Spread Spectrum Communication

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1 Introduction

1.1 General Introduction

This course is a common course by IENAC students in 3rd year of SAT track (Space and Aeronautical Telecommunications), and students in the 2nd year of the Master ASNAT (Aerospace Systems - Navigation and Telecommunications).
This course is composed of 12 hours of lectures given in 6 two-hour periods (Christophe MACABIAU), a collective work session (BE) with an evaluated report.
The exam will take place on end January 2020. The final mark for this course will be final_exam_mark * 1.5 + BE_mark * 1.

Communication using spread spectrum techniques consists in transmitting a useful signal over a bandwidth that is significantly greater than the bandwidth strictly necessary to transmit this information. Such techniques are used, for instance, for increasing resistance to interference, to allow confidential communication, to improve resistance to multipath and fading, to allow multiple access.

The objective of this course is to present the temporal and frequency structure of the spread spectrum signals (in particular the Pseudo Random Noise codes), the techniques for processing these signals (correlator, carrier tracking loop, code tracking loop), and the performance of these processing steps in the presence of noise, multipath and interference. Spread spectrum transmission allows a new sharing mode of the transmission channel called CDMA (Code Division Multiple Access). The course structure is as follows:

1. Introduction
2. Structure and properties of pseudo-random codes
3. Temporal and spectral structure of the emitted CDMA signal
4. Received DS-SS signal model
5. Correlation operation
6. Acquisition
7. Received signal carrier tracking within noise
8. Received signal code tracking within noise
9. Impact of multipath

Impact of interference will be addressed through a supervised exercise on this topic.

There are 2 main ways to increase the useful signal bandwidth:

- Direct Sequence (DS), which consists in multiplying the useful signal with a high frequency binary pseudo-random signal based on a pseudo random binary sequence. This results in an overall signal with a wider spectrum than the useful signal.
- Frequency Hopping (FH), which consists in changing the carrier frequency of the useful signal over many frequency channels (thus widening the spectral occupation of the signal) according to a pseudo-random sequence

In both cases, the pseudo-random sequence is key and has to be known by the emitter and the receiver.
The present course will focus on Direct Sequence Spread Spectrum (DS-SS) as it is widely used in telecommunication and navigation (GNSS).

1.2 Course Dependencies

This course provides the description of a signal design and processing for modern telecommunication and navigation.
Other courses building on this course in the IENAC SAT track or Master ASNAT are Space Technologies, GNSS, and potentially the project, and the internship.

2 Structure and properties of pseudo-random codes

The pseudo random codes that we will study in this course are binary sequences having specific individual properties whose typical examples are periodic binary sequences generated by linear feedback shift registers (LFSR), that are a linear combination of the past values of the same sequence.

2.1 Definition of Pseudo-Random Binary Sequence

A pseudo random noise (PRN) code is a known (deterministic) binary sequence that possesses statistical properties similar to those of a random white sequence: autocorrelation is close to a Dirac function, and Power Spectral Density is almost constant over all frequencies.
In the case of a DS-SS, these sequences are used to spread the useful signal spectral occupation.

In DS-SS, these sequences can also be used to distinguish the useful signal of several emitters of a same system sharing the same frequency band. This type of multiple access is referred to Code Division Multiple Access (CDMA), and is based on the assignment to each emitter of a specific PRN code that is as uncorrelated as possible with the other PRNs used by the other emitters. In this case, a PRN code can then be seen as an identifier modulating the useful signal.

DS-SS systems use PRN codes that are finite and periodic. The statistical properties of the PRN sequences are key to the performance of a DS-SS system.
The most used pseudo-random sequences are

- m-sequences
- Gold codes
- Kasami codes
- Barker codes.

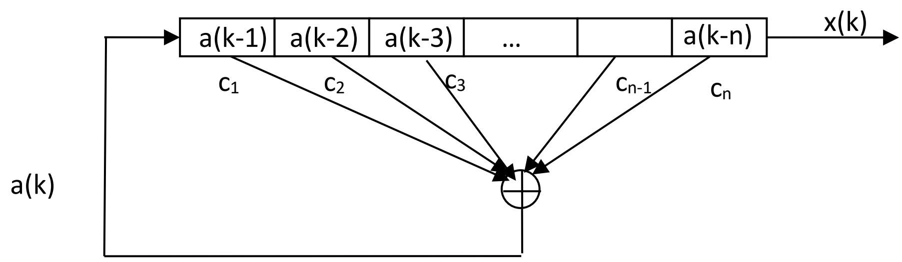
There are different ways to generate PRN sequences:

- Memory codes (codes are stored in emitter/receiver memory). These sequences can be optimally designed for the required length and for the desired number of family elements to provide the appropriate autocorrelation, cross-correlation properties. However, these memory codes require memory at the transmitter and receiver side.
- Codes generated by a Linear Feedback Shift Register (LFSR). These codes restrict the design choice, but can still provide high correlation performance, and it is very efficient to generate the codes «on the fly».

The next section will focus on codes generated by LFSR.

2.2 Linear Feedback Shift Registers Definition

Let us consider the Linear Feedback Shift Register presented in the figure below :



where:

- k is the current time index at which we consider the LFSR output signal $x(k)$ ($k \geq 0$)
- c_i is the coefficient associated to the feedback from the cell i ($c_i = 1$ or $c_i = 0$)
- n is the number of register bins (size of the register)
- $a(k-i)$ is the value stored in cell i of the register at epoch k.
- $x(k)$ is the value of the register output at epoch k: $x(k) = a(k-n)$

As we can see the value of cell i at epoch $k+1$ is equal to the value of cell $i-1$ at epoch k, and we can write $a((k+1)-i) = a(k-(i-1))$

Note that the following constraint c_n=1 must be true, because if this is not true, the register would be considered as having size $n-1$.
We can see that we have :

$$a(k) = \oplus_{i=1}^{n} c_i a(k-i), \forall k > 0$$

However, we can write the following relationship between the value of cell n and the output of the register :

$$\forall k, x(k) = a(k-n)$$

Therefore we can write

$$x(k+n) = a(k) = \oplus_{i=1}^{n} c_i a(k-i), \forall k > 0$$

If we offset these expressions by n, we can see that:

$$x(k) = \oplus_{i=1}^{n} c_i a(k-n-i), \forall k > 0$$

And then also :

$$x(k) = \oplus_{i=1}^{n} c_i x(k-i), \forall k > 0$$

This kind of register, as well as the binary sequence produced are said to be linear as its output at epoch k is a linear combination of its n previous outputs.

Example : determine the series of all the states for different initialization states such as 001 or 101.

The binary word formed by the n bins of the register is denoted $X(k)$:

$$X(k) = [\mathrm{a(k-1)} \quad \mathrm{a(k-2)} \quad \ldots \quad \mathrm{a(k-n+1)} \quad \mathrm{a(k-n)}]$$

It can be shown that the succession of values of $X(k)$ will always be periodic. Indeed, the number of distinct states is limited, so that the state will always return to a state already reached and from that point the same sequence will unfold. Therefore, the generated output sequence, constituted of the last bit of the register state, is thus also periodic.

The maximum number of different words $X(k)$ is the maximum number of words encodable in the register. The state 0…0 is however not possible as a word in a sequence without initializing to that null state because it is absorbing. Indeed, if the LFSR was falling into the state 0…0 without being initialized to 0…0, that would mean that the previous state would be 0…0X. If X=1, as $c_n = 1$, that would mean that the next state built by the feedback can not be 0…0. If X=0, that would mean that the previous state, and all previous states are 0…0. Therefore, the only situation leading to 0…0 is initializing with 0…0.
Note also that in order to avoid that 1…1 is also an absorbing state and that the sequence has a maximum length of $2^n - 1$, the number of connections for feedback must be even.

The maximum number of bits in the period of the output sequence is thus $L = 2^n - 1$. The output $X(k)$ is made of the last bit of this word.
In this general case, the sequence is thus periodic, the exact sequence depending on the following parameters:

- the LFSR size
- the weights associated to each register bin c=[c1 c2 … cn] where ci=0 or ci=1
- the initial state X(0) of the register

If the length of the period of the output sequence is $L = 2^n - 1$, we then say that the sequence is a maximum length sequence, also called m-sequence.

2.3 M-sequence from an LFSR

Note that in order to obtain an m-sequence, the number of connections needs to be even. Indeed, if the number of connections is odd, then there exists a second absorbing state which is the state 1…1, and in this case the maximum number of possible states is $L = 2^n - 2$.

If X(k) encodes the state of the register at epoch k, then we have:

$$X(k+1) = AX(k)$$

where A is the binary transition matrix with dimension nxn :

$$\mathrm{A} = \begin{bmatrix} c_1 & c_2 & c_3 & \dots & c_n \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}$$

We can see that A^{-1} is the following matrix

$$\mathrm{A}^{-1} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & & & \vdots \\ 1 & c_1 & \dots & 1 & c_{n-1} \end{bmatrix}$$

as $A \times A^{-1} = I$.

We can also see that

$$X(k+m) = A^m X(k)$$

and

$$A^{2n-1} = I$$

The eigen values λ are then defined from the characteristic equation:

$$\det(\mathrm{A} - \lambda \mathrm{I}) = 0$$

But we can see that the characteristic polynomial $\mathrm{P}(\lambda) = \det(\mathrm{A} - \lambda\mathrm{I})$ can be expressed as :

$$\det(\mathrm{A} - \lambda\mathrm{I}) = \lambda^{\mathrm{n}} + c_1\lambda^{\mathrm{n}-1} + c_2\lambda^{\mathrm{n}-2} + \cdots + c_{\mathrm{n}-1}\lambda + 1$$

and if we note $c_0 = 1$, then we can write

$$\det(\mathrm{A} - \lambda\mathrm{I}) = \oplus_{i=0}^{n} c_{\mathrm{i}}\lambda^{\mathrm{n}-\mathrm{i}}$$

Let us remind that the matrix A must comply with its characteristic equation, therefore

$$\mathrm{A}^{\mathrm{n}} + c_1\mathrm{A}^{\mathrm{n}-1} + c_2\mathrm{A}^{\mathrm{n}-2} + \cdots + c_{\mathrm{n}-1}\mathrm{A} + \mathrm{I} = 0$$

We then define the generating function of the sequence a(k), which is the function:

$$G(r) = \oplus_{k=0}^{\infty} a(k) r^k$$

As x(k)=a(k-n), we see that this function describes all the values of the output of the register from x(n) up to the infinite.

As $a(k) = \oplus_{i=1}^{\mathrm{n}} c_{\mathrm{i}} a(k-i)$, then we can write :

$$G(r) = \oplus_{k=0}^{\infty} \oplus_{i=1}^{\mathrm{n}} c_{\mathrm{i}} a(k-i) r^k$$

If we swap the 2 summations, we get:

$$G(r) = \oplus_{i=1}^{\mathrm{n}} c_{\mathrm{i}} r^i \oplus_{k=0}^{\infty} a(k-i) r^{k-i}$$

And if we explicit the last summation, we get:

$$G(r) = \oplus_{i=1}^{\mathrm{n}} c_{\mathrm{i}} r^i \left(a(-i) r^{-i} + a(-i+1) r^{-i+1} + \cdots + a(-1) r^{-1} + \oplus_{k=\mathrm{i}}^{\infty} a(k-i) r^{k-i}\right)$$

and thus :

$$G(r) = \oplus_{i=1}^{\mathrm{n}} c_{\mathrm{i}} r^i \left(a(-i) r^{-i} + a(-i+1) r^{-i+1} + \cdots + a(-1) r^{-1} + \oplus_{k=0}^{\infty} a(k) r^{k}\right)$$

where [a(-1) … a(-n)]=X(0) is the initial state of the register.

The last term of this summation is precisely $G(r)$, thus

$$G(r) = \oplus_{i=1}^{\mathrm{n}} c_{\mathrm{i}} r^i \left(a(-i) r^{-i} + a(-i+1) r^{-i+1} + \cdots + a(-1) r^{-1} + G(r)\right)$$

We can then solve for this equation to determine $G(r)$:

$$G(r) + \left(\oplus_{i=1}^{\mathrm{n}} c_{\mathrm{i}} r^i\right) G(r) = \oplus_{i=1}^{\mathrm{n}} c_{\mathrm{i}} r^i \left(a(-i) r^{-i} + a(-i+1) r^{-i+1} + \cdots + a(-1) r^{-1}\right)$$

And thus

$$G(r) = \frac{\oplus_{i=1}^{\mathrm{n}} c_{\mathrm{i}} r^i \left(a(-i) r^{-i} + a(-i+1) r^{-i+1} + \cdots + a(-1) r^{-1}\right)}{\oplus_{i=0}^{\mathrm{n}} c_{\mathrm{i}} r^i}$$

$G(r)$, the function that describes all the values of the output sequence x(k) since k=n appears thus as the division of 2 polynomials.

The necessary and sufficient condition for this sequence to be an m-sequence is that the polynomial $Q(r) = \oplus_{i=0}^{n} c_i r^i$ is a primitive polynomial (minimal polynomial which has at least one root which is primitive). Any polynomial which is non primitive with the same order would generated a shorter sequence. The primitive polynomial thus generates the 2^{n}-1 non null elements of the Galois Field GF(2n).
Furthermore, the 2^{n}-1 words generated are, in their vector form, the 2^{n}-1 successive powers of a, root of the primitive polynomial $Q(r)$ (Q(a)=0).

We can consider the non zero elements of CG(2n), in their vector representation, as the set of the integer numbers of the interval $[1,2^n-1]$. This set includes thus 2^n-1 odd numbers and 2^{n-1}-1 even numbers. An m-sequence includes thus 2^{n-1}-1 zeros and 2^{n-1} ones.

Let us note as well that the reciprocal polynomial $Q^R(r) = r^n Q(\frac{1}{r})$ of a primitive polynomial is also a primitive polynomial. In addition, $Q^R(r) = r^n Q(\frac{1}{r}) = r^n \oplus_{i=0}^{n} c_i \frac{1}{r^i} = \oplus_{i=0}^{n} c_i r^{n-i} = P(r)$ where $P(r)$ is the characteristic polynomial of A.

It can be shown that the number of distinct m-sequences that can be generated by a register of size n is :

$$N_m = \frac{\Phi(2^n - 1)}{n}$$

où

$$\Phi(n) = \begin{cases} 1 & \text{if } n = 1 \\ \prod_{i=1}^{k} p_i^{\alpha_i - 1}(p_i - 1) & \text{if } n > 1 \\ p - 1 & \text{if n is a prime number} \end{cases}$$

where p_i and α_ii are respectively the prime numbers and their power composing the integer n:

$$n = \prod_{i=1}^{k} p_i^{\alpha_i}$$

Note that N_m is always an even number, denombering the primitive polynomials and their reciprocal.

ex : For n=3, we have $\Phi(7) = 7 - 1 = 6$ thus $N_m = 2$ and for n=4 we have $\Phi(15) 2 * 4 = 8$, thus $N_m = 2$.
For n=10, we have $N_m = 60$.

The following table presents the list of primitive polynomials until n=11, not including the reciprocal polynomials:

L	$N_c=2^L$-1	Feedback Taps for m-sequences	# m-sequences
2	3	[2,1]	2
3	7	[3,1]	2
4	15	[4,1]	2
5	31	[5,3] [5,4,3,2] [5,4,2,1]	6
6	63	[6,1] [6,5,2,1] [6,5,3,2]	6
7	127	[7,1] [7,3] [7,3,2,1] [7,4,3,2] [7,6,4,2] [7,6,3,1] [7,6,5,2] [7,6,5,4,2,1] [7,5,4,3,2,1]	18
8	255	[8,4,3,2] [8,6,5,3] [8,6,5,2] [8,5,3,1] [8,6,5,1] [8,7,6,1] [8,7,6,5,2,1] [8,6,4,3,2,1]	16
9	511	[9,4] [9,6,4,3] [9,8,5,4] [9,8,4,1] [9,5,3,2] [9,8,6,5] [9,8,7,2] [9,6,5,4,2,1] [9,7,6,4,3,1] [9,8,7,6,5,3]	48
10	1023	[10,3] [10,8,3,2] [10,4,3,1] [10,8,5,1] [10,8,5,4] [10,9,4,1] [10,8,4,3] [10,5,3,2] [10,5,2,1] [10,9,4,2] [10,6,5,3,2,1] [10,9,8,6,3,2] [10,9,7,6,4,1] [10,7,6,4,2,1] [10,9,8,7,6,5,4,3] [10,8,7,6,5,4,3,1]	60
11	2047	[11,2] [11,8,5,2] [11,7,3,2] [11,5,3,2] [11,10,3,2] [11,6,5,1] [11,5,3,1] [11,9,4,1,] [11,8,6,2,] [11,9,8,3] [11,10,9,8,3,1]	176

2.4 Discrete signal corresponding to an m-sequence and properties

We can associate any discrete binary sequence $x(k)$ composed of the elements {1,0} and governed by the modulo-2 addition operator to an equivalent discrete bipolar signal composed of the elements {-1,1} and governed by the traditional multiplication. Indeed, there is an isomorphism between the 2 fields ({1,0},⊕) and ({-1,1},x).
Although it is important to differentiate between the sequence (composed of elements {0,1}) and its associated binary antipodal digital signal (composed of elements {-1,1}), the name 'sequence' will also be used to refer to the binary digital signal, for simplicity.
In order to determine the randomness properties of an m-sequence (to be considered as a PRN sequence), its autocorrelation and Power Spectral Density (PSD) must now be calculated.
We can then define the autocorrelation of the digital signal corresponding to the periodic binary sequence as:

$$R(m) = \frac{1}{N}\sum_{i=0}^{N-1} c(i)c(i-m)$$

where c(k) takes the value –1 or 1. Since the digital signal is periodic, then the autocorrelation function is also periodic with the same period as the digital signal.

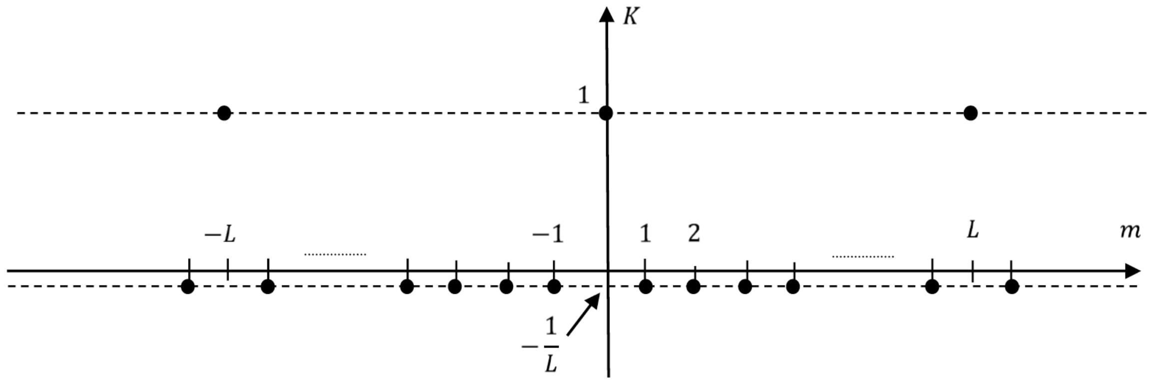
For an m-sequence, $N = L = 2^n - 1$ we then have the two following situations :

- m=0: $R(0) = \frac{1}{L}\sum_{i=0}^{L-1} c^2(i) = 1$

- $1 \le m \le L-1$: as there is isomorphism between the 2 fields ({1,0},⊕) and ({-1,1},x), then we can write $c(i)c(i-m) \leftrightarrow x(i) \oplus x(i-m)$ for any $1 \le i \le L-1$. Thus the binary sequence corresponding to the product $c(i)c(i-m)$ for any i is a binary sequence resulting from the sum of an m-sequence with this shifted m-sequence shifted by m. However, $x(i) = \bigoplus_{\mathrm{j}=1}^{\mathrm{n}} c_j x(i-j)$ and $x(i-m) = \bigoplus_{\mathrm{j}=1}^{\mathrm{n}} c_j x(i-m-j)$ thus $x(i) \oplus x(i-m) = \bigoplus_{\mathrm{j}=1}^{\mathrm{n}} c_j \big(x(i-j) \oplus x(i-m-j)\big)$. Thus the sum of this m-sequence and this shifted m-sequence shifted by m is also itself a shifted version of the m-sequence: $\exists p$ such that $x(i) \oplus x(i-m) = x(i-m+p)$. Thus

$$R(m) = \frac{1}{L}\sum_{i=0}^{L-1} c(i)c(i-m) = \frac{1}{L}\sum_{i=0}^{L-1} c(i-m+p) = -\frac{1}{L}$$

The autocorrelation function of the discrete signal associated to an m-sequence thus has the following shape:



If the length of the m-sequence increases, the autocorrelation of the m-sequence will look like a Dirac (impulse) function. Such type of autocorrelation function is characteristic of white noise. Interpretation: A long m-sequence does not look like any shifted version of itself (except for shifts equal to a multiple of the period).

The PSD of a digital signal can be obtained by applying a Discrete Fourier Transform (DFT) to its autocorrelation function. In the case of a periodic signal based on an m-sequence of length L, this means:

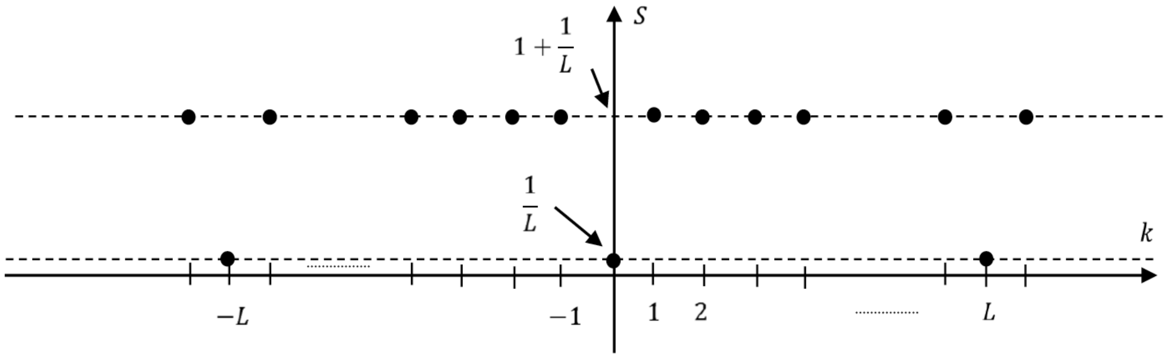
$$S(k) = \sum_{m=0}^{L-1} R(m) e^{-i2\pi\frac{km}{L}}$$

Due to the inherent behavior of the DFT, the PSD of a periodic signal of period L is periodic with a period L.
As the autocorrelation function $R(m)$ only has 2 values, we can write

$$S(k) = 1 - \frac{1}{L}\sum_{m=1}^{L-1} e^{-i2\pi\frac{km}{L}} = 1 + \frac{1}{L} - \frac{1}{L}\sum_{m=0}^{L-1} e^{-i2\pi\frac{km}{L}} = 1 + \frac{1}{L} - \delta(k) \quad for\ 0 \leq k \leq L-1$$

Indeed, $\sum_{m=0}^{L-1} e^{-i2\pi\frac{km}{L}} = \frac{1-e^{-i2\pi\frac{kL}{L}}}{1-e^{-i2\pi\frac{k}{L}}} = \frac{1-e^{-i2\pi k}}{1-e^{-i2\pi\frac{k}{L}}} = 0$ except if k=0.

The power spectrum density function takes then the value $\frac{1}{L}$ in 0 and $1 + \frac{1}{L}$ elsewhere.



The PSD of the digital signal associated with an m-sequence is, as expected, very similar to the PSD of a white noise (constant over the frequencies).

An m-sequence can thus be considered as a PRN sequence since the digital signal associated to this m-sequence has properties similar to those of white noise. This is especially true when the m-sequence used is long. In the continuation, the discrete periodic signal associated to a PRN sequence will be referred to as discrete PRN signal.

When DS-SS is used in CDMA, the PRN sequences used by each emitter also have to have excellent cross-correlation properties. A family of PRN sequences then has to be selected. M-sequences are then insufficient, but can still be used to find such PRN sequences that belong to a family. Gold codes (used by GPS) are, for instance, a type of code families generated based on the summation of 'preferred' m-sequences.

3 Temporal and Spectral Structure of a Transmitted Signal using DS-SS

Now that the discrete PRN signal has been presented, it is necessary to understand the process to build the continuous DS-SS signal that will be transmitted. This is based on 2 main operations:

- The physical shaping of the PRN sequence
- The modulation of the useful information by the PRN signal

The next sections provide

- the method used to obtain the physical shaping of the discrete PRN signal and the new properties of the resulting continuous signal, which will be referred to as the PRN signal.
- the way the data is modulated by the PRN signal and transmitted to the user

The bits of a PRN sequence will be referred to as chips (a piece of the PRN sequence). This allows to distinguish PRN chips from useful data bits.
When generating the PRN sequence, the duration between two PRN bits will be referred to as PRN bit duration or chip duration. It will be noted as T_c. The inverse of T_c is referred to as chipping rate and is noted F_c.

3.1 Pulse shaping of a PRN sequence

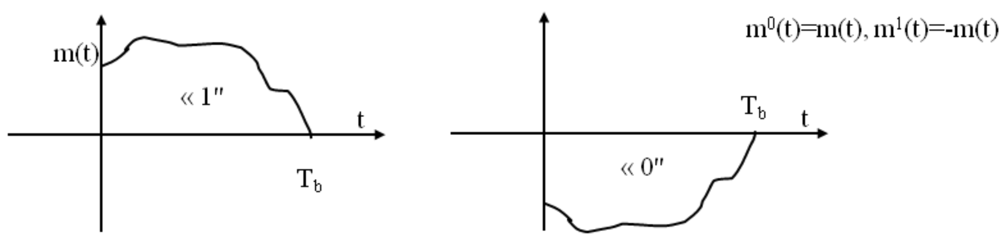
To use this signal in a communication system, it is important to transmit the PRN sequence via an analogue signal, or in other words, to represent these discrete binary values using a continuous shape, also referred to as a waveform.

The action of providing a physical support or a waveform to the PRN chips is referred to as pulse shaping.

To facilitate the upcoming mathematical developments, the discrete PRN signal without any pulse shaping can be expressed in the time-continuous domain as a series of weighted Dirac impulse trains by:

$$c(t) = \sum_{k=-\infty}^{+\infty} c_k \delta(t - kT_c)$$

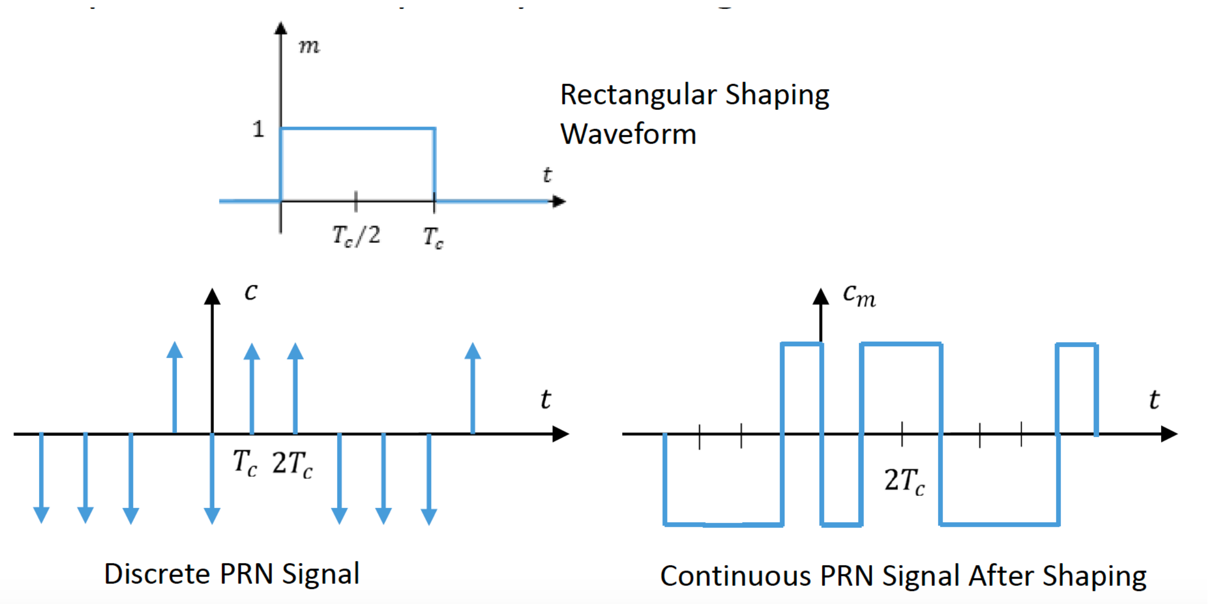
The physical pulse shaping of a binary sequence is achieved through the association of specific waveforms for the encoding of zero and one. Let us denote m0(t) and m1(t) the respective pulses encoding the zero and the one. We only consider here the case where the waveforms are antipodal : m0(t)=m(t), m1(t)=-m(t).



Examples of well-known waveforms are:

- the rectangular function (time-limited)
- the raised cosine shape (not time-limited)
- the square root raised cosine (not time-limited)

Example: Case of a discrete PRN signal based on an m-sequence and shaped by a rectangular waveform



The pulse shaping process of the discrete PRN signal c_k results in a continuous signal $c_m(t)$, referred to as the PRN signal, that can be modeled as

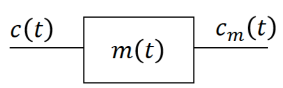
$$c_m(t) = \sum_{k=-\infty}^{+\infty} c_k m(t - kT_c) = \left[\sum_{k=-\infty}^{+\infty} c_k \delta(t - kT_c)\right] * m(t)$$

where
- T_c is PRN bit duration
- c_k represents the sequence of the discrete PRN signal bits

The shaping process thus consists in the convolution between the waveform $m(t)$ and the discrete PRN signal:

$$c_m(t) = (c * m)(t)$$

The process of shaping the PRN sequence bits is thus equivalent to filtering the discrete PRN, where the filter impulse response is the waveform $m(t)$.



The time and frequency properties of the PRN signal will thus also be influenced by the choice of the shaping waveform $m(t)$.

Due to the random-like nature of the PRN sequence, the literature usually models the PRN signal in two different ways:
- as a periodic and deterministic signal, which is the truth
- as a random binary signal (the PRN sequence is assumed infinite and random). This represents a common simplification generally acceptable as long as the PRN sequence is long

The properties of the PRN signal will be investigated considering these 2 models.

Ce signal peut être modélisé comme un signal déterministe périodique, ou bien comme un signal aléatoire car constitué d'une suite de valeurs c_k aléatoires.

We will now try to characterize the time and frequency structure of the transmitted signal $s(t)$. We will then search for the autocorrelation function of this sequence and the Fourier Transform of this autocorrelation function, which is the power spectrum density.

3.2 PRN Signal Properties

3.2.1 Case where $c_m(t)$ is assumed deterministic and periodic

In the case where c_k is deterministic and periodic with period L bits, we have thus

$$c(t) = \left[\sum_{k=0}^{L-1} c_k \delta(t - kT_c)\right] * \sum_{j=-\infty}^{+\infty} \delta(t - jT_R)$$

where $T_R = LT_C$ is the PRN code repetition period and $F_R = \frac{1}{T_R}$ is the PRN code repetition rate.
And so

$$c_m(t) = \left[\left[\sum_{k=0}^{L-1} c_k \delta(t - kT_c)\right] * \sum_{j=-\infty}^{+\infty} \delta(t - jT_R)\right] * m(t)$$

This is also equivalent to

$$c_m(t) = \left[\left[\sum_{k=0}^{L-1} c_k \delta(t - kT_c)\right] * m(t)\right] * \sum_{j=-\infty}^{+\infty} \delta(t - jT_R)$$

We remind that the autocorrelation function of a deterministic real signal $s(t)$ is

$$K_s(\tau) = \int_{-\infty}^{+\infty} s(t)s(t-\tau)dt$$

We also remind that if a signal $s(t)$ is real and periodic with period T, then its autocorrelation is

$$K_s(\tau) = \frac{1}{\mathrm{T}} \int_0^{\mathrm{T}} s(t)s(t-\tau)dt$$

In our case, as $c_m(t) = (c * m)(t)$, then we can write

$$K_{c_m}(\tau) = K_c(\tau) * K_m(\tau)$$

where $K_m(\tau)$ is the autocorrelation function of the deterministic signal $m(t)$:

$$K_m(\tau) = \int_{-\infty}^{+\infty} m(t)m(t-\tau)dt = m(\tau) * m^*(-\tau)$$

and $K_c(\tau)$ is the autocorrelation function of the periodic signal $c(t) = \left[\sum_{k=0}^{L-1} c_k \delta(t - kT_c)\right] * \sum_{j=-\infty}^{+\infty} \delta(t - jT_R)$.
As expected, the autocorrelation of the PRN signal depends upon the autocorrelation of the continuous time discrete PRN signal and the autocorrelation of the selected shaping waveform.

Because $c(t)$ represents the discrete PRN signal in the continuous domain, its autocorrelation function $K_c(\tau)$ is thus related to that of the discrete PRN sequence $R(m) = \frac{1}{N}\sum_{i=0}^{N-1} c(i)c(i-m)$ expressed in section 1.
We can thus obtain

$$K_c(\tau) = \frac{1}{T_R} \int_0^{T_R} c(t)c(t-\tau)dt$$

$$K_c(\tau) = \frac{1}{T_R}\int_0^{T_R} \left(\left[\sum_{k=0}^{L-1} c_k \delta(t - kT_c) \right] * \sum_{j=-\infty}^{+\infty} \delta(t - jT_R) \right) \left(\left[\sum_{k=0}^{L-1} c_k \delta(t - \tau - kT_c) \right] * \sum_{j=-\infty}^{+\infty} \delta(t - \tau - jT_R) \right) dt$$

We can drop the comb in the first term because the integration is over the duration of the pattern $0 \dots T_R$, so no need to consider other repetitions of the pattern. In addition, the second term, which is not properly expressed in the above equation because of the use of usual expression of the convolution between 2 functions, can be rewritten more properly in order to reflect the proper time shifht that will apply to the whole repeated sequence of weighted Dirac impulses.

$$K_c(\tau) = \frac{1}{T_c}\int_0^{T_R} \frac{1}{\mathrm{L}} \left(\left[\sum_{k=0}^{L-1} c_k \delta(t - kT_c) \right] \right) \left(\left[\sum_{k=0}^{L-1} c_k \delta(t - kT_c) \right] * \sum_{j=-\infty}^{+\infty} \delta(t - jT_R) \right) (t - \tau - jT_R) dt$$

The second term in this integral will also describe a full period of the +1/-1 weighted Dirac impulses however staring with a specific shift in time. We will therefore also drop the comb in the second term once we have considered the cyclic unfolding of the pattern with a specific shift in time. Also, as the autocorrelation is periodic with period T_R, we can identify a pattern that is repeated, so that we can write:

$$K_c(\tau) = \frac{1}{T_C} \left[\sum_{m=0}^{N-1} \mathrm{R}(m) \delta(\tau - mT_C) \right] * \sum_{j=-\infty}^{+\infty} \delta(\tau - jT_R)$$

Finally, as $K_{c_m}(\tau) = K_c(\tau) * K_m(\tau)$, the closed-form expression of the PRN signal autocorrelation is:

$$K_{c_m}(\tau) = \frac{1}{T_C} \left[\sum_{m=0}^{N-1} \mathrm{R}(m) \delta(\tau - mT_C) \right] * \sum_{j=-\infty}^{+\infty} \delta(\tau - jT_R) * K_m(\tau)$$

Or also

$$K_{c_m}(\tau) = \frac{1}{T_C} \left[\sum_{m=0}^{N-1} \mathrm{R}(m) \delta(\tau - mT_C) \right] * K_m(\tau) * \sum_{j=-\infty}^{+\infty} \delta(\tau - jT_R)$$

or

$$K_{c_m}(\tau) = \frac{1}{T_C} \left[\sum_{m=0}^{N-1} \mathrm{R}(m) K_m(\tau - mT_C) \right] * \sum_{j=-\infty}^{+\infty} \delta(\tau - jT_R)$$

The power spectrum density of $c_m(t)$, denoted $S_{c_m}(f)$, is such that:

$$K_{c_m}(\tau) \leftrightarrows S_{c_m}(f)$$

We can see therefore that

$$S_{c_m}(f) = \frac{1}{T_C} \left[\sum_{m=0}^{N-1} \mathrm{R}(m) e^{-i2\pi f m T_C} \right] \times S_m(f) \times \frac{1}{T_R} \sum_{j=-\infty}^{+\infty} \delta\left(f - j\frac{1}{T_R}\right)$$

where

$K_m(\tau) \leftrightarrows S_m(f)$ with $S_m(f) = |M(f)|^2$, with $m(t) \leftrightarrows M(f)$.

We can thus write:

$$S_{c_m}(f) = \frac{1}{T_R} |M(f)|^2 \sum_{j=-\infty}^{+\infty} \frac{1}{T_C} \left[\sum_{m=0}^{N-1} \mathrm{R}(m) e^{-i2\pi j F_R m T_C} \right] \times \delta(f - jF_R)$$

As $T_R = NT_C$, we can see that

$$\sum_{m=0}^{N-1} \mathrm{R}(m) e^{-i2\pi j F_R m T_C} = \sum_{m=0}^{N-1} \mathrm{R}(m) e^{-i2\pi \frac{jm}{N}}$$

is the Discrete Fourier Transform of the periodic discrete sequence with period L $\mathrm{R}(m)$.

We note

$$S_C(j) = \sum_{m=0}^{N-1} \mathrm{R}(m) e^{-i2\pi \frac{jm}{N}}$$

Note that $S_C(j)$ is itself periodic with period L.

We can thus write

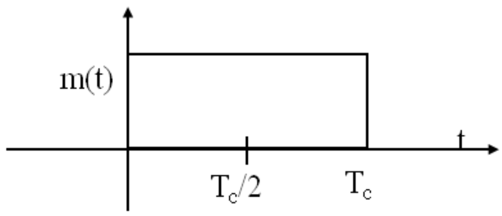
$$S_{c_m}(f) = \frac{1}{T_R} |M(f)|^2 \sum_{j=-\infty}^{+\infty} \frac{1}{T_C} S_C(j) \times \delta(f - jF_R)$$

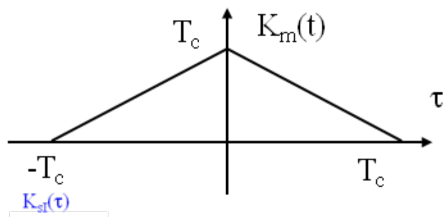
Therefore the spectrum is a line spectrum as the PRN code discrete and analogue versions are periodic.

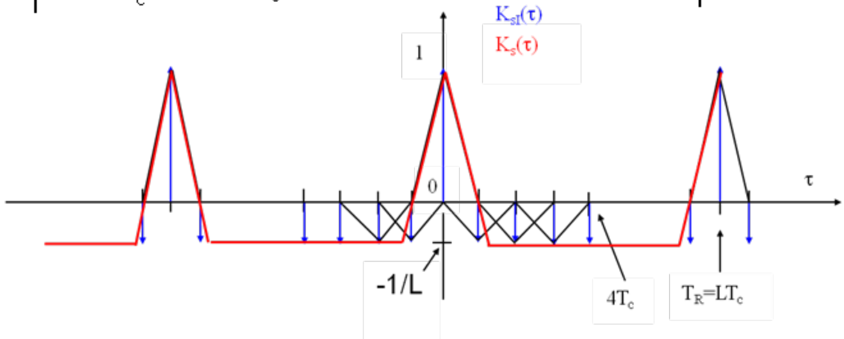
<u>Example</u>: case of an m-sequence with rectangular shaping.

This is the case where $m(t) = \mathrm{rect}\left(\frac{t - \frac{T_C}{2}}{T_C}\right)$, and where c_k is an m-sequence with period N.

In that case, $K_m(\tau) = T_C \mathrm{tri}\left(\frac{\tau}{T_C}\right)$, and $M(f) = T_C e^{-i\pi f T_C} \times \mathrm{sinc}(\pi f T_C)$, where $\mathrm{sinc}(\pi f T_C) = \frac{\sin(\pi f T_C)}{\pi f T_C}$ when $f \neq 0$ and $\mathrm{sinc}(\pi f T_C) = 1$ when $f = 0$.





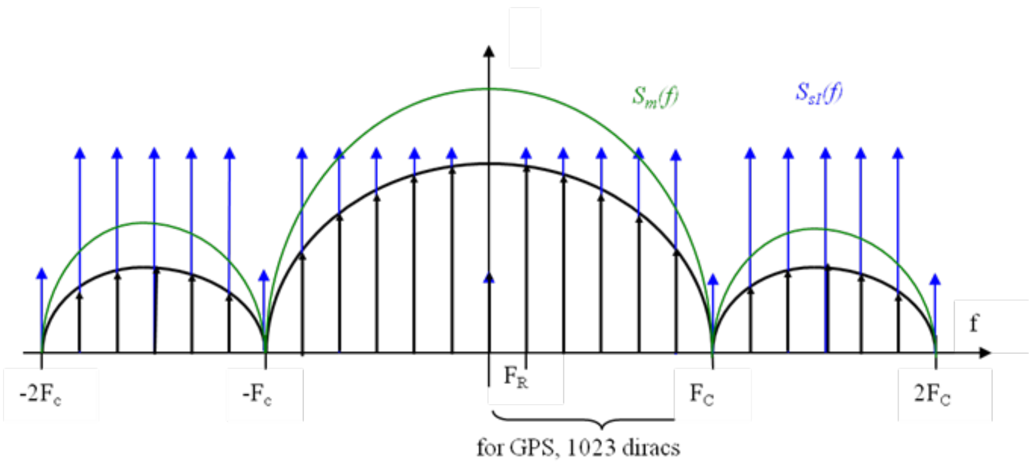


As c_k is an m-sequence with period N, we also have

$$S_C(j) = \begin{cases} \frac{1}{N} & si\ j = kN \\ \frac{N+1}{N} & si\ j \neq kN \end{cases}$$

We thus obtain the following PSD:

$$S_{c_m}(f) = \left(\frac{N+1}{N^2}\right) (\mathrm{sinc}(\pi f T_C))^2 \sum_{\substack{j=-\infty \\ j \neq 0}}^{+\infty} \delta(f - jF_R) + \frac{1}{N^2} \delta(f)$$



3.2.2 Case where $c_m(t)$ is assumed random

Let us consider the case where the PRN sequence c_k is now assumed random, uncorrelated, binary and without periodicity. This assumption is taken when the PRN sequence is long, typically several hundreds of bits.

In that case, the autocorrelation and the power spectrum density are given by the results related to the Pulse Amplitude Modulation model. The signal $c_m(t)$ is modeled as :

$$c_m(t) = \sum_{k=-\infty}^{+\infty} c_k m(t - kT_c)$$

where $E\left[c_i c_j\right] = \delta(i - j)$.

The autocorrelation is thus

$$K_{c_m}(\tau) = E[c_m(t) c_m^*(t - \tau)]$$

That type of signal model needs to be changed by inserting a uniformly distributed random phase θ over [0, Tc] in the expression of $m(t - kT_c)$ in order for this random signal to be stationary. The signal is then modeled as $c_m(t) = \sum_{k=-\infty}^{+\infty} c_k m(t - kT_c - \theta)$

As a reminder, if we have $s(t) = A \sum_{k=-\infty}^{k=+\infty} c_k h(t - kT_0 - \theta)$, then we have $E[s(t)] = \frac{A\, m_c}{T_0} H(0)$, and

$$E[s(t)s^*(t - \tau)] = \frac{A^2}{T_0} \sum_{m=-\infty}^{m=+\infty} R_c(m) R_h(\tau - mT_0)$$

In our case, we have then $K_{c_m}(\tau) = \frac{1}{T_c} K_m(\tau)$

where

$$K_m(\tau) = \int_{-\infty}^{+\infty} m(t) m(t - \tau) dt = m(\tau) * m^*(-\tau)$$

The resulting PRN signal autocorrelation function can be approximated by the autocorrelation of the shaping waveform (with a coefficient).

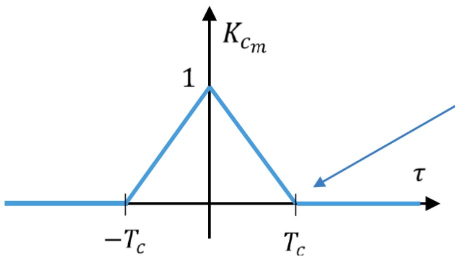
Thus the power spectrum density is expressed as:

$$S_{c_m}(f) = \frac{1}{T_c} |M(f)|^2$$

<u>Example</u> : in the case where $m(t) = \mathrm{rect}\left(\frac{t - \frac{T_C}{2}}{T_C}\right)$, where c_k is an independent random sequence.

We have, $K_m(\tau) = T_C \mathrm{tri}\left(\frac{\tau}{T_C}\right)$, where $M(f) = T_C e^{-i\pi f T_C} \operatorname{sinc}(\pi f T_C)$.

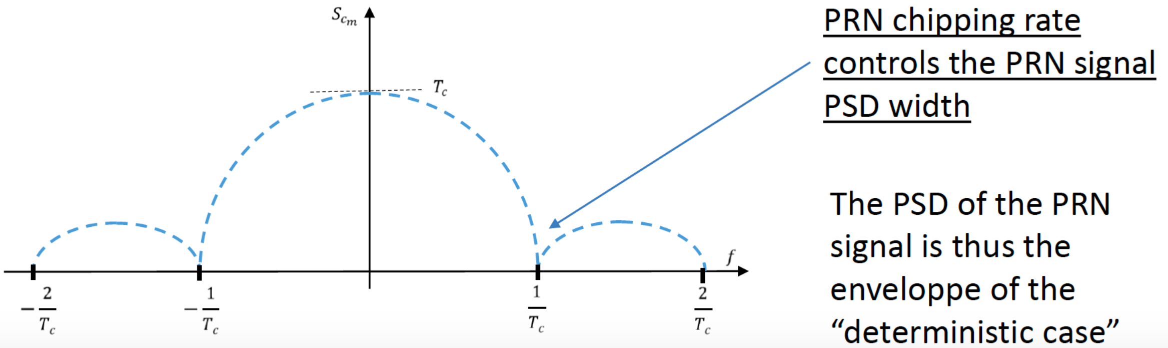
Therefore $K_{c_m}(\tau) = \mathrm{tri}\left(\frac{\tau}{T_C}\right)$.



The PRN signal is assumed not correlated with a version of itself shifted by more than one chip. The shape of this autocorrelation corresponds to the autocorrelation of a periodic PRN code when L grows to the infinite.

We then also have :

$$S_{c_m}(f) = T_C|\text{sinc}(\pi f T_C)|^2$$



The resulting PRN signal PSD can be approximated by the PSD of the shaping waveform (with a coefficient). Note that this PSD function is a continuous function (not discrete).

3.3 Transporting useful information using DS-SS

Now that the properties of the PRN sequence and its associated PRN signal have been presented, let us look at how it is used to transmit the useful information.

Let us consider the digital sequence d(k) to be transmitted through a propagation channel. The signal to be transmitted through the propagation channel is a digital sequence that is the product of d(k) by a PRN code c(k) with much faster rhythm, or much shorter bit duration than d(k).

We will consider here the case of transmission where the digital sequence is shaped by a waveform noted m(t).

The transmitted signal is thus modeled as :

$$s_T(t) = \Re\{Ad(t)c_m(t)e^{i2\pi f_L t}\}$$

$$s_T(t) = Ad(t)c_m(t)\cos(2\pi f_L t)$$

We assume that the useful data bits are shaped with a rectangular waveform:

$$d(t) = \left[\sum_{k=-\infty}^{+\infty} d_k \delta(t - kT_D)\right] * \text{rect}\left(\frac{t - \frac{T_D}{2}}{T_D}\right)$$

We also note

$$c_m(t) = \left[\sum_{k=-\infty}^{+\infty} c_k \delta(t - kT_c)\right] * m(t)$$

where

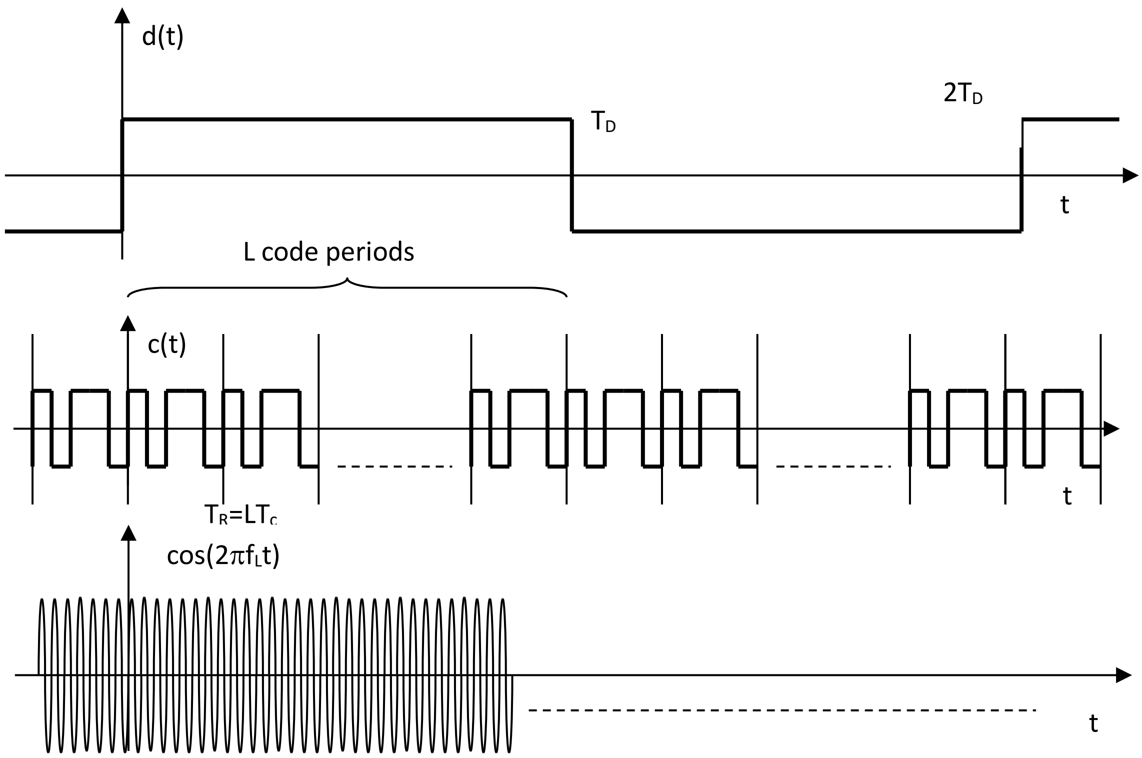
- d(k) is the digital sequence to be transmitted (d(k)=+/- 1)
- T_D is the duration of a data bit d(k)
- c(k) is the digital signal used for spreading (d(k)=+/-1)

• T_c is the chip duration of c(k)
In practice, we choose:

- $T_D = KT_R$. This means that there is an integer number of PRN sequences within one data symbol.
- $f_L = MF_C$ with M integer significantly greater than 1.
- $T_R = LT_C$

The 3 components (data, PRN sequence and carrier) are synchronous. The data bits start and end systematically at the same instant as a PRN sequence (transitions of data bits occur at transitions of code bits). There must also be an integer number of carrier periods within one PRN code bit.

Ex : $m(t) = \mathrm{rect}\left(\frac{t-\frac{T_C}{2}}{T_C}\right)$
The time domain shape of the signal is then :



If we assume that $c_m(t)$ is fully random, and independent from $d(t)$, then the model of the transmitted signal needs to be modified to

$$s_T(t) = Ad(t)c_m(t)\cos(2\pi f_{\mathrm{L}}t - \varphi)$$

In that case, we know that we have

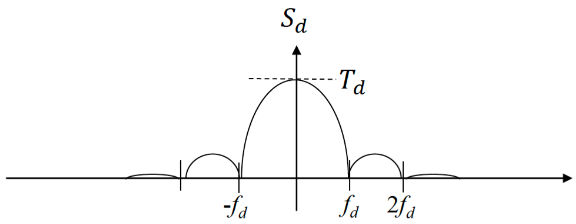
$$K_{s_T}(\tau) = \frac{A^2}{2}K_d(\tau)K_{c_m}(\tau)\cos(2\pi f_{\mathrm{L}}\tau)$$

If we note $K_{dc}(\tau) = K_d(\tau)K_{c_m}(\tau)$, then we have

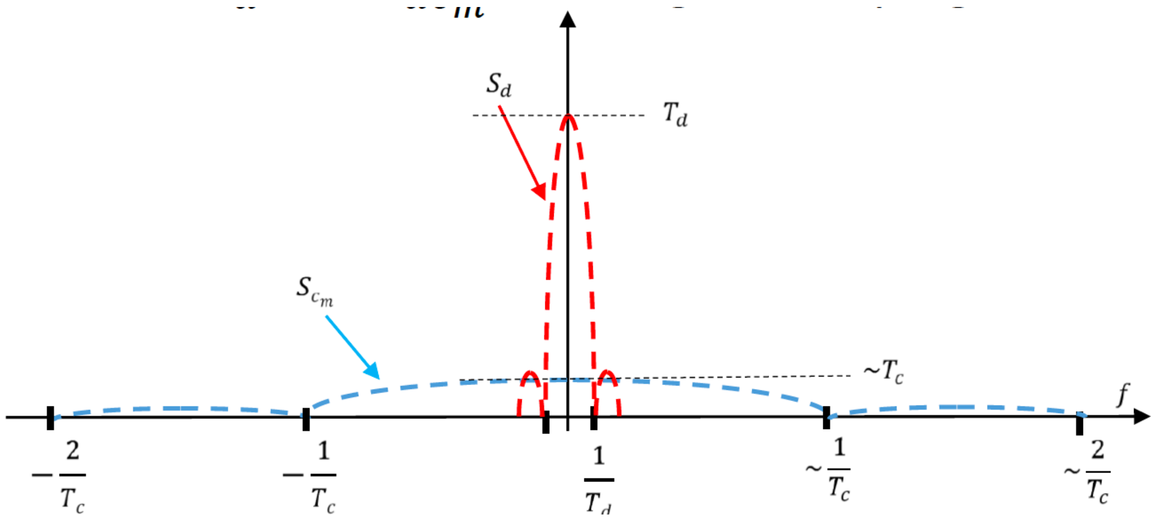
$$S(f) = \frac{A^2}{4}S_{DC}(f - f_{\mathrm{L}}) + \frac{A^2}{4}S_{DC}(-f - f_{\mathrm{L}})$$

where $S_{DC}(f) = S_D(f) * S_{c_m}(f)$.
In the case where the PRN code is considered as fully random, we have then $S_{c_m}(f) = \frac{1}{T_c}|M(f)|^2$, and for example $S_{c_m}(f) = T_C|\mathrm{sinc}(\pi f T_C)|^2$ in the case of a rectangular pulse shaping.

Let us recall that $S_D(f) = T_D(\mathrm{sinc}(\pi f T_D))^2$ as the useful data bits are shaped with a rectangular waveform. the PSD of the data signal is thus mainly contained within the main lobe of the sinc function: $-fd;fd$Hz



In that case we then have the following shape of the power spectrum density:



We can see that the bandwidth increase is approximately KL = $Td\ Tc$, which is typically a factor of several hundreds or thousands. We can also see that the maximum amplitude of $Sdcm$i s significantly lower than that of Sd, approximately by the same factor KL.

In the case where the PRN code is periodic, we have seen that

$$S_{C_m}(f) = \frac{1}{T_R}|M(f)|^2 \sum_{j=-\infty}^{+\infty} \frac{1}{T_C}\left[\sum_{m=0}^{N-1} R_C(m) e^{-i2\pi j F_R m T_C}\right] \times \delta(f - jF_R)$$

Therefore, as $S_{DC}(f) = S_D(f) * S_{C_m}(f)$, the data spectrum $S_D(f)$ will be reproduced every time there is a line in the spectrum $S_{C_m}(f)$. This will result in the existence of thousands of replicas of the data spectrum $S_D(f)$ over the complete spread bandwidth.

Example:

As an example, in the case of an m-sequence, for a rectangular pulse shaping, we have

$m(t) = \mathrm{rect}\left(\frac{t - \frac{T_C}{2}}{T_C}\right)$ and $|M(f)|^2 = T_C^2\left(\frac{\sin(\pi f T_C)}{\pi f T_C}\right)^2$, dont la TFI est $K_m(\tau) = T_C \mathrm{tri}\left(\frac{\tau}{T_C}\right)$.

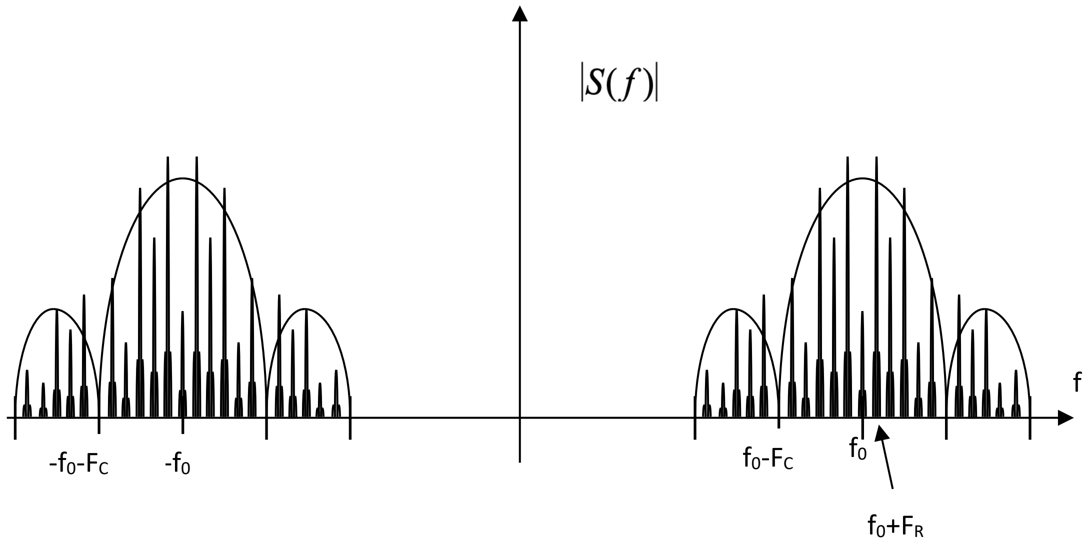
Indeed we know that the Fourier Transform of $T_0 \times \mathrm{tri}\left(\frac{t}{T_0}\right)$ is $T_0^2\left(\frac{\sin(\pi f T_0)}{\pi f T_0}\right)^2$.

In this particular case, we also have :

$$S_C(f) = \left(\frac{N+1}{N^2}\right)\left(\frac{\sin(\pi f T_C)}{\pi f T_C}\right)^2 \sum_{\substack{j=-\infty \\ j\neq 0}}^{+\infty} \delta(f - jF_R) + \frac{1}{N^2}\delta(f)$$

avec $F_R = \frac{1}{T_R}$, et $T_R = NT_C$

In this particular case, considering the large value of the ratios between the different rhythms, the shape of the spectrum is thus the following :



4 Received DS-SS signal model

The transmitted CDMA signal described above propagates in the propagation channel then is processed by the receiver.

To process optimally the received signal, it is important to know how the transmitted signal has been distorted until reaching the processing module. This includes the distortions brought by

- the transmitter including its antenna,
- the propagation medium including the obstacles,
- the user antenna, and
- the receiver RF front end

It is also important to know what kind of additive disturbances enter the antenna together with the useful signal (interference).

These elements are all part of what is considered as the propagation channel. It is thus important to identify and model this propagation channel before deciding upon the processing technique

In addition, in order to retrieve the transmitted information d_k, the receiver must achieve some processing steps such as incoming carrier and code synchronisation with the local carrier codes signals (acquisition and tracking of code and carrier), d_k demodulation. All these processing tasks use the value of the correlation between the incoming signal and local copies of this signal.

4.1 DS-SS RF signal model at the RF Front-End output

The signal to be processed is thus modeled as:

$$r(t) = h(t) * [g(t) * s_T(t) + b(t)]$$

where

$g(t)$ is the impulse response of the propagation medium

$h(t)$ is the impulse response of the equivalent RF front-end

$b(t)$ is an additive perturbation

The transmitted signal is modeled as :

$$s_T(t) = Ad(t)c_m(t)\cos(2\pi f_{\mathrm{L}} t)$$

where

- A is the amplitude of the GNSS signal at the RF front-end input
- $d(t)$ is the navigation message carried by the GNSS signal
- $c_m(t)$ is the PRN code signal
- f_{L} is the carrier frequency of the signal at the RF front-end input

We assume in this course that the propagation medium behaves only as a pure delay (representative of the propagation time) which can vary in time. It entails that the equivalent propagation filter can be expressed by:

$$g(t) = \delta\big(t - \tau(t)\big)$$

As such, this model does not then consider the presence of multipath or ionospheric disturbance, but could be augmented later.

The equivalent RF front-end filter can be modelled as an equivalent filter with impulse response $h(t) = h_{RF}(t)$, such that

$$H_{RF}(f) = \mathrm{rect}\left(\frac{f}{B}\right) * [\delta(f - f_0) + \delta(f + f_0)]$$

The equivalent RF Front-End filter at baseband is thus:

$$H_{RF,BB}(f) = \mathrm{rect}\left(\frac{f}{B}\right)$$

We assume that the dual sided bandwidth *B* of the equivalent selection filter is sufficiently large to let the complete useful signal components enter without significant deformation.

The equivalent RF Front-End filter model could also be enhanced to better reflect a more complex frequency selectivity.

The only considered additive perturbation is the noise created by the RF front-end. It creates an additional white Gaussian (thermal) noise n that is modeled as:

$$S(f) = \frac{N_0}{2} \text{ W.Hz}^{-1}$$

We then do not consider at this stage of the model the interferences due to other transmitters of the same system, other transmitters of similar systems, external non intentional or intentional interferences, nor multipath.
The received signal model can then be expressed as:

$$r(t) = \left[s_T\big(t-\tau(t)\big) + n(t)\right] * h_{RF}(t)$$

This is then equivalent to

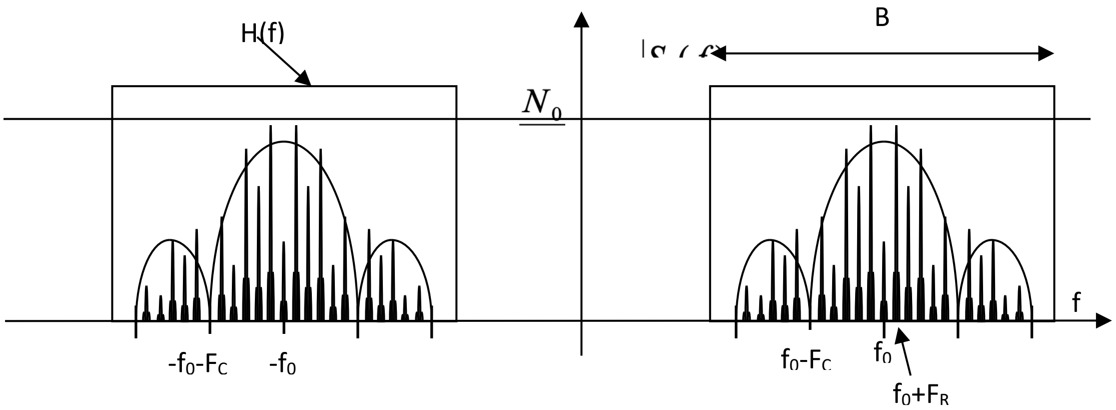
$$r(t) = s_T\big(t-\tau(t)\big) * h_{RF}(t) + n(t) * h_{RF}(t) = r_{u_f}(t) + n_f(t)$$

$$s_T\big(t-\tau(t)\big) = Ad\big(t-\tau(t)\big)c_m\big(t-\tau(t)\big)\cos(2\pi f_{\mathrm{L}}[t-\tau(t)] + \theta_0)$$

Where θ_0 is a sudden phase shift imposed by the antennas gain pattern.

Note that, in certain environments such as urban or indoor environments, in the case of a mobile application, the rapid variation in the characteristics of the multipaths (amplitude, delay, phase shift) is equivalent to a rapid deformation of the distortion brought by the environment. of propagation. The equivalent propagation channel is then characterized according to its coherence time, the spreading of the multipaths, or even its coherence band.

We can then illustrate the PSD of the received signal as:



4.2 SNR at the selective filter output

The power of the useful signal component at the entrance of the RF front-end $s_T\big(t-\tau(t)\big)$ where $s_T(t) = Ad(t)c_m(t)\cos(2\pi f_{\mathrm{L}} t)$ can be obtained by modeling $d(t)$ et $c(t)$ as signals with autocorrelation $K_D(\tau)$=tri$\left(\frac{\tau}{T_D}\right)$ and $K_{c_m}(\tau)$ having a unit power when $\tau = 0$ (ex: $K_{c_m}(\tau)$=tri$\left(\frac{\tau}{T_C}\right)$ for rectangular pulse shaping). Assuming that $d(t)$, $c(t)$ and the random phase of $\cos(2\pi f_0 t)$ that it is necessary to add to make the signal stationary are independent, we then have $K_{s_T}(\tau) = \frac{A^2}{2}K_D(\tau) \times K_{c_m}(\tau) \times \cos(2\pi f_0 \tau)$, so $\mathrm{C} = K_r(0) = \frac{A^2}{2}$.

Also, it is assumed that

$$r_{u_f}(t) = s_T\big(t-\tau(t)\big) * h_{RF}(t) = Ad\big(t-\tau(t)\big)c_{m_f}\big(t-\tau(t)\big)\cos(2\pi f_{\mathrm{L}}[t-\tau(t)]+\theta_0)$$

Where $c_{m_f}(t) = c_m(t) * h_{RF}(t)$

We assume here that $K_{c_m}(\tau) \approx K_{c_{m_f}}(\tau)$, so that the power of the useful signal component is $\mathrm{C} = \frac{A^2}{2}$.

The power spectral density of the filtered noise $S_{b_f}(f)$ is:

$$S_{b_f}(f) = |H(f)|^2 S_b(f) = \frac{N_0}{2}\mathrm{rect}\left(\frac{f}{B}\right) * [\delta(f-f_0)+\delta(f+f_0)]$$

therefore $P_{n_f} = N_0 B$

The signal to noise ratio at the selective filter output is :

$$\frac{P_{r_{u_f}}}{P_{n_f}} = \frac{\frac{A^2}{2}}{N_0 B}$$

4.3 Doppler shift of received signal

We can express the useful component of the received signal as:

$$s(t-\tau) = Ad(t-\tau)c(t-\tau)\cos(2\pi f_0 t - \theta)$$

where θ is the carrier phase offset, which varies with time. The variation with time of this phase offset induces a Doppler offset of the frequency of the received signal carrier.

We can write

$$\psi(t) = 2\pi f_0(t-\tau) + \theta_0$$

The carrier frequency at any time t will therefore be :

$$f = \frac{1}{2\pi}\frac{d\psi}{dt} = f_0 - f_0\frac{d\tau}{dt}$$

We model the propagation delay τ as $r = c\tau$. We then have $\frac{d\tau}{dt} = \frac{1}{c}\frac{dr}{dt} = \frac{1}{c}v$. We have:

$$f = \frac{1}{2\pi}\frac{d\theta}{dt} = f_0 - f_0\frac{1}{c}v$$

We note $f_d = -f_0\frac{1}{c}v$ the Doppler offset. We see that $f_d = -\frac{v}{\lambda}$ where v is the velocity of evolution of the distance between the transmitter and the receiver.

We see that if v is positive, i.e. if the distance r between the transmitter and the receiver increases, then the Doppler offset is negative, which reflects the fact that that the time interval between 2 wave fronts is larger than the nominal value driven by f_0.

5 Correlation operation

5.1 Presentation of correlator

In order to recover the useful information $\mathrm{d}(k)$ present in the transmitted signal, the receiver must perform several steps such as:

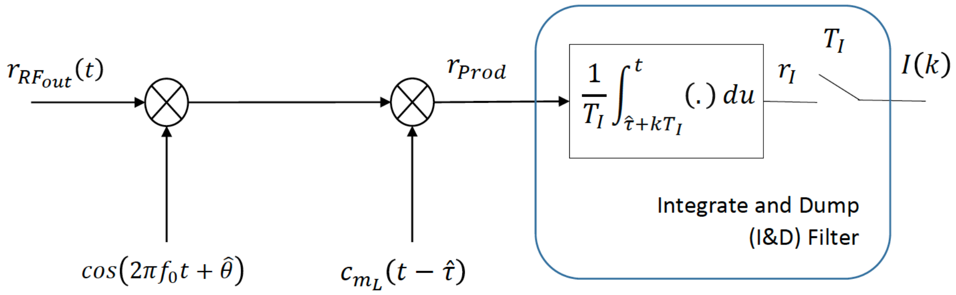
- Generating a local carrier that is synchronized with the incoming carrier phase
- Generating a local PRN code that is synchronized with the incoming PRN signal
- Perform symbol demodulation

These steps are all based on a specific operation referred to as the correlation operation.

The signal to be processed is :

$$r(t) = [s_T(t-\tau(t)) + n(t)] * h_{RF}(t) = r_{u_f}(t) + n_f(t)$$
$$r_{u_f}(t) = s_T(t-\tau(t)) * h_{RF}(t) = Ad(t-\tau(t))c_{m_f}(t-\tau(t))\cos(2\pi f_L[t-\tau(t)] + \theta_0)$$
$$r_{u_f}(t) = Ad(t-\tau(t))c_{m_f}(t-\tau(t))\cos(2\pi f_L t - \theta)$$
$$\theta(t) = 2\pi f_L \tau(t) - \theta_0$$

The global structure of a correlator is therefore:



where:

- T_I is the integration (or correlation) duration.
- $c_{m_L}(t) = [\sum_{k=-\infty}^{+\infty} c_k \delta(t-kT_c)] * m(t)$

Correlator output samples are obtained every T_Is, which is a much lower rate than the sampling frequency at the correlator input.

5.2 Modeling the integrator as a filter

The result of the integration over T_I can be considered as the result of filtering observed at epcoh $\hat{\tau} + (k+1)T_I$. Indeed:

$$\mathrm{I}(k) = r_i(\hat{\tau} + (k+1)T_I) = \frac{1}{T_I}\int_{\hat{\tau}+kT_I}^{\hat{\tau}+(k+1)T_I} r_{prod}(u)du$$
$$= \frac{1}{T_I}\int_{-\infty}^{+\infty} rect\left(\frac{u - \left(\hat{\tau} + (k+1)T_I - \frac{T_I}{2}\right)}{T_I}\right) r_{prod}(u)du$$

And we know

$$rect\left(\frac{u - \left(\hat{\tau} + (k+1)T_I - \frac{T_I}{2}\right)}{T_I}\right) = rect\left(\frac{\hat{\tau} + (k+1)T_I - u - \frac{T_I}{2}}{T_I}\right)$$

therefore

$$r_i(\hat{\tau} + (k+1)T_I) = \frac{1}{T_I}\int_{-\infty}^{+\infty} rect\left(\frac{\hat{\tau} + (k+1)T_I - u - \frac{T_I}{2}}{T_I}\right) r_b(u)du$$
$$r_i(\hat{\tau} + (k+1)T_I) = \int_{-\infty}^{+\infty} f\left(\frac{\hat{\tau} + (k+1)T_I - u}{T_I}\right) r_b(u)du$$

where

$f(t) = \frac{1}{T_I} rect\left(\frac{t - \frac{T_I}{2}}{T_I}\right)$, which is equivalent to $|F(f)|^2 = sinc^2(\pi f T_I)$

Therefore :

$$r_i(\hat{\tau} + (k+1)T_I) = (f * r_{prod})(\hat{\tau} + (k+1)T_I)$$

5.3 Signal model at correlator output

The signal after base band conversion and multiplication by the local code can be expressed as :

$$r_{prod}(t) = r_{u_f}(t) \times \cos\left(2\pi f_{\mathrm{L}} t - \hat{\theta}\right) \times c_{m_L}(t - \hat{\tau})$$

$$r_{prod}(t) = \left[A d\left(t - \tau(t)\right) c_{m_f}\left(t - \tau(t)\right) \cos(2\pi f_{\mathrm{L}} t - \theta) + n_f(t)\right] \times \cos\left(2\pi f_{\mathrm{L}} t - \hat{\theta}\right) \times c_{m_L}(t - \hat{\tau})$$

Therefore

$$r_{prod}(t) = A d(t - \tau) c_{m_f}(t - \tau) \cos(2\pi f_{\mathrm{L}} t - \theta) \cos\left(2\pi f_{\mathrm{L}} t - \hat{\theta}\right) c_{m_L}(t - \hat{\tau}) + n_f(t) \cos\left(2\pi f_{\mathrm{L}} t - \hat{\theta}\right) c_{m_L}(t - \hat{\tau})$$

or

$$r_{prod}(t) = \frac{A}{2} d(t - \tau) \times c_{m_f}(t - \tau) c_{m_L}(t - \hat{\tau}) \times \left[\cos\left(4\pi f_{\mathrm{L}} t - \theta - \hat{\theta}\right) + \cos\left(\theta - \hat{\theta}\right)\right] + n_f(t) \cos\left(2\pi f_{\mathrm{L}} t - \hat{\theta}\right) \times c_{m_L}(t - \hat{\tau})$$

In the rest, the high frequency components are omitted as they will be eliminated by the integrator.

$$r_{prod}(t) = \frac{A}{2} d(t - \tau) c_{m_f}(t - \tau) c_{m_L}(t - \hat{\tau}) \cos\left(\theta - \hat{\theta}\right) + n_f(t) \cos\left(2\pi f_{\mathrm{L}} t - \hat{\theta}\right) \times c_{m_L}(t - \hat{\tau})$$

We note $r_{n_I}(t) = n_f(t) \cos\left(2\pi f_{\mathrm{L}} t - \hat{\theta}\right) \times c(t - \hat{\tau})$.

We note $\varepsilon_\theta = \theta - \hat{\theta}$, and we assume that this deviation is constant over the integration duration, so that we have, at the correlator output :

$$\mathrm{I}(k) = \frac{1}{T_I} \int_{\hat{\tau}+kT_I}^{\hat{\tau}+(k+1)T_I} r_{prod}(u) du = \frac{A}{2T_I} \cos(\varepsilon_\theta) d(k) \int_{\hat{\tau}+kT_I}^{\hat{\tau}+(k+1)T_I} c_{m_f}(u - \tau) c_{m_L}(u - \hat{\tau}) du + n_I(k)$$

$$\mathrm{I}(k) = \frac{A}{2} d(\mathrm{k}) \cos(\varepsilon_\theta) K_{c_{m_f} c_{m_L}}(\varepsilon_\tau) + n_i(\mathrm{k})$$

Where $\varepsilon_\tau = \tau - \hat{\tau}$.

$$n_I(\mathrm{k}) = \frac{1}{T_I} \int_{\hat{\tau}+kT_I}^{\hat{\tau}+(k+1)T_I} r_{n_I}(u) du = \frac{1}{T_I} \int_{\hat{\tau}+kT_I}^{\hat{\tau}+(k+1)T_I} n_f(u) \cos\left(2\pi f_{\mathrm{L}} u - \hat{\theta}\right) \times c_{m_L}(u - \hat{\tau})\, du$$

Note that this correlator output is called the In-phase correlator output because the incoming signal is multiplied with an in-phase replica of the carrier, modeled as a local cosine signal when the incoming signal carrier is also modeled as a cosine.

A second correlator branch can exist in the receiver, called the Quadrature correlator output, where the incoming signal is multiplied with a sine signal. In that case, the Q correlator output is

$$\mathrm{Q}(k) = \frac{A}{2} d(\mathrm{k}) \sin(\varepsilon_\theta) K_{c_{m_f} c_{m_L}}(\varepsilon_\tau) + n_q(\mathrm{k})$$

where $n_q(\mathrm{k}) = \frac{1}{T_I} \int_{\hat{\tau}+kT_I}^{\hat{\tau}+(k+1)T_I} n_f(u) \sin\left(2\pi f_{\mathrm{L}} u - \hat{\theta}\right) \times c_{m_L}(u - \hat{\tau})\, du$.

We can also see that if noise is negligible $\mathrm{I}^2(k) + \mathrm{Q}^2(k) = \frac{A^2}{4}$ if $\varepsilon_\tau = 0$. Therefore, considering both the I and Q branches enables to recover all recoverable power even if ε_θ is different from 0.

If the deviation $\theta - \hat{\theta}$ is not constant but with linear variation over the integration interval, it can be shown that :

$$\mathrm{I}(k) = \frac{A}{2} \mathrm{sinc}(\pi \Delta f T_I)\, d(\mathrm{k}) K_{c_{m_f} c_{m_L}}(\tau - \hat{\tau}) \cos\left(\theta_0 - \hat{\theta}_0\right) + n_I(\mathrm{k})$$

Where:

- $\Delta f = f_d - \widehat{f_d}$ is the Doppler difference between the Doppler offset of the incoming carrier and the Doppler offset of the local carrier.
- $\theta_0 - \widehat{\theta}_0$ is the carrier phase tracking error in the middle of the integration interval

Observing this signal $\mathrm{I}(k)$ at the end of the integration time enables then to identify the transmitted bit.
The power of the useful signal at correlator output is then $\frac{C}{2} = \frac{A^2}{4}$.

5.4 Noise power at correlator output

We search for the power of the noise at correlatour output, which is the power of the random signal

$$n_I(\mathrm{k}) = \frac{1}{T_I} \int_{\hat{\tau}+kT_I}^{\hat{\tau}+(k+1)T_I} r_{n_I}(u) du = \frac{1}{T_I} \int_{\hat{\tau}+kT_I}^{\hat{\tau}+(k+1)T_I} n_f(u) \cos\left(2\pi f_{\mathrm{L}} u - \hat{\theta}\right) \times c_{m_L}(u - \hat{\tau})\, du$$

We see that $n_I(\mathrm{k}) = \frac{1}{T_I} \int_{\hat{\tau}+kT_I}^{\hat{\tau}+(k+1)T_I} r_{n_I}(u) du = \left(f * r_{n_I}\right)(\hat{\tau} + (k+1)T_I)$
where
$f(t) = \frac{1}{T_I} rect\left(\frac{t - \frac{T_I}{2}}{T_I}\right)$, which is equivalent to $|F(f)|^2 = sinc^2(\pi f T_I)$.

We assume that $\hat{\theta}$ is a random variable UD over [0 ;2π], that $\hat{\tau}$ is a random variable UD over *[0 ;T_R]*. We also assume that b_f, $\hat{\theta}$ and $\hat{\tau}$ are independent.
We then have

$$R_{r_{n_I}}(\tau) = E\left[r_{n_I}(t) r_{n_I}(t-\tau)\right]$$
$$R_{r_{n_I}}(\tau) = E\left[n_f(t)\cos\left(2\pi f_0 t - \hat{\theta}\right) c_{m_L}(t-\hat{\tau}) \times n_f(t-\tau)\cos\left(2\pi f_0 (t-\tau) - \hat{\theta}\right) c_{m_L}(t-\tau-\hat{\tau})\right]$$
$$R_{r_{n_I}}(\tau) = E\left[\cos\left(2\pi f_0 t - \hat{\theta}\right)\cos\left(2\pi f_0 (t-\tau) - \hat{\theta}\right)\right] E\left[n_f(t) n_f(t-\tau)\right] E\left[c_{m_L}(t-\hat{\tau}) c_{m_L}(t-\tau-\hat{\tau})\right]$$
$$R_{nr_{n_I}}(\tau) = \frac{1}{2}\cos(2\pi f_0 \tau)\, K_{n_f}(\tau) K_{c_{m_L}}(\tau)$$

We then have :

$$S_{r_{n_I}}(f) = S_{n_f}(f) * S_{c_{m_L}}(f) * \left[\frac{1}{4}\delta(f - f_0) + \frac{1}{4}\delta(f + f_0)\right]$$

We know that

$$S_{n_f}(f) = \frac{N_0}{2} rect\left(\frac{f}{B}\right)\left(\delta(f - f_{\mathrm{L}}) + \delta(f + f_{\mathrm{L}})\right)$$

We then have

$$S_{r_{n_I}}(f) = \frac{N_0}{8} rect\left(\frac{f}{B}\right)\left(\delta(f - 2f_{\mathrm{L}}) + 2\delta(f) + \delta(f + 2f_{\mathrm{L}})\right) * S_c(f)$$

Using the Wiener-Lee relationships, we can determine the PSD at the correlator output:

$$S_{n_i}(f) = |F(f)|^2 S_{r_{n_I}}(f)$$

where $|F(f)|^2 = sinc^2(\pi f T_I)$ is a filter with very narrow band *[-f_I ;f_I]*, negligible compared to *B*.

We can thus write

$$S_{n_i}(f) = |F(f)|^2 S_{r_{n_I}}(0)$$

where $S_{r_{n_I}}(0)$ is the result in 0 of the convolution of $\frac{N_0}{8} rect\left(\frac{f}{B}\right)\left(\delta(f - 2f_0) + 2\delta(f) + \delta(f + 2f_0)\right)$ with $S_c(f)$.

Therefore

$$S_{r_{n_I}}(0) = \int_{-\infty}^{+\infty} \frac{N_0}{8} rect\left(\frac{f}{B}\right)\left(\delta(f-2f_0)+2\delta(f)+\delta(f+2f_0)\right)S_c(-f)\mathrm{d}f$$

Because S_c is even:

$$S_{r_{n_I}}(0) = \frac{N_0}{8}\int_{-\infty}^{+\infty} rect\left(\frac{f}{B}\right)\left(\delta(f-2f_0)+2\delta(f)+\delta(f+2f_0)\right)S_c(f)\mathrm{d}f$$

We then have :

$$S_{r_{n_I}}(0) = \frac{N_0}{4}\int_{-B/2}^{B/2} S_c(f)\mathrm{d}f = \frac{N_0}{4}\times 1$$

Therefore $S_{n_i}(f) = \frac{N_0}{4}|F(f)|^2$

$$P_{n_i} = \int_{-\infty}^{+\infty} S_{n_i}(f)\mathrm{d}f$$

Thus,

$$P_{n_i} = \frac{N_0}{4}\int_{-\infty}^{+\infty} |F(f)|^2\mathrm{d}f$$

En utilisant

$$P_{n_i} = \frac{N_0}{4}\int_{-\infty}^{+\infty} |f(t)|^2\mathrm{d}t$$

$$P_{n_i} = \frac{N_0}{4\mathrm{T_I}}$$

As we can see, when $\mathrm{T_I}$ is getting larger, P_{n_i} is decreased.

The signal to noise ratio at correlator output is therefore

$$\frac{P_{s_i}}{P_{n_i}} = \frac{\mathrm{A}^2\mathrm{T_I}}{N_0}$$

The improvement brought to the signal to noise ratio between the output of RF front-end filter and correlator output is:

$$\frac{\frac{P_{s_i}}{P_{n_i}}}{\frac{P_r}{P_{b_f}}} = \frac{\frac{\mathrm{A}^2\mathrm{T_I}}{N_0}}{\frac{\frac{A^2}{2}}{N_0 B}} = 2\mathrm{T_I}B$$

The minimum value of this improvement factor is for *B=2f_c* and is then equal to $4\mathrm{T_I}f_c$.
The largest duration of coherent integration is when $\mathrm{T_I} = \mathrm{T_D}$, so that in that case the minimum SNR improvement factor is $4\frac{F_C}{F_D}$.
We note $G_T = \frac{f_c}{f_D}$ the processing gain, defined as the ratio between the rhtyhm of the spreading code by the rhythm of the data message. As this ratio is very largethe gain brough to the SNR can be

enormous so the initial SNR can be very weak. In practice, this ratio is sufficiently large so that the received useful signal has a PSD level strictly lower than the noise level.

5.5 Data demodulation

Let us note that the correlator allows to compute the correlation between the local code and the received signal, but is also the matched filter for decision on the binary values conveyed by the data message *d(t)*. The bit error rate matching the P/NRZ/L pulse shaping waveform is thus:

$$P_e = \frac{1}{2}\mathrm{erfc}\left(\sqrt{\frac{\mathrm{C}T_i}{N_0}}\right) = \frac{1}{2}\mathrm{erfc}\left(\sqrt{\frac{E_\mathrm{b}}{N_0}}\right)$$

where E_b is the RF energy of one bit $E_\mathrm{b} = \frac{\mathrm{A}^2}{2}\mathrm{T_D}$ for P/NRZ/L.

6 Acquisition

We showed that the correlator output in presence of Doppler shift is modeled as:

$$\mathrm{I}(k) = \frac{A}{2}\mathrm{sinc}(\pi\Delta\mathrm{f}T_I)\, d(\mathrm{k})K_{c_{m_f}c_{m_L}}(\varepsilon_\tau)\cos\left(\theta_0 - \hat{\theta}_0\right) + n_i(k)$$
$$\mathrm{Q}(k) = \frac{A}{2}\mathrm{sinc}(\pi\Delta\mathrm{f}T_I)\, d(\mathrm{k})K_{c_{m_f}c_{m_L}}(\varepsilon_\tau)\sin\left(\theta_0 - \hat{\theta}_0\right) + n_q(k)$$

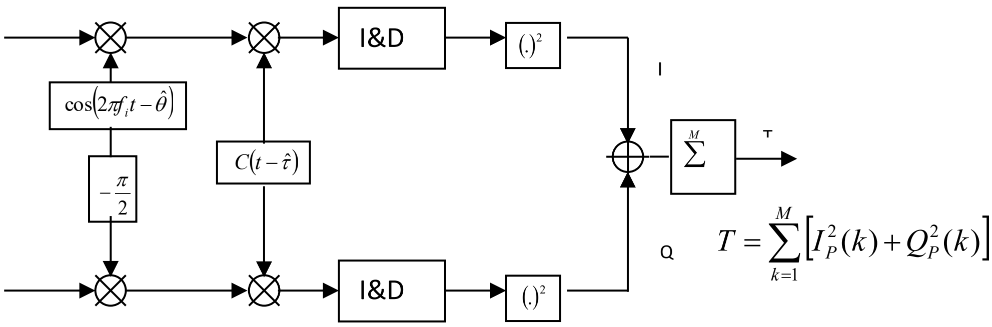
As a consequence, we see that if $|\Delta\mathrm{f}| > \frac{1}{T_i}$ or $|\varepsilon_\tau| > T_c$, then the I and Q correlator outputs will reflect a negligible part of the useful signal component, and observation of their value will not provide any information of the proximity of τ to τ, or $\hat{\theta}$ to θ.

A simple technique is therefore to carry an acquisition under the form of a search over pairs of values $\hat{\tau}$ and $\widehat{f_d}$.

Also, extraction of the data message values $d(\mathrm{k})$ can only be achieved if $\Delta\mathrm{f} = f_d - \widehat{f_d} \sim 0$ and $\varepsilon_\tau = \tau - \hat{\tau} \sim 0$.

6.1 Acquisition detector

The sequential acquisition detector has the following structure:



If we re-use the previous models, we have:

$$\mathrm{I}(k) = \frac{A}{2}\mathrm{sinc}(\pi\Delta\mathrm{f}T_I)\, d(\mathrm{k})K_{c_{m_f}c_{m_L}}(\varepsilon_\tau)\cos\left(\theta_0 - \hat{\theta}_0\right) + n_i(k)$$
$$\mathrm{Q}(k) = \frac{A}{2}\mathrm{sinc}(\pi\Delta\mathrm{f}T_I)\, d(\mathrm{k})K_{c_{m_f}c_{m_L}}(\varepsilon_\tau)\sin\left(\theta_0 - \hat{\theta}_0\right) + n_q(k)$$

It can be shown that the noises n_I and n_Q are independent and with identical power

$$P_{n_I} = P_{n_Q} = \frac{N_0}{4\mathrm{T_I}}.$$

I_P is called the in-phase correlator output for the prompt local code replica, and Q_P is called the quadrature-phase correlator output for the prompt local code replica.

The classical sequential detector is then written as:

$$T = \sum_{k=1}^{M}\left(I_P^2(k) + Q_P^2(k)\right)$$

The parameters that drive that sequential detector performance are then $\mathrm{T_I}$ which is called the coherent integration time, and M, which is called the number of non-coherent summations. Overall, $M\mathrm{T_I}$ is then called the dwell time.

Detection is declared achieved either for the cell such that this criterion T exceeds a predefined threshold Th, or for the cell such that the ratio of the highest value of T is larger than the 2nd highest value of T.

Acquisition tests are modeled as decision tests making a decision between the 2 hypotheses:

- Hypothesis H_0: the useful signal is absent
- Hypothesis H_1: the useful signal is present

This test can be specified by requiring a $P_{FA_{MAX}}$. In that case, the optimum test is the Neyman-Pearson test.

6.2 Detector performance

Let us consider that H_0 is true, that is that the useful signal is absent.

The entering signal is therefore only made of noise. We then can determine the value of the threshold Th.

The detector can be expressed as:

$$T = \sum_{k=1}^{M} \left(n_i^2(k) + n_q^2(k)\right)$$

We can normalize this detector by the value of the noise power $P_{n_I} = P_{n_Q} = \frac{N_0}{4\mathrm{T_I}} = \sigma_n^2$.

$$\tilde{T} = \frac{T}{\sigma_n^2} = \sum_{k=1}^{M} \left(\bar{n}_i^2(k) + \bar{n}_q^2(k)\right)$$

The sum of 2M squared centered normalized independent gaussian noises has a Chi2 distribution with 2M degrees of freedom.

As the distribution of the detection criterion $\tilde{T}$ is known, it is then possible to determine the threshold Th such that

$$P\left(\tilde{T} > \widetilde{T_h}\middle|H_0\right) = P(T > T_h|H_0) < P_{FA_{MAX}}$$

Once the threshold is determined, we need to determine the probability to detect the signal called P_d. We are then in hypothesis H_1 where the signal is present. In this case, the detector is expressed as:

$$T = \sum_{k=1}^{M} \left(\left(\frac{A}{2} d(\mathrm{k}) \operatorname{sinc}(\pi \Delta f T_I) \cos(\varepsilon_\theta) K_{c_{m_f} c_{m_L}}(\varepsilon_\tau) + n_I(\mathrm{k}) \right)^2 \right.$$
$$\left. + \left(\frac{A}{2} d(\mathrm{k}) \operatorname{sinc}(\pi \Delta f T_I) \sin(\varepsilon_\theta) K_{c_{m_f} c_{m_L}}(\varepsilon_\tau) + n_Q(\mathrm{k}) \right)^2 \right)$$

After normalization, and knowing that $\mathrm{C} = \frac{A^2}{2}$

$$\tilde{T} = \frac{T}{\sigma_n^2} = \sum_{k=1}^{M} \left(\left(\sqrt{\frac{2\mathrm{CT_I}}{N_0}} d(\mathrm{k}) \,\mathrm{sinc}(\pi \Delta f T_I) \cos(\varepsilon_\theta) \, K_{c_{m_f} c_{m_L}}(\varepsilon_\tau) + n_I(\mathrm{k}) \right)^2 + \left(\sqrt{\frac{2\mathrm{CT_I}}{N_0}} d(\mathrm{k}) \,\mathrm{sinc}(\pi \Delta f T_I) \sin(\varepsilon_\theta) \, K_{c_{m_f} c_{m_L}}(\varepsilon_\tau) + n_Q(\mathrm{k}) \right)^2 \right)$$

The sum of 2M squared normalized independent non-centered gaussian noises has a Chi2 distribution with 2M degrees of freedom and a non centrality parameter equal to:

$$\lambda = \sum_{k=1}^{M} \left(\left(\sqrt{\frac{2\mathrm{CT_I}}{N_0}} d(\mathrm{k}) \,\mathrm{sinc}(\pi \Delta f T_I) \cos(\varepsilon_\theta) \, K_{c_{m_f} c_{m_L}}(\varepsilon_\tau) \right)^2 + \left(\sqrt{\frac{2\mathrm{CT_I}}{N_0}} d(\mathrm{k}) \,\mathrm{sinc}(\pi \Delta f T_I) \sin(\varepsilon_\theta) \, K_{c_{m_f} c_{m_L}}(\varepsilon_\tau) \right)^2 \right)$$

Thus, if we assume that entering signal parameters along the $\mathrm{M}T_I$ seconds of detection, then:

$$\lambda = 2\mathrm{M}T_I \cdot \frac{\mathrm{C}}{N_0} \cdot \mathrm{sinc}^2(\pi \Delta \mathrm{f} T_I) \, K_{c_{m_f} c_{m_L}}^2(\varepsilon_\tau)$$

Once this distribution is determined, we can then evaluate the probability of detection as:

$$P\left(\tilde{T} > \widetilde{T_h}\right) = P(T > Th) = P_d$$

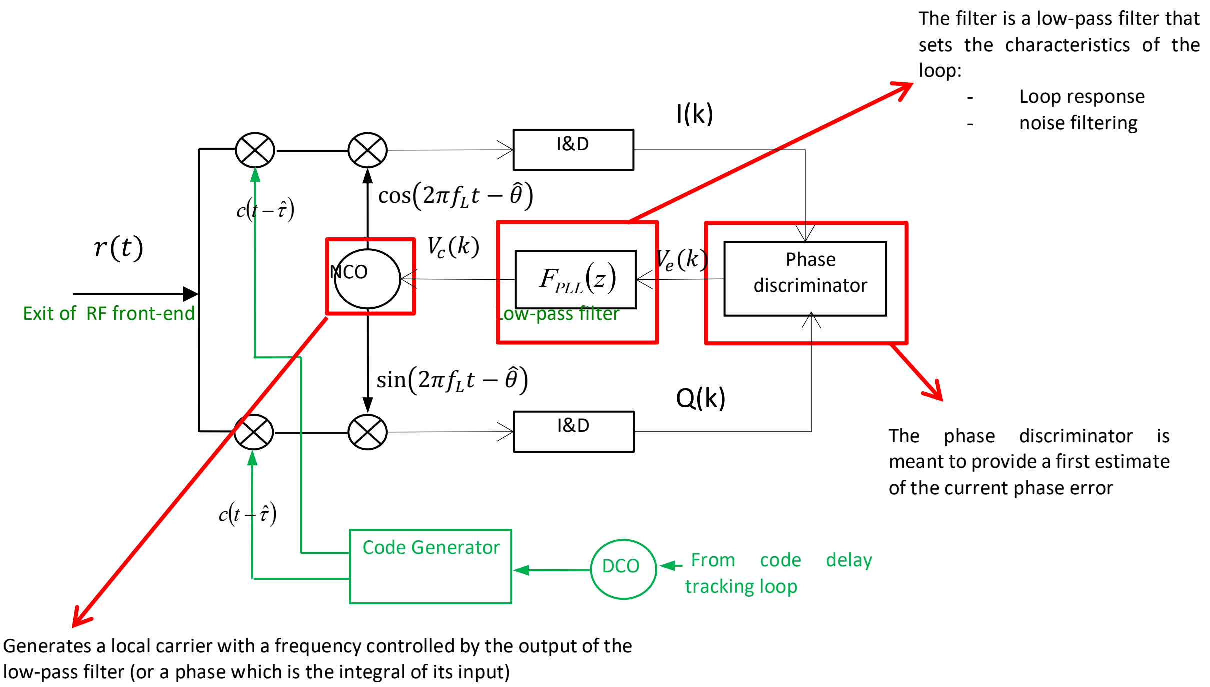
7 DS-SS Carrier phase tracking

In order to retrieve $d(\mathrm{k})$ from $\mathrm{I}(k)$, the receiver must be able to generate $\cos\left(2\pi f_{\mathrm{L}} t - \hat{\theta}\right)$ with a local phase $\hat{\theta}$ that is almost equal to θ.

The objective of a PLL is to generate a local carrier that has a phase identical to the phase of the incoming carrier at any time t. The use of a classical PLL, requiring a pure carrier residual, on this type of received signal is impossible as the sudden phase rotations due to the data signal *d(t)* will be interpreted by the PLL as sudden changes in the sign of θ.

7.1 PLL general architecture

Specific suppressed carrier PLLs are implemented, such as the Costas loop whose scheme is the following :



We know that a VCO is such that it provides an output signal $\cos\left(2\pi f_L t - \hat{\theta}\right)$, where $\hat{\theta}(t) = \hat{\theta}(t_0) + K_{VCO}\int_{t_0}^{t} V_c(u)du$, which is also equivalent to the fact that $\hat{\theta}(s) = K_{DCO}\frac{V_c(s)}{s}$

The digital version of the VCO is the DCO where we have :

$$\hat{\theta}(z) = \frac{z^{-1}K_{DCO}}{1-z^{-1}}V_c(z)$$

The feedback of the digital PLL introduces a delay z^{-1} on the Z-transform as the generation of the phase $\hat{\theta}(k)$ at epoch k can only be done using the phase comparison at epoch k-1, contrary to the analog PLL which can react instantaneously to an observed phase error.

We can see that $V_c(z) = F_{PLL}(z)V_e(z)$.

Different types of carrier phase discriminators can be implemented, among which are:

- Costas, or product, or IQ discriminator: $V_e(k) = I_P(k)Q_P(k)$
- Arctan discriminator: $V_e(k) = \arctan\frac{Q_P(k)}{I_P(k)}$

Note that the use of the Costas discriminator $V_e(k) = I_P(k)Q_P(k)$ requires the normalization of this discriminator output to received useful signal power as $I_P(k)Q_P(k)$ will strongly depend on useful signal power.

Therefore, if we can relate $V_e(k)$ to the phase tracking error ε_θ, we can build an equivalent model for the PLL involving a feedback loop on $\hat{\theta}(z)$ using the comparison ε_θ between $\theta(t)$ and $\hat{\theta}(t)$.

7.2 PLL equivalent linear model

We recall that

$$\begin{cases} I_P(k) = \dfrac{A}{2} d(\mathrm{k}) \operatorname{sinc}(\pi \Delta f T_I) \cos(\varepsilon_\theta) K_{c_{m_f} c_{m_L}}(\varepsilon_\tau) + n_I(\mathrm{k}) \\ Q_P(k) = \dfrac{A}{2} d(\mathrm{k}) \operatorname{sinc}(\pi \Delta f T_I) \sin(\varepsilon_\theta) K_{c_{m_f} c_{m_L}}(\varepsilon_\tau) + n_Q(\mathrm{k}) \end{cases}$$

Assuming that $\Delta f \sim 0$, we have then

$$\begin{cases} I_P(k) = \dfrac{A}{2} d(\mathrm{k}) K_{c_{m_f} c_{m_L}}(\varepsilon_\tau) \cos(\varepsilon_\theta) + n_i(k) \\ Q_P(k) = \dfrac{A}{2} d(\mathrm{k}) K_{c_{m_f} c_{m_L}}(\varepsilon_\tau) \sin(\varepsilon_\theta) + n_q(k) \end{cases}$$

If the discriminator is a product or Costas discriminator, then

$$V_e(k) = I_P(k)Q_P(k)$$

$$V_e(k) = \left(\frac{A}{2} d(\mathrm{k}) K_{c_{m_f} c_{m_L}}(\varepsilon_\tau) \cos(\varepsilon_\theta) + n_i(k)\right)\left(\frac{A}{2} d(\mathrm{k}) K_{c_{m_f} c_{m_L}}(\varepsilon_\tau) \sin(\varepsilon_\theta) + n_q(k)\right)$$

We can see that we can write

$$V_e(k) = \frac{A^2}{8} K^2_{c_{m_f} c_{m_L}}(\varepsilon_\tau) \sin(2\varepsilon_\theta) + n_{IQ}(k)$$

The useful signal component of this discriminator output still depends on $C = \frac{A^2}{2}$.

We can normalize this discrimator as:

$$V_e(k) = \frac{I_P(k)Q_P(k)}{I_P^2(k) + Q_P^2(k)}$$

Therefore,

$$V_e(k) = \frac{\left(\frac{A}{2} d(\mathrm{k}) K_{c_{m_f} c_{m_L}}(\varepsilon_\tau) \cos(\varepsilon_\theta) + n_i(k)\right)\left(\frac{A}{2} d(\mathrm{k}) K_{c_{m_f} c_{m_L}}(\varepsilon_\tau) \sin(\varepsilon_\theta) + n_q(k)\right)}{\left(\frac{A}{2} d(\mathrm{k}) K_{c_{m_f} c_{m_L}}(\varepsilon_\tau) \cos(\varepsilon_\theta) + n_i(k)\right)^2 + \left(\frac{A}{2} d(\mathrm{k}) K_{c_{m_f} c_{m_L}}(\varepsilon_\tau) \sin(\varepsilon_\theta) + n_q(k)\right)^2}$$

We can see that we can write

$$V_e(k) \approx \frac{1}{2} \sin(2\varepsilon_\theta) + N_e(k)$$

where n_e is the resulting noise at discriminator output.
if $n_i(k)$ and $n_q(k)$ are weak against the useful signal components at the denominator.
Finally, we have

$$V_e(k) \approx \varepsilon_\theta + N_e(k)$$

We know that:

$$V_c(z) = F_{PLL}(z)V_e(z)$$

And also $\hat{\theta}(z) = \frac{z^{-1}K_{DCO}}{1-z^{-1}}V_c(z)$.
Therefore we have:

$$\hat{\theta}(z) = \frac{z^{-1}K_{DCO}}{1-z^{-1}}F_{PLL}(z)\varepsilon_\theta(z) + \frac{z^{-1}K_{DCO}}{1-z^{-1}}F_{PLL}(z)N_e(z)$$

As we know that $\varepsilon_\theta(z) = \theta(z) - \hat{\theta}(z)$, then we have :

$$\hat{\theta}(z) = \underbrace{\frac{K_{DCO}\frac{z^{-1}}{1-z^{-1}}F_{PLL}(z)}{1 + K_{DCO}\frac{z^{-1}}{1-z^{-1}}F_{PLL}(z)}}_{H_{PLL}(z)}\theta(z) + \frac{K_{DCO}\frac{z^{-1}}{1-z^{-1}}F_{PLL}(z)}{1 + K_{DCO}\frac{z^{-1}}{1-z^{-1}}F_{PLL}(z)}N_e(z)$$

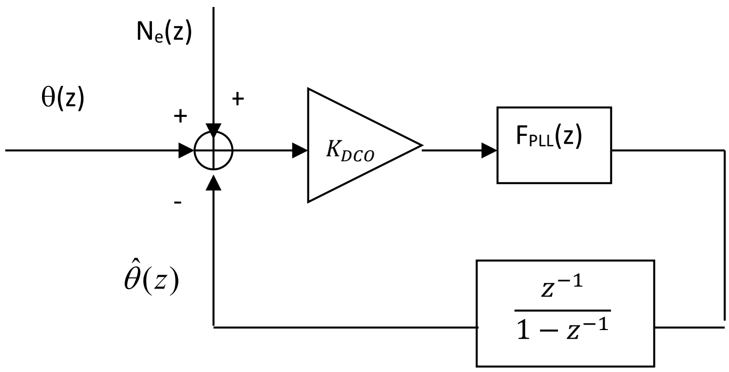
We note $H_{PLL}(z) = \frac{K_{DCO}\frac{z^{-1}}{1-z^{-1}}F_{PLL}(z)}{1+K_{DCO}\frac{z^{-1}}{1-z^{-1}}F_{PLL}(z)}$ the closed loop transfer function relating $\hat{\theta}$ to θ.

We then have :

$$\hat{\theta}(z) = H_{PLL}(z)\theta(z) + H_{PLL}(z)N_e(z)$$

This relationship shows that $\hat{\theta}(t)$ is the sum of 2 components, the first one reflecting the time domain evolution of $\theta(t)$ and the filtered version of the noise affecting the discriminator output.

This enables to view the operation of the PLL as an equivalent system illustrated below:



This model is called the equivalent linear model of the PLL.
It is characterized by the equivalent closed loop transfer function $H_{PLL}(z)$, driving the performance of the equivalent loop.

As $F_{PLL}(z)$ is a low pass filter, the filter $H_{PLL}(z)$ is also a low pass filter. We denote B_{PLL} the 3dB bandwidth of that low-pass filter.
From the relationship above, we can then see that in order to make sure that $\hat{\theta}(t)$ copies all evolutions of $\theta(t)$ and in particular the high frequency transients reflecting the dynamics, B_{PLL} needs to be large. However, in order to reduce the power of filtered noise from discriminator output, B_{PLL} needs to be low. There is therefore a compromise to be striken in order to reduce noise and at the same time track the desired dynamics.

In the same way, we can relate $\hat{\theta}(z) - \theta(z)$ to $\theta(z)$ as

$$\hat{\theta}(z) - \theta(z) = [H_{PLL}(z) - 1]\theta(z) + H_{PLL}(z)N_e(z)$$

We can therefore determine the phase tracking error due to dynamics and due to noise.

We can for example determine the phase tracking error ε_θ for any input of the form $\theta = a_n t^n U(t)$.

We can also determine the variance of the phase tracking error due to noise $H_{PLL}(z)N_e(z)$ as:

$$\sigma_{\varepsilon_\theta}^2 = \frac{B_{PLL}}{\frac{C}{N_0}}\left(1 + \frac{1}{2T_I\frac{C}{N_0}}\right)\ (rad^2)$$

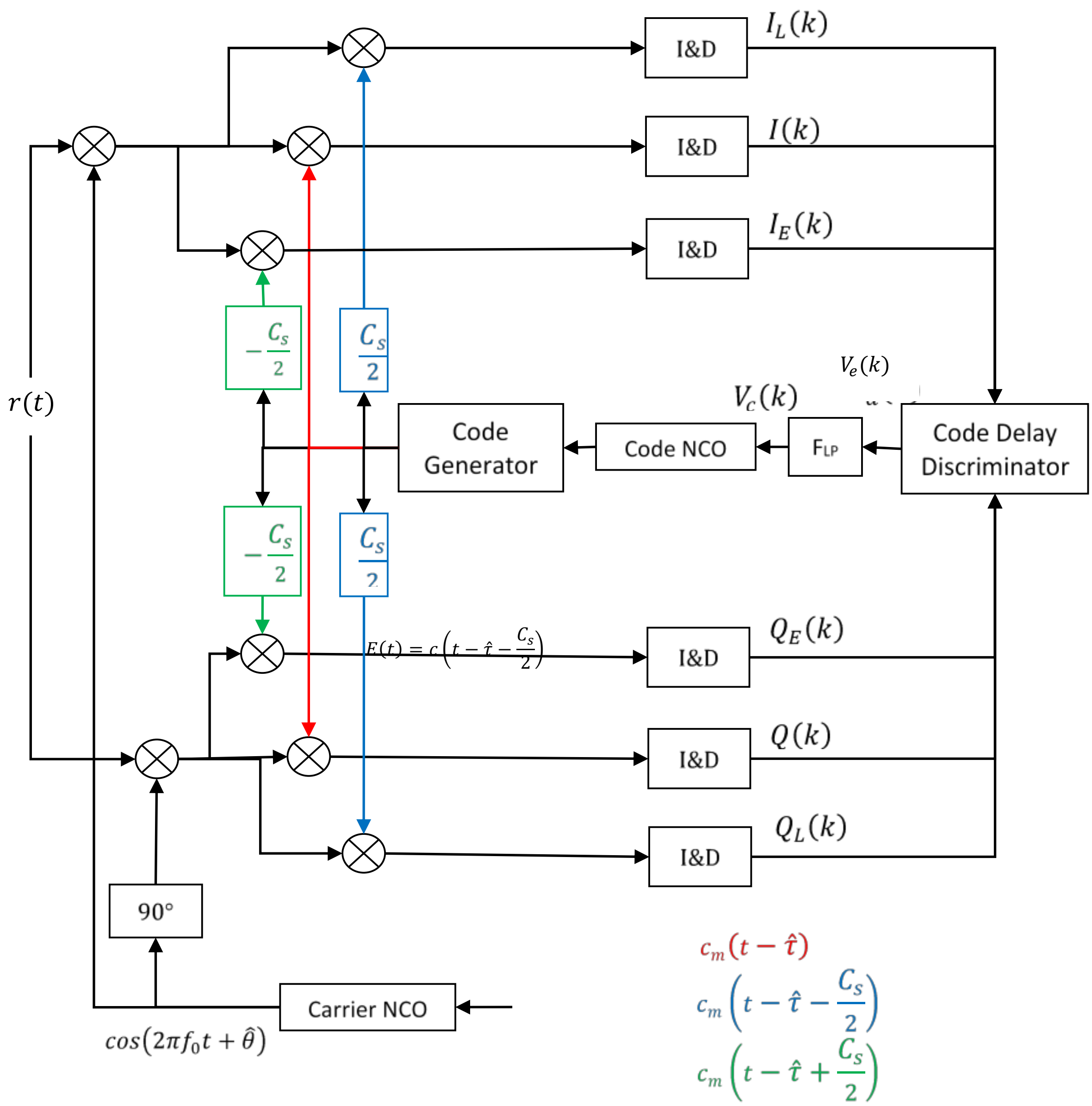
The second term in this variance of the phase tracking error due to noise is due to the non linear operations (multiplications and squaring) of the correlator noise.

8 DS-SS Code Delay Tracking

In order to generate a local code replica which is in phase with the incoming code, the receiver needs to implement a Delay Lock Loop.
The DLL uses the properties of the PRN code in order to observe the time offset between the local and incoming code replicas by observing the differential amplitude of the correlator output.

8.1 DLL general architecture

The typical architecture of a DLL is :



C_s is the Early minus Late correlator spacing between the early and the late local code replicas.
Note that $C_s \leq T_c$.

The model of combination of the code NCO and code generator block is such that $\hat{\tau}(t) = \hat{\tau}(t_0) + T_c K_{VCO} \int_{t_0}^{t} V_c(u)du$, which is also equivalent to the fact that $\hat{\tau}(s) = T_c K_{DCO} \frac{V_c(s)}{s}$

The digital version of the VCO + code generator is the DCO+code generator where we have:

$$\hat{\tau}(z) = \frac{z^{-1} T_c K_{DCO}}{1 - z^{-1}} V_c(z)$$

Different types of code delay discriminators can be implemented, among which are:

- Early minus Late discriminator: $V_e(k) = I_E(k) - I_L(k)$
- Dot Product discriminator: $V_e(k) = I_P(k)\big(I_E(k) - I_L(k)\big) + Q_P(k)\big(Q_E(k) - Q_L(k)\big)$
- Early minus Late power discriminator: $V_e(k) = \frac{\left(\frac{C_S}{T_c} - 1\right)\left[\left(I_E^2(k) + Q_E^2(k)\right) - \left(I_L^2(k) + Q_L^2(k)\right)\right]}{2\left[\left(I_E^2(k) + Q_E^2(k)\right) + \left(I_L^2(k) + Q_L^2(k)\right)\right]}$

The performance of these discriminators is bounded by the performance of the EMLP discriminator, which is therefore further elaborated below.

8.2 DLL equivalent tracking loop model

If we assume Δf is negligible, we can write:

$$\begin{cases} I_E(k) = \frac{A}{2} d(\mathrm{k}) K_{c_{m_f} c_{m_L}}\left(\varepsilon_\tau + \frac{C_s}{2}\right)\cos(\varepsilon_\theta) + n_{Ei}(k) \\ Q_E(k) = \frac{A}{2} d(\mathrm{k}) K_{c_{m_f} c_{m_L}}\left(\varepsilon_\tau + \frac{C_s}{2}\right)\sin(\varepsilon_\theta) + n_{Eq}(k) \\ I_L(k) = \frac{A}{2} d(\mathrm{k}) K_{c_{m_f} c_{m_L}}\left(\varepsilon_\tau - \frac{C_s}{2}\right)\cos(\varepsilon_\theta) + n_{Li}(k) \\ Q_L(k) = \frac{A}{2} d(\mathrm{k}) K_{c_{m_f} c_{m_L}}\left(\varepsilon_\tau - \frac{C_s}{2}\right)\sin(\varepsilon_\theta) + n_{Lq}(k) \end{cases}$$

In the case where the discriminator is the Early Minus Late Power:

$$V_e(k) = \frac{\left(\frac{C_S}{T_c} - 1\right)\left[\left(I_E^2(k) + Q_E^2(k)\right) - \left(I_L^2(k) + Q_L^2(k)\right)\right]}{2\left[\left(I_E^2(k) + Q_E^2(k)\right) + \left(I_L^2(k) + Q_L^2(k)\right)\right]}$$

Then if the noise at correlator output is negligible compared to the useful signal components:

$$V_e(k) = \frac{\left(\frac{C_S}{T_c} - 1\right)\left(\frac{A^2}{4} K^2_{c_{m_f} c_{m_L}}\left(\varepsilon_\tau + \frac{C_S}{2}\right) - \frac{A^2}{4} K^2_{c_{m_f} c_{m_L}}\left(\varepsilon_\tau - \frac{C_S}{2}\right)\right)}{2\left(\frac{A^2}{4} K^2_{c_{m_f} c_{m_L}}\left(\varepsilon_\tau + \frac{C_S}{2}\right) + \frac{A^2}{4} K^2_{c_{m_f} c_{m_L}}\left(\varepsilon_\tau - \frac{C_S}{2}\right)\right)} + N_{EMLP}(k)$$

We can see that this discriminator does not depend on the phase tracking error ε_θ, provided that Δf Is negligible. This discriminator is a non-coherent discriminator.
In the case of a pure rectangular correlation function:

$$K_{c_{m_f} c_{m_L}}(\varepsilon) = \begin{cases} 1 - \frac{|\varepsilon|}{T_c} & if \ \ |\varepsilon| < T_c \\ 0 & elsewhere \end{cases}$$

If we assume that $|\varepsilon_\tau| + \left|\frac{C_S}{2}\right| < T_c$, then the values of $K_{c_{m_f} c_{m_L}}$ are all taken in the triangular part, then:

$$\begin{cases} K_{c_{m_f}c_{m_L}}\left(\varepsilon_\tau + \frac{C_s}{2}\right) = 1 - \frac{\left|\varepsilon_\tau + \frac{C_s}{2}\right|}{T_c} \\ K_{c_{m_f}c_{m_L}}\left(\varepsilon_\tau - \frac{C_s}{2}\right) = 1 - \frac{\left|\varepsilon_\tau - \frac{C_s}{2}\right|}{T_c} \end{cases}$$

If we consider that the code tracking error is such that $|\varepsilon_\tau| < \frac{C_s}{2}$, then

$$\begin{cases} K_{c_{m_f}c_{m_L}}\left(\varepsilon_\tau + \frac{C_s}{2}\right) = 1 - \frac{\left(\varepsilon_\tau + \frac{C_s}{2}\right)}{T_c} \\ K_{c_{m_f}c_{m_L}}\left(\varepsilon_\tau - \frac{C_s}{2}\right) = 1 - \frac{\left(\frac{C_s}{2} - \varepsilon_\tau\right)}{T_c} \end{cases}$$

Therefore

$$V_e(k) \approx \frac{\varepsilon_\tau}{T_c} + N_{EMLP}(k)$$

Knowing that

$$\hat{\tau}(z) = \frac{z^{-1}T_c K_{DCO}}{1 - z^{-1}} V_c(z)$$

and $V_c(z) = F_{DLL}(z)V_e(z)$, then we have

$$\hat{\tau}(z) = \frac{z^{-1}T_c K_{DCO}}{1 - z^{-1}} F_{DLL}(z)\left(\frac{\varepsilon_\tau(z)}{T_c} + N_{EMLP}(z)\right)$$

As $\varepsilon_\tau(z) = \tau(z) - \hat{\tau}(z)$, we then have:

$$\hat{\tau}(z) = \frac{z^{-1}K_{DCO}}{1 - z^{-1}} F_{DLL}(z)\varepsilon_\tau(z) + \frac{z^{-1}K_{DCO}}{1 - z^{-1}} F_{DLL}(z)T_c N_{EMLP}(k)$$

We note $T_c N_{EMLP}(k) = \widetilde{N}_{EMLP}(k)$.

$$\hat{\tau}(z) = \frac{z^{-1}K_{DCO}}{1 - z^{-1}} F_{DLL}(z)\varepsilon_\tau(z) + \frac{z^{-1}K_{DCO}}{1 - z^{-1}} F_{DLL}(z)\widetilde{N}_{EMLP}(k)$$

As $\varepsilon_\tau(z) = \tau(z) - \hat{\tau}(z)$, we then have:

$$\hat{\tau}(z) = \underbrace{\frac{K_{DCO}\frac{z^{-1}}{1 - z^{-1}} F_{DLL}(z)}{1 + K_{DCO}\frac{z^{-1}}{1 - z^{-1}} F_{DLL}(z)}}_{H_{PLL}(z)} \tau(z) + \frac{K_{DCO}\frac{z^{-1}}{1 - z^{-1}} F_{DLL}(z)}{1 + K_{DCO}\frac{z^{-1}}{1 - z^{-1}} F_{DLL}(z)} \widetilde{N}_{EMLP}(z)$$

We note $H_{DLL}(z) = \frac{K_{DCO}\frac{z^{-1}}{1 - z^{-1}} F_{DLL}(z)}{1 + K_{DCO}\frac{z^{-1}}{1 - z^{-1}} F_{DLL}(z)}$ the closed loop transfer function relating $\hat{\tau}$ to τ.

We then have :

$$\hat{\tau}(z) = H_{DLL}(z)\tau(z) + H_{DLL}(z)\widetilde{N}_{EMLP}(z)$$

This relationship shows that $\hat{\tau}(t)$ is the sum of 2 components, the first one reflecting the time domain evolution of $\tau(t)$ and the filtered version of the noise affecting the discriminator output.
As $F_{DLL}(z)$ is a low-pass filter, $H_{DLL}(z)$ is also a low-pass filter with 3dB bandwidth B_{DLL}.

We can then establish a linear equivalent model, similar to the linear equivalent model built for the PLL.

It can be shown that the code tracking error due to noise has the variance:

- EML: $\sigma_{\varepsilon_\tau}^2 = \frac{B_{DLL}\frac{C_S}{T_C}}{2\frac{C}{N_0}}\ (chip^2)$
- EMLP: $\sigma_{\varepsilon_\tau}^2 = \frac{B_{DLL}\frac{C_S}{T_C}}{2\frac{C}{N_0}}\left(1+\frac{1}{T_I\frac{C}{N_0}}\right)\ (chip^2)$
- Dot Product: $\sigma_{\varepsilon_\tau}^2 = \frac{B_{DLL}\frac{C_S}{T_C}}{2\frac{C}{N_0}}\left(1+\frac{2}{\left(2-\frac{C_S}{T_C}\right)T_I\frac{C}{N_0}}\right)\ (chip^2)$

We can see that when the Early-Late spacing C_s is narrow, the variance reduces. However, it may not be possible to choose a value of C_s as low as desired because that requires the capability to distinguish the Early and the Late local code replicas over the integration duration. In a digital receiver implementation, distinction between the Early and Late replicas over the integration duration requires the capacity to sample with a sufficient sampling frequency in order to have a sufficient number of samples that would be different between the Early and the Late replicas over T_I.

Notice as well that this variance is expressed in chip^2, so that the standard deviation is expressed in units of chip. Therefore, conversion of this error in seconds requires the multiplication by T_c. Therefore, when the chipping rate is increased, this standard deviation is reduced as T_c will be shorter.

9 Impact of Multipath

Let us consider the following received signal model:

$$r(t) = \mathrm{A}_0 d(t-\tau_0)c_{m_f}(t-\tau_0)\cos(2\pi f_{\mathrm{L}}t-\theta_0) + \mathrm{A}_1 d(t-\tau_1)c_{m_f}(t-\tau_1)\cos(2\pi f_{\mathrm{L}}t-\theta_1) + n_f(t)$$

where $\mathrm{A}_0, \tau_0, \theta_0$ are the signal parameters of the Line Of Sight component, and $\mathrm{A}_1, \tau_1, \theta_1$ are the signal parameters of the multipath component.
We denote $\alpha = \frac{\mathrm{A}_1}{\mathrm{A}_0}$, $\Delta\tau = \tau_1 - \tau_0$, $\Delta\theta = \theta_1 - \theta_0$.
We note $\varepsilon_\tau = \tau_0 - \hat{\tau}$, and $\varepsilon_\theta = \theta_0 - \hat{\theta}$.

As the correlation operation is linear, we can then write

$$\mathrm{I}_P(k) = \frac{\mathrm{A}_0}{2} d(\mathrm{k}) K_{c_{m_f}c_{m_L}}(\varepsilon_\tau)\cos(\varepsilon_\theta) + \alpha\frac{\mathrm{A}_0}{2} d(\mathrm{k}) K_{c_{m_f}c_{m_L}}(\varepsilon_\tau+\Delta\tau)\cos(\varepsilon_\theta+\Delta\theta) + n_i(k)$$

We can search for the steady state tracking error in presence of multipath and in absence of noise, if we assume that the multipath parameters $\alpha = \frac{\mathrm{A}_1}{\mathrm{A}_0}$, $\Delta\tau = \tau_1 - \tau_0$, $\Delta\theta = \theta_1 - \theta_0$ are constant over the time constant of the tracking loops.
The situation of reaching the steady state of the tracking operation is defined as the situation where $\hat{\theta}$ and $\hat{\tau}$ do not evolve any more, that is when the code and phase discriminator outputs is equal to 0.

9.1 Carrier Phase Tracking Error Envelope

If we consider a PLL with arctan discriminator

$$V_e = \arctan\left(\frac{Q_P(k)}{I_P(k)}\right)$$

then the situation where $V_e = 0$ is the situation where $Q_P(k) = 0$.
This situation is defined with the phase tracking error due to multipath as:

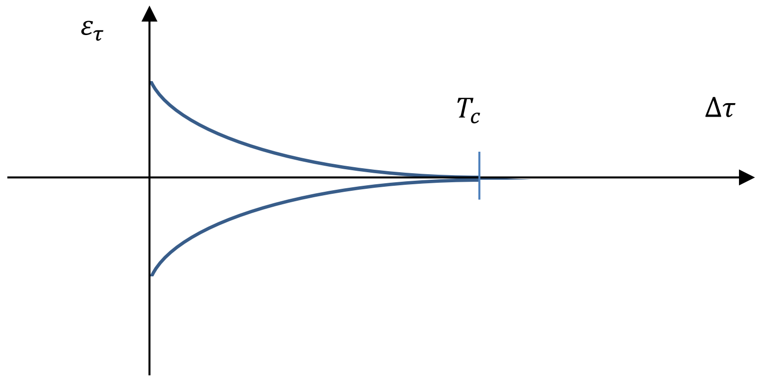
$$\tan\varepsilon_\theta = \frac{-\alpha K_{c_{m_f}c_{m_L}}(\varepsilon_\tau+\Delta\tau)\sin(\Delta\theta)}{K_{c_{m_f}c_{m_L}}(\varepsilon_\tau) + \alpha K_{c_{m_f}c_{m_L}}(\varepsilon_\tau+\Delta\tau)\cos(\Delta\theta)}$$

The largest magnitude of this multipath tracking erros is when

$$\alpha = 1, \Delta\tau \to 0, \Delta\theta \to \pm\pi$$

as in that case $\tan\varepsilon_\theta \to \pm\infty$ and therefore $\varepsilon_\theta \to \pm\frac{\pi}{2}$ or equivalently a quarter of a wavelength.

If $\alpha < 1$, then for $\varepsilon_\tau = 0$, we would have a phase tracking error envelope with the following characteristics:



Note that the phase tracking error decreases to 0 when $\Delta\tau \geq T_c$ as the multipath ray becomes uncorrelated with the locally generated code replica if it matches the LOS ray.

9.2 Code Tracking Error Envelope

The impact of multipath on the code tracking error depends on the DLL implemented.

However it can be shown that the code tracking error due to multipath is bounded by the code tracking error envelope due to multipath for the EML discriminator.

We know that

$$V_e = I_E(k) - I_L(k)$$

The steady state tracking error is the multipath parameters are constant over the DLL time constant is therefore when

$$V_e = 0$$

This can be rewritten as:

$$V(\varepsilon_\tau)\cos\varepsilon_\theta + \alpha V(\varepsilon_\tau + \Delta\tau)\cos(\varepsilon_\theta + \Delta\theta) = 0$$

where

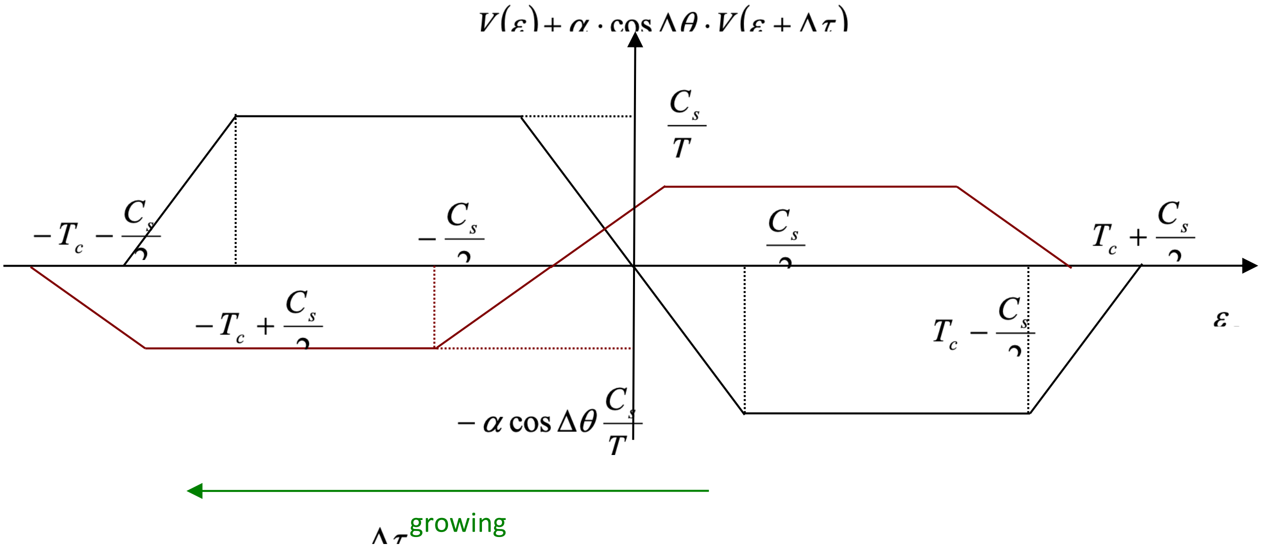
$$V(\varepsilon) = K_{c_{m_f}c_{m_L}}\left(\varepsilon_\tau + \frac{C_s}{2}\right) - K_{c_{m_f}c_{m_L}}\left(\varepsilon_\tau - \frac{C_s}{2}\right)$$

If we assume that

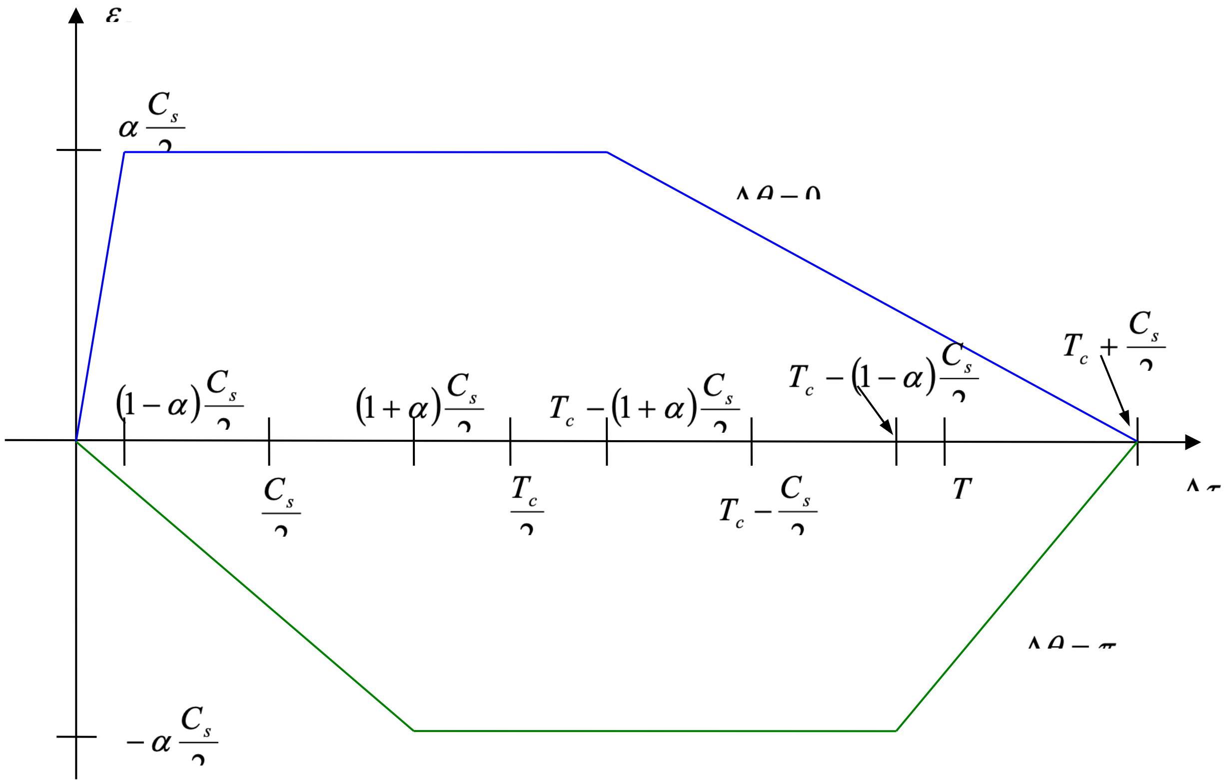
$$K_{c_{m_f}c_{m_L}}(\varepsilon) = \begin{cases} 1 - \frac{|\varepsilon|}{T_c} & if \ \ |\varepsilon| < T_c \\ 0 & elsewhere \end{cases}$$

then we can determine that searching for ε_τ such that $V_e = 0$ can be equivalently viewed as searching for the intersection between $V(\varepsilon_\tau)\cos\varepsilon_\theta$ and -$\alpha V(\varepsilon_\tau + \Delta\tau)\cos(\varepsilon_\theta + \Delta\theta)$.

This is plotted in the figure below when $\varepsilon_\theta = 0$:



We then obtain the code tracking error envelope:



Notice that when C_S reduces the multipath error reduces as well. Also, the influence of multipath does not extend further that $\Delta\tau = T_c + \frac{C_S}{2}$. Therefore, if the chipping rate is increased, that envelope will also shorten because T_c will be shorter.