# Dynamic Systems with Python

Prof. Seungchul Lee
iSystems (http://isystems.unist.ac.kr/)
UNIST

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# 1. Natural response to non-zero initial conditions

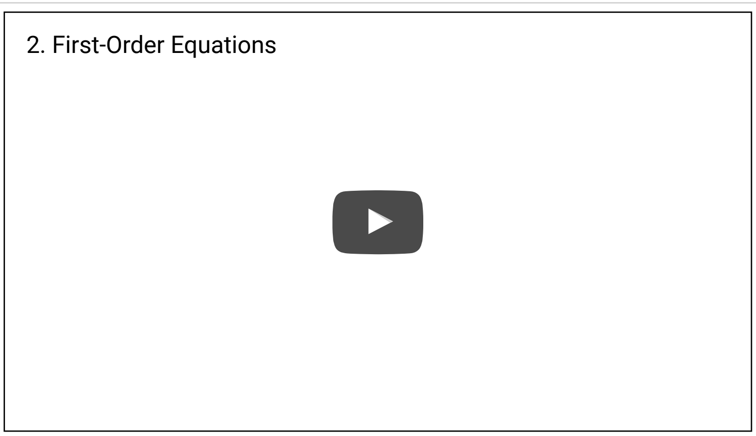
## 1.1. The First Order ODE

- MIT 2.087 Engineering Mathematics: Linear Algebra and ODEs, Fall 2014, by Gilbert Strang

$$\frac{dx(t)}{dt} = kx(t), \qquad x(0) = x_0$$
$$\rightarrow \; x(t) = x_0 e^{\,kt}$$

In [1]:

```
%%html
<iframe src="https://www.youtube.com/embed/4X0SGGrXDiI?start=399&end=459"
width="560" height="315" frameborder="0" allowfullscreen></iframe>
```

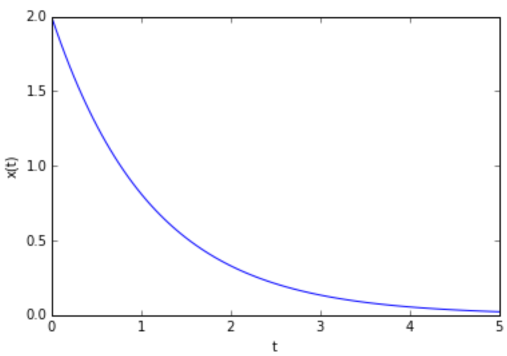


In [2]:

```
import numpy as np
import matplotlib.pyplot as plt
import matplotlib.patches as mpatches
from control import *
from scipy import *
from scipy import linalg as la
from scipy.ndimage.filters import convolve

%matplotlib inline
```

In [3]:

```
# plot an analytic solution

k = -0.9
x0 = 2

t = linspace(0,5,100)
x = x0*exp(k*t)

plt.plot(t,x)
plt.xlabel('t')
plt.ylabel('x(t)')

plt.show()

# but, all we did is just plotting (not computing)
```

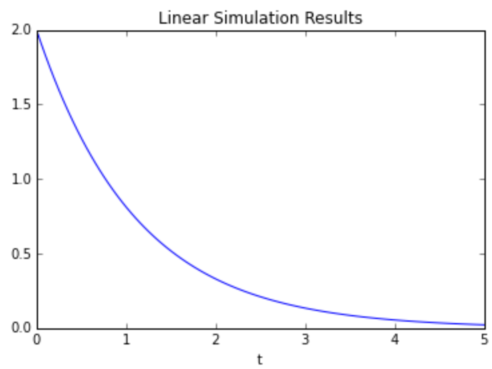



To use `lsim` command, transform it to the state space representation

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

In [4]:

```
# numerically solve an ODE using lsim

A = k
B = 0
C = 1
D = 0

G = ss(A,B,C,D)

x0 = 2

t = linspace(0,5,100)
u = np.zeros(t.shape)

[y,tout,x] = lsim(G,u,t,x0)

plt.plot(tout,y)
plt.title('Linear Simulation Results')
plt.xlabel('t')

plt.show()
```

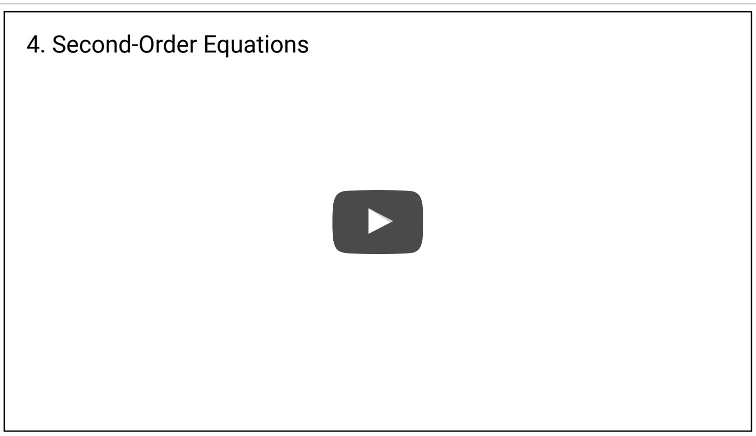



## 1.2. The Second Order ODE

- MIT 2.087 Engineering Mathematics: Linear Algebra and ODEs, Fall 2014, by Gilbert Strang

In [5]:

```
%%html
<iframe src="https://www.youtube.com/embed/xvTYUnqn2wY"
width="560" height="315" frameborder="0" allowfullscreen></iframe>
```




Generic form

$$a\frac{d^2x(t)}{dt^2} + b\frac{dx(t)}{dt} + cx(t) = 0, \qquad \dot{x}(0) = v_0, x(0) = x_0$$

One of examples is the mass-spring-damper system

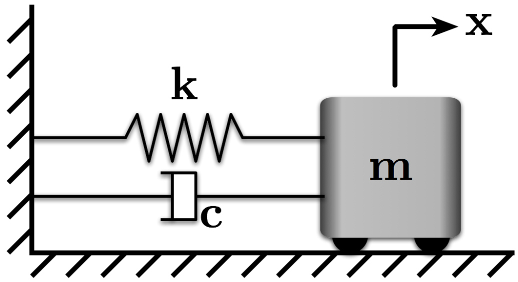



**Figure 1: A Mass-Spring-Damper System**

This notebook simluates the free vibration of a simple mass-spring-damper system like the one shown in Figure 1. More specifically, we'll look at how system response to non-zero initial conditions.

The equation of motion for the system is:

$$m\ddot{x} + c\dot{x} + kx = 0$$

We could also write this equation in terms of the damping ratio, $\zeta$, and natural frequency $\omega_n$.

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$$

We'll use the solution to the differential equation that we developed in class to plot the response. The solution for the underdamped case is:

$$x(t) = e^{-\zeta\omega_n t}\left(a_1 e^{i\omega_d t} + a_2 e^{-i\omega_d t}\right)$$

*or*

$$x(t) = e^{-\zeta\omega_n t}\left(b_1 \cos\omega_d t + b_2 \sin\omega_d t\right)$$

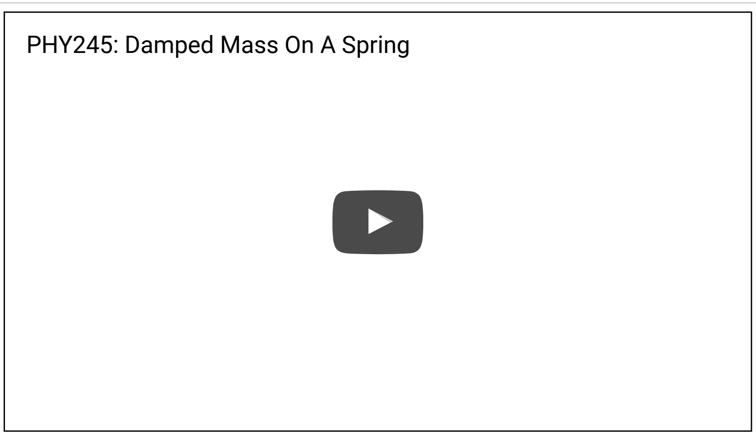
To use this equation, we need to solve for $a_1$ and $a_2$ or $b_1$ and $b_2$ using the initial conditions. Here, let's use the sin/cosine form. Solving the equation for generic intial velocity, $\dot{x} = v_0$, and a generic initial displacement, $x = x_0$, we find:

$$x(t) = e^{-\zeta\omega_n t}\left(x_0 \cos\omega_d t + \frac{\zeta\omega_n x_0 + v_0}{\omega_d}\sin\omega_d t\right)$$

**Experiment**

In [6]:

```
%%html
<iframe src="https://www.youtube.com/embed/ZqedDWEAUN4?start=80&end=114"
width="560" height="315" frameborder="0" allowfullscreen></iframe>
```

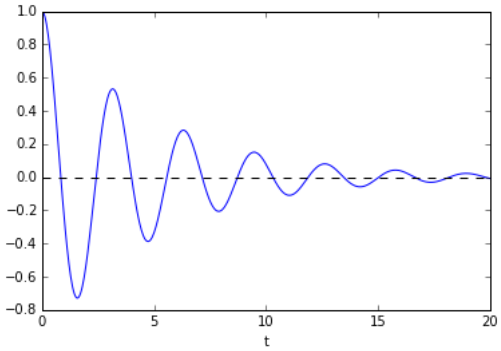



## 1.3. State Space Representation

$$\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix}$$
$$y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix}$$

In [7]:

```
# method 1: use 'lsim'

wn = 2
zeta = 0.1

A = np.array([[0,1],[-wn**2,-2*zeta*wn]])
B = [[0],[0]]
C = [1,0]
D = 0

G = ss(A,B,C,D)

x0 = np.array([[1],[0]])

t = linspace(0,20,500)
u = np.zeros(t.shape)

[y,tout,non] = lsim(G,u,t,x0)

plt.plot(tout,y)
plt.plot(tout,np.zeros(t.shape),'k--')
plt.xlabel('t')

plt.show()
```



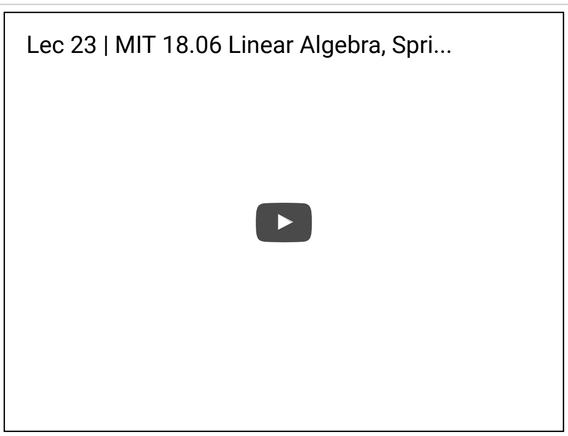

## 1.4. Matrix Exponentials

$$y(t) = e^{At}x(0)$$

- Lec 23 | MIT 18.06 Linear Algebra, Spring 2005
- Matrix Exponentials | MIT 18.03SC Differential Equations, Fall 2011
- Differential Equations and exp (At) | MIT 18.06SC Linear Algebra, Fall 2011

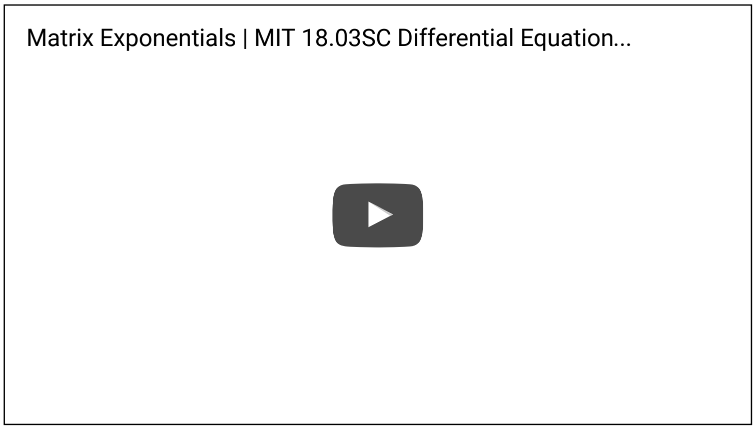
In [8]:

```
%%html
<iframe src="https://www.youtube.com/embed/IZqwi0wJovM"
width="420" height="315" frameborder="0" allowfullscreen></iframe>
```



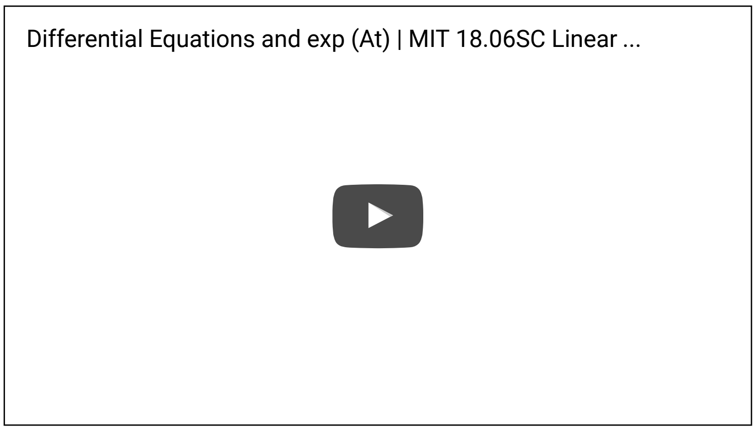
In [9]:

```
%%html
<iframe src="https://www.youtube.com/embed/qbyeQum8qTE"
width="560" height="315" frameborder="0" allowfullscreen></iframe>
```



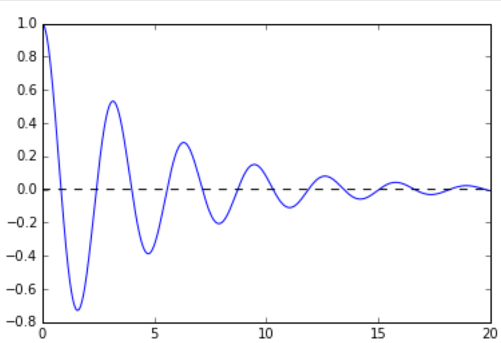

In [10]:

```
%%html
<iframe src="https://www.youtube.com/embed/dZfdKXxhnTM"
width="560" height="315" frameborder="0" allowfullscreen></iframe>
```




In [11]:

```
# method 2: use expm (matrix exponential)
x0 = np.array([[1],[0]])
x = np.array([[],[]])

t = np.arange(0.0,20.0,0.01)

for i in range(0,t.shape[0]):
    add = la.expm(t[i]*A)
    add_= np.dot(add,x0)
    x   = np.hstack((x,add_))

plt.plot(t,x[0])
plt.plot(t,np.zeros(t.shape),'k--')

plt.show()
```

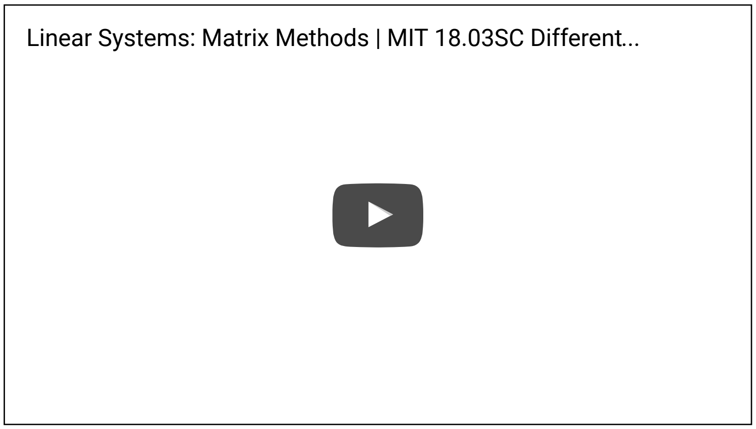



## 1.5. Systems of Differential Equations (Matrix Differential Equation)

- Matrix Methods | MIT 18.03SC Differential Equations, Fall 2011

In [12]:

```
%%html
<iframe src="https://www.youtube.com/embed/YUjdyKhWt6E"
width="560" height="315" frameborder="0" allowfullscreen></iframe>
```




Given

$$\dot{\vec{u}} = A\vec{u}, \qquad \vec{u}(0) = \vec{u}_0$$

- Eigenanalysis

$$A\vec{x}_1 = \lambda_1\vec{x}_1$$
$$A\vec{x}_2 = \lambda_2\vec{x}_2$$

- General solution:

$$\vec{u}(t) = c_1\, e^{\lambda_1 t}\, \vec{x}_1 + c_2\, e^{\lambda_2 t}\, \vec{x}_2$$

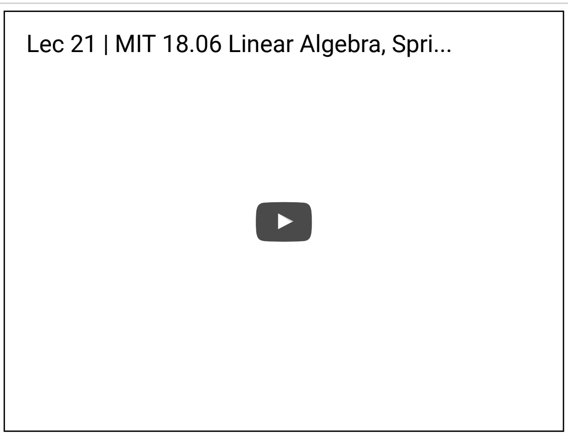
where

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = [\,\vec{x}_1 \;\; \vec{x}_2\,]^{-1} \begin{bmatrix} u_1(0) \\ u_2(0) \end{bmatrix}.$$

- Systems of differential equations $\leftrightarrow$ Eigenanalysis (Gilbert Strang Lecture 21, video below)
- System stability $\leftrightarrow$ eigenvalues

In [13]:

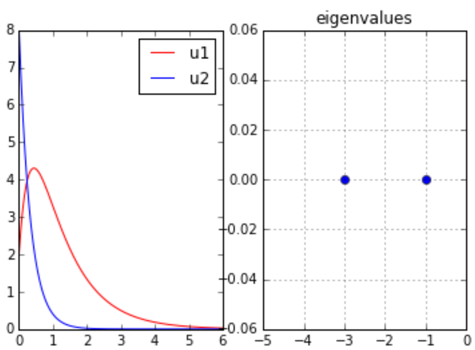
```
%%html
<iframe src="https://www.youtube.com/embed/lXNXrLcoerU"
width="420" height="315" frameborder="0" allowfullscreen></iframe>
```




### 1.5.1. Real eigenvalues

In [14]:

```
A = np.array([[-1,2],[0,-3]])

## eigen-analysis
####################################################################
V = la.eig(A)[1]
lamb = la.eig(A)[0]
D = diag(lamb)

t = np.arange(0.0,6.0,0.01)
U0 = np.array([[2,8]]).T
C = dot(la.inv(V),U0)

u = C[0][0]*dot(V[:,[0]],exp(lamb[0]*np.array([t]))) + \
    C[1][0]*dot(V[:,[1]],exp(lamb[1]*np.array([t])))
####################################################################

# plot u1 and u2 as a function of time
plt.subplot(1,2,1)
plt.plot(t,real(u[0,:]),'r')
plt.plot(t,real(u[1,:]),'b')
plt.axis('tight')
plt.legend(['u1','u2'])

plt.subplot(1,2,2)
plt.plot(real(lamb),imag(lamb),'o')
plt.xlim([-5,0])
plt.grid(True)
plt.title('eigenvalues')

plt.show()
```

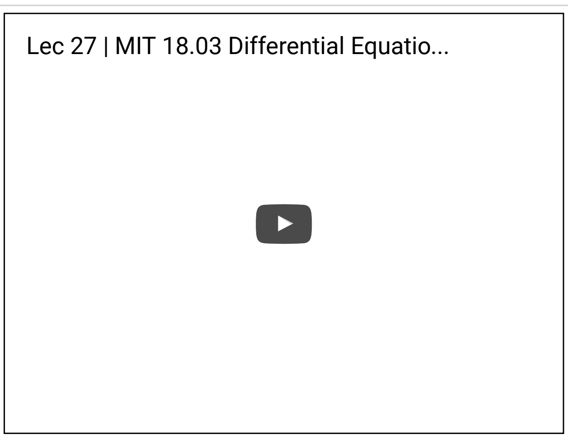



**Phase portrait**

- https://en.wikipedia.org/wiki/Phase_portrait (https://en.wikipedia.org/wiki/Phase_portrait)
- http://tutorial.math.lamar.edu/Classes/DE/PhasePlane.aspx (http://tutorial.math.lamar.edu/Classes/DE/PhasePlane.aspx)
- Lec 27 | MIT 18.03 Differential Equations, Spring 2006

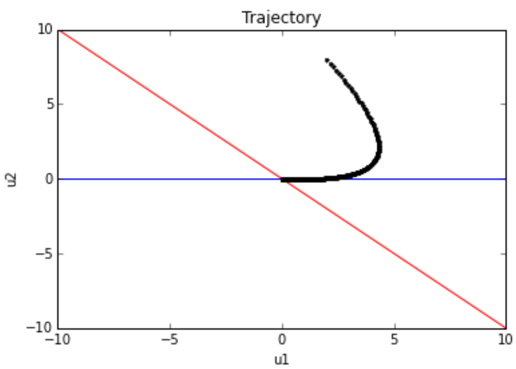
In [15]:

```
%%html
<iframe src="https://www.youtube.com/embed/e3FfmXtkppM"
width="420" height="315" frameborder="0" allowfullscreen></iframe>
```



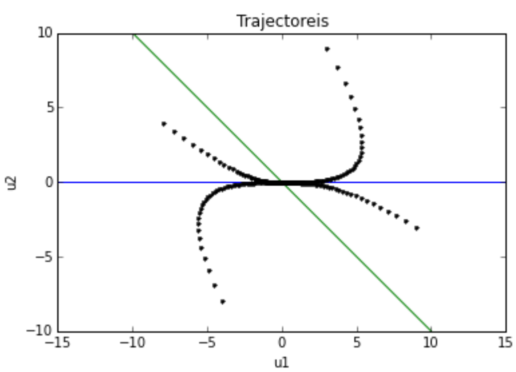
In [16]:

```
# plot eigenvectors (X1 and X2)

k = np.arange(-20.0,20.0,0.1)
y1 = np.dot(V[:,[0]],[k])
y2 = np.dot(V[:,[1]],[k])

plt.plot(real(y1[0,:]),real(y1[1,:]),'b')
plt.plot(real(y2[0,:]),real(y2[1,:]),'r')
plt.xlabel('u1')
plt.ylabel('u2')
plt.title('Trajectory')
plt.xlim(-10,10)
plt.ylim(-10,10)

# plot a trajectory of u1 and u2
for i in range(0, t.shape[0]):
    plt.plot(real(u[0,i]),real(u[1,i]),'k.')

plt.show()
```



In [17]:

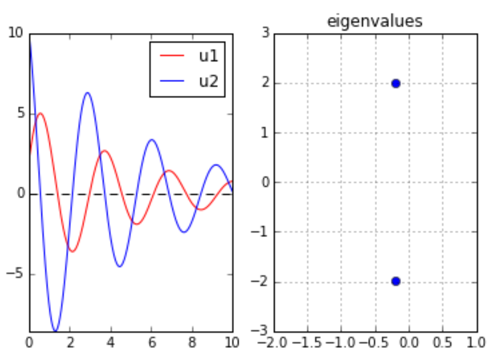
```
# with multiple initial conditions

A = np.array([[-1,2],[0,-3]])
#A = np.array([[-2,0],[0,-4]])
#A = np.array([[-2,0],[1,-4]])

##
V = la.eig(A)[1]
lamb = la.eig(A)[0]
D = diag(lamb)

t = np.arange(0.0,5.0,0.05)
U0 = np.array([[3, 9,-8, -4],
               [9,-3, 4, -8]])

# plot eigenvectors (X1 and X2)
k = np.arange(-20.0,20.0,0.1)
y1 = dot(V[:,[0]],[k])
y2 = dot(V[:,[1]],[k])

plt.plot(y1[0,:],y1[1,:])
plt.plot(y2[0,:],y2[1,:])

for m in range(0, U0.shape[1]):
    C = dot(la.inv(V),U0[:,[m]])
    u = C[0][0]*dot(V[:,[0]],exp(lamb[0]*np.array([t]))) + \
        C[1][0]*dot(V[:,[1]],exp(lamb[1]*np.array([t])))

    # plot a trajectory of u1 and u2
    for i in range(0, len(t)):
        plt.plot(real(u[0,i]),real(u[1,i]),'k.')

plt.xlabel('u1')
plt.ylabel('u2')
plt.title('Trajectoreis')
plt.axis('equal')
plt.axis([-10,10,-10,10])

plt.show()
```




### 1.5.2. Complex eigenvalues (starting oscilation)

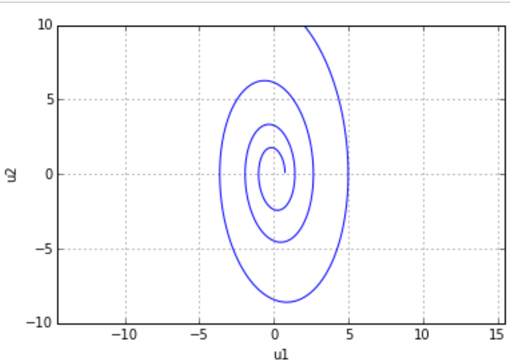
In [18]:

```
#A = np.array([[0,1],[-1,0]])
wn = 2
zeta = 0.1
A = np.array([[0,1],[-wn**2.0,-2*zeta*wn]])

###################################################################
V = la.eig(A)[1]
lamb = la.eig(A)[0]
D = diag(lamb)

t = np.arange(0.0,10.0,0.01)
U0 = np.array([[2,10]]).T
C = dot(la.inv(V),U0)

u = C[0][0]*dot(V[:,[0]],exp(lamb[0]*np.array([t]))) + \
    C[1][0]*dot(V[:,[1]],exp(lamb[1]*np.array([t])))

###################################################################

# plot u1 and u2 as a function of time
plt.subplot(1,2,1)
plt.plot(t,real(u[0,:]),'r')
plt.plot(t,real(u[1,:]),'b')
plt.plot(t,np.zeros(t.shape),'k--')
plt.axis('tight')
plt.legend(['u1','u2'])

plt.subplot(1,2,2)
plt.plot(real(lamb),imag(lamb),'o')
plt.xlim(-2,1)
plt.ylim(-3,3)
plt.grid(True)
plt.title('eigenvalues')

plt.show()
```




In [19]:

```
# Phase portrait
plt.plot(real(u[0,:]),real(u[1,:]),'b')
plt.axis('equal')
plt.grid(True)
plt.xlabel('u1')
plt.ylabel('u2')

plt.show()
```




# 2. Response to General Inputs

## 2.1. Step response

Start with a ***step response*** example

$$\dot{x} + 5x = 1 \quad \text{for} \quad t \geq 0, \qquad x(0) = 0$$

or

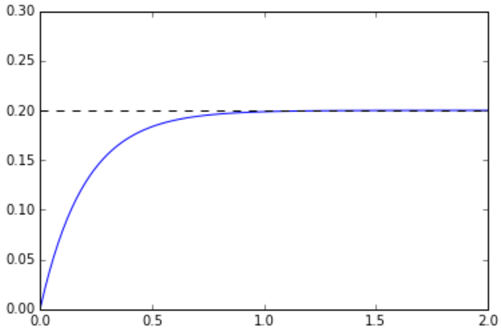
$$\dot{x} + 5x = u(t), \qquad x(0) = 0$$

The solution is given:

$$x(t) = \frac{1}{5}\left(1 - e^{-5t}\right)$$

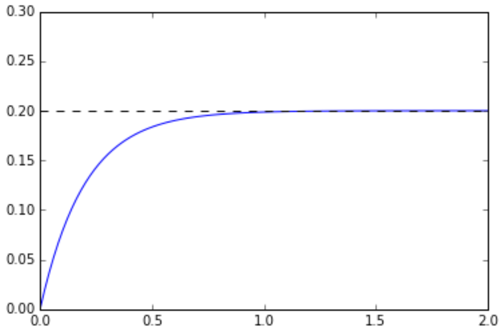
In [20]:

```
t = linspace(0,2,100)
x = 1./5*(1 - exp(-5*t))

plt.plot(t,x)
plt.plot(t,0.2*np.ones(t.shape),'k--')
plt.ylim(0,0.3)

plt.show()
```



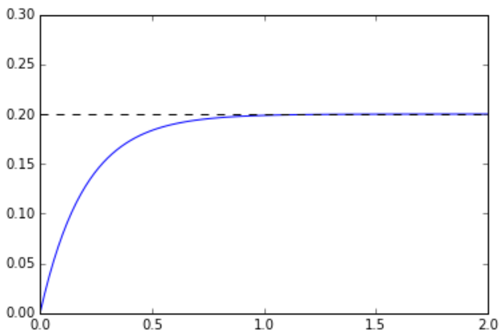

In [21]:

```
# method 1
# define a system first

num = 1
den = [1,5]

G = tf(num,den)

[y,tout] = step(G,t)

plt.plot(tout,y)
plt.plot(tout,0.2*np.ones(tout.shape),'k--')
plt.ylim([0,0.3])

plt.show()
```



In [22]:

```
# method 2
# define a system first
A = -5
B = 1
C = 1
D = 0

G = ss(A,B,C,D)

t = linspace(0,2,100)
u = np.ones(t.shape)
x0 = 0

[y,tout,non] = lsim(G,u,t,x0)

plt.plot(tout,y)
plt.plot(tout,0.2*ones(tout.shape),'k--')
plt.ylim([0, 0.3])

plt.show()
```




## 2.2. Impulse response

Now think about the ***impulse response***

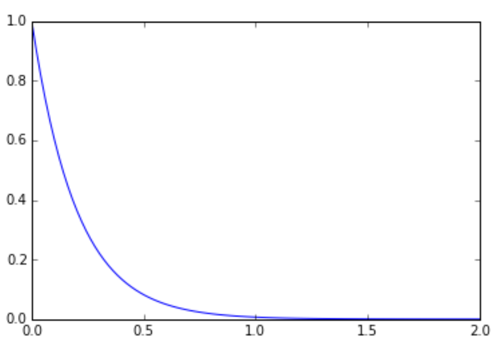
$$\dot{x} + 5x = \delta(t), \qquad x(0) = 0$$

The solution is given: (why?)

$$h(t) = e^{-5t}, \quad t \geq 0$$

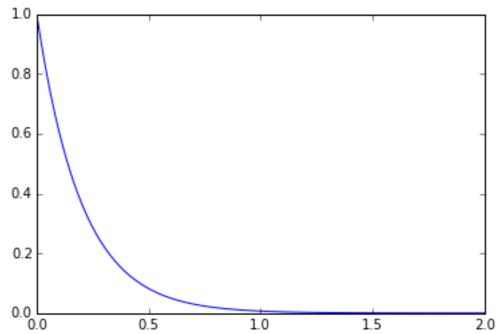
In [23]:

```
t = linspace(0,2,100)
h = exp(-5*t)

plt.plot(t,h)
plt.ylim([0,1])

plt.show()
```



In [24]:

```
# method 1
# define a system first
num = 1
den = [1,5]

G = tf(num,den)

[h,tout] = impulse(G,t)

plt.plot(tout,h)
plt.ylim([0,1])

plt.show()
```

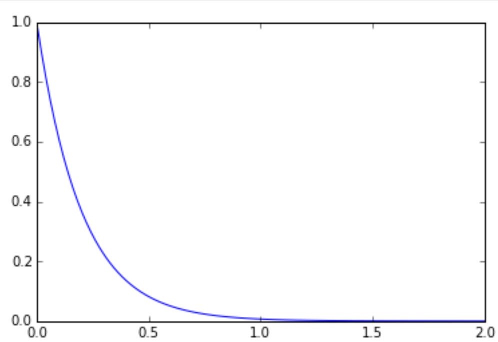


Impulse input can be equivalently changed to zero input with non-zero initial condition (by the impulse and momentum theory)

$$\int_{0^-}^{0^+} \delta(t)dt = u(0^+) - u(0^-) = 1$$

In [25]:

```
# method 2
# define a system first
A = -5
B = 1
C = 1
D = 0

G = ss(A,B,C,D)

t = linspace(0,2,100)

##########################################
u = np.zeros(t.shape)
x0 = 1
##########################################

[h,tout,non] = lsim(G,u,t,x0)

plt.plot(tout,h)
plt.ylim([0,1])

plt.show()
```



## 2.3. Response to a general input

Response to a ***general input***

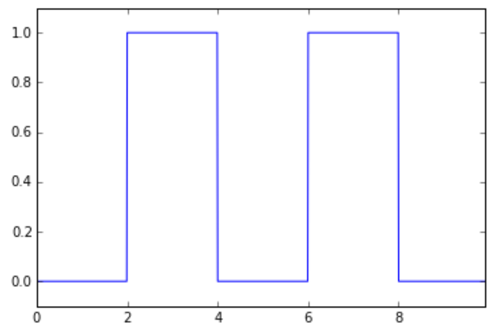
$$\dot{x} + 5x = f(t), \qquad x(0) = 0$$

The solution is given:

$$x(t) = h(t) * f(t), \quad t \geq 0$$

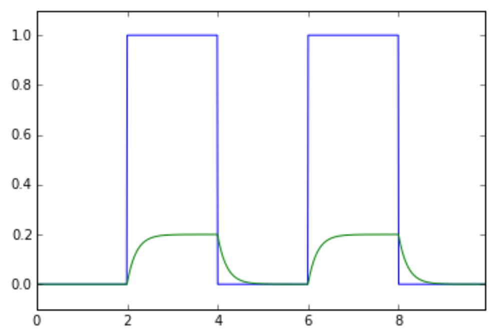
In [26]:

```
# generate a general input

t = np.arange(0,10,0.01)
f = np.zeros(200)
for i in range(0, 2):
    f = np.hstack((f, np.ones(200)))
    f = np.hstack((f, np.zeros(200)))

plt.plot(t,f)
plt.axis([0,9.9,-0.1,1.1])

plt.show()
```




In [27]:

```
# use lsim

A = -5
B = 1
C = 1
D = 0

G = ss(A,B,C,D)

x0 = 0
[y, tout, non] = lsim(G,f,t,x0)

plt.plot(t,f)
plt.plot(tout, y)
plt.axis([0,9.9,-0.1,1.1])

plt.show()
```




In [28]:

```
# cconv function does not exist in python
```

## 2.4. Reponse to a sinusoidal input

- only focus on *steady-state* solution
- transient solution is not our interest any more

Assume:

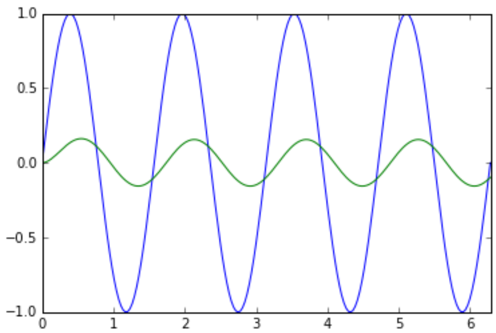
$$y(t) = \sin(\omega t)$$

Then, the solution $x(t)$ should have the form:

$$x(t) = a\sin(\omega t) + b\cos(\omega t) = A\sin(\omega t + \phi)$$

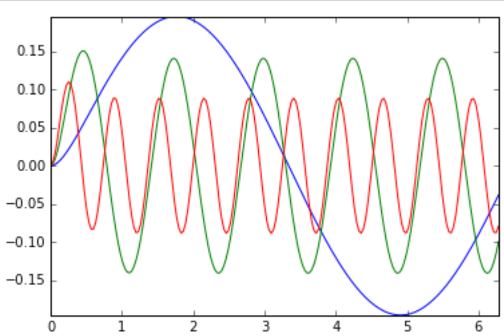
In [29]:

```
# use lsim

A = -5
B = 1
C = 1
D = 0

G = ss(A,B,C,D)

x0 = 0
w = 4

t = linspace(0,2*pi,200)
f = sin(w*t)

[y,tout,non] = lsim(G,f,t,x0)

plt.plot(t,f)
plt.plot(tout,y)
plt.axis('tight')
plt.show()
```




In [30]:

```
A = -5
B = 1
C = 1
D = 0

G = ss(A,B,C,D)

x0 = 0
t = linspace(0,2*pi,200)

W = [1,5,10]
for w in W:
    f = sin(w*t);
    [y,tout,non] = lsim(G,f,t,x0)
    plt.plot(tout,y)

plt.axis('tight')

plt.show()
```




## 3. Frequency response (frequency sweep)

Given input $e^{j\omega t}$

$$\dot{z} + 5z = e^{j\omega t}$$

If $z = Ae^{j(\omega t + \phi)}$

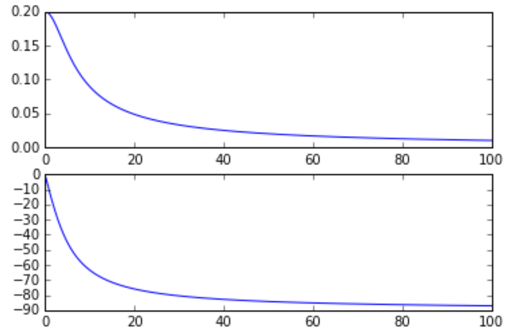
$$
\begin{aligned}
j\omega A e^{j(\omega t+\phi)} + 5 A e^{j(\omega t+\phi)} &= e^{j\omega t}\\
(j\omega + 5)\, A e^{j\phi} &= 1
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
A &= \frac{1}{|j\omega + 5|}\\
\phi &= -\angle(j\omega + 5)
\end{aligned}
$$

In [31]:

```
w = np.arange(0.1,100.0,0.1)
A = 1./abs(1j*w+5)
Ph = -angle(1j*w+5)*180/pi

plt.subplot(2,1,1)
plt.plot(w,A)

plt.subplot(2,1,2)
plt.plot(w,Ph)

plt.show()

# later, we will see that this is kind of a bode plot
```




$$
\begin{aligned}
\ddot{z} + 2\zeta\omega_n \dot{z} + \omega_n^2 z &= \omega_n^2 \ f(t)\\
\ddot{z} + 2\zeta\omega_n \dot{z} + \omega_n^2 z &= \omega_n^2 \ A_0 e^{j\Omega t}
\end{aligned}
$$

- We know that $z$ is in the form of

$$
z = A e^{j(\Omega t + \phi)}
$$

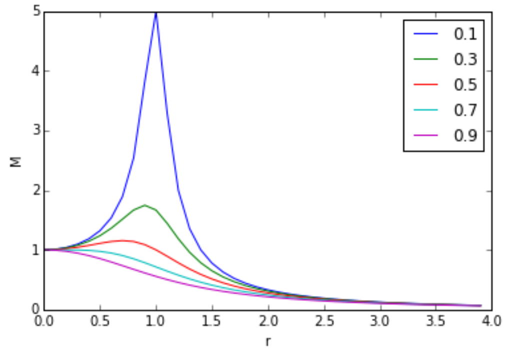
- Then

$$
\begin{aligned}
\left(-\Omega^2 + j2\zeta\omega_n\Omega + \omega_n^2\right) A e^{j\phi} e^{j\Omega t} &= \omega_n^2 A_0 e^{j\Omega t}\\
A e^{j\phi} &= \frac{\omega_n^2 A_0}{-\Omega^2 + j2\zeta\omega_n\Omega + \omega_n^2} = A_0 \frac{1}{1 - \left(\frac{\Omega}{\omega_n}\right)^2 + j2\zeta\left(\frac{\Omega}{\omega_n}\right)}
\end{aligned}
$$

$$
\begin{aligned}
\frac{A}{A_0} &= \frac{1}{\sqrt{\left(1 - \left(\frac{\Omega}{\omega_n}\right)^2\right)^2 + 4\zeta^2\left(\frac{\Omega}{\omega_n}\right)^2}} = \frac{1}{\sqrt{(1-\gamma^2)^2 + 4\zeta^2\gamma^2}}, \quad \left(\gamma = \frac{\Omega}{\omega_n}\right)\\
\phi &= -\tan^{-1}\left(\frac{2\zeta\frac{\Omega}{\omega_n}}{1 - \left(\frac{\Omega}{\omega_n}\right)^2}\right) = -\tan^{-1}\left(\frac{2\zeta\gamma}{1-\gamma^2}\right)
\end{aligned}
$$

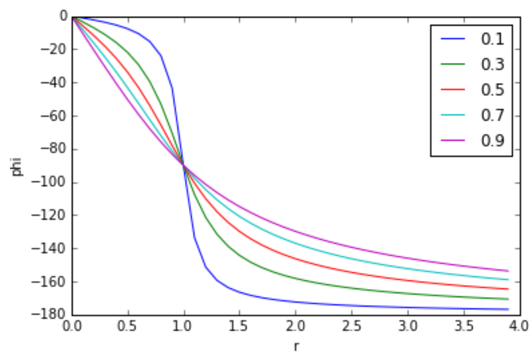
In [32]:

```
r = np.arange(0.0,4.0,0.1)

zeta = np.arange(0.1,1.0,0.2)

for i in range(0,zeta.shape[0]):
    M = 1./sqrt((1-r*r)**2 + (2*zeta[i]*r)**2) # ignore a constant of F0/k
    plt.plot(r, M)

plt.xlabel('r')
plt.ylabel('M')
plt.legend(['0.1','0.3','0.5','0.7','0.9'])

plt.show()
```




In [33]:

```
for i in range(0,zeta.shape[0]):
    phi = []
    for j in range(0,r.shape[0]):
        temp1 = 2*zeta[i]*r[j]
        temp2 = 1-r[j]** 2
        temp3 = math.atan2(temp1,temp2)
        phi.append(-temp3*180/pi)
    plt.plot(r, phi)

plt.xlabel('r')
plt.ylabel('phi')
plt.legend(['0.1', '0.3', '0.5', '0.7', '0.9'])

plt.show()
```



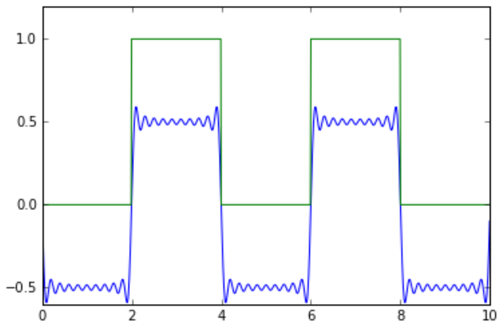

## 3.1. Fourier Series of square wave

- decompose a genernal signal (ex. square wave) to a linear combinarion of sinusoidal signals

$$x(t) = \frac{1}{2} - \frac{2}{\pi} \sum_{n=1,2,3,} \frac{1}{n} \sin\left(\frac{n\pi t}{L}\right)$$

In [34]:

```
L = 2

t = np.arange(0.0,10.0,0.01)
f = np.zeros(200)
for i in range(0,2):
    f = np.hstack((f,np.ones(200)))
    f = np.hstack((f,np.zeros(200)))

x = 1/2*np.ones(t.shape)
for n in np.arange(1.0, 20.0, 2.0):
    x = x - 2/pi*1/n*sin(n*pi/L*t)

plt.plot(t,x)
plt.plot(t,f)

plt.show()
```

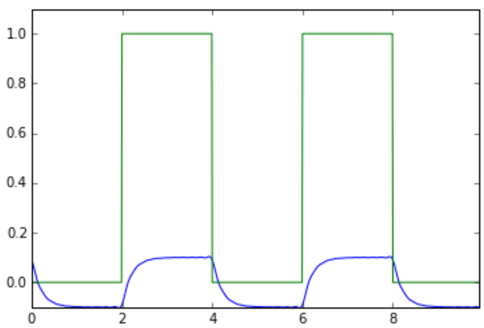



The output response of LTI

Linearity: input $\sum_k c_k x_k(t)$ produces $\sum_k c_k y_k(t)$

In [35]:

```
L = 2;

x = 1/2*np.ones(t.shape)
x2 = 0.2*x

for n in np.arange(1.0,20.0,2.0):
    w = n*pi/L
    A = 1./abs(1j*w+5)
    Ph = -angle(1j*w+5)
    x2 = x2 - 2/pi*1/n*A*sin(w*t + Ph)

plt.plot(t,x2)
plt.plot(t,f)
plt.axis([0,9.9,-0.1,1.1])

plt.show()
```




In [36]:

```
%%javascript
$.getScript('https://kmahelona.github.io/ipython_notebook_goodies/ipython_notebook_toc.js')
```