Dynamic Systems with Matlab

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iSystems (http://isystems.unist.ac.kr/)
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1. Natural response to non-zero initial conditions

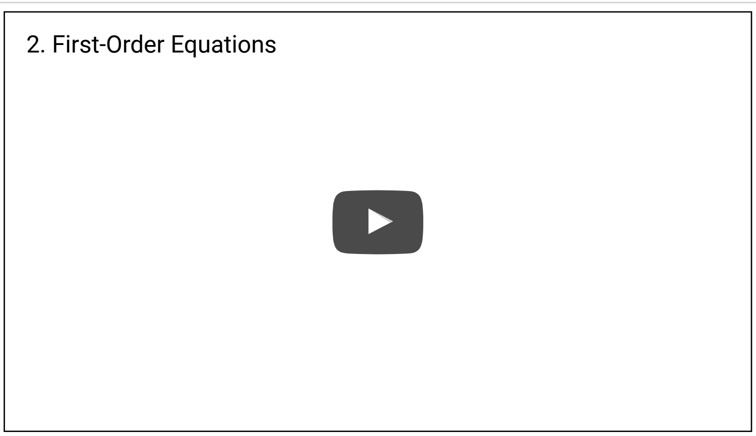
1.1. The First Order ODE

- MIT 2.087 Engineering Mathematics: Linear Algebra and ODEs, Fall 2014, by Gilbert Strang

$$\frac{dx(t)}{dt} = kx(t), \qquad x(0) = x_0$$
$$\rightarrow \; x(t) = x_0 e^{kt}$$

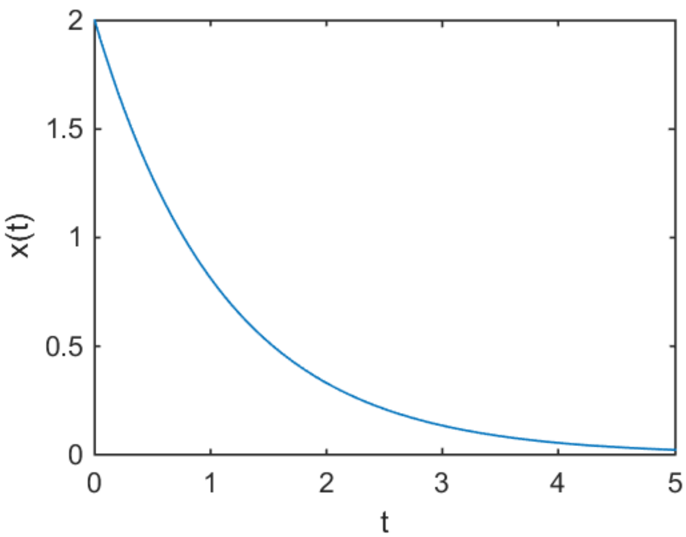
In [1]:

```
%%html
<iframe width="560" height="315"
src="https://www.youtube.com/embed/4X0SGGrXDiI?start=399&end=459" frameborder="0"
allowfullscreen></iframe>
```



In [2]:

```
% plot an analytic solution

k = -0.9;
x0 = 2;

t = linspace(0,5,100);
x = x0*exp(k*t);

plot(t,x), xlabel('t'), ylabel('x(t)')

% but, all we did is just plotting (not computing)
```



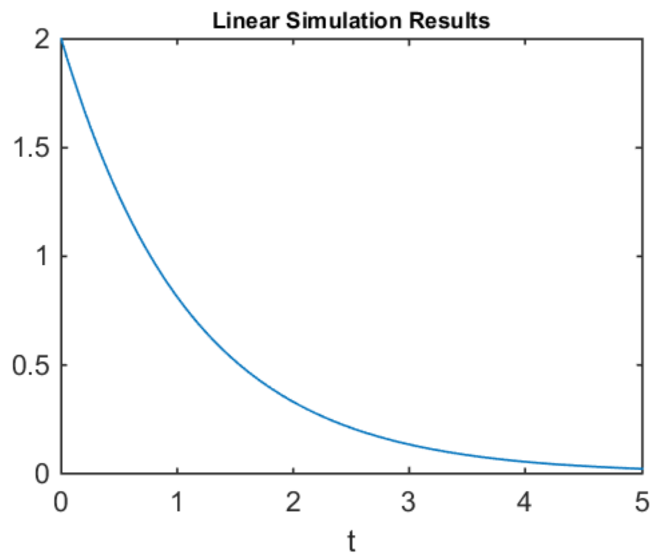
Out[2]:

To use `lsim` command, transform it to the state space representation

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

In [3]:

```
% numerically solve an ODE using lsim
A = k;
B = 0;
C = 1;
D = 0;
G = ss(A,B,C,D);

x0 = 2;
t = linspace(0,5,100);
u = zeros(size(t));
[y,tout] = lsim(G,u,t,x0);
plot(tout,y)
title('Linear Simulation Results','fontsize',8)
xlabel('t')
```



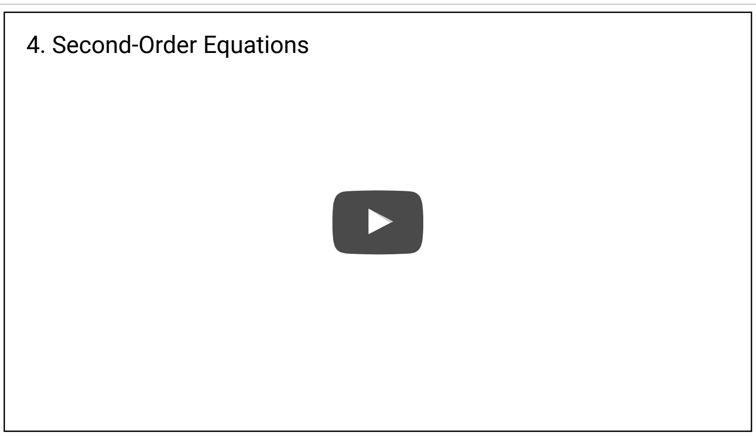
Out[3]:

1.2. The Second Order ODE

- MIT 2.087 Engineering Mathematics: Linear Algebra and ODEs, Fall 2014, by Gilbert Strang

In [4]:

```
%%html
<iframe src="https://www.youtube.com/embed/xvTYUnqn2wY"
width="560" height="315" frameborder="0" allowfullscreen></iframe>
```



Generic form

$$a\frac{d^2x(t)}{dt^2} + b\frac{dx(t)}{dt} + cx(t) = 0, \qquad \dot{x}(0) = v_0, x(0) = x_0$$

One of examples is the mass-spring-damper system

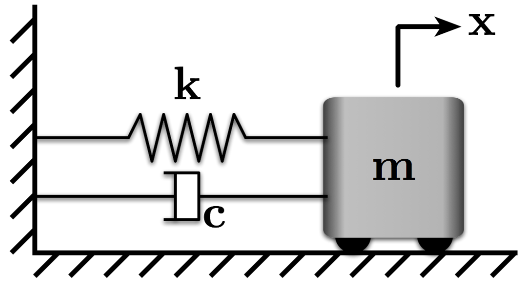


Figure 1: A Mass-Spring-Damper System

This notebook simluates the free vibration of a simple mass-spring-damper system like the one shown in Figure 1. More specifically, we'll look at how system response to non-zero initial conditions.

The equation of motion for the system is:

$$m\ddot{x} + c\dot{x} + kx = 0$$

We could also write this equation in terms of the damping ratio, ζ, and natural frequency ω_n.

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2x = 0$$

We'll use the solution to the differential equation that we developed in class to plot the response. The solution for the underdamped case is:

$$x(t) = e^{-\zeta\omega_nt}\left(a_1 e^{i\omega_dt} + a_2 e^{-i\omega_dt}\right)$$

or

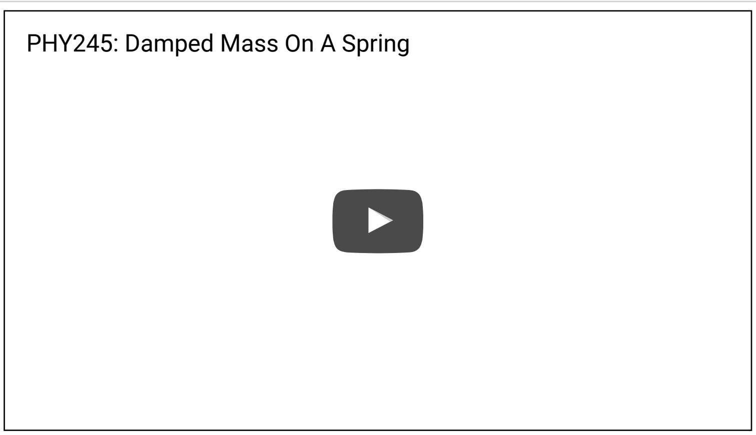
$$x(t) = e^{-\zeta\omega_nt}\left(b_1 \cos{\omega_dt} + b_2 \sin{\omega_dt}\right)$$

To use this equation, we need to solve for a_1 and a_2 or b_1 and b_2 using the initial conditions. Here, let's use the sin/cosine form. Solving the equation for generic intial velocity, $\dot{x} = v_0$, and a generic initial displacement, $x = x_0$, we find:

$$x(t) = e^{-\zeta\omega_nt}\left(x_0 \cos{\omega_dt} + \frac{\zeta\omega_nx_0 + v_0}{\omega_d}\sin{\omega_dt}\right)$$

Experiment

In [5]:
```
%%html
<iframe width="560" height="315"
src="https://www.youtube.com/embed/ZqedDWEAUN4?start=80&end=114"
frameborder="0" allowfullscreen></iframe>
```



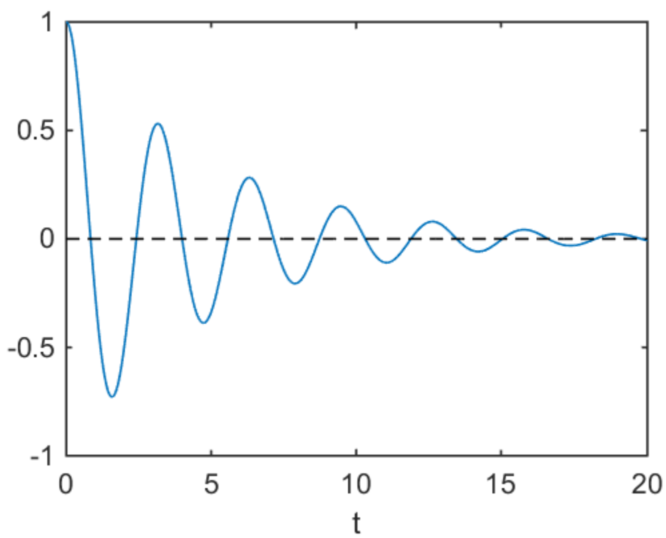
1.3. State Space Representation

$$\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix}$$

$$y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix}$$

In [6]:

```
% method 1: use 'lsim'
wn = 2;
zeta = 0.1;

A = [0 1;-wn^2 -2*zeta*wn];
B = [0 0]';
C = [1 0];
D = 0;
G = ss(A,B,C,D);

x0 = [1 0]';

t = linspace(0,20,500);
u = zeros(size(t));
[y, tout] = lsim(G,u,t,x0);
plot(tout,y,tout,zeros(size(tout)),'k--'), xlabel('t')
```



Out[6]:

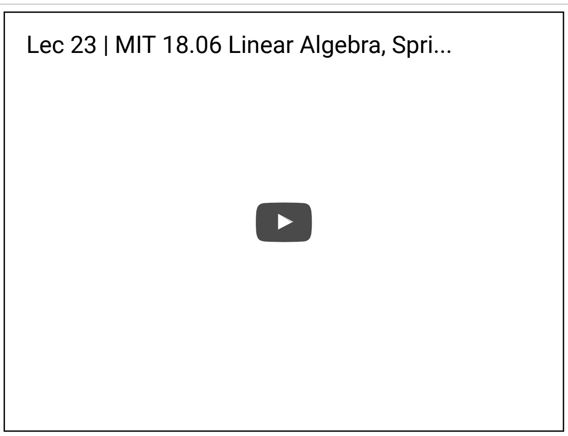
1.4. Matrix Exponentials

$$y(t) = e^{At}x\left(0\right)$$

- Lec 23 | MIT 18.06 Linear Algebra, Spring 2005
- Matrix Exponentials | MIT 18.03SC Differential Equations, Fall 2011
- Differential Equations and exp (At) | MIT 18.06SC Linear Algebra, Fall 2011

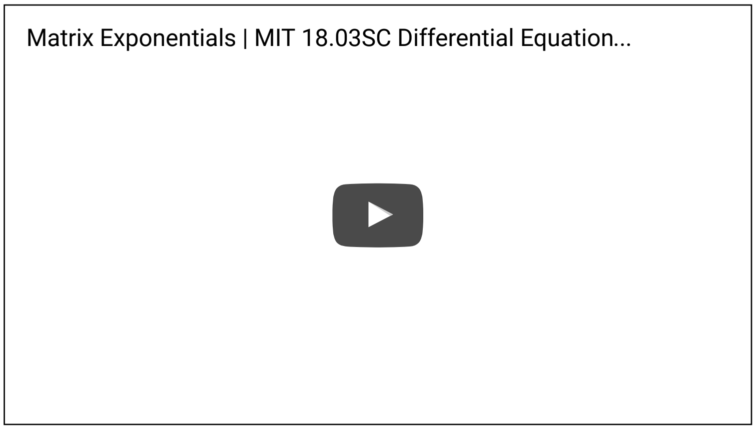
In [7]:

```
%%html
<iframe src="https://www.youtube.com/embed/IZqwi0wJovM"
width="420" height="315" frameborder="0" allowfullscreen></iframe>
```



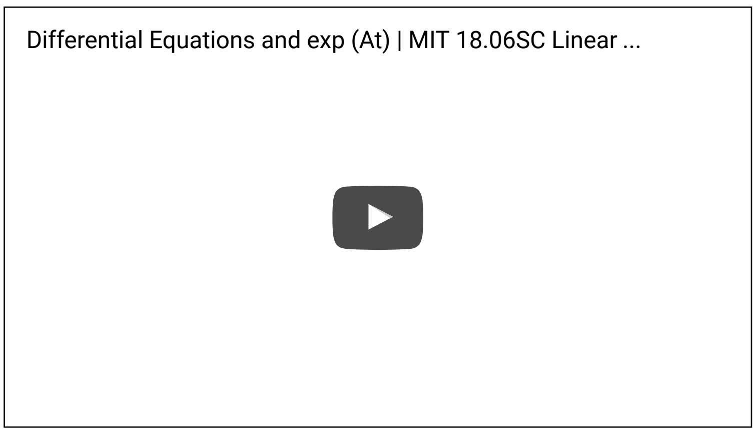
In [8]:

```
%%html
<iframe src="https://www.youtube.com/embed/qbyeQum8qTE"
width="560" height="315" frameborder="0" allowfullscreen></iframe>
```



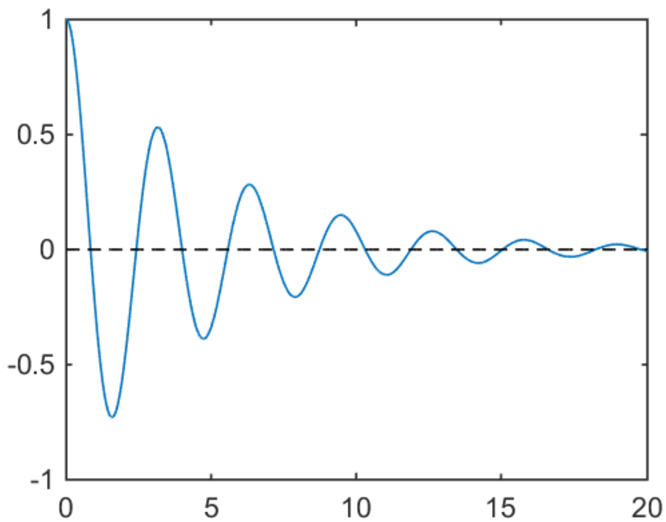
In [9]:

```
%%html
<iframe src="https://www.youtube.com/embed/dZfdKXxhnTM"
width="560" height="315" frameborder="0" allowfullscreen></iframe>
```



In [10]:

```
% method 2: use expm (matrix exponential)
x0 = [1 0]';
x = [];
t = 0:0.01:20;

for i = 1:length(t)
    x = [x expm(t(i)*A)*x0];
end

plot(t,x(1,:),t,zeros(size(t)),'k--')
```



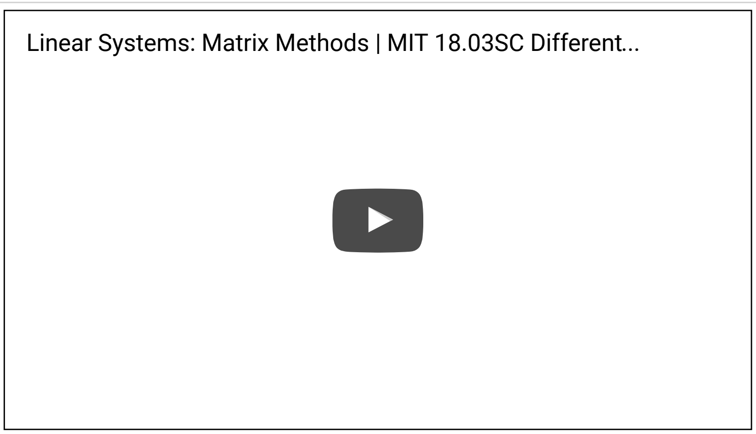
Out[10]:

1.5. Systems of Differential Equations (Matrix Differential Equation)

- Matrix Methods | MIT 18.03SC Differential Equations, Fall 2011

In [11]:

```
%%html
<iframe src="https://www.youtube.com/embed/YUjdyKhWt6E"
width="560" height="315" frameborder="0" allowfullscreen></iframe>
```



Given

$$\dot{\vec{u}} = A\vec{u}, \qquad \vec{u}(0) = \vec{u}_0$$

- Eigenanalysis

$$\begin{aligned} A\vec{x}_1 &= \lambda_1 \vec{x}_1 \\ A\vec{x}_2 &= \lambda_2 \vec{x}_2 \end{aligned}$$

- General solution:

$$\vec{u}(t) = c_1\, e^{\lambda_1 t}\, \vec{x}_1 + c_2\, e^{\lambda_2 t}\, \vec{x}_2$$

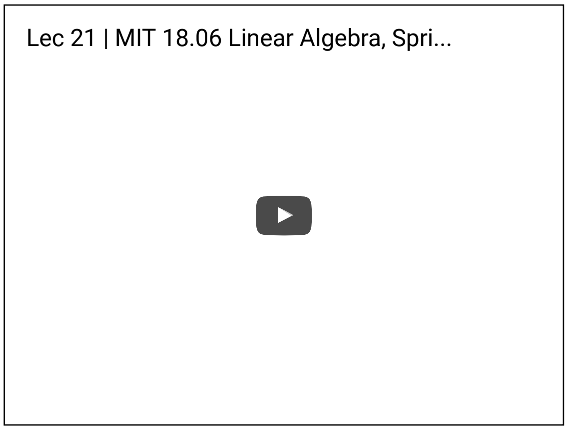
where

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = [\,\vec{x}_1\ \vec{x}_2\,]^{-1} \begin{bmatrix} u_1(0) \\ u_2(0) \end{bmatrix}.$$

- Systems of differential equations ↔ Eigenanalysis (Gilbert Strang Lecture 21, video below)
- System stability ↔ eigenvalues

In [12]:

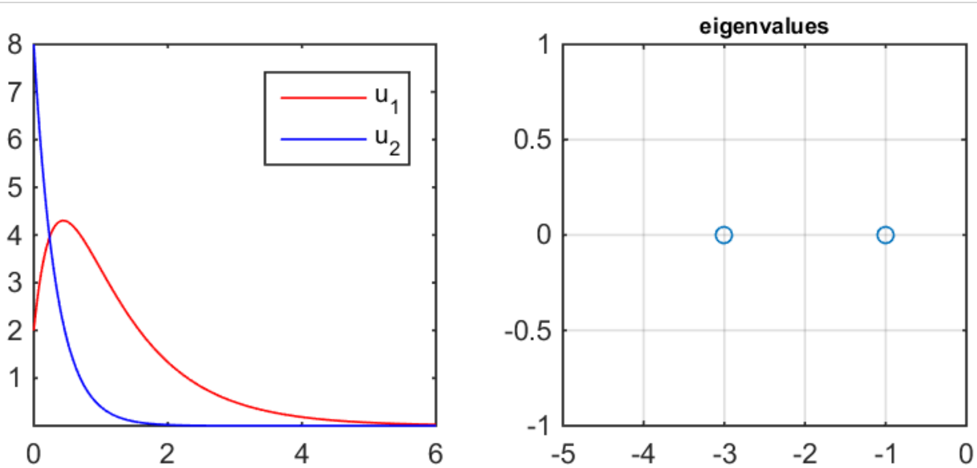
```
%%html
<iframe src="https://www.youtube.com/embed/lXNXrLcoerU"
width="420" height="315" frameborder="0" allowfullscreen></iframe>
```



1.5.1. Real eigenvalues

In [13]:

```
%plot -s 900,350
A = [-1  2
     0 -3];

%% eigen-analysis
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
[V,D] = eig(A);
lambda = diag(D);

t = 0:0.01:6;
U0 = [2 8]';

C = inv(V)*U0;
u = C(1)*V(:,1)*exp(lambda(1)*t) + C(2)*V(:,2)*exp(lambda(2)*t);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% plot u1 and u2 as a function of time
subplot(1,2,1),  plot(t,u(1,:),'r',t,u(2,:),'b'), axis tight, legend('u_1','u_2')
subplot(1,2,2),  plot(real(lambda),imag(lambda),'o'), xlim([-5,0]), grid on, title('eigenvalues','fontsize',8)
```



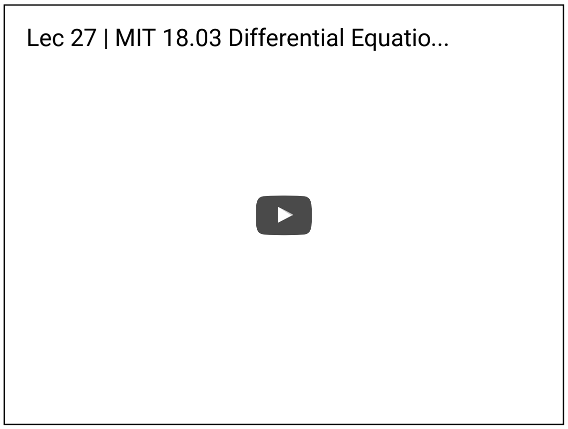
Out[13]:

Phase portrait

- https://en.wikipedia.org/wiki/Phase_portrait (https://en.wikipedia.org/wiki/Phase_portrait)
- http://tutorial.math.lamar.edu/Classes/DE/PhasePlane.aspx (http://tutorial.math.lamar.edu/Classes/DE/PhasePlane.aspx)
- Lec 27 | MIT 18.03 Differential Equations, Spring 2006

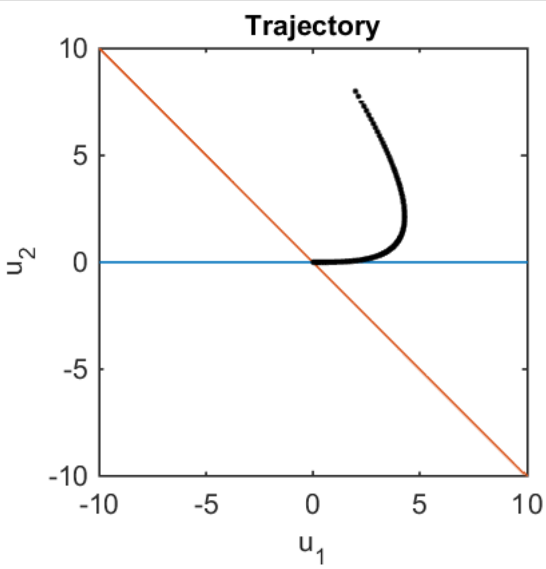
In [14]:

```
%%html
<iframe src="https://www.youtube.com/embed/e3FfmXtkppM"
width="420" height="315" frameborder="0" allowfullscreen></iframe>
```



In [15]:

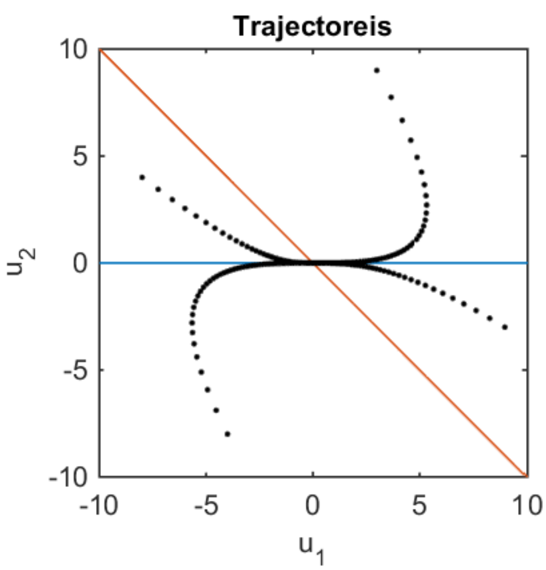
```
%plot -s 560,420
% plot eigenvectors (X1 and X2)
k = -20:0.1:20;
y1 = V(:,1)*k;
y2 = V(:,2)*k;

plot(y1(1,:),y1(2,:));   hold on
plot(y2(1,:),y2(2,:));
xlabel('u_1','fontsize',10)
ylabel('u_2','fontsize',10)
title('Trajectory','fontsize',10)
axis equal
axis([-10 10 -10 10])

% plot a trajectory of u1 and u2
for i = 1:length(t)
    plot(u(1,i),u(2,i),'k.');
end
hold off
```



Out[15]:

In [16]:

```
% with multiple initial conditions

A = [-1  2;0 -3];
%A = [-2  0;0 -4];
%A = [-2  0;1 -4];

%%
[V,D] = eig(A);
lambda = diag(D);

t = 0:0.05:5;
U0 = [3  9 -8  -4;
      9 -3  4 -8 ];

% plot eigenvectors (X1 and X2)
k = -20:0.1:20;
y1 = V(:,1)*k;
y2 = V(:,2)*k;

plot(y1(1,:),y1(2,:));   hold on
plot(y2(1,:),y2(2,:));

for m = 1:length(U0)
    C = inv(V)*U0(:,m);
    u = C(1)*V(:,1)*exp(lambda(1)*t) + C(2)*V(:,2)*exp(lambda(2)*t);

    % plot a trajectory of u1 and u2
    for i = 1:length(t)
        plot(u(1,i),u(2,i),'k.');
    end
end

hold off
xlabel('u_1','fontsize',10)
ylabel('u_2','fontsize',10)
title('Trajectoreis','fontsize',10)
axis equal
axis([-10 10 -10 10])
```

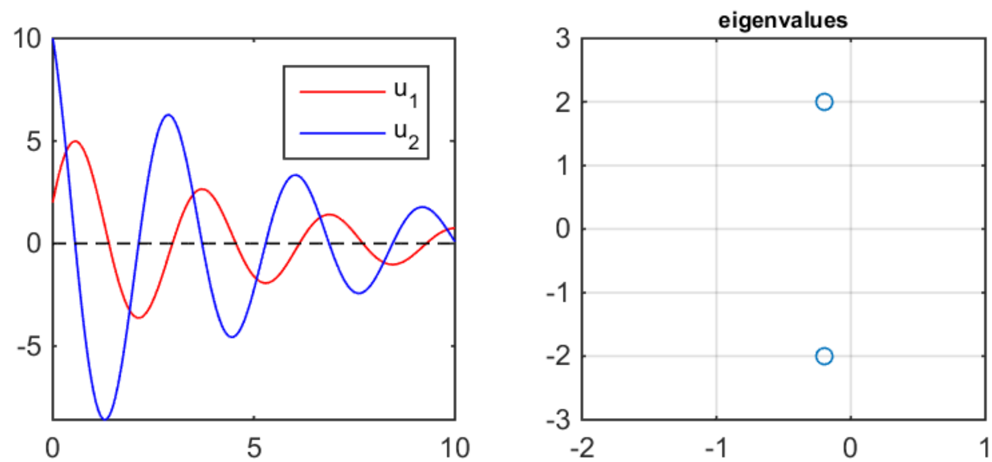


Out[16]:

1.5.2. Complex eigenvalues (starting oscilation)

In [17]:

```
%plot -s 900,350

%A = [0 1;-1 0];
wn = 2; zeta = 0.1; A = [0 1;-wn^2 -2*zeta*wn];

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
[V,D] = eig(A);
lambda = diag(D)

t = 0:0.01:10;
U0 = [2 10]';
C = inv(V)*U0;

u = C(1)*V(:,1)*exp(lambda(1)*t) + C(2)*V(:,2)*exp(lambda(2)*t);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% plot u1 and u2 as a function of time
subplot(1,2,1),  plot(t,u(1,:),'r',t,u(2,:),'b',t,zeros(size(t)),'k--'), axis tight, legend('u_1','u_2')
subplot(1,2,2),  plot(real(lambda),imag(lambda),'o'), ylim([-3,3]), grid on, title('eigenvalues','fontsize',8)
```

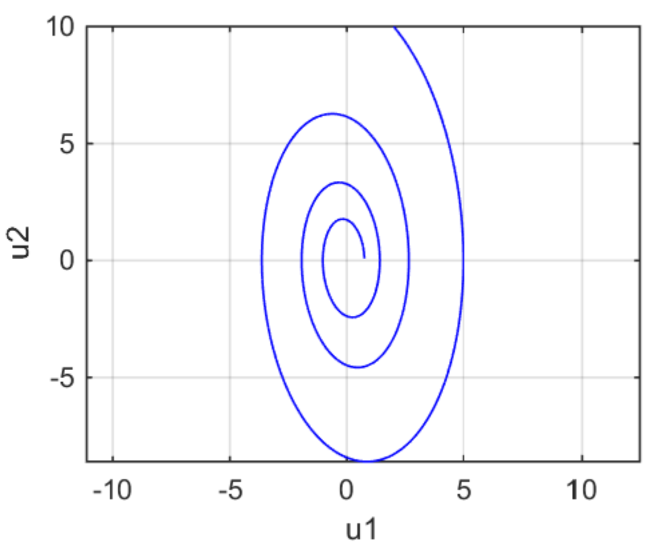


Out[17]:

```
lambda =

  -0.2000 + 1.9900i
  -0.2000 - 1.9900i
```

In [18]:

```
%plot -s 560,420

% Phase portrait
plot(u(1,:),u(2,:),'b')
axis equal, grid on
xlabel('u1')
ylabel('u2')
```



Out[18]:

2. Response to General Inputs

2.1. Step response

Start with a ***step response*** example

$$\dot{x} + 5x = 1 \quad \text{for} \quad t \geq 0, \qquad x(0) = 0$$

or

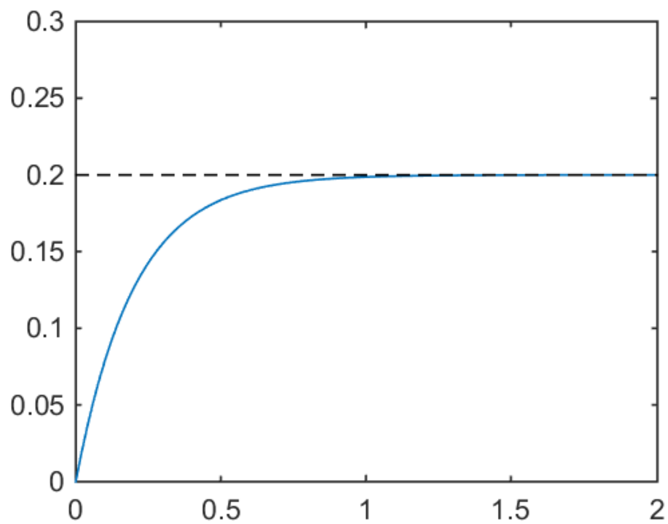
$$\dot{x} + 5x = u(t), \qquad x(0) = 0$$

The solution is given:

$$x(t) = \frac{1}{5}\left(1 - e^{-5t}\right)$$

In [19]:

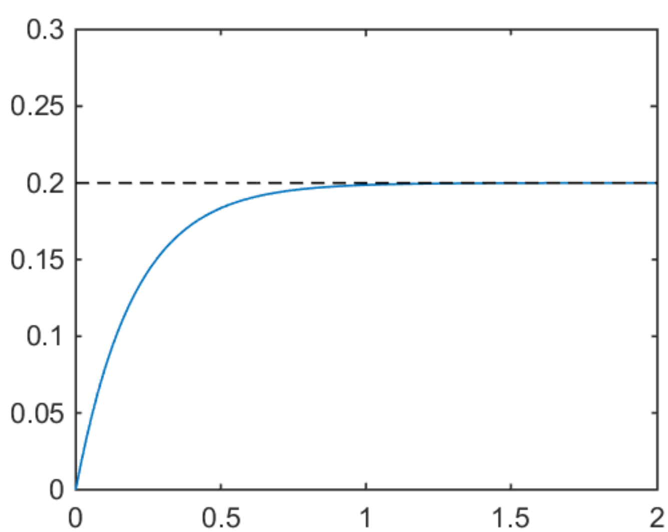
```
t = linspace(0,2,100);
x = 1/5*(1-exp(-5*t));
plot(t,x,t,0.2*ones(size(t)),'k--'), ylim([0,0.3])
```



Out[19]:

In [20]:

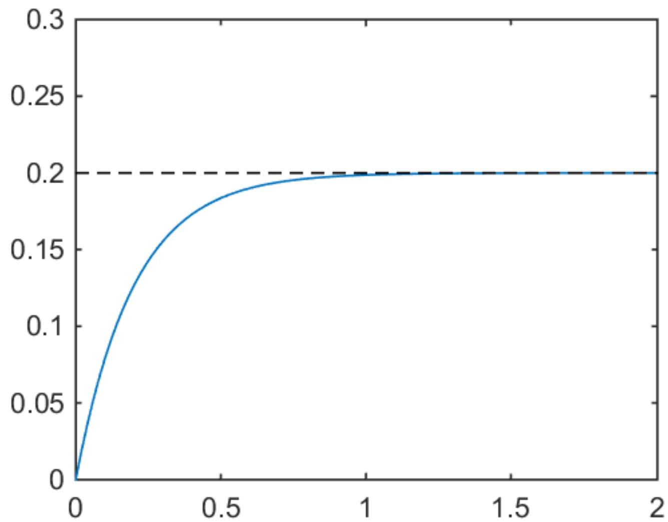
```
% method 1
% define a system first
num = 1;
den = [1 5];
G = tf(num,den);

[y,tout] = step(G,2);
plot(tout,y,tout,0.2*ones(size(tout)),'k--'), ylim([0,0.3])
```



Out[20]:

In [21]:

```
% method 2
% define a system first
A = -5;
B = 1;
C = 1;
D = 0;
G = ss(A,B,C,D);

t = linspace(0,2,100);
u = ones(size(t));
x0 = 0;

[y,tout] = lsim(G,u,t,x0);
plot(tout,y,tout,0.2*ones(size(tout)),'k--'), ylim([0,0.3])
```



Out[21]:

2.2. Impulse response

Now think about the ***impulse response***

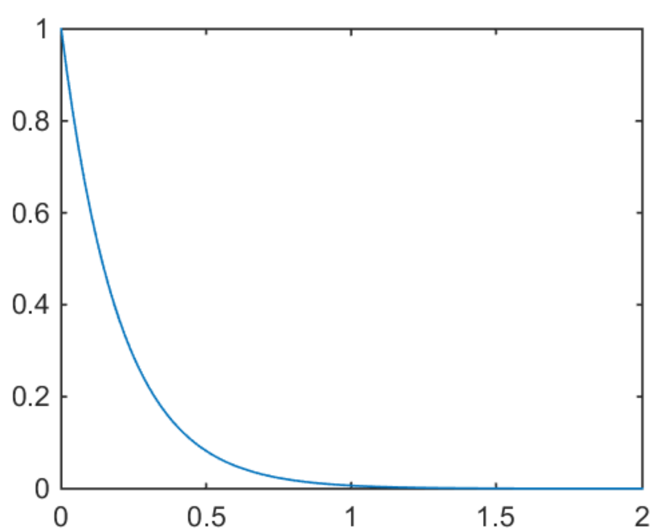
$$\dot{x} + 5x = \delta(t), \qquad x(0) = 0$$

The solution is given: (why?)

$$h(t) = e^{-5t}, \quad t \geq 0$$

In [22]:

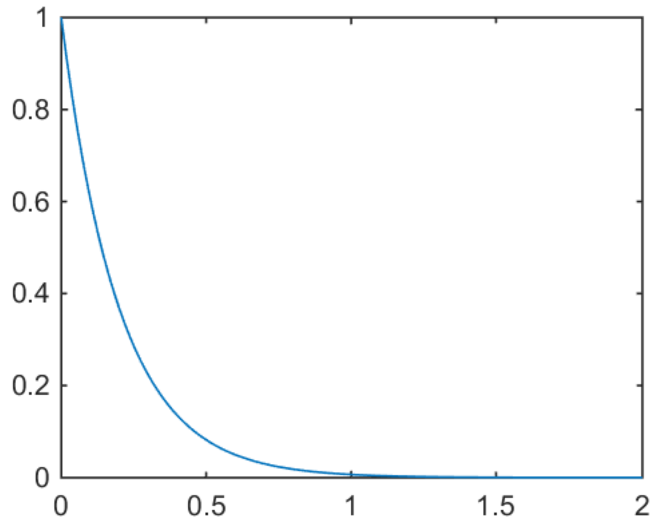
```
t = linspace(0,2,100);
h = exp(-5*t);
plot(t,h), ylim([0,1])
```



Out[22]:

In [23]:

```
% method 1
% define a system first
num = 1;
den = [1 5];
G = tf(num,den);

[h,tout] = impulse(G,2);
plot(tout,h), ylim([0,1])
```



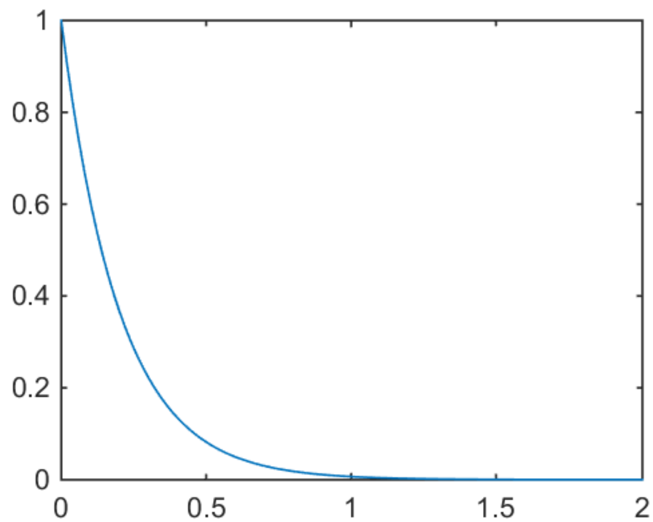
Out[23]:

Impulse input can be equivalently changed to zero input with non-zero initial condition (by the impulse and momentum theory)

$$\int_{0^-}^{0^+} \delta(t)dt = u(0^+) - u(0^-) = 1$$

In [24]:

```
% method 2
% define a system first
A = -5;
B = 1;
C = 1;
D = 0;
G = ss(A,B,C,D);

t = linspace(0,2,100);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
u = zeros(size(t));
x0 = 1;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

[h,tout] = lsim(G,u,t,x0);
plot(tout,h), ylim([0,1])
```



Out[24]:

2.3. Response to a general input

Response to a ***general input***

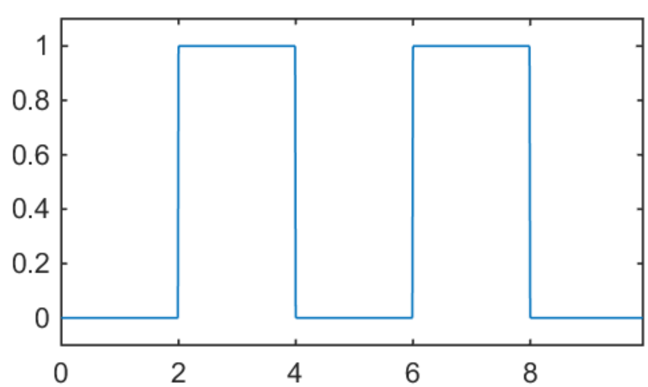
$$\dot{x} + 5x = f(t), \qquad x(0) = 0$$

The solution is given:

$$x(t) = h(t) * f(t), \quad t \geq 0$$

In [25]:

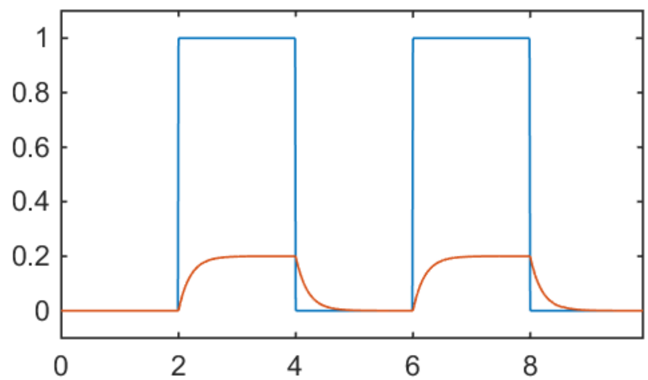
```
%plot -s 560,300
% generate a general input

[f,t] = gensig('square',4,10,0.01);
plot(t,f), axis([0,9.9,-0.1,1.1])
```



Out[25]:

In [26]:

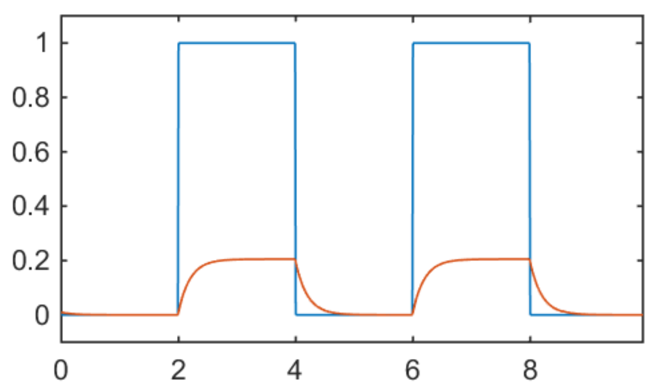
```
%plot -s 560,300
% use lsim

A = -5;
B = 1;
C = 1;
D = 0;
G = ss(A,B,C,D);

x0 = 0;
[f,t] = gensig('square',4,10,0.01);
[y,tout] = lsim(G,f,t,x0);

plot(t,f), hold on
plot(tout,y), hold off, axis([0,9.9,-0.1,1.1])
```



Out[26]:

In [27]:

```
%plot -s 560,300
% convolution

[f,t] = gensig('square',4,10,0.01);
h = exp(-5*t);
y = cconv(f,h,length(t))/100;

plot(t,f), hold on
plot(t,y), hold off, axis([0,9.9,-0.1,1.1])
```



Out[27]:

2.4. Reponse to a sinusoidal input

- only focus on *steady-state* solution
- transient solution is not our interest any more

Assume:

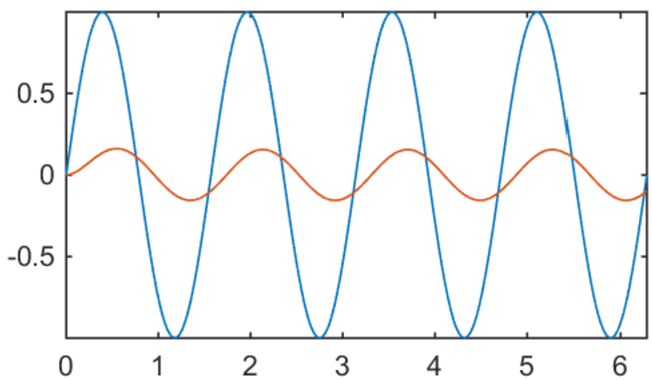
$$y(t) = \sin(\omega t)$$

Then, the solution $x(t)$ should have the form:

$$x(t) = a\sin(\omega t) + b\cos(\omega t) = A\sin(\omega t + \phi)$$

In [28]:

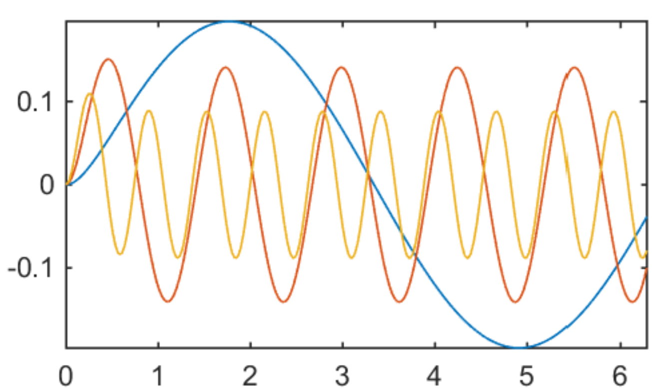
```
%plot -s 560,300
% use lsim

A = -5;
B = 1;
C = 1;
D = 0;
G = ss(A,B,C,D);

x0 = 0;
w = 4;

t = linspace(0,2*pi,200);
f = sin(w*t);
[y,tout] = lsim(G,f,t,x0);

plot(t,f), hold on
plot(tout,y), hold off, axis tight
```



Out[28]:

In [29]:

```
%plot -s 560,300
% use lsim

A = -5;
B = 1;
C = 1;
D = 0;
G = ss(A,B,C,D);

x0 = 0;

t = linspace(0,2*pi,200);

W = [1,5,10];
for w = W
    f = sin(w*t);
    [y,tout] = lsim(G,f,t,x0);

    plot(tout,y), hold on
end
hold off, axis tight
```



Out[29]:

3. Frequency response (frequency sweep)

Given input $e^{j\omega t}$

$$\dot{z} + 5z = e^{j\omega t}$$

If $z = Ae^{j(\omega t+\phi)}$

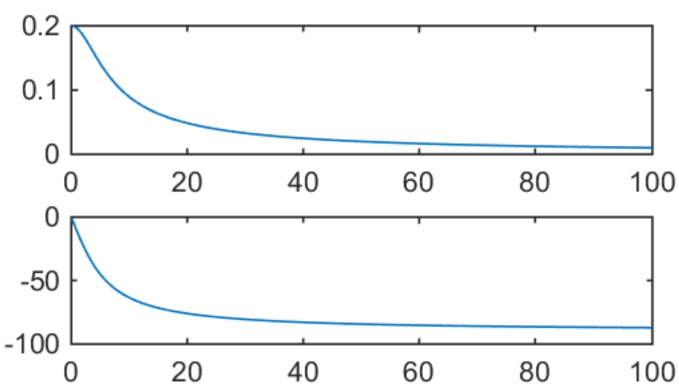
$$\begin{aligned} j\omega A e^{j(\omega t+\phi)} + 5Ae^{j(\omega t+\phi)} &= e^{j\omega t} \\ (j\omega + 5)\, Ae^{j\phi} &= 1 \end{aligned}$$

Therefore,

$$\begin{aligned} A &= \frac{1}{|j\omega + 5|} \\ \phi &= -\angle(j\omega + 5) \end{aligned}$$

In [30]:

```
w = 0.1:0.1:100;
A = 1./abs(1j*w+5);
Ph = -angle(1j*w+5)*180/pi;

subplot(2,1,1), plot(w,A)
subplot(2,1,2), plot(w,Ph)

% later, we will see that this is kind of a bode plot
```



Out[30]:

$$\begin{aligned} \ddot{z} + 2\zeta\omega_n\dot{z} + \omega_n^2 z &= \omega_n^2 \ f(t) \\ \ddot{z} + 2\zeta\omega_n\dot{z} + \omega_n^2 z &= \omega_n^2 \ A_0 e^{j\Omega t} \end{aligned}$$

- We know that z is in the form of

$$z = Ae^{j(\Omega t+\phi)}$$

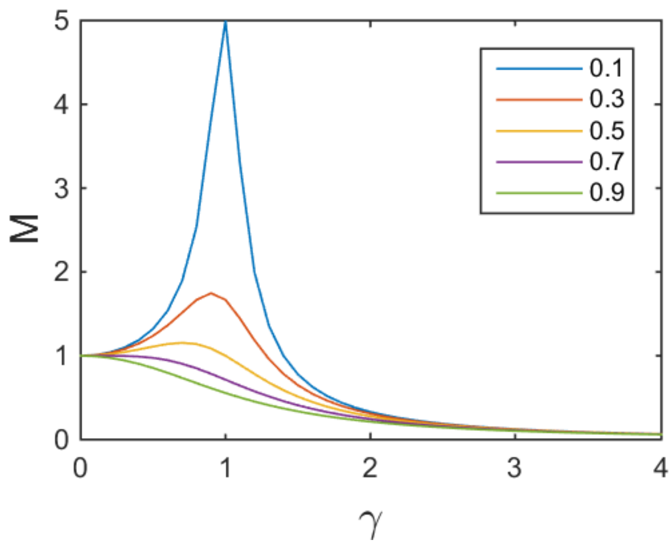
- Then

$$\begin{aligned} \left(-\Omega^2 + j2\zeta\omega_n\Omega + \omega_n^2\right) Ae^{j\phi}e^{j\Omega t} &= \omega_n^2 A_0 e^{j\Omega t} \\ Ae^{j\phi} &= \frac{\omega_n^2 A_0}{-\Omega^2 + j2\zeta\omega_n\Omega + \omega_n^2} = A_0 \frac{1}{1 - \left(\frac{\Omega}{\omega_n}\right)^2 + j2\zeta\left(\frac{\Omega}{\omega_n}\right)} \end{aligned}$$

$$\begin{aligned} \frac{A}{A_0} &= \frac{1}{\sqrt{\left(1 - \left(\frac{\Omega}{\omega_n}\right)^2\right)^2 + 4\zeta^2\left(\frac{\Omega}{\omega_n}\right)^2}} = \frac{1}{\sqrt{(1-\gamma^2)^2 + 4\zeta^2\gamma^2}}, \qquad \left(\gamma = \frac{\Omega}{\omega_n}\right) \\ \phi &= -\tan^{-1}\left(\frac{2\zeta\frac{\Omega}{\omega_n}}{1 - \left(\frac{\Omega}{\omega_n}\right)^2}\right) = -\tan^{-1}\left(\frac{2\zeta\gamma}{1-\gamma^2}\right) \end{aligned}$$

In [31]:

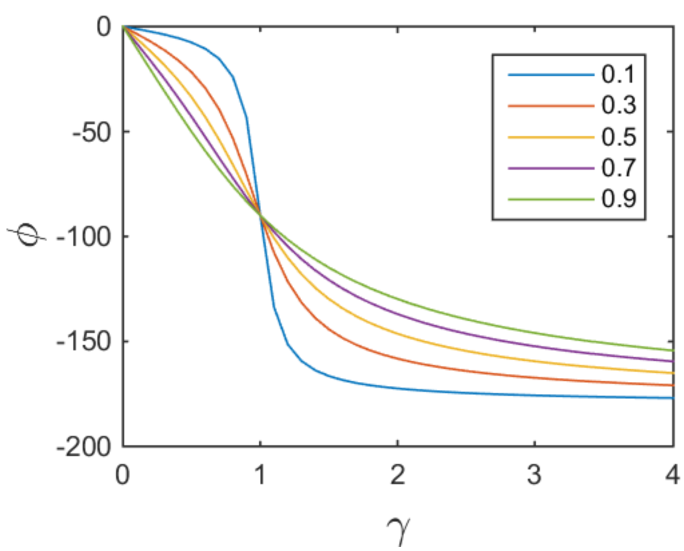
```
%plot -s 560,420
r = 0:0.1:4;

zeta = 0.1:0.2:1;
M = [];
for i = 1:length(zeta)
    M(i,:) = 1./sqrt((1-r.^2).^2 + (2*zeta(i)*r).^2); % ignore a constant of F0/k
end

plot(r,M)
xlabel('\gamma','fontsize',16)
ylabel('M','fontsize',14)
legend('0.1','0.3','0.5','0.7','0.9')
```



Out[31]:

In [32]:

```
%plot -s 560,420
phi = [];
for i = 1:length(zeta)
    phi(i,:) = -atan2((2*zeta(i).*r),(1-r.^2));
end

plot(r,phi*180/pi)
xlabel('\gamma','fontsize',16)
ylabel('\phi','fontsize',16)
legend('0.1','0.3','0.5','0.7','0.9')
```



Out[32]:

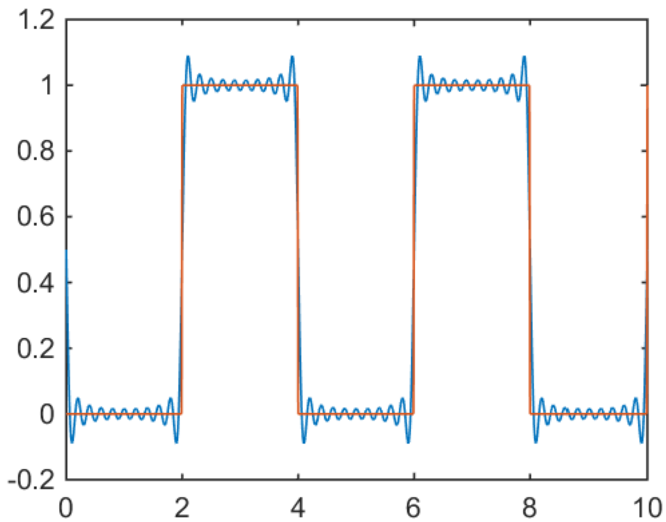
3.1. Fourier Series of square wave

- decompose a genernal signal (ex. square wave) to a linear combinarion of sinusoidal signals

$$x(t) = \frac{1}{2} - \frac{2}{\pi}\sum_{n=1,2,3,}\frac{1}{n}\sin\left(\frac{n\pi t}{L}\right)$$

In [33]:

```
[f,t] = gensig('square',4,10,0.01);
L = 2;
x = 1/2*ones(size(t));

for n = 1:2:20
    x = x - 2/pi*1/n*sin(n*pi/L*t);
end

plot(t,x,t,f)
```



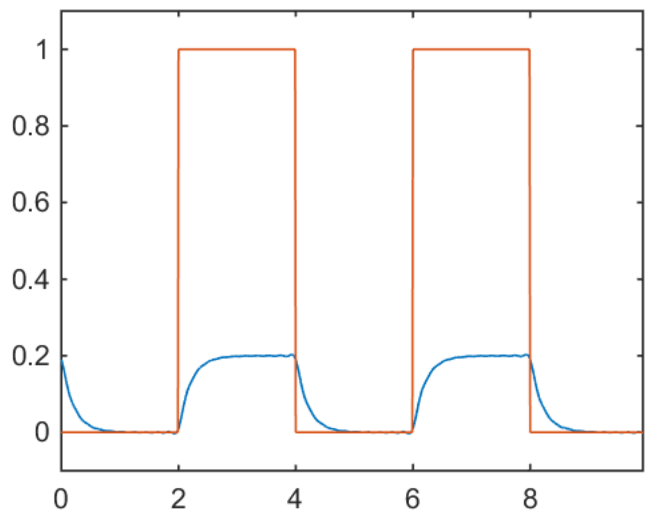
Out[33]:

The output response of LTI

Linearity: input $\sum_k c_k x_k(t)$ produces $\sum_k c_k y_k(t)$

In [34]:

```
[f,t] = gensig('square',4,10,0.01);
L = 2;
x = 1/2*ones(size(t));
x2 = 0.2*x;

for n = 1:2:20
    w = n*pi/L;
    A = 1./abs(1j*w+5);
    Ph = -phase(1j*w+5);
    x2 = x2 - 2/pi*1/n*A*sin(w*t + Ph);
end

plot(t,x2,t,f), axis([0,9.9,-0.1,1.1])
```



Out[34]:

In [35]:

```
%%javascript
$.getScript('https://kmahelona.github.io/ipython_notebook_goodies/ipython_notebook_toc.js')
```