Ecuații Diferențiale Liniare de ordinul I

Ideea de lucru:

1. Aducem ec dif la forma: $\frac{dy}{dx} + P(x) \cdot y = Q(x)$

P(x) și Q(x) sunt date
funcții de x

2. Trebuie să găsim $\psi(x)$:

$$\psi(x) = e^{\int P(x)\, dx}$$

3. Înmulțim fiecare membru cu $\psi(x)$:

$$\psi(x) \cdot \frac{dy}{dx} + \psi(x) \cdot P(x) \cdot y = \psi(x) \cdot Q(x)$$

4. Tratăm ca pe un produs $(A \cdot B' + A' \cdot B)$ și integrăm de ambele părți: $D_x[A \cdot B]$

$$\int D_x[\psi(x) \cdot y] = \int \psi(x) \cdot Q(x)$$

Obținem din

$$\boxed{\psi(x)} \cdot \frac{dy}{dx} + \psi(x) \cdot P(x) \cdot \boxed{y} = \boxed{\psi(x) \cdot Q(x)}$$

5. Integrala anulează derivata $\int D_x$ și obținem:

$$\psi(x) \cdot y = \int \psi(x) \cdot Q(x)$$

6. Rezolvăm

Exercitii:

① $x^3 \frac{dy}{dx} + 3x^2 \cdot y = 2 \quad \Big| \cdot \frac{1}{x^3}$ (Exercitiul este mai mult pt exemplificarea produsului)

$\Rightarrow \frac{dy}{dx} + \underbrace{\boxed{\frac{3}{x}}}_{P(x)} y = \frac{2}{x^3}$ $\quad \psi(x) = e^{\int P(x)\,dx} = e^{\int \frac{3}{x}\,dx} = e^{3\ln|x|}$
$= \cancel{e}^{\cancel{\ln}|x|^3} = |x|^3$

$x^3 \cdot \frac{dy}{dx} + x^{\cancel{3}} \frac{3}{\cancel{x}} y = \cancel{x^3} \cdot \frac{2}{\cancel{x^3}} \Rightarrow \int D_x[x^3 \cdot y]\,dx = \int 2\,dx$

$\Rightarrow x^3 y = 2x + C$

② $\frac{dy}{dx} + y = 2, \quad y(0) = 0$

$\frac{dy}{dx} + \underbrace{\boxed{}}_{P(x)} y = 2$ $\quad \psi(x) = e^{\int P(x)\,dx} = e^{\int 1\,dx} = e^x$

$\Rightarrow e^x \cdot \frac{dy}{dx} + e^x \cdot y = 2 \cdot e^x \Rightarrow \int D_x[e^x \cdot y]\,dx = \int 2e^x\,dx$

$\Rightarrow e^x \cdot y = 2e^x + C$

pt $y(\underset{x}{0}) = 0 \Rightarrow e^0 \cdot 0 = 2e^0 + C \Rightarrow 1 \cdot 0 = 2 \cdot 1 + C \Rightarrow C = -2$

$\Rightarrow e^x \cdot y = 2e^x - 2 \quad \Big| \cdot \frac{1}{e^x} \Rightarrow \cancel{e^x} \cdot y \cdot \frac{1}{\cancel{e^x}} = \frac{2\cancel{e^x}}{\cancel{e^x}} - \frac{2}{e^x} \Rightarrow$

$\Rightarrow y(x) = 2 - \frac{2}{e^x} \Rightarrow y(x) = 2 - 2 \cdot e^{-x}$

③ $\frac{dy}{dx} + \boxed{3}y = 2xe^{-3x}$

$\hookrightarrow P(x) \qquad \Psi(x) = e^{\int P(x)\,dx} = e^{\int 3\,dx} = e^{3x}$

$e^{3x} \cdot \frac{dy}{dx} + e^{3x} \cdot 3 \cdot y = 2xe^{-3x} \cdot e^{3x} \Rightarrow$

$e^{3x}\frac{dy}{dx} + 3e^{3x} \cdot y = 2x \Rightarrow \int D_x[e^{3x} \cdot y]\,dx = \int 2x\,dx$

$\Rightarrow e^{3x} \cdot y = \cancel{2} \cdot \frac{x^2}{\cancel{2}} + c \Rightarrow e^{3x} \cdot y = x^2 + c \Rightarrow$

$\Rightarrow y = \frac{x^2+c}{e^{3x}}$ sau $y = (x^2+c) \cdot e^{-3x}$

Obs: Se ia și semnul

④ $\frac{dy}{dx} \boxed{-2x} y = e^{x^2}$

$\hookrightarrow P(x) \qquad \Psi(x) = e^{\int P(x)\,dx} = e^{\int -2x\,dx} = e^{-\cancel{2}\frac{x^2}{\cancel{2}}} = e^{-x^2}$

$e^{-x^2} \cdot \frac{dy}{dx} - e^{-x^2} \cdot 2xy = e^{-x^2} \cdot e^{x^2} \Rightarrow e^{-x^2}\frac{dy}{dx} - 2xe^{-x^2} \cdot y = 1$

$\Rightarrow \int D_x[e^{-x^2} \cdot y]\,dx = \int 1\,dx \Rightarrow e^{-x^2} \cdot y = x + c \Rightarrow$

$\Rightarrow y = \frac{x+c}{e^{-x^2}}$ sau $y = (x+c) \cdot e^{x^2}$

⑤ $x\frac{dy}{dx} + 5y = 7x^2, \quad y(2) = 5$

$x\frac{dy}{dx} + 5y = 7x^2 \,\Big|\cdot \frac{1}{x} \Rightarrow \frac{1}{\cancel{x}} \cdot \cancel{x} \cdot \frac{dy}{dx} + \frac{1}{x} \cdot 5y = \frac{1}{\cancel{x}} \cdot 7x^{\cancel{2}} \Rightarrow$

$\hookrightarrow$ Trebuie aduse la forma $\frac{dy}{dx} + P(x) \cdot y = Q(x)$

$$\Rightarrow \frac{dy}{dx} + \boxed{\frac{5}{x}} \cdot y = 7x$$

$\hookrightarrow P(x)$

$$\Psi(x) = e^{\int P(x)\,dx} = e^{\int \frac{5}{x}\,dx} = e^{\overbrace{5\ln|x|}} = e^{\ln|x|^5} = |x|^5$$

$$\Rightarrow x^5 \cdot \frac{dy}{dx} + \cancel{x^5} \cdot \frac{5}{\cancel{x}} \cdot y = 7x \cdot x^5 \Rightarrow x^5 \frac{dy}{dx} + 5x^4 \cdot y = 7x^6$$

$$\Rightarrow \int D_x[x^5 \cdot y]\,dx = \int 7x^6 \Rightarrow x^5 \cdot y = \cancel{7} \cdot \frac{x^{6+1}}{\cancel{7}} + C \Rightarrow$$

$$x^5 \cdot y = x^7 + C$$

pt $y(2) = 5 \Rightarrow 2^5 \cdot 5 = 2^7 + C \Rightarrow C = 32$

Deci: $x^5 \cdot y = x^7 + 32 \;\Big|\cdot \frac{1}{x^5} \Rightarrow \frac{1}{\cancel{x^5}} \cdot \cancel{x^5} \cdot y = \frac{x^{\cancel{7}}}{\cancel{x^5}} + \frac{32}{x^5}$

$$\Rightarrow y = x^2 + \frac{32}{x^5}$$

⑥ $2x \frac{dy}{dx} + y = 10\sqrt{x}$

$$2x \frac{dy}{dx} + y = 10\sqrt{x} \;\Big|\cdot \frac{1}{2x} \Rightarrow \frac{1}{\cancel{2x}} \cdot \cancel{2x} \cdot \frac{dy}{dx} + \frac{1}{2x} \cdot y = \frac{1}{\cancel{2}x} \cdot \cancel{10}^{5}\sqrt{x}$$

$$\Rightarrow \frac{dy}{dx} + \boxed{\frac{1}{2x}} \cdot y = \frac{5\sqrt{x}}{x}$$

$\hookrightarrow P(x)$

$$\Psi(x) = e^{\int P(x)\,dx} = e^{\int \frac{1}{2x}\,dx} = e^{\frac{1}{2}\int \frac{1}{x}\,dx} = e^{\overbrace{\frac{1}{2}\ln|x|}} = e^{\ln|x|^{1/2}} = |x|^{1/2}$$

$$\Rightarrow \sqrt{x} \cdot \frac{dy}{dx} + \sqrt{x} \cdot \frac{1}{2x} \cdot y = \cancel{\sqrt{x}} \cdot \frac{5\sqrt{\cancel{x}}}{\cancel{x}} \Rightarrow \int D_x[\sqrt{x} \cdot y]\,dx = \int 5\,dx$$

$$\Rightarrow \sqrt{x} \cdot y = 5x + C \;\Big|\cdot \frac{1}{\sqrt{x}} \Rightarrow \frac{1}{\cancel{\sqrt{x}}} \cdot \cancel{\sqrt{x}} \cdot y = \frac{5x}{\sqrt{x}} + \frac{C}{\sqrt{x}} \Rightarrow$$

$$\Rightarrow y = \frac{5x + C}{\sqrt{x}}$$

(7) $2x \cdot \frac{dy}{dx} - 3y = 9x^3$

$2x \cdot \frac{dy}{dx} - 3y = 9x^3 \;\Big|\cdot \frac{1}{2x} \Rightarrow \frac{1}{\cancel{2x}} \cdot \cancel{2x} \cdot \frac{dy}{dx} - \frac{1}{2x} \cdot 3y =$

$= \frac{1}{2x} \cdot 9x^{\cancel{3}} \Rightarrow \frac{dy}{dx} \boxed{-\frac{3}{2x}} \cdot y = \frac{9x^2}{2}$

$\hookrightarrow P(x)$

$\psi(x) = e^{\int -\frac{3}{2x} dx} = e^{-\frac{3}{2}\int \frac{1}{x} dx}$

$= e^{-\frac{3}{2} \ln|x|} = \cancel{e}^{\cancel{\ln}|x|^{-3/2}}$

$= |x|^{-3/2}$

$x^{-\frac{3}{2}} \cdot \frac{dy}{dx} - x^{-\frac{3}{2}} \cdot \frac{3}{2x} \cdot y = x^{-\frac{3}{2}} \cdot \frac{9x^2}{2}$

$x^{-\frac{3}{2}} \cdot x^2 = x^{-\frac{3}{2} + \frac{2}{2}}$

$= x^{\frac{-3+4}{2}} = x^{1/2}$

$\Rightarrow \int D_x\left[x^{-\frac{3}{2}} \cdot y\right] dx = \int \frac{9}{2} x^{\frac{1}{2}} dx \Rightarrow$

$\Rightarrow x^{-\frac{3}{2}} \cdot y = \frac{\cancel{9}^{3}}{\cancel{2}} \cdot \frac{x^{\frac{1}{2}+\frac{2}{2}}}{\frac{\cancel{3}_1}{\cancel{2}}} + C \Rightarrow x^{-\frac{3}{2}} \cdot y = 3x^{\frac{3}{2}} + C \;\Big|\cdot x^{\frac{3}{2}} \Rightarrow$

$\Rightarrow \cancel{x^{-\frac{3}{2}}} \cdot \cancel{x^{\frac{3}{2}}} \cdot y = \left(3x^{\frac{3}{2}} + C\right) x^{\frac{3}{2}} \Rightarrow y = 3x^3 + Cx^{\frac{3}{2}}$

(8) $\frac{dy}{dx} + \boxed{}y = e^x \qquad \psi(x) = e^{\int 1\, dx} = e^x$

$\hookrightarrow P(x) = 1$

$e^x \cdot \frac{dy}{dx} + e^x \cdot y = e^x \cdot e^x \Rightarrow \int D_x\left[e^x \cdot y\right] dx = \int e^{2x} dx$

$\Rightarrow e^x \cdot y = \frac{1}{2} e^{2x} + C \;\Big|\cdot \frac{1}{e^x}$

$\int e^{2x} dx \qquad u = 2x$

$du = 2dx \;|\cdot \frac{1}{2}$

$\frac{du}{2} = dx$

$= \int e^u \cdot \frac{du}{2} = \frac{1}{2}\int e^u du$

$= \frac{1}{2} e^u + C = \frac{1}{2} e^{2x} + C$

$\Rightarrow \frac{1}{\cancel{e^x}} \cdot \cancel{e^x} \cdot y = \frac{1}{\cancel{e^x}} \cdot \frac{1}{2} \cdot e^{\cancel{2}x} + \frac{C}{e^x}$

$\Rightarrow y = \frac{1}{2} e^x + C e^{-x}$

9) $\frac{dy}{dx} = (1-y)\cos x$; $y(\pi) = 2$

Este un exemplu ce se poate rezolva atât prin metoda variabilelor separabile dar și prin metoda pt ec dif de ordinul I (Liniare).

A. variabile separabile

$\frac{dy}{dx} = (1-y)\cos x \,\Big|\cdot \frac{dx}{(1-y)} \Rightarrow \frac{dy}{\cancel{dx}} \cdot \frac{\cancel{dx}}{(1-y)} = \cancel{(1-y)}\cos x \cdot \frac{dx}{\cancel{(1-y)}}$

$\Rightarrow \frac{dy}{(1-y)} = \cos x \frac{dx}{1} \Rightarrow \underbrace{\int \frac{1}{1-y}\,dy}_{} = \underbrace{\int \cos x\, dx}_{= \sin x + C_1}$

$\int \frac{1}{1-y}\,dy$ $\quad u = 1-y$, $du = -dy \,|\cdot(-1)$, $-du = dy$

$= \int \frac{1}{u} \cdot - du$

$= -\ln|u| = -\ln|1-y|$

$\Rightarrow -\ln|1-y| = \sin x + C_1 \,|(-1)$

$\Rightarrow \ln|1-y| = -\sin x - C_1$

$\Rightarrow e^{\ln|1-y|} = e^{-\sin x} \cdot e^{-C_1}$

$\Rightarrow |1-y| = e^{-C_1} \cdot e^{-\sin x}$

$\Rightarrow 1-y = \pm e^{-C_1} \cdot e^{-\sin x}$

Fie $C = \pm e^{-C_1} \Rightarrow 1-y = C e^{-\sin x}$

Pentru $y(\pi) = 2 \Rightarrow 1-2 = C e^{-\sin \pi} \Rightarrow -1 = C e^{0} \Rightarrow$

$\Rightarrow C = -1$

Obținem: $1-y = -1\, e^{-\sin x} \Rightarrow -y = -e^{-\sin x} - 1 \,|\cdot(-1) \Rightarrow$

$\Rightarrow y = e^{\sin x} + 1$

B. ec dif liniară de ordinul I

Trebuie să o aducem la forma $\frac{dy}{dx} + P(x) \cdot y = Q(x)$

$\frac{dy}{dx} = (1-y)\cos x \Rightarrow \frac{dy}{dx} = \cos x - y\cos x \Rightarrow$

$\Rightarrow \frac{dy}{dx} + \underbrace{\boxed{\cos x}}_{\to P(x)} \cdot y = \cos x \qquad \Psi(x) = e^{\int \cos x \, dx} = e^{\sin x}$

$\Rightarrow e^{\sin x} \cdot \frac{dy}{dx} + e^{\sin x} \cdot \cos x \cdot y = e^{\sin x} \cdot \cos x \Rightarrow$

$\Rightarrow \int D_x[e^{\sin x} \cdot y]\,dx = \int e^{\sin x} \cdot \cos x \, dx \Rightarrow e^{\sin x} \cdot y = e^{\sin x} + c$

$\Rightarrow y = \frac{\cancel{e^{\sin x}}}{\cancel{e^{\sin x}}} + \frac{c}{e^{\sin x}}$

$\int e^{\sin x} \cdot \cos x \, dx \qquad u = \sin x, \quad du = \cos x \, dx$

$\Rightarrow y = 1 + \frac{c}{e^{\sin x}}$

$= \int e^u \cdot du = e^u + c = e^{\sin x} + c$

$\Rightarrow y = 1 + c e^{-\sin x}$

$\Rightarrow$ Observăm că obținem același rezultat ca cel obținut atunci când am rezolvat prin prima metodă.

(10) $\frac{dy}{dx} + y\underbrace{\boxed{\cot(x)}}_{\to P(x)} = \cos x \qquad \Psi(x) = e^{\int P(x)\,dx} = e^{\int \cot(x)\,dx} = e^{\ln|\sin x|} = |\sin x|$

$\sin(x) \cdot \frac{dy}{dx} + \sin(x) \cdot \cot(x) \cdot y = \sin(x) \cdot \cos(x)$

$\Rightarrow \int D_x[\sin(x) \cdot y]\,dx = \int \sin(x) \cdot \cos(x)\,dx \Rightarrow$

$$\int \sin x \cos x \, dx \qquad u = \sin x$$
$$du = \cos x \, dx$$

$$= \int u \cdot du = \frac{u^2}{2} + c = \frac{\sin^2 x}{2} + c$$

$$\Rightarrow \sin(x) \cdot y = \frac{\sin^2 x}{2} + c \;\Big|\; \cdot \frac{1}{\sin(x)} \Rightarrow \frac{1}{\cancel{\sin(x)}} \cdot \cancel{\sin(x)} \cdot y =$$

$$= \frac{\sin^{\cancel{2}} x}{2} \cdot \frac{1}{\cancel{\sin(x)}} + \frac{c}{\sin(x)} \Rightarrow y = \frac{\sin(x)}{2} + \frac{c}{\sin(x)} \Rightarrow$$

$$\Rightarrow y = \frac{1}{2} \sin(x) + C \cdot \sin x^{-1}$$

(11) $\frac{dy}{dx} = 1 + x + y + xy, \quad y(0) = 0$

$= (1+x) + y(1+x) \Rightarrow$ rezolvăm ca pt ec dif lin de ord I

$$\Rightarrow \frac{dy}{dx} \boxed{-(1+x)} \cdot y = 1 + x \qquad \psi(x) = e^{\int P(x)\,dx} = e^{\int -(1+x)\,dx}$$
$$\hookrightarrow P(x) \qquad = e^{-(x+\frac{1}{2}x^2)}$$

$$\int (-1-x)\,dx = -\int 1\,dx - \int x\,dx \qquad \Rightarrow e^{-(x+\frac{1}{2}x^2)} \cdot \frac{dy}{dx} - e^{-(x+\frac{1}{2}x^2)} \cdot$$

$$= -x - \frac{x^2}{2} + c = -\left(x + \frac{x^2}{2}\right) + c \qquad \cdot (1+x) \cdot y = (1+x) \cdot e^{-(x+\frac{1}{2}x^2)}$$

$$\Rightarrow \int D_x\left[e^{-(x+\frac{1}{2}x^2)} \cdot y\right] dx = \int (1+x) \cdot e^{-(x+\frac{1}{2}x^2)} \, dx$$

$$\int (1+x) \cdot e^{-(x+\frac{1}{2}x^2)} \, dx \qquad u = x + \frac{1}{2}x^2 \qquad \left(x + \frac{1}{2}x^2\right)' = x' + \left(\frac{x^2}{2}\right)'$$
$$du = (1+x)\,dx \qquad = 1 + \frac{(x^2)' \cdot 2 - x^2 \cdot 2'}{4}$$
$$= 1 + \frac{\cancel{2}x \cdot \cancel{2}}{\cancel{4}} = 1 + x$$

$$\Rightarrow \int e^{-u}\cdot du = -e^{-u} = -e^{-(x+\frac{1}{2}x^2)} + c$$

$s = -u$

$ds = -du \mid (-1)$

$-ds = du$

$$\Rightarrow \int e^{s}\cdot -ds = -\int e^{s}ds = -e^{s}$$

$$= -e^{-u} + c$$

$$\Rightarrow e^{-(x+\frac{1}{2}x^2)}\cdot y = -e^{-(x+\frac{1}{2}x^2)} + c$$

$$\text{pt } y(0)=0 \Rightarrow \underbrace{e^{-(0+\frac{1}{2}\cdot 0^2)}\cdot 0}_{0} = -e^{-(0+\frac{1}{2}\cdot 0^2)} + c$$

$$\Rightarrow 0 = -1 + c \Rightarrow c = 1$$

Obtinem: $e^{-(x+\frac{1}{2}x^2)}\cdot y = -e^{-(x+\frac{1}{2}x^2)} + 1 \Big| \cdot \frac{1}{e^{-(x+\frac{1}{2}x^2)}}$

$$\Rightarrow y = -1 + e^{(x+\frac{1}{2}x^2)}$$

(12) $x\frac{dy}{dx} + (2x-3)y = 4x^4$

Aducem la forma $\frac{dy}{dx} + P(x)\cdot y = Q(x)$

$$x\frac{dy}{dx} + (2x-3)y = 4x^4 \Big| \cdot \frac{1}{x} \Rightarrow \frac{1}{\cancel{x}}\cdot\cancel{x}\cdot\frac{dy}{dx} + \frac{1}{x}\cdot(2x-3)\cdot y = \frac{1}{\cancel{x}}\cdot 4x^{\cancel{4}} \Rightarrow \frac{dy}{dx} + \underbrace{\boxed{\frac{2x-3}{x}}}_{P(x)}\cdot y = 4x^3$$

$$\psi(x) = e^{\int\frac{2x-3}{x}dx} = e^{2x-3\ln|x|} = \frac{e^{2x}}{\cancel{e^{3\ln|x|}}} = \frac{e^{2x}}{|x|^3}$$

$$\int\frac{2x-3}{x}dx = \int\frac{2\cancel{x}}{\cancel{x}}dx - \int\frac{3}{x}dx = 2x - 3\ln|x|$$

$$\Rightarrow \frac{e^{2x}}{x^3} \cdot \frac{dy}{dx} + \frac{e^{2x}}{x^3} \cdot \left(\frac{2x-3}{x}\right) \cdot y = \frac{e^{2x}}{\cancel{x^3}} \cdot 4\cancel{x^3}$$

$$\Rightarrow \int D_x\left[\frac{e^{2x}}{x^3} \cdot y\right]dx = \int 4e^{2x}dx \Rightarrow \frac{e^{2x}}{x^3} \cdot y = 2e^{2x} + c \;\Big|\cdot \frac{x^3}{e^{2x}}$$

$$\Rightarrow \cancel{\frac{e^{2x}}{x^3}} \cdot y \cdot \cancel{\frac{x^3}{e^{2x}}} = 2\cancel{e^{2x}} \cdot \frac{x^3}{\cancel{e^{2x}}} + c \cdot \frac{x^3}{e^{2x}}$$

$$\Rightarrow y = 2x^3 + \frac{cx^3}{e^{2x}}$$

$$\int 4e^{2x}dx \qquad u = 2x$$

$$du = 2dx \;\Big|\; \frac{1}{2}$$

$$\frac{du}{2} = dx$$

$$= \int \cancel{4}^{2} \cdot e^u \cdot \frac{du}{\cancel{2}_1}$$

$$= 2\int e^u du = 2e^u + c = 2e^{2x} + c$$

(13) $x\frac{dy}{dx} + y = 3xy$

$$x\frac{dy}{dx} + y - 3xy = 0 \Rightarrow x\frac{dy}{dx} + y(1-3x) = 0 \;\Big|\cdot \frac{1}{x} \Rightarrow$$

$$\Rightarrow \frac{1}{\cancel{x}} \cdot \cancel{x} \cdot \frac{dy}{dx} + \underbrace{\boxed{\frac{1}{x} \cdot (1-3x)}}_{P(x)} \cdot y = 0$$

$$\psi(x) = e^{\int \frac{1-3x}{x}dx}$$

$$= e^{\ln|x| - 3x}$$

$$= \frac{e^{\ln|x|}}{e^{3x}} = \frac{|x|}{e^{3x}}$$

$$\int \frac{1-3x}{x}dx = \int \frac{1}{x}dx - 3\int \frac{\cancel{x}}{\cancel{x}}dx = \ln|x| - 3x$$

$$\Rightarrow \frac{x}{e^{3x}} \cdot \frac{dy}{dx} + \frac{x}{e^{3x}} \cdot \frac{1}{x} \cdot (1-3x) \cdot y = 0$$

$$\Rightarrow \int D_x\left[\frac{x}{e^{3x}} \cdot y\right]dx = \int 0\,dx \Rightarrow \frac{x}{e^{3x}} \cdot y = C \Rightarrow y = \frac{Ce^{3x}}{x}$$

$$\int 0\,dx = C$$