated to understand and implement. Nothing comes for free after all. I have also left out the details of the adjoint method. You can find these details on the web, in my course notes or in our SIGGRAPH 2004 paper.

To summarize: A&A proved me wrong but I did not use FORTRAN. I wrote my own stuff in my loft in downtown Seattle in Pioneer Square using the IDE† called Visual Studio 6 from Microsoft. I also used automatic differentiation.

* You can find the paper at http://www.autodeskresearch.com/publications/adjointcontrol.

† **IDE** stands for **I**ntegrated **D**evelopment **E**nvironment. It helps you to write code more efficiently. Look it up. Hipster coders on the other hand like to use the **vi** editor and run **Makefiles** on LINUX. Well they also listen to LPs. I like to think that I was a hipster in the 1980s.

CHAPTER 15

Real Experiments, Computer Experiments, and Validation

> Real knowledge is to know the extent of one's ignorance.
>
> CONFUCIUS (CHINESE PHILOSOPHER, -551, -479)

I promise this section will be brief.

When I first started to talk about my fluid work, I would always get questions like "how accurate are your simulations?" "What is the Reynolds number?" And other stuff that experts seemed to pull out of their back packs. Then I bought Van Dyke's book. Remember from earlier that I mentioned his wonderful book called an *Album of Fluid Motion*. It has black and white pictures of famous fluid experiments. I decided to reproduce them with my fluid solver. But animated and interactive! A user could always alter the simulation because it was running in real time for reasonable grid sizes.

My simulations did not *exactly* reproduce the experiments but they did so *qualitatively*.

What does *qualitatively* mean?

This means that you get the same overall behavior, which does not necessarily match the values from an experiment. No existing fluid solver can reproduce the *exact* flow behind a large complicated structure anyway.

This is because the computer representation of the fluid is too coarse to capture all the details of the fluid. Therefore, there is a need to model the smaller scales not captured by the discrete computer representation. Remember turbulence models?

How do you validate computer code in the *serious world*? Well, you usually show that it works on simple examples and then reproduce some emergent properties. As an example, the *drag* behind a sphere.

15.1 SPHERES ARE SUCH A DRAG

> Avoid the world, it's just a lot of dust and drag and means nothing in the end.
>
> JACK KEROUAC (AMERICAN BEAT POET, 1922–1969)

What is drag?

Well it is something that drags you down, slows you down, stops you from moving faster, and so on. That is how the word *drag* is commonly used in English. Drag queens and drag car racing stretch the meaning in interesting ways. *Drag* is a multifaceted word in the English language. But for our purposes, we'll go with the first meaning mentioned.

For a sphere, drag is a single number that varies with the Reynolds Number. Remember the Swiss dude being thrown into a gigantic pool? The drag value for spheres has been experimentally measured in a lab. That is of course tested limited by the precision of the measuring devices involved. What you get is the curve shown in Figure 15.1. The curve should be thicker, to convey the amount of uncertainty. No measurement is absolutely precise. Curves in real life have width, unlike their skinny mathematical cousins who are of measure zero. You can never be too thin. In the mathematical world you can. Math won't make you too rich on the other hand, unless you win the Clay Prize.

The curve shown in Figure 15.1 is very interesting. The first part is what you expect. Let's keep the size of the experiment fixed: say the size of an Olympic-sized pool filled with a fluid.* It is pretty obvious that swimming in a pool full of honey is more challenging than swimming in a pool filled with water. The Reynolds number of honey is lower than that of water. Therefore, according to the experimental results, drag is higher for honey than for water. This I think is pretty obvious to anyone who did some

* I want to jump in my bathtub again. But no one except the superrich can swim in their bathtubs and fill it up with honey. At least not that I know of.

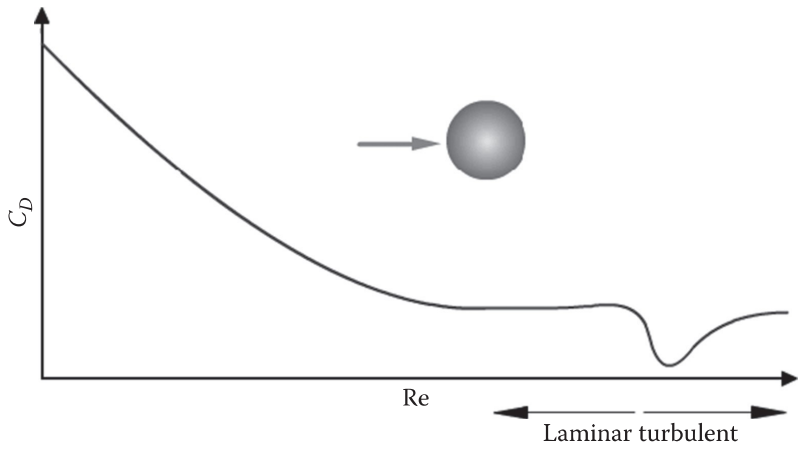


FIGURE 15.1 Plot of drag as a function of the Reynolds number.

swimming both in honey and water. I know of no one who has done both. I should get out more. (I have heard that there is a *MythBusters* episode about swimming in syrup, though). But no worries experiments confirm this observation.

The second part of the curve shown in Figure 15.1 is not really what most people expect. Most people are used to a linear world. One thing increases causing another thing to decrease and vice versa. Drag is not like that. There is a point when you decrease viscosity that the drag starts to increase again. Why? That is because of turbulence, we are told. Turbulence, that elusive phenomenon that no one really understands, creates drag.

If the goal is to reduce drag, for example in the car industry, you have to build a shape that hits the sweet spot, the valley, as in Figure 15.1. Less drag means a more fuel-efficient car.

Drag allows you to validate your simulation if it can reproduce the curve shown in Figure 15.1. You run your simulation with a pretend sphere and fix the Reynolds number. By the way, real-life spheres are also pretend spheres. Then through a magic formula that I will not mention, you compute the drag value. By varying the Reynolds number you get different drag values. And there you go: you get a curve. And if it matches the experimental curve closely enough, you can light up a cigar and claim that your solver is accurate.

That is how folks validate their solvers.

Of course this is just one example. There are other examples and many other criteria to validate a fluid simulation. It is a bit of a public relation coup to get customers to trust a piece of software. But it is also a sanity check for any piece of serious CFD code.

This is the *quantitative* approach: reproduce experimental results by comparing them to numerical values resulting from the simulation.

I am interested in the *qualitative* approach: try to create animations and compare them to pictures from experiments. That is why I love Van Dyke's book mentioned earlier. It is all pictures: no boring curves.

Remember that the book you are reading is all about whatever it takes to create animations of fluids. But to me less is more: try to find simple models that can create and control complex fluid motion. That is the gist of my research.

To summarize: Spheres actually are not a drag. They are just spheres after all. The drag they create when being blown on can be useful to validate fluid code. Even though drag is a simple measure, it reveals an interesting link to turbulence theory.

15.2 CURLY FLOWS BEHIND SPHERES, WAVY FLOWS IN TUBES, AND TURBULENT PLUMES BETWEEN PLATES

> You may have heard the world is made up of atoms and molecules, but it's really made up of stories. When you sit with an individual that's been here, you can give quantitative data a qualitative overlay.
>
> WILLIAM TURNER (ENGLISH SCIENTIST 1508–1568)

Figure 15.2 shows three experiments documented in Van Dyke's book. These are 3D experiments. I used my simple solver in two dimensions to

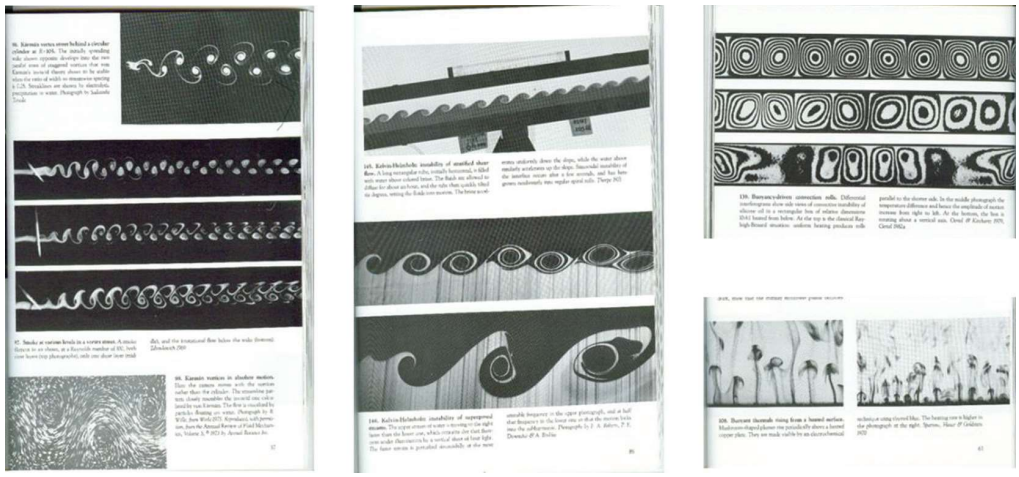


FIGURE 15.2 Three famous experiments from Van Dyke's book.

recreate the visuals of these experiments. I will go over all three of them. I just cherry-picked these three because they are cool.

First, I want to make an important point.

All the qualitatively *visual phenomena are* emergent. *They are a result of the basic laws of fluid dynamics and the boundary conditions. You set it up and then you get these results that* qualitatively *match the experimental results. That's it.*

There is no cheating at the equation level. There are no secret spices and sauces. The reason I created these simulations was to prove that I was indeed using the Navier–Stokes equations. Not just some hacked together computer graphics–specific model.

Let's go over the experiments.

15.2.1 Experiment Number One: Curly Flows behind Spheres

The visuals of this experiment are shown on the left-hand side of Figure 15.2.

This is the setup. A sphere, well a circle actually, is placed near the left-hand side boundary. Its size shouldn't be too large. The boundary conditions are as follows:

Left edge: an inflow

Right edge: an outflow

Top and bottom edges: a slip condition

Think of a pipe with slippery boundaries with air being blown from the left side being allowed to escape from the right side. At the front, there is a tiny marble. And that's it. That is the setup.

Since the pipe size and the viscosity are fixed, one can change the Reynolds number through the inflow velocity. The higher the velocity the higher the Reynolds number, and the lower the velocity the lower the Reynold's number.

Figure 15.3 shows a snapshot of the simulation using my simulation code.

The animation is much cooler of course. These patterns oscillate just like in the experiment. Because the solver is fast, a user can interact with it and *mess it up*.

15.2.2 Experiment Number Two: Wavy Flows in Tubes

The visuals of this experiment are shown in the middle of Figure 15.2.

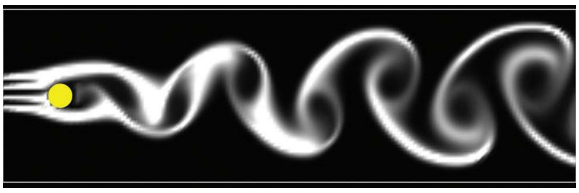


FIGURE 15.3 A frame generated with my code simulating the Von Karmann experiment.

In this case, a tube is initially filled with a fluid injected with a dye in the bottom half. Then the tube is slowly tilted and wave-like patterns appear at the boundary of the density. Sometimes you can observe this phenomenon up in the sky when two layers of different densities collide. Figure 15.4 shows an example. Next time you are on a flight try to look for them. It is a truly cool emergent phenomenon: waves in the sky.

This is the setup. Fill the bottom half with a density field. The density of the fluid is still constant. The bottom density is some substance put into the fluid. Apply a slowly varying rotating gravity force field to the velocity. The boundary conditions are as follows:

Left and right edge: periodic

Top and bottom: slip

That is it. The curly waves emerge automatically. Figure 15.5 shows a snapshot of a frame from a simulation with this setup.

The results are not as crisp as the experimental result, but they are pretty close to the real-world clouds. At any rate they exhibit the qualitative behavior that is to be expected. Actually, our simulations show waves within waves. Cool!

FIGURE 15.4 Kelvin–Helmholtz in the sky. For real. (Copyright: Michael deLeon Photography.)

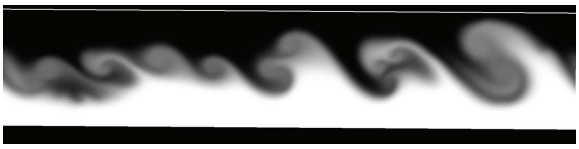


FIGURE 15.5 A frame generated with my code simulating the Kelvin–Helmholtz experiment.

15.2.3 Experiment Number Three: Turbulent Plumes between Plates

The visuals of this experiment are shown on the right-hand side of Figure 15.2.

This is another example of a *Hopf Bifurcation* discussed earlier for the lid-driven cavity problem. As a parameter is increased, the system goes through bifurcations resulting in more complex systems. In this case, the bifurcation parameter is the difference between the temperatures between two plates.

This is the setup. Fix the difference in temperature between the top and the bottom edge. This is our bifurcation dial. As we increase the dial the flow becomes more chaotic. This is clearly shown in Figure 15.2 (bottom right). If the difference in temperature is zero, nothing happens. Like a glass of water just sitting on your kitchen counter: Boring.

Figure 15.2 (top right). For certain ranges of the dial one gets what are called *Rayleigh–Benard* cells. That is probably the coolest regime. Regular cells form. They are even cooler looking in three dimensions as depicted in Figure 15.6. The image on the left is done in a controlled laboratory environment, while the one on the right is an image of clouds exhibiting this behavior. They are *Voronoi diagrams* computed by nature using

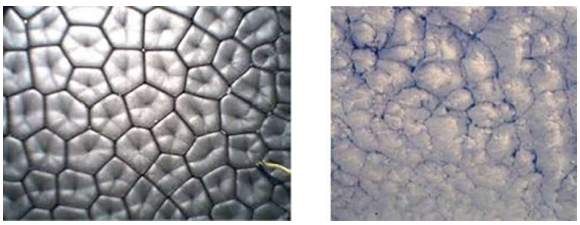


FIGURE 15.6 Two examples of real-world Rayleigh–Benard cells. (From Maroto, J.A., Perez-Munuzuri, V., and Romero-Cano, M.S., 2007, Introductory analysis of Bernard-Marangoni convection, *European Journal of Physics*, 28, 311–320, http://oiswww.eumetsat.org/WEBOPS/iotm/iotm/20010823_benard_cells/20010823_benard_cells.html. With permission.)

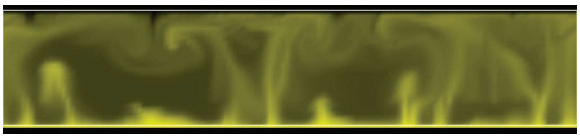


FIGURE 15.7 A frame generated with my code simulating the Rayleigh–Benard experiment.

fluids! How amazing is that. No space here to explain this connection. I just mention this connection to point out that some mathematical concepts do occur in nature. Look it up if you're interested.

When you increase the temperature difference dial you get into a cool fiery regime as depicted in Figure 15.2 (bottom right). No need for fancy turbulence or fancy computer graphics noise functions. This behavior emerges from the equations of fluids and the boundary conditions.

The boundary conditions are as follows:

Left and right edge: periodic

Top and bottom: slip

Figure 15.7 shows a snapshot of our simulation of this case when the temperature difference is cranked way up. You get into the interesting chaotic zone. Again it is a bit chalky compared to the pictures in Figure 15.2. But qualitatively they show the same behavior.

The point again is that this behavior completely emerged from just an initial state and the boundary conditions.

It is time for another analogy.

Grab a pencil or anything that has a sharp end. Try to make it stand still on its sharp end on a flat surface. Not easy. At least I cannot do it. Mathematically it is possible. In practice, however, it is hard to achieve. Why am I mentioning this? Because it happens that the zero velocity solution is a valid solution for any of the values of the temperature difference dial. When I launch my program, nothing happens until I apply a tiny disturbance. Then the show is in full swing.

The point I am making here is the following.

Depending on how you stir the fluid, you will get a different result. No matter how hard you try to place your pencil straight, you will get a different final state when it falls down and hits the plane. The qualitative

conclusion is: *the pencil will fall down and come to a rest state on the plane but we cannot predict exactly where it will end up.*

To summarize: I pointed out the difference between *qualitative* results and *quantitative* results. In these experiments, it is hopeless to predict all the exact values of the fluid. Emergent values like drag, for example, can be simulated *quantitatively*. But these emergent values obfuscate all the visual details of the fluid. And visual details are what we want in animation. I have presented three of our simulations that *qualitatively* agree with the visuals of three experiments. *Qualitative* agreement is easier than *quantitative* agreement. That is why I chose the former. But still it is a challenge.

CHAPTER 16

Epilogue

Let's Call It Quits

> You have your way. I have my way. As for the right way, the correct way, and the only way, it does not exist.
>
> FRIEDRICH NIETZSCHE
> (GERMAN PHILOSOPHER, 1844–1900)

I have a lot more material that I want to share with you, my dear reader who got this far, but I have to call it quits. My editor is knocking on my door and my safe is empty and I already paid off my landlord.

I sincerely hope that you enjoyed this book and that you learned some about the art of fluid animation. I tried to convey to the reader some of the basic magic and science behind fluid effects in movies, games, web apps, etc.

I also provided concrete code. I hope it compiles in your hood. No arguing. It either runs or it doesn't. Coders have a straightforward religion. You are not coding for a supernatural being but for a dumb speedy piece of hardware.

There is not going to be a grand *finale*!

No fireworks.

I always joke that we will always be employed in this business of ours. As soon as you create better data structures or improve the speed of your

software, animators will increase the size of their simulations.* And then we are back in business. As far as I can tell, it is an endless process. This makes Fluid Animation distinct from parts of Mathematics, where when a result is proven it is proven once and for all.

More is more.

But I strive to create more from less.

Thanks for listening.

* There is an upper limit however. The number of atoms in the universe is estimated to be roughly 10^{80} atoms.

Index

D

E

F

N

O

P

Q

R

S

Computer Graphics

The Art of Fluid Animation

Fluid simulation is a computer graphic used to develop realistic animation of liquids in modern games. **The Art of Fluid Animation** describes visually rich techniques for creating fluid-like animations that do not require advanced physics or mathematical skills. It explains how to create fluid animations like water, smoke, fire, and explosions through computer code in a fun manner.

The book presents concepts that drive fluid animation and gives a historical background of the computation of fluids. It covers many research areas that include stable fluid simulation, flows on surfaces, and control of flows. It also gives one-paragraph summaries of the material after each section for reinforcement.

This book includes computer code from the author's website that readers can download and run on several platforms so they can extend their work beyond what is described in the book. The material provided here is designed to serve as a starting point for aspiring programmers to begin creating their own programs using fluid animation.

K24510

CRC Press
Taylor & Francis Group
an **informa** business
www.crcpress.com

6000 Broken Sound Parkway, NW
Suite 300, Boca Raton, FL 33487
711 Third Avenue
New York, NY 10017
2 Park Square, Milton Park
Abingdon, Oxon OX14 4RN, UK

ISBN: 978-1-4987-0020-7
90000
9 781498 700207