PARTIAL DIFFERENTIAL EQUATIONS

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MAT4022 Notebook

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Contents

Acknowledgments

This book is from the MAT4022 in fall semester, 2019. In addition, some material is cited from *Partial Differential Equations* 4^{th} edition by F.John.

CUHK(SZ)

Chapter 1

Week1

1.1. Tuesday

1.1.1. Introduction and Examples

$u(x,y)$ a smooth function

$u_x = \frac{\partial u}{\partial x}$

$u_y = \frac{\partial u}{\partial y}$

$u_{xx} = \frac{\partial^2 u}{\partial x \partial x}$

$u_{xy} = \frac{\partial^2 u}{\partial y \partial x}$

In this course, as u is smooth, we don't fuss about difference between u_{xy} and u_{yx}.

Definition 1.1 A PDE is a relation for $F(x,y,u,u_x,y_y,u_{xy},\dots) = 0$ linear equation. If F is linear in u, u_x, u_y, u_{xx}, $\cdots$ (Not necessarily in x, y)

Example 1.1 Laplace equation: $u_{xx} + u_{xy} = 0$

$(x^2 + y^2)u_{xx} + e^{xy}u_{yy} = 0$

In the second case, it is still linear as it is a linear equation for u_{xx} and u_{yy}.

Other examples

Cauchy-Riemann.

$$\begin{cases} u_x - v_y = 0 \\ u_y + v_x = 0 \end{cases} \quad \text{1st order system}$$

$$\begin{cases} u_{xx} - v_{yx} = 0 \\ u_{yy} + v_{yx} \end{cases} \rightarrow u_{xx} + y_{yy} = 0 \text{ harmonic function } (2^{\text{st}} \text{ order}).$$

Order: highest order that partial derivatives are taken.

Notation for laplace operator:

$$\Delta u = u_{xx} + u_{yy},\ \Delta u = u_{xx},\ \Delta u = u_{x_1 x_1} + \cdots + u_{x_n x_n}$$

■ **Example 1.2** Wave equation. $u_{tt} = c^2 \Delta u$, $c > 0$ wave speed a constant, t: time.

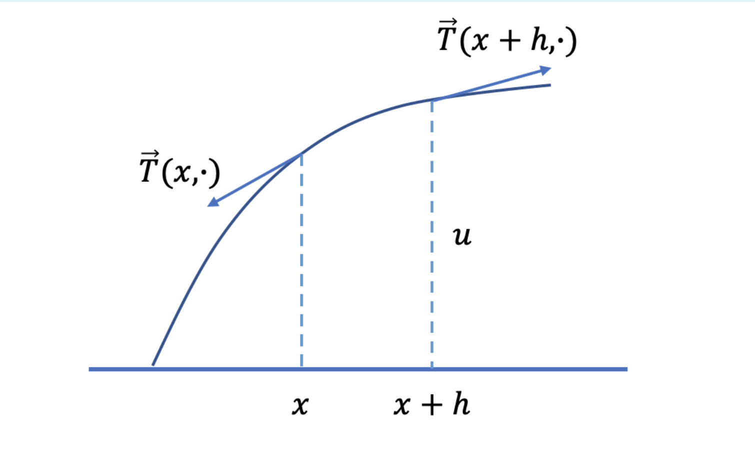
$n = 1$ $u_t t = c^2 u_{xx}$: vibration of a string

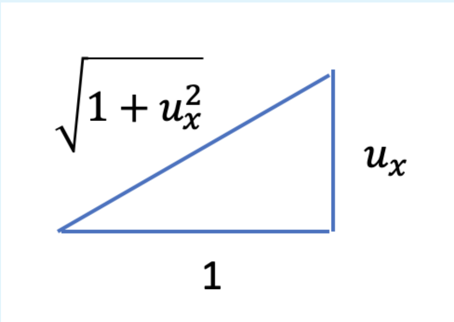
$n = 2$ $u_{tt} = c^2(u_{xx} + u_{yy})$: water wave

$n = 3$ $u_{tt} = c^2(u_{x_1 x_1} + u_{x_2 x_2} + u_{x_3 x_3})$: sound wave

n: space dimension

n=1 A string, flexible, elastic, homogeneous with density ρ





When we magnefy part of the string and let h to be small enough. The horizontal parts of forces of $\vec{T}(x,\cdot)$ and $\vec{T}(x+h,\cdot)$ are equal to each other. Vertical forces is equal to am according to Newton's second law. Therefore, we have the following.

$$\begin{cases} \dfrac{|T|(x+h,\cdot)}{\sqrt{1+u_x^2(x+h,\cdot)}} = \dfrac{|T|(x,\cdot)}{\sqrt{1+u_x^2(x,\cdot)}} \\ \dfrac{|T|u_x}{\sqrt{1+u_x^2}}(x+h,\cdot) - \dfrac{|T|u_x}{\sqrt{1+u_x^2}}(x,\cdot) = h\rho u_{tt} \end{cases}$$

When h is small enough,

$$\frac{1}{\sqrt{1+u_x^2}} = (1+u_x^2)^{-\frac{1}{2}} = 1 - \frac{1}{2}u_x^2 + \cdots \approx 1$$

$$|T|(x+h,\cdot) = |T|(x,\cdot)$$

$$\rho u_{tt} = |T| \cdot [u_x(x+h,\cdot) - u_x(x,\cdot)]\frac{1}{h} \to |T|u_{xx}$$

$$u_{tt} = \frac{|T|}{\rho}u_{xx},\ c = \sqrt{\frac{|T|}{\rho}}$$

■

■ **Example 1.3** [Heat equation] $u_t = \Delta u$, u: temprature
$H(t) =$ total amount of heat in $\Omega \subset \mathbb{R}^3$

$$= \iiint\limits_{\Omega} c\rho u \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

$$\iint\limits_{\partial\Omega} \kappa \bigtriangleup u\nu \,\mathrm{d}s = \frac{\mathrm{d}H}{\mathrm{d}t} = \iiint\limits_{\Omega} c\rho u \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

κ is heat conduction constant. ν is unit outward normal.

$$\vec{w} = \bigtriangleup u = (u_x, u_y, u_z)$$

$$\mathrm{dir}\vec{w} = u_{xx} + u_{yy} + u_{zz} = \Delta u$$

By divergent theorem:

$$\int\!\!\int\!\!\int_{\Omega} c\rho u_t = \int\!\!\int\!\!\int_{\Omega} \kappa \Delta u$$

As this is the case for all Ω we can tell $c\rho u_t = \kappa \Delta u$

$$u_t = \frac{\kappa}{c\rho} \Delta u$$

■

Theorem 1.1 — Divergent theorem. $\vec{w} = (w_1, w_2, w_3)$: vector field

$$\int\!\!\int\!\!\int_{\Omega} \operatorname{dir} \vec{w} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = \int\!\!\int_{\partial\Omega} \vec{w} v \, \mathrm{d}s$$

where $\operatorname{dir} \vec{w} = \frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial y} + \frac{\partial w_3}{\partial z}$

1.2. Thursday

1.2.1. Examples

Minimal surface equation.

Given a curve Γ in $\mathbb{R}^3$ to find a surface spanning Γ and has the smallest possible area.

$$(1+u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1+u_x^2)u_{yy} = 0$$

This is a 2^{nd} order linear in u_{xx}, u_{yy} equation. It is called quasi-linear equation.

Now, before we take a look at solving some partial differential equations. Let's review a simple ode case.

In ODE,

$$v' + cv = 0$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = v' = -cv$$

$$\frac{\mathrm{d}v}{v} = -c\,\mathrm{d}t$$

At the moment, just ignore $|v|$

$$\ln v = -ct + \alpha$$

$$v(t) = e^{-ct}e^{\alpha} = v(0)e^{-ct}, t = 0\ v(0) = e^{\alpha}$$

In PDE,

$$u_y + cu_x = 0\ c\text{: constant}$$

This is called transport equation. Later on, we will talk about this name.

$$\frac{\mathrm{d}}{\mathrm{d}t}u(x(t), y(t)) = u_x\frac{\mathrm{d}x}{\mathrm{d}t} + u_y\frac{\mathrm{d}y}{\mathrm{d}t} = cu_x + u_y = 0$$

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = c \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = 1 \end{cases} \Rightarrow \begin{cases} x = ct + X(0) = ct + \xi \\ y = t + y(0) = t \end{cases}$$

$$\Rightarrow \xi = x - ct \qquad \xi = x - cy$$

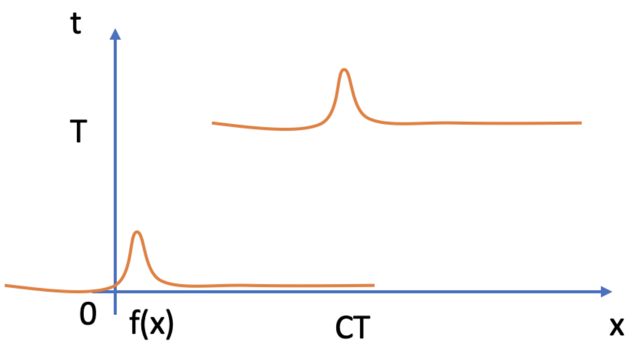
On Γ_ξ, $\frac{\mathrm{d}t}{\mathrm{d}u}(x(t),y(t)) \equiv 0\ \forall t \Rightarrow$, $u(x(t),y(t)) \equiv$ constant on Γ_ξ.

$$u(x,y) = f(x - cy)$$

As convention, let t denote y, transport equation is $\begin{cases} u_t + cu_x = 0 \\ u(x,0) = f(x) \end{cases} \Rightarrow u(x,t) = f(x - ct)$.

The following is a graph used to illustrate why it is called transport equation:

$$u(cT,T) = f(cT - cT) = f(0) = u(0,0)$$



■ **Example 1.4**

$$\begin{cases} xu_x + yu_y = \alpha u \\ u(x,1) = f(x) \end{cases}$$

Γ : characteristic curve $\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = x \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = y \end{cases} \Rightarrow \begin{cases} x = x_0 e^t = se^t \\ y = y_0 e^t = 1e^t \rightarrow t = \ln y \end{cases}$.

Initial curve $\begin{cases} x = s \\ y = 1 \end{cases}$.

$$s = \frac{x}{e^t} = \frac{x}{y}$$

On Γ, we have $\frac{\mathrm{d}u}{\mathrm{d}t} = u_x \frac{\mathrm{d}x}{\mathrm{d}t} + u_y \frac{\mathrm{d}y}{\mathrm{d}t} = xu_x + yu_y = \alpha u \Rightarrow u = ce^{\alpha t} = f(s)e^{\alpha}$ About the last equal sign of above equation, it is because $u(x_0, y_0) = u(s, 1) = c$ when t=0.

$$u(x,y) = f(\frac{x}{y})y^{\alpha}$$

Next lecture will discuss quasi-linear equation.

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u)$$

Chapter 2

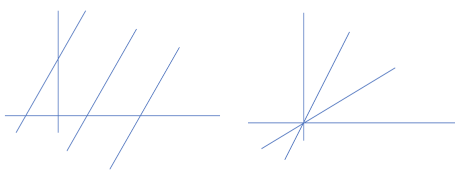
week2

2.1. Tuesday

2.1.1. Quasi-linear Equations

Review: last week we have learn how to solve the following PDE and get the solution along characteristic curves.

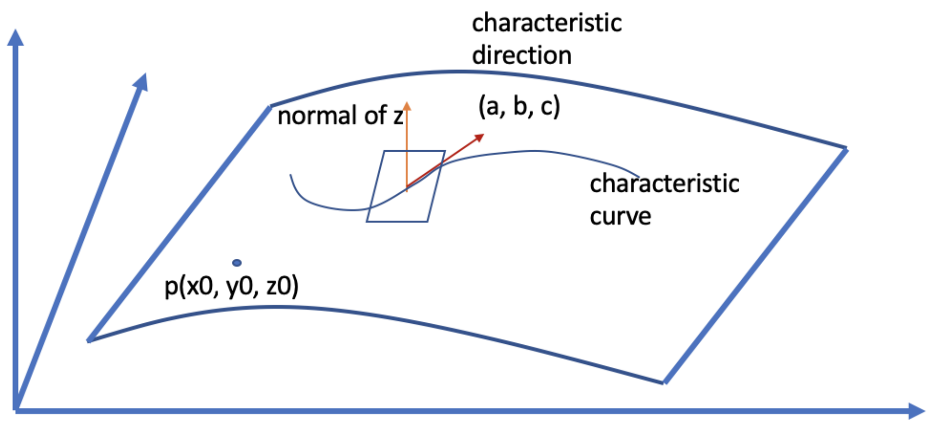
$$u_y + cu_x = 0 \qquad xu_x + yu_y = u$$



$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u)$$

This is what we called quasi-linear equation. At this time, I have to quote our textbook p9 since I cannot explain what professor said better than it does.

We represent the function $u(x,y)$ by a surface $z=u(x,y)$ in xyz-space. Surfaces corresponding to solutions of a P.D.E. are called *integral surfaces* of the P.D.E. The prescribed functions $a(x,y,z), b(x,y,z), c(x,y,z)$ define a field of vectors in xyz-space (or in a portion Ω of that space). Obviously only the direction of the vector, the *characteristic direction*, matters for the P.D.E. (4.2). Since $(u_x, u_y, -1)$ constitute direction numbers of the normal of the surface $z=u(x,y)$, we see that (4.2) is just the condition that the normal of an integral surface at any point is perpendicular to the direction of the vector (a,b,c) corresponding to that point. Thus integral surfaces are surfaces that at each point are tangent to the characteristic direction.



Next we are going to show that the integral surface, that is the solution u we want, is the union of characteristic curves.

Theorem 2.1 If a characteristic curve $\mathscr{C}$ has one point on a integral surface $z = u(x,y)$ then $\mathscr{C}$ lies entirely on the surface.

Proof.

$$z_0 = u(x_0, y_0)$$

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = a(x(t), y(t), u(x_0, y_0)) \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = b(x(t), y(t), u(x_0, y_0)) \\ x(0) = x_0 \\ y(0) = y_0 \end{cases}$$

This is a system of odes, there is a solution $(x(t), y(t))$ when t is around 0.

This gives us a curve $(x(t), y(t), u(x(t), y(t)))$ lies on the integral surface.

Claim: It's a characteristic curve.

$$z(t) = u(x(t), y(t))$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = u_x \frac{\mathrm{d}x}{\mathrm{d}t} + u_y \frac{\mathrm{d}y}{\mathrm{d}t} = au_x + bu_y = c$$

■

2.1.2. Cauchy Problem

"Finding the function $u(x,y)$ for given data $f(s)$, $g(s)$, $h(s)$ constitutes the *Cauchy problem* for quasi-linear equation." (John, 1982, p.9)

The following φ, ψ, ρ are f, g, h correspondingly.

To solve the Cauchy problem, it suffices to require that $(\psi', \varphi')\&(a,b)$ are linearly independent.

The

$$\begin{cases} \frac{\mathrm{d}\bar{X}}{\mathrm{d}t} = a(\bar{X}, \bar{Y}, Z) \\ \frac{\mathrm{d}\bar{Y}}{\mathrm{d}t} = b(\bar{X}, \bar{Y}, Z) \\ \frac{\mathrm{d}\bar{z}}{\mathrm{d}t} = c(\bar{X}, \bar{Y}, Z) \\ \bar{X}(s,0) = \varphi(s) \\ \bar{Y}(s,0) = \psi(s) \\ \bar{Z}(s,0) = \rho(s) \end{cases}$$

Existence and uniqueness theorem for ODE.

$\Rightarrow$ For every s, $\exists$ one solution $(\bar{X}(s,t), \bar{Y}(s,t), z(s,t))$. In order to make $u(x,y) = z(s(x,y), t(x,y))$ is the case. We need implicit function theorem which makes us able to write s,t as a function of (x,y)

$$\begin{vmatrix} \frac{\partial \bar{X}}{\partial s} & \frac{\partial \bar{X}}{\partial t} \\ \frac{\partial \bar{Y}}{\partial s} & \frac{\partial \bar{Y}}{\partial t} \end{vmatrix} = \begin{vmatrix} \varphi' & a \\ \psi' & b \end{vmatrix} \neq 0$$

■ **Example 2.1**

$$u_y + uu_x = 0, \quad u(x,0) = f(x)$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = z, \ \frac{\mathrm{d}y}{\mathrm{d}t} = 1, \ \frac{\mathrm{d}z}{\mathrm{d}t} = 0$$

z is constant in t. $z(s,0) = f(s) \Rightarrow z(s,t) = f(s)$.

$$\begin{cases} x(s,0) = s \ \Rightarrow \ x(s,t) = f(s)t + s \\ y(s,0) = 0 \ \Rightarrow \ y(s,t) = t \end{cases} \rightarrow \ s = x - f(s)t = x - f(s)y = x - zy$$

$$z(s,0) = f(s) \Rightarrow z(s,t) = f(s) = f(x - zy)$$

$$u(x,y) = f(x - uy)$$

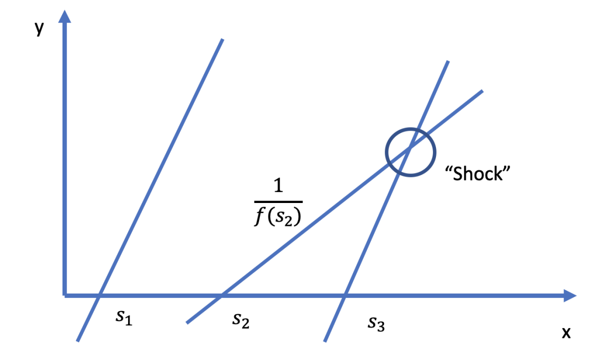
Characteristic curve

$$f(s_2) > f(s_3)$$

$$\frac{1}{f(s_2)} < \frac{1}{s_3}$$

Why it is called a "Shock"?

The reason is that: for same (x,y) z should be the same. However, from the graph we can see there are two z corresponds to the same point marked by the circle. Therefore, the solution does't exist.



y
1
f(s2)
"Shock"
s1
s2
s3
x

2.2. Thursday

2.2.1. 2nd order Quasi-linear

$$au_{xx} + 2bu_{xy} + cu_{yy} = d$$

a, b, c, d are functions of (x, y, u, u_x, u_y)

Initial curve $f(s)$, $g(s)$. Initial value $u = h(s), u_x = \varphi(s), u_y = \psi(s)$.

$$u(f(s), g(s)) = h(s)$$

$$u_x f' + u_y g' = h'$$

$$\varphi f' + \psi g' = h'$$

Assume the above equation is satisfied.

$$u_x(f(s), g(s)) = \varphi(s) \qquad u_y(f(s), g(s)) = \psi(s)$$

$$u_{xx} f' + u_{xy} g' = \varphi' \qquad u_{yx} f' + u_{yy} g' = \psi'$$

$$\begin{pmatrix} a & 2b & c \\ f' & g' & 0 \\ 0 & f' & g' \end{pmatrix} \begin{pmatrix} u_{xx} \\ u_{xy} \\ u_{yy} \end{pmatrix} = \begin{pmatrix} d \\ \varphi' \\ \psi' \end{pmatrix}$$

$$0 = \begin{vmatrix} a & 2b & c \\ f' & g' & 0 \\ 0 & f' & g' \end{vmatrix} = ag'^2 - 2bf'g' + cf'^2$$

Personal understanding (may not be correct):

This is called characteristic curve for when determinant is equal to zero above system will not have a solution. This situation will on happen when the initial curve $(f, g, h)$is tangent to a charateristic curve at every point. Therefore, $ag'^2 - 2bf'g' + cf'^2$.

$$\begin{cases} x = f(s) \\ y = g(s) \end{cases} \quad \begin{cases} \frac{\mathrm{d}x}{\mathrm{d}s} = f' \\ \frac{\mathrm{d}y}{\mathrm{d}s} = g' \end{cases} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{g'}{f'}$$

$$a(\frac{\mathrm{d}y}{\mathrm{d}x})^2 - 2b\frac{\mathrm{d}y}{\mathrm{d}x} + c = 0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2b \pm \sqrt{2b^2 - 4ac}}{2a}$$

- $b^2 - ac > 0$: Hyperbolic
- $b^2 - ac = 0$: Parabolic
- $b^2 - ac < 0$: Elliptic

Example 2.2
- $u_{yy} - c^2 u_{xx} = 0 \; A = -c^2, B = 0, C = 1 \Rightarrow \; B^2 - AC = C^2$
- heat equation $u_y = c^2 = u_{xx} = 0 \; A = -C^2, B = 0, C = 0 \; \Rightarrow \; B^2 - AC = 0$
- laplace equation $u_{yy} + u_{xx} = 0 \; A = 1 = C, B = 0 \; \Rightarrow \; B^2 - AC = -1 < 0$

Wave equation

$$u_{yy} - c^2 u_{xx} = 0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{-c^2}(\pm c) = \mp\frac{1}{c}$$

$$y = \mp\frac{1}{c}x + \mathrm{const}$$

$$y \pm \frac{1}{c}x = \mathrm{const}$$

$$\xi = y + \frac{1}{c}x$$

$$\eta = y - \frac{1}{c}x$$

$$v(\xi, \eta) = u(x(\xi, \eta), y(\xi, \eta))$$

$$u(x, y) = v(\xi(x, y), \eta(x, y))$$

$$u_x = v_\xi \xi_x + v_\eta \eta_x = v_\xi \frac{1}{c} - v_\eta \frac{1}{c}$$

$$u_{xx} = v_{\xi\xi}\frac{1}{c^2} + v_{\xi\eta}(-\frac{1}{c^2}) - v_{\eta\xi}\frac{1}{c^2} + v_{\eta\eta}\frac{1}{c^2}$$

$$u_y = v_\xi \xi_y + v_\eta \eta_y = v_\xi + v_\eta$$

$$u_{yy} = v_{\xi\xi} + 2v_{\xi\eta} + v_{yy}$$

$$0 = u_{yy} - c^2 u_{xx} = v_{\xi\xi} + 2v_{\xi\eta} + v_{yy} - v_{\xi\xi} + 2v_{\xi\eta} - v_{\eta\eta} = 4v_{\xi\eta}$$

$$v_{\eta\xi} = 0 \;\Rightarrow\; (v_\xi)_\eta = 0 \;\Rightarrow\; v_\xi = f'(\xi)$$

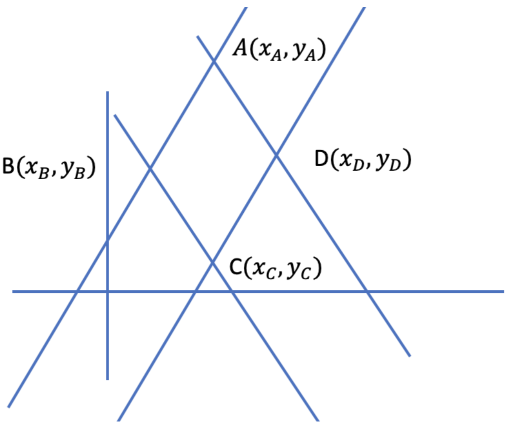
$$v = f(\xi) = \text{const in } \xi$$

$$v(\xi,\eta) = f(\xi) + g(\eta) = u(x,y) = f(y + \frac{1}{c}x) + g(y - \frac{1}{c}x)$$

An observation:

$$u(A) + u(C) = f(y_A + \frac{1}{c}x_A) + g(\underline{y_A - \frac{1}{c}x_A}) + f(y_C \frac{1}{c}x_C) + g(y_C - \frac{1}{c}x_C)$$

$$u(B) + u(D) = f(y_B + \frac{1}{c}x_B) + g(\underline{y_B - \frac{1}{c}x_B}) + f(y_D \frac{1}{c}x_D) + g(y_D - \frac{1}{c}x_D)$$



Take $c = 1$ for simplicity

$$\begin{cases} u_{yy} - u_{xx} = 0 \\ u(x,0) = \varphi(x) \\ u_y(x,0) = \psi(x) \end{cases}$$

$$u(x,y) = f(y+x) + g(y-x)$$

$$\varphi(x) = u(x,0) = f(x) + g(-x)$$

$$\psi(x) = u_y(x,0) = f'(x) + g'(-x)$$

$$\varphi(x) = f(x) + g(-x) \;\Rightarrow\; \varphi'(x) = f'(x) - g'(-x)$$

$$\begin{cases} \varphi(x) = f(x) = g(-x) \\ \psi(x) = f'(x) = g'(x) \end{cases} \Rightarrow \psi'(x) = f'(x) - g'(-x)$$

$$\frac{\varphi'(x) + \psi(x)}{2} = f'(x) \Rightarrow \; f(x) = \frac{1}{2}\psi(x) + \frac{1}{2}\int_0^x \psi(s)\,\mathrm{d}s + \text{const}\alpha$$

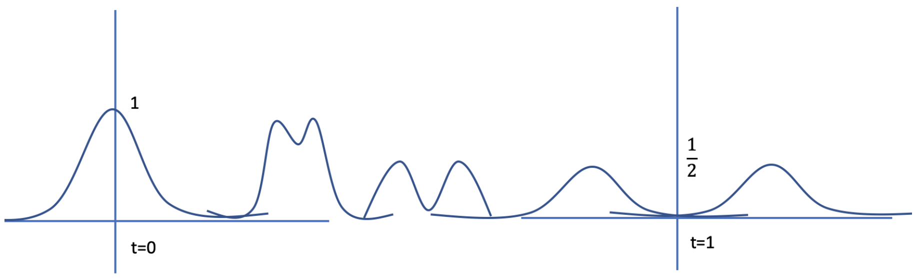
$$\frac{-\varphi'(x) + \psi(x)}{2} = g'(-x) \;\Rightarrow\; g(-x) = \frac{1}{2}\varphi(x) - \frac{1}{2}\int_0^x \psi(s)\,\mathrm{d}s + \text{const}\beta$$

$$\begin{aligned} u(x,y) &= f(x+y) + g(y-x) \\ &= \frac{1}{2}\varphi(x+y) + \frac{1}{2}\int_0^{x+y} \psi(s)\,\mathrm{d}s + \alpha + \frac{1}{2}\varphi(x-y) - \frac{1}{2}\int_0^{x-y} \psi(s)\,\mathrm{d}s + \beta \\ &= \frac{1}{2}[\varphi(x+y) + \varphi(x-y)] + \frac{1}{2}\int_{x-y}^{x+y} \psi(s)\,\mathrm{d}s + \alpha + \beta \end{aligned}$$

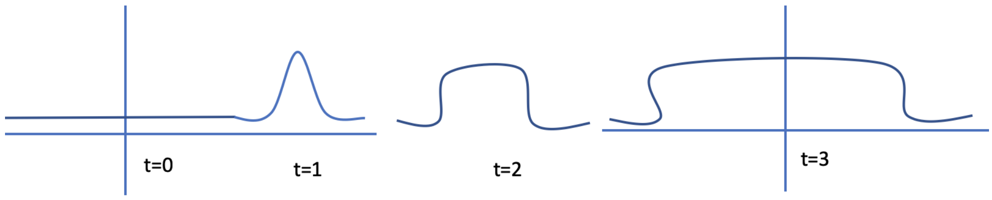
$$u(x,y) = \frac{1}{2}[\varphi(x+y) + \varphi(x-y)] + \frac{1}{2}\int_{x-y}^{x+y} \psi(s)\,\mathrm{d}s$$

$$\begin{cases} u_{yy} - u_{xx} = 0 \\ u(x,0) = \varphi(x)\text{–initial displacement} \\ u_y(x,0) = \psi(x)\text{–initial velocity} \end{cases}$$

case(i) $\psi \equiv 0$ $t = y = 1$ $u(x,1) = \frac{1}{2}\varphi(x+1) + \frac{1}{2}\varphi(x-1)$



case(ii)$\psi = 0$



Chapter 3

week3

3.1. Tuesday

3.1.1. Initial-boundary Value Problem

$$
\begin{cases}
u_{tt} = c^2 u_{xx} \\
u(x,0) = \varphi(x) \\
u_t(x,0) = \psi(x) \\
u(x,t) = \dfrac{1}{2}[\psi(x+ct) + \psi(x-ct)] + \dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct} \psi(\xi)\,\mathrm{d}\xi
\end{cases}
$$