Complex Number

Complex Number

$a + ib$, $a + jb$

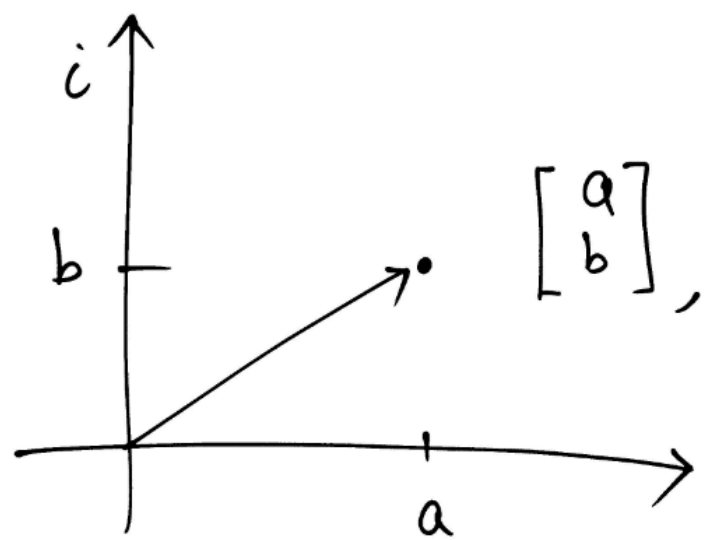
$a + bi$, $a + bj$

a : real number

b : imaginary number

Complex number

= vector in 2D

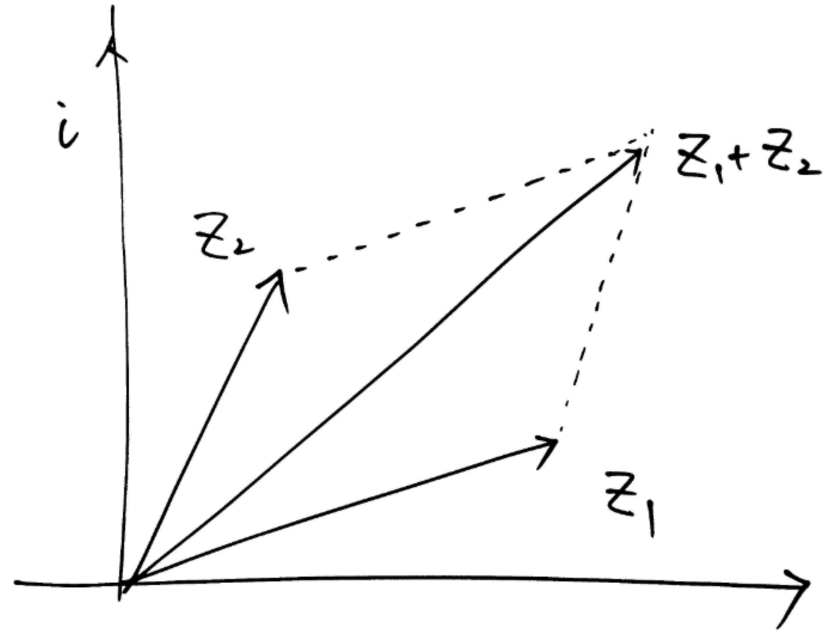


$\begin{bmatrix} a \\ b \end{bmatrix}$, $a + bi$

Adding Complex Numbers

$z_1 = a_1 + b_1 i$, $\vec{z}_1 = \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}$

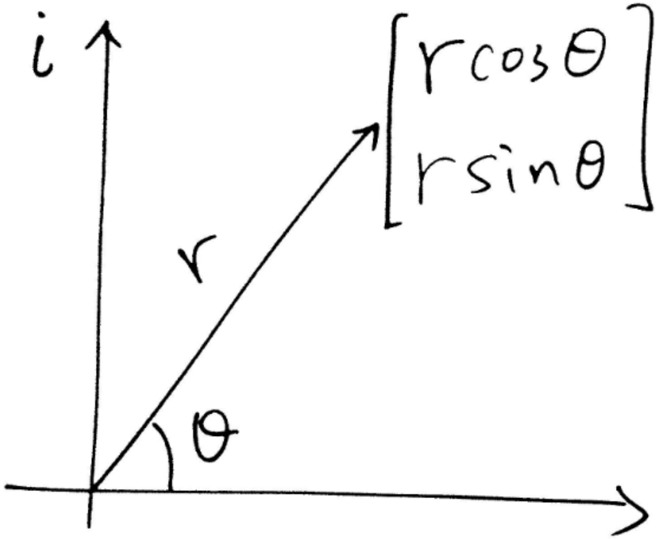
$z_2 = a_2 + b_2 i$, $\vec{z}_2 = \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}$



$$z = z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2) i$$

$$\vec{z} = \vec{z}_1 + \vec{z}_2 = \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} + \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 \\ b_1 + b_2 \end{bmatrix}$$

Complex Exponential Function (Euler's Formula)



vector $\qquad$ complex number

$$\begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix} \quad \text{or} \quad r\cos\theta + ir\sin\theta$$

$$\vec{r} = re^{i\theta} = r(\cos\theta + i\sin\theta)$$

$$r\, e^{i\theta}$$

$r \downarrow$ magnitude (length)

$\theta \downarrow$ phase (angle)

Multiplying/Dividing Complex Numbers

$z_1 = r_1 e^{i\theta_1}$ (like an exponential func.)

$z_2 = r_2 e^{i\theta_2}$

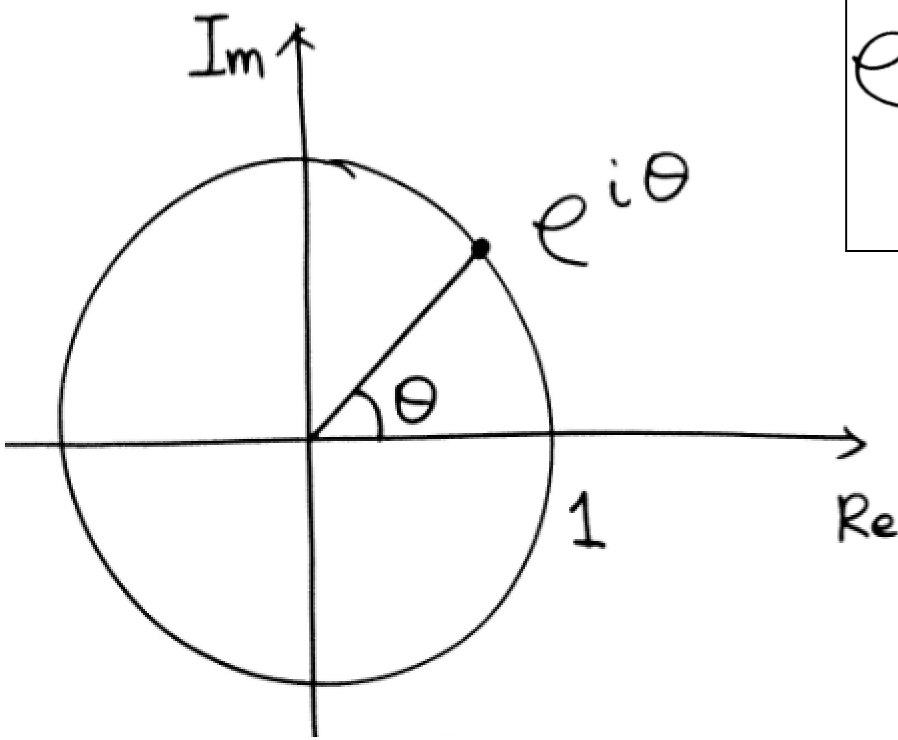
$$\Rightarrow \quad z_1 \cdot z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

Circular Motion Represented by Complex Number

Geometrical Meaning of $e^{i\theta}$

$$e^{i\theta} = \cos\theta + i\sin\theta$$



$e^{i\theta}$: point on the unit circle with angle of θ

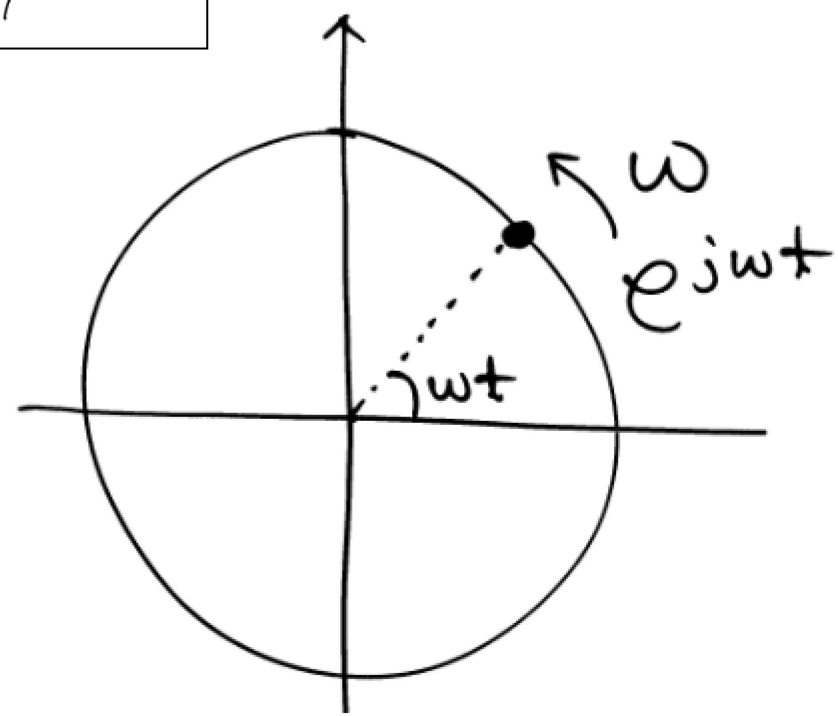
$$e^{i\theta} = e^{i(\theta+2\pi)}$$

Geometrical Meaning of $e^{i\omega t}$

$\theta = \omega t$

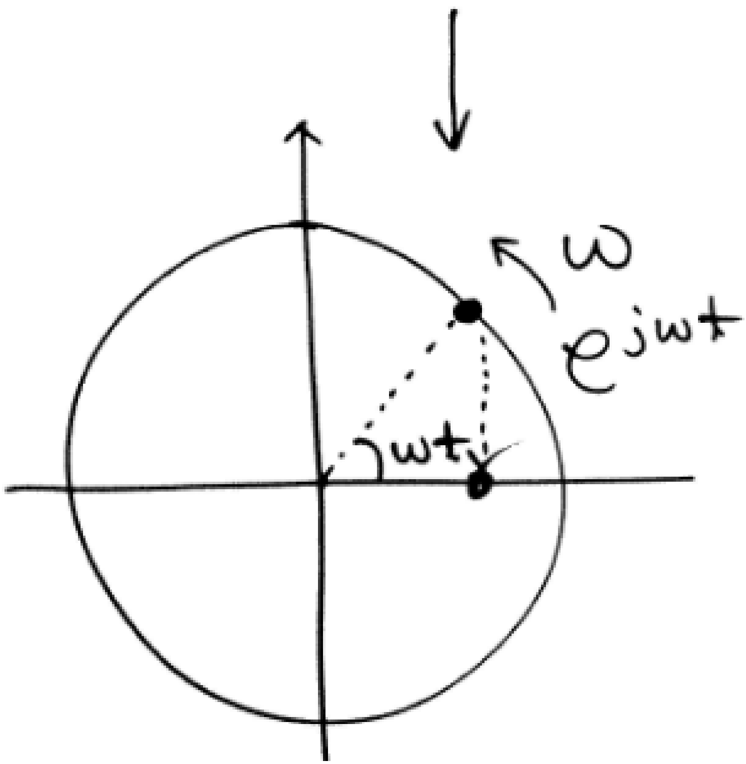
$e^{i\omega t}$: rotating on an unit circle

with angular velocity of ω



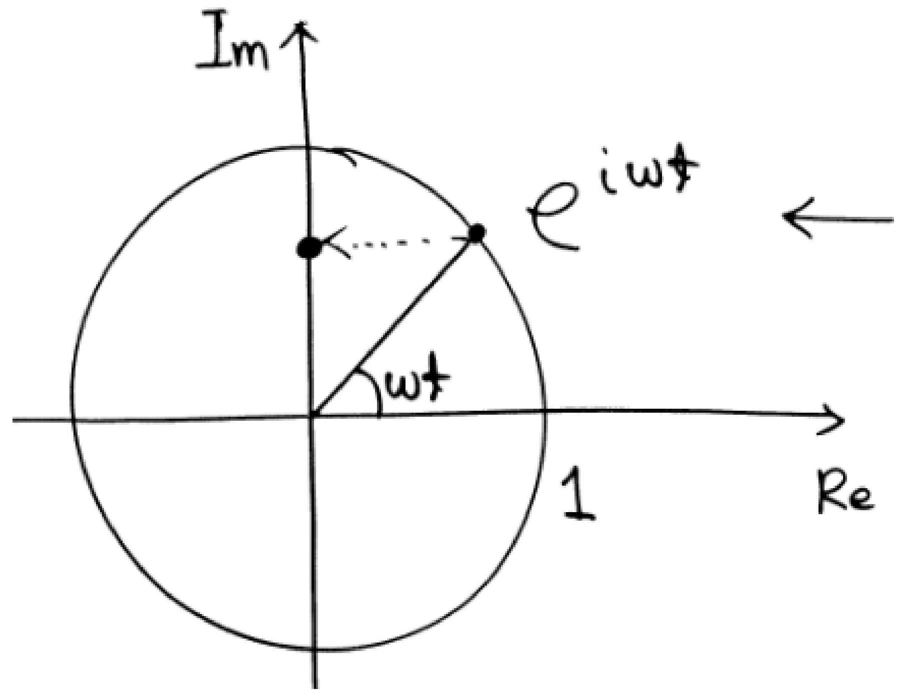
Sinusoidal functions comes from circular motions.
(sin & cos)

projection of $e^{j\omega t}$ onto Re-axis



$\cos \omega t$

projection of $e^{j\omega t}$ onto Im-axis



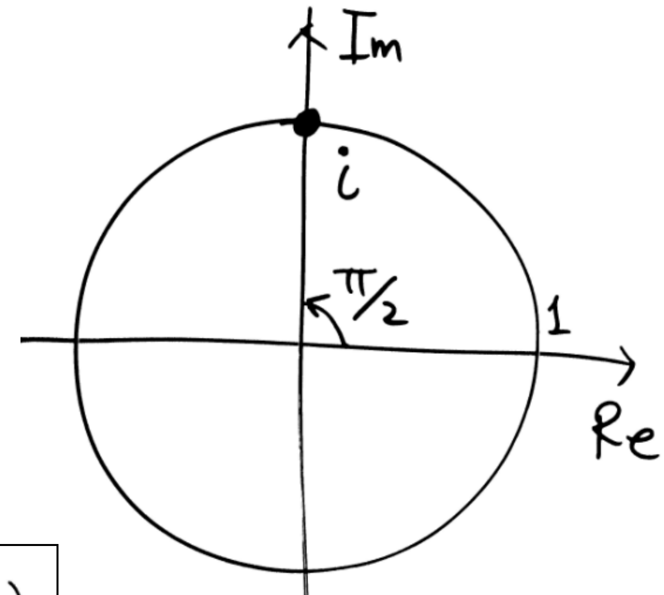
$\sin \omega t$

i Multiplying

$$i\, e^{i\theta} = ?$$

$$z_1 = i \quad = e^{i\frac{\pi}{2}}$$

$$z_2 = e^{i\theta}$$



$$z_1 z_2 = e^{i\left(\frac{\pi}{2} + \theta\right)}$$

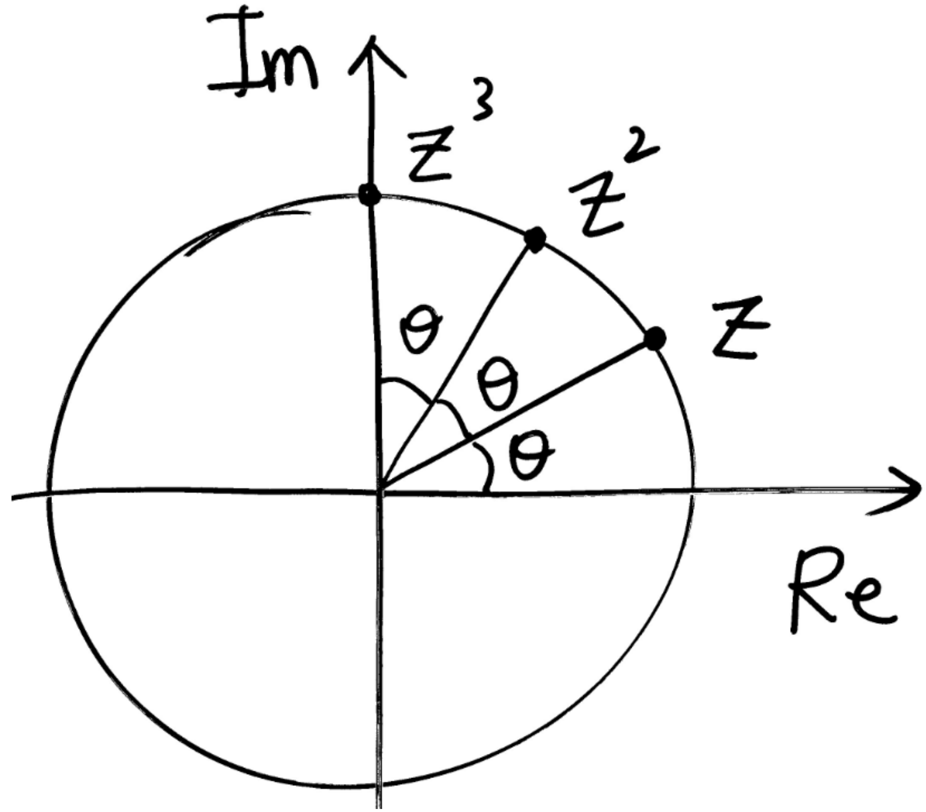
i multiplication

$\Rightarrow$ 90° rotation

Example

$$z = e^{i\theta}$$

$$z^n = (e^{i\theta})^n = e^{in\theta}$$

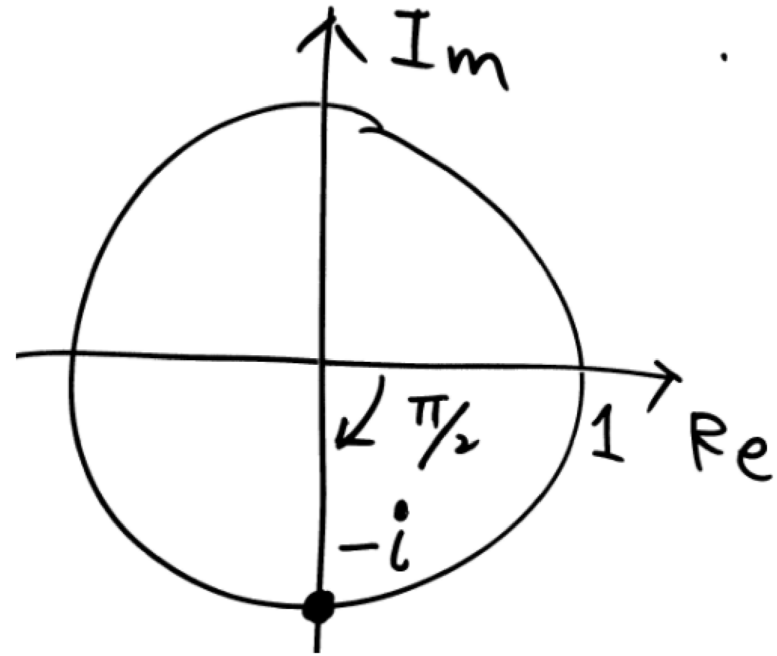


i Dividing

$$\frac{e^{i\theta}}{i} = -ie^{i\theta}$$

$$\frac{z_2}{z_1} = e^{i\left(\theta - \frac{\pi}{2}\right)}$$

$\boxed{\Rightarrow\ -90^\circ \text{ rotation}}$



Circular Motion

Particle rotates on the unit circle

with angular velocity of ω

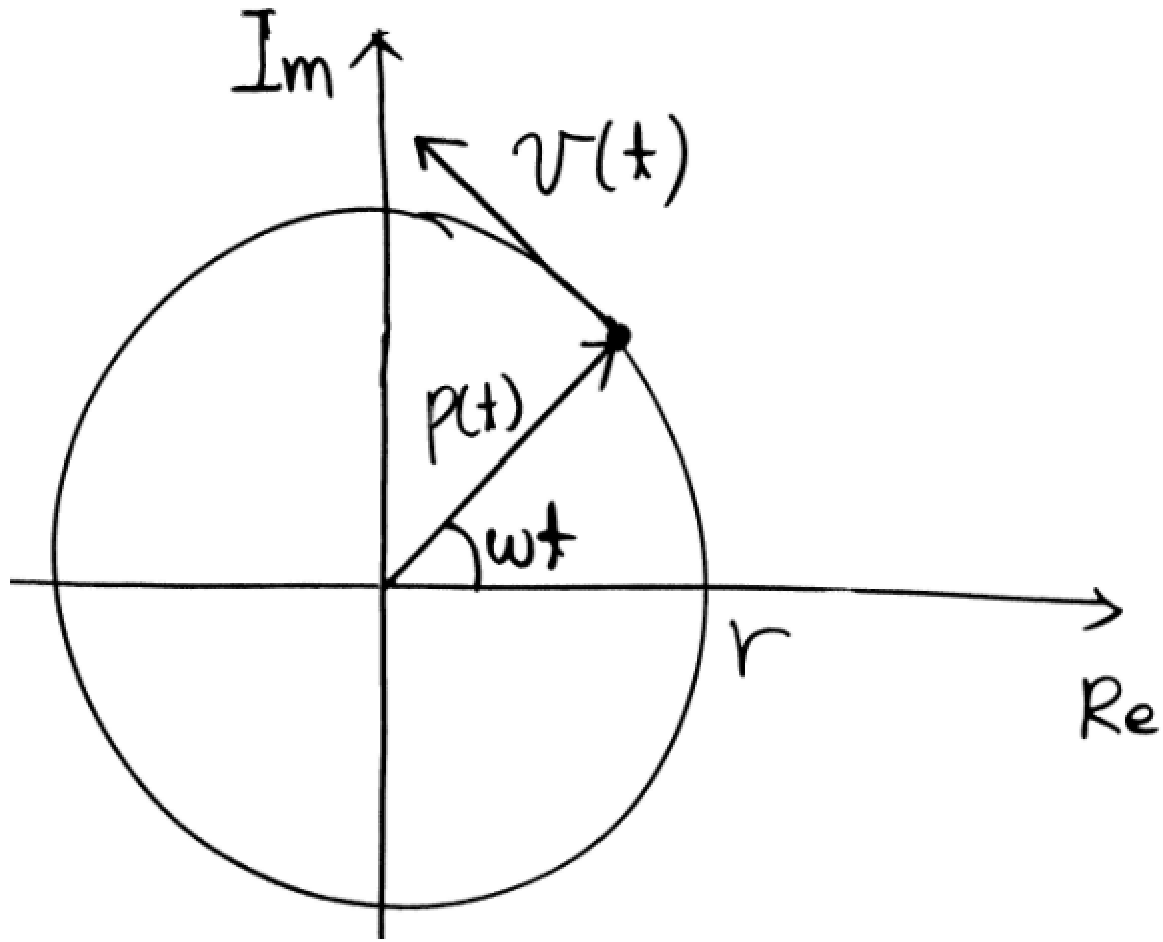
$$p(t) = re^{i\omega t}$$

$$v(t) = \frac{d\,p(t)}{dt} = r \cdot i\omega e^{i\omega t}$$

$$= r \cdot \omega \cdot i e^{i\omega t}$$

$$\boxed{\begin{aligned} |v(t)| &= r\omega \\ \angle v(t) &= \omega t + \frac{\pi}{2} \end{aligned}}$$

Velocity in Circular Motion

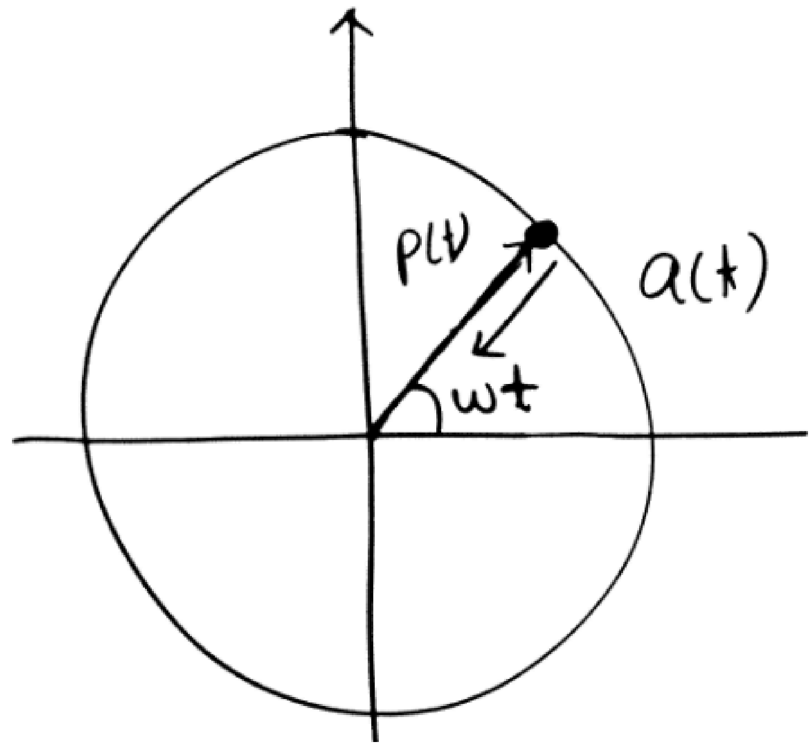


Acceleration in Circular Motion

$$a(t) = \frac{d\,v(t)}{dt} = r\omega\, i\, i\omega e^{i\omega t}$$

$$= -r\omega^2 e^{i\omega t}$$

$$\boxed{\begin{aligned} |a(t)| &= r\omega^2 \\ \angle a(t) &= \omega t + \pi \end{aligned}}$$

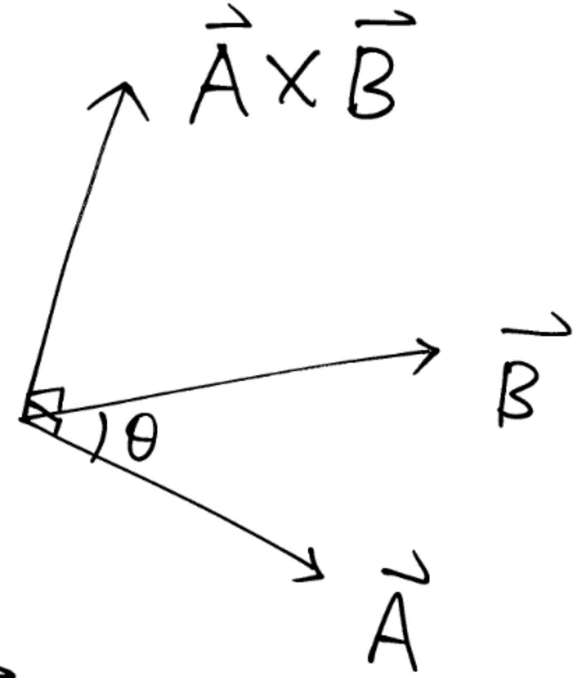


Circular Motion Represented by Vectors

Cross Product

cross product

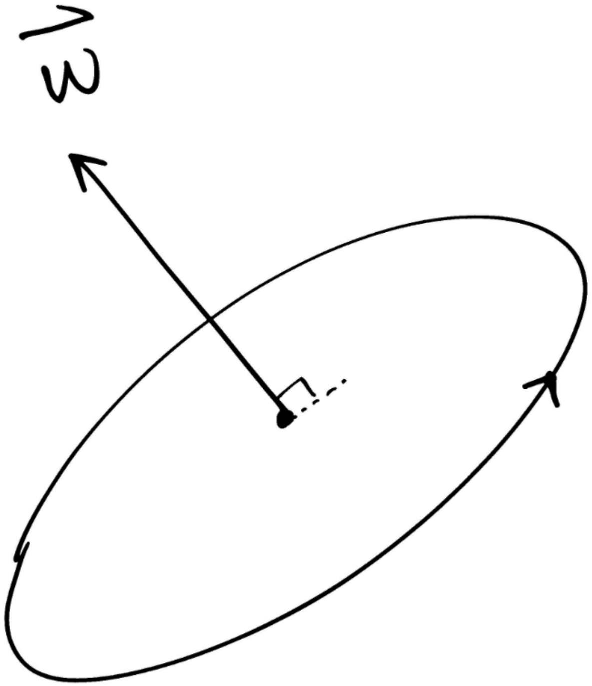
$$\vec{A} \times \vec{B}$$

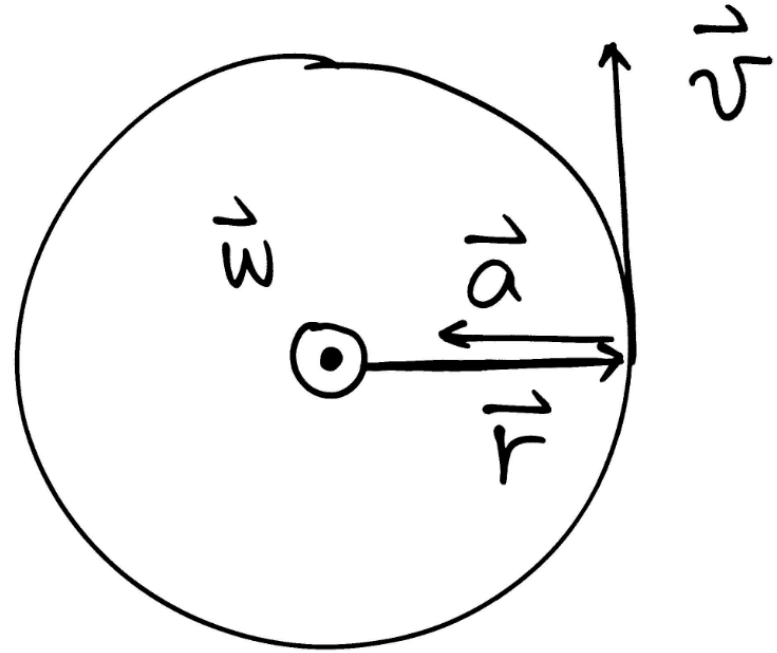


$$|\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}| \sin\theta$$

Why do we need to know the cross product?

Vector Representation in Circular Motion





How to represent *v* and *a* in a cross product form

$$\vec{v} = \vec{\omega} \times \vec{r}$$

$$\vec{a} = \vec{\omega} \times \vec{v} = \vec{\omega} \times (\vec{\omega} \times \vec{r})$$