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Hamze

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(54) BACKPROPAGATION USING PARAMETRIZING ANGLES OF UNITARY MATRIX

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(51) **Int. Cl.**
G06N 3/084 (2023.01)
G06F 17/16 (2006.01)

(52) **U.S. Cl.**
CPC ***G06N 3/084*** (2013.01); ***G06F 17/16*** (2013.01)

(58) **Field of Classification Search**
CPC G06N 3/084; G06N 3/044; G06N 3/045; G06N 20/00; G06F 17/16
See application file for complete search history.

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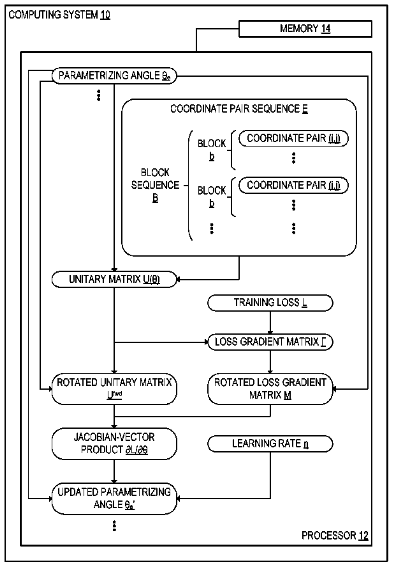
(57) ABSTRACT

A computing system, including a processor configured to train a machine learning model in a plurality of backpropagation iterations. Each backpropagation iteration may include generating a coordinate pair sequence. Each coordinate pair may be unique within the coordinate pair sequence and may include non-matching coordinates. The backpropagation iteration may further include receiving parametrizing angles respectively associated with the coordinate pairs. The backpropagation iteration may further include computing a unitary matrix parametrized by the parametrizing angles, computing a loss gradient matrix, and computing a Jacobian-vector product (JVP). Computing the JVP may include computing a rotated unitary matrix and a rotated loss gradient matrix for each coordinate pair. The JVP may be computed from the rotated unitary matrix and the rotated loss gradient matrix. The backpropagation iteration may further include updating the parametrizing angles based at least in part on the JVP.

16 Claims, 18 Drawing Sheets



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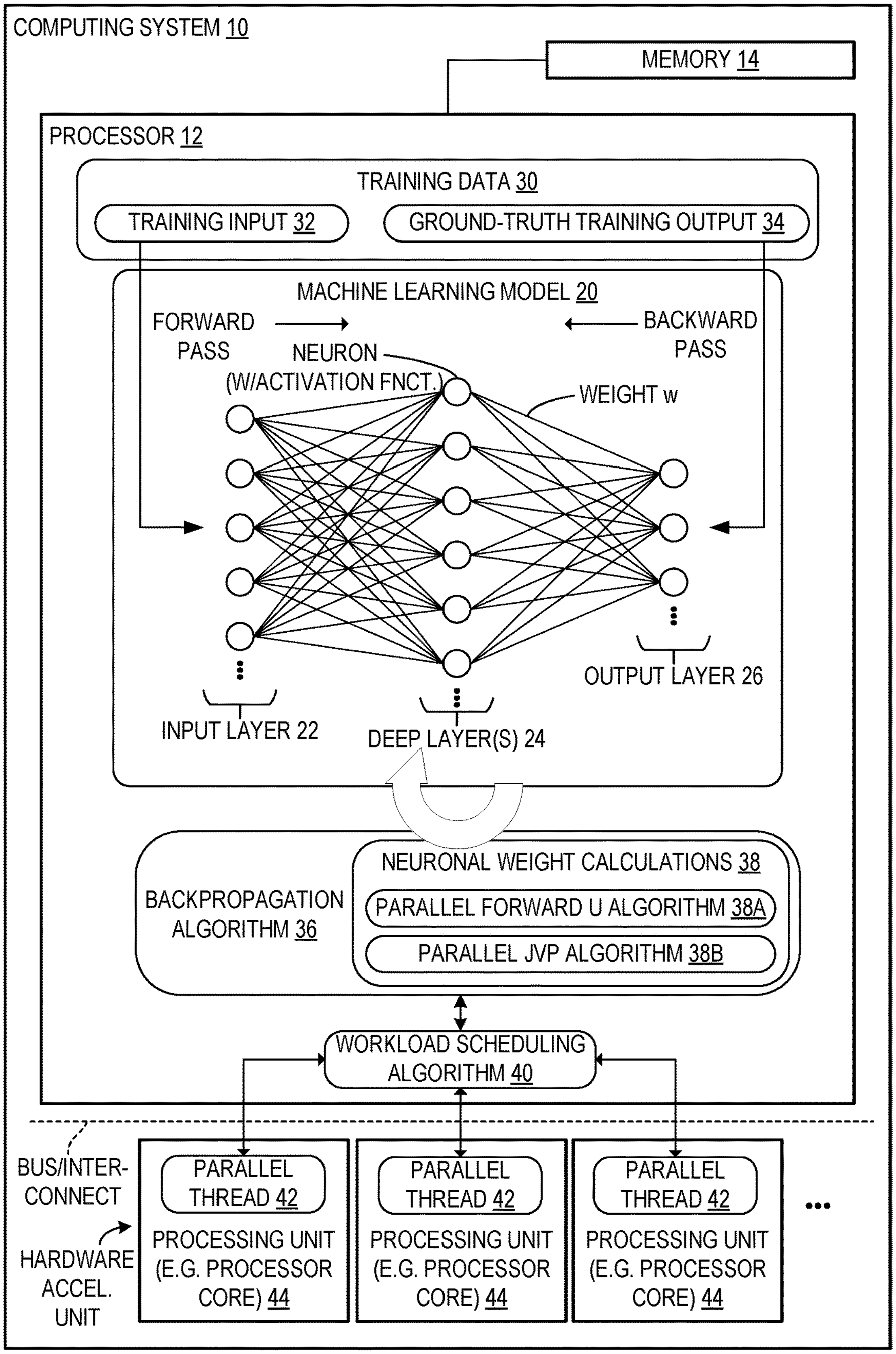

COMPUTING SYSTEM 10
MEMORY 14
PROCESSOR 12
TRAINING DATA 30
TRAINING INPUT 32
GROUND-TRUTH TRAINING OUTPUT 34
MACHINE LEARNING MODEL 20
FORWARD PASS
BACKWARD PASS
NEURON (W/ACTIVATION FNCT.)
WEIGHT w
INPUT LAYER 22
DEEP LAYER(S) 24
OUTPUT LAYER 26
BACKPROPAGATION ALGORITHM 36
NEURONAL WEIGHT CALCULATIONS 38
PARALLEL FORWARD U ALGORITHM 38A
PARALLEL JVP ALGORITHM 38B
WORKLOAD SCHEDULING ALGORITHM 40
BUS/INTER-CONNECT
HARDWARE ACCEL. UNIT
PARALLEL THREAD 42
PROCESSING UNIT (E.G. PROCESSOR CORE) 44
PARALLEL THREAD 42
PROCESSING UNIT (E.G. PROCESSOR CORE) 44
PARALLEL THREAD 42
PROCESSING UNIT (E.G. PROCESSOR CORE) 44
...

FIG. 1A

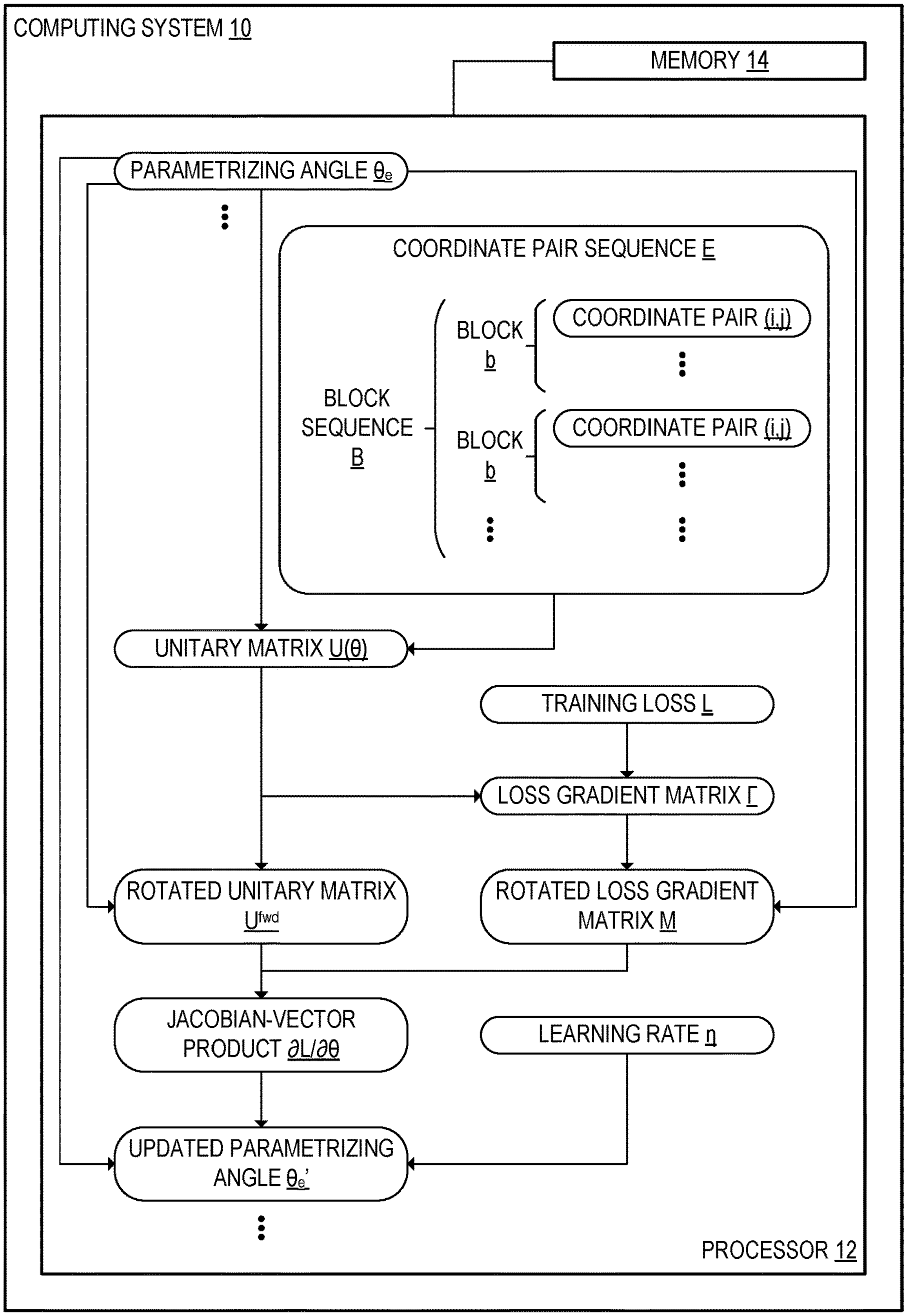

COMPUTING SYSTEM 10
MEMORY 14
PARAMETRIZING ANGLE θ_e
COORDINATE PAIR SEQUENCE E
BLOCK SEQUENCE B
BLOCK b
COORDINATE PAIR (i,j)
BLOCK b
COORDINATE PAIR (i,j)
UNITARY MATRIX $U(\theta)$
TRAINING LOSS L
LOSS GRADIENT MATRIX Γ
ROTATED UNITARY MATRIX U^{fwd}
ROTATED LOSS GRADIENT MATRIX M
JACOBIAN-VECTOR PRODUCT $\partial L/\partial\theta$
LEARNING RATE η
UPDATED PARAMETRIZING ANGLE θ_e'
PROCESSOR 12

FIG. 1B

100

Algorithm 1 Forward $\boldsymbol{U}$

```
Input:
    Sequence of coordinate pairs $E = (e_1, \ldots, e_{\mathcal{N}})$
    Parametrizing angles $\boldsymbol{\theta} = (\theta_{e_1}, \ldots, \theta_{e_{\mathcal{N}}})$
Output:
    $\boldsymbol{U}(\boldsymbol{\theta})$
$\boldsymbol{U} \leftarrow \boldsymbol{I}_n$
for $e \in$ reversed$(E)$ do
    $(i, j) \leftarrow e$
    $\boldsymbol{r}_i \leftarrow \cos\theta_{ij}\boldsymbol{U}_{i:} - \sin\theta_{ij}\boldsymbol{U}_{j:}$
    $\boldsymbol{r}_j \leftarrow \sin\theta_{ij}\boldsymbol{U}_{i:} + \cos\theta_{ij}\boldsymbol{U}_{j:}$
    $\boldsymbol{U}_{i:} \leftarrow \boldsymbol{r}_i$
    $\boldsymbol{U}_{j:} \leftarrow \boldsymbol{r}_j$
end for
```

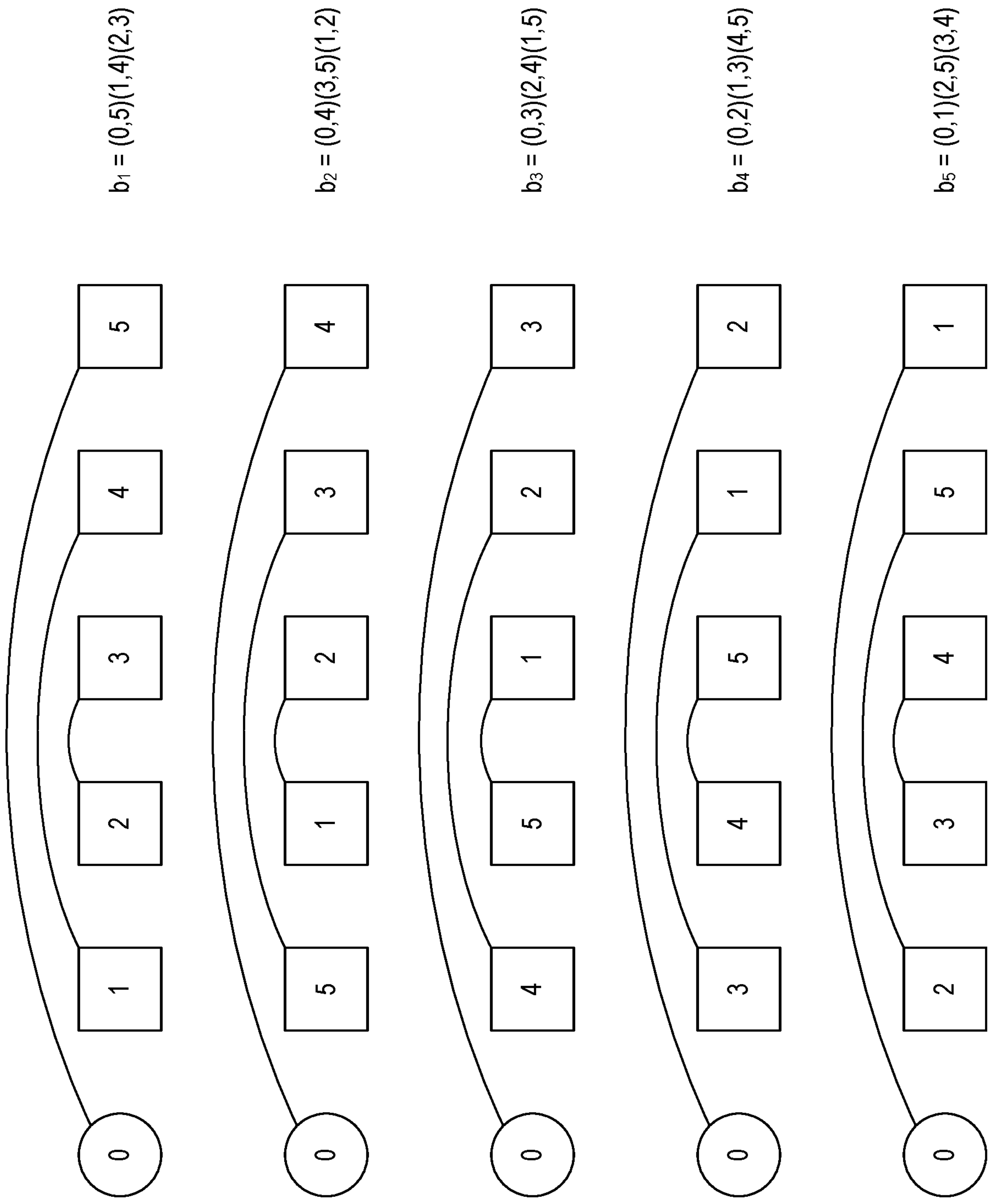
FIG. 2

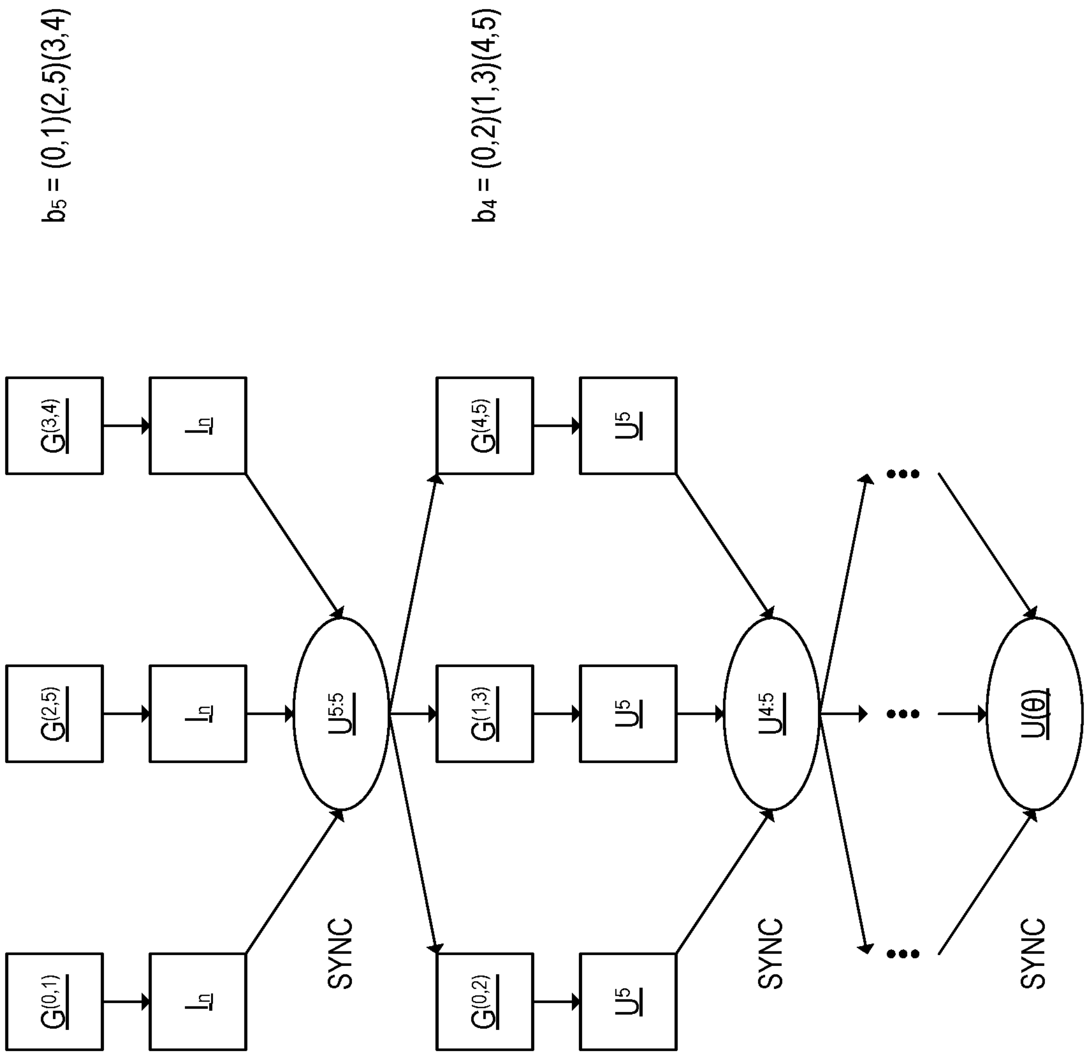
FIG. 3

110

```
Algorithm 2 Parallel Forward $\boldsymbol{U}$
Input:
    Block sequence $B = (b_1, \ldots, b_{n-1})$ such that $E$ is round-robin
    Parametrizing angles $\boldsymbol{\theta} = (\theta_{e_1}, \ldots, \theta_{e_N})$
Output:
    $\boldsymbol{U}(\boldsymbol{\theta})$
$\boldsymbol{U} \leftarrow \boldsymbol{I}_n$
for $b \in$ reversed($B$) do
    for $e \in b$ do                                        ▷ This block is fully parallel
        $(i, j) \leftarrow e$
        $\boldsymbol{r}_i \leftarrow \cos\theta_{ij}\boldsymbol{U}_{i:} - \sin\theta_{ij}\boldsymbol{U}_{j:}$
        $\boldsymbol{r}_j \leftarrow \sin\theta_{ij}\boldsymbol{U}_{i:} + \cos\theta_{ij}\boldsymbol{U}_{j:}$
        $\boldsymbol{U}_{i:} \leftarrow \boldsymbol{r}_i$
        $\boldsymbol{U}_{j:} \leftarrow \boldsymbol{r}_j$
    end for
    Synchronize parallel operations
end for
```

FIG. 4


$b_1 = (0,5)(1,4)(2,3)$
$b_2 = (0,4)(3,5)(1,2)$
$b_3 = (0,3)(2,4)(1,5)$
$b_4 = (0,2)(1,3)(4,5)$
$b_5 = (0,1)(2,5)(3,4)$
0 1 2 3 4 5
0 5 1 2 3 4
0 4 5 1 2 3
0 3 4 5 1 2
0 2 3 4 5 1



$b_5 = (0,1)(2,5)(3,4)$
$b_4 = (0,2)(1,3)(4,5)$
$\underline{G}^{(0,1)}$
$\underline{G}^{(2,5)}$
$\underline{G}^{(3,4)}$
$\underline{I}_n$
$\underline{I}_n$
$\underline{I}_n$
$\underline{U}^{5:5}$
SYNC
$\underline{G}^{(0,2)}$
$\underline{G}^{(1,3)}$
$\underline{G}^{(4,5)}$
$\underline{U}^5$
$\underline{U}^5$
$\underline{U}^5$
$\underline{U}^{4:5}$
SYNC
$\underline{U}(\theta)$
SYNC

FIG. 5

120

120A

```
Algorithm 3 Parallel JVP
Input:
    Block sequence $B = (b_1, \ldots, b_{n-1})$ such that $E$ is round-robin
    Parametrizing angles $\boldsymbol{\theta} = (\theta_{e_1}, \ldots, \theta_{e_{\mathcal{N}}})$
    $\boldsymbol{U}$ computed by Forward Algorithm 2
    Loss gradient with respect to outputs $\boldsymbol{\Gamma}$
Output:
    Loss gradient with respect to parameters: $\frac{\partial \mathcal{L}}{\partial \boldsymbol{\theta}}$
Initialize:
    $\boldsymbol{U}^{\text{fwd}} \leftarrow \boldsymbol{U}$
    $\boldsymbol{M} \leftarrow \boldsymbol{\Gamma}^T$
    $\boldsymbol{A} \leftarrow$ empty $N/2 \times N$ matrix
for $b \in$ reversed$(B)$ do
    for $e \in b$ do                                        ▷ Parallel $\boldsymbol{U}^{\text{fwd}}$ update: $O(1)$
        $(i, j) \leftarrow e$
        $\boldsymbol{c}_i \leftarrow \cos\theta_{ij} \boldsymbol{U}^{\text{fwd}}_{:i} - \sin\theta_{ij} \boldsymbol{U}^{\text{fwd}}_{:j}$
        $\boldsymbol{c}_j \leftarrow \sin\theta_{ij} \boldsymbol{U}^{\text{fwd}}_{:i} + \cos\theta_{ij} \boldsymbol{U}^{\text{fwd}}_{:j}$
        $\boldsymbol{U}^{\text{fwd}}_{:i} \leftarrow \boldsymbol{c}_i$
        $\boldsymbol{U}^{\text{fwd}}_{:j} \leftarrow \boldsymbol{c}_j$
    end for
    Synchronize
```

TO FIG. 6B

FIG. 6A

FROM FIG. 6A

120

120B

```
for $e \in b$ do                                              ▷ Parallel $\boldsymbol{M}$ update: $O(1)$
    $(i, j) \leftarrow e$
    $\boldsymbol{r}_i \leftarrow \cos\theta_{ij}\boldsymbol{M}_{i:} - \sin\theta_{ij}\boldsymbol{M}_{j:}$
    $\boldsymbol{r}_j \leftarrow \sin\theta_{ij}\boldsymbol{M}_{i:} + \cos\theta_{ij}\boldsymbol{M}_{j:}$
    $\boldsymbol{M}_{i,:} \leftarrow \boldsymbol{r}_i$
    $\boldsymbol{M}_{j,:} \leftarrow \boldsymbol{r}_j$
end for
Synchronize
for $e \in b$ do                                              ▷ Parallel $\boldsymbol{A}$ assignment: $O(1)$
    $(i, j) \leftarrow e$
    $m \leftarrow m(e)$
    for $l \in \{0, \ldots, n-1\}$ do                          ▷ Parallel
        $A_{ml} \leftarrow M_{il}u_{lj} - M_{jl}u_{li}$
    end for
end for
Synchronize
$\boldsymbol{d} \leftarrow \boldsymbol{A}1_n$                  ▷ Sum rows of $\boldsymbol{A}$: $O(\log n)$
for $e \in b$ do                                              ▷ Parallel JVP assignment: $O(1)$
    $(i, j) \leftarrow e$
    $m \leftarrow m(e)$
    $\frac{\partial \mathcal{L}}{\partial \theta_e} \leftarrow d_m$
end for
end for
```

FIG. 6B

130

Algorithm 4 Parallel Forward $\boldsymbol{U} \in \mathrm{U}(n)$

```
Input:
    Block sequence $B = (b_1, \ldots, b_{n-1})$ such that $E$ is round-robin
    Parametrizing angles $\boldsymbol{\theta} = (\theta_{e_1}, \ldots, \theta_{e_N})$
    Parametrizing phase angles $\boldsymbol{\phi} = (\phi_{e_1}, \ldots, \phi_{e_N})$
Output:
    $\boldsymbol{U}(\boldsymbol{\theta})$
$U \leftarrow \boldsymbol{I}_n$
for $b \in \mathrm{reversed}(B)$ do
    for $e \in b$ do
        $(i, j) \leftarrow e$
        $\boldsymbol{r}_i \leftarrow e^{i\phi_{ij}} \cos\theta_{ij} \boldsymbol{U}_{i:} - \sin\theta_{ij} \boldsymbol{U}_{j:}$
        $\boldsymbol{r}_j \leftarrow e^{i\phi_{ij}} \sin\theta_{ij} \boldsymbol{U}_{i:} + \cos\theta_{ij} \boldsymbol{U}_{j:}$
        $\boldsymbol{U}_{i:} \leftarrow \boldsymbol{r}_i$
        $\boldsymbol{U}_{j:} \leftarrow \boldsymbol{r}_j$
    end for
    Synchronize parallel operations
end for
```

FIG. 7

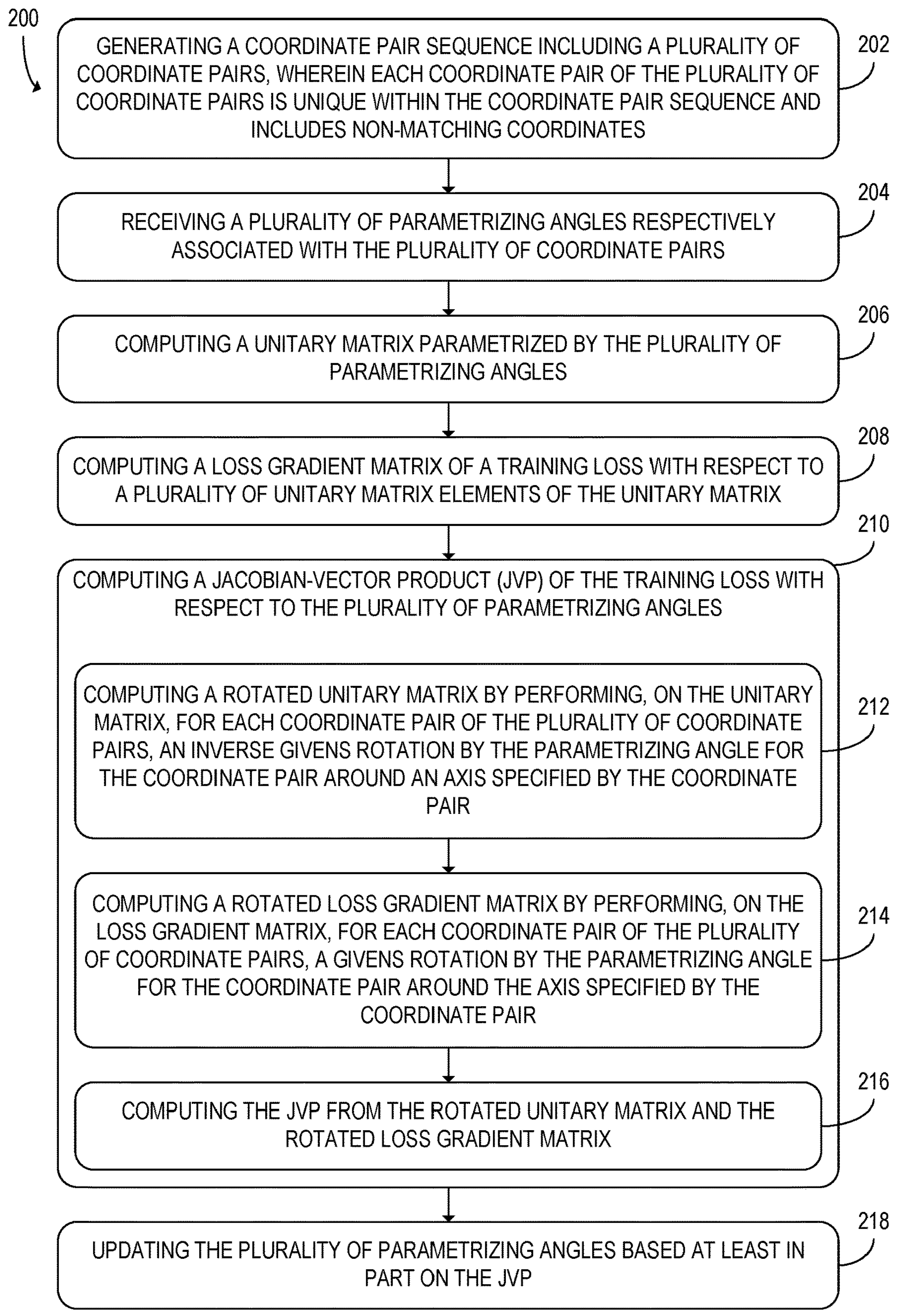

200
GENERATING A COORDINATE PAIR SEQUENCE INCLUDING A PLURALITY OF COORDINATE PAIRS, WHEREIN EACH COORDINATE PAIR OF THE PLURALITY OF COORDINATE PAIRS IS UNIQUE WITHIN THE COORDINATE PAIR SEQUENCE AND INCLUDES NON-MATCHING COORDINATES
202
RECEIVING A PLURALITY OF PARAMETRIZING ANGLES RESPECTIVELY ASSOCIATED WITH THE PLURALITY OF COORDINATE PAIRS
204
COMPUTING A UNITARY MATRIX PARAMETRIZED BY THE PLURALITY OF PARAMETRIZING ANGLES
206
COMPUTING A LOSS GRADIENT MATRIX OF A TRAINING LOSS WITH RESPECT TO A PLURALITY OF UNITARY MATRIX ELEMENTS OF THE UNITARY MATRIX
208
COMPUTING A JACOBIAN-VECTOR PRODUCT (JVP) OF THE TRAINING LOSS WITH RESPECT TO THE PLURALITY OF PARAMETRIZING ANGLES
210
COMPUTING A ROTATED UNITARY MATRIX BY PERFORMING, ON THE UNITARY MATRIX, FOR EACH COORDINATE PAIR OF THE PLURALITY OF COORDINATE PAIRS, AN INVERSE GIVENS ROTATION BY THE PARAMETRIZING ANGLE FOR THE COORDINATE PAIR AROUND AN AXIS SPECIFIED BY THE COORDINATE PAIR
212
COMPUTING A ROTATED LOSS GRADIENT MATRIX BY PERFORMING, ON THE LOSS GRADIENT MATRIX, FOR EACH COORDINATE PAIR OF THE PLURALITY OF COORDINATE PAIRS, A GIVENS ROTATION BY THE PARAMETRIZING ANGLE FOR THE COORDINATE PAIR AROUND THE AXIS SPECIFIED BY THE COORDINATE PAIR
214
COMPUTING THE JVP FROM THE ROTATED UNITARY MATRIX AND THE ROTATED LOSS GRADIENT MATRIX
216
UPDATING THE PLURALITY OF PARAMETRIZING ANGLES BASED AT LEAST IN PART ON THE JVP
218

FIG. 8A

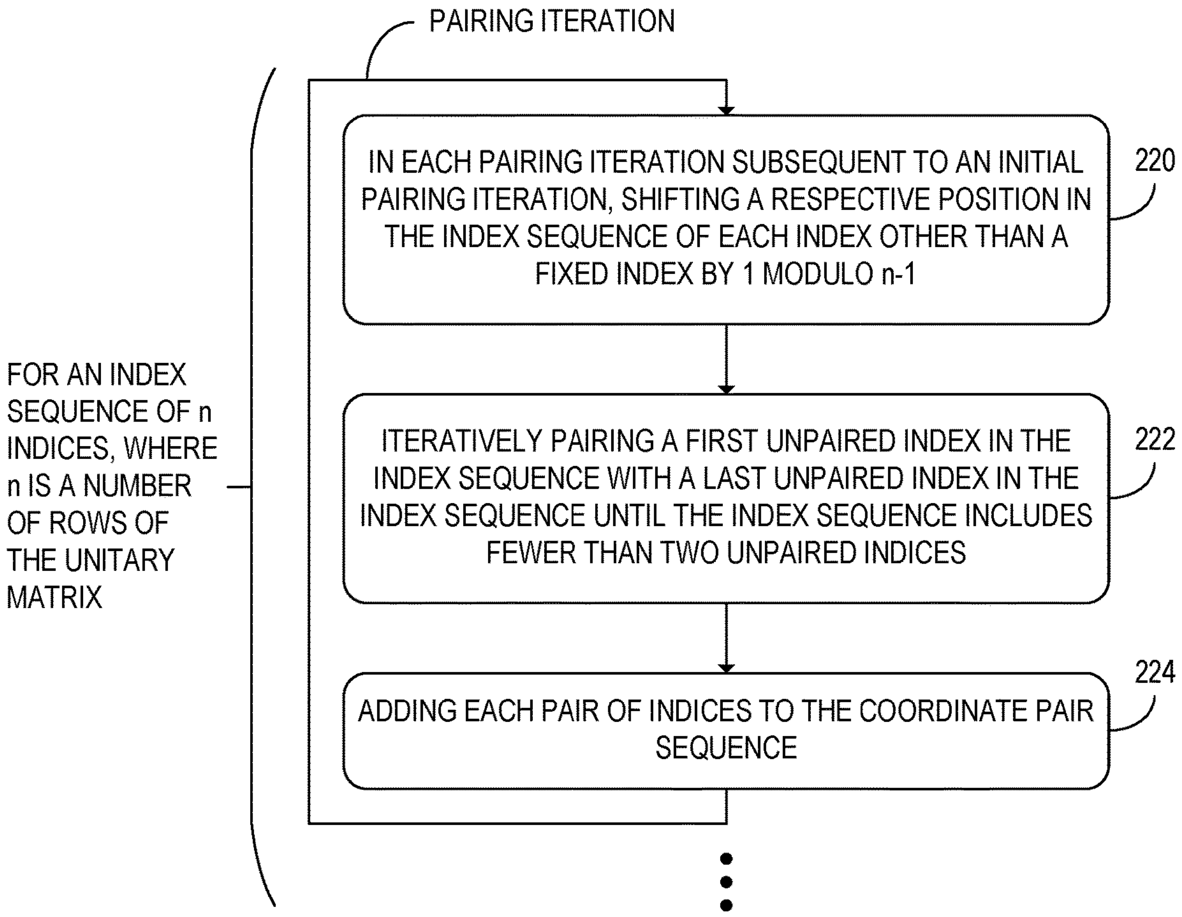

PAIRING ITERATION
IN EACH PAIRING ITERATION SUBSEQUENT TO AN INITIAL PAIRING ITERATION, SHIFTING A RESPECTIVE POSITION IN THE INDEX SEQUENCE OF EACH INDEX OTHER THAN A FIXED INDEX BY 1 MODULO n-1
220
ITERATIVELY PAIRING A FIRST UNPAIRED INDEX IN THE INDEX SEQUENCE WITH A LAST UNPAIRED INDEX IN THE INDEX SEQUENCE UNTIL THE INDEX SEQUENCE INCLUDES FEWER THAN TWO UNPAIRED INDICES
222
ADDING EACH PAIR OF INDICES TO THE COORDINATE PAIR SEQUENCE
224
FOR AN INDEX SEQUENCE OF n INDICES, WHERE n IS A NUMBER OF ROWS OF THE UNITARY MATRIX

FIG. 8B

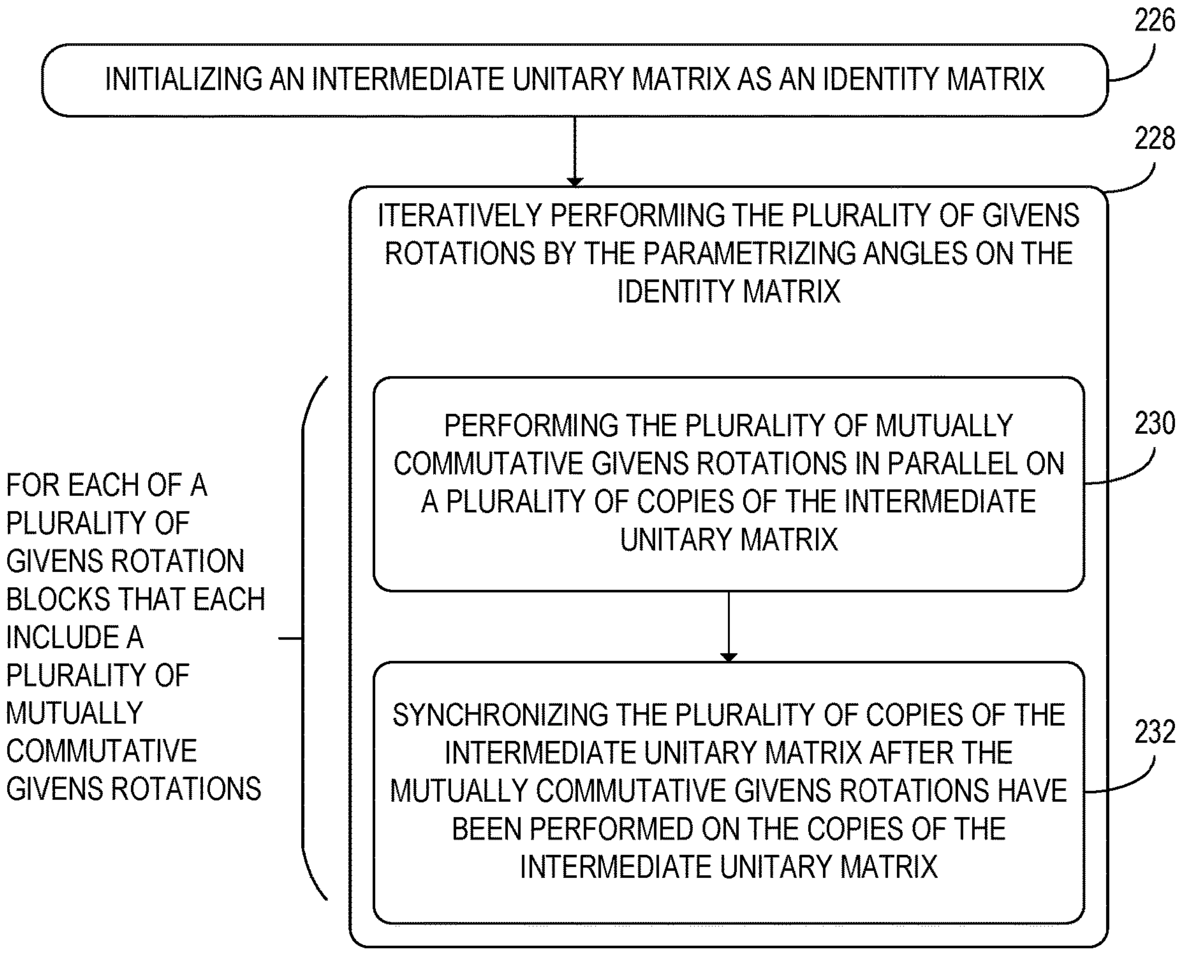

226
INITIALIZING AN INTERMEDIATE UNITARY MATRIX AS AN IDENTITY MATRIX
228
ITERATIVELY PERFORMING THE PLURALITY OF GIVENS ROTATIONS BY THE PARAMETRIZING ANGLES ON THE IDENTITY MATRIX
FOR EACH OF A PLURALITY OF GIVENS ROTATION BLOCKS THAT EACH INCLUDE A PLURALITY OF MUTUALLY COMMUTATIVE GIVENS ROTATIONS
230
PERFORMING THE PLURALITY OF MUTUALLY COMMUTATIVE GIVENS ROTATIONS IN PARALLEL ON A PLURALITY OF COPIES OF THE INTERMEDIATE UNITARY MATRIX
232
SYNCHRONIZING THE PLURALITY OF COPIES OF THE INTERMEDIATE UNITARY MATRIX AFTER THE MUTUALLY COMMUTATIVE GIVENS ROTATIONS HAVE BEEN PERFORMED ON THE COPIES OF THE INTERMEDIATE UNITARY MATRIX

FIG. 8C

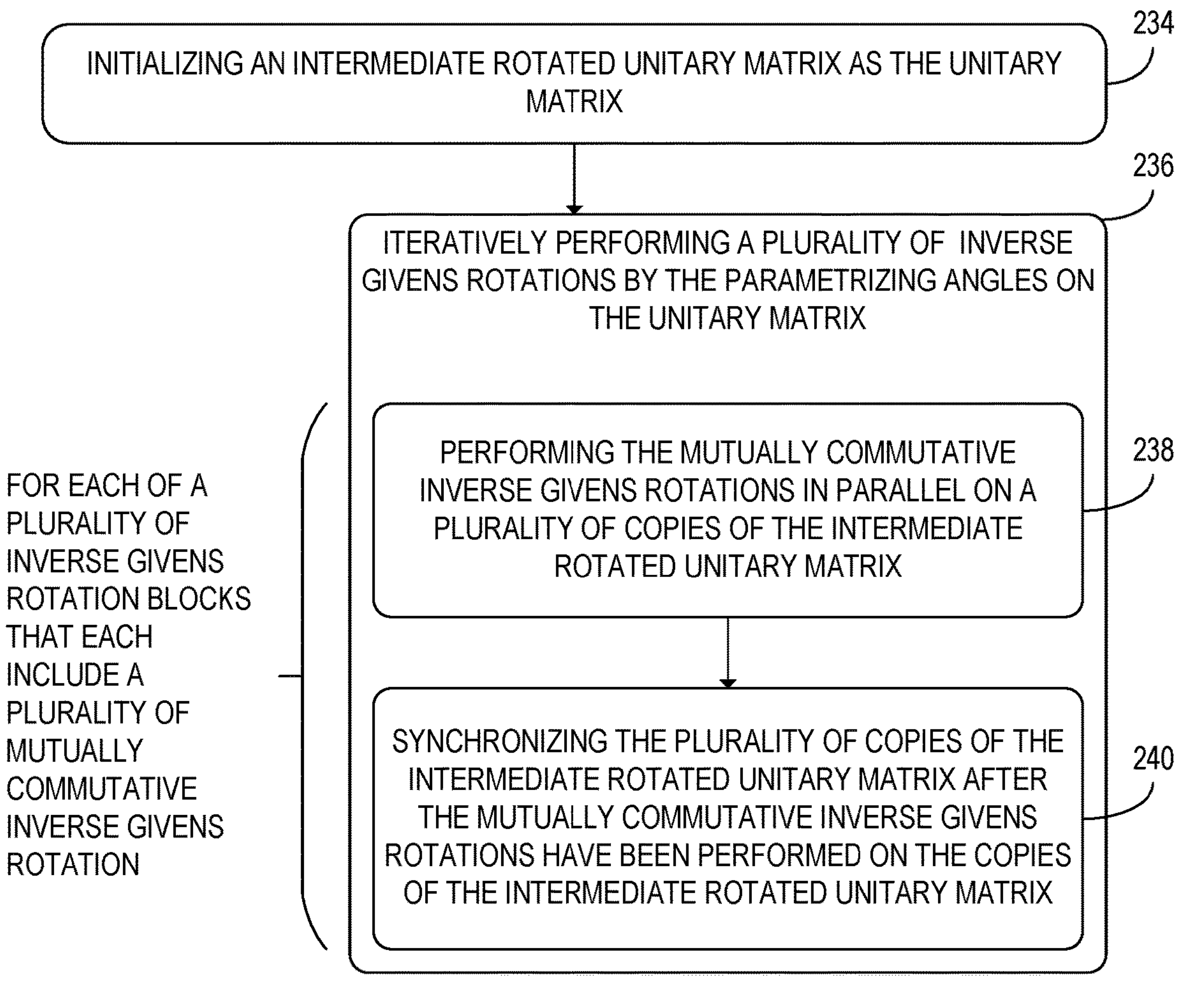
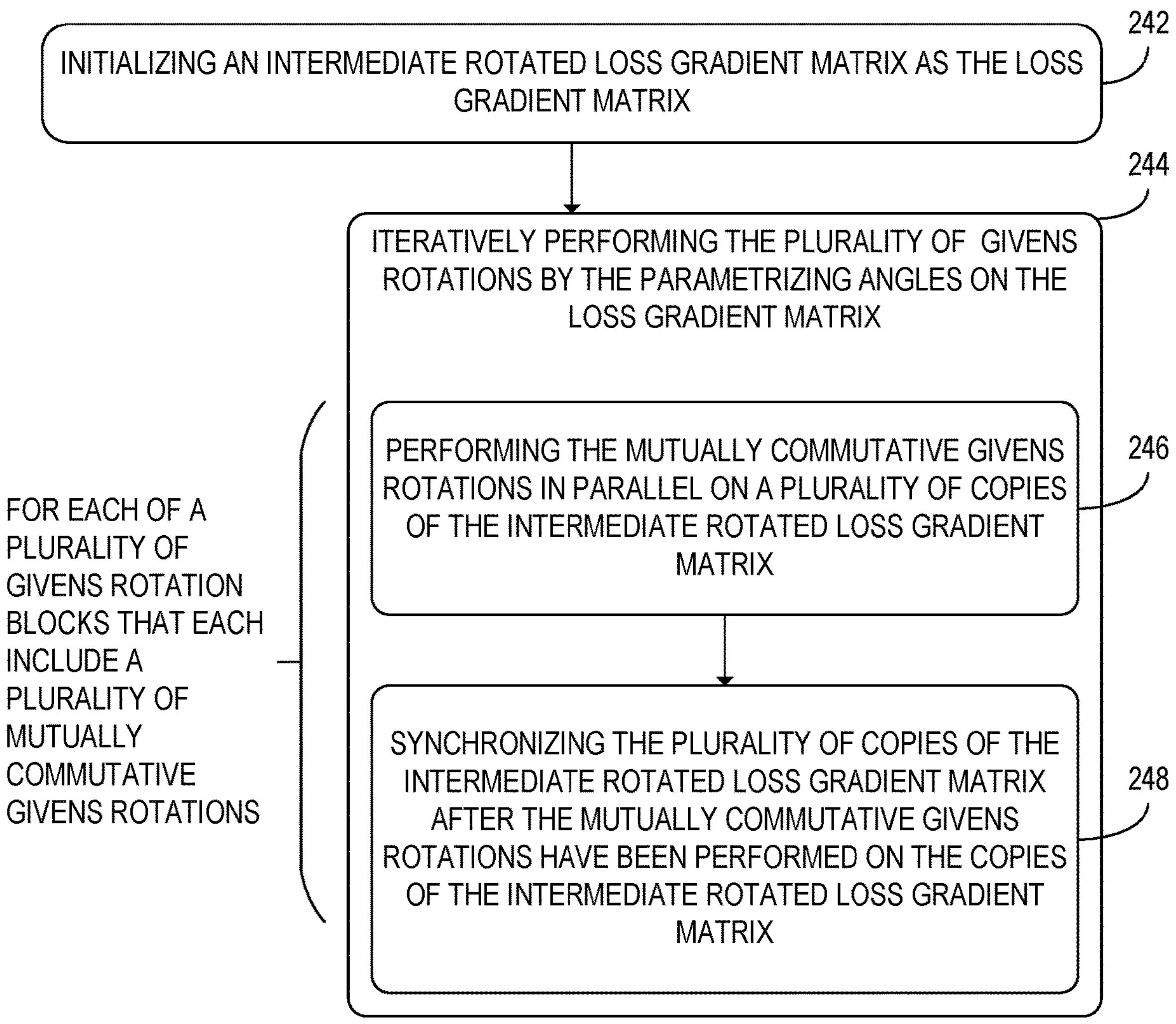
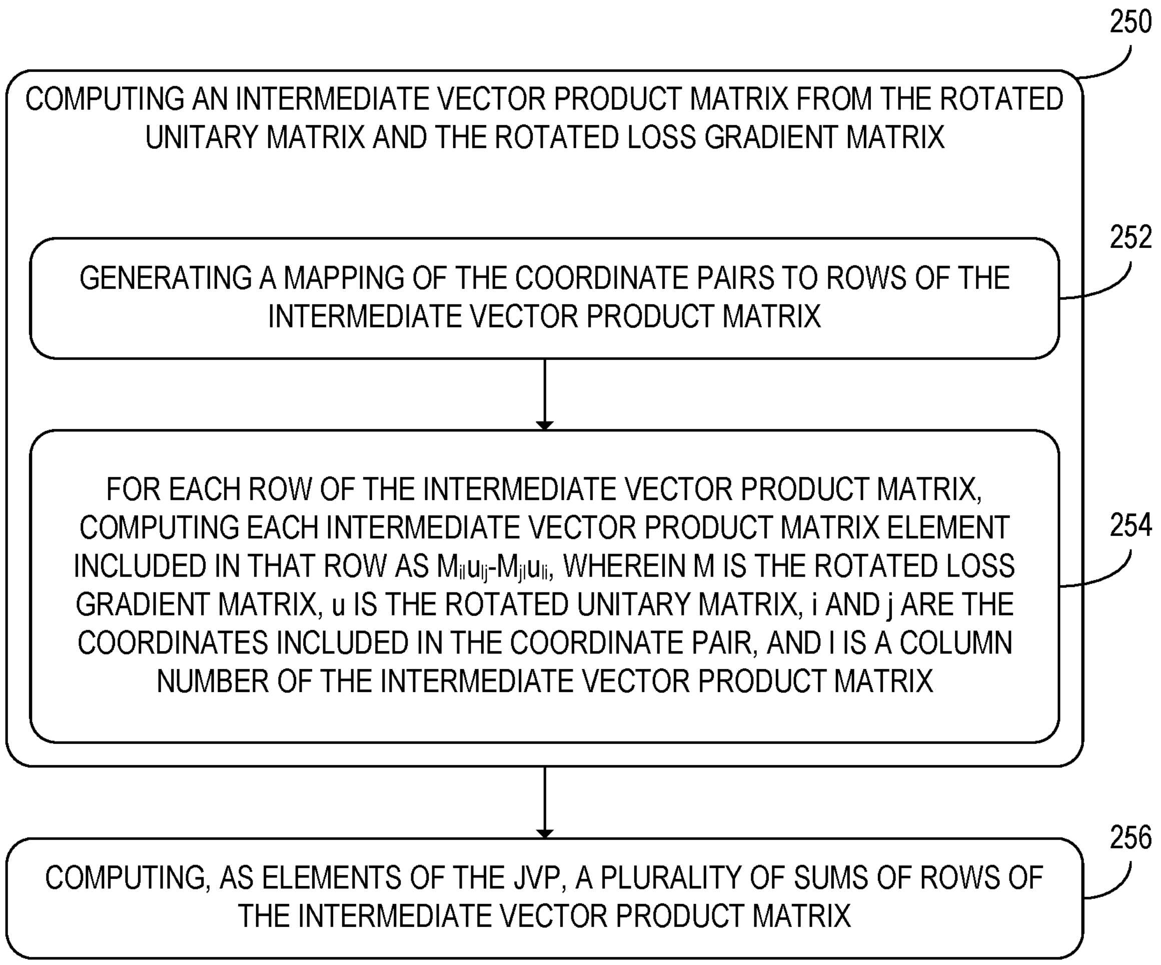

INITIALIZING AN INTERMEDIATE ROTATED UNITARY MATRIX AS THE UNITARY MATRIX
234
ITERATIVELY PERFORMING A PLURALITY OF INVERSE GIVENS ROTATIONS BY THE PARAMETRIZING ANGLES ON THE UNITARY MATRIX
236
FOR EACH OF A PLURALITY OF INVERSE GIVENS ROTATION BLOCKS THAT EACH INCLUDE A PLURALITY OF MUTUALLY COMMUTATIVE INVERSE GIVENS ROTATION
PERFORMING THE MUTUALLY COMMUTATIVE INVERSE GIVENS ROTATIONS IN PARALLEL ON A PLURALITY OF COPIES OF THE INTERMEDIATE ROTATED UNITARY MATRIX
238
SYNCHRONIZING THE PLURALITY OF COPIES OF THE INTERMEDIATE ROTATED UNITARY MATRIX AFTER THE MUTUALLY COMMUTATIVE INVERSE GIVENS ROTATIONS HAVE BEEN PERFORMED ON THE COPIES OF THE INTERMEDIATE ROTATED UNITARY MATRIX
240

FIG. 8D



INITIALIZING AN INTERMEDIATE ROTATED LOSS GRADIENT MATRIX AS THE LOSS GRADIENT MATRIX
242
ITERATIVELY PERFORMING THE PLURALITY OF GIVENS ROTATIONS BY THE PARAMETRIZING ANGLES ON THE LOSS GRADIENT MATRIX
244
FOR EACH OF A PLURALITY OF GIVENS ROTATION BLOCKS THAT EACH INCLUDE A PLURALITY OF MUTUALLY COMMUTATIVE GIVENS ROTATIONS
PERFORMING THE MUTUALLY COMMUTATIVE GIVENS ROTATIONS IN PARALLEL ON A PLURALITY OF COPIES OF THE INTERMEDIATE ROTATED LOSS GRADIENT MATRIX
246
SYNCHRONIZING THE PLURALITY OF COPIES OF THE INTERMEDIATE ROTATED LOSS GRADIENT MATRIX AFTER THE MUTUALLY COMMUTATIVE GIVENS ROTATIONS HAVE BEEN PERFORMED ON THE COPIES OF THE INTERMEDIATE ROTATED LOSS GRADIENT MATRIX
248

FIG. 8E



COMPUTING AN INTERMEDIATE VECTOR PRODUCT MATRIX FROM THE ROTATED UNITARY MATRIX AND THE ROTATED LOSS GRADIENT MATRIX

GENERATING A MAPPING OF THE COORDINATE PAIRS TO ROWS OF THE INTERMEDIATE VECTOR PRODUCT MATRIX

FOR EACH ROW OF THE INTERMEDIATE VECTOR PRODUCT MATRIX, COMPUTING EACH INTERMEDIATE VECTOR PRODUCT MATRIX ELEMENT INCLUDED IN THAT ROW AS $M_{il}u_{lj}$-$M_{jl}u_{li}$, WHEREIN M IS THE ROTATED LOSS GRADIENT MATRIX, u IS THE ROTATED UNITARY MATRIX, i AND j ARE THE COORDINATES INCLUDED IN THE COORDINATE PAIR, AND l IS A COLUMN NUMBER OF THE INTERMEDIATE VECTOR PRODUCT MATRIX

COMPUTING, AS ELEMENTS OF THE JVP, A PLURALITY OF SUMS OF ROWS OF THE INTERMEDIATE VECTOR PRODUCT MATRIX

FIG. 8F

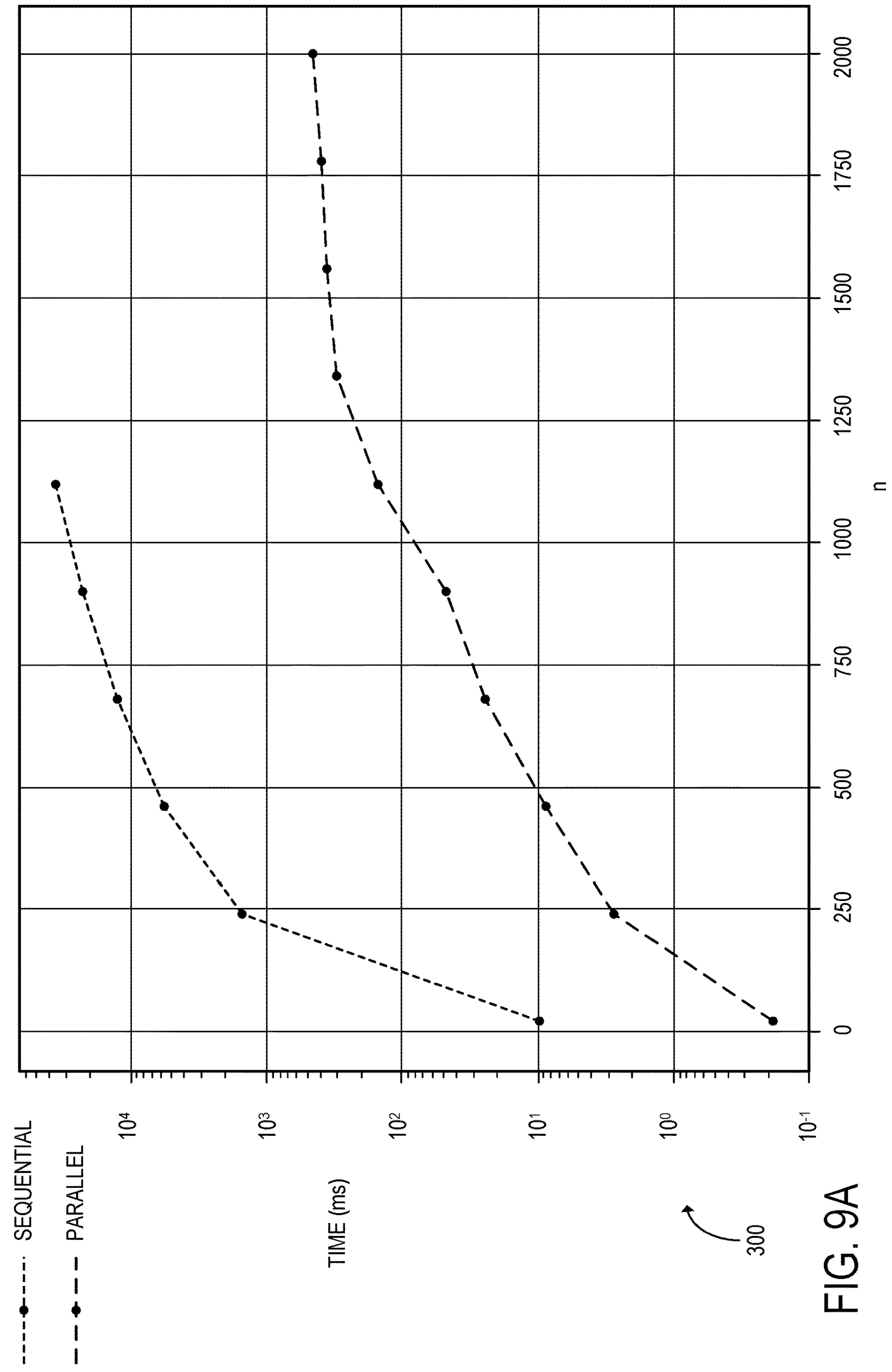

SEQUENTIAL
PARALLEL
TIME (ms)
10^4
10^3
10^2
10^1
10^0
10^{-1}
0
250
500
750
1000
1250
1500
1750
2000
n
300

FIG. 9A

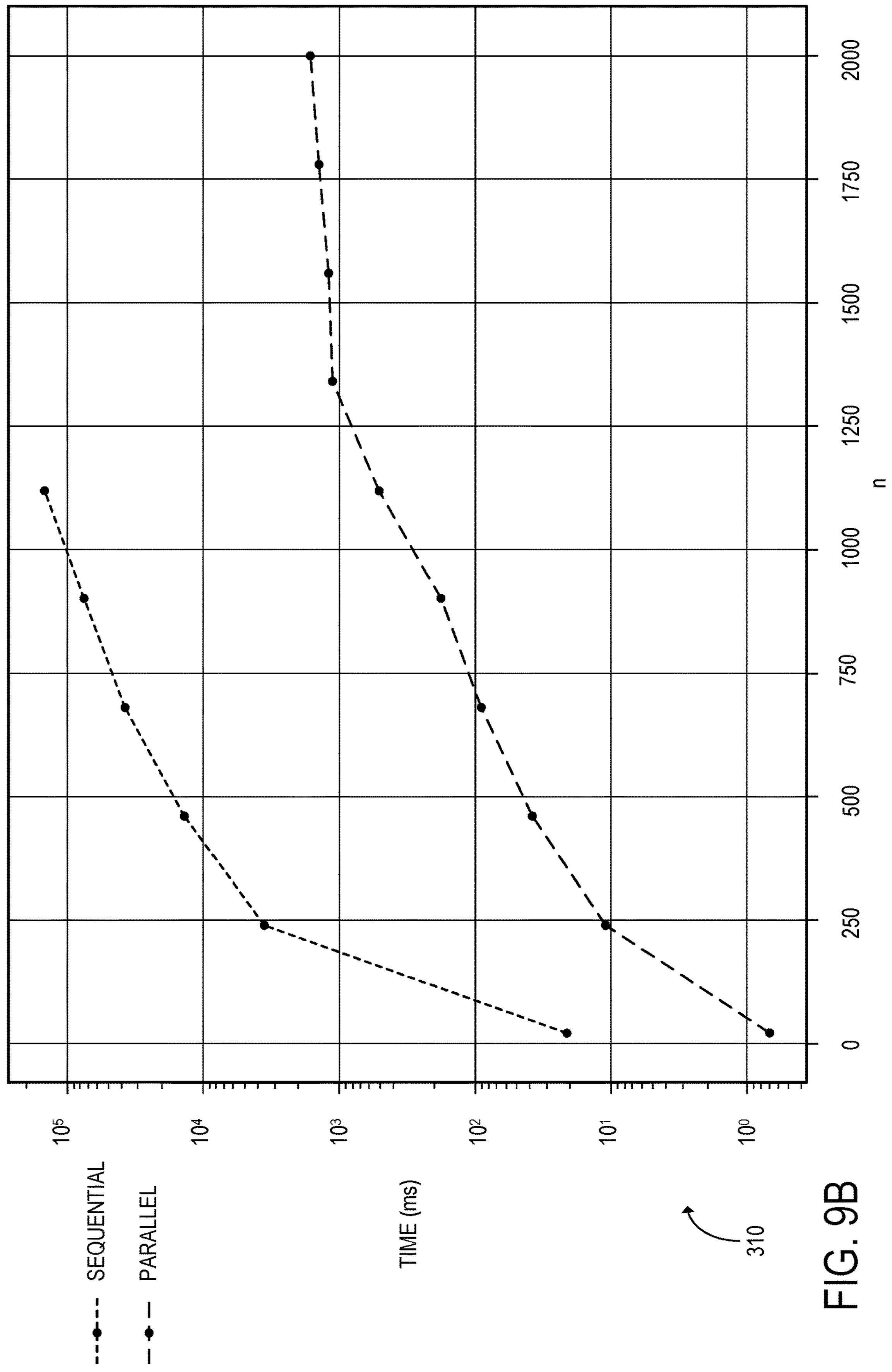

SEQUENTIAL
PARALLEL
TIME (ms)
10^5
10^4
10^3
10^2
10^1
10^0
0
250
500
750
1000
1250
1500
1750
2000
n
310

FIG. 9B

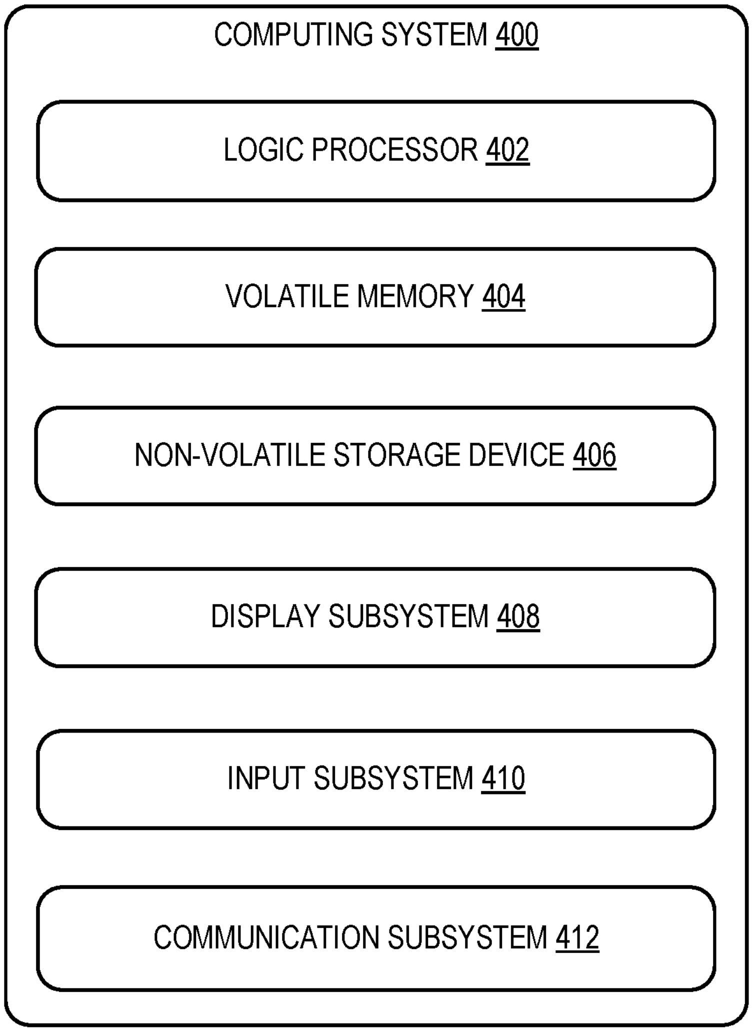

COMPUTING SYSTEM 400
LOGIC PROCESSOR 402
VOLATILE MEMORY 404
NON-VOLATILE STORAGE DEVICE 406
DISPLAY SUBSYSTEM 408
INPUT SUBSYSTEM 410
COMMUNICATION SUBSYSTEM 412

FIG. 10

BACKPROPAGATION USING PARAMETRIZING ANGLES OF UNITARY MATRIX

BACKGROUND

In some recent work in machine learning, orthogonal matrices have been used to encode the weights of neurons in neural networks. An orthogonal matrix is a square matrix in which the columns of the matrix specify a set of orthogonal vectors. Orthogonal matrices satisfy the property $U^T = U^{-1}$, where U^T is the transpose of the matrix U and U^{-1} is the inverse of the matrix U. In addition, the set of orthogonal matrices is the subset of the set of unitary matrices for which each matrix element is real-valued. The set of unitary matrices is the set of matrices for which $U^\dagger = U^{-1}$, where $U^\dagger$ is the conjugate transpose of the matrix U.

By using an orthogonal matrix to encode the weights of the neurons in a neural network, the vanishing gradient problem and the exploding gradient problem may be avoided when training the neural network via backpropagation. The exploding gradient problem and the vanishing gradient problem, respectively, occur when a gradient of a loss function becomes very large or very small when backpropagation is performed. Thus, the weights of the neural network may be updated by amounts that are too large or too small to achieve the desired behavior of the neural network. Representing the weights in an orthogonal matrix may prevent the exploding and vanishing gradient problems by normalizing the lengths of the column vectors.

SUMMARY

According to one aspect of the present disclosure, a computing system is provided, including a processor configured to train a machine learning model in a plurality of backpropagation iterations. Each backpropagation iteration may include generating a coordinate pair sequence including a plurality of coordinate pairs. Each coordinate pair of the plurality of coordinate pairs may be unique within the coordinate pair sequence and may include non-matching coordinates. The backpropagation iteration may further include receiving a plurality of parametrizing angles respectively associated with the plurality of coordinate pairs. The backpropagation iteration may further include computing a unitary matrix parametrized by the plurality of parametrizing angles. The backpropagation iteration may further include computing a loss gradient matrix of a training loss with respect to a plurality of unitary matrix elements of the unitary matrix. The backpropagation iteration may further include computing a Jacobian-vector product (JVP) of the training loss with respect to the plurality of parametrizing angles. Computing the JVP may include computing a rotated unitary matrix by performing, on the unitary matrix, for each coordinate pair of the plurality of coordinate pairs, an inverse Givens rotation by the parametrizing angle for the coordinate pair around an axis specified by the coordinate pair. Computing the JVP may further include computing a rotated loss gradient matrix by performing, on the loss gradient matrix, for each coordinate pair of the plurality of coordinate pairs, a Givens rotation by the parametrizing angle for the coordinate pair around the axis specified by the coordinate pair. The JVP may be computed from the rotated unitary matrix and the rotated loss gradient matrix. The backpropagation iteration may further include updating the plurality of parametrizing angles based at least in part on the JVP.

This Summary is provided to introduce a selection of concepts in a simplified form that are further described below in the Detailed Description. This Summary is not intended to identify key features or essential features of the claimed subject matter, nor is it intended to be used to limit the scope of the claimed subject matter. Furthermore, the claimed subject matter is not limited to implementations that solve any or all disadvantages noted in any part of this disclosure.

BRIEF DESCRIPTION OF THE DRAWINGS

FIG. **1**A schematically shows an example computing system including a processor at which a machine learning model may be trained in parallel at a plurality of processor threads, according to one example embodiment.

FIG. **1**B schematically shows the example computing system of FIG. **1**A when a backpropagation iteration is performed at the processor.

FIG. **2** shows example pseudocode for a first algorithm by which the processor may be configured to compute a unitary matrix based on a plurality of parametrizing angles and a coordinate pair sequence, according to the example of FIG. **1**A.

FIG. **3** shows example pseudocode for a second algorithm by which the processor may be configured to compute a unitary matrix based on a plurality of parametrizing angles and a coordinate pair sequence arranged in a plurality of blocks, according to the example of FIG. **1**A.

FIG. **4** schematically shows the generation of an example block sequence including a plurality of blocks of coordinate pairs, according to the example of FIG. **1**A.

FIG. **5** schematically shows an example computation of the unitary matrix using the second algorithm of FIG. **3** when a plurality of Givens rotations are performed in parallel within the plurality of blocks generated in the example of FIG. **4**.

FIGS. **6**A-**6**B show example pseudocode for a third algorithm by which the processor may be configured to compute a Jacobian-vector product (JVP) for the machine learning model, according to the example of FIG. **1**A.

FIG. **7** shows example pseudocode for a fourth algorithm by which the processor may be configured to compute a unitary matrix based on a plurality of parametrizing angles, a plurality of parametrizing phase angles, and a coordinate pair sequence arranged in a plurality of blocks, according to the example of FIG. **1**A.

FIG. **8**A shows an example method for use with a computing system to train a machine learning model in a plurality of backpropagation iterations, according to the example of FIG. **1**A.

FIG. **8**B shows additional steps of the method of FIG. **8**A that may be performed when generating a coordinate pair sequence.

FIG. **8**C shows additional steps of the method of FIG. **8**A that may be performed when computing a unitary matrix.

FIG. **8**D shows additional steps of the method of FIG. **8**A that may be performed when computing a rotated unitary matrix.

FIG. **8**E shows additional steps of the method of FIG. **8**A that may be performed when computing a rotated loss gradient matrix.

FIG. **8**F shows additional steps of the method of FIG. **8**A that may be performed when computing a JVP from a rotated unitary matrix and a rotated loss gradient matrix.

FIG. **9**A shows a plot of execution time when computing a unitary matrix using a sequential algorithm and using a parallel algorithm, according to the example of FIG. **1**A.

FIG. **9**B shows a plot of execution time when computing a JVP using a sequential algorithm and using a parallel algorithm, according to the example of FIG. **1**A.

FIG. **10** shows a schematic view of an example computing environment in which the computing system of FIG. **1**A may be enacted.

DETAILED DESCRIPTION

In previous work on the use of orthogonal matrices in machine learning, methods of training neural networks when weights are represented in orthogonal matrices have been difficult to generalize. Some of the previously existing methods have been limited to specific subsets of the set of unitary matrices. Other existing methods have included steps with high computational complexity, such as an $O(n^3)$ matrix inversion that may make such methods impractical to use for training large models. In addition, existing methods of using orthogonal matrices when training machine learning models have not been amenable to high levels of parallelization.

In order to overcome the above challenges, systems and methods for training machine learning models are discussed below. In the following examples, the matrices used to represent weights are members of the set of special orthogonal matrices $\{U \in SO(n)\}$, which are real-valued matrices that satisfy the properties $UU^T = U^T U = I$ and det $U=1$, where I is an identity matrix. However, as discussed in further detail below, the following systems and methods may be extended such that any unitary matrices may be used. In addition, the following examples assume that the number of columns n included in the matrix U is even. Extensions of the following techniques to odd values of n are also discussed below.

FIG. **1**A schematically shows an example computing system **10** at which a machine learning model **20** may be trained, according to one embodiment of the present disclosure. The computing system **10** may include a processor **12** and memory **14** that are communicatively coupled. In the example of FIG. **1**A, the processor **12** and memory **14** are shown within a single physical computing device. However, in other examples, the functionality of the processor **12** and/or the memory **14** may be distributed across a plurality of communicatively coupled physical computing devices. The plurality of physical computing devices may, in such examples, be a plurality of server computing devices located in a data center. The processor **12** may, for example, be instantiated in one or more central processing units (CPUs), graphical processing units (GPUs), field-programmable gate arrays (FPGAs), specialized hardware accelerators, or processing devices of some other type.

The processor **12** may be configured to execute a plurality of processor threads **42** in parallel when training the machine learning model **20**. As shown in the example of FIG. **1**A, the processor **12** may be configured to execute a workload scheduling algorithm **40** to assign computational tasks to the plurality of processor threads **42**. The workload scheduling algorithm **40** may, in such examples, be configured to transmit instructions to a plurality of processing units **44** to execute the plurality of processor threads **42** in parallel. The plurality of processor threads **42** may then be executed at the respective plurality of processing units **44**. The plurality of processing units **44** may be instantiated at a single physical computing device or at a plurality of physical computing devices included in the computing system **10**. The processing units may be particular cores of a multi-core processor (e.g., multi-core central processing unit (CPU), multi-core graphics processing unit (GPU), or multi-core tensor processing unit (TPU), or may be embodied by single core CPUs, GPUs, or TPUs, or may be field programmable gate array (FPGA) acceleration hardware, as some examples. In the example of FIG. **1**A, the plurality of processor threads **42** are depicted as being executed at a plurality of processor cores of a hardware acceleration unit coupled to the processor **12** by an interconnect.

The machine learning model **20** may be a deep neural network, as shown in the example of FIG. **1**A. For example, the machine learning model **20** may be a convolutional neural network, a recurrent neural network, or a generative model. The machine learning model **20** may include a plurality of neurons with respective activation functions. In addition, the connections between the neurons may have respective weights w that may form the parameters of the machine learning model **20**. The plurality of neurons may be organized into an input layer **22**, one or more deep layers **24**, and an output layer **26**. Each neuron may be connected to at least one other neuron in each adjacent layer. When the machine learning model **20** receives input, that input may be loaded into the input layer **22** and passed through the one or more deep layers **24** such that one or more outputs are produced at the output layer **26**.

The machine learning model **20** may be trained using training data **30** including a plurality of training inputs **32** and a corresponding plurality of ground-truth training outputs **34**. The processor **12** may be configured to train the machine learning model **20** using a backpropagation algorithm **36**. When the processor **12** executes the backpropagation algorithm **36**, the processor **12** may be configured to perform a plurality of iterations that each include a forward pass and a backward pass through the machine learning model **20**. In each forward pass, the processor **12** may be configured to input one or more training inputs **32** into the input layer **22** and compute one or more outputs. During the backward pass, the processor **12** may be configured to compare the one or more outputs produced during the forward pass to corresponding ground-truth training outputs **34** using a loss function, of which the processor **12** may be configured to compute a gradient. The processor **12** may then be configured to modify the weights of the machine learning model **20** based on the gradient of the loss function. Thus, when the backpropagation algorithm **36** is executed, the processor **12** may be configured to perform a plurality of neuronal weight calculations **38** for the machine learning model **20**.

Each neuronal weight calculation **38** of the backpropagation algorithm may include performing a parallel forward U algorithm **38**A and a parallel Jacobian-vector product (JVP) algorithm **38**B. FIG. **1**B shows the computing system **10** in additional detail when the parallel forward U algorithm **38**A and the parallel JVP algorithm **38**B are performed at the processor **12**. As discussed above, the processor **12** of the computing system **10** may be configured to train a machine learning model in a plurality of backpropagation iterations that each include a forward pass and a backward pass. In the forward pass, the processor **12** may be configured to compute a unitary matrix U parametrized by a plurality of parametrizing angles θ, as discussed in further detail below. During the backward pass, the processor **12** may be configured to compute a Jacobian-vector product (JVP) defined as

$$\frac{\partial L}{\partial \theta} = \left(\frac{\partial U}{\partial \theta}\right)^T \frac{\partial L}{\partial U}$$

where L is a training loss of the machine learning model. In each backpropagation iteration, the processor **12** may be further configured to update the parameters of the machine learning model based at least in part on the JVP. Thus, the processor **12** may be configured to train the machine learning model by updating the model parameters over the course of the plurality of backpropagation iterations.

A unitary matrix U may be parametrized via composition of a plurality of Givens rotations. When the unitary matrix U is parametrized using Givens rotations, $N \triangleq n(n-1)/2$ Givens rotations may be performed in the planes spanned by pairs of coordinate axes of the unitary matrix U. The processor **12** may be configured to define an order in which the Givens rotations are applied to parametrize SO(n). The order in which the Givens rotations are applied may be indicated as a coordinate pair sequence $E=(e_1 \ldots e_N)$ including a plurality of coordinate pairs $e_k=(i, j)$, with $i<j$. Each coordinate pair e_k of the plurality of coordinate pairs may be unique within the coordinate pair sequence E and may include non-matching coordinates. The coordinates of the unitary matrix U may be labeled with indices in $\{0, \ldots n-1\}$. The elements e_k of the coordinate pair sequence E may be used to define the order in which the Givens rotations are applied, as discussed in further detail below.

Each backpropagation iteration may include receiving a plurality of parametrizing angles θ_e respectively associated with the plurality of coordinate pairs e_k. Thus, the processor **12** may be configured to receive N parametrizing angles θ_e, which may be indicated collectively as $\theta=(\theta_1, \ldots \theta_N)$. The plurality of parametrizing angles θ_e may be used as a basis in which the weights of the neurons included in the machine learning model are parametrized. The plurality of parametrizing angles θ_e may each parametrize a rotation in a corresponding plane spanned by unit vectors for the i and j coordinates. Each parametrizing angle θ_e may be the angle by which a corresponding Givens rotation is performed, as discussed in further detail below.

Each Givens matrix of the plurality of Givens matrices may differ from an identity matrix by only four elements. For the coordinate pair e=(i, j), the elements of the corresponding Givens matrix that may differ from those of the identity matrix are (i, i), (i, j), (j, i), and (j, j). The nonzero elements of the Givens matrix $G^e(\theta_e)$ may be defined as follows:

$$G_{ii}^e = G_{jj}^e = \cos\theta_{ij}$$

$$G_{ij}^e = -\sin\theta_{ij}$$

$$G_{ji}^e = \sin\theta_{ij}$$

$$G_{ll}^e = 1 \text{ for } l \neq i, l \neq j.$$

Thus, multiplication by $G^e(\theta_e)$ may only modify rows i and j of the multiplicand. The unitary matrix U may be parametrized as follows using the plurality of Givens matrices:

$$U = \prod_{e \in E} G^e(\theta_e)$$

In the above parametrization, the angles $\theta=(\theta_{e_1}, \ldots, \theta_{e_N}) \in \mathbb{R}^N$ associated with the coordinate pairs e may be the parameters that trace the set of matrices in SO(n).

Real-valued orthogonal matrices with dimension n, forming what is known as the orthogonal group, may belong to either SO(n) or to a set of real-valued orthogonal matrices with determinant −1, also known as reflections. Reflections may be parametrized using Givens matrices similarly to matrices in SO(n), with an additional step of negating an arbitrary fixed column following construction of the matrix from Givens rotations.

FIG. **2** shows pseudocode of an example first algorithm **100** via which the processor **12** may be configured to compute the unitary matrix U(θ) based on the plurality of parametrizing angles θ. In the first algorithm **100** of FIG. **2**, the unitary matrix U is initialized as an n×n identity matrix I_n. The processor **12** may be further configured to compute the unitary matrix U at least in part by iteratively performing a plurality of Givens rotations by the parametrizing angles θ on the identity matrix I_n. In the example first algorithm **100** of FIG. **2**, the plurality of Givens rotations by the angles θ_{ij} associated with the coordinate pairs e=(i, j) are iteratively performed on the identity matrix I_n in a reverse order from the order in which the coordinate pairs occur in E. After all N Givens rotations have been performed, the first algorithm **100** returns the resulting matrix as U(θ).

In the example of FIG. **2**, each Givens rotation may be performed in O(n) sequential operations, which may be parallelized. However, the first algorithm **100** includes $N=O(n^2)$ Givens rotations. Since these Givens rotations do not typically commute, the $N=O(n^2)$ Givens rotations are also not parallelizable. In order to address these shortcomings of the first algorithm **100**, pseudocode of a second algorithm **110** is shown in the example of FIG. **3**. The second algorithm **110** is one example of the parallel forward U algorithm **38**A shown in FIG. **1**A. In the second algorithm **110** of FIG. **3**, the coordinate pairs are arranged into a block sequence $B=(b_1, \ldots b_{n-1})$. Within each block b, each index included in $\{0, \ldots n-1\}$ may occur once. Thus, the block sequence may define a plurality of Givens rotation blocks b that each include a plurality of mutually commutative Givens rotations. Since the Givens rotations included in each block b commute with each other, those Givens rotations may be computed in parallel.

The processor **12** may be configured to compute block sequence B using a round-robin scheduling algorithm known as the circle method. The circle method may be performed for an index sequence of n indices and may include n−1 pairing iterations. An example of the circle method when n=6 is shown in FIG. **4**. In each pairing iteration of the plurality of pairing iterations, the processor **12** may be configured to hold one index fixed while shifting the positions of the other indices by one, in a loop that connects the ends of the index sequence without including the fixed index. Thus, in each pairing iteration subsequent to an initial pairing iteration, the pairing iteration may include shifting a respective position in the index sequence of each index other than the fixed index by 1 modulo n−1. In the example of FIG. **4**, the fixed index is 0. Each pairing iteration may further include iteratively pairing a first unpaired index in the index sequence with a last unpaired index in the index sequence until the index sequence includes fewer than two unpaired indices.

Subsequently to pairing the indices, the pairing iteration may further include adding each pair of indices to the coordinate pair sequence E. The pairs added to the coordinate pair sequence E in the pairing iteration may collectively form a block b. Thus, within the coordinate pair sequence E, the coordinate pairs e may be grouped into a plurality of blocks b that form the block sequence B. The block sequence B in the example of FIG. **4** includes the blocks b_1, b_2, b_3, b_4, and b_5.

In examples in which n is even, the circle method may leave no unpaired indices in each of the pairing iterations. Thus, each index included in the index sequence may be included in each block b. However, in examples in which n is odd, an index may be left unpaired in each pairing iteration. The index left unpaired may be different for each pairing iteration of the plurality of pairing iterations.

In some examples, when the processor **12** computes the block sequence B, the processor **12** may be configured to independently and lazily evaluate a current block at each of a plurality of processor threads. When a block is lazily executed, the processor **12** computes the coordinate pairs included in the block when an instruction to perform a computation using the block is executed, rather than computing the plurality of blocks in advance. The processor **12** may be configured to determine the current block b based at least in part on an iteration number of the current pairing iteration. Accordingly, the plurality of processor threads may compute individual blocks b of the block sequence B without computing the entire block sequence B.

Returning to the example second algorithm **110** of FIG. **3**, the processor **12** may be further configured to initialize an intermediate unitary matrix as an identity matrix I_n. Iteratively, for each block b of mutually commutative Givens rotations, the processor **12** may be configured to perform the plurality of mutually commutative Givens rotations in parallel on a plurality of copies of the intermediate unitary matrix. The plurality of mutually commutative Givens rotations may, for example, be performed in parallel at a plurality of processor threads, a GPU, an FPGA, or a specialized hardware accelerator. The processor **12** may be further configured to synchronize the plurality of copies of the intermediate unitary matrix after the mutually commutative Givens rotations have been performed on the copies of the intermediate unitary matrix. In some examples, as shown in FIG. **3**, the corresponding Givens rotations for the plurality of blocks b may be computed in a reverse order from the order in which the blocks b occur in the block sequence B.

FIG. **5** shows example Givens rotations that may be performed in parallel as indicated by the Givens rotation blocks b_5 and b_4 of the example of FIG. **4**. The Givens rotations may be performed in parallel according to the second algorithm **110** of FIG. **3**. In the example of FIG. **5**, the intermediate unitary matrix is indicated as $U^{k:n-1}$, where b_k is the most recent block for which Givens rotations were performed on the intermediate unitary matrix. As shown in the example of FIG. **5**, the processor **12** may be configured to perform the Givens rotations $G^{(0,1)}$, $G^{(2,5)}$ and $G^{(3,4)}$ on respective copies of the identity matrix I_n at an iteration corresponding to the Givens rotation block b_5. The processor **12** may be further configured to synchronize the plurality of copies of the identity matrix I_n after the Givens rotations have been performed. The processor **12** may be configured to thereby generate $U^{5:5}$. When the processor synchronizes the rotated identity matrices, the processor **12** may be configured to multiply the rotated identity matrices together. Since the rotated identity matrices commute with each other, the rotated identity matrices may be multiplied in any order.

At the Givens rotation block b_4, the processor **12** may be further configured to perform the Givens rotations $G^{(0,2)}$, $G^{(1,3)}$, and $G^{(4,5)}$ on respective copies of $U^{5:5}$. Thus, the Givens rotations included in the plurality of blocks b may be applied iteratively. The processor **12** may be further configured to synchronize the plurality of copies of $U^{5:5}$ after the Givens rotations have been performed in order to generate $U^{4:5}$. Although not shown in FIG. **5**, the processor **12** may be further configured to perform the Givens rotations included in the blocks b_3, b_2, and b_1 on the intermediate unitary matrix and to synchronize the copies of the intermediate unitary matrix after applying the Givens rotations included in each of the above blocks. After the plurality of Givens rotations included in each block of the block sequence B have been applied to the intermediate unitary matrix, the resulting intermediate unitary matrix may be used as the unitary matrix U(0).

Using the second algorithm **110** shown in FIG. **3**, the processor **12** may be configured to compute the unitary matrix U(0) with a computational complexity of O(n). In contrast, the first algorithm **100** shown in FIG. **2** may have a computational complexity of $O(n^2)$. Thus, using the second algorithm **110** may allow for significant speedups when computing the unitary matrix U(0) for large values of n.

Returning to FIG. **1B**, the processor **12** may be further configured to compute a loss gradient matrix Γ of a training loss L with respect to a plurality of unitary matrix elements of the unitary matrix U(0). The processor **12** may be configured to compute the training loss L by computing the value of a loss function, which may be a classification loss function or a regression loss function. The loss gradient matrix Γ may be an n×n matrix defined as:

$$\Gamma \triangleq \frac{\partial L}{\partial \theta}$$

In some examples, the processor **12** may be configured to compute the loss gradient matrix Γ via automatic differentiation of the training loss L with respect to the unitary matrix elements of the unitary matrix U(0). The processor **12** may, for example, be configured to perform automatic differentiation by executing a function included in a machine learning library.

The processor **12** may be further configured to compute the JVP of the training loss L with respect to the plurality of parametrizing angles θ. The processor **12** may be configured to compute the JVP directly rather than computing the two partial derivatives on the righthand side of the equation defining the JVP as discussed above. When the processor **12** computes the JVP, the processor **12** may be configured to compute a rotated unitary matrix U^{fwd} and a rotated loss gradient matrix M, as discussed in further detail below. The processor **12** may be further configured to compute the JVP from the rotated unitary matrix U^{fwd} and the rotated loss gradient matrix M.

FIGS. **6A-6B** show pseudocode of a third algorithm **120**. The third algorithm **120** of FIGS. **6A-6B** is one example of the parallel JVP algorithm **38A** of FIG. **1A** by which the processor **12** may be configured to compute the JVP. A first portion **120A** of the third algorithm **120** is shown in FIG. **6A**. As shown in FIG. **6A**, the inputs of the third algorithm **120** may be the block sequence $B=(b_1, \ldots b_{n-1})$, the parametrizing angles $\theta=(\theta_{e_1}, \ldots, \theta_{e_N})$, the unitary matrix U, and the loss gradient matrix Γ.

In the first portion **120A** of the third algorithm **120** shown in FIG. **6A**, the processor **12** may be configured to compute the rotated unitary matrix U^{fwd}. The processor **12** may be configured to compute the rotated unitary matrix U^{fwd} at least in part by initializing an intermediate rotated unitary matrix as the unitary matrix U. Computing the rotated unitary matrix U^{fwd} may further include performing, on the unitary matrix U, for each coordinate pair e=(i, j) of the plurality of coordinate pairs e, an inverse Givens rotation by the parametrizing angle θ_e for the coordinate pair e around an axis specified by the coordinate pair e. The axis specified by the coordinate pair may be an axis perpendicular to a plane spanned by the i and j unit vectors, as discussed above. An inverse Givens rotation by an angle θ is defined herein as a Givens rotation by the angle −θ. The inverse Givens rotations may be performed iteratively, for each of a plurality of inverse Givens rotation blocks that each include a plurality of mutually commutative inverse Givens rotations. As shown in the example of FIG. **6**A, the inverse Givens rotation blocks may be the Givens rotation blocks b as defined above.

Within each inverse Givens rotation block, the processor **12** may be configured to perform the mutually commutative inverse Givens rotations in parallel on a plurality of copies of the intermediate rotated unitary matrix. The processor **12** may be further configured to synchronize the plurality of copies of the intermediate rotated unitary matrix after the mutually commutative inverse Givens rotations have been performed on the copies of the intermediate rotated unitary matrix. Thus, within each inverse Givens rotation block, the plurality of inverse Givens rotations may be performed in parallel and the results of the parallel computations may be synchronized to update the intermediate rotated unitary matrix. The blocks of inverse Givens rotations may be applied to the intermediate rotated unitary matrix in reverse block sequence order. After the inverse Givens rotations have been performed for each inverse Givens rotation block, the resulting intermediate rotated unitary matrix may be used as the rotated unitary matrix U^{fwd}.

FIG. **6**B shows pseudocode for a second portion **120**B of the third algorithm **120** as a continuation of the first portion **120**A shown in FIG. **6**A. FIG. **6**B shows pseudocode for the computation of the rotated loss gradient matrix M. When the processor **12** computes the rotated loss gradient matrix M, the processor **12** may be configured to initialize an intermediate rotated loss gradient matrix as the loss gradient matrix F. The processor **12** may be further configured to perform, on the loss gradient matrix F, for each coordinate pair e=(i, j) of the plurality of coordinate pairs e, a Givens rotation by the parametrizing angle θ_e for the coordinate pair e around the axis specified by the coordinate pair e. The processor **12** may be configured to iteratively perform the plurality of Givens rotations in a plurality of Givens rotation blocks that each include a plurality of mutually commutative Givens rotations. The plurality of Givens rotation blocks may be the blocks b discussed above, which may be applied to the intermediate rotated loss gradient matrix in reverse block sequence order.

In each Givens rotation block b, the processor **12** may be configured to perform the mutually commutative Givens rotations included in the block b in parallel on a plurality of copies of the intermediate rotated loss gradient matrix. The processor **12** may be further configured to synchronize the plurality of copies of the intermediate rotated loss gradient matrix after the mutually commutative Givens rotations have been performed on the copies of the intermediate rotated loss gradient matrix. After the plurality of Givens rotations have been performed for each Givens rotation block b, the resulting intermediate rotated loss gradient matrix may be used as the rotated loss gradient matrix M.

When the processor **12** computes the JVP from the rotated unitary matrix U^{fwd} and the rotated loss gradient matrix M, the processor **12** may be configured to compute an intermediate vector product matrix A from the rotated unitary matrix U^{fwd} and the rotated loss gradient matrix M. The intermediate vector product matrix A may be an N/2×N matrix and may be initialized as an empty matrix when the third algorithm **120** is performed. The processor **12** may be configured to compute the intermediate vector product matrix A at least in part by generating a mapping of the coordinate pairs e to rows m(e) of the intermediate vector product matrix A. The processor **12** may be configured to compute N elements of the intermediate vector matrix A for each coordinate pair e. For each row of the intermediate vector product matrix A, the processor **12** may be further configured to compute each intermediate vector product matrix element included in that row as follows:

$$A_{ml} \leftarrow M_{il}u_{li} - M_{jl}u_{li}$$

where M is the rotated loss gradient matrix, u is the rotated unitary matrix, i and j are the coordinates included in the coordinate pair, and l is a column number of the intermediate vector product matrix A.

Subsequently to computing the elements of the intermediate vector product matrix A, the processor **12** may be further configured to compute, as elements of the JVP, a plurality of sums of rows of the intermediate vector product matrix A. The processor **12** may be configured to compute the sums of the rows of the intermediate vector product matrix A by multiplying A by a vector of ones:

$$d \leftarrow A1_n$$

The processor **12** may be further configured to set the elements of the JVP to corresponding elements of d as specified by the mapping between the rows of A and the coordinate pairs e:

$$\frac{\partial L}{\partial \theta_e} \leftarrow d_{m(e)}$$

Thus, the processor **12** may be configured to compute the corresponding element of the JVP for each parametrizing angle θ_e.

Using the third algorithm **120** shown in FIGS. **6**A-**6**B, the processor **12** may be configured to compute the JVP with O(n log n) complexity. The summation of the rows of the intermediate vector product matrix A may be performed in O(log n) time, and each of the other steps performed for a block b in the third algorithm **120** may be performed in O(1) time. Since the number of blocks b in the block sequence B is n−1 when the circle method is used to compute the block sequence B, the overall computational complexity of computing the JVP using the third algorithm **120** may be O(n log n).

Returning to FIG. **1**B, subsequently to computing the JVP, the processor **12** may be further configured to update the plurality of parametrizing angles θ_e based at least in part on the JVP. Thus, the processor **12** may be configured to compute a plurality of updated parametrizing angles θ_e'. The processor **12** may be configured to compute the plurality of updated parametrizing angles θ_e' via gradient descent, which may be further based at least in part on a learning rate η. The plurality of updated parametrizing angles θ_e' may be computed for each backpropagation iteration executed at the processor **12** when training the machine learning model. In some examples, the processor **12** may be configured to dynamically modify the learning rate η between backpropagation iterations.

Although, in the above examples, U(θ) is a real-valued orthogonal matrix, the techniques discussed above may be extended to unitary matrices that include imaginary- or complex-valued elements, as shown in the example fourth algorithm **130** of FIG. **7**. The example fourth algorithm **130** shown in FIG. **7** is an algorithm by which the unitary matrix U may be computed when the elements of the unitary matrix U are not assumed to be real-valued. As shown in the example of FIG. **7**, the processor **12** may be configured to receive a plurality of parametrizing phase angles $\varphi=(\varphi_{e_1}, \ldots \varphi_{e_N})$ in addition to the block sequence B and the plurality of parametrizing angles θ. Each parametrizing phase angle φ_e may contribute a respective complex phase factor $e^{\iota\varphi_e}$ to the corresponding Givens rotation for that coordinate pair e. Thus, the elements of the Givens rotation $G^e(\theta_e, \varphi_e)$ matrix may be as follows:

$$G_{ii}^e = G_{jj}^e = e^{\iota\varphi_{ij}} \cos\theta_{ij}$$

$$G_{ij}^e = -\sin\theta_{ij}$$

$$G_{ji}^e = e^{\iota\varphi_{ij}} \sin\theta_{ij}$$

$$G_{ll}^e = 1 \text{ for } l \neq i, l \neq j.$$

In the fourth algorithm **130** shown in FIG. **7**, the processor **12** may be further configured to initialize an intermediate unitary matrix as an n×n identity matrix I_n. The processor **12** may be further configured to iteratively perform the plurality of Givens rotations $G^e(\theta_e, \varphi_e)$ on the intermediate unitary matrix in a plurality of Givens rotation blocks b that each include a plurality of mutually commutative Givens rotations, as in the second algorithm **110** of FIG. **3**. Within each Givens rotation block b, the plurality of Givens rotations may be performed in parallel on a respective plurality of copies of the intermediate unitary matrix. The processor **12** may be further configured to synchronize the plurality of copies of the intermediate unitary matrix after the plurality of Givens rotations in the block have been performed on the copies of the intermediate unitary matrix.

Backpropagation is now discussed for examples in which the elements of U(θ) are not assumed to be real-valued. A forward matrix U^{fwd} and a backward matrix U^{bck} may be defined as the following running matrix products:

$$U^{fwd} = U^{1:k-1} \triangleq \prod_{b \in b_1, \ldots, b_{k-1}} G^b$$

$$U^{bck} = U^{k:n-1} \triangleq \prod_{b \in b_k, \ldots, b_{n-1}} G^b$$

The rows and columns of the forward matrix U^{fwd} and the backward matrix U^{bck} may be expressed as follows:

$$U^{fwd} = [\, u_1 \;\cdots\; u_n \,]$$

$$U^{bck} = \begin{bmatrix} v_1^\dagger \\ \vdots \\ v_n^\dagger \end{bmatrix}$$

The processor **12** may be configured to compute a gradient of the unitary matrix U with respect to the parametrizing angles θ and a gradient of the unitary matrix U with respect to the parametrizing phase angles φ:

$$\frac{\partial U}{\partial \theta_e} = u_j v_i^\dagger - u_i v_j^\dagger$$

$$\frac{\partial U}{\partial \varphi_e} = \iota\cos\theta_e(\cos\theta_e u_i + \sin\theta_e u_j)v_i^\dagger + \iota\sin\theta_e(\cos\theta_e u_i + \sin\theta_e u_j)v_j^\dagger = \iota(\cos\theta_e u_i + \sin\theta_e u_j)(\cos\theta_e v_i + \sin\theta_e v_j)^\dagger$$

The processor **12** may be further configured to compute gradients of the training loss L relative to the parametrizing angles θ and the parametrizing phase angles φ by applying the chain rule:

$$\frac{\partial L}{\partial \theta} = \left(\frac{\partial U}{\partial \theta}\right)^T \frac{\partial L}{\partial U}$$

$$\frac{\partial L}{\partial \varphi} = \left(\frac{\partial U}{\partial \varphi}\right)^T \frac{\partial L}{\partial U}$$

In this example, the gradient of the training loss L with respect to the unitary matrix elements of the unitary matrix U may be computed as discussed above. Thus, the processor **12** may be configured to compute a JVP with respect to the parametrizing angles θ and a JVP with respect to the parametrizing phase angles cp.

In another extension of the techniques discussed above, a subset of coordinates may be specified such that during each backpropagation iteration, no rotations are performed in planes indicated by coordinate pairs in which both coordinates are included in the subset. Without loss of generality, for an integer m≤n−1, the following quantities may be defined:

$$S \triangleq (0, \ldots, m-1)$$

$$\overline{S} \triangleq (m, \ldots, n-1)$$

$$K \triangleq n-m$$

With the above definitions, the number of excluded coordinate pairs may be equal to κ(κ−1)/2, and the number of remaining coordinate pairs may be equal to mn−m(m+1)/2. For example, when n=8 and m=4, the pairs {(4,5), (4,6), (4,7), (5,6), (5,7), (6,7)} may be excluded from the full set of 28 pairs, leaving 22 remaining pairs. Exclusion of pairs of a subset of coordinates, as discussed above, may be used to parametrize arbitrary m×n matrices.

In examples in which pairs of a subset of coordinates are excluded, the processor **12** may be configured to partition the remaining coordinate pairs into a block sequence of a plurality of blocks. Partitioning the remaining coordinate pairs may be equivalent to solving an edge-coloring problem for a graph in which the first m nodes are adjacent to all other nodes and in which the last K nodes are not connected to each other. In order to adapt the second algorithm **110** of FIG. **3** to be used in examples in which pairs K coordinates are excluded, the inner "for" loop may be augmented with a condition instructing the processor **12** to skip performing the Givens rotation for a coordinate pair when i≥m. Since, by convention, i<j, such a condition would be met when (i, j) is an excluded pair. Similarly to the inner "for" loop of the second algorithm **110**, conditions to skip performing the Givens rotations for coordinate pairs where i≥m may also be added to the parallel "for" loops included in the third algorithm **120** and the fourth algorithm **130** in examples in

which pairs of a subset of the coordinates are excluded. Skipping Givens rotations in this manner may leave the arithmetic instruction count of sequential portions of the second algorithm **110**, third algorithm **120**, and fourth algorithm **130** unchanged. In addition, in examples in which m scales linearly with n such that m=αn for a fixed value of $\alpha \in (0,1]$, the total number of coordinate pairs N may remain $O(n^2)$. Thus, parallelizing the Givens rotations within blocks may still result in the speedups discussed above in examples in which coordinate pairs are excluded.

FIG. **8**A shows a flowchart of a method **200** of training a machine learning model at a computing system. The method **200** may, for example, be performed at the computing system **10** depicted in FIG. **1**A. The steps of the method **200** may be performed in each of a plurality of backpropagation iterations that may be performed to determine the weights of neurons included in the machine learning model.

At step **202**, the method **200** may include generating a coordinate pair sequence including a plurality of coordinate pairs. Each coordinate pair of the plurality of coordinate pairs may be unique within the coordinate pair sequence and may include non-matching coordinates. At step **204**, the method **200** may further include receiving a plurality of parametrizing angles respectively associated with the plurality of coordinate pairs. The parametrizing angle for a coordinate pair may parametrize a rotation in a plane defined by unit vectors corresponding to the coordinates of the coordinate pair.

At step **206**, the method **200** may further include computing a unitary matrix parametrized by the plurality of parametrizing angles. The unitary matrix may be computed at least in part by iteratively performing a plurality of Givens rotations by the parametrizing angles on an identity matrix. At step **208**, the method **200** may further include computing a loss gradient matrix of a training loss with respect to a plurality of unitary matrix elements of the unitary matrix. The loss gradient matrix may be computed for a predefined loss function of the machine learning model. In some examples, the loss gradient matrix may be computed via automatic differentiation of the training loss with respect to the unitary matrix elements of the unitary matrix.

At step **210**, the method **200** may further include computing a JVP of the training loss with respect to the plurality of parametrizing angles. At step **212**, computing the JVP at step **210** may include computing a rotated unitary matrix by performing, on the unitary matrix, for each coordinate pair of the plurality of coordinate pairs, an inverse Givens rotation by the parametrizing angle for the coordinate pair. The inverse Givens rotation for a coordinate pair may be performed around an axis specified by the coordinate pair. At step **214**, computing the JVP at step **210** may further include computing a rotated loss gradient matrix by performing, on the loss gradient matrix, for each coordinate pair of the plurality of coordinate pairs, a Givens rotation by the parametrizing angle for the coordinate pair. The Givens rotation for a coordinate pair may be performed around the axis specified by the coordinate pair. At step **216**, computing the JVP at step **210** may further include computing the JVP from the rotated unitary matrix and the rotated loss gradient matrix.

The method **200** may further include, at step **218**, updating the plurality of parametrizing angles based at least in part on the JVP. The plurality of parametrizing angles may be updated via gradient descent, which may be performed based at least in part on a learning rate as well as on the JVP. Thus, each backpropagation iteration may be performed to update the parametrizing angles that parametrize the weights of the neurons included in the machine learning model.

FIG. **8**B shows additional steps of the method **200** that may be performed in some examples when generating the coordinate pair sequence at step **202**. The steps shown in FIG. **8**B may be performed for an index sequence of n indices, where n is a number of rows of the unitary matrix. In addition, the steps of FIG. **8**B may be performed in each of a plurality of pairing iterations. At step **220**, the method **200** may further include, in each pairing iteration subsequent to an initial pairing iteration, shifting a respective position in the index sequence of each index other than a fixed index by 1 modulo n−1. At step **222**, the method **200** may further include iteratively pairing a first unpaired index in the index sequence with a last unpaired index in the index sequence until the index sequence includes fewer than two unpaired indices. At step **224**, the method **200** may further include adding each pair of indices to the coordinate pair sequence. Thus, in each pairing iteration of the plurality of pairing iterations, a block of coordinate pairs corresponding to mutually commutative Givens rotations may be generated. The order of the blocks in the coordinate pair sequence may indicate a block sequence.

FIG. **8**C shows additional steps of the method **200** that may be performed in some examples when computing the unitary matrix at step **206**. At step **226**, the method **200** may include initializing an intermediate unitary matrix as an identity matrix. The identity matrix may be of size n×n. At step **228**, the method **200** may further include iteratively performing the plurality of Givens rotations by the parametrizing angles on the identity matrix. Step **228** may include, at step **230**, performing the plurality of mutually commutative Givens rotations in parallel on a plurality of copies of the intermediate unitary matrix. Step **228** may further include, at step **232**, synchronizing the plurality of copies of the intermediate unitary matrix after the mutually commutative Givens rotations have been performed on the copies of the intermediate unitary matrix. Step **230** and step **232** may be performed for each of a plurality of Givens rotation blocks that each include a plurality of mutually commutative Givens rotations. After step **230** and step **232** have been performed for each Givens rotation block, the resulting intermediate unitary matrix may be used as the unitary matrix.

FIG. **8**D shows additional steps of the method **200** that may be performed in some examples when computing the rotated unitary matrix at step **212**. At step **234**, the method **200** may further include initializing an intermediate rotated unitary matrix as the unitary matrix. At step **236**, the method **200** may further include iteratively performing a plurality of inverse Givens rotations by the plurality of parametrizing angles on the unitary matrix. Step **236** may include, at step **238**, performing the mutually commutative inverse Givens rotations in parallel on a plurality of copies of the intermediate rotated unitary matrix. Step **236** may further include, at step **240**, synchronizing the plurality of copies of the intermediate rotated unitary matrix after the mutually commutative inverse Givens rotations have been performed on the copies of the intermediate rotated unitary matrix. Step **238** and step **240** may be performed for each of a plurality of inverse Givens rotation blocks that each include a plurality of mutually commutative inverse Givens rotations. The plurality of inverse Givens rotation blocks may be the Givens rotation blocks determined when computing the coordinate pair sequence. After step **238** and step **240** have been performed for each inverse Givens rotation block, the resulting intermediate rotated unitary matrix may be used as the rotated unitary matrix.

FIG. **8**E shows additional steps of the method **200** that may be performed in some examples when computing the rotated loss gradient matrix at step **216**. At step **242**, the method **200** may include initializing an intermediate rotated loss gradient matrix as the loss gradient matrix. At step **244**, the method **200** may further include iteratively performing the plurality of Givens rotations by the parametrizing angles on the loss gradient matrix. Step **244** may include, at step **246**, performing the mutually commutative Givens rotations in parallel on a plurality of copies of the intermediate rotated loss gradient matrix. Step **244** may further include, at step **248**, synchronizing the plurality of copies of the intermediate rotated loss gradient matrix after the mutually commutative Givens rotations have been performed on the copies of the intermediate rotated loss gradient matrix. Step **246** and step **248** may be performed for each of a plurality of Givens rotation blocks that each include a plurality of mutually commutative Givens rotations. After step **246** and step **248** have been performed for each Givens rotation block, the resulting intermediate rotated loss gradient matrix may be used as the rotated loss gradient matrix.

FIG. **8**F shows additional steps of the method **200** that may be performed in some examples when computing the JVP from the rotated unitary matrix and the rotated loss gradient matrix at step **216**. At step **250**, the method **200** may include computing an intermediate vector product matrix from the rotated unitary matrix and the rotated loss gradient matrix. Computing the intermediate vector product matrix at step **250** may include, at step **252**, generating a mapping of the coordinate pairs to rows of the intermediate vector product matrix. Computing the intermediate vector product matrix at step **250** may further include, at step **254**, for each row of the intermediate vector product matrix, computing each intermediate vector product matrix element included in that row as $M_{il}u_{lj}-M_{jl}u_{li}$. In this expression for the intermediate vector product matrix elements, M is the rotated loss gradient matrix, u is the rotated unitary matrix, i and j are the coordinates included in the coordinate pair, and l is a column number of the intermediate vector product matrix. At step **256**, the method **200** may further include computing, as elements of the JVP, a plurality of sums of rows of the intermediate vector product matrix. The sums of the rows of the intermediate vector product matrix may, for example, be computed by multiplying the intermediate vector product matrix by a vector of ones.

The techniques discussed above may allow the exploding and vanishing gradient problems to be avoided without relying on computationally expensive steps. Using the systems and methods discussed above, a machine learning model may be represented at least in part by a unitary matrix and may be trained in a plurality of backpropagation iterations. A unitary matrix may be generated from a plurality of parametrizing angles in O(n) time using the systems and methods discussed above. In addition, each of the backpropagation iterations may be performed in O(n log n) time, where n is the number of rows of the unitary matrix. Thus, the systems and methods discussed above may allow backpropagation to be performed efficiently even for large unitary matrices.

FIG. **9**A shows a plot **300** of execution time, in milliseconds, as a function of n when computing the unitary matrix U based on the plurality of parametrizing angles θ. In the example of FIG. **9**A, corresponding execution times are shown for the first algorithm **100** (a sequential algorithm) and the second algorithm **110** (a parallel algorithm). In addition, FIG. **9**B shows a plot **310** of execution time (in milliseconds) as a function of n when computing the JVP. In the example of FIG. **9**B, corresponding execution times are shown for a sequential algorithm as well as for the third algorithm **120** (a parallel algorithm). The sequential algorithms were executed on a CPU and the parallel algorithms (the second algorithm **110** and the third algorithm **120**) were executed on a GPU. As shown in the example of FIG. **9**A, the parallel algorithms may allow computation of U and the JVP to be sped up by multiple orders of magnitude compared to the sequential algorithms for the tested values of n.

In some embodiments, the methods and processes described herein may be tied to a computing system of one or more computing devices. In particular, such methods and processes may be implemented as a computer-application program or service, an application-programming interface (API), a library, and/or other computer-program product.

FIG. **10** schematically shows a non-limiting embodiment of a computing system **400** that can enact one or more of the methods and processes described above. Computing system **400** is shown in simplified form. Computing system **400** may embody the computing system **10** described above and illustrated in FIG. **1**A. Components of the computing system **400** may be included in one or more personal computers, server computers, tablet computers, home-entertainment computers, network computing devices, gaming devices, mobile computing devices, mobile communication devices (e.g., smart phone), and/or other computing devices, and wearable computing devices such as smart wristwatches and head mounted augmented reality devices.

Computing system **400** includes a logic processor **402** volatile memory **404**, and a non-volatile storage device **406**. Computing system **400** may optionally include a display subsystem **408**, input subsystem **410**, communication subsystem **412**, and/or other components not shown in FIG. **10**.

Logic processor **402** includes one or more physical devices configured to execute instructions. For example, the logic processor may be configured to execute instructions that are part of one or more applications, programs, routines, libraries, objects, components, data structures, or other logical constructs. Such instructions may be implemented to perform a task, implement a data type, transform the state of one or more components, achieve a technical effect, or otherwise arrive at a desired result.

The logic processor may include one or more physical processors (hardware) configured to execute software instructions. Additionally or alternatively, the logic processor may include one or more hardware logic circuits or firmware devices configured to execute hardware-implemented logic or firmware instructions. Processors of the logic processor **402** may be single-core or multi-core, and the instructions executed thereon may be configured for sequential, parallel, and/or distributed processing. Individual components of the logic processor optionally may be distributed among two or more separate devices, which may be remotely located and/or configured for coordinated processing. Aspects of the logic processor may be virtualized and executed by remotely accessible, networked computing devices configured in a cloud-computing configuration. In such a case, these virtualized aspects are run on different physical logic processors of various different machines, it will be understood.

Non-volatile storage device **406** includes one or more physical devices configured to hold instructions executable by the logic processors to implement the methods and processes described herein. When such methods and processes are implemented, the state of non-volatile storage device **406** may be transformed—e.g., to hold different data.

Non-volatile storage device **406** may include physical devices that are removable and/or built-in. Non-volatile storage device **406** may include optical memory (e.g., CD, DVD, HD-DVD, Blu-Ray Disc, etc.), semiconductor memory (e.g., ROM, EPROM, EEPROM, FLASH memory, etc.), and/or magnetic memory (e.g., hard-disk drive, floppy-disk drive, tape drive, MRAM, etc.), or other mass storage device technology. Non-volatile storage device **406** may include nonvolatile, dynamic, static, read/write, read-only, sequential-access, location-addressable, file-addressable, and/or content-addressable devices. It will be appreciated that non-volatile storage device **406** is configured to hold instructions even when power is cut to the non-volatile storage device **406**.

Volatile memory **404** may include physical devices that include random access memory. Volatile memory **404** is typically utilized by logic processor **402** to temporarily store information during processing of software instructions. It will be appreciated that volatile memory **404** typically does not continue to store instructions when power is cut to the volatile memory **404**.

Aspects of logic processor **402**, volatile memory **404**, and non-volatile storage device **406** may be integrated together into one or more hardware-logic components. Such hardware-logic components may include FPGAs, program- and application-specific integrated circuits (PASIC/ASICs), program- and application-specific standard products (PSSP/ASSPs), system-on-a-chip (SOC), and complex programmable logic devices (CPLDs), for example.

The terms "module," "program," and "engine" may be used to describe an aspect of computing system **400** typically implemented in software by a processor to perform a particular function using portions of volatile memory, which function involves transformative processing that specially configures the processor to perform the function. Thus, a module, program, or engine may be instantiated via logic processor **402** executing instructions held by non-volatile storage device **406**, using portions of volatile memory **404**. It will be understood that different modules, programs, and/or engines may be instantiated from the same application, service, code block, object, library, routine, API, function, etc. Likewise, the same module, program, and/or engine may be instantiated by different applications, services, code blocks, objects, routines, APIs, functions, etc. The terms "module," "program," and "engine" may encompass individual or groups of executable files, data files, libraries, drivers, scripts, database records, etc.

When included, display subsystem **408** may be used to present a visual representation of data held by non-volatile storage device **406**. The visual representation may take the form of a graphical user interface (GUI). As the herein described methods and processes change the data held by the non-volatile storage device, and thus transform the state of the non-volatile storage device, the state of display subsystem **408** may likewise be transformed to visually represent changes in the underlying data. Display subsystem **408** may include one or more display devices utilizing virtually any type of technology. Such display devices may be combined with logic processor **402**, volatile memory **404**, and/or non-volatile storage device **406** in a shared enclosure, or such display devices may be peripheral display devices.

When included, input subsystem **410** may comprise or interface with one or more user-input devices such as a keyboard, mouse, touch screen, or game controller. In some embodiments, the input subsystem may comprise or interface with selected natural user input (NUI) componentry. Such componentry may be integrated or peripheral, and the transduction and/or processing of input actions may be handled on- or off-board. Example NUI componentry may include a microphone for speech and/or voice recognition; an infrared, color, stereoscopic, and/or depth camera for machine vision and/or gesture recognition; a head tracker, eye tracker, accelerometer, and/or gyroscope for motion detection and/or intent recognition; as well as electric-field sensing componentry for assessing brain activity; and/or any other suitable sensor.

When included, communication subsystem **412** may be configured to communicatively couple various computing devices described herein with each other, and with other devices. Communication subsystem **412** may include wired and/or wireless communication devices compatible with one or more different communication protocols. As non-limiting examples, the communication subsystem may be configured for communication via a wireless telephone network, or a wired or wireless local- or wide-area network, such as a HDMI over Wi-Fi connection. In some embodiments, the communication subsystem may allow computing system **400** to send and/or receive messages to and/or from other devices via a network such as the Internet.

The following paragraphs discuss several aspects of the present disclosure. According to one aspect of the present disclosure, a computing system is provided, including a processor configured to train a machine learning model in a plurality of backpropagation iterations. Each backpropagation iteration may include generating a coordinate pair sequence including a plurality of coordinate pairs. Each coordinate pair of the plurality of coordinate pairs may be unique within the coordinate pair sequence and may include non-matching coordinates. The backpropagation iteration may further include receiving a plurality of parametrizing angles respectively associated with the plurality of coordinate pairs. The backpropagation iteration may further include computing a unitary matrix parametrized by the plurality of parametrizing angles. The backpropagation iteration may further include computing a loss gradient matrix of a training loss with respect to a plurality of unitary matrix elements of the unitary matrix. The backpropagation iteration may further include computing a Jacobian-vector product (JVP) of the training loss with respect to the plurality of parametrizing angles. Computing the JVP may include computing a rotated unitary matrix by performing, on the unitary matrix, for each coordinate pair of the plurality of coordinate pairs, an inverse Givens rotation by the parametrizing angle for the coordinate pair around an axis specified by the coordinate pair. Computing the JVP may further include computing a rotated loss gradient matrix by performing, on the loss gradient matrix, for each coordinate pair of the plurality of coordinate pairs, a Givens rotation by the parametrizing angle for the coordinate pair around the axis specified by the coordinate pair. The JVP may be computed from the rotated unitary matrix and the rotated loss gradient matrix. The backpropagation iteration may further include updating the plurality of parametrizing angles based at least in part on the JVP.

According to this aspect, the processor may be configured to generate the coordinate pair sequence at least in part by, for an index sequence of n indices, where n is a number of rows of the unitary matrix, in each of a plurality of pairing iterations subsequent to an initial pairing iteration, shifting a respective position in the index sequence of each index other than a fixed index by 1 modulo n−1. In each pairing iteration of the plurality of pairing iterations, the processor may be further configured to generate the coordinate pair sequence at least in part by iteratively pairing a first unpaired index in the index sequence with a last unpaired index in the index sequence until the index sequence includes fewer than two unpaired indices. In each pairing iteration of the plurality of pairing iterations, the processor may be further configured to generate the coordinate pair sequence at least in part by adding each pair of indices to the coordinate pair sequence.

According to this aspect, the processor may be configured to compute the unitary matrix at least in part by iteratively performing the plurality of Givens rotations by the parametrizing angles on an identity matrix.

According to this aspect, the processor may be configured to compute the unitary matrix at least in part by initializing an intermediate unitary matrix as the identity matrix. The processor may be further configured to compute the unitary matrix at least in part by iteratively, for each of a plurality of Givens rotation blocks that each include a plurality of mutually commutative Givens rotations, performing the plurality of mutually commutative Givens rotations in parallel on a plurality of copies of the intermediate unitary matrix and synchronizing the plurality of copies of the intermediate unitary matrix after the mutually commutative Givens rotations have been performed on the copies of the intermediate unitary matrix.

According to this aspect, the processor may be configured to compute the rotated unitary matrix at least in part by initializing an intermediate rotated unitary matrix as the unitary matrix. The processor may be further configured to compute the rotated unitary matrix at least in part by iteratively, for each of a plurality of inverse Givens rotation blocks that each include a plurality of mutually commutative inverse Givens rotations, performing the mutually commutative inverse Givens rotations in parallel on a plurality of copies of the intermediate rotated unitary matrix and synchronizing the plurality of copies of the intermediate rotated unitary matrix after the mutually commutative inverse Givens rotations have been performed on the copies of the intermediate rotated unitary matrix.

According to this aspect, the processor may be configured to compute the rotated loss gradient matrix at least in part by initializing an intermediate rotated loss gradient matrix as the loss gradient matrix. The processor may be further configured to compute the rotated loss gradient matrix at least in part by iteratively, for each of a plurality of Givens rotation blocks that each include a plurality of mutually commutative Givens rotations, performing the mutually commutative Givens rotations in parallel on a plurality of copies of the intermediate rotated loss gradient matrix and synchronizing the plurality of copies of the intermediate rotated loss gradient matrix after the mutually commutative Givens rotations have been performed on the copies of the intermediate rotated loss gradient matrix.

According to this aspect, the processor may be configured to compute the loss gradient matrix via automatic differentiation of the training loss with respect to the unitary matrix elements of the unitary matrix.

According to this aspect, the processor may be configured to compute the JVP from the rotated unitary matrix and the rotated loss gradient matrix at least in part by computing an intermediate vector product matrix from the rotated unitary matrix and the rotated loss gradient matrix. The processor may be further configured to compute the JVP from the rotated unitary matrix and the rotated loss gradient matrix at least in part by computing, as elements of the JVP, a plurality of sums of rows of the intermediate vector product matrix.

According to this aspect, the processor may be configured to compute the intermediate vector product matrix at least in part by generating a mapping of the coordinate pairs to rows of the intermediate vector product matrix. The processor may be further configured to compute the intermediate vector product matrix at least in part by, for each row of the intermediate vector product matrix, computing each intermediate vector product matrix element included in that row as $M_{il}m_{lj}-M_{jl}u_{li}$, wherein M is the rotated loss gradient matrix, u is the rotated unitary matrix, i and j are the coordinates included in the coordinate pair, and l is a column number of the intermediate vector product matrix.

According to this aspect, the processor may be configured to compute the JVP with O(n log n) complexity.

According to another aspect of the present disclosure, a method of training a machine learning model is provided. The method may include, in each of a plurality of backpropagation iterations, generating a coordinate pair sequence including a plurality of coordinate pairs. Each coordinate pair of the plurality of coordinate pairs may be unique within the coordinate pair sequence and may include non-matching coordinates. The method may further include, in each of the plurality of backpropagation iterations, receiving a plurality of parametrizing angles respectively associated with the plurality of coordinate pairs. The method may further include, in each of the plurality of backpropagation iterations, computing a unitary matrix parametrized by the plurality of parametrizing angles. The method may further include, in each of the plurality of backpropagation iterations, computing a loss gradient matrix of a training loss with respect to a plurality of unitary matrix elements of the unitary matrix. The method may further include, in each of the plurality of backpropagation iterations, computing a Jacobian-vector product (JVP) of the training loss with respect to the plurality of parametrizing angles. Computing the JVP may include computing a rotated unitary matrix by performing, on the unitary matrix, for each coordinate pair of the plurality of coordinate pairs, an inverse Givens rotation by the parametrizing angle for the coordinate pair around an axis specified by the coordinate pair. Computing the JVP may further include computing a rotated loss gradient matrix by performing, on the loss gradient matrix, for each coordinate pair of the plurality of coordinate pairs, a Givens rotation by the parametrizing angle for the coordinate pair around the axis specified by the coordinate pair. The JVP may be computed from the rotated unitary matrix and the rotated loss gradient matrix. The method may further include, in each of the plurality of backpropagation iterations, updating the plurality of parametrizing angles based at least in part on the JVP.

According to this aspect, generating the coordinate pair sequence may include, for an index sequence of n indices, where n is a number of rows of the unitary matrix, in each of a plurality of pairing iterations subsequent to an initial pairing iteration, shifting a respective position in the index sequence of each index other than a fixed index by 1 modulo n−1. Generating the coordinate pair sequence may further include, in each of the plurality of pairing iterations, iteratively pairing a first unpaired index in the index sequence with a last unpaired index in the index sequence until the index sequence includes fewer than two unpaired indices. Generating the coordinate pair sequence may further include, in each of the plurality of pairing iterations, adding each pair of indices to the coordinate pair sequence.

According to this aspect, computing unitary matrix may include iteratively performing the plurality of Givens rotations by the parametrizing angles on an identity matrix.

According to this aspect, computing the unitary matrix may include initializing an intermediate unitary matrix as the identity matrix. Computing the unitary matrix may further include iteratively, for each of a plurality of Givens rotation blocks that each include a plurality of mutually commutative Givens rotations, performing the plurality of mutually commutative Givens rotations in parallel on a plurality of copies of the intermediate unitary matrix and synchronizing the plurality of copies of the intermediate unitary matrix after the mutually commutative Givens rotations have been performed on the copies of the intermediate unitary matrix.

According to this aspect, computing the rotated unitary matrix may include initializing an intermediate rotated unitary matrix as the unitary matrix. Computing the rotated unitary matrix may further include iteratively, for each of a plurality of inverse Givens rotation blocks that each include a plurality of mutually commutative inverse Givens rotations, performing the mutually commutative inverse Givens rotations in parallel on a plurality of copies of the intermediate rotated unitary matrix and synchronizing the plurality of copies of the intermediate rotated unitary matrix after the mutually commutative inverse Givens rotations have been performed on the copies of the intermediate rotated unitary matrix.

According to this aspect, computing the rotated loss gradient matrix may include initializing an intermediate rotated loss gradient matrix as the loss gradient matrix. Computing the rotated loss gradient matrix may further include iteratively, for each of a plurality of Givens rotation blocks that each include a plurality of mutually commutative Givens rotations, performing the mutually commutative Givens rotations in parallel on a plurality of copies of the intermediate rotated loss gradient matrix and synchronizing the plurality of copies of the intermediate rotated loss gradient matrix after the mutually commutative Givens rotations have been performed on the copies of the intermediate rotated loss gradient matrix.

According to this aspect, the loss gradient matrix may be computed via automatic differentiation of the training loss with respect to the unitary matrix elements of the unitary matrix.

According to this aspect, computing the JVP from the rotated unitary matrix and the rotated loss gradient matrix may include computing an intermediate vector product matrix from the rotated unitary matrix and the rotated loss gradient matrix. Computing the JVP from the rotated unitary matrix and the rotated loss gradient matrix may further include computing, as elements of the JVP, a plurality of sums of rows of the intermediate vector product matrix.

According to this aspect, computing the intermediate vector product matrix may include generating a mapping of the coordinate pairs to rows of the intermediate vector product matrix. Computing the intermediate vector product matrix may further include, for each row of the intermediate vector product matrix, computing each intermediate vector product matrix element included in that row as $M_{il}u_{lj}-M_{jl}u_{li}$, wherein M is the rotated loss gradient matrix, u is the rotated unitary matrix, i and j are the coordinates included in the coordinate pair, and l is a column number of the intermediate vector product matrix.

According to another aspect of the present disclosure, a computing system is provided, including a processor configured to train a machine learning model in a plurality of backpropagation iterations. Each backpropagation iteration may include, in a forward pass, computing a unitary matrix parametrized by a plurality of parametrizing angles at least in part by performing a plurality of Givens rotations in parallel. Each backpropagation iteration may further include, in a backward pass, computing a Jacobian-vector product (JVP) of a training loss with respect to the plurality of parametrizing angles at least in part by performing, in parallel, the plurality of Givens rotations on a loss gradient matrix and a plurality of inverse Givens rotations on the unitary matrix. Each backpropagation iteration may further include updating the plurality of parametrizing angles based at least in part on the JVP.

"And/or" as used herein is defined as the inclusive or v, as specified by the following truth table:

A	B	A ∨ B
True	True	True
True	False	True
False	True	True
False	False	False

It will be understood that the configurations and/or approaches described herein are exemplary in nature, and that these specific embodiments or examples are not to be considered in a limiting sense, because numerous variations are possible. The specific routines or methods described herein may represent one or more of any number of processing strategies. As such, various acts illustrated and/or described may be performed in the sequence illustrated and/or described, in other sequences, in parallel, or omitted. Likewise, the order of the above-described processes may be changed.

The subject matter of the present disclosure includes all novel and non-obvious combinations and sub-combinations of the various processes, systems and configurations, and other features, functions, acts, and/or properties disclosed herein, as well as any and all equivalents thereof.

The invention claimed is:

1. A computing system comprising:

a processor configured to:

train a machine learning model in a plurality of backpropagation iterations, wherein each backpropagation iteration includes:

generating a coordinate pair sequence including a plurality of coordinate pairs, wherein each coordinate pair of the plurality of coordinate pairs is unique within the coordinate pair sequence and includes non-matching coordinates;

receiving, as parameters of the machine learning model, a plurality of parametrizing angles respectively associated with the plurality of coordinate pairs;

computing a unitary matrix parametrized by the plurality of parametrizing angles, wherein computing the unitary matrix includes:

initializing an intermediate unitary matrix as an identity matrix;

iteratively, for each of a plurality of Givens rotation blocks, wherein each of the Givens rotation blocks is a sequence of mutually commutative Givens rotations:

performing the mutually commutative Givens rotations in parallel on a plurality of copies of the intermediate unitary matrix; and

synchronizing the plurality of copies of the intermediate unitary matrix after the mutually commutative Givens rotations have been performed on the copies of the intermediate unitary matrix;

computing a loss gradient matrix of a training loss with respect to a plurality of unitary matrix elements of the unitary matrix;

computing a Jacobian-vector product (JVP) of the training loss with respect to the plurality of parametrizing angles, wherein computing the JVP includes:

computing a rotated unitary matrix by performing, on the unitary matrix, for each coordinate pair of the plurality of coordinate pairs, an inverse Givens rotation by the parametrizing angle for the coordinate pair around an axis specified by the coordinate pair;

computing a rotated loss gradient matrix by performing, on the loss gradient matrix, for each coordinate pair of the plurality of coordinate pairs, a Givens rotation by the parametrizing angle for the coordinate pair around the axis specified by the coordinate pair; and

computing the JVP from the rotated unitary matrix and the rotated loss gradient matrix; and

updating the plurality of parametrizing angles based at least in part on the JVP to thereby train the machine learning model.

2. The computing system of claim **1**, wherein the processor is configured to generate the coordinate pair sequence at least in part by:

for an index sequence of n indices, where n is a number of rows of the unitary matrix:

in each of a plurality of pairing iterations:

in each pairing iteration subsequent to an initial pairing iteration, shifting a respective position in the index sequence of each index other than a fixed index by 1 modulo n−1;

iteratively pairing a first unpaired index in the index sequence with a last unpaired index in the index sequence until the index sequence includes fewer than two unpaired indices; and

adding each pair of indices to the coordinate pair sequence.

3. The computing system of claim **1**, wherein the processor is configured to compute the rotated unitary matrix at least in part by:

initializing an intermediate rotated unitary matrix as the unitary matrix; and

iteratively, for each of a plurality of inverse Givens rotation blocks that each include a plurality of mutually commutative inverse Givens rotations:

performing the mutually commutative inverse Givens rotations in parallel on a plurality of copies of the intermediate rotated unitary matrix; and

synchronizing the plurality of copies of the intermediate rotated unitary matrix after the mutually commutative inverse Givens rotations have been performed on the copies of the intermediate rotated unitary matrix.

4. The computing system of claim **1**, wherein the processor is configured to compute the rotated loss gradient matrix at least in part by:

initializing an intermediate rotated loss gradient matrix as the loss gradient matrix; and

iteratively, for each of a plurality of Givens rotation blocks that each include a plurality of mutually commutative Givens rotations:

performing the mutually commutative Givens rotations in parallel on a plurality of copies of the intermediate rotated loss gradient matrix; and

synchronizing the plurality of copies of the intermediate rotated loss gradient matrix after the mutually commutative Givens rotations have been performed on the copies of the intermediate rotated loss gradient matrix.

5. The computing system of claim **1**, wherein the processor is configured to compute the loss gradient matrix via automatic differentiation of the training loss with respect to the unitary matrix elements of the unitary matrix.

6. The computing system of claim **1**, wherein the processor is configured to compute the JVP from the rotated unitary matrix and the rotated loss gradient matrix at least in part by:

computing an intermediate vector product matrix from the rotated unitary matrix and the rotated loss gradient matrix; and

computing, as elements of the JVP, a plurality of sums of rows of the intermediate vector product matrix.

7. The computing system of claim **6**, wherein the processor is configured to compute the intermediate vector product matrix at least in part by:

generating a mapping of the coordinate pairs to rows of the intermediate vector product matrix; and

for each row of the intermediate vector product matrix, computing each intermediate vector product matrix element included in that row as $M_{il}u_{lj}-M_{jl}u_{li}$, wherein M is the rotated loss gradient matrix, u is the rotated unitary matrix, i and j are the coordinates included in the coordinate pair, and I is a column number of the intermediate vector product matrix.

8. The computing system of claim **1**, wherein the processor is configured to compute the JVP with O(n log n) complexity.

9. A method of training a machine learning model, the method comprising, in each of a plurality of backpropagation iterations:

generating a coordinate pair sequence including a plurality of coordinate pairs, wherein each coordinate pair of the plurality of coordinate pairs is unique within the coordinate pair sequence and includes non-matching coordinates;

receiving, as parameters of the machine learning model, a plurality of parametrizing angles respectively associated with the plurality of coordinate pairs;

computing a unitary matrix parametrized by the plurality of parametrizing angles, wherein computing the unitary matrix includes:

initializing an intermediate unitary matrix as an identity matrix;

iteratively, for each of a plurality of Givens rotation blocks, wherein each of the Givens rotation blocks is a sequence of mutually commutative Givens rotations:

performing the mutually commutative Givens rotations in parallel on a plurality of copies of the intermediate unitary matrix; and

synchronizing the plurality of copies of the intermediate unitary matrix after the mutually commutative Givens rotations have been performed on the copies of the intermediate unitary matrix;

computing a loss gradient matrix of a training loss with respect to a plurality of unitary matrix elements of the unitary matrix;

computing a Jacobian-vector product (JVP) of the training loss with respect to the plurality of parametrizing angles, wherein computing the JVP includes:
computing a rotated unitary matrix by performing, on the unitary matrix, for each coordinate pair of the plurality of coordinate pairs, an inverse Givens rotation by the parametrizing angle for the coordinate pair around an axis specified by the coordinate pair;
computing a rotated loss gradient matrix by performing, on the loss gradient matrix, for each coordinate pair of the plurality of coordinate pairs, a Givens rotation by the parametrizing angle for the coordinate pair around the axis specified by the coordinate pair; and
computing the JVP from the rotated unitary matrix and the rotated loss gradient matrix; and
updating the plurality of parametrizing angles based at least in part on the JVP to thereby train the machine learning model.

10. The method of claim **9**, wherein generating the coordinate pair sequence includes:
for an index sequence of n indices, where n is a number of rows of the unitary matrix:
in each of a plurality of pairing iterations:
in each pairing iteration subsequent to an initial pairing iteration, shifting a respective position in the index sequence of each index other than a fixed index by 1 modulo n−1;
iteratively pairing a first unpaired index in the index sequence with a last unpaired index in the index sequence until the index sequence includes fewer than two unpaired indices; and
adding each pair of indices to the coordinate pair sequence.

11. The method of claim **9**, wherein computing the rotated unitary matrix includes:
initializing an intermediate rotated unitary matrix as the unitary matrix; and
iteratively, for each of a plurality of inverse Givens rotation blocks that each include a plurality of mutually commutative inverse Givens rotations:
performing the mutually commutative inverse Givens rotations in parallel on a plurality of copies of the intermediate rotated unitary matrix; and
synchronizing the plurality of copies of the intermediate rotated unitary matrix after the mutually commutative inverse Givens rotations have been performed on the copies of the intermediate rotated unitary matrix.

12. The method of claim **9**, wherein computing the rotated loss gradient matrix includes:
initializing an intermediate rotated loss gradient matrix as the loss gradient matrix; and
iteratively, for each of a plurality of Givens rotation blocks that each include a plurality of mutually commutative Givens rotations:
performing the mutually commutative Givens rotations in parallel on a plurality of copies of the intermediate rotated loss gradient matrix; and
synchronizing the plurality of copies of the intermediate rotated loss gradient matrix after the mutually commutative Givens rotations have been performed on the copies of the intermediate rotated loss gradient matrix.

13. The method of claim **9**, wherein the loss gradient matrix is computed via automatic differentiation of the training loss with respect to the unitary matrix elements of the unitary matrix.

14. The method of claim **9**, wherein computing the JVP from the rotated unitary matrix and the rotated loss gradient matrix includes:
computing an intermediate vector product matrix from the rotated unitary matrix and the rotated loss gradient matrix; and
computing, as elements of the JVP, a plurality of sums of rows of the intermediate vector product matrix.

15. The method of claim **14**, wherein computing the intermediate vector product matrix includes:
generating a mapping of the coordinate pairs to rows of the intermediate vector product matrix; and
for each row of the intermediate vector product matrix, computing each intermediate vector product matrix element included in that row as $M_{il}u_{lj}-M_{jl}u_{li}$, wherein M is the rotated loss gradient matrix, u is the rotated unitary matrix, i and j are the coordinates included in the coordinate pair, and I is a column number of the intermediate vector product matrix.

16. A computing system comprising:
a processor configured to:
train a machine learning model in a plurality of backpropagation iterations, wherein each backpropagation iteration includes:
in a forward pass, computing a unitary matrix parametrized by a plurality of parametrizing angles, wherein:
the parametrizing angles are parameters of the machine learning model; and
computing the unitary matrix includes:
initializing an intermediate unitary matrix as an identity matrix;
iteratively, for each of a plurality of Givens rotation blocks, wherein each of the Givens rotation blocks is a sequence of mutually commutative Givens rotations:
performing the mutually commutative Givens rotations in parallel on a plurality of copies of the intermediate unitary matrix; and
synchronizing the plurality of copies of the intermediate unitary matrix after the mutually commutative Givens rotations have been performed on the copies of the intermediate unitary matrix; and
in a backward pass:
computing a Jacobian-vector product (JVP) of a training loss with respect to the plurality of parametrizing angles at least in part by performing, in parallel, the plurality of Givens rotations on a loss gradient matrix and a plurality of inverse Givens rotations on the unitary matrix; and
updating the plurality of parametrizing angles based at least in part on the JVP to thereby train the machine learning model.

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