Реляти-ий коллапс.

$M > 3 \cdot M_\odot$; $(10 \div 30)\, M_\odot$ в некоторых моделях.

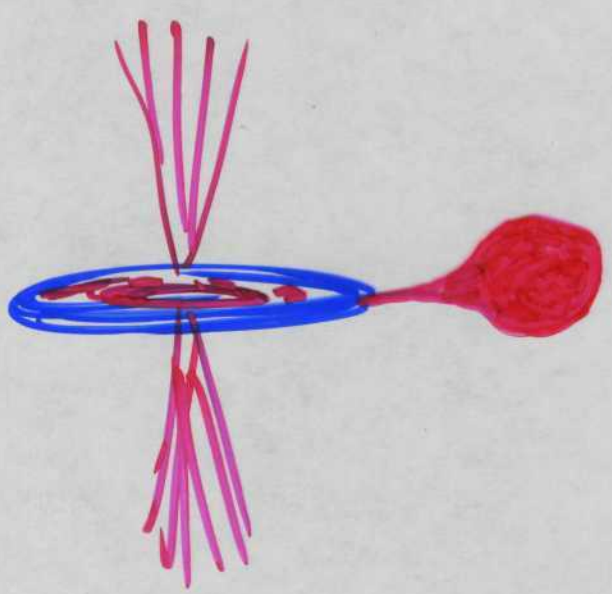
Взрыв сверхновой:

$$\rho \to \rho_{max} = 2 \cdot 10^{16} \left(\frac{M_\odot}{M} \right)^2, \ \frac{г}{см^3}$$

$$L \sim L_0 \cdot \exp[-\alpha \cdot r / r_g]; \quad r_g = \frac{2GM}{c^2}$$

$$\alpha \sim \frac{r_g}{c} = 10^{-5} с = 10 \text{ мкс}!$$

Наблюдаемы только внешние проявления: аккреция в-ва, излучение диска (рентген, γ), выбросы из компактного центрального объекта.



Чёрные дыры.

В совре-ой АФ рассматривают:

- 1. ЧД с $M \simeq M_\odot$
- 2. Сверхмассивные ЧД с $M \gtrsim 10^9 M_\odot$.
- 3. Первичные ЧД с $M \gtrsim 10^{15}$ г $\simeq 10^{-18} \cdot M_\odot$ (испарение)

ЧД с массой $\sim M_\odot$

- "Когда все источники термо-ядерной энергии иссякнут массивная звезда начинает сжиматься".

(PR, Опенгеймер, Волков, 1939г.)

$$M_{OB} > 3 M_\odot \rightarrow \text{предсказали обра-ие ЧД.}$$

$$M_{OB}^{совр} > (10 \div 30) \cdot M_\odot, \text{ (кварковая звезда)}$$

Количество ЧД в Галактике: при $M_{OB} \simeq (12 \div 30) M_\odot$

$$\dot{N}_{ЧД} \simeq 0{,}01 \frac{шт}{год}, \quad T_{гал} \simeq 10^{10} \text{ лет} \Rightarrow N_{ЧД} \simeq 10^8$$

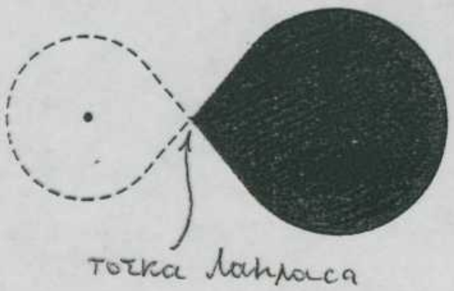
Обнаружение ЧД.

Проявляет себя только через грави-ое поле ($Q \to 0$)

аккреция!

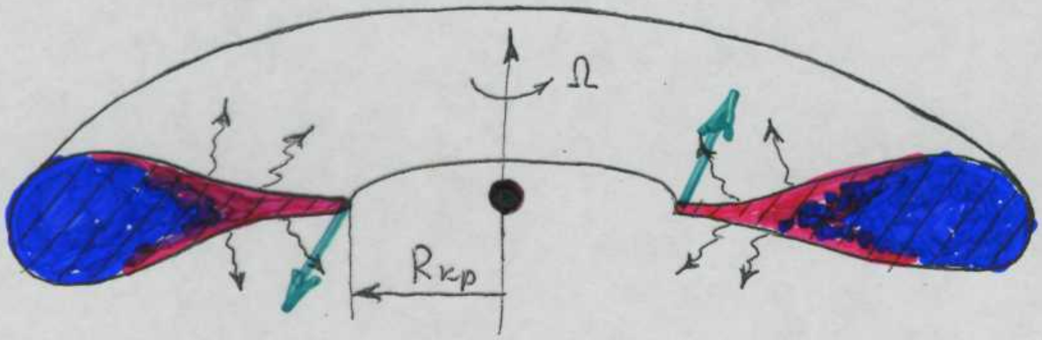
- двойные системы
- сверх массивные ЧД в центре Галактики

Двойные системы



В фазе красного гиганта через точку Лапласа прои-т перетекание газа

$R_g = \frac{2GM}{c^2}$, $R_g^{\odot} = 3$ км, $R_{кр} = 3 \cdot R_g$, $E_{св} = 0.057 \cdot mc^2$



Основные процессы в диске:

- вращение газа + вязкость.
- ⇒ замедление, нагрев газа, переход на < r.
- ⇒ повышение температуры, излучение
- ⇒ падение в ЧД ($\Delta E = 0.057 \cdot mc^2$, ~6% M)

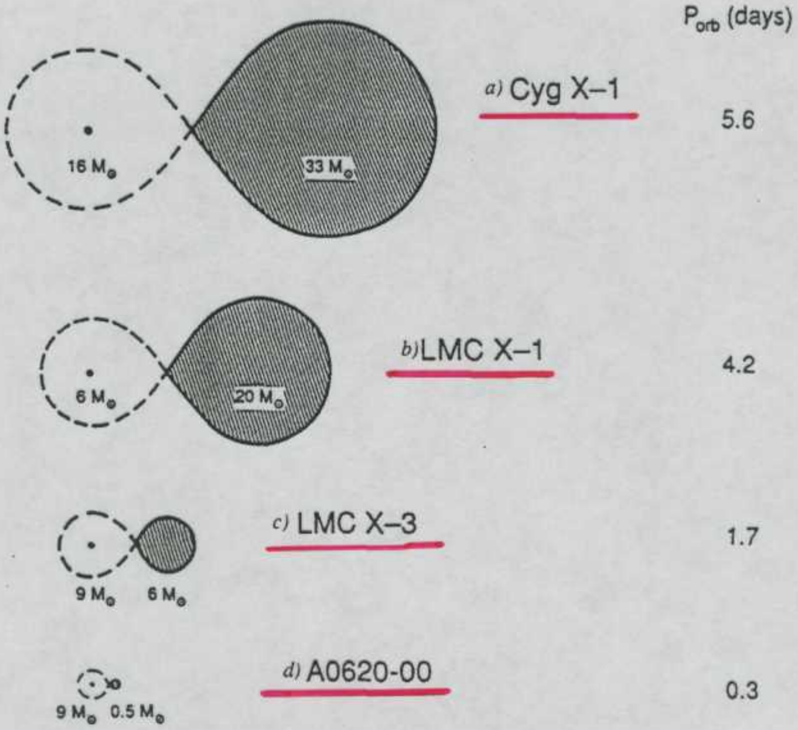
$$L_{диска} = 0.057\, \dot{M} c^2 = \left(3 \cdot 10^{36} \frac{эрг}{с}\right)\left[\frac{\dot{M}^{акк.}}{10^{-9} \cdot M_{\odot}/год}\right]$$

$\dot{m} \doteq 10^{-9} \cdot M_{\odot}/год$ – типичное значение ($\simeq M_{земли}$)

$T_{диска} \simeq 10^7 \div 10^8$ K.

$L_{диска} \approx 10^3 \cdot L_{\odot}$

Schematic sketches to scale of plausible models for the four black hole candidates



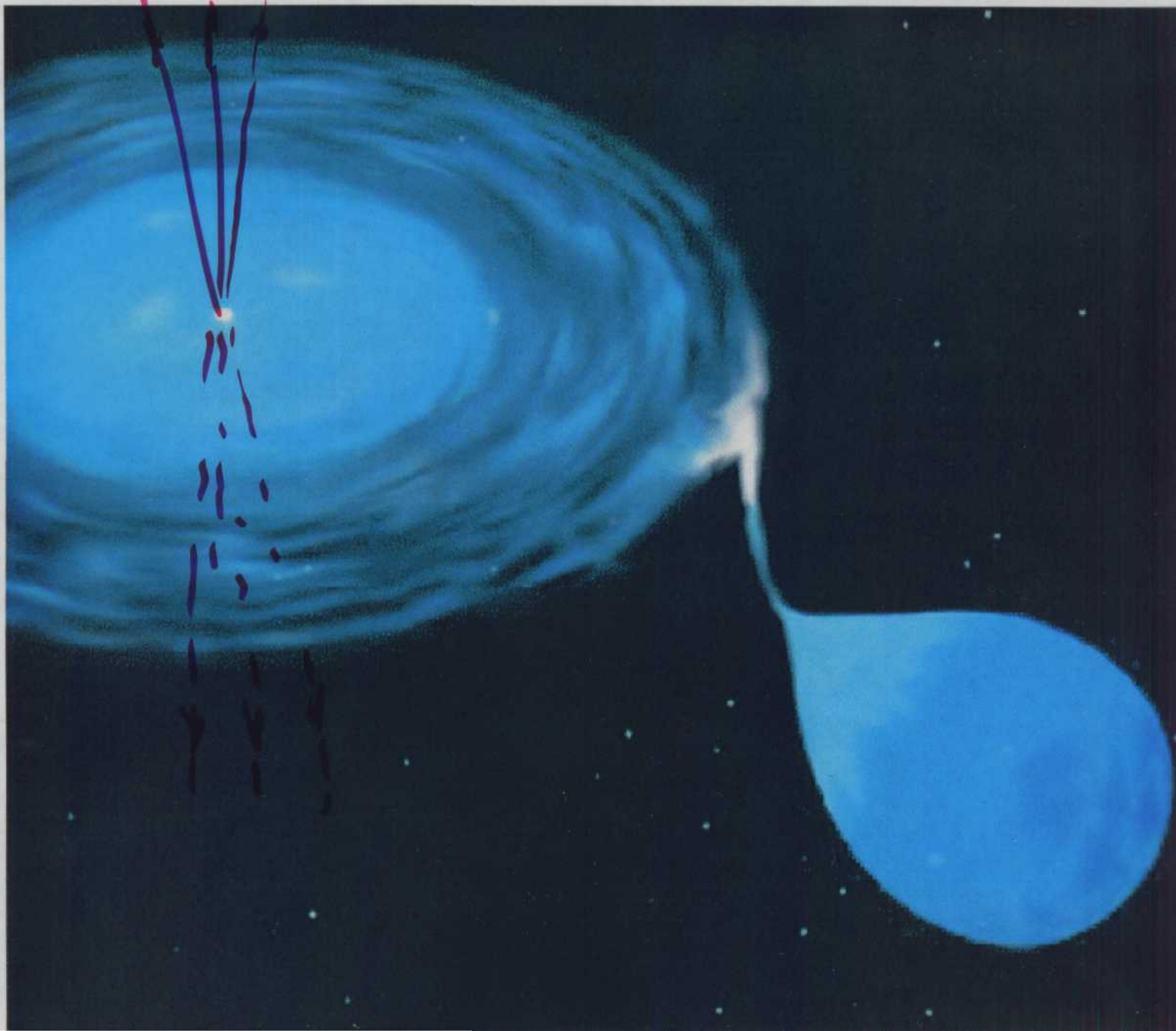
Note: The optical companions are indicated by the shaded regions and are shown filling their critical Roche lobes.

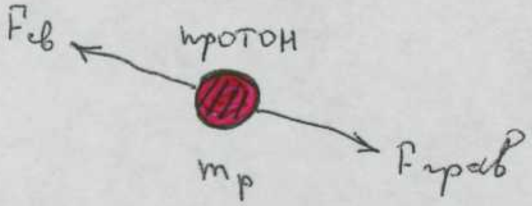
a) Gies and Bolton (1986);
b) Hutchings *et al.* (1987);
c) Cowley *et al.* (1983);
d) McClintock and Remillard (1986).

Source: McClintock (1991).

В настоящее время более 30 кандидатов в ЧД в двойных системах.

Малое количество как ⇒ малого времени нахождения в фазе красного гиганта. ($t \approx 10^4$ лет)





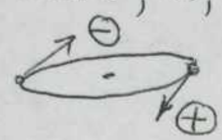
Поток в-ва на ЧД внешний параметр $\Rightarrow$

$P > P_{Эддингтона}$ — выброс в-ва

$P < P_{Эддингтона}$ — тяготение доминирует.

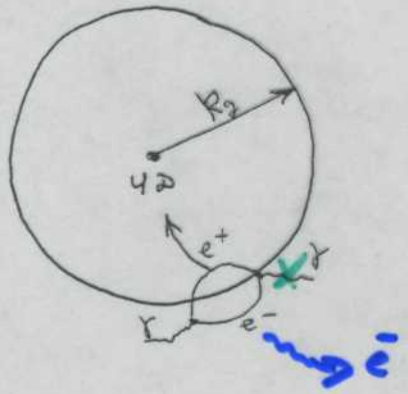
При падении протона ($m \simeq 10^3$ МэВ) в ЧД излучается $\Delta E \simeq 0{,}06 \cdot 10^3 \simeq 60$ МэВ !

В двойных системах наблюдается:

- быстропеременное изл-ие (оптическое, X, γ) $\ell \simeq \frac{\tau}{c} \sim r_g$
- доплеровское смещение ($\pm$)  $\vartheta \sim \vartheta_{ОТО}$.
- переменность видимой звезды.
- невидимый компонент с $>$ой массой.
- выброс струй вещества из 2х систем и галактик.

Испарение чёрных дыр.

$$\varphi(r \to R_g) \to \infty$$

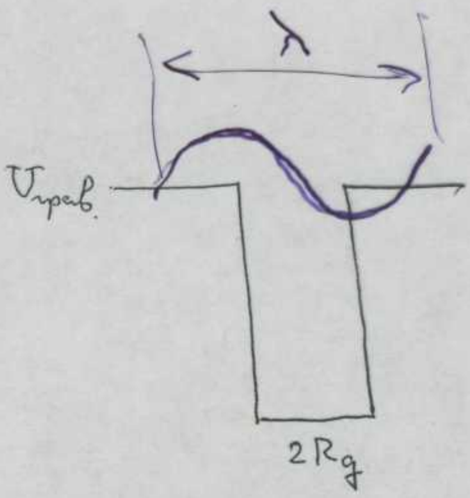


Зельдович: излучение частиц ЧД.

Хокинг: ЧД излучает как АЧТ.!

с температурой:

$$T_{ЧД} \simeq \frac{1}{M}$$



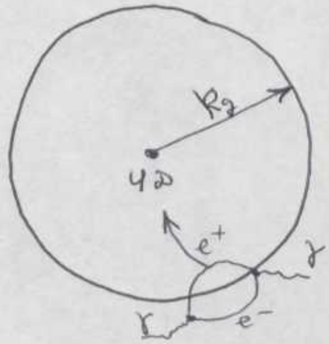
$$t_{исп} = 10^{10}\left(\frac{M}{10^{15}\,гр}\right)^3, \text{ лет.}$$

$$\lambda_{изл} \simeq R_g$$

$\lambda > 2R_g \to$ не связана ЧД.

Испарение чёрных дыр.

$$\varphi(r \to R_g) \to \infty$$

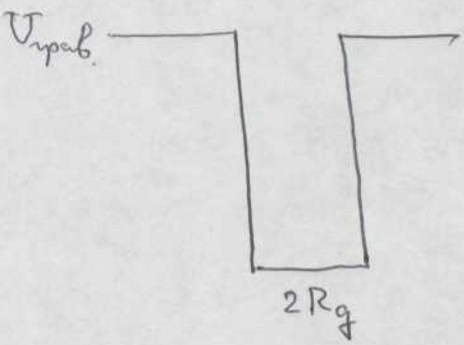


Зельдович: излучение частиц ЧД.

Хокинг: ЧД излучает как АЧТ.

с температурой:

$$T_{ЧД} =$$



$$t_{исп} = 10^{10}\left(\frac{M}{10^{15}\,гр}\right)^3, \text{ лет.}$$

$\lambda > 2R_g \to$ не связана ЧД.

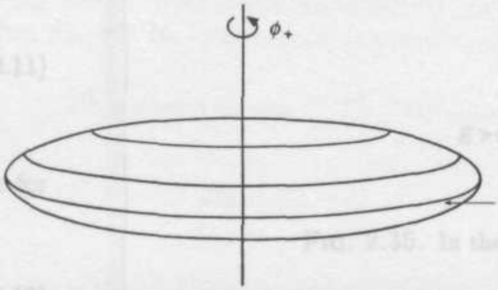


FIG. 2.33. An embedding diagram in flat Euclidean three-space of the intrinsic geometry of a Kerr black hole. Note the flattening at the poles.

$$\begin{aligned}\mathcal{A} &= \int\int d\theta d\phi_+ \{\det[g_{ab}(\theta,\phi_+)]\}^{\frac{1}{2}} \\ &= 8\pi M r_+ \\ &= 8\pi M[M + (M^2 - a^2)^{\frac{1}{2}}].\end{aligned} \tag{2.5.2.11}$$

The quantity $\mathcal{A}$ is called the *area of the event horizon*; note that is the same for any section through the horizon, of the form $\{\theta, \phi_+, u_+ = f(\theta,\phi)\}$, namely

$$\int_0^{\pi}\int_0^{2\pi} d\theta d\phi_+ (2Mr_+ \sin\theta - a\frac{\partial f}{\partial \phi_+}\sin\theta) = 8\pi M r_+. \tag{2.5.2.12}$$

The area and its variations under small perturbations of the black hole (such as infalling test particles) will be studied in the following subsection 2.5.3.

2.5.3 *Extraction of rotational energy; reversible and irreversible transformations*

Consider now a process known as the Penrose process for extracting energy from rotating black holes. Note that in a stationary space–time, with time-translation Killing vector field k^a, the quantity $E = -k_a p^a$ is constant along the geodesic path of a test particle with four-momentum p^a. This can be verified directly from the geodesic equation and Killing vector equation:

$$\begin{aligned}p^b(k_a p^a)_{;b} &= p^b p^a k_{a;b} + k_a p^b p^a{}_{;b} \\ &= 0,\end{aligned} \tag{2.5.3.1}$$

where the first term vanishes by the Killing equation $k_{(a;b)} = 0$ and the second term vanishes by the geodesic equation. For a particle at large distances from the black hole, where the geometry is nearly flat, E is essentially the relativistic energy (mass) γ of the particle, where $\gamma = \gamma(v) = (1 - v^2)^{-1/2}$, with v being the speed of the particle near infinity. Since E is conserved in the particle's

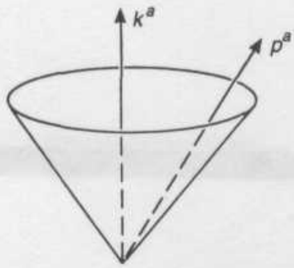


FIG. 2.34. Outside the ergosurface, the energy $E = -k_a p^a > 0$ for future-directed momentum p^a.

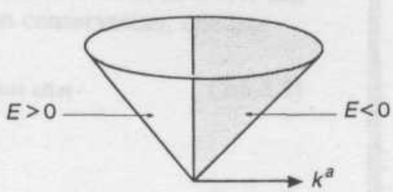


FIG. 2.35. In the ergoregion, $E = -k_a p^a$ can have either sign.

motion, one also defines E to be the particle's energy in general. Now k^a is timelike outside the ergosurface, so that $E > 0$ there for a future-directed timelike momentum p^a (see Fig. 2.34). But in the ergoregion k^a is spacelike, and $E = -k_a p^a$ can have either sign (see Fig. 2.35). The condition $E < 0$ means that the particle's binding energy exceeds its rest energy. One can now consider a process which utilizes such a particle.

As in Fig. 2.36, take the process in which a particle A is sent on a timelike

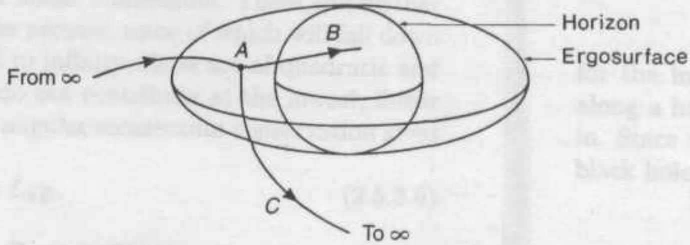


FIG. 2.36. This process will extract energy from the black hole, provided that $E_B < 0$.

geodesic from infinity, so that it has positive energy E_A, such that it enters the ergoregion. There it explodes into two particles, B and C, where B then falls inside the horizon, while C moves back out to infinity. As described above, it can be arranged that $E_B < 0$. By local energy–momentum conservation, one has

$$(p_A^a)_{\text{just before explosion}} = (p_B^a + p_C^a)_{\text{just after}}. \tag{2.5.3.2}$$

Hence

$$\begin{aligned} E_A &= E_B + E_C, \\ L_{zA} &= L_{zB} + L_{zC}, \end{aligned} \tag{2.5.3.3}$$

where $L_z = p_a m^a$, with m^a being the rotational Killing vector; here L_z is the angular momentum of the test particle. The angular momentum result will be needed shortly. One can also use global energy conservation (Misner *et al.* 1973), noting that energy is additive for widely separated non-interacting systems, to write

$$M_{\text{hole,initial}} + E_A = M_{\text{hole,final}} + E_C \tag{2.5.3.4}$$

and hence deduce

$$\delta M_{hole} = E_B. \tag{2.5.3.5}$$

There is also a global conservation law for linear momentum. There will further be gravitational waves generated during the process, some of which will fall down the hole, and others which will be emitted to infinity; these are of quadratic and higher order in the small masses, and so do not contribute at the lowest, linear order being studied here. Similarly, global angular momentum conservation gives

$$\delta L_{z\text{hole}} = L_{zB}. \tag{2.5.3.6}$$

As remarked above, one can arrange that $E_B < 0$. Then

$$\delta M_{\text{hole}} < 0 \tag{2.5.3.7}$$

and

$$E_C > E_A. \tag{2.5.3.8}$$

In such a process, observers near infinity can gain energy from the rotating black hole.

One can now find efficiency bounds on a process of the above type. That is, one can find constraints on the change of the black hole's energy and angular momentum, when a particle falls through the horizon on a geodesic. Let ℓ^a be the Killing vector

$$\ell^a = k^a + \Omega m^a \tag{2.5.3.9}$$

tangent to the null generators of the event horizon, as in Fig. 2.37(a). One sees

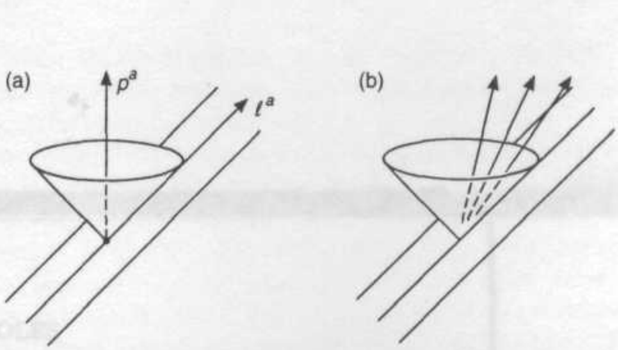


FIG. 2.37. In (a), a massive particle falling into the Kerr black hole leads to an irreversible transformation with $\delta\mathcal{A} > 0$, where $\mathcal{A}$ is the area of the black hole. In (b), a sequence of irreversible transformations can only approximate reversibility.

from Fig. 2.37(a) that

$$p_a \ell^a \leq 0, \tag{2.5.3.10}$$

with equality only if p^a is directed along ℓ^a. From the definitions $E = -p_a k^a$ and $L_z = p_a m^a$ and from eqn (2.5.3.9), one finds

$$E \geq \Omega L_z \tag{2.5.3.11}$$

for the infalling particle. Equality holds only for a massless particle directed along a horizon null generator, in which case the particle is not actually falling in. Since $\Omega = a/2Mr_+$, one has for an infinitesimal transformation of a Kerr black hole by means of test particles:

$$\delta M \geq \frac{\mathbf{a}.\,\delta\mathbf{S}}{2Mr_+}, \tag{2.5.3.12}$$

where

$$\mathbf{S} = M\mathbf{a}. \tag{2.5.3.13}$$

In the Penrose process, $\delta M < 0$ implies that $\mathbf{a}.\,\delta\mathbf{S} < 0$. Thus one can only gain energy by slowing down the black hole's rotation; the extracted energy comes from the rotational energy of the black hole. The inequality (2.5.3.12) can most effectively be expressed in terms of variations of the area of the black hole. As shown in subsection 2.5.2, the area of the horizon of a Kerr geometry is

$$\mathcal{A} = 8\pi M[M + (M^2 - a^2)^{\frac{1}{2}}]. \tag{2.5.3.14}$$

The variation of $\mathcal{A}$ is then

$$\delta\mathcal{A} = \frac{8\pi}{(M^2 - a^2)^{\frac{1}{2}}}(2Mr_+\delta M - \mathbf{a}.\,\delta\mathbf{S}). \tag{2.5.3.15}$$

Hence, from the inequality (2.5.3.12), one has

$$\delta\mathcal{A} \geq 0 \tag{2.5.3.16}$$

in any process involving test particles. A transformation with $\delta\mathcal{A} = 0$ is *reversible*. A transformation with $\delta\mathcal{A} > 0$ is *irreversible*. As can be seen from Fig. 2.37b, transformations using massive particles must be irreversible; one can only approximate reversibility.

The expression (2.5.3.14) for the area can be rewritten as

$$M^2 = \frac{\mathcal{A}}{16\pi} + \frac{4\pi S^2}{\mathcal{A}}. \tag{2.5.3.17}$$

The first term on the right-hand side is the 'irreducible contribution' to M^2; this can be held almost constant by making almost reversible transformations. The second term on the right-hand side of eqn (2.5.3.17) is the 'rotational energy contribution' to M^2; this can be reduced to zero reversibly. Hence, if we define the irreducible mass of the black hole as

$$M_{ir} = \sqrt{(\mathcal{A}/16\pi)}, \tag{2.5.3.18}$$

we cannot reduce M below M_{ir}. We can only extract the maximum energy $M - M_{ir}$ by a sequence of reversible transformations leading to $a = 0$ (the Schwarzschild metric). Thus the Schwarzschild geometry can be regarded as a ground state of the Kerr geometries.

One can consider more general uncharged rotating stationary black holes, which might have an equilibrium distribution of rotating matter outside the horizon. It can be shown (Carter 1973, 1979) then that

$$\delta M = \frac{\kappa}{8\pi}\delta\mathcal{A} + \mathbf{\Omega}.\delta\mathbf{S}, \tag{2.5.3.19}$$

the *first law of black hole mechanics* (which can be generalized to the charged case). Here κ is known as the *surface gravity*, and is constant over the horizon (the *zeroth law* of black-hole mechanics). For the Kerr case, κ can be read off from eqn (2.5.3.15). For general stationary black holes, the angular velocity Ω is also constant over the horizon ('rigidity'). Up to constant factors, κ corresponds to a temperature for the black hole, and $\mathcal{A}$ corresponds to an entropy. Further, for any dynamical black hole, $\delta\mathcal{A} \geq 0$, the *second law*, from the global theory of black holes (Penrose 1971; Hawking 1972a; Hawking and Ellis 1973). The *third law* states that one cannot attain an extremal ($\kappa = 0$) black hole such as the Kerr metric with $a = M$ by a finite number of steps, starting from $a < M$.

Quantum field theory (Hawking 1975) shows that black holes emit particles thermally, with temperature

$$T = \kappa\hbar/2\pi kc, \tag{2.5.3.20}$$

where k is Boltzmann's constant and κ has dimensions of acceleration. The corresponding entropy is

$$S = c^3 A / 4G\hbar. \tag{2.5.3.21}$$

These results show that a black hole can be in equilibrium with a heat bath (Hartle and Hawking 1976), and one can determine the temperature of the black hole and its contribution to the entropy.

The area increase theorem $\delta A \geq 0$ (Hawking and Ellis 1973) makes the assumption of cosmic censorship, i.e. that all singularities of the space–time are hidden inside the black hole. One can use this theorem, assuming cosmic censorship, to set bounds on the fraction of the initial mass–energy which is emitted in gravitational waves in a collision and merger of two black holes. Suppose (for simplicity) that one starts with a pair of widely separated Schwarzschild black holes at rest, each having mass M_1, and that finally one has a single black hole at rest, of mass M_2, together with gravitational radiation propagating out towards future null infinity $\mathcal{I}^+$. Let $A_{\text{final}} = 16\pi(M_2)^2$ be the area of the event horizon of the final black hole, which is assumed to be a Schwarzschild black hole, and let $A_1 = 16\pi(M_1)^2$ be the initial area of each black hole. Then the area increase theorem gives $A_{\text{final}} > 2A_1$, i.e.

$$(M_2)^2 > 2(M_1)^2, \tag{2.5.3.22}$$

i.e. $M_2 > \sqrt{2}M_1$. The energy M_2 must also be less than or equal to the initial energy $2M$, since the sum of M_2 and the final energy in gravitational waves must equal $2M$. The efficiency ϵ of production of gravitational waves is given by

$$\epsilon = \frac{(2M_1 - M_2)}{2M_1} = 1 - \frac{M_2}{2M_2} < 1 - \frac{1}{\sqrt{2}} \simeq 29\%. \tag{2.5.3.23}$$

The numerical calculations of Smarr (1979) show that, for initial data given by two time-symmetric black holes (i.e. two black holes initially at rest), the true efficiency of gravitational wave production is several orders of magnitude smaller than the bound (2.5.3.23) on ϵ.

If one takes the other extreme, in which two black holes collide at or near the speed of light, then Penrose (1974) (see Chapter 6) has shown that there is the same bound of 29% on the efficiency of production of gravitational waves, assuming cosmic censorship. However, in this case the bound of 29% based on initial data gives a much more reasonable estimate of the exact efficiency. It is found in Chapter 8 that a good estimate of the radiation described by the first two angular harmonics corresponds to 16.8% efficiency for the speed-of-light head-on collision.

2.6 Geodesic equations in the Kerr metric

Geodesics in the Kerr metric are an example of perturbations of the Kerr geometry. For example, as we shall see in Chapter 3, if one has a low-mass ('small') Kerr black hole moving in a background universe, then the 'small' black hole moves approximately on a timelike geodesic of the background universe, and its