练习28

第6章，第201页

Introductory Combinatorics

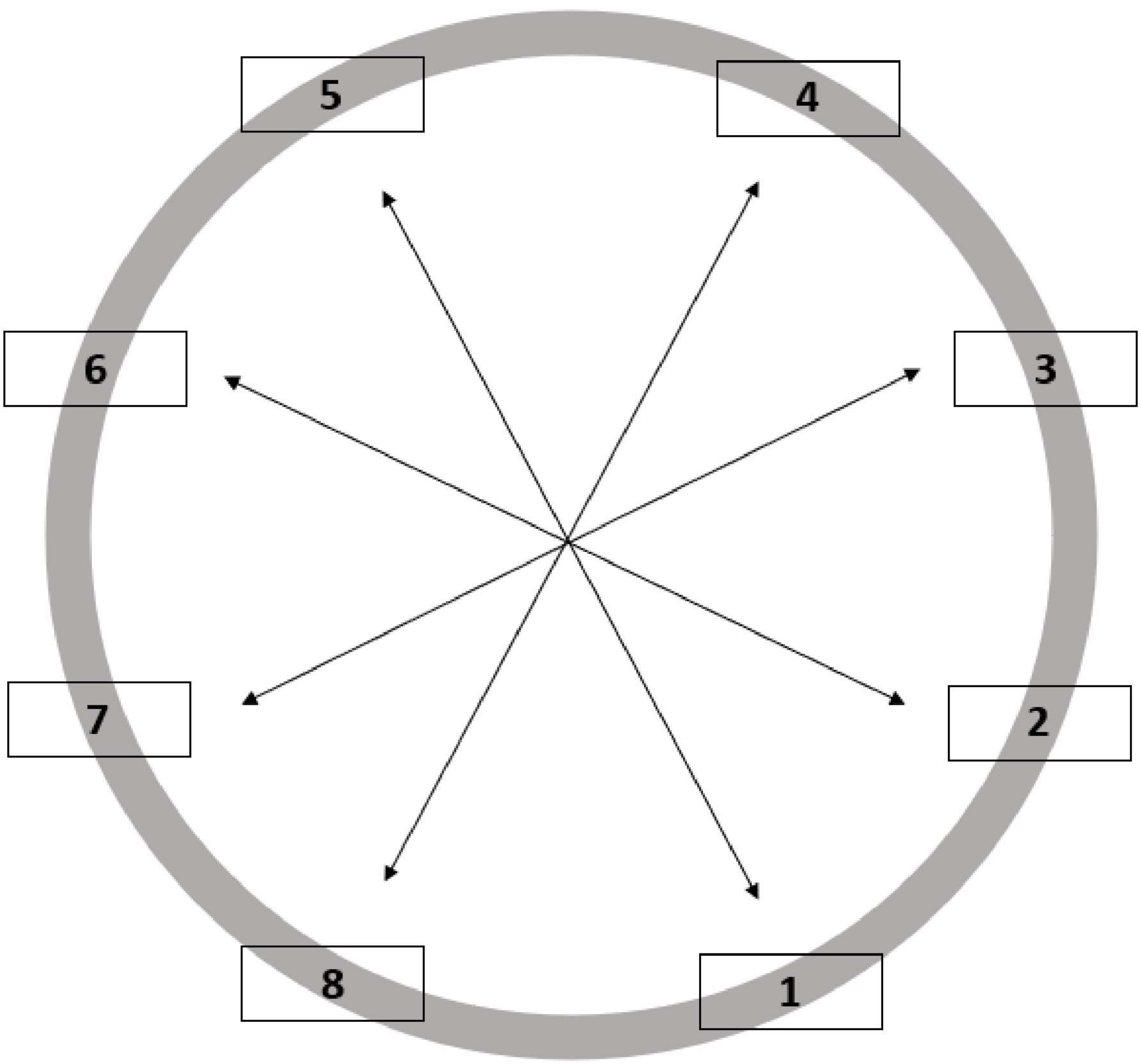
ISBN：9780136020400

目录

解答 已验证

步骤1

The carousel has eight distinguishable seats. Label them with counting numbers $1, 2, \ldots, 7, 8$ around the carousel in such a way that seat k is across seat $k+4$ for $1 \leq k \leq 4$:



步骤2 步骤2/10

Each of the eight boys takes one seat at the carousel. For $1 \leq k \leq 8$ the boy in seat k will move to another seat labeled s_k. Therefore, $s_1 s_2 \ldots s_8$ is a permutation of $\{1, 2, \ldots, 8\}$ such that neither of the boy has the same boy oppsote of him. That is, seat s_k is not across seat s_{k+1} for $1 \leq k \leq 8$.

步骤3 步骤3/10

Let P be the set of permuations of set $\{1, 2, \ldots, 8\}$ and let A_k for $1 \leq k \leq 4$ be the set of permutations of $s_1 s_2 \ldots s_8$ in P such that seat s_k is facing inward to seat s_{k+4}.

步骤4 步骤4/10

The goal of the task is to find how many ways can the boys change seats so that each has a different boy opposite of him. That is, the number of permutations of set $\{1, 2, \ldots, 8\}$ such that seat s_k is not across seat s_{k+4} for $1 \leq k \leq 4$.

步骤5 步骤5/10

Now, as set A_1 denotes the undesirable set of permutations in P such that s_1 is across seat $s_{1+4} = s_5$, it follows that its complement $\overline{A_1}$ denotes the number of desirable permutations.

Similarly, sets $\overline{A_2}, \overline{A_3}, \overline{A_4}$ also represent the number of specified desirable permutations.

Since sets A_k represent undesirable outcomes, it follows that the intersection of their complements will yield the wanted number of permutations:

$$\left|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4}\right|$$

步骤6

Define A_s as the interesection of s consecutive sets. That is, for $s \subseteq \{1,2,3,4\}$ set A_s is defined as:

$$A_s = \bigcap_{i \in s} A_i$$

Use **Theorem 6.5.1.** to obtain that the number of elements in A_s for $s \leq 4$ is given by:

$$|A_0| = 8! \qquad |A_1| = 8 \cdot 6! \qquad |A_2| = 8 \cdot 6 \cdot 4!$$
$$|A_3| = 8 \cdot 6 \cdot 4 \cdot 2! \qquad |A_4| = 8 \cdot 6 \cdot 4 \cdot 2 \cdot 0!$$

步骤7

Use the **Inclusion - Exclusion Principle** to obtain that:

$$\left|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4}\right| = \sum_{s \subseteq \{1,...,4\}} |A_s|(-1)^s$$

Now, using the information from step 6 it follows that:

$$\begin{aligned}\left|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_8}\right| &= 8! - 4 \cdot 8 \cdot 6! + 6 \cdot 8 \cdot 6 \cdot 4! - \\ &\quad 4 \cdot 8 \cdot 6 \cdot 2! + 8 \cdot 6 \cdot 4 \cdot 2 \cdot 0! \\ &= 23040\end{aligned}$$

步骤8

Therefore, there are exactly **23040** ways in which the boy can change seats such that each one of them has a different boy across him.

步骤9

步骤9/10

On the other hand, if all of the given seats are identical to one another, then any seating arrangements that can be obtained from the other by a circular permutation becomes indistinguishable. As there are eight seats if follows that the number of ways in which the boys can change seats such that each one has a different person across of him is eight times smaller than the one in each every seat is different:

$$\frac{23040}{8} = \mathbf{2880}$$

结果

步骤10/10

Use the Inclusion-Exclusion Principle to find that there are 23040 ways in which the boys can change seats so that each has a different one across him, or 2880 ways if the seats are indistinguishable.

为此解答评分

☆ ☆ ☆ ☆ ☆