练习2

第8章，第315页

Introductory Combinatorics
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目录

解答 已验证

步骤1 步骤1/7

Let M_n be the set of 2-by-n arrays:

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} & \dots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \dots & x_{2n} \end{pmatrix}$$

that can be made from the numbers $1, 2, \dots, 2n$ in such a way that:

$$x_{11} < x_{12} < x_{13} < \cdots < x_{1n} \quad \text{and} \quad x_{21} < x_{22} < x_{23} < \cdots < x_{2n}$$

步骤2 步骤2/7

Let R_1 and R_2 be two sets representing the rows of the array. Then:

$$R_1 = \{i \,;\, 1 \le j \le 2n, a_i = 1\}$$

$$R_2 = \{i \,;\, 1 \le j \le 2n, a_i = -1\}$$

Note that the first row of the array corresponds to the elements in R_1 and that the second row of the array corresponds to the elements of R_2, all listed in increasing order.

步骤3 步骤3/7

Furthermore, let P_n represent the set of permutations of the multiset $\{n \cdot 1, n \cdot (-1)\}$ and let $\mathbf{a_1 a_2 a_3 \ldots a_{2n}}$ be a permutation in P_n. This permutation is represented by an array in M_n. The relation $P_n \to M_n$ is a bijection.

步骤4 步骤4/7

The Catalan number C_n counts the number of permumations $a_1 a_2 a_3 \ldots a_{2n}$ in P_n such that for $1 \leq k \leq 2n$ the partial sum is a nonnegative number:

$$a_1 + a_2 + a_3 + \cdots + a_k \geq 0$$

步骤5 步骤5/7

Hence, the Catlan number C_n counts the number of arrays in M_n such that for every $1 \leq k \leq n$ the following is true:

- the kth 1 is before the kth -1 in the sequence
- the $(1, k)$ of the array is less that the $(2, k)$ element

步骤6 步骤6/7

Therefore, it is proven that the number of 2-by-n arrays

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} & \dots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \dots & x_{2n} \end{pmatrix}$$

that can be made from $1, 2, 3, \dots, 2n$ such that

$$x_{11} < x_{12} < x_{13} < \cdots < x_{1n} \quad \text{and} \quad x_{21} < x_{22} < x_{23} < \cdots < x_{2n}$$

$$x_{11} < x_{21}, x_{12} < x_{22}, x_{13} < x_{23}, \dots, x_{1n} < x_{2n}$$

is equal to the nth Catalan number:

$$C_n = \frac{1}{n+1}\binom{2n}{2}$$

结果 步骤7/7

Use Theorem 8.1.1. to prove that the number of 2-by-n arrays with the given conditions equals the nth Catalan number.

为此解答评分

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‹ 练习1 练习3 ›