Exercițiul cu sisteme de la pregătire:

Să se rezolve pt sistemul

$$\begin{cases} x + 2y + 3z = 6 \\ x - 3y + 5z = 3 \\ 7x - y - z = 5 \end{cases} \qquad x + y + z = ?$$

Varianta 1: aducem Matricea la REF (forma eșalon)

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 1 & -3 & 5 & 3 \\ 7 & -1 & -1 & 5 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -5 & 2 & -3 \\ 7 & -1 & -1 & 5 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -5 & 2 & -3 \\ 0 & -15 & -22 & -37 \end{array}\right]$$

$R_2 = (-1)R_1 + R_2$ $\qquad$ $R_3 = (-7)R_1 + R_3$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -5 & 2 & -3 \\ 0 & 0 & -28 & -28 \end{array}\right]$$

$R_3 = (-3)R_2 + R_3$

Deci:

$$x + 2y + 3z = 6$$
$$-5y + 2z = -3$$
$$-28z = -28$$

$-28z = -28 \Rightarrow \boxed{z = 1}$

$-5y + 2 \cdot 1 = -3$

$-5y = -5$

$\boxed{y = 1}$

$x + 2 + 3 = 6$

$x = 6 - 5 \Rightarrow \boxed{x = 1}$

$x + y + z = \underline{\underline{3}}$

Soluțiile sunt: $\boxed{\begin{array}{c} x = 1 \\ y = 1 \\ z = 1 \end{array}}$

Varianta 2: folosim formulele lui Cramer din Curs

! Când $\det A \neq 0$

"Varianta Oficială"

$\det A = \det (a_{i,j})_{1 \le i,j \le n} \neq 0$

avem formulele:

$$x_j = \frac{\det A_j}{\det A}, \quad \forall j = \overline{1,n}$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 1 & -3 & 5 \\ 7 & -1 & -1 \end{vmatrix} = (-1)^{1+1} \cdot \begin{vmatrix} -3 & 5 \\ -1 & -1 \end{vmatrix} + 2 \cdot (-1)^{1+2} \begin{vmatrix} 1 & 5 \\ 7 & -1 \end{vmatrix} + 3(-1)^{3+1} \begin{vmatrix} 1 & -3 \\ 7 & -1 \end{vmatrix}$$

$$= 8 - 2(-36) + 3(20) = 8 + 72 + 60 = 140$$

$$x = \frac{\begin{vmatrix} 6 & 2 & 3 \\ 3 & -3 & 5 \\ 5 & -1 & 1 \end{vmatrix}}{140} = \frac{140}{140} = 1$$

$$\begin{vmatrix} 6 & 2 & 3 \\ 3 & -3 & 5 \\ 5 & -1 & 1 \end{vmatrix} = 6 \cdot \begin{vmatrix} -3 & 5 \\ -1 & -1 \end{vmatrix} - 2 \begin{vmatrix} 3 & 5 \\ 5 & -1 \end{vmatrix} + 3 \cdot \begin{vmatrix} 3 & -3 \\ 5 & -1 \end{vmatrix} = 6(3+5) - 2(-3-25) + 3(-3+15) = 48 + 56 + 36 = 140$$

$$y = \frac{\begin{vmatrix} 1 & 6 & 3 \\ 1 & 3 & 5 \\ 7 & 5 & -1 \end{vmatrix}}{140} = \frac{140}{140} = 1$$

$$\begin{vmatrix} 1 & 6 & 3 \\ 1 & 3 & 5 \\ 7 & 5 & -1 \end{vmatrix} = \begin{vmatrix} 3 & 5 \\ 5 & -1 \end{vmatrix} + (-6) \begin{vmatrix} 1 & 5 \\ 7 & -1 \end{vmatrix} + 3 \begin{vmatrix} 1 & 3 \\ 7 & 5 \end{vmatrix} = -28 + 216 - 48 = 188 - 48 = 140$$

$$z = \frac{\begin{vmatrix} 1 & 2 & 6 \\ 1 & -3 & 3 \\ 7 & -1 & 5 \end{vmatrix}}{140} = \frac{140}{140} = 1$$

$$\begin{vmatrix} 1 & 2 & 6 \\ 1 & -3 & 3 \\ 7 & -1 & 5 \end{vmatrix} = \begin{vmatrix} -3 & 3 \\ -1 & 5 \end{vmatrix} - 2 \begin{vmatrix} 1 & 3 \\ 7 & 5 \end{vmatrix} + 6 \begin{vmatrix} 1 & -3 \\ 7 & -1 \end{vmatrix} = -12 - 2(-16) + 6(22) = -12 + 32 + 120 = 140$$

Deci: $x + y + z = 3$

Exercițiul cu inversa unei matrici de la pregătire

$$A = \begin{pmatrix} \bar{1} & \bar{2} & \bar{0} \\ \bar{1} & \bar{1} & \bar{2} \\ \bar{0} & \bar{1} & \bar{1} \end{pmatrix}, \mathbb{Z}_5 \qquad A^{-1} = ?$$

$\exists A^{-1}$ dacă $\det A \neq 0$

$$\begin{vmatrix} \bar{1} & \bar{2} & \bar{0} \\ \bar{1} & \bar{1} & \bar{2} \\ \bar{0} & \bar{1} & \bar{1} \end{vmatrix} = \bar{1} \begin{vmatrix} \bar{1} & \bar{2} \\ \bar{1} & \bar{1} \end{vmatrix} - \bar{2} \begin{vmatrix} \bar{1} & \bar{2} \\ \bar{0} & \bar{1} \end{vmatrix} + \bar{0} \cdot \underbrace{\begin{vmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{1} \end{vmatrix}}_{0} = \bar{1}(\bar{1} - \bar{2}) - \bar{2}(\bar{1})$$

$$= -\bar{1} - \bar{2} = -\bar{3}$$

$$= \bar{2} \neq \bar{0}$$

$-3 = 5 \cdot (-?) + ?$

$-3 = 5 \cdot (-1) + \underline{\underline{2}}$

A^{-1} = inversul lui $\det A \cdot (\text{adjuncta lui } A)^T$

Elementele din adj A se obțin astfel:

$(-1)^{i+j} \cdot \det_{ij}$ (determinantul obținut prin tăierea liniei i și coloanei j)

$$\text{adj} A = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} \bar{4} & \bar{4} & \bar{1} \\ \bar{3} & \bar{1} & \bar{4} \\ \bar{4} & \bar{3} & \bar{4} \end{pmatrix}$$

$b_{11} = (-1)^{1+1} \begin{vmatrix} \bar{1} & \bar{2} \\ \bar{1} & \bar{1} \end{vmatrix} = \bar{1} - \bar{2} = -\bar{1} = \bar{4};$

$b_{12} = (-1)^{1+2} \begin{vmatrix} \bar{1} & \bar{2} \\ \bar{0} & \bar{1} \end{vmatrix} = -\bar{1} = \bar{4};$

$b_{13} = (-1)^{1+3} \begin{vmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{1} \end{vmatrix} = \bar{1};$

$b_{21} = (-1)^{2+1} \begin{vmatrix} \bar{2} & \bar{0} \\ \bar{1} & \bar{1} \end{vmatrix} = -\bar{2} = \bar{3};$

$b_{22} = (-1)^{2+2} \begin{vmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{1} \end{vmatrix} = \bar{1}$

$b_{23} = (-1)^{2+3} \begin{vmatrix} \bar{1} & \bar{2} \\ \bar{0} & \bar{1} \end{vmatrix} = -\bar{1} = \bar{4}$

$b_{31} = (-1)^{3+1} \begin{vmatrix} \bar{2} & \bar{0} \\ \bar{1} & \bar{2} \end{vmatrix} = \bar{4}$

$b_{32} = (-1)^{3+2} \begin{vmatrix} \bar{1} & \bar{0} \\ \bar{1} & \bar{2} \end{vmatrix} = -\bar{2} = \bar{3}$

$b_{33} = (-1)^{3+3} \begin{vmatrix} \bar{1} & \bar{2} \\ \bar{1} & \bar{1} \end{vmatrix} = -\bar{1} = \bar{4}$

$$A^{-1} = (\text{inversul lui } \bar{2} \text{ in } \mathbb{Z}_5) \cdot \begin{pmatrix} \bar{4} & \bar{4} & \bar{1} \\ \bar{3} & \bar{1} & \bar{4} \\ \bar{4} & \bar{3} & \bar{4} \end{pmatrix}^T$$

$\bar{2} \cdot \bar{x} \equiv \bar{1} \pmod 5$

$\bar{2} \cdot \bar{3} \equiv \bar{1} \pmod 5$

$\bar{6} \equiv \bar{1} \pmod 5$

$\bar{1} \equiv \bar{1} \pmod 5$

$$A^{-1} = \bar{3} \begin{pmatrix} \bar{4} & \bar{3} & \bar{4} \\ \bar{4} & \bar{1} & \bar{3} \\ \bar{1} & \bar{4} & \bar{4} \end{pmatrix} = \begin{pmatrix} \overline{12} & \bar{9} & \overline{12} \\ \overline{12} & \bar{3} & \bar{9} \\ \bar{3} & \overline{12} & \overline{12} \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} \bar{2} & \bar{4} & \bar{2} \\ \bar{2} & \bar{3} & \bar{4} \\ \bar{3} & \bar{2} & \bar{2} \end{pmatrix}$$

Verificare: $A \cdot A^{-1} = I_3$

$$A \cdot A^{-1} = \begin{pmatrix} \bar{1} & \bar{2} & \bar{0} \\ \bar{1} & \bar{1} & \bar{2} \\ \bar{0} & \bar{1} & \bar{1} \end{pmatrix} \begin{pmatrix} \bar{2} & \bar{4} & \bar{2} \\ \bar{2} & \bar{3} & \bar{4} \\ \bar{3} & \bar{2} & \bar{2} \end{pmatrix} = \begin{pmatrix} \bar{1} & \bar{0} & \bar{0} \\ \bar{0} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix} \checkmark$$

Calcul determinanți pe $\mathcal{M}_3$, $\mathcal{M}_4$

Exemplu: $\underline{\underline{\mathcal{M}_3}}$

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 6 & 2 & 5 \\ 8 & 2 & 0 \end{pmatrix} \qquad \det A = ?$$

Obținem 0-uri pe coloană (rând) prin operații pe rânduri/coloane pt a simplifica calculele.

$$\begin{bmatrix} 2 & 1 & 3 \\ 6 & 2 & 5 \\ 8 & 2 & 0 \end{bmatrix} \overset{R_2=(-3)R_1+R_2}{\sim} \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & -4 \\ 8 & 2 & 0 \end{bmatrix} \overset{R_3=(-4)R_1+R_3}{\sim} \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & -4 \\ 0 & -2 & -12 \end{bmatrix}$$

$$\det A = 2\cdot(-1)^{1+1}\cdot\begin{vmatrix} -1 & -4 \\ -2 & -12 \end{vmatrix} = 2(12-8) = 2\cdot 4 = \underline{\underline{8}}$$

$\underline{\underline{\mathcal{M}_4}}$

$$B = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & 5 & 3 & 0 \\ 1 & 2 & 3 & 4 \\ 0 & 2 & 6 & 5 \end{pmatrix} \qquad \det B = ?$$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 5 & 3 & 0 \\ 1 & 2 & 3 & 4 \\ 0 & 2 & 6 & 5 \end{bmatrix} \overset{R_2=(-2)R_1+R_2}{\sim} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & -3 & 0 \\ 1 & 2 & 3 & 4 \\ 0 & 2 & 6 & 5 \end{bmatrix} \overset{R_3=(-1)R_1+R_3}{\sim} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 2 & 6 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 4 \\ 2 & 6 & 5 \end{bmatrix} \underset{\sim}{\overset{(-)}{R_2 \rightleftarrows R_3}} \begin{bmatrix} 1 & -3 & 0 \\ 2 & 6 & 5 \\ 0 & 0 & 4 \end{bmatrix} \overset{R_2=(-2)R_1+R_2}{\sim} \begin{bmatrix} 1 & -3 & 0 \\ 0 & 12 & 5 \\ 0 & 0 & 4 \end{bmatrix}$$

schimbăm semnul determinantului

$$\begin{vmatrix} 1 & 2 & 5 \\ & 0 & 4 \end{vmatrix} = 48 \xrightarrow{\text{pt cō } R_2 \rightleftharpoons R_3} \underline{\underline{-48}}$$

Inversul unui element in $\mathbb{Z}_n$

Ex: Inversul lui 7 in $\mathbb{Z}_{53}$ = ?

$gcd(53, 7) = 1 \Rightarrow \exists$ invers

Bezout $53s + 7t = 1$

$53 = 7 \cdot 7 + 4$
$7 = 4 \cdot 1 + 3$
$4 = 3 \cdot 1 + 1$
$3 = 1 \cdot 3 + 0$ STOP

$4 = 53 - 7 \cdot 7$
$3 = 7 - 4 \cdot 1$
$1 = 4 - 3 \cdot 1$

$$\begin{aligned}
1 &= 4 - 1 \cdot (7 - 4 \cdot 1) \\
&= 4 - 7 + 4 \cdot 1 \\
&= 4 \cdot 2 - 7 \\
1 &= 2(53 - 7 \cdot 7) - 7 \\
&= 2 \cdot 53 - 14 \cdot 7 - 7 \\
&= 53 \cdot 2 - 15 \cdot 7 \\
&= 53 \cdot 2 + 7 \cdot (-15)
\end{aligned}$$

Avem $53 - 15 = 38$

Inversul lui 7 in 53 este 38

Ex: Inversul lui 2 in $\mathbb{Z}_{33}$ = ?

$gcd(33, 2) = 1 \Rightarrow \exists$ invers

Bézout $33s + 2t = 1$

$33 = 2 \cdot 16 + 1$
$2 = 1 \cdot 2 + 0$ STOP

$$\begin{aligned}
1 &= 33 - 2 \cdot 16 \\
&= 33 \cdot 1 + 2(-16)
\end{aligned}$$

Avem $33 - 16 = 17$

Inversul lui 2 in 33 este 17

Exercițiul cu polinoame de la pregatire

$$x^4 + x^3 + 2x^2 + 7x + 5 = 0$$

$x_1 + x_2 + x_3 + x_4 = ?$

$x_1^2 + x_2^2 + x_3^2 + x_4^2 = ?$

$$\underset{a_0}{1x^4} + \underset{a_1}{1x^3} + \underset{a_2}{2x^2} + \underset{a_3}{7x} + \underset{a_4}{5} = 0$$

$$x_1 + x_2 + x_3 + x_4 = -\frac{a_1}{a_0} = -\frac{1}{1} = -1$$

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = (x_1 + x_2 + x_3 + x_4)^2 - 2(x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4) = (-1)^2 - 2 \cdot 2 = \underline{\underline{-3}}$$

$$(x_1 + x_2 + x_3 + x_4)^2 = (-1)^2 = 1$$

$$(x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4) = \frac{a_2}{a_0} = \frac{2}{1} = 2$$

Formule:

$x_1 + x_2 + x_3 + x_4 = -\frac{a_1}{a_0}$; $x_1x_2x_3x_4 = +\frac{a_4}{a_0}$;

$x_1^2 + x_2^2 + x_3^2 = (x_1 + x_2 + x_3)^2 - 2(x_1x_2 + x_2x_3 + x_3x_1)$

$x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 = +\frac{a_2}{a_0}$;

$x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 = -\frac{a_3}{a_0}$

$(x_1 + x_2)^2 = x_1^2 + x_2^2 + 2x_1x_2$;

$(x_1 + x_2 + x_3)^2 = (x_1 + x_2 + x_3)^2 - 2(x_1x_2 + x_1x_3 + x_2x_3)$;

$x_1^2 + x_2^2 + x_3^2 + x_4^2 = (x_1 + x_2 + x_3 + x_4)^2 - 2(x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4)$

Rangul unei matrici

Rangul = Nr de coloane | rânduri liniar independente

Ex: $\begin{bmatrix} -1 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \longrightarrow \underset{a}{\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}}, \underset{b}{\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}}, \underset{c}{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}}$

$(-1)\cdot a = (-1)\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \neq b, c$

$(-1)\cdot b = (-1)\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \neq a$ dar $= c$

$\Rightarrow$ c nu este liniar independent

$\Rightarrow$ Matricea are 2 coloane liniar independente a și b

$\Rightarrow$ Rangul = 2

Dacă nu se observă direct, atunci se aduce matricea la forma eșalon și Rangul = nr de rânduri $\neq 0$.

Ex: $\begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & 1 & 4 \end{bmatrix} \overset{R_4 \rightleftarrows R_3}{\sim} \begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & 1 \\ 1 & 1 & 4 \\ 0 & 2 & -1 \end{bmatrix} \overset{R_2 = (-1)R_3 + R_2}{\sim} \begin{bmatrix} 2 & -1 & 3 \\ 0 & -1 & -3 \\ 1 & 1 & 4 \\ 0 & 2 & -1 \end{bmatrix}$

$\overset{R_3 = R_1 + (-2)R_3}{\sim} \begin{bmatrix} 2 & -1 & 3 \\ 0 & -1 & -3 \\ 0 & -3 & -5 \\ 0 & 2 & -1 \end{bmatrix} \overset{R_3 = (-3)R_2 + R_3}{\sim} \begin{bmatrix} 2 & -1 & 3 \\ 0 & -1 & -3 \\ 0 & 0 & 4 \\ 0 & 2 & -1 \end{bmatrix} \overset{R_4 = 2R_2 + R_4}{\sim}$

$\begin{bmatrix} 2 & -1 & 3 \\ 0 & -1 & -3 \\ 0 & 0 & 4 \\ 0 & 0 & -7 \end{bmatrix} \overset{R_4 = 7R_3 + R_4}{\sim} \begin{bmatrix} 2 & -1 & 3 \\ 0 & -1 & -3 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$ Rangul este $\underline{3}$