Ecuații Diferențiale cu variabile Separabile

Ideea de lucru: grupăm "y" + dy de o parte și
"x" + dx de cealaltă parte
după aceea le integrăm pe amândouă

Funcționează pt $\frac{dy}{dx} = f(x,y)$ atunci când $f(x,y)$ poate fi scrisă ca un <u>produs</u>.

<u>Obs</u>: Păstrăm constantele în partea lui "x"
Rezolvăm pt "y" explicit <u>DACĂ</u> este posibil

Exerciții:

① $\frac{dy}{dx} + 2xy = 0$

$$\frac{dy}{dx} = -2xy \Big| \cdot \frac{dx}{y} \Rightarrow \frac{dy}{\cancel{dx}} \cdot \frac{\cancel{dx}}{y} = -2x\cancel{y} \cdot \frac{dx}{\cancel{y}} \Rightarrow$$

$$\Rightarrow \frac{1}{y}\,dy = -2x\,dx \Rightarrow \int \frac{1}{y}\,dy = \int -2x\,dx \Rightarrow$$

$$\Rightarrow \ln|y| = -\cancel{2} \cdot \frac{x^2}{\cancel{2}} + C_1 \Rightarrow \ln|y| = -x^2 + C_1 \Rightarrow$$

$$\Rightarrow e^{\ln|y|} = e^{-x^2 + C_1} \Rightarrow |y| = e^{-x^2} \cdot e^{C_1}$$

$$\Rightarrow |y| = e^{C_1} \cdot e^{-x^2} \Rightarrow y = \pm e^{C_1} \cdot e^{-x^2}$$

Fie $C = \pm e^{C_1}$

$$\Rightarrow y = C e^{-x^2}$$

Pentru a scăpa de logaritm, ne folosim de e cu proprietatea că $e^{\ln \square} = \square$

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

② $\frac{dy}{dx} + 2xy^2 = 0$

$\frac{dy}{dx} = -2xy^2 \Big| \cdot \frac{dx}{y^2} \Rightarrow \frac{dy}{\cancel{dx}} \cdot \frac{\cancel{dx}}{y^2} = -2x\cancel{y^2} \cdot \frac{dx}{\cancel{y^2}} \Rightarrow$

$\Rightarrow \frac{1}{y^2}\, dy = -2x\, dx \Rightarrow \int \frac{1}{y^2}\, dy = \int -2x\, dx \Rightarrow$

$\Rightarrow \int y^{-2}\, dy = -2\int x\, dx \Rightarrow \frac{y^{-2+1}}{-2+1} = \cancel{-2} \cdot \frac{x^2}{\cancel{2}} + C_1 \Rightarrow$

$\frac{y^{-1}}{-1} = -x^2 + C_1 \Rightarrow -\frac{1}{y} = -x^2 + C_1 \Big| \cdot (-1) \Rightarrow$

$\Rightarrow \frac{1}{y} = x^2 - C_1 \Rightarrow y = \frac{1}{x^2 - C_1}$

Fie $C = -C_1 \Rightarrow y = \frac{1}{x^2 + C}$

③ $\frac{dy}{dx} = y \cdot \sin(x)$

$\frac{dy}{dx} = y \sin(x) \Big| \cdot \frac{dx}{y} \Rightarrow \frac{dy}{\cancel{dx}} \cdot \frac{\cancel{dx}}{y} = \cancel{y} \cdot \sin(x) \cdot \frac{dx}{\cancel{y}}$

$\Rightarrow \frac{1}{y}\, dy = \sin(x)\, dx \Rightarrow \int \frac{1}{y}\, dy = \int \sin(x)\, dx$

$\Rightarrow \ln|y| = -\cos x + C_1 \Rightarrow e^{\ln|y|} = e^{-\cos x + C_1} \Rightarrow$

$\Rightarrow |y| = e^{-\cos x} \cdot e^{C_1} \Rightarrow y = \pm e^{C_1} \cdot e^{-\cos x}$

Fie $C = \pm e^{C_1} \Rightarrow y = C e^{-\cos x}$

④ $(1+x)\cdot\frac{dy}{dx}=4y$

$$(1+x)\cdot\frac{dy}{dx}=4y \;\Big|\cdot\frac{dx}{(1+x)y} \Rightarrow \cancel{(1+x)}\cdot\frac{dy}{\cancel{dx}}\cdot\frac{\cancel{dx}}{\cancel{(1+x)}y}=4\cancel{y}\cdot\frac{dx}{(1+x)\cancel{y}}$$

$$\Rightarrow \frac{1}{y}\,dy=\frac{4}{(1+x)}\,dx \Rightarrow \int\frac{1}{y}\,dy=\int\frac{4}{(1+x)}\,dx \Rightarrow$$

$$\Rightarrow \int\frac{1}{y}\,dy=4\int\frac{1}{(1+x)}\,dx \Rightarrow \ln|y|=4\ln|1+x|+C_1$$

$$\Rightarrow \cancel{e}^{\cancel{\ln}|y|}=e^{4\ln|1+x|+C_1}$$

$$\int\frac{1}{(1+x)}dx \qquad u=1+x,\quad du=1\,dx$$

$$\Rightarrow |y|=e^{\ln(|1+x|)^4+C_1}$$

$$\Rightarrow |y|=\cancel{e}^{\cancel{\ln}(|1+x|)^4}\cdot e^{C_1}$$

$$=\int\frac{1}{u}\,du=\ln|u|+C=\ln|1+x|+C$$

$$\Rightarrow |y|=e^{C_1}(1+x)^4$$

$$\Rightarrow y=\pm e^{C_1}(1+x)^4$$

Fie $C=\pm e^{C_1} \Rightarrow y=C(1+x)^4$

Renunțăm la modul când avem $(|1+x|)^4$ pt că rezultatul nu poate fi altfel decât pozitiv

$4\ln|1+x| = \ln(|1+x|)^4$

$n\ln x = \ln x^n$

⑤ $2\sqrt{x}\,\frac{dy}{dx}=\sqrt{1-y^2}$

$$2\sqrt{x}\,\frac{dy}{dx}=\sqrt{1-y^2}\;\Big|\cdot\frac{dx}{2\sqrt{x}\cdot(\sqrt{1-y^2})} \Rightarrow \cancel{2\sqrt{x}}\,\frac{dy}{\cancel{dx}}\cdot\frac{\cancel{dx}}{\cancel{2\sqrt{x}}(\sqrt{(1-y)^2})}=$$

$$=\cancel{\sqrt{1-y^2}}\cdot\frac{dx}{2\sqrt{x}\cdot(\cancel{\sqrt{1-y^2}})} \Rightarrow \frac{1}{\sqrt{(1-y)^2}}\,dy=\frac{1}{2\sqrt{x}}\,dx \Rightarrow$$

$$\Rightarrow \underbrace{\int \frac{1}{\sqrt{1-y^2}}\,dy}_{\sin^{-1}(y)} = \int \frac{1}{2\sqrt{x}}\,dx \Rightarrow \sin^{-1}(y) = \sqrt{x} + C_1$$

$$\Rightarrow y = \sin(\sqrt{x} + C)$$

$$\int \frac{1}{2\sqrt{x}} = \frac{1}{2}\int x^{-\frac{1}{2}} = \frac{1}{2} \cdot \frac{x^{-\frac{1}{2}+\frac{2}{1}}}{-\frac{1}{2}+\frac{2}{1}}$$

$$= \frac{1}{\cancel{2}} \cdot \frac{x^{\frac{1}{2}}}{\frac{1}{\cancel{2}}} = \sqrt{x} + C$$

$\arcsin y = \sin^{-1} y$

$\arcsin(y) = \sqrt{x} + c$

aplicăm prop inv trig:

$\arcsin(x) = a \Rightarrow x = \sin(a)$

$y = \sin(\sqrt{x} + c$

⑥ $$\frac{dy}{dx} = (64xy)^{\frac{1}{3}}$$

$$\frac{dy}{dx} = 64^{\frac{1}{3}} \cdot x^{\frac{1}{3}} \cdot y^{\frac{1}{3}} \quad \Big| \cdot \frac{dx}{y^{\frac{1}{3}}} \Rightarrow$$

$$\Rightarrow \frac{dy}{\cancel{dx}} \cdot \frac{\cancel{dx}}{y^{\frac{1}{3}}} = 4 \cdot x^{\frac{1}{3}} \cdot \cancel{y^{\frac{1}{3}}} \cdot \frac{dx}{\cancel{y^{\frac{1}{3}}}} \Rightarrow \frac{1}{y^{\frac{1}{3}}}\,dy = 4x^{\frac{1}{3}}\,dx \Rightarrow$$

$$\Rightarrow \int \frac{1}{y^{\frac{1}{3}}}\,dy = 4\int x^{\frac{1}{3}}\,dx \Rightarrow \frac{y^{-\frac{1}{3}+\frac{3}{1}}}{-\frac{1}{3}+\frac{3}{1}} = 4 \cdot \frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1} + C_1 \Rightarrow$$

$$\Rightarrow \frac{3}{2} \cdot y^{\frac{2}{3}} = 3x^{\frac{4}{3}} + C_1 \quad \Big| \cdot \frac{2}{3} \Rightarrow \frac{\cancel{3}}{\cancel{2}} \cdot \frac{\cancel{2}}{\cancel{3}} \cdot y^{\frac{2}{3}} = \frac{2}{\cancel{3}} \cdot \cancel{3}x^{\frac{4}{3}} + \frac{2}{3}C_1$$

$$\text{Fie } C = \frac{2}{3}C_1 \Rightarrow y^{\frac{2}{3}} = 2x^{\frac{4}{3}} + C \quad \Big|^{3/2} \Rightarrow \left(y^{\frac{\cancel{2}}{\cancel{3}}}\right)^{\frac{\cancel{3}}{\cancel{2}}} = \left(2x^{\frac{4}{3}} + c\right)^{\frac{3}{2}}$$

$$\Rightarrow y = \left(2x^{\frac{4}{3}} + c\right)^{\frac{3}{2}}$$

(7) $\frac{dy}{dx} = 2x \sec y$

$\frac{dy}{dx} = 2x \cdot \sec y \;\Big|\cdot \frac{dx}{\sec y} \Rightarrow \frac{dy}{\cancel{dx}} \cdot \frac{\cancel{dx}}{\sec y} = 2x\,\cancel{\sec y}\,\frac{dx}{\cancel{\sec y}}$

$= \frac{1}{\sec y} dy = 2x\,dx \Rightarrow \int \underbrace{\frac{1}{\sec y}}_{\cos y} dy = 2\int x\,dx \Rightarrow$

$\boxed{\sec y = \frac{1}{\cos y} \quad \frac{1}{\sec y} = \cos y}$

$\Rightarrow \sin y = \cancel{2}\cdot\frac{x^2}{\cancel{2}} + C_1 \Rightarrow \sin y = x^2 + C_1$

$\Rightarrow y = \sin^{-1}(x^2 + C)$

(8) $(1-x^2)\frac{dy}{dx} = 2y$

$(1-x^2)\frac{dy}{dx} = 2y \;\Big|\cdot \frac{dx}{(1-x^2)\cdot y} \Rightarrow \cancel{(1-x^2)}\frac{dy}{\cancel{dx}} \cdot \frac{\cancel{dx}}{\cancel{(1-x^2)}\cdot y} = 2\cancel{y}\frac{dx}{\cancel{y}(1-x^2)}$

$\Rightarrow \frac{1}{y} dy = \frac{2}{(1-x^2)} dx \Rightarrow \int \frac{1}{y} dy = \int \frac{2}{1-x^2} dx$

$\frac{2}{1-x^2} \to \frac{2}{(1-x)(1+x)} \Rightarrow \frac{2}{(1-x)(1+x)} = \frac{A}{1-x} + \frac{B}{1+x}$

$2 = A(1+x) + B(1-x)$ Dacă $x = 1 \Rightarrow 2 = A\cdot 2 + \underbrace{B\cdot 0}_{0}$

$\Rightarrow A = 1$

Dacă $x = -1 \Rightarrow 2 = A\cdot 0 + B\cdot 2$

$\Rightarrow B = 1$

$\Rightarrow \frac{1}{1-x} + \frac{1}{1+x}$

$\Rightarrow \int \frac{1}{y} dy = \int\left(\frac{1}{1-x} + \frac{1}{1+x}\right) dx \Rightarrow$

$$\Rightarrow \ln|y| = -\ln|1-x| + \ln|1+x| + C_1$$

$$\int \frac{1}{1-x}\,dx \quad \begin{aligned} u &= 1-x \\ du &= -dx \;\; |\cdot(-1) \\ -du &= dx \end{aligned}$$

$$\int \frac{1}{1+x}\,dx \quad \begin{aligned} u &= 1+x \\ du &= 1\,dx \end{aligned}$$

$$\int \frac{1}{u}\cdot du = \ln|u| + c$$
$$= \ln|1+x| + c$$

$$= \int \frac{1}{u}\cdot -du = -\ln|u| + c$$
$$= -\ln|1-x| + c$$

$$\Rightarrow \ln|y| = \ln|1+x| - \ln|1-x| + C_1$$

$$\boxed{\ln|a| - \ln|b| = \ln\left|\frac{a}{b}\right|}$$

$$\Rightarrow \ln|y| = \ln\left|\frac{1+x}{1-x}\right| + C_1 \Rightarrow$$

$$\Rightarrow e^{\cancel{\ln}|y|} = e^{\ln\left|\frac{1+x}{1-x}\right| + C_1} \Rightarrow |y| = \cancel{e}^{\cancel{\ln}\left|\frac{1+x}{1-x}\right|}\cdot e^{C_1} \Rightarrow$$

$$\Rightarrow y = \underbrace{\pm e^{C_1}}_{=C}\left(\frac{1+x}{1-x}\right) \Rightarrow y = C\left(\frac{1+x}{1-x}\right)$$

9. $(1+x)^2 \dfrac{dy}{dx} = (1+y)^2$

$$(1+x)^2\frac{dy}{dx} = (1+y)^2 \;\Big|\cdot \frac{dx}{(1+x)^2(1+y)^2} \Rightarrow$$

$$\Rightarrow \cancel{(1+x)^2}\frac{dy}{\cancel{dx}}\cdot\frac{\cancel{dx}}{\cancel{(1+x)^2}(1+y)^2} = \cancel{(1+y)^2}\cdot\frac{dx}{(1+x)^2\cancel{(1+y)^2}} \Rightarrow$$

$$\Rightarrow \frac{1}{(1+y)^2}\,dy = \frac{1}{(1+x)^2} \Rightarrow \int\frac{1}{(1+y)^2}\,dy = \int\frac{1}{(1+x)^2}\,dx$$

$$\int \frac{1}{(1+y)^2}\,dy \qquad \begin{matrix} u = 1+y \\ du = dy \end{matrix}$$

$$= \int \frac{1}{u^2}\,du = \int u^{-2}\,du = \frac{u^{-2+1}}{-2+1} + C$$

$$= \frac{u^{-1}}{-1} + C = -\frac{1}{u} + C = -\frac{1}{1+y} + C$$

$$\int \frac{1}{(1+x)^2}\,dx \qquad \begin{matrix} u = 1+x \\ du = dx \end{matrix}$$

$$\int \frac{1}{u^2}\,du = \int u^{-2}\,du$$

$$\frac{u^{-2+1}}{-2+1} + C = \frac{u^{-1}}{-1} + C =$$

$$= -\frac{1}{u} + C = -\frac{1}{1+x} + C$$

$$\Rightarrow -\frac{1}{1+y} = -\frac{1}{1+x} + C_1 \quad \Big| \cdot (-1)$$

$$\Rightarrow \frac{1}{1+y} = \frac{1}{1+x} \overset{1+x}{-} C_1 \Rightarrow \frac{1}{1+y} = \frac{1}{1+x} - \frac{C_1(1+x)}{1+x} \Rightarrow$$

$$\Rightarrow \frac{1}{1+y} = \frac{1 - C_1(1+x)}{1+x} \Rightarrow 1+y = \frac{1+x}{1-C_1(1+x)}$$

Für $C = -C_1 \Rightarrow y = \dfrac{1+x}{1+C(1+x)} - 1$

(10) $y^3 \dfrac{dy}{dx} = (y^4+1)\cos x$

$$y^3 \frac{dy}{dx} = (y^4+1)\cos x \;\Big|\cdot \frac{dx}{(y^4+1)} \Rightarrow y^3 \frac{dy}{\cancel{dx}} \cdot \frac{\cancel{dx}}{(y^4+1)} = \cancel{(y^4+1)}\cos x \frac{dx}{\cancel{(y^4+1)}}$$

$$\Rightarrow \frac{y^3}{y^4+1}\,dy = \cos x\,dx \Rightarrow \int \frac{y^3}{y^4+1}\,dy = \underbrace{\int \cos x\,dx}_{\sin x + C_1}$$

$$\int \frac{y^3}{y^4+1}\,dy \qquad \begin{matrix} u = y^4+1 \\ du = 4y^3\,dy \;|:4 \\ \frac{du}{4} = y^3\,dy \end{matrix} \qquad \Rightarrow \int \frac{1}{u} \cdot \frac{du}{4} = \frac{1}{4}\int \frac{1}{u}\,du = \frac{1}{4}\ln|u| + C$$

$= \frac{1}{4}\ln(y^4+1)$

$\Rightarrow \frac{1}{4}\ln(y^4+1) = \sin x + C_1 \,\Big|\cdot 4 \Rightarrow \cancel{\frac{1}{4}}\cdot\cancel{4}\ln(y^4+1) = 4\sin x$

$+ 4C_1$ $\Rightarrow$ Lăsăm ecuația implicit, nu rezolvăm pt y

(11) $\frac{dy}{dx} = \frac{(x-1)y^5}{x^2(2y^3-y)}$

$$\frac{dy}{dx} = \frac{(x-1)y^5}{x^2(2y^3-y)} \,\Big|\cdot \frac{dx(2y^3-y)}{y^5}$$

$$\Rightarrow \frac{dy}{\cancel{dx}}\cdot\frac{\cancel{dx}(2y^3-y)}{y^5} = \frac{(x-1)\cancel{y^5}}{x^2\cancel{(2y^3-y)}}\cdot\frac{dx\cancel{(2y^3-y)}}{\cancel{y^5}} \Rightarrow$$

$$\Rightarrow \frac{2y^3-y}{y^5}\,dy = \frac{x-1}{x^2}\,dx \Rightarrow \int\frac{2y^3-y}{y^5}\,dy = \int\frac{x-1}{x^2}\,dx \Rightarrow$$

$$\Rightarrow \int\frac{2\cancel{y^3}}{\cancel{y^5}}dy - \int\frac{\cancel{y}}{\cancel{y^5}}dy = \int\frac{\cancel{x}}{\cancel{x^2}}dx - \int\frac{1}{x^2}dx \Rightarrow 2\int\frac{1}{y^2}dy - \int\frac{1}{y^4}dy =$$

$$= \int\frac{1}{x}dx - \int\frac{1}{x^2}dx \Rightarrow 2\cdot\frac{y^{-2+1}}{-2+1} - \frac{y^{-4+1}}{-4+1} = \ln|x| - \frac{x^{-2+1}}{-2+1} + C_1$$

$$\Rightarrow 2\frac{y^{-1}}{-1} - \frac{y^{-3}}{-3} = \ln|x| - \frac{x^{-1}}{-1} + C_1 \Rightarrow -2\frac{1}{y} + \frac{1}{3y^3} = \ln|x| + \frac{1}{x} + C_1$$

$\Rightarrow$ O lăsăm implicit, nu rezolvăm pt y

(12) $(x^2+1)\,tg(y)\frac{dy}{dx} = x$

$$(x^2+1)\,tg(y)\cdot\frac{dy}{dx} = x \,\Big|\cdot\frac{dx}{(x^2+1)} \Rightarrow$$

$$\cancel{(x^2+1)}\, tg(y) \cdot \frac{dy}{\cancel{dx}} \cdot \frac{\cancel{dx}}{\cancel{(x^2+1)}} = x \frac{dx}{(x^2+1)} \Rightarrow$$

$$\Rightarrow tg(y)\, dy = \frac{x}{x^2+1}\, dx \Rightarrow \underbrace{\int tg(y)\, dy}_{\ln|\sec y|} = \int \frac{x}{x^2+1}\, dx$$

$$\int \frac{x\, dx}{x^2+1} \quad u = x^2+1$$

$$du = 2x\, dx \mid :2 \qquad \Rightarrow \ln|\sec y| = \frac{1}{2} \ln(x^2+1) + C_1$$

$$\frac{du}{2} = x\, dx$$

$$= \int \frac{1}{u} \cdot \frac{du}{2} = \frac{1}{2} \int \frac{1}{u}\, du = \frac{1}{2} \ln|u| + c$$

$$= \frac{1}{2} \ln(x^2+1) + c$$

$$\Rightarrow \ln|\sec y| = \ln(\sqrt{x^2+1}) + C_1 \Rightarrow e^{\ln|\sec y|} = e^{\ln(\sqrt{x^2+1}) + C_1}$$

$$\Rightarrow |\sec y| = \cancel{e^{\ln}}(\sqrt{x^2+1}) \cdot e^{C_1} \Rightarrow \sec y = \pm e^{C_1} (\sqrt{x^2+1})$$

$$\text{Für } C = \pm e^{C_1} \Rightarrow \sec y = C(\sqrt{x^2+1}) \Rightarrow y = \sec^{-1}(c\sqrt{x^2+1})$$

(13) $$x^2 \frac{dy}{dx} = 1 - x^2 + y^2 - x^2 y^2 \qquad (1-x^2) + y^2(1-x^2) = (1-x^2)(1+y^2)$$

$$x^2 \frac{dy}{dx} = (1-x^2)(1+y^2) \;\Big|\cdot \frac{dx}{x^2(1+y^2)}$$

$$\Rightarrow \cancel{x^2} \frac{dy}{\cancel{dx}} \cdot \frac{\cancel{dx}}{\cancel{x^2}(1+y^2)} = (1-x^2)\cancel{(1+y^2)} \cdot \frac{dx}{x^2 \cancel{(1+y^2)}} \Rightarrow$$

$$\Rightarrow \frac{1}{(1+y^2)}\, dy = \frac{(1-x^2)}{x^2}\, dx \Rightarrow \underbrace{\int \frac{1}{(1+y^2)}\, dy}_{\tan^{-1}(y)} = \int \frac{1-x^2}{x^2}\, dx$$

$$\int \frac{1-x^2}{x^2}\, dx = \int \frac{1}{x^2} - \frac{\cancel{x^2}}{\cancel{x^2}}\, dx = \int \frac{1}{x^2}\, dx - \int 1\, dx = \frac{x^{-2+1}}{-2+1} - x + c$$

$= \frac{x^{-1}}{-1} - x + C = -x^{-1} - x + C$

$\Rightarrow \tan^{-1}(y) = -x^{-1} - x + C_1$

Füi $C = C_1 \Rightarrow y = \tan\left(-\frac{1}{x} - x + C\right)$