Assigment1

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1 Probabilities (Theoritical)

1.1 Probability of n couples have 1 daughter and 1 son

Each couple have 2 kids so all the outcomes for the genders of their kids are :

- Boy,Boy (BB)
- Boy,Girl (BG)
- Girl,Boy (GB)
- Girl,Girl (GG)

Since the genders are independent .The probability of each gender to occur is equal. Thus $P(B)$=$P(G)$= 1/2:

- $\mathbb{P}(BB) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$
- $\mathbb{P}(BG) = \frac{1}{4}$
- $\mathbb{P}(GB) = \frac{1}{4}$
- $\mathbb{P}(GG) = \frac{1}{4}$

The probability that a couple has 1 son and 1 daughter is either BG or GB:

$$\mathbb{P}(\text{son and daughter}) = \mathbb{P}(BG) + \mathbb{P}(GB) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

In the end the probability of all n couples to have one boy and one girl is :

$$\mathbb{P}(BB) = \left(\frac{1}{2}\right)^n$$

1.2 Fair coin

To find the probability that all n flips result in heads, you need to multiply the probability of getting heads on each individual flip.
Since the coin is fair the probability of the outcome heads is:

$$\mathbb{P}(H) = \frac{1}{2}$$

Thus the probability that all n throws are heads is :

$$\mathbb{P}(\text{all heads}) = \left(\frac{1}{2}\right)^n$$

1.3 Hollow Beads

To find the probability of drawing a hollow bead randomly from a box leveraging the law of total probability the law of total probability.

- Let R,G and B represent drawing red ,green and blue respectively
- Let H represent drawing a hollow bead

All the necessary probabilities for the law of total probability :

- $\mathbb{P}(R) = 0.3$
- $\mathbb{P}(G) = 0.5$
- $\mathbb{P}(R) = 0.2$
- $\mathbb{P}(H \mid R) = \frac{1}{2}$
- $\mathbb{P}(H \mid G) = \frac{2}{3}$
- $\mathbb{P}(H \mid B) = \frac{2}{3}$

So the probability of drawing a hollow bead is :

$$\mathbb{P}(H) = \mathbb{P}(H \mid R) \cdot \mathbb{P}(R) + \mathbb{P}(H \mid G) \cdot \mathbb{P}(G) + \mathbb{P}(H \mid B) \cdot \mathbb{P}(B)$$

Thus

$$\mathbb{P}(H) \approx 0.61666$$

2 Bayes Theorem

2.1 Photon packages

The the probability that a photon package was actually received, given that the detector reported a detection is an application of the Bayes Rule :

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(B \mid A) \cdot \mathbb{P}(A)}{\mathbb{P}(B)} \tag{1}$$

where $\mathbb{P}(B)$ is :

$$\mathbb{P}(B) = \mathbb{P}(B \mid A) \cdot \mathbb{P}(A) + \mathbb{P}(B \mid \neg A) \cdot \mathbb{P}(\neg A)$$

Lets define :

- A :the event that photon package was actually received.
- D : he event that the detector reports a detection.

And the information given is :

- $\mathbb{P}(A) = 1 \times 10^{-7}$
- $\mathbb{P}(D \mid A) = 0.85$
- $\mathbb{P}(D \mid \neg A) = 0.1$
- And naturally $\mathbb{P}(\neg A) = 1 - \mathbb{P}(A) \approx 1$

The requested probability is $\mathbb{P}(A \mid D)$. So using (1) we have that :

$$\mathbb{P}(A \mid D) \approx 8.5 \times 10^{-7}$$

2.2 Normal Distribution of photon packages

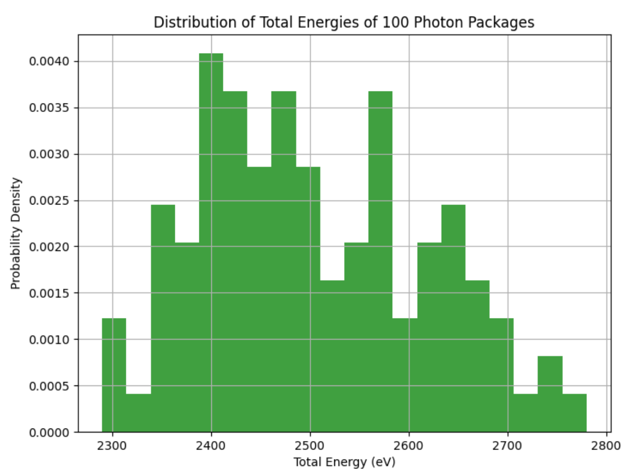

Figure 1: Histogram for the total energy of 100 photons

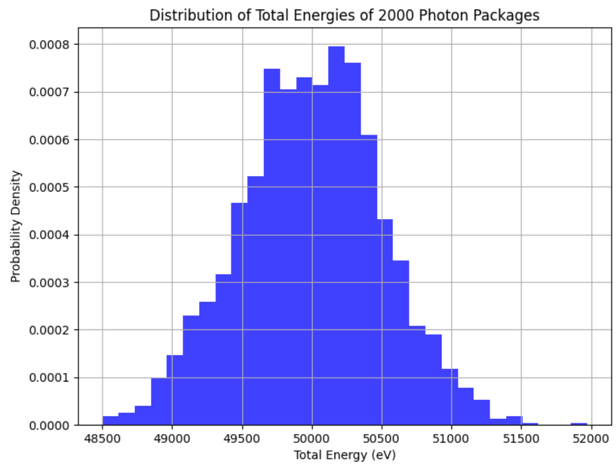

Figure 2: Histogram for the total energy of 2000 photons

This distribution resembles a normal (Gaussian) distribution. This is an instance of the Central Limit Theorem (CLT) in action. The CLT states that when independent random variables (in this case, the energies of photons) are summed, their sum tends to follow a normal distribution, even if the original variables are not normally distributed. Since the energy of each photon follows a discrete uniform distribution, the sum of the energies across many photons results in an approximately normal distribution.

It is clear that as the sample size increases, the distribution of the total energy becomes more evidently normal, confirming the CLT's role in the sum of independent random variables.

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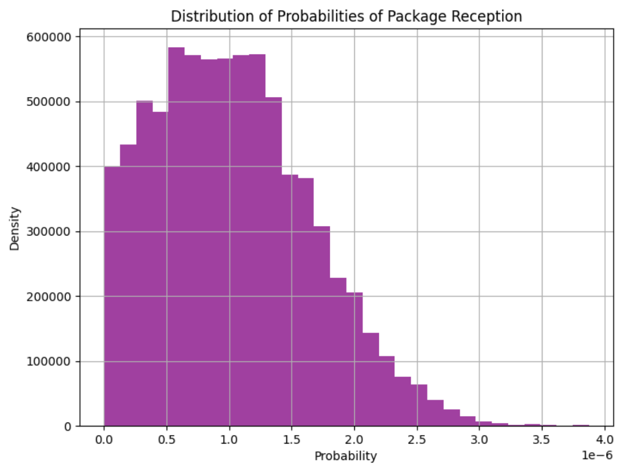

Figure 3: Distribution of the probability that a photon package was actually received, given a detection by the detector

Instead of having a fixed probability of a photon package reaching the detector, the process now follows a stochastic model. This randomness is captured by the normal distribution, and after applying Bayes' theorem to each sample from the normal distribution, we see that the resulting probabilities follow a distinct pattern.

The distribution of these probabilities is skewed toward smaller values, reflecting the fact that the photon packages are rare events, and the true positive rate is relatively high compared to the very small prior probabilities.