Discrete Fourier Transformation (DFT)

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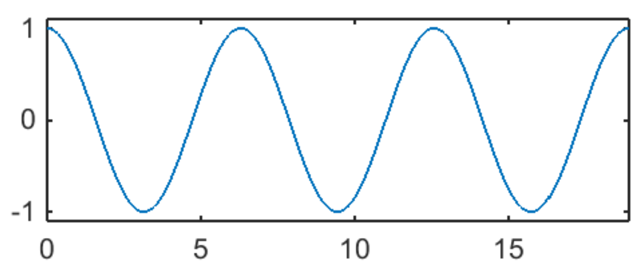
1. Fourier Series

1.1. Delta Dirac Function

$$\sum_{n=1,2,3,\cdots} \cos(n\theta) = \cos\theta + \cos 2\theta + \cos 3\theta + \cdots$$

In [4]:

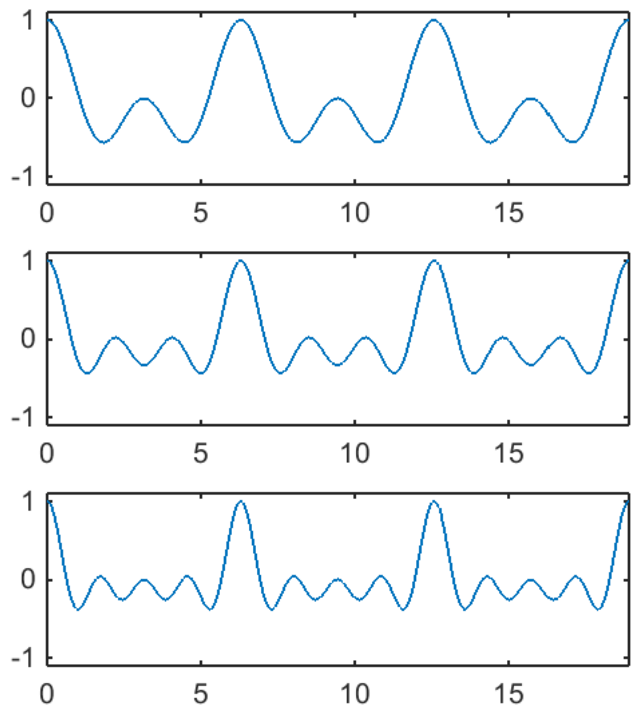
```
%plot -s 560,200

theta = linspace(0,6*pi,2^8);
x1 = cos(theta);
plot(theta,x1), axis tight, ylim([-1.1 1.1])
```



Out[4]:

In [11]:

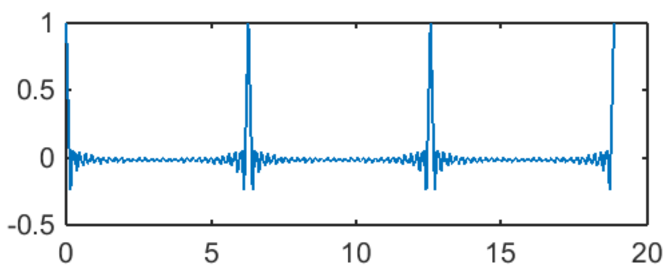
```
%plot -s 560,600

x2 = x1 + cos(2*theta);
x3 = x2 + cos(3*theta);
x4 = x3 + cos(4*theta);
subplot(3,1,1), plot(theta, x2/2), axis tight, ylim([-1.1 1.1])
subplot(3,1,2), plot(theta, x3/3), axis tight, ylim([-1.1 1.1])
subplot(3,1,3), plot(theta, x4/4), axis tight, ylim([-1.1 1.1])
```



Out[11]:

In [5]:

```
%plot -s 560,200

x = zeros(1,2^8);

N = 30;
for n = 1:N
    x = x + cos(n*theta);
end
plot(theta,x/N)
```



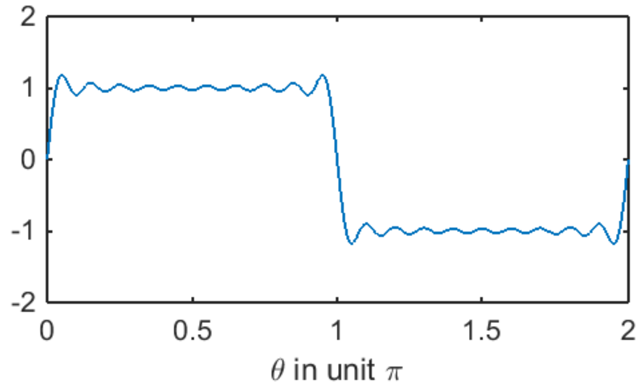
Out[5]:

1.2. Square Wave

$$\sum_{n=1,3,5,\cdots} \frac{4}{\pi n}\sin(n\theta) = \frac{4}{\pi}\sin\theta + \frac{4}{3\pi}\sin 3\theta + \frac{4}{5\pi}\sin 5\theta + \cdots$$

In [6]:

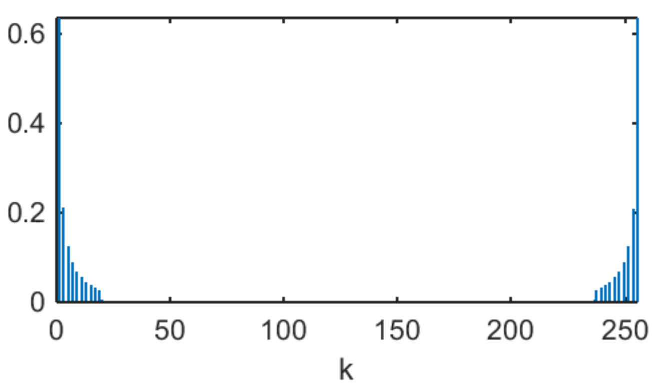
```
%plot -s 560,300

N = 2^8;
x = zeros(1,N);
theta = linspace(0,2*pi,N);

for n = 1:2:20
    x = x + 4/(pi*n)*sin(n*theta);
end

plot(theta/pi,x), xlabel('\theta in unit \pi')
```



Out[6]:

In [11]:

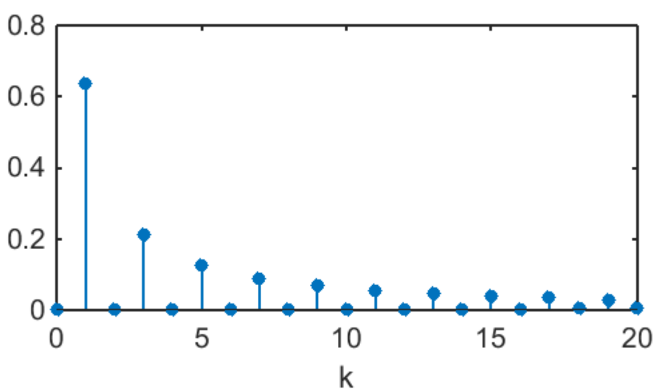
```
%plot -s 560,300

X = dft(x,N)/N;
stem(0:N-1,abs(X),'marker','none'), axis tight, xlabel('k')
```



Out[11]:

In [13]:

```
%plot -s 560,300

stem(0:N-1,abs(X),'filled','markersize',4), xlabel('k')
xlim([0,20])
```



Out[13]:

In [15]:

```
% sound of square wave

N = 2^10;
x = zeros(1,N);
theta = linspace(0,2*pi,N);

for n = 1:2:30
    x = x + 4/(pi*n)*sin(n*theta);
end

x_rep = repmat(x,1,100);
Fs = 44100;

sound(x_rep,Fs);
```

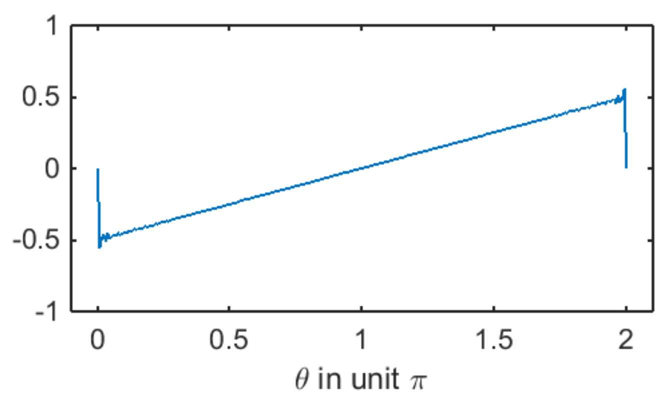
Out[15]:

1.3. Sawtooth wave

$$\frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin(\pi\theta)$$

In [19]:

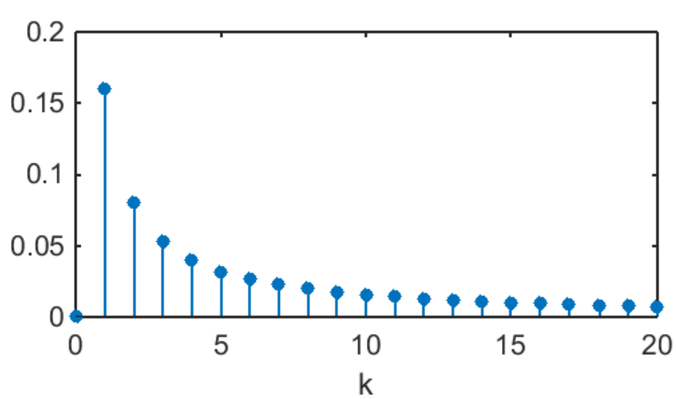
```
%plot -s 560,300

N = 2^8;
x = zeros(1,N);
theta = linspace(0,2*pi,N);


for n = 1:100
    x = x - 1/(pi*n)*sin(n*theta);
end

plot(theta/pi,x),
xlim([-0.1 2.1]), xlabel('\theta in unit \pi')
```



Out[19]:

In [21]:

```
%plot -s 560,300
X = fft(x)/N;

stem(0:N-1,abs(X),'filled','markersize',4), xlabel('k')
xlim([0,20])
```



Out[21]:

In [23]:

```
% sound of sawtooth wave

N = 2^10;
x = zeros(1,N);
theta = linspace(0,2*pi,N);

for n = 1:30
    x = x - 1/(pi*n)*sin(n*theta);
end

x_rep = repmat(2*x,1,50); % make a signal -1 to 1 and repeat (periodic)
Fs = 44100;

sound(x_rep,Fs);
```

Out[23]:

2. Eigen-analysis

2.1. Basis

A **basis** $\{b_k\}$ for a vector space V is a collection of vectors from V that **linearly independent** and **span** V

Basis matrix: Stack the basis vectors b_k as columns

$$B \;=\; [b_0 \,|\, b_1 \,|\, b_2 \,|\, \cdots \,|\, b_{N-1}]$$

Using this matrix, We can now write a linear combination of basis elements as the matrix/vector product

$$x \;=\; B\,a \;=\; a_0 b_0 + a_1 b_1 + \,\cdots\, + a_{N-1} b_{N-1} \;=\; \sum_{k=0}^{N-1} a_k b_k \;=\; [b_0 \,|\, b_1 \,|\, \cdots \,|\, b_{N-1}] \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{N-1} \end{bmatrix}$$

An **orthogonal basis** $\{b_k\}_{k=0}^{N-1}$ for a vector space V is a basis whose elements are mutually orthogonal

$$\langle b_k, b_l \rangle \;=\; 0\,, \quad k \,\neq\, l$$

An **orthonormal basis**$\{b_k\}_{k=0}^{N-1}$ for a vector space V is a basis whose elements are mutually orthogonal and normalized (in the 2-norm)

$$\begin{aligned} \langle b_k, b_l \rangle \;&=\; 0\,, \quad k \,\neq\, l \\ \|b_k\|_2 \;&=\; 1 \end{aligned}$$

2.2. Orthogonal Bases

Signal Representation by Orthonormal Basis

$$x = B\alpha = \sum_{k=0}^{N-1} \alpha_k b_k$$

$$ \alpha = B^Hx \qquad \text{or} \qquad \alpha_k = \big $$

- **Synthesis**: Build up the signal x as a linear combination of the basis elements b_k weighted by the weights α_k
- **Analysis**: Compute the weights α_k such that the synthesis produces x; the weight α_k measures the similarity between x and the basis element b_k

2.3. Eigenvector and eigenvalues

$$Av = \lambda v$$

Using this, we can find this equation

$$AV \;=\; V\Lambda$$

$$V \;=\; [v_0|v_1|\cdots|v_{N-1}],\; \Lambda = \begin{bmatrix} \lambda_0 & & & \\ & \lambda_1 & & \\ & & \ddots & \\ & & & \lambda_{N-1} \end{bmatrix}$$

We can change to

$$\begin{aligned} A = V\Lambda V^{-1} \Rightarrow \; Eigendecomposition \\ V^{-1}AV \;=\; \Lambda \end{aligned}$$

- Eigenvectors v are input signals that emerge at the system output unchanged (except for a scaling by the eigenvalue \lambda) and so are somehow "fundamental" to the system
- Eigenanalysis of LTI Systems (Finite-Length Signals)

- Goal: Calcuate the **eigenvectors** and **eigenvalues** of **H**

- Fact: The eigenvectors of a circulent matrix (LTI system) are the complex **harmonic sinusoids**
- Fact: The eigenvalue $\lambda \in C$ corresponding to the sinusoid eigenvector s_k is called the frequency response at frequency k since it measures how the system "responds" to s_k

$$\lambda_k = \sum_{n=0}^{N-1} h[n] e^{-j\frac{2\pi}{N}kn} = \langle h, s_k \rangle = H_u[k]$$

2.4. Eigenvector Matrix of Harmonic Sinusoids

$$S = [s_0 | s_1 | \cdots | s_{N-1}]$$
$$s_k[n] = \frac{1}{\sqrt{N}} e^{j\frac{2\pi}{N}kn}$$

Stack N normalized harmonic sinusoid $\{s_k\}_{k=0}^{N-1}$ as columns into an $N \times N$ complex orthonormal basis matrix

$$s_k \longrightarrow \boxed{\mathcal{H}} \longrightarrow \lambda_k s_k$$

$$\begin{aligned} s_k[n] * h[n] &= \sum_{m=0}^{N-1} s_k[(n-m)_N]\, h[m] = \sum_{m=0}^{N-1} \frac{e^{j\frac{2\pi}{N}k(n-m)_N}}{\sqrt{N}} h[m] \\ &= \sum_{m=0}^{N-1} \frac{e^{j\frac{2\pi}{N}k(n-m)}}{\sqrt{N}} h[m] = \sum_{m=0}^{N-1} \frac{e^{j\frac{2\pi}{N}kn}}{\sqrt{N}} e^{-j\frac{2\pi}{N}km}\ h[m] \\ &= \left(\sum_{m=0}^{N-1} e^{-j\frac{2\pi}{N}km}\, h[m] \right) \frac{e^{j\frac{2\pi}{N}kn}}{\sqrt{N}} = \lambda_k\, s_k[n] \end{aligned}$$

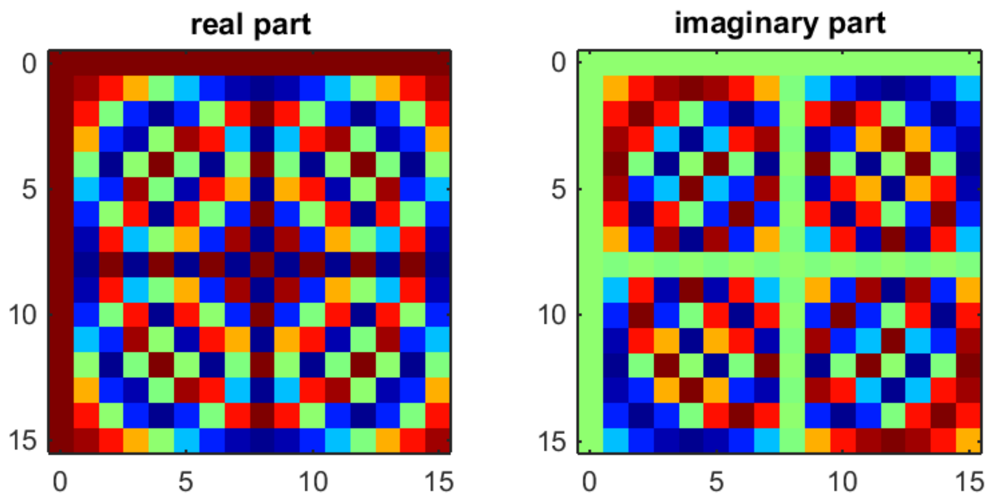
2.5. Eigenvalues of LTI Systems

$$\lambda_k = \sum_{m=0}^{N-1} h[n]\, e^{-j\frac{2\pi}{N}kn} = \langle h, s_k \rangle = H_u[k]$$

λ_k means the number of s_k in $h[n] \implies$ similarity

In [11]:

```
%plot -s 900,500

% visualize eigenvector matrix

N = 16;
n = 0:N-1;
k = 0:N-1;

S = 1/sqrt(N)*exp(1j*2*pi/N).^(n'*k);

subplot(1,2,1), imagesc(n,k,real(S)); colormap('jet'); axis('square');
title('real part')
subplot(1,2,2), imagesc(n,k,imag(S)); colormap('jet'); axis('square');
title('imaginary part')
```

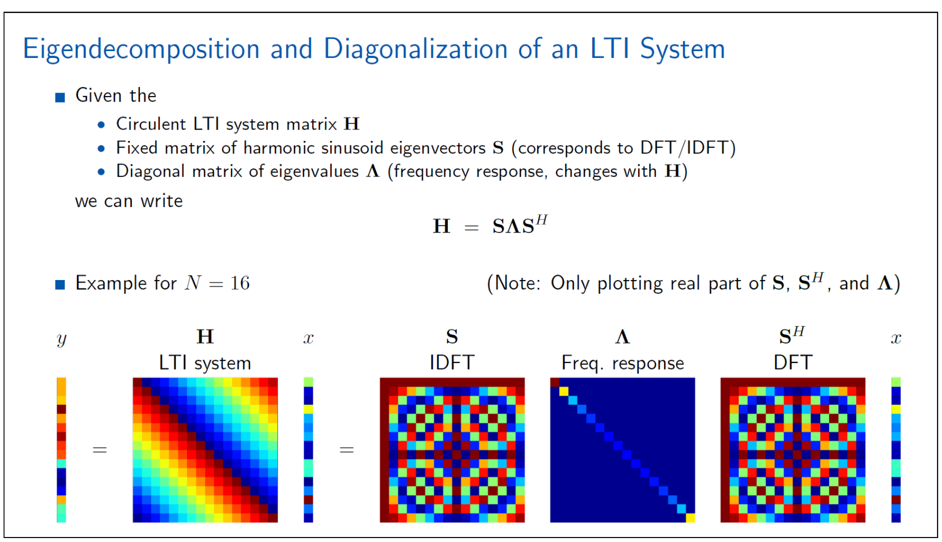


Out[11]:

2.6. Eigendecomposition and Diagonalization of an LTI Systems

- H is LTI System matrix
- S is harmonic sinusoid eigenvectors matrix
- Λ is eigenvalue diagonal matrix

$$H = S\Lambda S^H$$



3. Discrete Fourier Transform (DFT)

3.1 Harmonic Sinusoids are an Orthonormal Basis

$$s_k[n] = \frac{e^{j\frac{2\pi}{N}kn}}{\sqrt{N}}$$

$$\langle s_k, s_l \rangle = 0, \quad k \neq l \qquad \|s_k\|_2 = 1$$

3.2 DFT & Inverse DFT

- DFT

$$X = S^H x$$

$$X[k] = \langle x, s_k \rangle = \sum_{k=0}^{N-1} X[k] \frac{e^{-j\frac{2\pi}{N}kn}}{\sqrt{N}}$$

It is finding how many certain frequncy

- Inverse DFT

$$x = SX$$

$$x[n] = \sum_{k=0}^{N-1} X[k] \frac{e^{j\frac{2\pi}{N}kn}}{\sqrt{N}}$$

It is returing to time domain

Signal Representation by Harmonic Sinusoids

- Given the normalized complex harmonic sinusoids $\{s_k\}_{k=0}^{N-1}$ and the orthonormal basis matrix $\mathbf{S}$, we define the (normalized) **discrete Fourier transform** (DFT) for any signal $x \in \mathbb{C}^N$
- **Analysis (Forward Normalized DFT)**

$$X = \mathbf{S}^H x$$

$$X[k] = \langle x, s_k \rangle = \sum_{n=0}^{N-1} x[n] \frac{e^{-j\frac{2\pi}{N}kn}}{\sqrt{N}}$$

- **Synthesis (Inverse Normalized DFT)**

$$x = \mathbf{S}X$$

$$x[n] = \sum_{k=0}^{N-1} X[k] \frac{e^{j\frac{2\pi}{N}kn}}{\sqrt{N}}$$

Interpretation: Signal Representation by Harmonic Sinusoids

- **Analysis (Forward DFT)**
  - Choose the DFT coefficients $X[k]$ such that the synthesis produces the signal x
  - The weight $X[k]$ measures the similarity between x and the harmonic sinusoid s_k
  - Therefore, $X[k]$ measures the "frequency content" of x at frequency k

$$X[k] = \langle x, s_k \rangle = \sum_{n=0}^{N-1} x[n] \frac{e^{-j\frac{2\pi}{N}kn}}{\sqrt{N}}$$

- **Synthesis (Inverse DFT)**
  - Build up the signal x as a linear combination of harmonic sinusoids s_k weighted by the DFT coefficients $X[k]$

$$x[n] = \sum_{k=0}^{N-1} X[k] \frac{e^{j\frac{2\pi}{N}kn}}{\sqrt{N}}$$

The Unnormalized DFT

- Normalized forward and inverse DFT

$$X[k] = \sum_{n=0}^{N-1} x[n] \frac{e^{-j\frac{2\pi}{N}kn}}{\sqrt{N}}$$

$$x[n] = \sum_{k=0}^{N-1} X[k] \frac{e^{j\frac{2\pi}{N}kn}}{\sqrt{N}}$$

- **Unnormalized** forward and inverse DFT is more popular in practice (we will use both)

$$X_u[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X_u[k]\, e^{j\frac{2\pi}{N}kn}$$

2.5. Unnormalized DFT

In [25]:

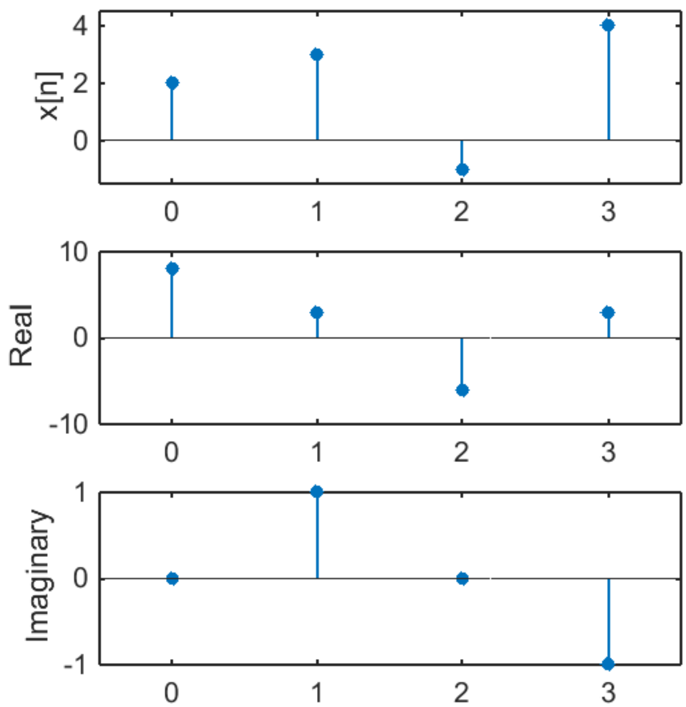
```
%% DFT without using the built-in function

x = [2 3 -1 4];
N = length(x);
X = zeros(1,N);

for k = 0:N-1
    for n = 0:N-1
        X(k+1) = X(k+1) + x(n+1)*exp(-1j*2*pi/N*n*k);
    end
end

X
```

Out[25]:

```
X =

   8.0000 + 0.0000i   3.0000 + 1.0000i  -6.0000 - 0.0000i   3.0000 - 1.0000i
```

In [28]:

```
%plot -s 560,600

% plot a discrete-time signal
xn = 0:N-1;
k = 0:N-1;

subplot(3,1,1), stem(xn,x,'filled','markersize',4), axis([-0.5,3.5,-1.5,4.5]), ylabel('x[n]')
subplot(3,1,2), stem(k,real(X),'filled','markersize',4), xlim([-0.5,3.5]), ylabel('Real')
subplot(3,1,3), stem(k,imag(X),'filled','markersize',4), xlim([-0.5,3.5]), ylabel('Imaginary')
```



Out[28]:

FFT algorithms are so commonly employed to compute DFTs that the term 'FFT' is often used to mean 'DFT': DFT refers to a mathematical transformation or function, whereas 'FFT' refers to a specific family of algorithms for computing DFTs.

- use fft command to compute dft
- fft (computationally efficient)

In [14]:

```
%% DFT using FFT
% fft(X) is the discrete Fourier transform (DFT) of vector X.
% get the same results

fft(x)
```

Out[14]:

```
ans =

   8.0000 + 0.0000i   3.0000 + 1.0000i  -6.0000 + 0.0000i   3.0000 - 1.0000i
```

- can use dft function defined below, but
- computationally expensive

In [15]:

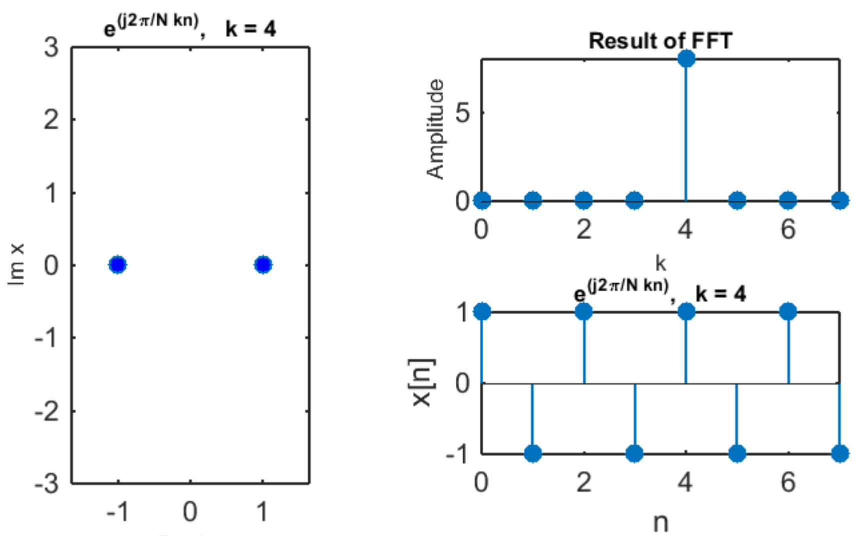
```
% see the below dft function
dft(x,4)
```

Out[15]:

```
ans =

   8.0000 + 0.0000i   3.0000 + 1.0000i  -6.0000 - 0.0000i   3.0000 - 1.0000i
```

In [4]:

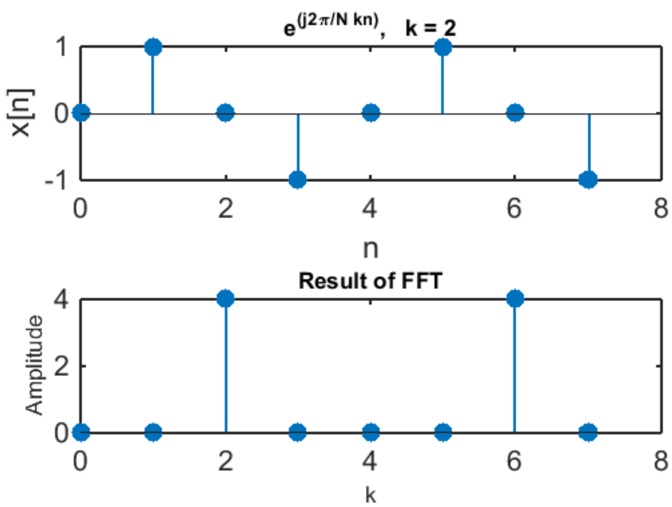
```
%plot -s 800,400

k = 4;                          % index for freqeuncy

N = 8;
n = 0:N-1;                      % sampling period

x = exp(1j*2*pi/N*k*n);           % harmonic complex exponential
X = dft(x,N);


subplot(2,2,[1,3]); plot(real(x),imag(x),'o','MarkerFaceColor','b')
axis equal; ylim([-3 3])
xlabel('Real {x}','Fontsize',8)
ylabel('Im {x}','Fontsize',8)
title(['e^{(j2\pi/N kn)},   k = ',num2str(k)],'Fontsize',8)

subplot(2,2,2); stem(n,abs(X),'filled'), axis tight
xlabel('k','Fontsize',8)
ylabel('Amplitude','Fontsize',8)
title('Result of FFT','fontsize',8)

subplot(2,2,4), stem(n,real(x),'filled'), axis tight
xlabel('n'), ylabel('x[n]')
title(['e^{(j2\pi/N kn)},   k = ',num2str(k)],'Fontsize',8)
```



Out[4]:

In [30]:

```
%plot -s 560,400

k = 2;                         % index for freqeuncy

N = 8;
n = 0:N-1;                    % sampling period

x = sin(2*pi/N*k*n);           % harmonic complex exponential
X = dft(x,N);

subplot(2,1,1), stem(n,real(x),'filled')
xlabel('n'), ylabel('x[n]')
title(['e^{(j2\pi/N kn)},   k = ',num2str(k)],'Fontsize',8)

subplot(2,1,2); stem(n,abs(X),'filled')
xlabel('k','Fontsize',8)
ylabel('Amplitude','Fontsize',8)
title('Result of FFT','fontsize',8)
```



Out[30]:

4. DFT Properties

DFT pair

$$x[n] \longleftrightarrow X[k]$$

DFT Frequencies

- $X[k]$ measures the similarity between the time signal $x[n]$ and the harmonic sinusoid $s_k[n]$
- $X[k]$ measures the "frequency content" of $x[n]$ at frequency

$$\omega_k = \frac{2\pi}{N}k$$

DFT and Circular Shift

$$x[(n-m)_N] \longleftrightarrow e^{-j\frac{2\pi}{N}km}X[k]$$

- no amplitude changed
- phase changed

DFT and Modulation

$$e^{-j\frac{2\pi}{N}r\,n}x[n] \longleftrightarrow X[(k-r)_N]$$

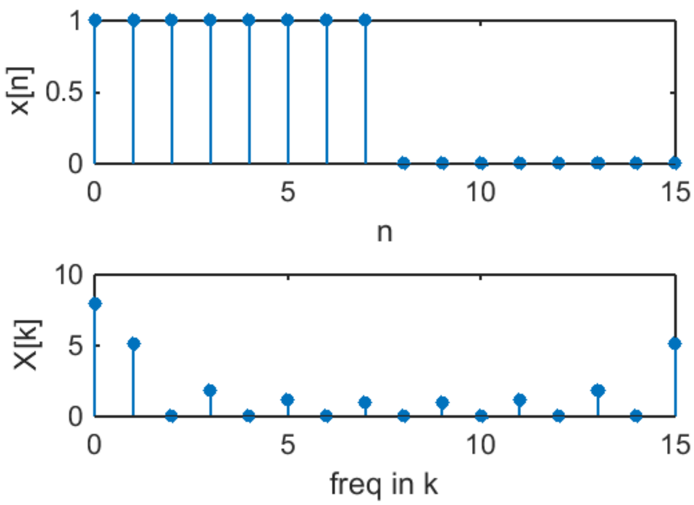
DFT and Circular Convolution

- Circular convolution in the time domain $=$ multiplication in the frequency domain

$$\begin{aligned} Y[k] &= H[k]X[k] \\ h[n] \star x[n] &\longleftrightarrow H[k]X[k] \\ y[n] &= \text{IDFT}(Y[k]) \end{aligned}$$

In [32]:

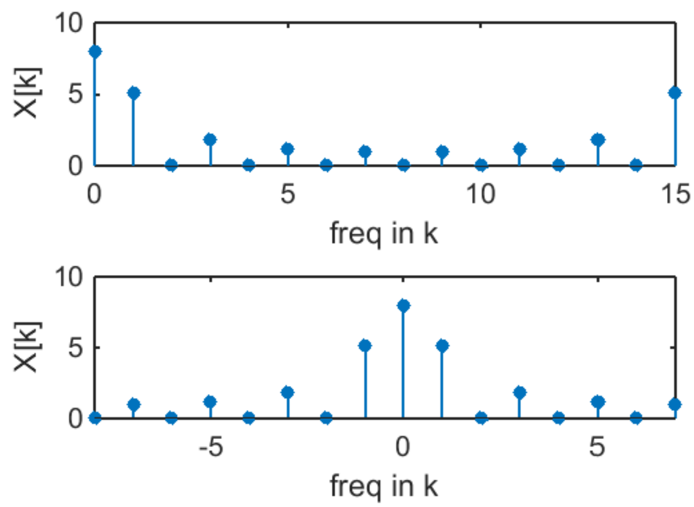
```
%plot -s 560,400

N = 16;
h = zeros(1,N);
h(1:8) = 1;
n = 0:N-1;
k = n;

X = dft(h,N);

subplot(2,1,1), stem(n,h,'filled','markersize',4), xlabel('n'), ylabel('x[n]')
subplot(2,1,2), stem(k,abs(X),'filled','markersize',4), xlabel('freq in k'), ylabel('X[k]')
```



Out[32]:

In [33]:

```
kr = [0:N/2-1 -N/2:-1];
ks = fftshift(kr);

Xs = fftshift(X);

subplot(2,1,1), stem(k,abs(X),'filled','markersize',4), xlabel('freq in k'), ylabel('X[k]')
subplot(2,1,2), stem(ks,abs(Xs),'filled','markersize',4), xlabel('freq in k'), ylabel('X[k]'), xlim([-N/2,N/2-1])
```

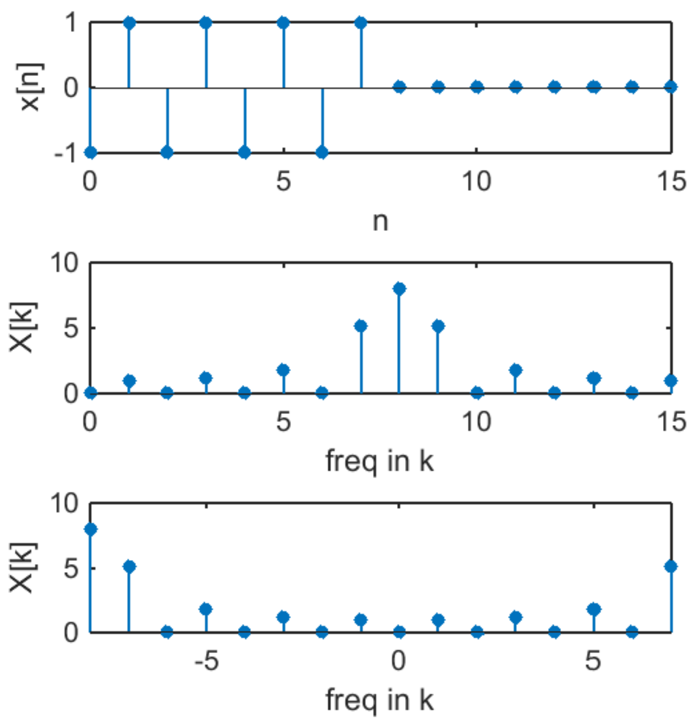


Out[33]:

$$e^{-j\frac{2\pi}{N}r\,n}x[n] \longleftrightarrow X[(k-r)_N]$$

In [34]:

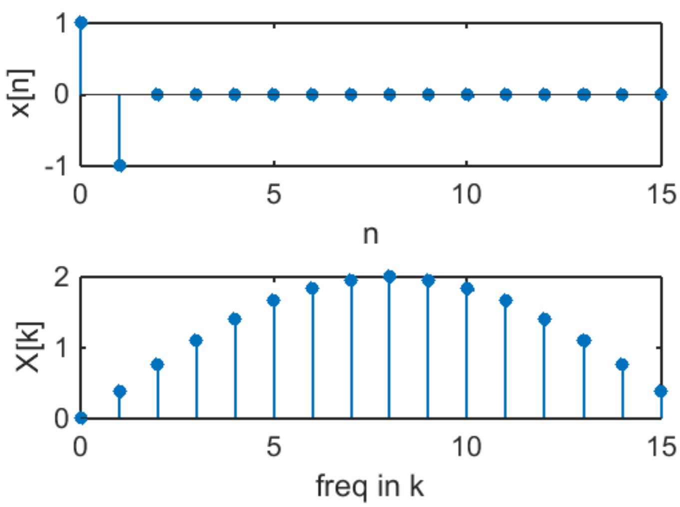
```
%plot -s 560,600

N = 16;
h = zeros(1,N);
h(1:8) = 1;

for i = 1:N
    h(i) = (-1)^i*h(i);
end

n = 0:N-1;
k = n;

X = dft(h,N);

kr = [0:N/2-1 -N/2:-1];
ks = fftshift(kr);

Xs = fftshift(X);

subplot(3,1,1), stem(n,h,'filled','markersize',4), xlabel('n'), ylabel('x[n]')
subplot(3,1,2), stem(k,abs(X),'filled','markersize',4), xlabel('freq in k'), ylabel('X[k]')
subplot(3,1,3), stem(ks,abs(Xs),'filled','markersize',4), xlabel('freq in k'), ylabel('X[k]'), xlim([-N/2,N/2-1])
```



Out[34]:

In [35]:

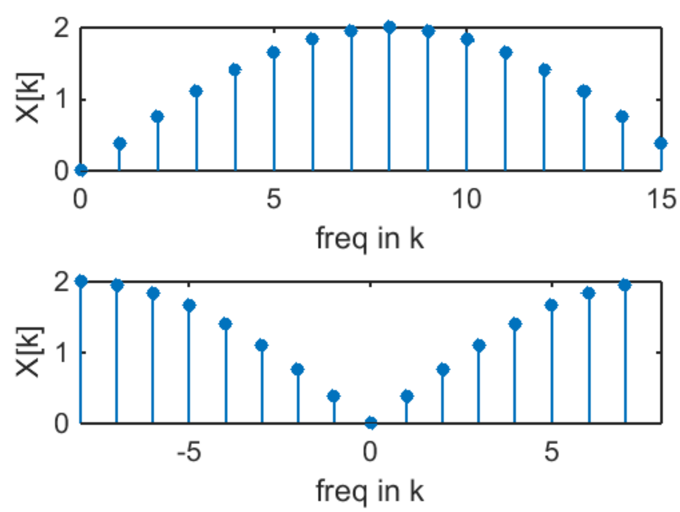
```
%plot -s 560,400

N = 16;
h = zeros(1,N);
h(1) = 1;
h(2) = -1;
n = 0:N-1;
k = n;

X = dft(h,N);

figure(1)
subplot(2,1,1), stem(n,h,'filled','markersize',4), xlabel('n'), ylabel('x[n]')
subplot(2,1,2), stem(k,abs(X),'filled','markersize',4), xlabel('freq in k'), ylabel('X[k]')
```



Out[35]:

In [36]:

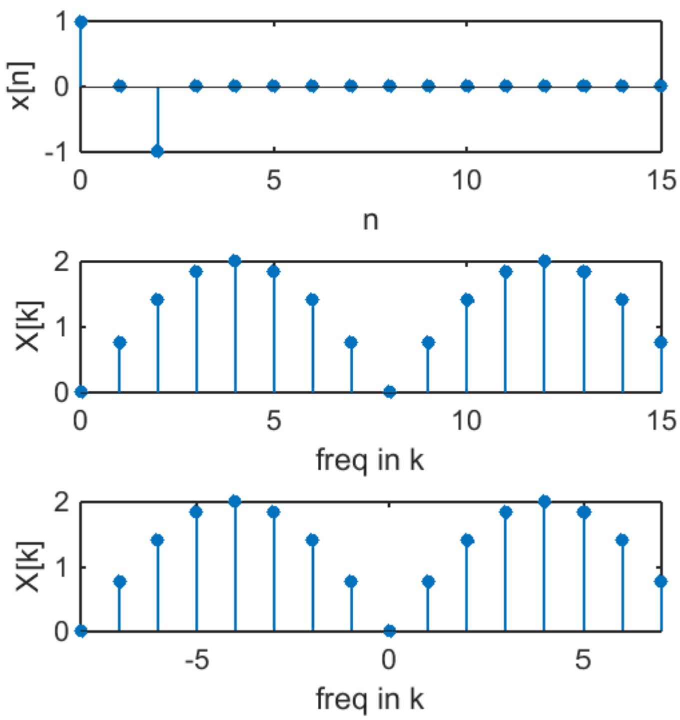
```
%plot -s 560,400

kr = [0:N/2-1 -N/2:-1];
ks = fftshift(kr);

Xs = fftshift(X);

subplot(2,1,1), stem(k,abs(X),'filled','markersize',4), xlabel('freq in k'), ylabel('X[k]')
subplot(2,1,2), stem(ks,abs(Xs),'filled','markersize',4), xlabel('freq in k'), ylabel('X[k]'), xlim([-N/2,N/2])
```



Out[36]:

In [37]:

```
%plot -s 560,600

N = 16;
h = zeros(1,N);
h(1) = 1;
h(2) = 0;
h(3) = -1;
n = 0:N-1;
k = n;

X = dft(h,N);

kr = [0:N/2-1 -N/2:-1];
ks = fftshift(kr);

Xs = fftshift(X);

subplot(3,1,1), stem(n,h,'filled','markersize',4), xlabel('n'), ylabel('x[n]')
subplot(3,1,2), stem(k,abs(X),'filled','markersize',4), xlabel('freq in k'), ylabel('X[k]')
subplot(3,1,3), stem(ks,abs(Xs),'filled','markersize',4), xlabel('freq in k'), ylabel('X[k]'), xlim([-N/2,N/2-1])
```

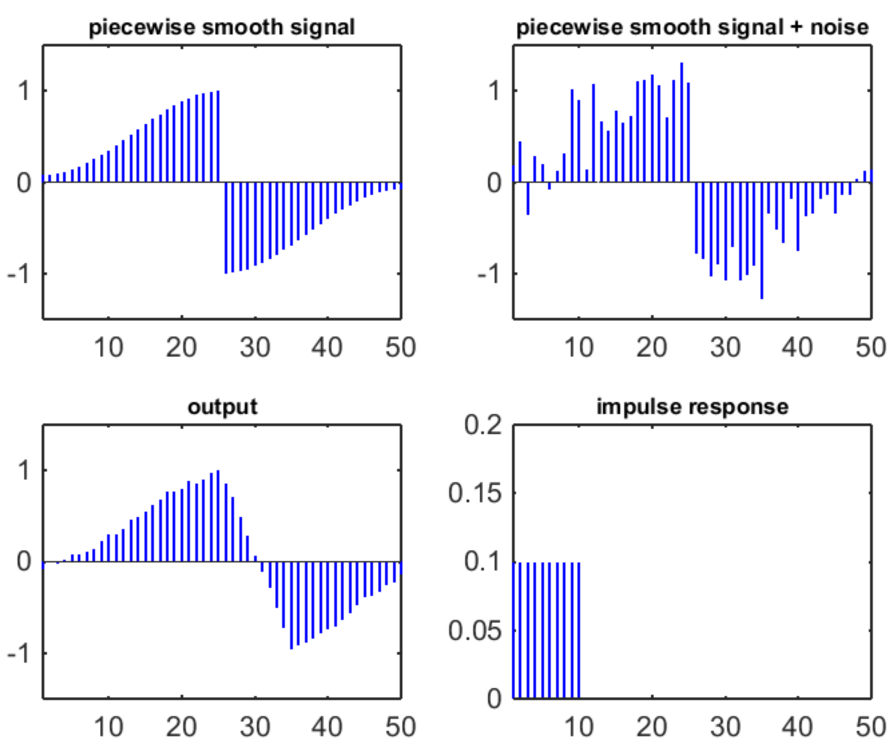


Out[37]:

5. Filtering in Frequency Domain

In [24]:

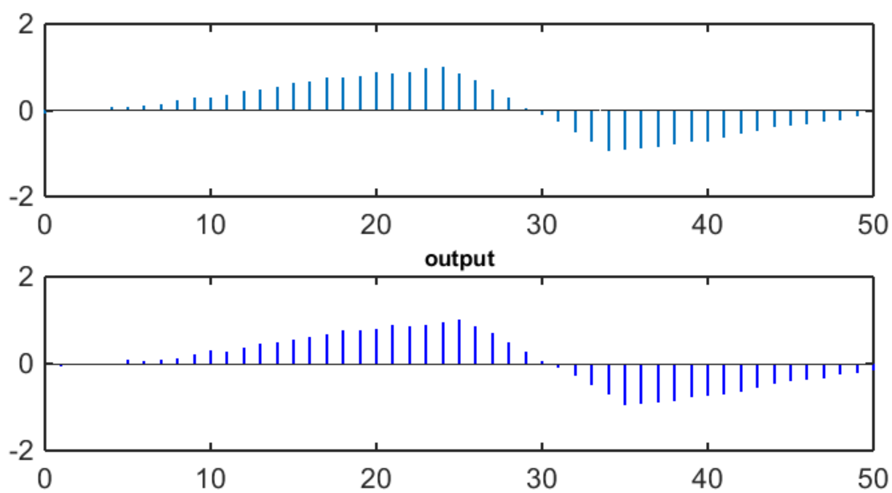
```
%plot -s 800,600

%---EXAMPLE 2
%---Denoising a piecewise smooth signal
figure('units','normalized','position',[0 0 1 1.5]);

% piecewise smooth signal
N = 50;
n = 0:N-1;
s = hamming(N) .* [ones(N/2,1); -ones(N/2,1)];

subplot(221)
stem(s,'b','Marker','none','LineWidth',1);
axis([1 N -1.5 1.5])
title('piecewise smooth signal','fontsize',8)

% add noise to the signal
x = s + 0.2*randn(N,1);

subplot(222)
stem(x,'b','Marker','none','LineWidth',1);
axis([1 N -1.5 1.5])
title('piecewise smooth signal + noise','fontsize',8)

% construct moving average filter impulse response of length M
M = 10;
%M = 3;
h = ones(M,1)/M;

h1 = h; h1(N) = 0;
subplot(224)
stem(h1,'b','Marker','none','LineWidth',1);
axis([1 N 0 0.2])
title('impulse response','fontsize',8)

% convolve noisy signal with impulse response
y = circonvt(x',h',50);

subplot(223)
%stem(y(M/2:N+M/2-1),'b','Marker','none','LineWidth',1);
stem(y,'b','Marker','none','LineWidth',1);
axis([1 N -1.5 1.5])
title('output','fontsize',8)
```



Out[24]:

In [25]:

```
%plot -s 800,400
H = fft(h,N);
k = n;
figure
stem(k,abs(H))

X = fft(x);

Y = H.*X;
yi = ifft(Y);

subplot(2,1,1), stem(n,(yi),'marker','none')
subplot(2,1,2),
%stem(y(M/2:N+M/2-1),'b','Marker','none','LineWidth',1);
stem(y,'b','Marker','none','LineWidth',1);
% axis([1 N -1.5 1.5])
title('output','fontsize',8)
```



Out[25]:

```
function [Xk] = dft(xn,N)

% Computes Discrete Fourier Transform
% [Xk] = dft(xn,N)
% Xk = DFT coeff. array over 0 <= k <= N-1
% xn = N-point finite-duration sequence
%  N = Length of DFT
%

n = [0:1:N-1];                    % row vector for n
k = [0:1:N-1];                    % row vecor for k
WN = exp(-1j*2*pi/N);             % Wn factor
nk = n'*k;                        % creates a N by N matrix of nk values
WNnk = WN.^nk;                  % DFT matrix
Xk = xn*WNnk;                   % row vector for DFT coefficients
```

In [1]:

```
%%javascript
$.getScript('https://kmahelona.github.io/ipython_notebook_goodies/ipython_notebook_toc.js')
```