TD N° 2

Exercice 1 (2.8)

a.) Demande quotidienne moyenne ;

soit X = v.a = demande quotidienne. $\mathbb{E}(X)$?

x	$\mathbb{P}(X^2 = x)$
1	3/5
0	1/10
4	3/10

x	$\mathbb{P}(X = x)$
-1	1/5
0	1/10
1	2/5
2	3/10

x^2	$\mathbb{P}(X = x)$
1	1/5
0	1/10
1	2/5
4	3/10

$R_X = \{-1, 0, 1, 2\}$, $\mathbb{E}(X) = \sum_{x \in R_X} x \mathbb{P}(X = x)$

$$\mathbb{E}(X) = -1 \times \frac{1}{5} + 0 \times \frac{1}{10} + 1 \times \frac{2}{5} + 2 \times \frac{3}{10} = \frac{4}{5} = 0.8$$

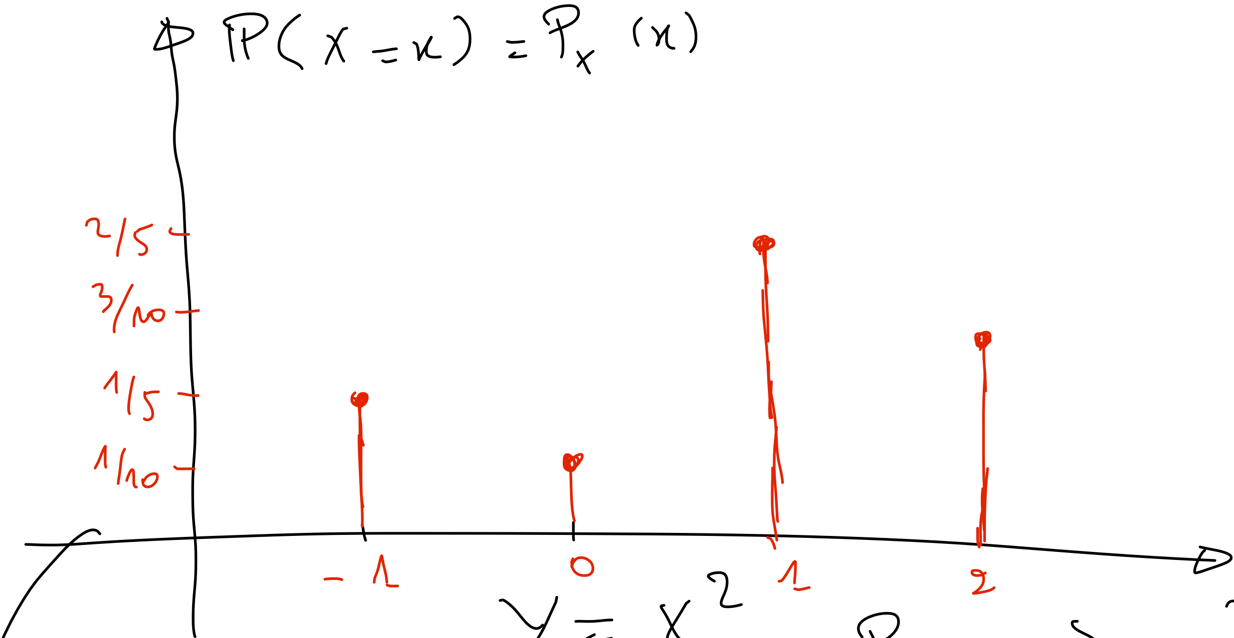
b.) soit $\mathbb{E}(X^2) = \frac{9}{5}$. on cherche σ_X :

$\sigma_X = \sqrt{Var(X)}$, où $Var(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2$

$$\Rightarrow Var(X) = \frac{9}{5} - \left(\frac{4}{5}\right)^2 = \frac{29}{25}$$

$$\Rightarrow \sigma_X = \sqrt{29/25} \simeq 1.077$$

c.) graphique fonction de masse ;



$$Y = X^2, \quad R_Y = \{0, 1, 4\}$$

$$y = 4$$

$$\mathbb{P}(Y = y) = \sum_{\{x \in R_X \,:\, x^2 = 4\} = \{2\}} \mathbb{P}(X = x) = \mathbb{P}(X = 2)$$

$$\mathbb{P}(Y = 4) = \mathbb{P}(X = 2) = 3/10$$

Exercice 2 ; (2.11)

$$f(x) = \begin{cases} kx & \text{si } 0 \leq x < 2 \\ k(4-x) & \text{si } 2 \leq x \leq 4 \\ 0 & \text{sinon.} \end{cases}$$

a-) k tel que f est une densité ;

Il faut que $\int_{\mathbb{R}} f(x)dx = 1$. on a que :

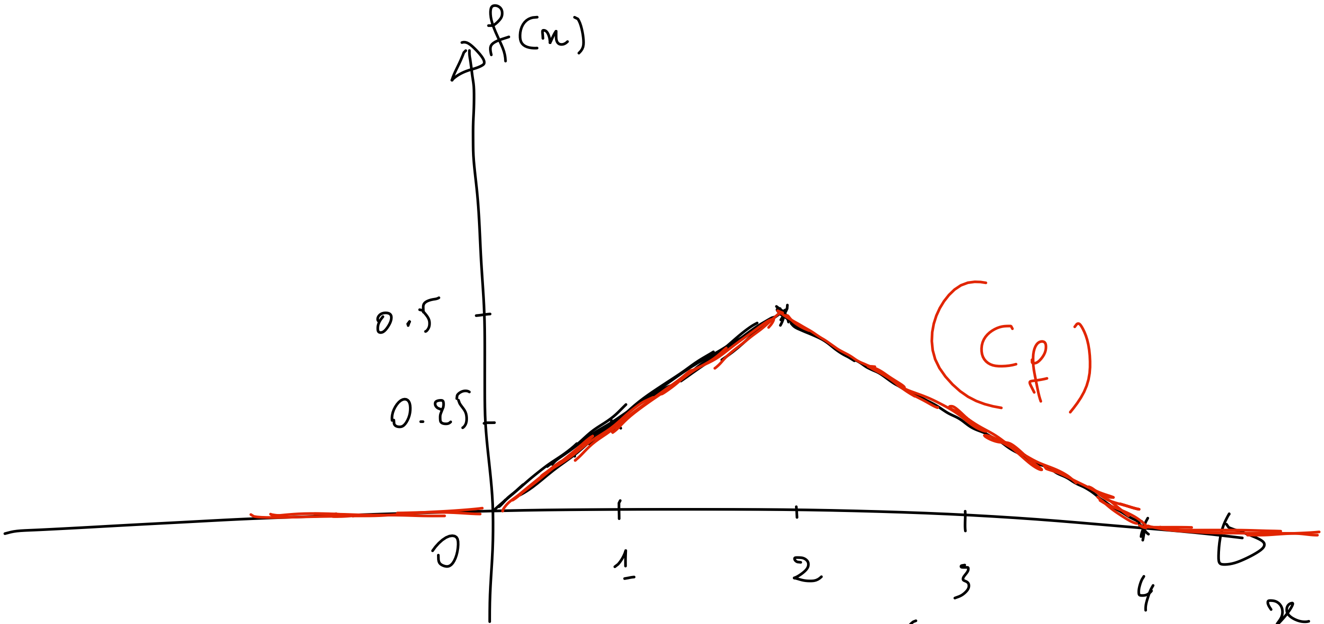
$$\int_{\mathbb{R}} f(x)dx = \int_{-\infty}^{0} \underbrace{f(x)}_{0}dx + \int_{0}^{2} f(x)dx + \int_{2}^{4} f(x)dx + \int_{4}^{+\infty} \underbrace{f(x)}_{0}dx$$

$$= \int_{0}^{2} kx\,dx + \int_{2}^{4} k(4-x)\,dx = 4k = 1$$

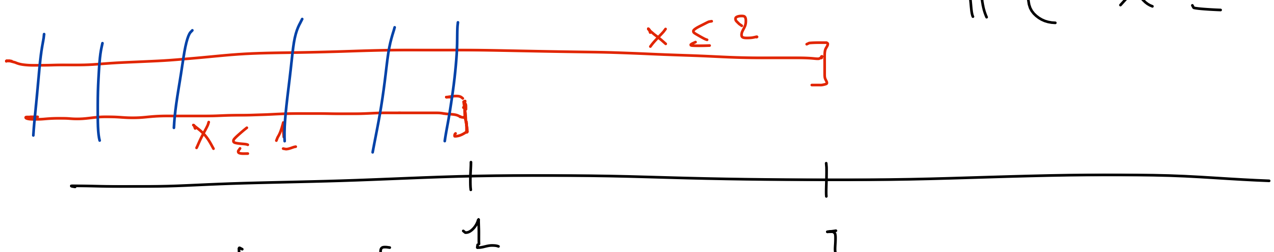
$$\Rightarrow \boxed{k = \frac{1}{4}}$$

b-) graphique densité ;

$$f(x) = \begin{cases} \frac{1}{4}x & \text{si } x \in [0,2[\\ 1 - \frac{x}{4} & \text{si } x \in [2,4[\\ 0 & \text{sinon} \end{cases}$$



c.) $\mathbb{P}(\{X \le 1\} | \{X \le 2\}) = \dfrac{\mathbb{P}(\{X \le 1\} \cap \{X \le 2\})}{\mathbb{P}(X \le 2)}$



$$\{X \le 1\} \cap \{X \le 2\} = \{X \le 1\}$$

$$\mathbb{P}(X \le 1 \mid X \le 2) = \frac{\mathbb{P}(X \le 1)}{\mathbb{P}(X \le 2)}, \text{ où}$$

$$\mathbb{P}(X \le 1) = \int_{-\infty}^{1} f_X(x)\,dx = \int_0^1 \frac{1}{4} x\,dx = \frac{1}{8}$$

$$\mathbb{P}(X \le 2) = \int_0^2 \frac{x}{4}\,dx = \frac{1}{2}$$

$$\mathbb{P}(X \le 1 \mid X \le 2) = \frac{1/8}{1/2} = \frac{1}{4}$$

d) $\mathbb{E}(X)$ et $Var(X)$;

$$* \mathbb{E}(X) = \int_{\mathbb{R}} x f_X(x)dx$$

$$= \int_0^2 x \times \frac{1}{4} x\, dx + \int_2^4 x \times \frac{1}{4}(4-x)dx$$

$$= 2$$

$$* Var(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 \text{ , où}$$

$$\mathbb{E}(X^2) = \int_{\mathbb{R}} x^2 f_X(x)dx$$

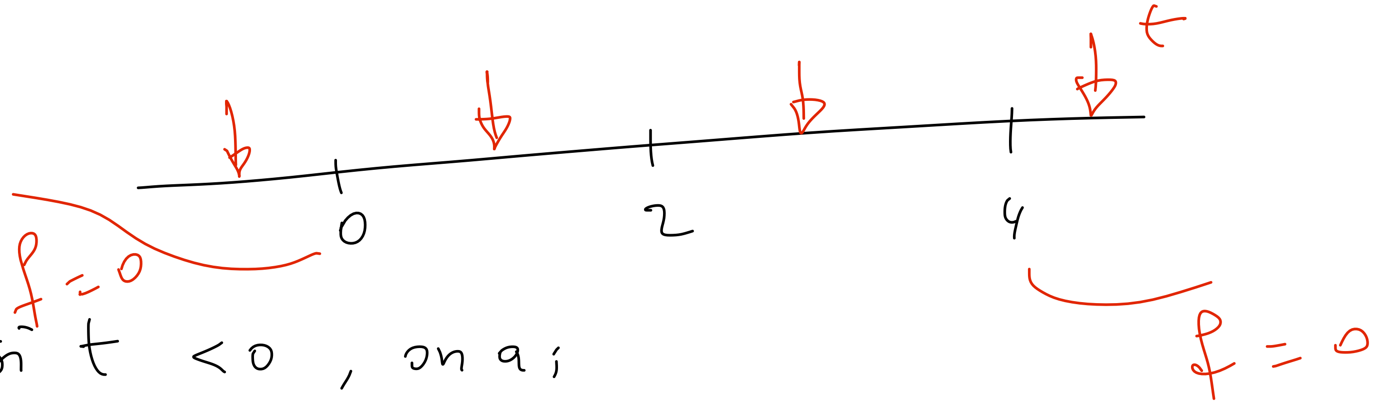
$$= \int_0^2 x^2 \times \frac{1}{4} x\, dx + \int_2^4 x^2 \times \frac{1}{4}(4-x)dx$$

$$= \frac{14}{3}$$

$$Var(X) = \frac{14}{3} - (2)^2 = \frac{2}{3}.$$

e.) Fonction de répartition F_X de x ;

$$F_X(t) = \mathbb{P}(X \leq t) = \int_{-\infty}^{t} f_X(x)\,dx, \ \forall t \in \mathbb{R}$$



* si $t < 0$, on a ;

$$F_X(t) = \int_{-\infty}^{t} 0\,dx = 0$$

* si $t \in [0, 2[$

$$F_X(t) = \int_{0}^{t} \frac{1}{4} x\,dx = \frac{t^2}{8}$$

* si $t \in [2, 4[$

$$F_X(t) = \int_{0}^{2} \frac{x}{4}\,dx + \int_{2}^{t} \left(1 - \frac{x}{4}\right) dx = -\frac{t^2}{8} + t - 1$$

* si $t \geq 4$ $\quad (t \in [4, +\infty[)$

$$F_X(t) = \int_{0}^{4} f_X(x)\,dx = 1$$

$$F_X(t) = \begin{cases} 0 & \text{si } t < 0 \\ \dfrac{t^2}{8} & \text{si } t \in [0,2[\\ -\dfrac{t^2}{8} + t - 1 & \text{si } t \in [2,4[\\ 1 & \text{si } t \geq 4 \end{cases}$$

Exercice 3

Soit X une v.a. telle que $\sigma_X = 1$.

$$f_X(x) = \begin{cases} \dfrac{1}{2\theta} & \text{si } -\theta \leq x \leq \theta \\ 0 & \text{sinon.} \end{cases}$$

a) Valeur de θ :

$$\int_{\mathbb{R}} f_X(x)\,dx = \int_{-\theta}^{\theta} \frac{1}{2\theta}\,dx = \frac{1}{2\theta}[x]_{-\theta}^{\theta} = 1$$

(ça n'aide pas).

on utilise alors le fait que $\sigma_X = 1$.

On a : $\sigma_X = \sqrt{Var(X)}$ où $Var(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2$

$$\mathbb{E}(X) = \int_{-\theta}^{\theta} x \times \frac{1}{2\theta}\,dx = 0$$

$$\mathbb{E}(X^2) = \int_{\mathbb{R}} x^2 f_X(x)\,dx$$

$$= \int_{-\theta}^{\theta} x^2 \times \frac{1}{2\theta}\,dx = \frac{\theta^2}{3}$$

$$Var(X) = \frac{\theta^2}{3} - 0^2 = \frac{\theta^2}{3}$$

$$\sigma_X = \sqrt{\frac{\theta^2}{3}} = \frac{\theta}{\sqrt{3}} = 1$$

$$\Rightarrow \boxed{\theta = \sqrt{3}}$$

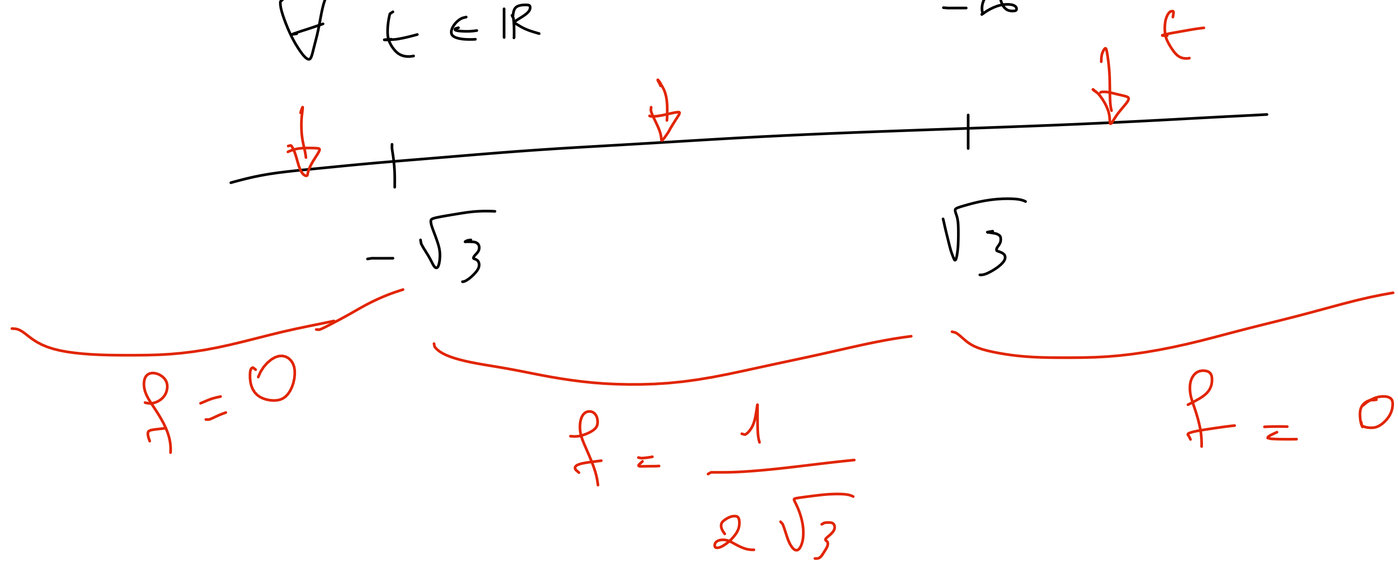
b) $\mathbb{E}(X) = 0$ (voir a))

c) Fonction de distribution cumulative (Fonction de répartition), F_X :

$$f_X(x) = \begin{cases} \frac{1}{2\sqrt{3}} & \text{si } -\sqrt{3} \leq x \leq \sqrt{3} \\ 0 & \text{sinon.} \end{cases}$$

$$F_X(t) = \mathbb{P}(X \le t) = \int_{-\infty}^{t} f_X(x)\,dx$$

$\forall\, t \in \mathbb{R}$



* si $t < -\sqrt{3}$

$$F_X(t) = \int_{-\infty}^{t} 0\,dx = 0$$

* si $t \in [-\sqrt{3}, \sqrt{3}[$ $\quad [-\sqrt{3}, \sqrt{3}]$

$$F_X(t) = \int_{-\sqrt{3}}^{t} \frac{1}{2\sqrt{3}}\,dx = \frac{t\sqrt{3}}{6} + \frac{1}{2}$$

* si $t \geq \sqrt{3}$ $\quad t > \sqrt{3}$

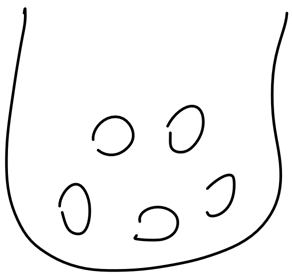
$$F_X(t) = \int_{-\sqrt{3}}^{\sqrt{3}} f_X(x)\,dx = 1$$

$$F_X(t) = \begin{cases} 0 & \text{si } t < -\sqrt{3} \\ \dfrac{t\sqrt{3}}{6} + \dfrac{1}{2} & \text{si } t \in [-\sqrt{3}, \sqrt{3}[\\ 1 & \text{si } t \geq \sqrt{3} \end{cases}$$

Exercice 4 confère 2.29 manuel.

Exercice 5

10 articles, 3 D et 7 C.



$$\underbrace{D\ C\ C\ C\ C\ D}_{X=6}$$

$$\underbrace{D\ D}_{X=2}$$

$R_X = \{2, 3, 4, \dots, 9\}$

a) Fonction de masse de X : $P(X=x)\ \forall x \in R_X$

$\boxed{\{D, C\} \mid D}$ $\qquad$ $\boxed{\{C, C, C, D, C, C\} \mid D}$

C D D

$\boxed{C \mid C \mid ?}$ $\qquad \{C, D\}$

* $x = 2$ $\quad \{D, D\}$, $P(X=2) = \dfrac{C_2^3}{C_2^{10}} = \dfrac{1}{15}$

* $x = 3$, $\boxed{\{D, C\} \mid D}$ $\quad \cancel{\{D, C, D\}}$

$$P(X=3) = \frac{C_1^3 \times C_1^7 \times C_1^2}{C_2^{10} \times C_1^8} = \frac{7}{60}$$

* $x = 4$

$$\boxed{\{C, D, C\} \;\big|\; D}$$

$$\mathbb{P}(X = 4) = \frac{C_2^7 \times C_1^3 \times C_1^2}{C_3^{10} \times C_1^7} = \frac{3}{20}$$

* Pour x quelconque dans $\{2, 3, \dots, 9\}$.

$$\boxed{\underbrace{\{C, \dots C, D\}}_{x-1 \text{ éléments}} \;\big|\; D}$$

$$C_1^2 = \frac{2!}{1!\,(2-1)!}$$

$$\mathbb{P}(X = x) = \frac{C_1^3 \times C_{x-2}^7 \times C_1^2}{C_{x-1}^{10} \times C_1^{10-(x-1)}}$$

$$= \frac{C_1^3 \times C_{x-2}^7}{C_{x-1}^{10}} \times \frac{2}{10-(x-1)}$$

$$= \frac{3 \times \dfrac{7!}{(x-2)!\,(7-x+2)!}}{\dfrac{10!}{(x-1)!\,(10-x+1)!}} \times \frac{2}{(11-x)}$$

$$= \frac{3 \times 7! \times 2 \times (x-1)!\,(11-x)!}{(x-2)!\,(9-x)!\;10!\;(11-x)}$$

or $n! = n \times (n-1)!$

$\Rightarrow (11-x)! = (11-x)(10-x)!$ et $(x-1)! = (x-1)(x-2)!$

$(10-x)! = (10-x)(9-x)!$

$\Rightarrow (11-x)! = (11-x)(10-x)(9-x)!$

$$= \frac{6 \times 7!\,(x-1)\cancel{(x-2)!}\,\cancel{(11-x)}(10-x)\cancel{(9-x)!}}{\cancel{(x-2)!}\,\cancel{(9-x)!}\;10!\;\cancel{(11-x)}}$$

$$= \frac{\cancel{6}^{\,2} \times \cancel{7!}\,(x-1)(10-x)}{\cancel{10}_{\,5} \times \cancel{9}_{\,3} \times 8 \times \cancel{7!}}$$ car $10! = 10 \times 9 \times 8 \times 7!$

$$\boxed{\mathbb{P}(X = x) = \frac{1}{120}(x-1)(10-x)}$$