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Kolych et al.

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(54) **TRACKING OF OBJECTS USING RECONSTRUCTION OF DATA CARRIED BY WIRELESS SENSING SIGNALS FROM LIMITED SENSING FREQUENCIES**

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H04L 25/02	(2006.01)
H04W 4/38	(2018.01)

(52) **U.S. Cl.**
CPC ***H04W 4/026*** (2013.01); ***H04L 25/0232*** (2013.01); ***H04W 4/027*** (2013.01); ***H04W 4/38*** (2018.02)

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CPC H04W 4/026; H04W 4/38; H04W 4/027; H04L 25/0232
USPC 342/350
See application file for complete search history.

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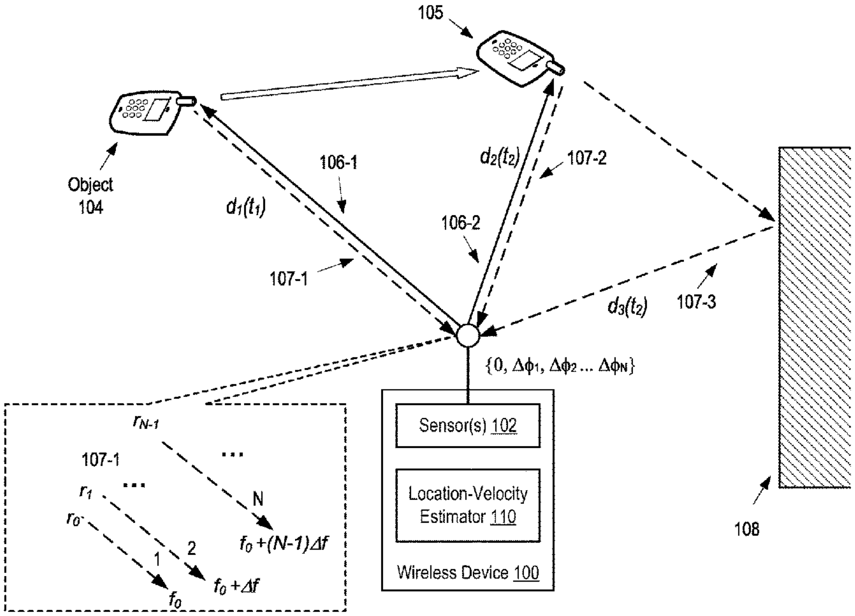
(57) **ABSTRACT**

Implementations disclosed describe techniques and systems for efficient determination and tracking of trajectories of objects in an environment of a wireless device. The disclosed techniques include, among other things, obtaining multiple sets of sensing values that characterize one or more radio signals received, during a respective sensing event, from an object in an environment of the wireless device. The sensing signals may be carried by waves with randomly selected frequencies representing a portion of all frequencies that are used as working sensing frequencies. Multiple techniques of efficient frequency interpolation and temporal interpolation are disclosed that reconstruct the sensing values to the full range of working frequencies. The reconstructed sensing values may then be used to track objects in the environment.

20 Claims, 12 Drawing Sheets



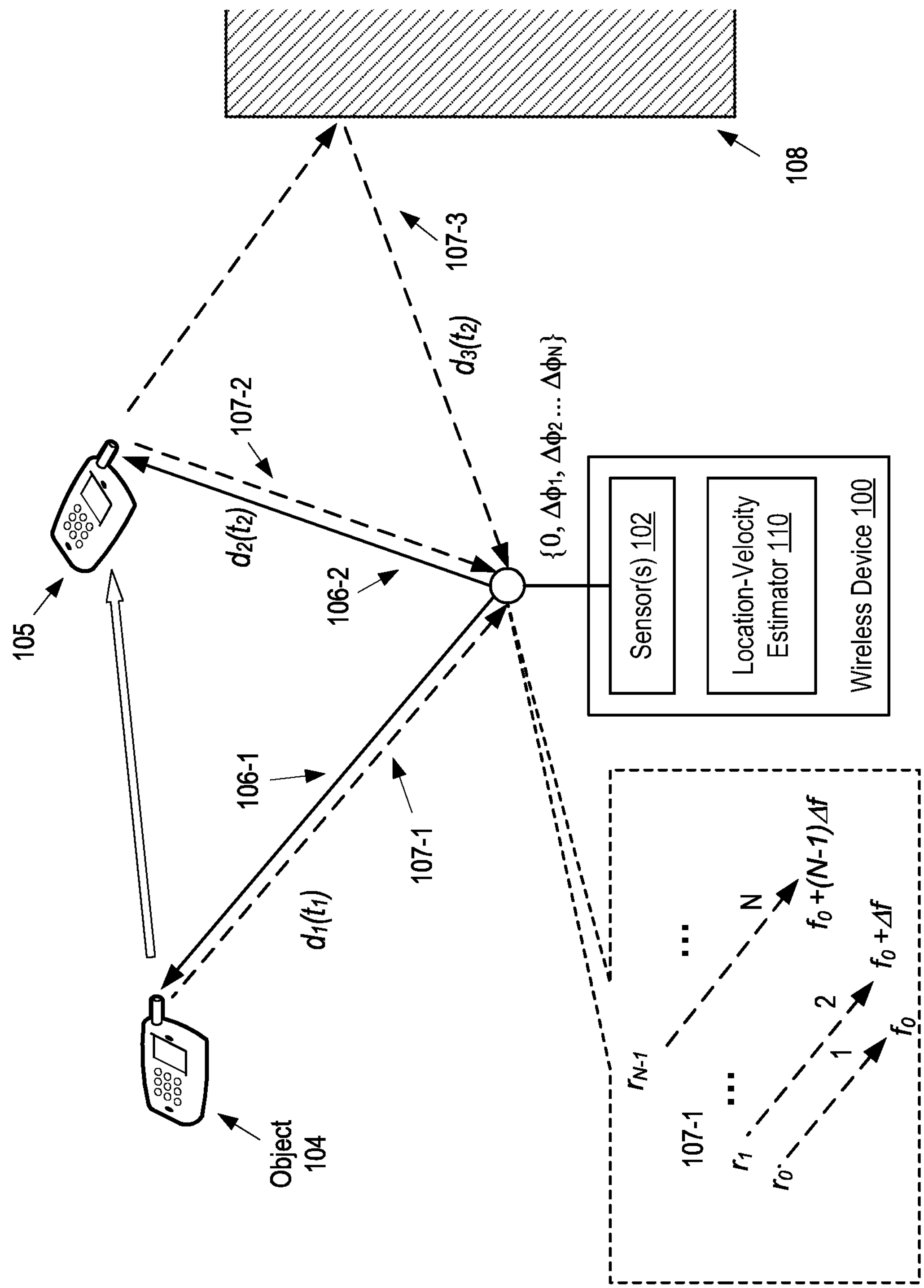

108
107-3
$d_3(t_2)$
$\{0, \Delta\phi_1, \Delta\phi_2 \ldots \Delta\phi_N\}$
107-2
$d_2(t_2)$
106-2
105
Sensor(s) 102
Location-Velocity Estimator 110
Wireless Device 100
106-1
107-1
$d_1(t_1)$
Object 104
107-1
r_0
r_1
r_{N-1}
...
1
2
N
f_0
$f_0+\Delta f$
$f_0+(N-1)\Delta f$

FIG. 1

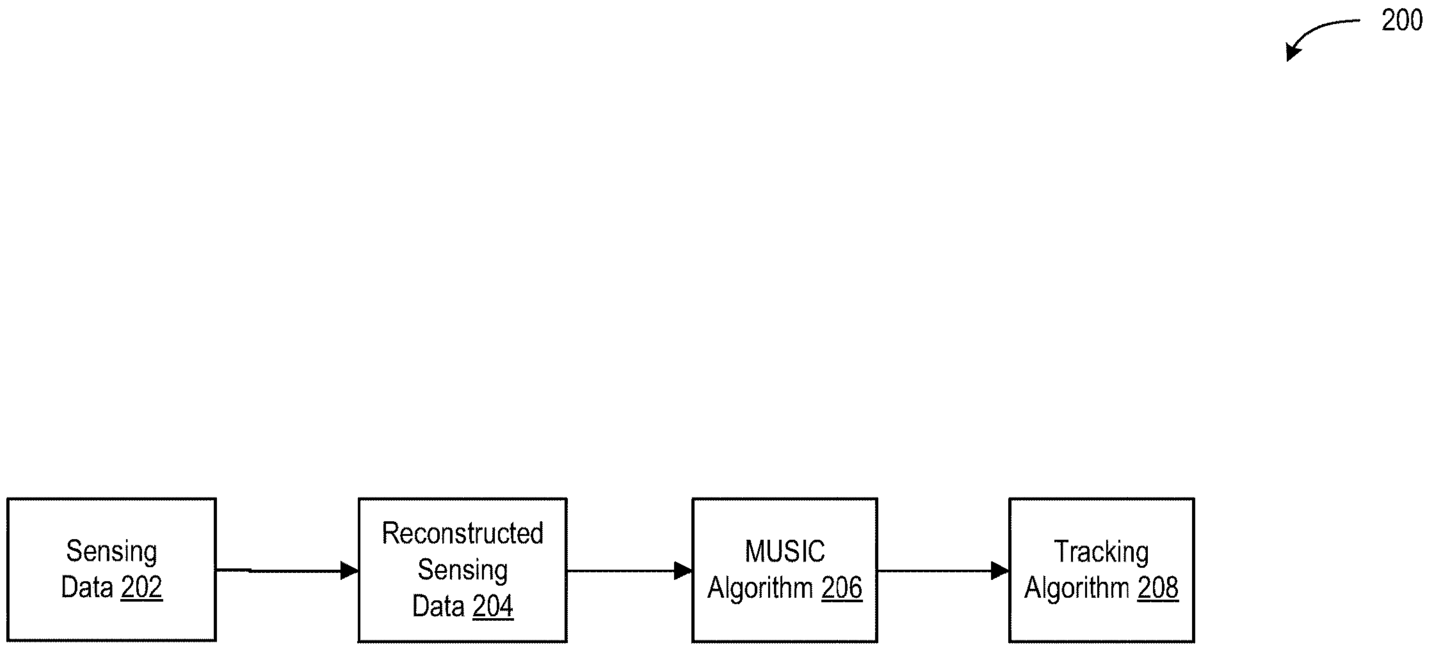

200
Sensing Data 202
Reconstructed Sensing Data 204
MUSIC Algorithm 206
Tracking Algorithm 208

FIG. 2

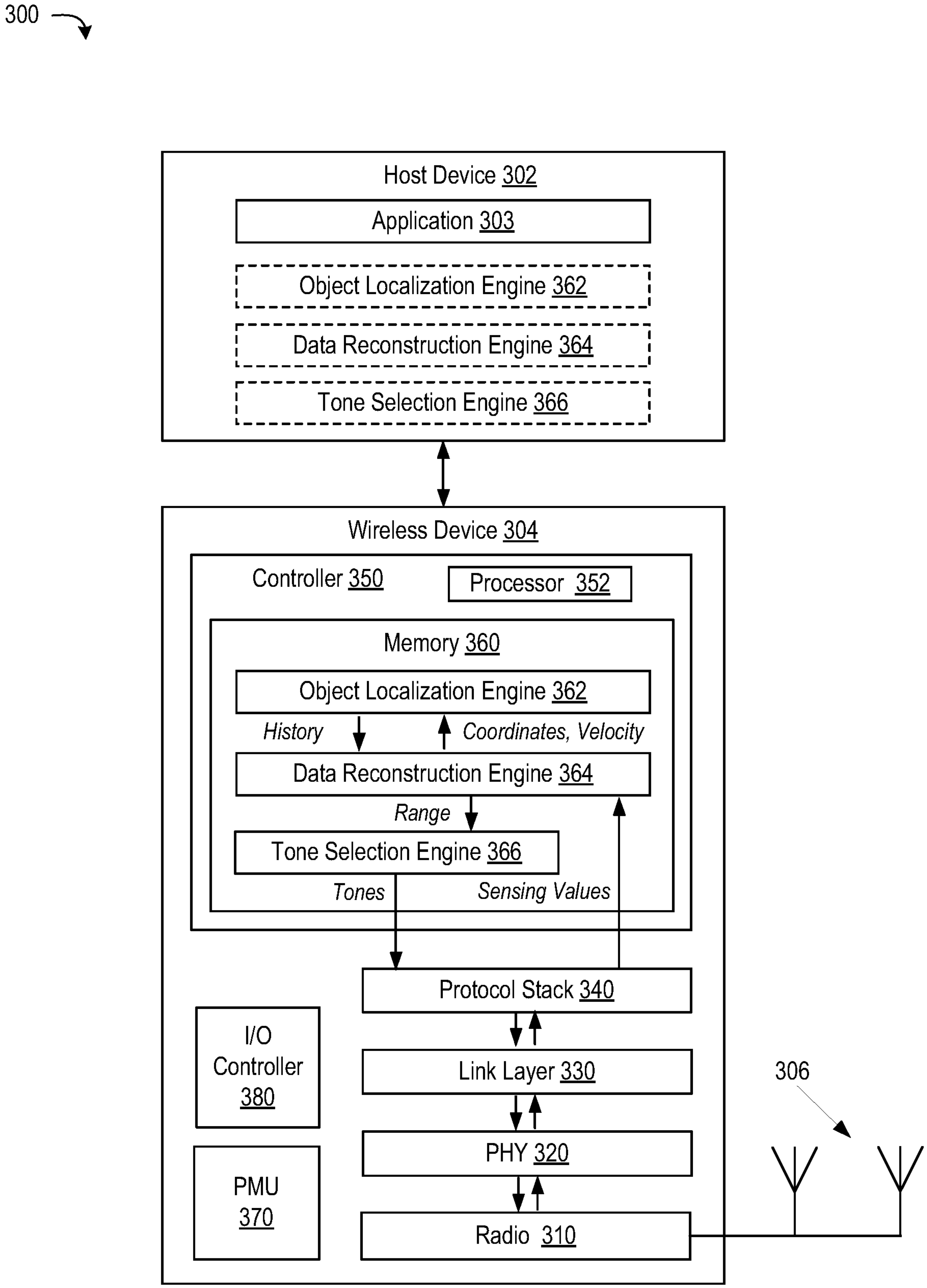
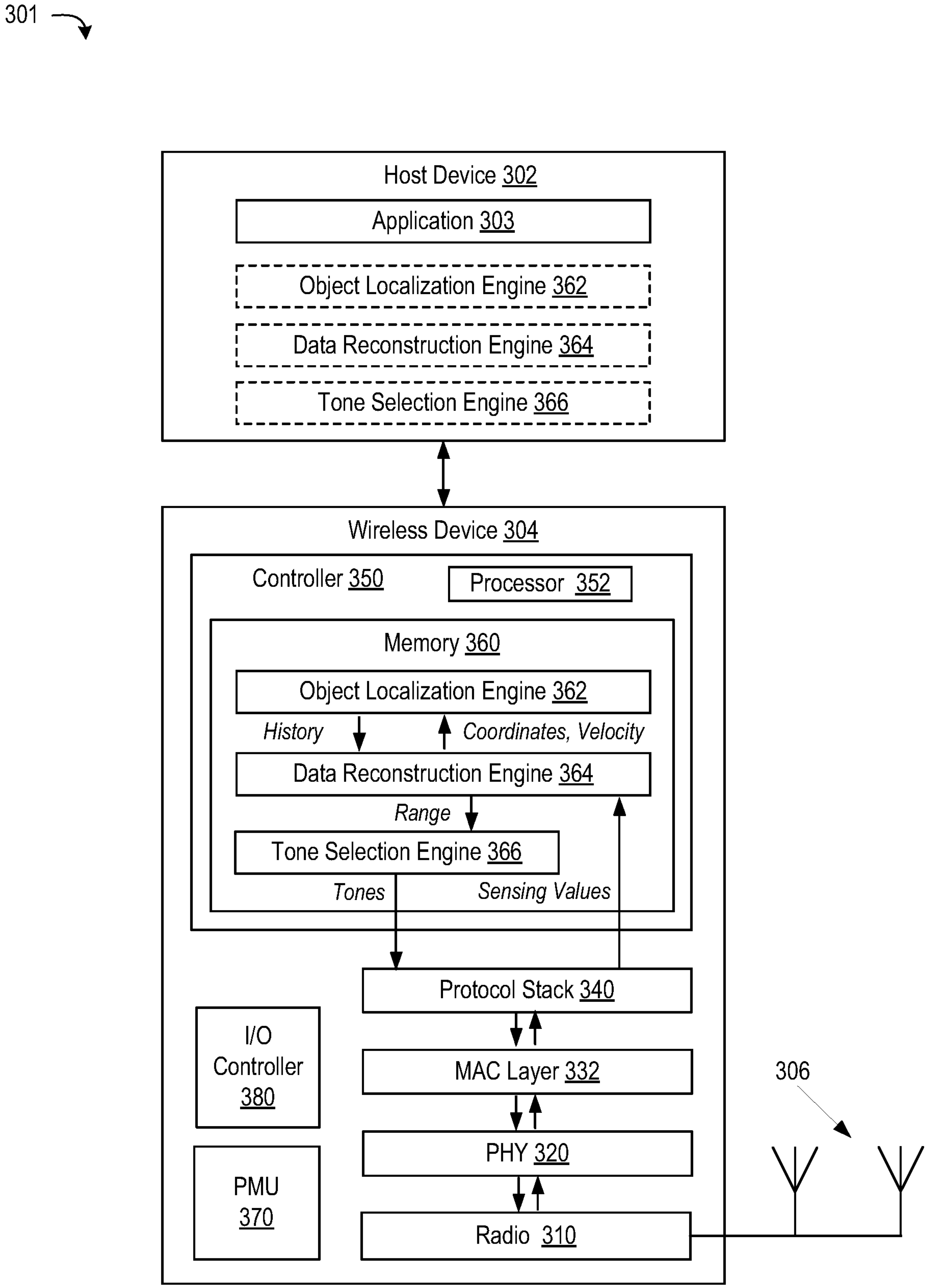

300
Host Device 302
Application 303
Object Localization Engine 362
Data Reconstruction Engine 364
Tone Selection Engine 366
Wireless Device 304
Controller 350
Processor 352
Memory 360
Object Localization Engine 362
History
Coordinates, Velocity
Data Reconstruction Engine 364
Range
Tone Selection Engine 366
Tones
Sensing Values
Protocol Stack 340
I/O Controller 380
Link Layer 330
306
PHY 320
PMU 370
Radio 310

FIG. 3A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Object Localization Engine 362
Data Reconstruction Engine 364
Tone Selection Engine 366
Wireless Device 304
Controller 350
Processor 352
Memory 360
Object Localization Engine 362
History
Coordinates, Velocity
Data Reconstruction Engine 364
Range
Tone Selection Engine 366
Tones
Sensing Values
Protocol Stack 340
I/O Controller 380
MAC Layer 332
306
PHY 320
PMU 370
Radio 310

FIG. 3B

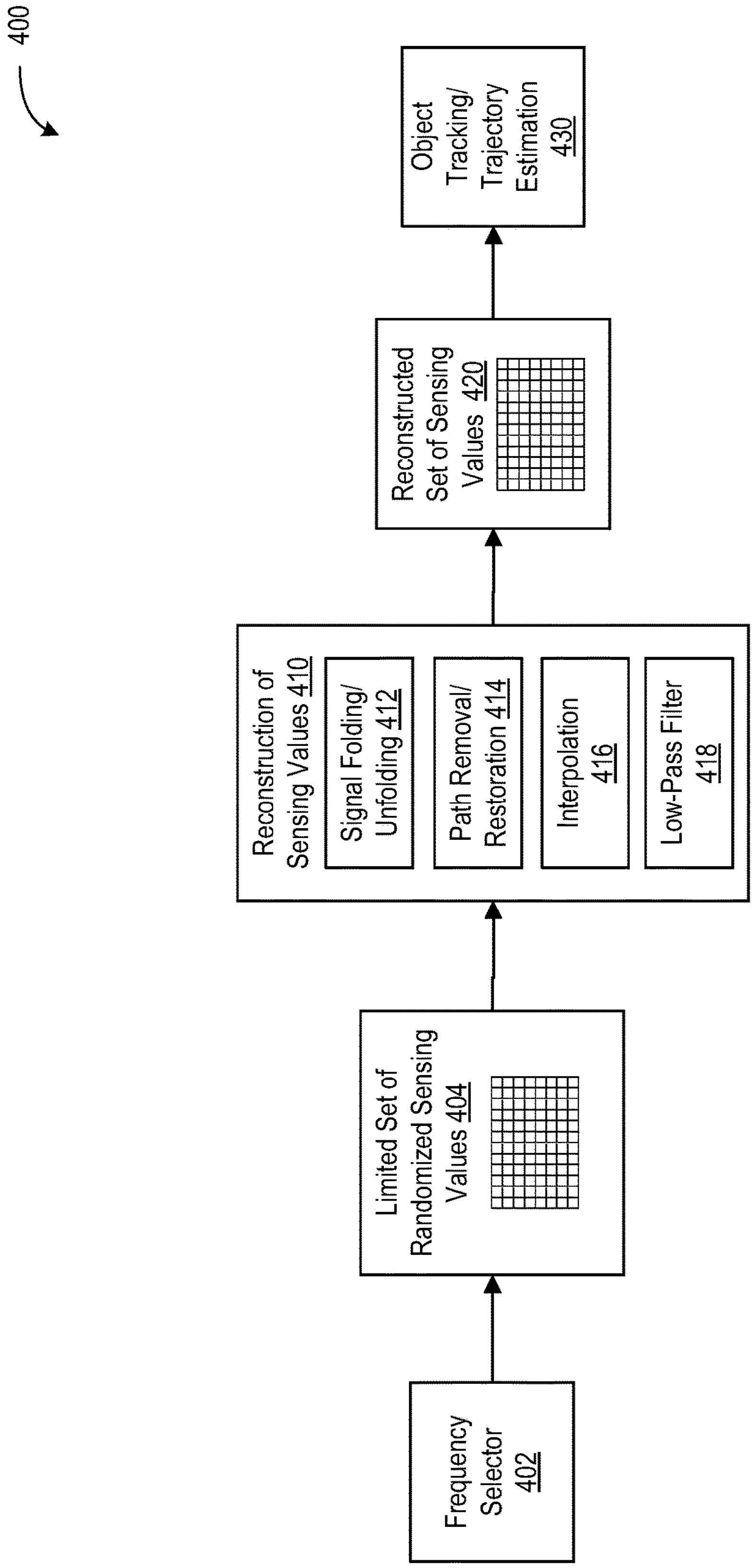

400
Frequency Selector 402
Limited Set of Randomized Sensing Values 404
Reconstruction of Sensing Values 410
Signal Folding/ Unfolding 412
Path Removal/ Restoration 414
Interpolation 416
Low-Pass Filter 418
Reconstructed Set of Sensing Values 420
Object Tracking/ Trajectory Estimation 430

FIG. 4

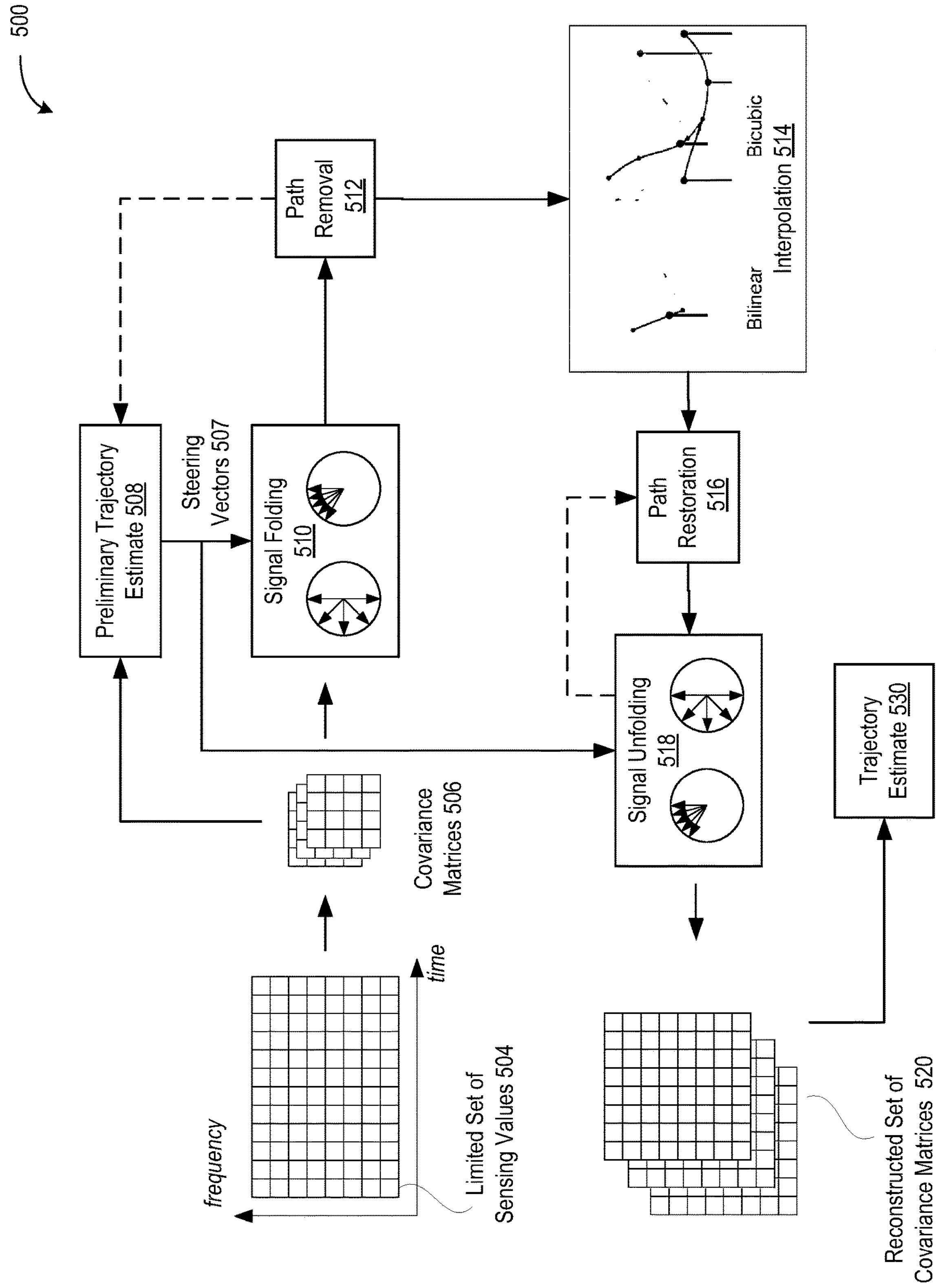
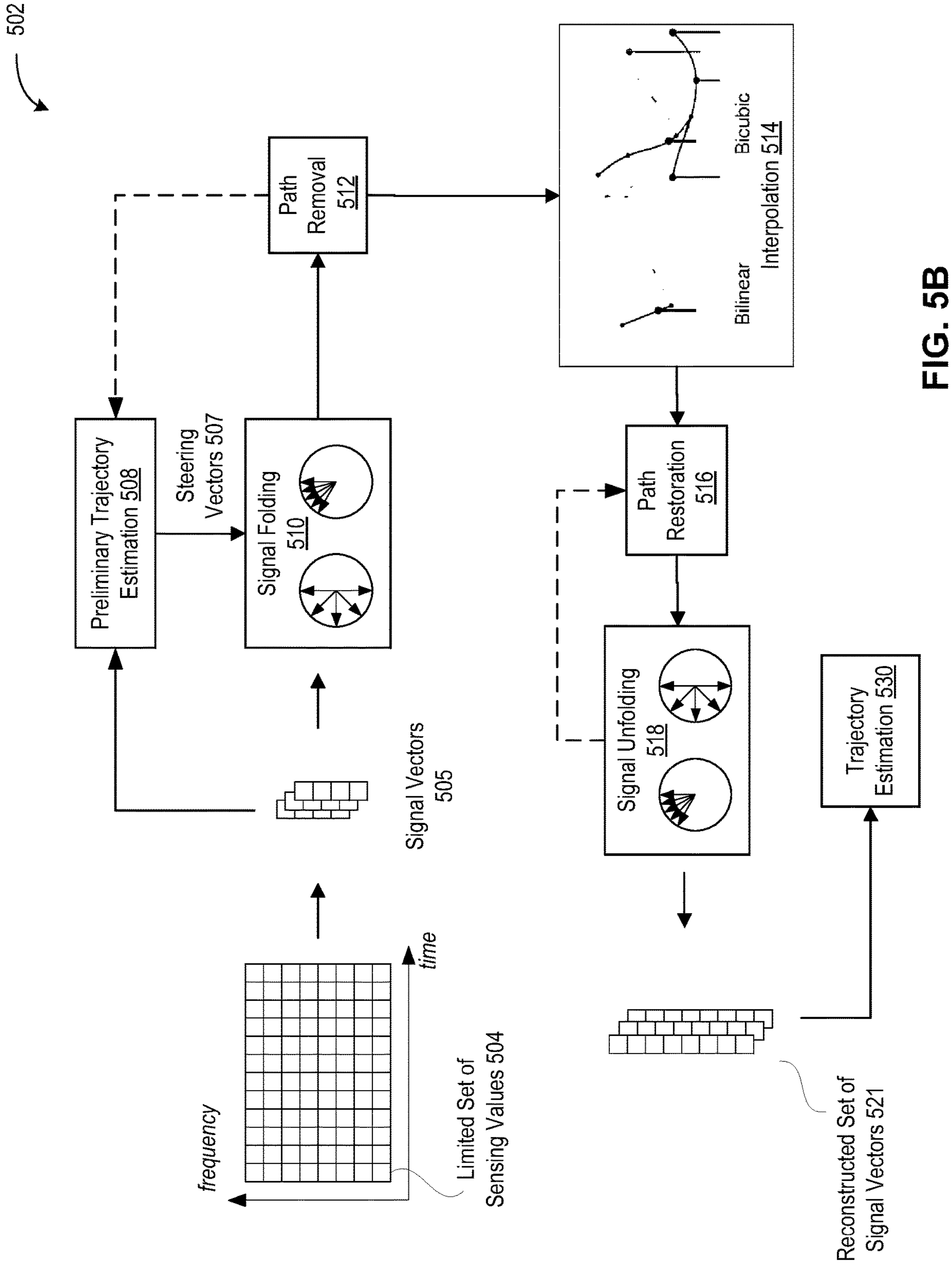

500
Preliminary Trajectory Estimate 508
Steering Vectors 507
Signal Folding 510
Path Removal 512
Bilinear
Bicubic
Interpolation 514
Path Restoration 516
Signal Unfolding 518
Trajectory Estimate 530
Covariance Matrices 506
frequency
time
Limited Set of Sensing Values 504
Reconstructed Set of Covariance Matrices 520

FIG. 5A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frequency
time
Limited Set of Sensing Values 504
Signal Vectors 505
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Steering Vectors 507
Signal Folding 510
Path Removal 512
Bilinear
Bicubic
Interpolation 514
Path Restoration 516
Signal Unfolding 518
Reconstructed Set of Signal Vectors 521
Trajectory Estimation 530

FIG. 5B

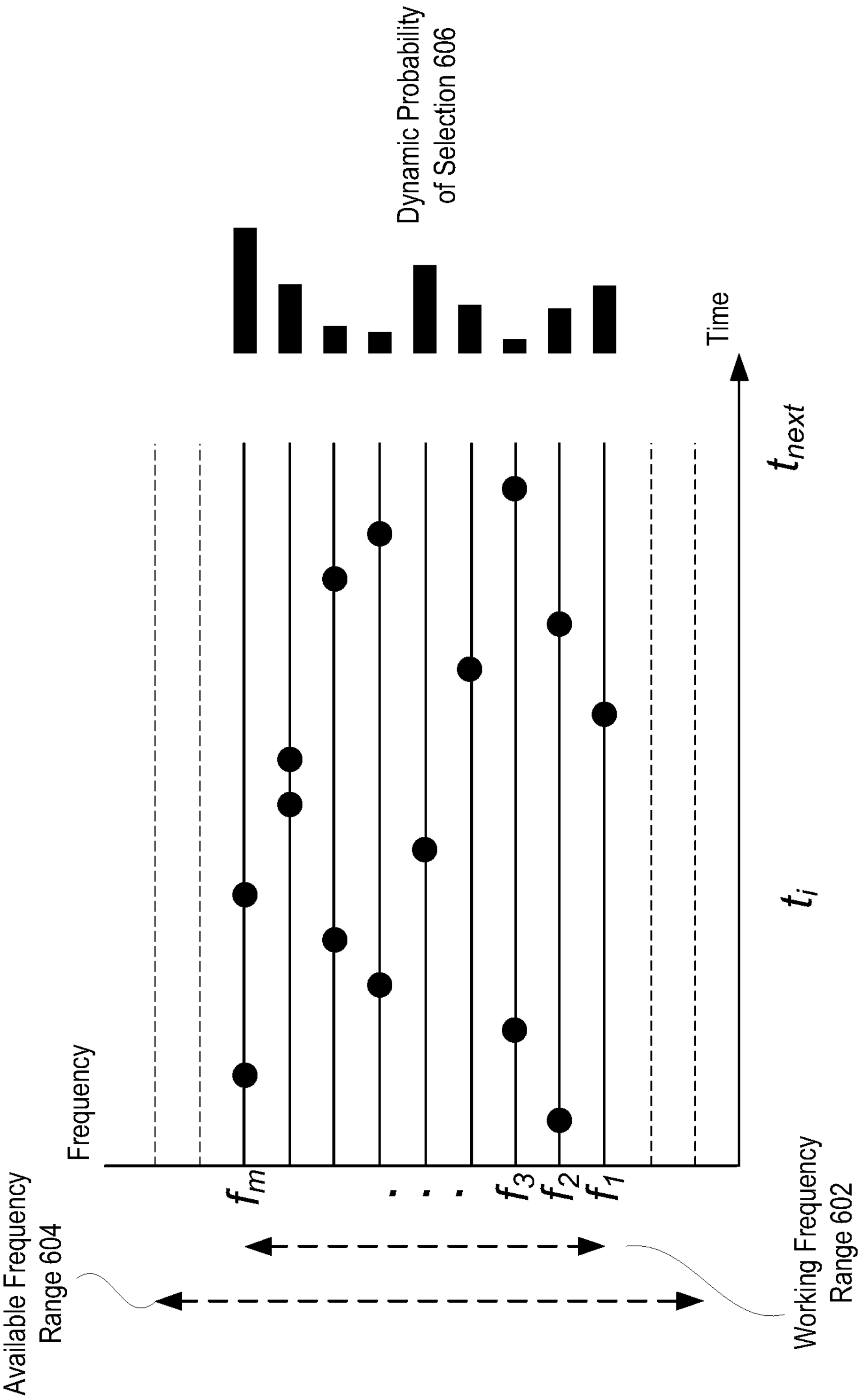
Dynamic Probability of Selection 606
Time
Frequency
t_{next}
t_i
f_m
. . .
f_3
f_2
f_1
Available Frequency Range 604
Working Frequency Range 602

FIG. 6A

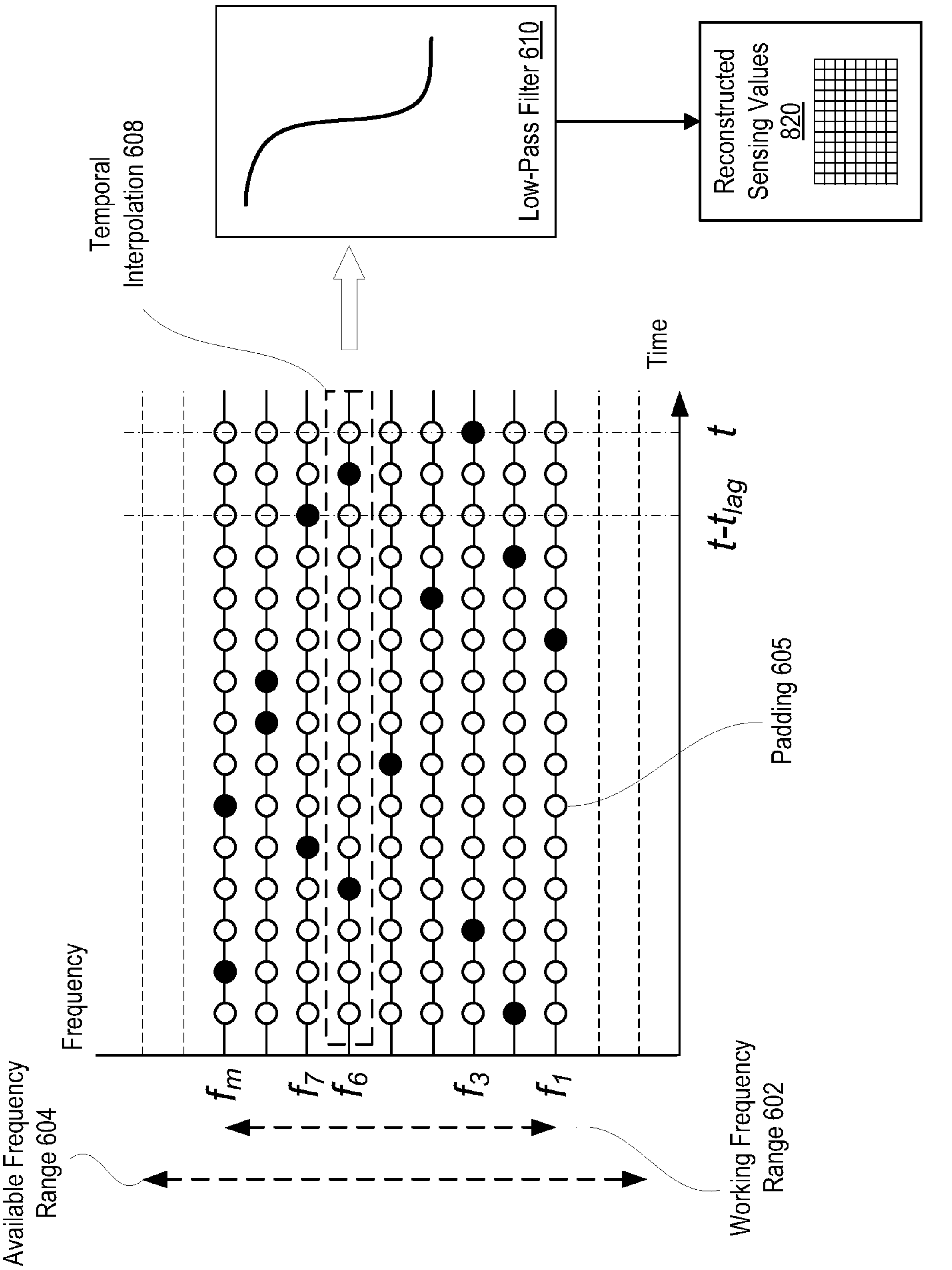
Temporal Interpolation 608
Low-Pass Filter 610
Reconstructed Sensing Values 820
Frequency
Time
t
$t\text{-}t_{lag}$
Padding 605
f_m
f_7
f_6
f_3
f_1
Available Frequency Range 604
Working Frequency Range 602

FIG. 6B

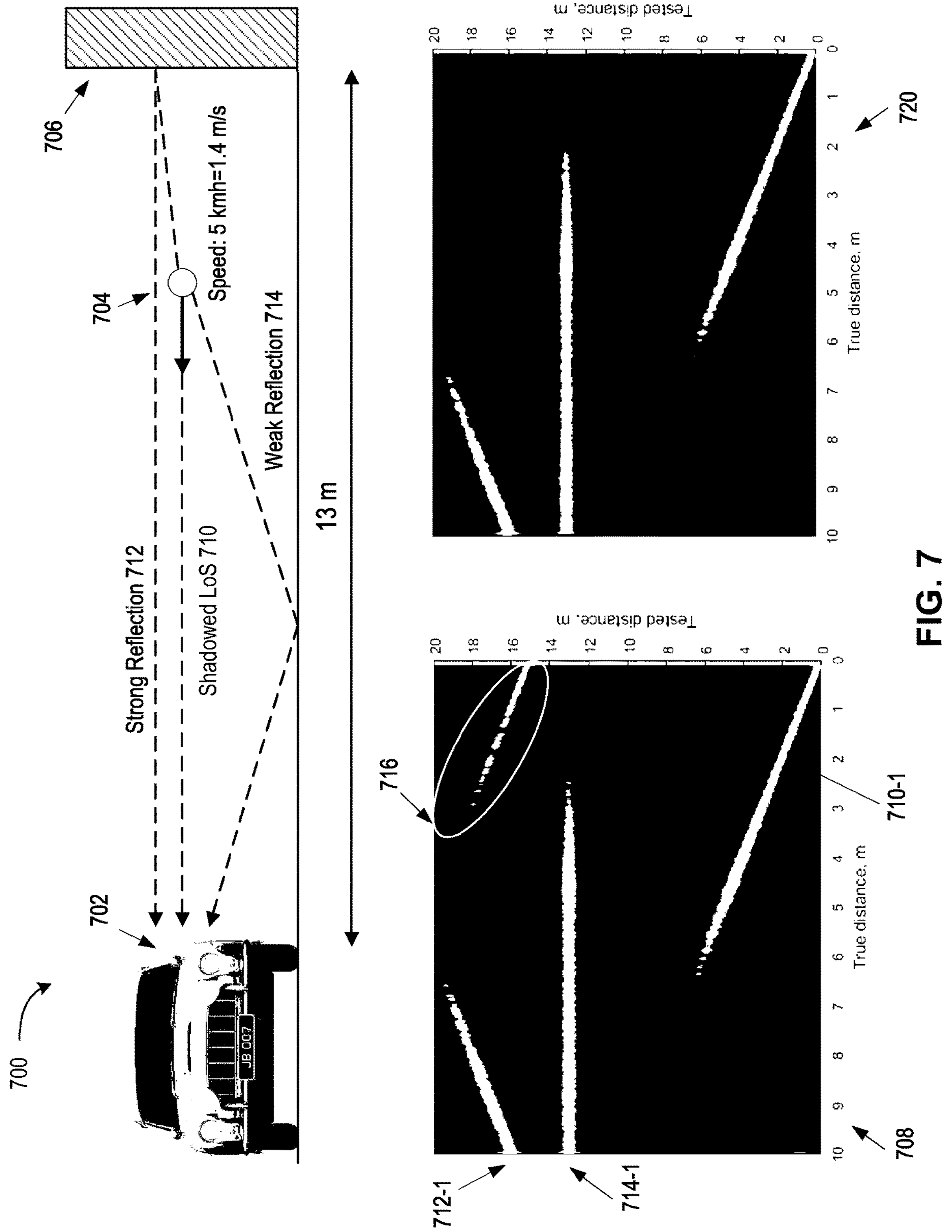
700
702
704
706
JB 007
Strong Reflection 712
Shadowed LoS 710
Weak Reflection 714
Speed: 5 kmh=1.4 m/s
13 m
708
720
716
710-1
712-1
714-1
Tested distance, m
0
2
4
6
8
10
12
14
16
18
20
True distance, m
0
1
2
3
4
5
6
7
8
9
10

FIG. 7

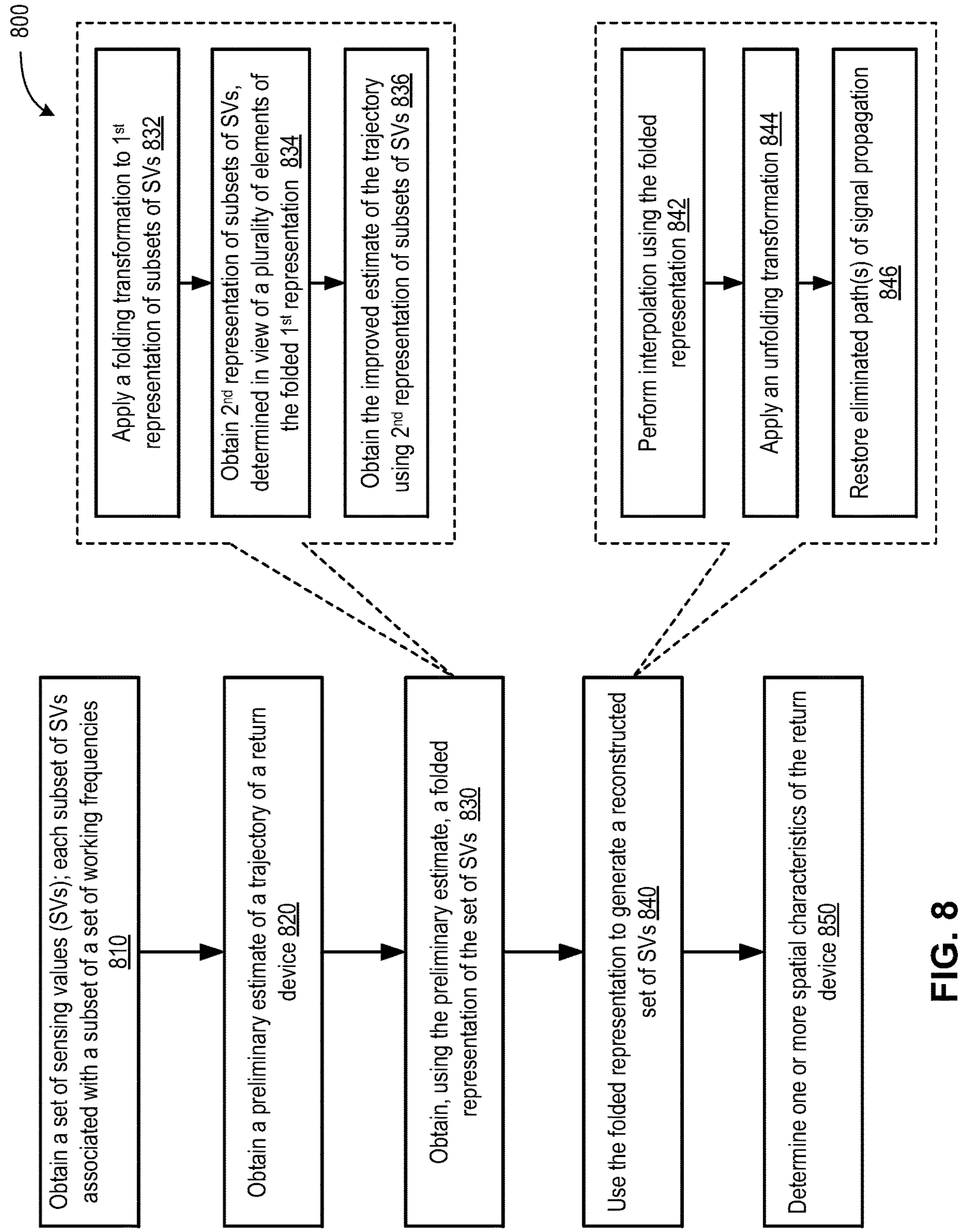
800
Obtain a set of sensing values (SVs); each subset of SVs associated with a subset of a set of working frequencies 810
Obtain a preliminary estimate of a trajectory of a return device 820
Obtain, using the preliminary estimate, a folded representation of the set of SVs 830
Use the folded representation to generate a reconstructed set of SVs 840
Determine one or more spatial characteristics of the return device 850
Apply a folding transformation to 1st representation of subsets of SVs 832
Obtain 2nd representation of subsets of SVs, determined in view of a plurality of elements of the folded 1st representation 834
Obtain the improved estimate of the trajectory using 2nd representation of subsets of SVs 836
Perform interpolation using the folded representation 842
Apply an unfolding transformation 844
Restore eliminated path(s) of signal propagation 846

FIG. 8

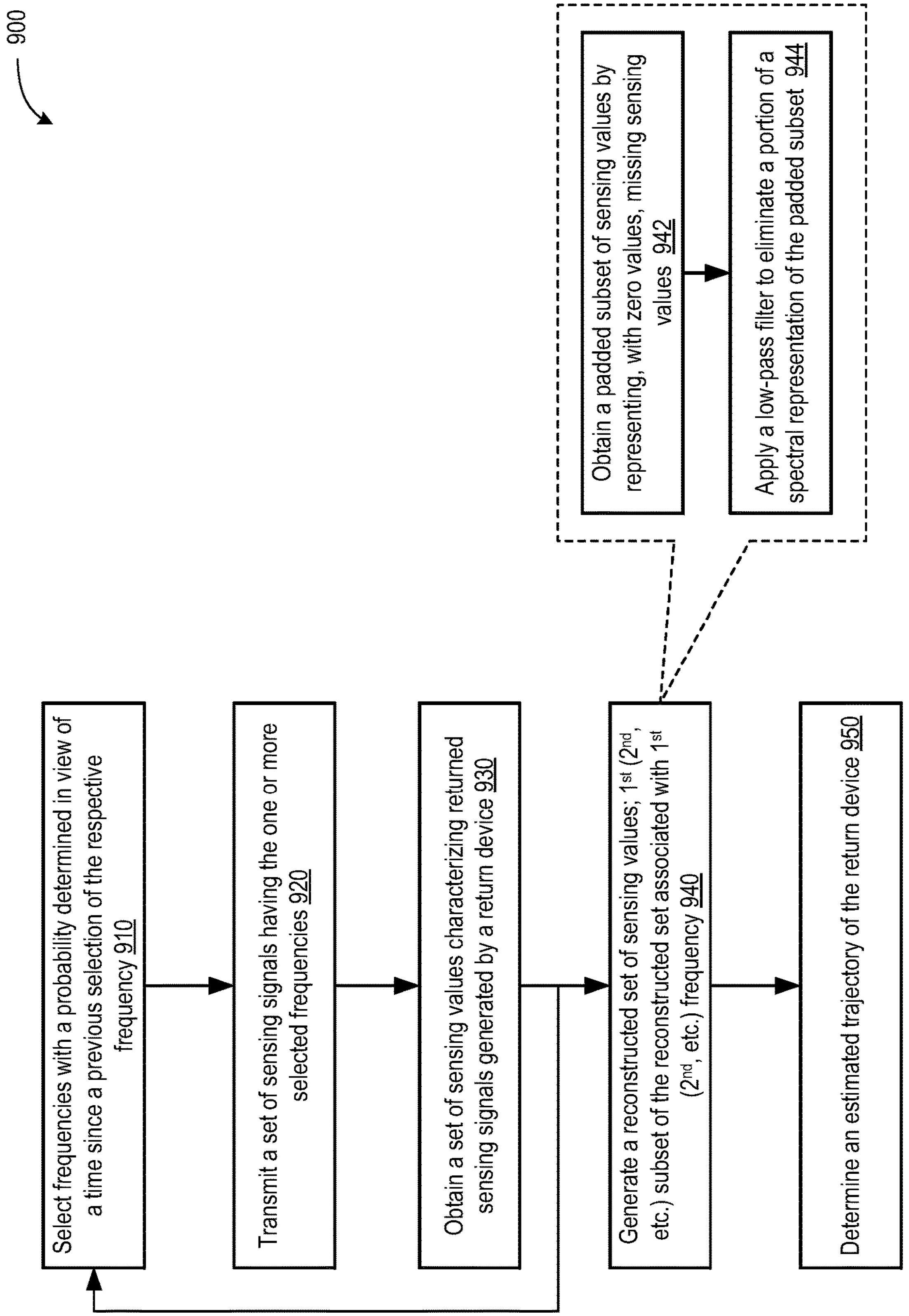
900
Select frequencies with a probability determined in view of a time since a previous selection of the respective frequency 910
Transmit a set of sensing signals having the one or more selected frequencies 920
Obtain a set of sensing values characterizing returned sensing signals generated by a return device 930
Generate a reconstructed set of sensing values; 1st (2nd, etc.) subset of the reconstructed set associated with 1st (2nd, etc.) frequency 940
Obtain a padded subset of sensing values by representing, with zero values, missing sensing values 942
Apply a low-pass filter to eliminate a portion of a spectral representation of the padded subset 944
Determine an estimated trajectory of the return device 950

FIG. 9

TRACKING OF OBJECTS USING RECONSTRUCTION OF DATA CARRIED BY WIRELESS SENSING SIGNALS FROM LIMITED SENSING FREQUENCIES

TECHNICAL FIELD

The present disclosure pertains to wireless networks. More specifically, the present disclosure pertains to tracking position and velocity of wireless devices by detecting sensing signals that carry information about distances to such devices. The present disclosure further pertains to techniques and systems capable of tracking wireless devices using a reconstructed set of sensing signals that is based on a limited number of sensing frequencies.

BACKGROUND

Personal area networks (PAN), such as Bluetooth (BT), Bluetooth Low Energy (BLE), and wireless local area networks (WLAN), such as Wi-Fi networks and other networks operating under the IEEE 802.11 or other wireless standards, provide wireless connection for various personal industrial, scientific, and medical applications. Many BT, BLE, and IEEE 802.11 applications use identification and secure communications that are predicated on correct localization of various objects that carry a wireless device. For example, automotive applications deploy passive keyless entry systems that localizes a key fob and locks/unlocks/starts the car based on the proximity of the key fob to the car. Similarly, a tire pressure monitoring system identifies a specific tire whose pressure falls below a certain reading. BLE specification defines a variety of techniques for performing object localization, such as by estimating signal strength of received wireless signals (e.g., received signal strength indication, RSSI), angle (direction) of arrival (AoA) of wireless signals, high-accuracy distance measurements (HADM) using time-of-flight (ToF) channel sensing, phase-based ranging (PBR), and other techniques. AoA uses multiple sensors (antennas) that exploit differences in phases of one or more unmodulated tones arriving at the sensors (positioned at different points in space) to estimate the directions of the wave propagation. Similarly, channel sensing (e.g., HADM) estimates a distance to an object (e.g., another BLE device) by measuring phase delays accumulated by a plurality of signals of different frequencies along a path from an initiator wireless device to return wireless device and back.

BRIEF DESCRIPTION OF THE DRAWINGS

FIG. **1** illustrates one example setup for trajectory determination and tracking by a wireless device, in accordance with some implementations.

FIG. **2** illustrates example flow of operations during distance-based object tracking, in accordance with some implementations.

FIG. **3**A illustrates one exemplary implementation of a wireless (BT or BLE) system capable of supporting efficient trajectory determination and tracking for fast and precise localization of objects, in accordance with some implementations.

FIG. **3**B illustrates one exemplary implementation of a wireless (WLAN) system capable of supporting efficient trajectory determination and tracking for fast and precise localization of objects, in accordance with some implementations.

FIG. **4** illustrates an example flow of operations of reconstruction of sensing values performed in the frequency and/or temporal domain, in accordance with some implementations.

FIGS. **5**A-B illustrates reconstruction of sensing values from a limited set of sensing signals of random frequencies, in accordance with some implementations.

FIGS. **6**A-B illustrate frequency selection and reconstruction of sensing values in the temporal domain, in accordance with some implementations.

FIG. **7** illustrates advantages of using a low-pass filter in reconstruction of sensing values in temporal domain, in accordance with some implementations.

FIG. **8** is a flow diagram of an example method that uses reconstruction of sensing values a limited set of sensing signals of random frequencies, in accordance with some implementations.

FIG. **9** is a flow diagram of an example method that uses frequency selection and reconstruction of sensing values in temporal domain for efficient tracking of wireless devices, in accordance with some implementations

DETAILED DESCRIPTION

In various applications, wireless devices and various moving (or movable) objects that carry wireless devices, e.g., people, vehicles, cell phones, key fobs, items stored in a warehouse, etc., may be tracked using wireless (e.g., radio) signals. A distance to an object may be measured (e.g., using PBR techniques) for a series of times t_i and the object's trajectory may be determined based on the measured distances $d(t_i)$. A distance to the object may be determined from the phase acquired by a sensing signal of some frequency f_j upon propagation of the signal to the object and back, e.g., $\Delta\phi_j=4\pi f_j d/c$. Optimal (accurate) distance estimation can utilize uniformly spaced sensing frequencies, $f_j=f_0+(j-1)\Delta f$, with j=1, 2 . . . and frequency spacing Δf. However, the use of uniformly spaced sensing frequencies can make wireless networks and devices vulnerable to outside (e.g., spoofing) attacks and can also be inconsistent with governmental regulations. On the other hand, selecting sensing frequencies randomly (or pseudo-randomly) can make application of various available distance tracking algorithms difficult and/or less accurate.

Aspects and implementations of the present disclosure address these and other limitations of the existing technology by enabling systems and methods of efficient tracking of moving objects using randomized sensing frequencies. In some implementations, during any given sensing event that occurs at time t_i, a number l of sensing frequencies f_j may be selected, which may be a fraction of all m frequencies (working frequencies) being used. A sensing signal may then be prepared and transmitted with each selected frequency and returned by a device whose motion is being tracked. The phase and amplitude of the returned signals $r(f_j, t_i)$, associated with frequency f_j and sensing time (event) t_i, may then be considered as a sensing signal data point (cell) within the frequency-time space (grid) of sensing signals. This frequency-time space may be partially filled with a fraction l/m of all sensing cells known from measurements and the remaining fraction 1−l/m of cells lacking measured data points (sensing values) $r(f_j, t_i)$. Various techniques disclosed herein enable interpolating the sensing values $r(f_j, t_i)$ from the randomized subset of known (measured) data points to the full set of data points for all used frequencies $\{f_j\}$.

In some implementations, a frequency interpolation may be used, e.g., interpolation within the same temporal slice (fixed t_i but different f_j). In some implementations, a temporal interpolation may be used, e.g., interpolation within the same frequency slice (fixed f_j but different t_i). In some implementations, both frequency interpolation and temporal interpolation may be used together. In some implementations, frequency and/or temporal interpolation may be performed based on the sensing values, $r(f_j, t_i)$ directly. In some implementations, frequency and/or temporal interpolation may be performed based on sensing values that undergo a certain transformation (folding) that allows identifying contributions into sensing signals from strong paths of wave propagation and performing interpolation that is aware of these strong paths.

In some implementations, during temporal interpolation, the missing data points may be padded with zero values (or otherwise interpolated), $r(f_j, t_i)=0$, transformed into a spectral representation (with respect to the time variable t_i), $r(f_j, t_i)\rightarrow r(f_j,\Omega)$, and filtered using a low-pass filter that removes the part of the spectral representation $r(f_j,\Omega)$ with the Fourier components that are faster than a certain predetermined threshold, $\Omega>\Omega_T$. The inverse Fourier transform, illustrated schematically with the Heaviside function, $r(f_j,\Omega)\Theta(\Omega_T-\Omega)\rightarrow R(f_j, t_i)$, may then recover reconstructed (interpolated) set of sensing values $R(f_j, t_i)$ for the entire range of sensing times and frequencies.

In some implementations, the sensing frequencies may be selected for each sensing event using a dynamic probability $P(t_{last})$, which depends on the last time t_{last} a particular frequency was selected into a set of sensing signals and increases with increasing t_{last}, e.g., so that each of the sensing frequencies is used at least once during a predetermined time interval τ_{max} This facilitates elimination of aliasing (ghost objects). More specifically, when a sensing signal of a given frequency f_j is repeated every τ seconds to detect an object moving with velocity v, consecutive sensing signals acquire the phases that differ by $\Delta\phi=4\pi v\tau/c$. This means that the maximum velocity that can be resolved by this sequence of sensing signals (determined from the condition $\Delta\phi=2\pi$) is

$$v_{max}=\frac{c}{2f_j\tau},$$

as objects having velocity in the interval $[-v_{max}, v_{max}]$ can appear similar to objects (velocity aliasing) having velocity in the intervals $[v_{max}, 3v_{max}]$, $[3v_{max}, 5v_{max}]$, $[-3v_{max}, v_{max}]$, $[-5v_{max}, -3v_{max}]$, etc. Accordingly, forcing each frequency f_j to be selected at least once every time interval τ_{max} allows to eliminate velocity aliasing if it is known that the speed of the device that is being tracked does not exceed $c/(2f_j\tau_{max})$. Additionally, the described techniques eliminate or reduce distance aliasing. Specifically, phases acquired by signals of frequency $f_j=f_0+(j-1)\Delta f$ over the distance 2d of travel to and from an object are $\Delta\phi_j=4\pi f_0 d/c+4\pi(j-1)\Delta f d/c$. These phases have periodicity with respect to distance d with the period (when the phases differ by 2π)

$$d_{max}=\frac{c}{2\Delta f},$$

which is also the maximum distance that can be resolved by the sensing signals with frequency spacing Δf, meaning that an object located within the interval $[0, d_{max}]$ can appear similar to an object located within the intervals of distances $[d_{max}, 2d_{max}]$, $[2d_{max}, 3d_{max}]$, etc.

The described techniques allow reconstruction of an expanded set of sensing values $\{R(f_j, t_i)\}$ using a limited set of measured sensing values $\{r(f_j, t_i)\}$ for pseudorandom frequencies $\{f_j\}$ selected as part of multiple sensing events t_i. As described in more detail below, the reconstructed (expanded) set of sensing values may be used to compute a likelihood tensor $P(d_0, v)$ that determines the likelihood (probability) that the return object's motion is described by a trajectory $d=d_0+vt$, characterized by parameters d_0 (reference distance at time t=0) and v (velocity).

In one example of BLE systems, during a sensing event, m waves of different frequencies (tones) f_j ($j\in[1, m]$), from the BT bandwidth (i.e., from 2.402 GHz to 2.480 GHz) may be transmitted by a wireless device that performs trajectory tracking of another wireless device or an object that transports such a wireless device. In another example of IEEE 802.11 wireless systems, tones of a sample set may be subcarriers transmitted simultaneously in a long training field (LTF) of a packet. The transmitted signals may be reflected by a return device (the device whose trajectory is being estimated) along the same path. A sensor (e.g., antenna) may detect arrival of m returned signals and extract a phase information from these signals, the phase information being representative of the length of the traveled path. In some instances, m signals reflected by the object may follow multiple, e.g., n, paths, including a path that correspond to the line-of-sight propagation and paths that involve reflections from walls, ceilings, and/or other objects, including multiple reflections. Such paths may be identified, e.g., as multiple maxima of the likelihood tensor that may further allow distinguishing the line-of-sight propagation from multipath reflections, as described below.

The advantages of the disclosed implementations include computationally efficient trajectory tracking for fast real-time monitoring of the locations of various objects in wireless network environments, which may include multiple wireless devices and various additional objects.

FIG. **1** illustrates one example setup for trajectory determination and tracking by a wireless device, in accordance with some implementations. A wireless device **100** may be any BT device, BLE device, or any other type of a device capable of generating, transmitting, receiving, and processing electromagnetic sensing signals. In some implementations, the sensing signals may be radio signals in the frequency range of the IEEE 802.11 wireless standard (e.g. 2.4 GHz band, 5 GHz band, 60 GHz band), or any other radio communication band. In some implementations, sensing signals may be compatible with one or more IEEE 802.11 wireless standards. Wireless device **100** may include one or more sensors **102** capable of transmitting and receiving sensing signals. Sensors **102** may include or be communicatively coupled to one or more antennas, such as dipole antennas, loop antennas, multiple-input multiple-output (MIMO) antennas, antenna arrays, or any other type of devices capable of receiving and transmitting electromagnetic signals. Localization system may consist or one or several wireless devices **100**. The environment of wireless device **100** may include one or more objects, e.g., object **104**, or additional objects not explicitly depicted in FIG. **1**. Object **104** may be a wireless device (or may carry or otherwise transport a wireless device) capable of communicating wirelessly with wireless device **100**. Although the references below are made to trajectory tracking being performed by wireless device **100**, it should be understood that similar techniques and implementations may be used for distance estimation and trajectory determination/tracking performed by object **104**, e.g., with trajectory of wireless device **100** (and/or any additional) devices being determined.

Wireless device **100** may generate and transmit a plurality of sensing signals. In some implementations, the sensing signals may have different frequencies (tones). More specifically, wireless device **100** may generate a signal **106-1** that includes multiple (e.g., N) tones, e.g., f_0, $f_0+\Delta f_1$, $f_0+\Delta f_2$ . . . , and may transmit the generated signal to object **104**, which may be a responding device belonging to the same wireless network as the wireless device **100**. The responding device may perform analysis of the received signal **106-1**, evaluate phase information that is used in returned signal **107-1**. Wireless device **100** may similarly perform evaluation of phase information of the returned signal **107-1** to estimate the distance between wireless device **100** and the responding device (object **104-1**) based on the total phase change. Each tone of the transmitted signal **106-1** (and, correspondingly, of the returned signal **107-1**) may carry its own phase information. In particular, the total phase change $\Delta\phi$ associated with the distance d_1 between wireless device **100** and object **104** traveled by signal **106-1** and the same distance d_1 traveled by returned signal **107-1** of frequency f_j is $\Delta\phi_j=4\pi f_j d_1/c$, where c is the speed of light. This phase change is representative of the distance d_1 (t_1) to object **104** at time t_1. The callout portion of FIG. **1** illustrates schematically a composition of signal **107-1** (illustrated with dashed lines) that is returned from object **104**.

At a later time t_2, object **104** may move to a different location **105**. A new sensing signal **106-2** (e.g., similarly having N sensing tones f_j) may be transmitted by wireless device **100** and may cause a returned signal **107-2** carrying phase information representative of the new distance $d_2(t_2)$ to object **104**. As depicted in FIG. **1**, an additional returned signal **107-3** may reach sensor(s) **102** of wireless device **100** over a different path that includes reflection from another object, e.g., a wall **108**. The distance $d_3(t_2)$ traveled by signal **107-3** may be larger than distance d_2 of the line-of-sight propagation. Although one returned signal path is indicated for time t_1 and two paths are indicated for time t_2, any number of returned signal paths may be present for any detection time (sensing event) t_i. Similarly, transmitted signal paths may also include multipath propagation (not depicted in FIG. **1**). In some instances, the direct line-of-site path may be blocked by other objects, such that only paths with one or more reflections may be present. As object **104** is moving relative to other objects of the environment, the number of paths may change with time, e.g., may be different at different detection times (sensing events) t_i.

The phase changes $\Delta\phi_j$ carried by the returned sensing signals may be exploited using the multiple signal classification (MUSIC) algorithm, generalized cross-correlation (GCC) algorithm, inverse Fourier transform algorithm, or any other suitable processing algorithms that are further improved according to implementations of the present disclosure. The following operations may be performed for each of the sensing events t_i to determine a respective likelihood vector $P_i(d)$, as described below. Likelihood vector $P_i(d)$ may be a vector (array) in a distance space with specific values of the likelihood vector indicating the likelihood (probability or be proportional to probability or has some relation to probability) of various distances to the wireless device that is being tracked. Multiple likelihood vectors $P_i(d)$ may then be combined into a likelihood tensors described in more detail below.

As illustrated in FIG. **1**, a sensing event initiated (at each of the detection times t_i) by wireless device **100** may include transmitting and then detecting N returned signals (each with a different frequency f_j). Each of the detected sensing values r_j may characterize a superposition of waves propagating along n paths; some (or all) of the paths may include one or more reflections:

$$r_j=\left(\sum_{k=1}^{n}a_j(d_k)S_k+n_j\right)\left(\sum_{l=1}^{n}a_j^*(d_l)S_l^*+n'_j\right)j\in[1,N],$$

where S_k represents the amplitude of the wave traveled over k-th path, n_j is the noise associated with forward propagation (and detection) of j-th frequency (tone, channel) f_j, n'_j is the noise associated with backward propagation (and detection) of j-th frequency, and a_j (d) is a steering vector (also denoted, in vector notations, as $\hat{a}(d)$) that describes phase change over distance d, which may take one of the values $d=d_1$ . . . d_n. In particular, for N equidistant sensing tones, $f_j=f_0+(j-1)\Delta f$, the steering vector may have the form, $a_j(d)=\exp[4\pi i(j-1)\Delta fd/c]$.

In MUSIC algorithm implementations, sensing values may be used to construct the N×N covariance matrix, $R_{jl}=\langle(r_j r_l^*)\rangle$, where the angular brackets $\langle$ . . . $\rangle$ denote statistical averaging and r_l^* stands for complex conjugation of r_l^*. In some implementations, covariance matrix may be formed using square roots (with suitably chosen sign value) of sensing values, e.g., $R_{jl}=\langle\sqrt{r_j}\sqrt{r_l^*}\rangle$. In some implementations, statistical averaging may be performed using smoothing in the frequency domain, e.g., using smooth-MUSIC algorithm. In some implementations, statistical averaging may include averaging in the time domain, e.g., by collecting multiple instances of data. In some implementations, time averaging is not performed. For uncorrelated noise, $\langle n_j n_l^*\rangle=\delta_{jl}\sigma^2$, where σ^2 is the noise variance in a single sensing channel.

Covariance matrix $\hat{R}$ may have n signal eigenvectors $\hat{s}_{(1)}$ . . . $\hat{s}_{(n)}$ and N−n noise eigenvectors $\hat{g}_{(n+1)}$ . . . $\hat{g}_{(N)}$ (which define what is commonly referred to as the noise subspace), with subscript a enumerating various eigenvectors. For uncorrelated noise, noise eigenvectors are orthogonal to the steering vector: $\hat{a}^\dagger(d_m)\cdot\hat{g}_{(\alpha)}=0$. Accordingly, the localization vector P(d) (often referred to in MUSIC and GCC applications as the pseudo-spectrum), defined using noise eigenvectors as,

$$P^{-1}(d)=\sum_{\alpha=n+1}^{N}\left|\hat{a}^\dagger(d)\cdot\hat{g}_{(\alpha)}\right|^2$$

has maxima for the actual distances $d=d_1$ . . . d_n of signal propagation, some of which may correspond to direct (line-of-sight) signal propagation and some may correspond to paths that include at least one reflection. In some implementations, the localization vector may be defined using signal eigenvectors, e.g., as

$$P(d)=\sum_{\alpha=1}^{n}\lambda_{(\alpha)}\left|\hat{a}^\dagger(d)\cdot\hat{s}_{(\alpha)}\right|^2,$$

where $\lambda_{(\alpha)}$ is the eigenvalue corresponding to signal eigenvector $\hat{s}_{(\alpha)}$.

The above example of the MUSIC localization vector is intended to be illustrative. In various implementations, the localization vector P(d) may be obtained using different procedures. For example, in the GCC method, the localization vector may be defined as,

$$P(d) = \left| \sum_{j=1}^{N} a_j^*(d) r_j \right|^2.$$

This vector may similarly have maxima for the actual distances d=d_1 . . . d_n, and may be computed using Inverse Fast Fourier algorithm techniques. Numerous other ways of defining the localization vector P(d) are also within the scope of the present disclosure.

In some implementations, a likelihood vector P(d) may be transformed into a likelihood tensor P(d_0, v). The transformation P(d)→P(d_0, v) may be performed in a variety of ways. This can be done because of replacement single independent variable by two dependent variables d→d_0, v in P(d). In some implementations, the likelihood vectors from multiple sensing events may be joined into the likelihood tensor as follows,

$$P(d_0,v)=(d_0+vt_1)+P_2(d_0+vt_2).$$

In some implementations, different likelihood vectors may be weighted differently, e.g.,

$$P(d_0,v)=(d_0+vt_1)+W_2P_2(d_0+vt_2),$$

with weights W_1 and W_2, which may be used for normalization of P(d) vector, so, total sum of P(d) is one, e.g., with likelihood vectors corresponding to closer ranges given higher weights. In these formulas, the likelihood vectors are identified with subscripts 1 and 2, to reference the sensing events whose measurements contribute data for the respective likelihood vectors, even though the function that is used to construct each likelihood vector may be the same, e.g., P(d). The likelihood tensor P(d_0, v) is a quantity that is defined on the two-dimensional space of distance d_0 and velocity v. The actual values of d_0 and v for the object being tracked may be determined by an optimization procedure, e.g., by finding the maxima of P(d_0, v), or, alternatively, the minima of P^{-1}(d_0, v). Although two likelihood vectors are combined in this example, any number of likelihood vectors may be combined in a similar fashion.

The above process may continue with the additional sensing data for each new sensing event, i=3, 4, . . . , being used to update the likelihood tensor using a new likelihood vector P_i(d_0+vt_i):

$$P(d_0,v)\rightarrow P(d_0,v)+W_i\cdot P_i(d_0+vt_i),$$

In some implementations, the number of sensing events that are counted towards the likelihood tensor may be limited to a predetermined number M of sensing events, such that after M contributions into the likelihood tensor are collected, when each additional contribution is added, the earliest contribution is subtracted from the likelihood tensor (e.g., the i=1 contribution in this example):

$$P(d_0,v)\rightarrow P(d_0,v)+W_{M+1}\cdot P_{M+1}(d_0+vt_i)-W_1\cdot P_1(d_0+vt_i).$$

The number M may be selected to include multiple sensing events but still be small enough so that the velocity of the object is unlikely to change substantially over the duration of the last M sensing events. For example, if one sensing event occurs every 0.1 sec, the maximum number of events included in the likelihood tensor may be M=10. As a result, the optimization of the likelihood tensor provides an accurate estimate of the average velocity (and reference distance) to the object over the sliding window of the last 0.5 sec, which under many practical conditions may be short enough for the constant velocity model to be sufficiently accurate.

In the implementation described above, the likelihood tensor for a combination of events is the harmonic mean of the likelihood vectors for each event, P=P_1+P_2+ . . . . In some implementations, the likelihood tensor for a combination of events may instead be the sum of likelihood vectors computed for each event, =P_1+P_2+ . . . , or any other suitable measure. In some implementations, e.g., where different sensing events have unequal number of sub-events, the likelihood vectors computed for individual events may be weighed with suitably chosen weights, e.g., weights proportional to the number of sub-events in each sensing event, weights that are empirically determined, and so on.

In the above examples, the location-velocity determination and tracking is illustrated using the two-dimensional (2D) likelihood tensor P(d_0, v) in location-velocity space. Similar techniques may be used for trajectory determination and tracking in higher dimensions, where multiple coordinates of the object and multiple components of velocity are being determined. For example, in the more general case, the trajectory may be determined using a vector model, $\vec{r}=\vec{r}_0+\vec{v}t$, with a vector reference location, $\vec{r}_0$=(x_0, y_0 . . .) and a vector velocity $\vec{v}$=(v_x, v_y . . .), where x, y . . . are any suitable coordinates including Cartesian coordinates, polar coordinates, elliptical coordinates, spherical coordinates, cylindrical coordinates, and so on. A higher-dimensional (HD) likelihood tensor may be a tensor in 2m-dimensional space of m (e.g., m=2 or m=3) coordinates x, y . . . and m velocity components, v_x, v_y . . . . More specifically, the likelihood tensor may be

$$P(\vec{r}_0, \vec{v}) = \sum_{Last\ M\ events} W_i P_i(|\vec{r}_0 + \vec{v}t_i|).$$

FIG. **2** illustrates example flow of operations **200** during distance-based object tracking using a single sensing station, in accordance with some implementations. As depicted in FIG. **2**, a wireless device may collect sensing data (block **202**) representative of the current distances d(t_i) to the object being tracked. The sensing data may be obtained using a limited set of sensing frequencies. At block **204**, the limited set of sensing data may be used to obtain an expanded (or reconstructed) set of sensing data for an expanded set of frequencies, as described in more detail below. The wireless device may then generate the 2D likelihood tensor (as described above). For example, the wireless device may obtain, at block **206**, a set of likelihood vectors P_i(d) using MUSIC algorithm, or any other algorithm that uses a uniform frequency step or a uniform time step. In some implementations, various other algorithms may be used, including but not limited to GCC, Inverse Fast Fourier Transform, or any other suitable algorithm. The wireless device may then use any suitable tracking algorithm **208** to track distance and/or velocity of the object. In one non-limiting example implementation, tracking algorithm **208** may include constructing the 2D likelihood tensor P(d_0, v). The 2D likelihood tensor may be constructed according to any techniques described above, e.g., using the discretization and interpolation techniques. The 2D likelihood tensor

$P(d_0, v)$ may be used to estimate the trajectory of the object, by finding one or more extrema of the respective likelihood tensor.

FIG. **3**A illustrates one exemplary implementation of a wireless (BT or BLE) system **300** capable of supporting efficient trajectory determination and tracking for fast and precise localization of objects, in accordance with some implementations. Wireless system **300** may be a BL network, a BLE network, a Wi-Fi, or any other type of a wireless network (e.g., PAN, WLAN, etc.). Wireless system **300** may include any number of host devices **302** (one host device is depicted for conciseness). A host device **302** may be any desktop computer, laptop computer, tablet, phone, smart TV, sensor, appliance, system controller (e.g., an air conditioning, heating, water heating controller), component of a security system, medical testing or monitoring equipment, automotive equipment, or any other type of a device. Host device **302** may be coupled (e.g., via a wired connection) to a respective wireless device **304**. For brevity, shown is a single wireless device **304**, but it should be understood that host device **302** may be coupled to any number of such wireless devices, e.g., a BLE wireless device and a Zigbee® wireless device. In some implementations, wireless device **304** may be implemented as an integrated circuit (IC) device (e.g., disposed on a single semiconductor die). In some implementations, various modules and components may be absent or shared between multiple wireless devices coupled to the host device (e.g., antenna(s) **306** and/or processor **352** may be shared between multiple wireless devices).

Wireless device **304** may use one or more antennas **306** to receive and transmit radio waves (e.g., sensing signals or radio frequency (RF) signals). A radio frequency (RF) signal received by antenna(s) **306** may be processed by radio **310** which may include filters (e.g., band-pass filters), low-noise radio-frequency amplifiers, down-conversion mixer(s), intermediate-frequency amplifiers, analog-to-digital converters, inverse Fourier transform modules, deparsing modules, interleavers, error correction modules, scramblers, and other (analog and/or digital) circuitry that may be used to process modulated signals received by antenna(s) **306**. Radio **310** may further include a tone (frequency) generator to generate radio signals at selected tones. Radio **310** may also include antenna control circuits to control access to one or more antennas **306**, including switching between antennas. Radio **310** may additionally include radio control circuits, such as phase measurement circuits and a tone selector circuit. The phase measurement circuits can perform phase measurements on received signals, e.g., IQ decomposition, which may include measuring a phase difference between the received signal and a local oscillator signal. The tone selector circuit can select tones for transmission.

Radio **310** may provide the received (and digitized) signals to a PHY **320** components. The received signals may carry information about radio wave propagation to and from one or more return devices (this information is referred to as sensing values or signal values herein). PHY **320** may support one or more operation modes, e.g., BLE operation modes. Although one PHY **320** is shown, any suitable number of PHY layers (supporting a respective number of operation modes) may be present. PHY **320** may convert the digitized signals received from radio **310** into frames that can be fed into a Link Layer **330**. Link Layer **330** may have a number of states, such as advertising, scanning, initiating, connection, standby. Link Layer **330** may transform frames into data packets. During transmission, data processing may occur in the opposite direction, with Link Layer **330** transforming data packets into frames that are then transformed by PHY **320** into digital signals provided to radio **310**. Radio **310** may convert digital signals into radio signals and transmit the radio signals using antenna(s) **306**. In some implementations, radio **310**, PHY **320**, and Link Layer **330** may be implemented as parts of a single integrated circuit.

Wireless device **304** may include a protocol stack **340**. The protocol stack **340** may include a number of protocols, e.g., Logical Link Control Adaptation Protocol (L2CAP), which may perform segmentation and reassembly of data packets that are generated by one or more applications **303** operating on host device **302**. Specifically, L2CAP may segment data packets of arbitrary size, as output by the application(s) **303**, into packets of the size and format that can be processed by Link Layer **330**. L2CAP may also perform error detection operations. The protocol stack **340** may also include generic access profile (GAP) and generic attribute profile (GATT). GAP may specify how wireless device **304** advertises itself on the wireless network, discovers other network devices, and establishes wireless links with the discovered devices. GATT may specify how a data exchange between communicating wireless devices is to occur once the connection between the two devices is established. The protocol stack **340** may further include a security manager (SM) that controls how pairing, signing, and encryption of data is performed. GATT may use attribute protocol (ATT) that specifies how units of data are transferred between devices. Wireless device **304** may also include other components not explicitly shown in FIG. **3**A, such as a host-controller interface.

Wireless device **304** may have a controller **350**, which may include one or more processors **352**, such as central processing units (CPUs), finite state machines (FSMs), field-programmable gate arrays (FPGAs), application-specific integrated circuits (ASIC), or the like. Processor(s) **352** may also include custom logic and/or programmable logic, or any combinations thereof. In some embodiments, controller **350** may be a single processing device that supports processes associated with data transmission and reception as well as distance (and/or angle) estimation computations. In some implementations, wireless device **304** may have a dedicated processor for distance (and/or angle) estimation computations that is separate from a processor that executes other operations on wireless device **304** (e.g., processes associated with data transmission and reception).

Wireless device **304** may also include a power management unit (PMU) **370**, which manages clock/reset and power resources. Wireless device **304** may further include an input/output (I/O) controller **380** to enable communications with other external devices (including non-network devices) and structures. In some implementations, I/O controller **380** may enable a general purpose I/O (GPIO) interface, a USB interface, a serial digital interface (SDI), a PCM digital audio module, a universal asynchronous receiver transmitter (UART), I2C, I2S, or any other I/O components.

Controller **350** may include a memory **360**, which may be (or include) a non-volatile, e.g., read-only (ROM) memory, and/or volatile, e.g., random-access (RAM), memory. Memory **360** may store codes and supporting data for an object localization engine **362**, a data reconstruction engine **364**, a tone selection engine **366**, and other suitable engines. In some implementations, any one or more of the engines may be located on host device **302**, as indicated with the respective dashed boxes in FIG. **3**A. The engines may be operating in conjunction with a domain-specific application **303**, which may be a device- or asset-tracking application, an indoor navigation application, an authentication application, or any other suitable application. Placement of the engines **362**-**366** on host device **302** or wireless device **304** may be based on domain-specific criteria as well as on power constraints. In those implementations, where low latency is of high priority, engines **362**-**366** may be placed on wireless device **304**. In other implementations, where reduced power consumption is advantageous, engines **362**-**366** may be placed on host device **302**. In some implementations, some engines (e.g., tone selection engine **366**) may be placed on wireless device **304** while other engines (e.g., object localization engine **362** and data reconstruction engine **364**) may be placed on host device **302**.

Application **303** may use information about various objects located in the environment of the host device **302**/wireless device **304** (which may, in some implementations, be mounted on a single platform or in proximity of each other). Such information may include distances to the objects, directions to the objects, orientations of the objects relative to host device **302**/wireless device **304**, or any other spatial characteristics data. The data may be provided by the object localization engine **362**, which receives and processes sensing data reconstructed by data reconstruction engine **364**, e.g., as described in more detail below. In some implementations, the object localization engine **362** provides an expected range of distances to the objects to the tone selection engine **366**. The expected range of distances may be dependent on specific application **303** that is supported by operations of wireless device **304**. For example, in key fob applications, the range of distances may be up to several meters, in warehouse product tracking applications, the range of distances may be tens of meters or even more. Based on the received range of distances, the tone selection engine **366** may select tones for a given sensing event, which may be tones separated by a uniform frequency increment, e.g. $f_0+(k-1)\times\Delta f$. The tone selection engine **366** may further specify the total number N of tones to be used, e.g., k=1 . . . N. In some implementations, the tone selection engine **366** may select tones that maximize the use of the available bandwidth (e.g., BT bandwidth), the tones that maximize the range of detected distances (e.g., closely spaced tones), the tones that maximize the accuracy of detected distances (e.g., widely spaced tones). In some implementations, the tone selection engine **366** may select tones randomly, or according to any predefined pattern.

Selected tones may be provided to protocol stack **340** (and link layer **330** and PHY **320**) that may cause radio **310** to generate signals at the selected tones and transmit the generated signals to the outside environment. Radio **310** may then receive the reflected (returned) signals from various objects (other wireless devices) of the environment and determine phase shifts experienced by the reflected signals, e.g., by comparing the phase information carried by the reflected signals with phase information of the local oscillator copies of the transmitted signals. Radio **310** may further determine amplitudes of the reflected signals. The amplitude and phase information may be provided to the data reconstruction engine **364** (e.g., in the form of sensing values), which computes the covariance matrix. The data reconstruction engine **364** may include the location-velocity estimator **110** (depicted in FIG. **1**). The object localization engine **362** may estimate trajectories and perform tracking of the objects, authentication of the objects, maintaining communications with the objects and so on.

FIG. 3B illustrates one exemplary implementation of a wireless (WLAN) system **301** capable of supporting efficient trajectory determination and tracking for fast and precise localization of objects, in accordance with some implementations. Although the wireless system **300** has been described above in conjunction with BT/BLE implementations, similar systems may be used in conjunction with any WLAN (e.g., Wi-Fi) implementations. In a WLAN wireless system **301**, an appropriate wireless media access control (MAC) layer **332** may be used instead of the Link Layer **330** in addition to a WLAN-specific PHY layer **320** and a protocol stack **340**. In Wi-Fi, and other WLAN systems, sensing tones may be transmitted in a single packet.

FIG. **4** illustrates an example flow of operations **400** of reconstruction of sensing values performed in the frequency and/or temporal domain, in accordance with some implementations. As depicted schematically in FIG. **4**, a frequency selector **402** may select a set of randomized frequencies $\{f_j\}$ for each of sensing times (events) t_i. In some implementations, frequency selector **402** can select a working range of frequencies, such as m frequencies of a total number M of available frequencies. For example, the available frequencies can include a set of 1 MHz channels within the [2402 MHz, 2480 MHz] range of BT frequencies. In some implementations, m=M, whereas in other implementations, m<M. For each sensing event, frequency selector **402** may select l frequencies to be used in sensing signals that are transmitted to a return device and received from the return device. In some implementations, l=m, whereas in other implementations, l<m. In some implementations, each frequency f_j is selected randomly. In some implementations, each frequency f_j is selected pseudo-randomly. In particular, selected frequencies may appear as random to an outside observer but may be generated deterministically, e.g., based on a secret key and/or in any other way that may be agreed upon by the communicating wireless devices (e.g., a device that perform tracking and a target device). In some implementations, as described in more detail below, the probability of selecting a particular frequency f_j may depend on the time that has elapsed since the last time this frequency was selected.

The wireless device hosting frequency selector **402** may then prepare and transmit sensing signals having the selected frequencies and may receive the returned signals with the same frequencies. The phase and amplitude information carried by the returned signals may be used by the wireless device to generate a limited set $\{r(f_j, t_i)\}$ of randomized sensing values **404**, in which a fraction of l/m sensing values is known while a fraction of 1−l/m of sensing values is missing. At block **410**, the wireless device may perform a reconstruction of sensing values to obtain a reconstructed set $\{R(f_j, t_i)\}$ of sensing values, as described in more detail below. In particular, reconstruction of sensing values may include signal folding/unfolding **412**, path removal/restoration **414**, and interpolation **416**. In the implementations that deploy frequency interpolation, sensing values $\{r(f_j, t_i)\}$ may first be transformed using an appropriate folding transformation, removing (temporarily) one or more strong paths of propagation, performing interpolation **416**, restoring the removed paths of propagation, and finally using an unfolding transformation to obtain a full set $\{R(f_j, t_i)\}$ of reconstructed signals **420**. Object tracking and/or trajectory estimation **430** may then be performed from the set of reconstructed signals **420** using any suitable techniques, including but not limited to any of the techniques described above. In the implementations that deploy temporal interpolation, sensing values $\{r(f_j, t_i)\}$ may first be padded (e.g., with zero values in place of the missing sensing values) or otherwise interpolated in the time domain before being then interpolated in the spectral representation using the Whittaker-Shannon techniques or any other suitable interpolation to obtain the full set $\{R(f_j, t_i)\}$ of reconstructed signals **420**.

In some implementations, interpolation in the spectral representation may include using a low-pass filter **418**.

FIG. **5**A illustrates reconstruction of sensing values **500** from a limited set of sensing signals of random frequencies, in accordance with some implementations. Shown schematically is a limited set of sensing values **504** (depicted as a grid) obtained using m sensing frequencies f_1, $f_2 \ldots f_m$ (the vertical axis) and multiple sensing events/times (the horizontal axis). Each shaded cell of the grid represents a sensing value $r(f_j, t_i)$ obtained using a sensing signal with frequency f_j during a sensing event that occurs at time t_i. During each sensing event, l sensing values may be obtained corresponding to l frequencies selected for that particular event. In some implementations, frequencies f_j may be selected randomly or pseudo-randomly. The white cells in the limited set of sensing values **504** depict sensing values (frequencies) that have not been selected (and measured) during a respective event and are, therefore, missing. In some implementations, to obtain a reconstructed set of sensing values **520**, a wireless device tracking another device (the return device), may perform the following operations.

In one non-limiting implementation, for each sensing event t_i (where $i=1 \ldots n$), a processing unit of the wireless device may construct an $l\times l$ covariance matrix $R_{jk}(t_i)=r(f_j, t_i)r^*(f_k, t_i)$. As a result, n covariance matrices **506** may be constructed. Covariance matrices **506** may then be used to obtain a preliminary trajectory estimate 508 of the responding device, $d(t)=d_0+vt$, e.g., as described above. For example, the processing unit may covariance matrices $R_{jk}(t_i)$ by steering vectors $a_j(d)=\exp[4\pi i f_j d/c]$, form a suitable evaluation measure (e.g., the likelihood tensor) and determine the trajectory parameters d_0 and v (or $\vec{r}_0$ and $\vec{v}$) from the optimization of this measure. In some implementations, the trajectory parameters can be determined using some other techniques, e.g., GCC method, correlating signals for different sensing frequencies and events. In some implementations, velocity and distance may be estimated using separate techniques, e.g., velocity may be estimated from correlations of signals obtained using the same sensing frequencies for different sensing events and distance may be estimated using time-of-flight techniques. In some implementations, velocity may be estimated based on distances measured at two (or more) different times. In some implementations, estimates of distances and/or velocities may be improved by filtering techniques, e.g., by using a Kalman filter or any similar filters. In some implementations, all n covariance matrices **506** may be used for the trajectory estimation. In some implementations, only some (e.g., two, three, etc.) of covariance matrices **506** may be used for the trajectory estimation.

The determined parameters d_0 and v may then be used to obtain the steering vectors **507**, e.g., $a_j(t_i)=\exp[4\pi i f_j i(d_0+v t_i)/c]$, for each of the n sensing events. The obtained steering vectors may then be used to transform (signal folding **510**) each of the n available $l\times l$ covariance matrices $R_{jk}(t_i)$ to obtain a set of n folded matrices. In one implementation, this may be done by computing the element-by-element (Hadamard) multiplication of the covariance matrices $R_{jk}(t_i)$ with the dyadic product $F_{jk}(t_i)$ of the steering vectors,

$$T_{jk}(t_i)=F_{jk}(t_i)R_{jk}(t_i),F_{jk}(t_i)=a_j(t_i)a^*_k(t_i).$$

Elements of the folded (transformed) covariance matrices $T_{jk}(t_i)$ may be dominated by one or more strong paths of signal propagation, e.g., the line-of-sight path, a strong reflection (e.g., from a nearby wall) path, and so on. Because such strong path(s) can skew reconstruction (interpolation) of weaker paths, a processing loop can perform path removal **512** of one or more strong paths of signal propagation. In some implementations, a strong path may be removed by determining a mean element of a given folded covariance matrix $T_{jk}(t_i)$ and subtracting that mean element from $T_{jk}(t_i)$, to obtain a reduced folded matrix, e.g.,

$$R'_{jk}(t_i) = T_{jk}(t_i) - \overline{T_{jk}}(t_i),\ \overline{T_{jk}}(t_i) = \frac{1}{l^2}\sum_{j,k=1}^{l} T_{jk}(t_i).$$

The obtained (first-order) reduced folded matrices $R'_{jk}(t_i)$ may then be used as part of a removal loop indicated by a dashed arrow in FIG. **5**A. More specifically, new eigenvectors of at least some of the reduced covariance matrices $R'_{jk}(t_i)$ may be computed and used to update the trajectory parameters d'_0 and v' (or $\vec{r}'_0$ and $\vec{v}'$). The updated parameters d'_0 and v' may subsequently be used to update the steering vectors **507**, e.g., $a'_j(t_i)=\exp[4\pi i f_j(d'_0+v't_i)/c]$, for each or at least some of the n sensing events. The obtained steering vectors may then be used to transform (signal folding **510**) each or at least some of the n available $l\times l$ covariance matrices $R_{jk}(t_i)$ to obtain a set of new folded matrices, e.g., $T'_{jk}(t_i)=F'_{jk}(t_i)R'_{jk}(t_i)$, using the dyadic product of the updated steering vectors, $F'_{jk}(t_i)=a'_j(t_i)a'^*_k(t_i)$. The second strong path may then similarly be removed by subtracting the mean element $\overline{T'_{jk}}(t_i)$ from the matrix $T'_{jk}(t_i)$, to obtain the second-order reduced covariance matrix $R''_{jk}(t_i)=T'_{jk}(t_i)-\overline{T'_{jk}}(t_i)$. This loop processing may be continued to remove any predetermined number (e.g., one, two, three) of strong paths and obtain consecutively-ordered reduced covariance matrices $R'_{jk}(t_i)$, $R''_{jk}(t_i)$, $R'''_{jk}(t_i)$, etc.

After the predetermined number of paths has been removed, the processing unit of the wireless device may perform interpolation **514**. For example, if two paths have been removed, interpolation **514** may be performed based on the second-order reduced covariance matrix $R''_{jk}(t_i)$. In some implementations, interpolation **514** may be performed both in the time domain and the frequency domains, e.g., using linear interpolation, cubic interpolation, and the like. For example, the covariance matrices may be presumed to be functions of the difference of frequencies f_j-f_k,

$$R''_{jk}(t_i)\approx R''(f_j-f_k,t_i).$$

Whereas for the folding matrices $F_{jk}(t_i)$ this property may be satisfied by construction, the same may be taken as an approximated for the measured covariance matrices. Accordingly, the expanded m×m reduced covariance matrix $R''(f_j-f_k, t_i)$ may be obtained by from the available values of n reduced covariance matrices $R''(f_j-f_k, t_i)$, of the smaller size $l\times l$. In particular, each covariance matrix $R''(f_j-f_k, t_i)$ may include information about $2l-1$ different values of $R''(f_j-f_k, t_i)$ corresponding to the frequency differences $f_j-f_k \in [-(l-1)\Delta f, (l-1)\Delta f]$. In some implementations, each value $R''(f_j-f_k, t_i)$ may be obtained by averaging over multiple matrix elements $R''_{jk}(t_i)$ with a given difference f_j-f_k (which may be visualized as elements belonging to various bottom-left/top-right diagonals of $R''_{jk}(t_i)$ matrices). For example, in an $l\times l$ covariance matrix there may be $l-|f_j-f_k|/\Delta f$ elements (with various f_j and f_k) with the same value of the difference f_j-f_k. Each value $R''(f_j-f_k, t_i)$ may be obtained by averaging over all (or a subset) of such elements.

Correspondingly, the expanded m×m reduced covariance matrix $R''_{jk}(t_i)$ may be interpolated from $2l-1$ different values (in general, non-contiguous) available for each of n sensing times t_i. Any suitable techniques of a two-dimensional interpolation (e.g., bilinear interpolation, bicubic interpolation) may be used. The output of interpolation **514** may be the expanded m×m reduced covariance matrix $R''_{jk}(t_i)$. The processing units of the wireless device may then perform another loop processing (as indicated with the dashed arrow) that includes path restoration **516** and signal unfolding **518**. More specifically, in the example where matrix R''_{jk} (t_i) is a second-order reduced matrix (obtained following removal of two paths), path restoration **516** and signal unfolding **518** may be performed as follows,

$$R_{jk}(t_i)=F^*_{jk}(t_i)[F'^*_{jk}(t_i)(R''_{jk}(t_i)+\overline{T'_{\mathrm{jk}}}(t_i))+\overline{T_{\mathrm{jk}}}(t_1)].$$

The unfolding transformation matrices $F'^*_{jk}(t_i)$ and $F^*_{jk}(t_i)$ are m×m matrices build from the same steering vectors as used in the respective folding matrices $F'_{jk}(t_i)$ and $F_{jk}(t_i)$, with one difference being that the unfolding matrices are full m×m matrices whereas the folding matrices are smaller l×l matrices. In particular, the unfolding matrix $F'^*_{jk}(t_i)$ may be determined by parameters d'_0 and v' and the unfolding matrix $F^*_{jk}(t_i)$ may be determined by parameters d_0 and v. The obtained unfolded matrix $R_{jk}(t_i)$ represents a characterization of the full reconstructed set of covariance matrices **520**, which may then be used to determine the trajectory of the return object, e.g., $d(t)=d_0+vt$, or $\vec{r}=\vec{r}_0+\vec{v}t$.

The implementations described in conjunction with FIG. **5**A deploy two-dimensional interpolation in the space of f_j-f_k and t_i with the additional approximation that the covariance matrices depend on the difference of frequencies. In some implementations, the interpolation may be performed in the three-dimensional space f_j, f_k, t_i without additional assumptions regarding the frequency dependence.

FIG. **5**A and the corresponding description above illustrate one possible technique of obtaining a reconstructed set of sensing values **520**, various other modifications of reconstruction of sensing values are within the scope of this disclosure. More specifically, as illustrated in FIG. **5**B, in some implementations, the processing units of the wireless device may perform reconstruction based on signal vectors **505** made of the sensing values $r(f_j, t_i)$ directly, without forming covariance matrices $R_{jk}(t_i)$. In such implementations, sensing values $r(f_i, t_i)$ may be treated as n signal vectors (e.g., l-component vectors) that are folded using steering vectors: $t(f_j, t_i)=r(f_j, t_i)a_j(t_i)$. One or more strong paths can then be removed, e.g., by subtracting one or more mean values, e.g., $r'(f_j, t_i)=t(f_j, t_i)-\overline{t(t_i)}$, where $\overline{t(t_i)}=\Sigma j=1^l t(f_j, t_i)$. The corresponding reduced sensing vectors, e.g. $r'(f_j, t_i)$, $r''(f_j, t_i)$, etc., may then be used to perform interpolation from l different sensing values (in general, non-contiguous) available for each of n sensing times t_i. The processing units of the wireless device may then perform another loop of path restoration **516** and signal unfolding **518**, e.g., using the inverse steering vectors a^*_j (t_i). The obtained unfolded signal vectors $r(f_j, t_i)$ represent a characterization of the full reconstructed set of signal vectors **521**, which may then be used to determine the trajectory of the return object, as described above.

FIGS. **6**A-B illustrate frequency selection and reconstruction of sensing values in the temporal domain, in accordance with some implementations. Depicted schematically (with solid horizontal lines) in FIG. **6**A is a working frequency range **602** having m frequencies f_1, f_2 . . . f_m, which may represent a subset of the whole available frequency range **604** (corresponding to both solid and dashed horizontal lines). Although the working frequency range **602** has a contiguous range of frequencies, this is not a requirement and any non-contiguous frequency range may be used.

During each of the sensing events t_j, one or more frequencies f_j may be selected. For brevity, conciseness, and ease of illustration, only a single frequency is shown to be selected (as indicated with a black circle) during each sensing event. It should be understood, however, that any number of frequencies may be selected during a single sensing event. Sensing signals that use the selected frequencies may be transmitted sequentially, such that a single frequency signal is transmitted and received over a certain time (e.g., 0.3 ms). In some implementations, sensing signals of different frequencies may be transmitted and received concurrently. In some implementations, frequency selection for a specific sensing event, e.g., event t_{next} may be performed probabilistically, using a dynamic probability of selection **606**. For example, the dynamic probability may be constructed in such a way as to ensure that each frequency f_j is selected at least once every time interval τ_{max}, during which N sensing signals are transmitted and received (e.g., $N=\tau_{max}/0.3$ ms). Accordingly, a particular frequency f_j has not been selected for time τ_j, the probability P_j to select this frequency f_j for the next sensing event t_{next} may be an increasing function of τ_j, e.g.

$$P_j = \frac{A}{\tau_{max} - \tau_j},$$

where A can be any suitably-chosen coefficient. In some implementations, or any other function of τ_{max} and τ_j can be used. In some implementations, when τ_j reaches or exceeds τ_{max}, to avoid large (infinite) values, the probability P_j can be replaced with some fixed value.

The lengths of horizontal bars in FIG. **6**A illustrate the concept of the dynamic probability. For example, frequency f_m has not been selected during any of the nine most recent sensing events; as a result, the probability to select frequency f_m during the next sensing event is large. On the other hand, frequency f_3 has been selected during the most recent sensing event; as a result, the probability to select frequency f_3 is small.

In some implementations, if none of n=N−1 most recent sensing events have selected a particular frequency f_j, this frequency may be selected with certainty during the next sensing event. In some implementations, the number N of sensing events during time interval τ_{max} may be sufficiently large (and, therefore, the time τ_{max} may be sufficiently long) that all m sensing frequencies are likely to be selected during N consecutive sensing events. For example, if l frequencies are selected during each sensing events, it would take on average N/l sensing events to select each sensing frequency. Hence, the number N of sensing events during time interval τ_{max} may be at least N≥m/l. In some implementations, the number N of sensing events during time interval τ_{max} may be substantially larger than m/l, to reduce the likelihood of frequency "collisions." A frequency collision refers to a situation where, during any given sensing event, more than l frequencies have not been selected during the last N−1 sensing events. Accordingly, the number N may be chosen sufficiently large so that the probability of a frequency collision is less than a target threshold, e.g., less than 5%, 2%, or any other value that is set in view of a target accuracy, e.g., based on a type and nature of application **303** that uses device/object tracking. In some implementations, the coefficient A in the above example formula can be selected to normalize the probabilities:

$$A=\sum_{k=1}^{m}\frac{1}{\tau_{max}-\tau_k},$$

so that one frequency is selected during each sensing event.

Each black circle in FIG. **6**A may be associated with a sensing value $r(f_j, t_i)$ measured upon the return of the corresponding sensing signal. The resulting set $\{r(f_j, t_i)\}$ represents the limited set of randomized sensing values **404** of FIG. **4**. FIG. **6**B illustrates interpolation from the limited set $\{r(f_j, t_i)\}$ to a reconstructed set of sensing values $\{R(f_j, t_i)\}$ **420** that are defined for all frequencies within the working range of frequencies **602**. It should be understood that various interpolation schemes may be used to accomplish reconstruction of the sensing values. In one non-limiting illustration, the interpolation may use the Whittaker-Shannon techniques. More specifically, the missing sensing values, depicted by the white circles in FIG. **6**B may be replaced with zeros (padding **605**). Subsequently, the padded sensing values $r_{pad}(f_j, t_i)$ associated with the same frequency f_j and different values of t_i may undergo temporal interpolation **608**, which may include transforming the padded sensing values $r_{pad}(f_j, t_i)$ to a spectral representation with respect to time t_i: $r_{pad}(f_j, t_i)\rightarrow r_{pad}(f_j,\Omega)$, applying a low-pass filter to limit the spectral representation to the values of spectral frequency Ω to those below a certain threshold, $\Omega<\Omega_0$, and then performing an inverse spectral transformation to obtain reconstructed sensing values, $r_{pad}(f_j, \Omega<\Omega_0)\rightarrow R(f_j, t_i)$. Any suitable low-pass filter may be used. For example, if a "brick-wall" low-pass filter is used, the reconstructed sensing values $R(f_j, t_i)$ may be represented with the convolution of the padded sensing values $r_{pad}(f_j, t_i)$ with the sinc-function, $\sin(\Omega_0 t_i)/[2\pi t_i]$. It should be understood that the brick-wall low-pass filter is merely one possible implementation and that any other known low-pass filters may be used instead. In some implementations, the low-pass filter may be adaptive. In some implementations, a Wiener filter may be used in addition to (or in place of) the low-pass filter. This process of interpolation and filtering may be performed for each temporal slice (indicated with the shaded box in FIG. **6**B) of the padded sensing values. The obtained reconstructed set $\{R(f_j, t_i)\}$ of sensing values **820** may then be used to determine the trajectory of the return object, e.g., $d(t)=d_0+vt$, or $\vec{r}=\vec{r}_0+\vec{v}t$, as described in more detail above. Numerous other variations of the temporal interpolation **608** and filtering using low-pass filter **610** may be used. For example, in some implementations, the inverse spectral transformation is not used and the trajectory determination may be performed directly based on the filtered spectral representation $r_{pad}(f_j, \Omega<\Omega_0)$ of the padded sensing values. In some implementations, any suitable interpolation in the time domain (e.g., a linear interpolation or a cubic interpolation) may be used instead of padding **605**.

In some implementations, the reconstruction may be performed with a time lag t_{lag}. For example, at time t, the reconstruction may be performed for up to time $t-t_{lag}$ using the sensing values obtained up to the current time t. This may increase the accuracy of reconstruction. More specifically, when reconstruction is performed up to time t, most frequencies f_j may have missing sensing values for the most recent sensing events. As illustrated in FIG. **6**B, only one frequency f_3 is associated with a sensing value obtained during the most recent sensing event. As a result, reconstruction of the sensing values for the most recent sensing events (for all frequencies except f_3) may involve extrapolation from the past sensing events (black dots). On the other hand, when reconstruction is performed for the sensing events up to time $t-t_{lag}$, additional frequencies f_6 and f_7 have sensing values that have been obtained between times $t-t_{lag}$ and t. As a result, reconstruction of more frequencies (e.g., f_3, f_6, and f_7) may involve interpolation between past sensing values (obtained before time time $t-t_{lag}$) and "future" sensing values (obtained between time $t-t_{lag}$ and time t). Such lag-based interpolation reconstruction may be more precise compared with the extrapolation reconstruction (without a lag), and may be preferred when accuracy of the trajectory determination is important. In some implementations, when the speed of trajectory determination is more important than accuracy, extrapolation reconstruction (without a lag) may be performed.

FIG. **7** illustrates advantages of using a low-pass filter in reconstruction of sensing values in temporal domain, in accordance with some implementations. The environment **700** of a wireless device **702** located on or at a vehicle includes a return device **704** (e.g., a key fob carried by the owner of the vehicle), and a wall **706** capable of reflecting radio signals. Return device **704** is moving towards wireless device **702** with the speed of 5 km per hour. Three paths of wireless signal propagation include a direct line-of-sight (LoS) path experiencing shadowing **710**, a strong reflection **712** from wall **706**, and a weak reflection **714** from the ground. FIG. **7** also shows estimations of distance from the wireless device to the return device **704** in a situation where the distance from wireless device to wall **706** is approximately 13 m. Left image **708** is a heat map depicting detected (tested) distances d to the return device **704** as a function of the true distance D between wireless device **702** and return device **704** detected using temporal interpolation of sensing signals without low-pass filtering. As a result of the existence of multiple paths of signal propagation, multiple distances d are detected. More specifically, the line **710-1** corresponds to the direct LoS path, $d_1=D$; the line **712-1** corresponds to the signal reflected from the wall, $d_2=26$ m−D; and the line **714-1** corresponds to a combination of lines **712-1** and **714**, $d_3=13$ m. Such a combination can arise in a situation where a sensing signal from wireless device **702** propagates to the return device **704** over the LoS path while the return signal reflects from wall **706** (or vice versa). An additional ghost line **716**, $d_2=15$ m+D, is caused by distance aliasing at the frequency step $\Delta f=10$ MHz, with the maximum resolved distance of $d_{max}=c/2\Delta f=15$ m. Right image **720** is a heat map depicting the detected (tested) distances d to the same return device **704** in an implementation that uses low-pass filtering. As illustrated by right image **720**, filtering out the low portion of the spectral representation of the reconstructed set of sensing values eliminates the ghost line **716** that is present in unfiltered left image **708**.

FIGS. **8-9** are flow diagrams of example methods **800-900** that use sensing signals with randomized frequencies for efficient determination and tracking of trajectories of objects in an environment of a wireless device, in accordance with some implementations. Methods **800-900** may be performed to identify distance from one or more objects in the outside environment. Methods **800-900** may be performed by a BT wireless device, a BLE wireless device, a WLAN wireless device, or any other suitable wireless device or apparatus. The wireless device may also include a radio configured to transmit, using one or more antennas, a plurality of sensing radio waves, e.g., Bluetooth sensing waves, Bluetooth Low Energy sensing waves, or Wi-Fi sensing waves. Methods **800-900** may be performed by a wireless controller of the wireless device, e.g., controller **350** of FIG. **3**A. The controller may include a memory (e.g., memory **360**) and a processing device (e.g., processor **352**) communicatively coupled to the memory. The processing device may include hardware (e.g., circuitry, dedicated logic, programmable logic, microcode, etc.), firmware, and/or software, or any combination thereof. The controller **350** may receive data from radio **310**, PHY **320**, and other components/modules. In some implementations, the processing device performing methods **800-900** may execute instructions of data reconstruction engine **364**. In certain implementations, each of methods **800-900** may be performed by a single processing thread. Alternatively, each of methods **800-900** may be performed by two or more processing threads, each thread executing one or more individual functions, routines, subroutines, or operations of the corresponding method. The processing threads implementing methods **800-900** may be synchronized (e.g., using semaphores, critical sections, and/or other thread synchronization mechanisms). Alternatively, the processing threads implementing methods **800-900** may be executed asynchronously with respect to each other. Various operations of methods **800-900** may be performed in a different order compared with the order shown in FIGS. **8-9**. Some operations of methods **800-900** may be performed concurrently with other operations. Some operations may be optional.

FIG. **8** is a flow diagram of an example method **800** that uses reconstruction of sensing values from a limited set of sensing signals of random frequencies, in accordance with some implementations. Method **800** may include collecting, by a wireless device, sensing data for a plurality of sensing events. More specifically, at block **810**, method **800** may include obtaining a set of sensing values (e.g., a set $\{r(f_j, t_i)\}$) characterizing returned sensing signals received from a return device. The set of sensing values may include a plurality of subsets of sensing values. Each subset of the sensing values may be selected for a respective one of a plurality of sensing events, e.g., sensing event t_1, sensing event t_2, and so on. Each subset of sensing values may be associated with a respective subset of a set of working frequencies. For example, a subset $r(f_1, t_1)$, $r(f_2, t_1)$ . . . of sensing values may be associated with a subset f_1, f_2 . . . of working frequencies (e.g., the BT frequency range) selected during a sensing event t_1. In some implementations, each subset of frequencies f_1, f_2 . . . is selected pseudo-randomly. Each subset of frequencies may be known to the return device (e.g., a device that is being tracked). In some implementations, the number of selected frequencies may be 8,10, etc., or any other number, per sensing event t_i.

At block **820**, method **800** may continue with obtaining a preliminary estimate of a trajectory of the return device (e.g., $d(t)=d_0+vt$). For example, the wireless device may use two or more subsets of sensing values to determine a likelihood tensor (e.g. $P(d, v)$), and use the likelihood tensor to obtain the preliminary estimate of the trajectory, as disclosed in more detail above. At block **830**, method **800** may continue with obtaining, using the preliminary estimate of the trajectory of the return device, a folded representation of the set of sensing values. The folded representation may be any suitable representation of sensing values, e.g., a folded covariance matrix of sensing values, a folded vector of sensing values, and so on. Additionally, the folded representation may be any of the above (or additional) examples from which one or more paths have been removed, e.g. the folded representation may include a set of matrices $R'_{jk}(t_i)$, provided that a single path has been removed, a set of matrices $R''_{jk}(t_i)$, provided that two paths have been removed, and so on.

In particular, as illustrated with the upper callout portion in FIG. **8**, obtaining the folded representation of the set of sensing values may include one or more iterations. Each of the one or more iterations may include performing, for each of the plurality of subsets of the sensing values, operations of blocks **832-836**. More specifically, at block **832**, the processing device of the wireless device may apply a folding transformation to a first representation of each subset (or at least some subsets) of the sensing values. For example, the first representation of a subset of the sensing values may include a covariance matrix for the respective subset of the sensing values, e.g., R_{jk} (t_i), or a vector of the sensing values, $r(f_j, t_i)$, as described in more detail above in conjunction with FIG. **9**A and FIG. **9**B. The folding transformation (e.g., transformation $T_{jk}(t_i)=R_{jk}(t_i)F_{jk}(t_i)$) may be based on the preliminary estimate of the trajectory. In particular, applying the folding transformation (e.g., $F_{jk}(t_i)$) to the first representation (e.g., R_{jk} (t_i)) of the respective subset of the sensing values may include applying a steering vector (e.g., $a_j(t_i)$, a^*_k (t_i)) to the respective subset of the sensing values, (e.g., $R_{jk}(t_i)F_{jk}(t_i)$, where $F_{jk}(t_i)=a_j(t_i)a^*_k$ (t_i)). In some implementations, a phase of the steering vector (e.g., $a_j(t_i)=\exp$ $[4\pi i f_j$ $(d_0+vt_i)/c]$) may depend on a distance (e.g., $d=d_0+vt_i$, at the time of the sensing event t_i) to the return device. The distance to the return device d_0+vt_i may be obtained using the preliminary estimate of the trajectory of the return device.

At block **834**, method **800** may include obtaining a second representation (e.g., $R'_{jk}(t_i)=T_{jk}(t_i)-\overrightarrow{T_{jk}}$) of each subset of the sensing values by subtracting a value, determined in view of a plurality of elements of the folded first representation of the respective subset, from each of the plurality of elements of the folded first representation of the respective subset of the sensing values. For example, the plurality of elements of the folded first representation may include the elements of the matrix $T_{jk}(t_i)$ and the subtracted value $\overline{T_{jk}}$ may be the average value of the elements of the matrix T_{jk} (t_i). The operations of block **834** may amount to removing a path of signal propagation between the wireless device and the return device. At block **836**, method **800** may include obtaining the improved estimate of the trajectory, e.g., updating the preliminary estimate of the trajectory (e.g., updating d_0, $v \rightarrow d'_0$, v') or otherwise computing the improved estimate of the trajectory using the second representation of two or more of the subsets of sensing values (e.g., using two or more matrices $R'_{jk}(t_i)$). In some implementations, the improved estimate of the trajectory may be computed using the same techniques as used for the preliminary estimate at block **830**. In some implementations, the improved estimate may be computed using different techniques. Operations **832-836** may be repeated any predetermined number of times, e.g., depending on the number of paths of propagation to be removed.

At block **840**, the processing device of the wireless device may use the folded representation of the set of sensing values to generate a reconstructed set of sensing values (e.g., matrix $R_{jk}(t_i)$). In some implementations, the folded representation of the set of sensing values may include a folded representation of each of the plurality of subsets of the sensing values (e.g., a set of matrices $R''_{jk}(t_i)$ for different sensing events t_i). The reconstructed set may include reconstructed sensing values for at least some (or all) of the set of working frequencies and for at least some of the plurality of sensing events. As illustrated with the lower callout portion in FIG. **8**, using the folded representation of the set of sensing values to generate a reconstructed set of sensing values may include, at block **842**, performing an interpolation using the folded representation for each of the plurality of subsets of the sensing values. In some implementations, the interpolation is performed between two or more frequencies of the set of working frequencies. In some implementations, the interpolation may be further performed between two or more times of the plurality of sensing events. At block **844**, the processing device may apply an unfolding transformation to the interpolated folded representation, and, at block **846**, may restoring the one or more paths of signal propagation between the wireless device and the return device. For example, restoring the one or more paths may include adding the previously subtracted mean values $\overline{T_{jk}}$, $\overline{T'_{jk}}$, etc., and applying an unfolding transformation may include applying the inverse folding transformation (e.g., $F^*_{jk}(t_i)$), as described in more detail in relation to FIG. **3**A.

At block **850**, method **800** may include determining, using the reconstructed set of sensing values, one or more spatial characteristics of the return device. The one or more spatial characteristics may include an improved estimate of the trajectory of the return device. In some implementations, operations of block **850** may any other operations that determine a distance to the return device, a velocity of the return device, a trajectory of the return device, e.g., a set of distances and/or velocities of the return device for a set of times, directions to the return device, and so on, using any suitable method, e.g., a method that uses a uniform frequency step or an uniform time step, such as MUSIC, and the like.

FIG. **9** is a flow diagram of an example method **900** that uses frequency selection and reconstruction of sensing values in temporal domain for efficient tracking of wireless devices, in accordance with some implementations. Method **900** may include collecting sensing data for a plurality of sensing events, with blocks **910**-**930** repeated for each sensing event. More specifically, at block **910**, method **900** may include probabilistically selecting, by a wireless device (or a processing unit of the wireless device), one or more frequencies f_j. In some implementations, the number of selected frequencies may be 8, 10, etc., or any other number, per sensing event t_i. In some implementations, each frequency of the one or more frequencies f_j is selected with a probability that is determined in view of a time (or a number of sensing events that occurred) since the previous selection of the respective frequency. For example, the probability to select a particular frequency may increase with the time since the previous selection of this frequency. In some implementations, the probability to select a particular frequency over a target interval of time (e.g., τ_{max}, as described in conjunction with FIGS. **6**A-B) is at least a threshold probability, e.g., 95%, 98%, etc. In some implementations, the threshold probability is a unity (100% probability). In some implementations, the target interval of time is determined in view of an expected maximum velocity v_{max} of the return device, e.g., $\tau_{max}=c/(2f_jv_{max})$. In some implementations, selection of frequencies may be performed stochastically, e.g., pseudo-randomly. In some implementations, method **900** may include communicating, to the return device, the one or more selected frequencies. For example, the wireless device may select a number (e.g., one, two, four, ten, etc.) of frequencies and notify the return device of the selected frequency (e.g., during one of exchanges of sensing signals with the return device or via a separate communication). In some implementations, the probabilistic selection

of frequencies may be performed during actual tracking of wireless devices. In some implementations, the probabilistic selection may be performed prior to tracking of wireless devices. For example, the initiator device may exchange, with a wireless device, previously selected frequencies or one or more parameters of the frequency selection algorithm. The parameters may include one or more seeds for a random generation function (or similar parameters), which define a starting point in a pseudorandom sequence.

At block **920**, method **900** may continue with the wireless device transmitting, e.g., using a radio module, a set of sensing signals having the one or more selected frequencies f_j. At block **930**, method **900** may include obtaining, by the wireless device, a set of sensing values characterizing returned sensing signals generated by a return device in response to receiving the set of sensing signals. All the sets of sensing values obtained for different sensing events t_i may constitute a randomized limited set of sensing values $\{r(f_j, t_i)\}$ **404** of FIG. **4**.

At block **940**, method **900** may include generating, using the obtained sets of sensing values (which together constitute the limited set $\{r(f_j, t_i)\}$), a reconstructed set of sensing values (e.g., $\{R(f_j, t_i)\}$). The reconstructed set of sensing values may include multiple subsets. For example, a first (second, etc.) subset of the reconstructed set may be associated with a first frequency f_1 (second frequency f_2, etc.) of the one or more frequencies $\{f_j\}$ and may be generated using the sensing values associated with the first frequency f_1 (second frequency f_2, etc.) and obtained during one or more of the plurality of sensing events (e.g., multiple sensing events t_1, t_2, etc. In some implementations, the first (second, etc.) subset of the reconstructed set of sensing values is generated using interpolation from the sensing values associated with the first frequency (second frequency, etc.) and obtained during the one or more of the plurality of sensing events. In some implementations, as depicted with the callout portion in FIG. **9**, the interpolation may include, at block **942**, obtaining a padded subset of sensing values by representing, with zero values, missing sensing values associated with the first (second, etc.) frequency. At block **944**, the interpolation may further include applying a low-pass filter to eliminate a portion of a spectral representation of the padded subset of sensing values.

At block **950**, method **900** may include determining, using the reconstructed set of sensing values, an estimated trajectory of the return device. In some implementations, operations of block **950** may include any other operations that determine a distance to the return device, a velocity of the return device, a trajectory of the return device, e.g., a set of distances and/or velocities of the return device for a set of times, and so on, using any suitable method, e.g., MUSIC, GCC, and the like.

It should be understood that the above description is intended to be illustrative, and not restrictive. Many other implementation examples will be apparent to those of skill in the art upon reading and understanding the above description. Although the present disclosure describes specific examples, it will be recognized that the systems and methods of the present disclosure are not limited to the examples described herein, but may be practiced with modifications within the scope of the appended claims. Accordingly, the specification and drawings are to be regarded in an illustrative sense rather than a restrictive sense. The scope of the present disclosure should, therefore, be determined with reference to the appended claims, along with the full scope of equivalents to which such claims are entitled.

The implementations of methods, hardware, software, firmware or code set forth above may be implemented via instructions or code stored on a machine-accessible, machine readable, computer accessible, or computer readable medium which are executable by a processing element. "Memory" includes any mechanism that provides (i.e., stores and/or transmits) information in a form readable by a machine, such as a computer or electronic system. For example, "memory" includes random-access memory (RAM), such as static RAM (SRAM) or dynamic RAM (DRAM); ROM; magnetic or optical storage medium; flash memory devices; electrical storage devices; optical storage devices; acoustical storage devices, and any type of tangible machine-readable medium suitable for storing or transmitting electronic instructions or information in a form readable by a machine (e.g., a computer).

Reference throughout this specification to "one implementation" or "an implementation" means that a particular feature, structure, or characteristic described in connection with the implementation is included in at least one implementation of the disclosure. Thus, the appearances of the phrases "in one implementation" or "in an implementation" in various places throughout this specification are not necessarily all referring to the same implementation. Furthermore, the particular features, structures, or characteristics may be combined in any suitable manner in one or more implementations.

In the foregoing specification, a detailed description has been given with reference to specific exemplary implementations. It will, however, be evident that various modifications and changes may be made thereto without departing from the broader spirit and scope of the disclosure as set forth in the appended claims. The specification and drawings are, accordingly, to be regarded in an illustrative sense rather than a restrictive sense. Furthermore, the foregoing use of implementation, implementation, and/or other examplar language does not necessarily refer to the same implementation or the same example, but may refer to different and distinct implementations, as well as potentially the same implementation.

The words "example" or "exemplary" are used herein to mean serving as an example, instance, or illustration. Any aspect or design described herein as "example' or "exemplary" is not necessarily to be construed as preferred or advantageous over other aspects or designs. Rather, use of the words "example" or "exemplary" is intended to present concepts in a concrete fashion. As used in this application, the term "or" is intended to mean an inclusive "or" rather than an exclusive "or." That is, unless specified otherwise, or clear from context, "X includes A or B" is intended to mean any of the natural inclusive permutations. That is, if X includes A; X includes B; or X includes both A and B, then "X includes A or B" is satisfied under any of the foregoing instances. In addition, the articles "a" and "an" as used in this application and the appended claims should generally be construed to mean "one or more" unless specified otherwise or clear from context to be directed to a singular form. Moreover, use of the term "an implementation" or "one implementation" or "an implementation" or "one implementation" throughout is not intended to mean the same implementation or implementation unless described as such. Also, the terms "first," "second," "third," "fourth," etc. as used herein are meant as labels to distinguish among different elements and may not necessarily have an ordinal meaning according to their numerical designation.

What is claimed is:

1. A method comprising:

obtaining, by a wireless device, a set of sensing values characterizing returned sensing signals received from a return device, wherein the set of sensing values comprises a plurality of subsets of sensing values, wherein each of the plurality of subsets of sensing values is associated with a respective subset of a set of working frequencies, the respective subset being selected for a respective one of a plurality of sensing events, each sensing event of the plurality of sensing events associated with a corresponding time of a plurality of times;

obtaining a plurality of distributions of possible locations of the return device, an individual distribution of the plurality of distributions obtained for a respective time of the plurality of times and using a respective subset of sensing values of the plurality of subsets of sensing values;

obtaining, using the plurality of distributions, a preliminary estimate of a trajectory of the return device characterizing motion of the return device across at least the plurality of times;

obtaining, using the preliminary estimate of the trajectory of the return device, a folded representation of the set of sensing values;

using the folded representation of the set of sensing values to generate a reconstructed set of sensing values, wherein the reconstructed set comprises reconstructed sensing values for at least some of the set of working frequencies and for at least some of the plurality of sensing events; and

determining, using the reconstructed set of sensing values, one or more spatial characteristics of the return device.

2. The method of claim **1**, wherein the respective subset of the set of working frequencies is selected pseudo-randomly, the selected subset being known to the return device.

3. The method of claim **1**, wherein obtaining the folded representation of the set of sensing values comprises one or more iterations, each of the one or more iterations comprising:

for each of the plurality of subsets of the sensing values,

applying a folding transformation to fold a first representation of a respective subset of the sensing values, wherein the folding transformation is based on the preliminary estimate of the trajectory; and

obtaining a second representation of the respective subset of the sensing values by subtracting a value, determined in view of a plurality of elements of the folded first representation of the respective subset, from each of the plurality of elements of the folded first representation of the respective subset of the sensing values; and

obtaining the one or more spatial characteristics of the return device using the second representation of two or more of the subsets of the sensing values.

4. The method of claim **3**, wherein the first representation of the respective subset of the sensing values comprises a covariance matrix for the respective subset of the sensing values.

5. The method of claim **3**, wherein applying the folding transformation to the first representation of the respective subset of the sensing values comprises:

applying a steering vector to the respective subset of the sensing values, wherein a phase of the steering vector depends on a distance to the return device, and wherein the distance to the return device is obtained using the preliminary estimate of the trajectory of the return device.

6. The method of claim **1**, wherein the folded representation of the set of sensing values comprises a folded representation of each of the plurality of subsets of the sensing values, and wherein using the folded representation of the set of sensing values to generate a reconstructed set of sensing values comprises:

performing an interpolation using the folded representation for each of the plurality of subsets of the sensing values.

7. The method of claim **6**, wherein the interpolation is performed between two or more frequencies of the set of working frequencies.

8. The method of claim **7**, wherein the interpolation is further performed between two or more times of the plurality of sensing events.

9. The method of claim **6**, wherein using the folded representation of the set of sensing values to generate the reconstructed set of sensing values further comprises:

applying an unfolding transformation; and

restoring one or more paths of signal propagation between the wireless device and the return device.

10. A method comprising:

for each of a plurality of sensing events associated with a corresponding time of a plurality of times,

selecting, by a wireless device, one or more frequencies;

transmitting, by the wireless device, a set of sensing signals having the one or more selected frequencies;

obtaining, by the wireless device, a set of sensing values characterizing returned sensing signals, wherein the returned sensing signals are received from a return device responsive to the transmitted set of sensing signals; and

obtaining, using the set of sensing values, a distribution of possible locations of the return device at the corresponding time;

generating, using the distributions of possible locations of the return device obtained for each of the plurality of sensing events, a reconstructed set of sensing values, wherein a first subset of the reconstructed set of sensing values is associated with a first frequency of the one or more frequencies and is generated using a first subset of the obtained sensing values, wherein the first subset of the obtained sensing values is associated with the first frequency and obtained during one or more of the plurality of sensing events; and

determining, using the reconstructed set of sensing values, one or more spatial characteristics of the return device.

11. The method of claim **10**, wherein a respective frequency of the one or more frequencies is selected with a probability that increases with a time since a previous selection of the respective frequency, the method further comprising:

communicating, to the return device, the one or more frequencies selected by the wireless device.

12. The method of claim **10**, wherein a respective frequency of the one or more frequencies is selected with a probability over a target interval of time that is at least a threshold probability.

13. The method of claim **12**, wherein the threshold probability is a unity.

14. The method of claim **12**, wherein the target interval of time is determined in view of an expected maximum velocity of the return device.

15. The method of claim **10**, wherein a second subset of the reconstructed set of sensing values is associated with a second frequency of the one or more frequencies and is generated using a second subset of the obtained sensing values, wherein the second subset of the obtained sensing values is associated with the second frequency and obtained during at least one of the plurality of sensing events.

16. The method of claim **10**, wherein the first subset of the reconstructed set of sensing values is generated using interpolation from the first subset of the obtained sensing values.

17. The method of claim **16**, wherein the interpolation comprises:

obtaining a padded subset of sensing values by representing, with zero values, missing sensing values associated with the first frequency; and

applying a low-pass filter to eliminate a portion of a spectral representation of the padded subset of sensing values.

18. An apparatus comprising:

a radio configured to transmit a plurality of radio frequency (RF) signals, wherein each of the plurality of RF signals is transmitted according to a Bluetooth, a Bluetooth Low Energy, or a Wi-Fi protocol;

a memory; and

a processing device coupled to the memory, the processing device configured to:

obtain a set of signal values characterizing returned RF signals received from a return device in response to the transmitted RF signals, wherein the set of signal values comprises a plurality of subsets of signal values, each subset of sensing values of the plurality of subsets of signal values associated with a corresponding time of a plurality of times;

obtain a plurality of distributions of possible locations of the return device, an individual distribution of the plurality of distributions obtained for a respective time of the plurality of times and using a respective subset of signal values of the plurality of subsets of signal values;

obtain, using the plurality of distributions, a preliminary estimate of a trajectory of the return device characterizing motion of the return device across at least the plurality of times;

obtain, using the preliminary estimate of the trajectory of the return device, a folded representation of the set of signal values;

use the folded representation of the set of signal values to generate a reconstructed set of signal values; and

determine, using the reconstructed set of signal values, one or more spatial characteristics of the return device.

19. The apparatus of claim **18**, wherein to obtain the folded representation of the set of signal values, the processing device to perform one or more iterations, each of the one or more iterations comprising:

for each of the plurality of subsets of the signal values,

apply a folding transformation to fold a first representation of a respective subset of the signal values, wherein the folding transformation is based on the preliminary estimate of the trajectory; and

obtain a second representation of the respective subset of the signal values by subtracting a value, determined in view of a plurality of elements of the folded first representation of the respective subset, from each of the plurality of elements of the folded first representation of the respective subset of the signal values; and obtain the one or more spatial characteristics of the return device using the second representation of two or more of the subsets of signal values.

20. The apparatus of claim **18**, wherein the folded representation of the set of signal values comprises a folded representation of each of the plurality of subsets of the signal values, and wherein to use the folded representation of the set of signal values to generate a reconstructed set of signal values, the processing device is to:

perform an interpolation using the folded representation for each of the plurality of subsets of the signal values;

apply an unfolding transformation; and

restore one or more paths of signal propagation between the radio and the return device.

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