A FIRST COURSE

IN

ORDINARY DIFFERENTIAL EQUATIONS

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MAT2002 Notebook

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Contents

Acknowledgments

This book is from the MAT2002 in spring semester, 2019. In addition, some material is cited from *Differential Equations and Their Applications* 4^{th} by Martin Braun.

CUHK(SZ)

Chapter 1

Week1

1.1. Monday

1.1.1. Motivations

- A rocket is set up and then fall down. Which time is longer going-up period or falling-down part considering air resistance? At the first, it is clear that the time rocket goes up is shorter because at any level goes-up velocity is higher than goes-down velocity with constant energy point of view. We can even figure out explicitly how faster it is at a specific position in the future with the help of ode.

- A hot coffee is added cream immediately. Another one is added ten minutes after the coffee was made. Which one is cooler after ten minutes?

- A lion wants to chase a deer. The velocity of lion is twice faster than deer. In which way can lion catch the deer within least time consumed?(Deer run in a straight line.)

- There are four people sitting at the corner of a square respectly. They look at each other in a clock wise way. Then they begin to walk towards the men they look at. Where will they meet? How long do they walk?

Those are all related to ode. If you are interested in those topic, study ode!

1.1.2. 1_{st} order linear differential equations

Prof Ni started directly with the material. However, some of the content from *Differential equations and their applications* will be cited to improve readers comprehension. As present in the textbook, a differential equation is a relationship (namely the equation) between a function of time and its derivatives. In addition, a solution of a differential equation means a continuous function y(t) which together with its derivatives satisfies the relationship.

Definition 1.1 [1_{st} order linear DE]

$$y'(t) + a(t)y = b(t)$$

R Assume the coefficient of first derivative is 1.

The following are methods to solve to different kinds of DE. When $b \equiv 0$, the equation above is called **homogeneous** 1_{st} order DE.

$$y'(t) + a(t)y = 0$$

$$\frac{y'}{y} = -a(t)$$

$$ln|y| = -\int a(t)\,\mathrm{d}t + c$$

$$|y| = e^{-\int a(t)\,\mathrm{d}t}e^{c}$$

With c̃> 0, y is a continuous function

$$|y| = \tilde{c}e^{-\int a(t)\,\mathrm{d}t}$$

With $\bar{c} \in \mathbb{R}$,

$$y = \bar{c}e^{-\int a(t)\,\mathrm{d}t}$$

The above procedure shows that $y = \bar{c}e^{-\int a(t)\,\mathrm{d}t}$ is a solution of homogeneous 1_{st} order DE. When $b \not\equiv 0$, it is called **inhomogeneous**,

$$\begin{aligned} y &= ce^{-\int a(t)\,\mathrm{d}t} && ,c \text{ constant in } \mathbb{R} \text{ (homog)} \\ y &= c(t)e^{-\int a(t)\,\mathrm{d}t} && ,c(t) : \text{to be determined} \ldots (1) \end{aligned}$$

Take derivative of both sides of (1),

$$\Rightarrow y' = c(t)e^{-\int a(t)\,\mathrm{d}t}(-a(t)) + c'(t)e^{-\int a(t)\,\mathrm{d}t} = -a(t)y + c'(t)e^{-\int a(t)\,\mathrm{d}t}$$

Move $-a(t)y$ to the other side,

$$y' + a(t)y = c'(t)c'(t)e^{-\int a(t)\,\mathrm{d}t} = b$$

$$c'(t) = be^{\int a(t)\,\mathrm{d}t}$$

Integrate both sides, we get

$$\boxed{\Rightarrow c(t) = \int (b(t)e^{\int a(t)\,\mathrm{d}t})\,\mathrm{d}t} + \tilde{c}$$

Take it in (1), we can get the solution:

$$y(t) = \tilde{c}e^{-\int a(t)\,\mathrm{d}t} + e^{-\int a(t)\,\mathrm{d}t}\int [b(t)e^{\int a(t)\,\mathrm{d}t}\,\mathrm{d}t]$$

Integrating factor is another method to solve inhomogenous equations. The intuition is:

after multiplying both side by μ, a function of t,

$$\mu y'(t) + \mu a(t)y = b(t)\mu$$

if the left hand side of the equation equals to the derivative of a particular function $(y\mu)'$. There is

$$(y\mu)' = \mu b$$

Integrate both sides, we get $y\mu = \int(b\mu) + c$. Move μ to right-hand side we will get the solution.

To find $\mu(t)$ s.t. $\mu(y' + a(t)y) = (y\mu)' = y'\mu + y\mu'$ is to find $\mu a(t) = \mu'$.

Solve $\mu(t)$ under condition $\mu a(t) = \mu'$ is homogenuous which we have discuss how to solve before. (See the following.)

$$\frac{\mu'}{\mu} = a$$

$$ln|\mu| = \int a(t)\,\mathrm{d}t + c$$

$$|\mu| = e^{\int a(t)\,\mathrm{d}t}e^{c}$$

$$\mu = \pm e^{\int a(t)\,\mathrm{d}t}e^{c} = \tilde{c}e^{\int a(t)\,\mathrm{d}t}$$

Substitude to $\mu(y' + a(t)y) = \mu b$,

$$(ye^{\int a(t)\,\mathrm{d}t})' = be^{\int a(t)\,\mathrm{d}t}$$

Integrate both sides,

$$ye^{\int a(t)\,\mathrm{d}t} = \int(be^{\int a(t)\,\mathrm{d}t}) + C$$

Move $e^{\int a(t)\,\mathrm{d}t}$ to the right hand side,

$$y = e^{-\int a(t)\,\mathrm{d}t}(\int(be^{\int a(t)\,\mathrm{d}t}) + C)$$

■ **Example 1.1** Find y s.t.

$$\begin{cases} y' - 2ty = t \\ y(0) = 1 \end{cases}$$

$$y' - 2ty = 0$$

$$\frac{y'}{y} = 2t$$

$$ln|y| = t^2$$

$$y = ce^{t^2}, c \in \mathbb{R}(\text{This is the } \mu \text{ we want.})$$

$$(y' - 2ty)e^{-t^2} = te^{-t^2}$$

$$(ye^{-t^2})' = te^{-t^2}$$

Integrate both sides and move e^{-t^2} to the right hand side:

$$\begin{aligned} y &= e^{t^2}(\int te^{-t^2}\,\mathrm{d}t + c) \\ &= e^{t^2}(-\frac{1}{2}e^{-t^2}) + ce^{t^2} \\ &= -\frac{1}{2} + ce^{t^2} \end{aligned}$$

$$y(0) = 1$$

$$c = \frac{3}{2}$$

■

1.2. Wednesday

1.2.1. Is the solution unique?

Solve the last lecture's remaining question: Why there can not be other solutions to DE except the one we got from previous lecture?

The uniqueness of linear I.V.P solution.

$$\begin{cases} y' + a(t)y = b(t) \\ y(0) = y_0 \end{cases}$$

Suppose there are two sols $y_1(t)$, $y_2(t)$. Define function $f(t) = y_1(t) - y_2(t)$.
First observe that:

$$f(0) = y_1(0) - y_2(0) = y_0 - y_0 = 0$$

$$f' + a(t)f = y_1' + ay_1 - y_2' - ay_2 = b(t) - b(t) = 0$$

Therefore, we have change the question to show that with the properties,

$$\begin{cases} f' + a(t)f = 0 \\ f(0) = 0 \end{cases} \Rightarrow f \equiv 0$$

Proof. Define a function $Z(t) = \int_0^t |f(s)| \, \mathrm{d}s$. First let's devide the problem into two case; $t \geq 0$ and $t < 0$. Observe that $Z(t) \geq 0$ when $t \geq 0$ (the absolute value).
Next we want to show that $Z(t) \leq 0$ when $t \geq 0$
Integrate the first equation with first property.

$$\int_0^t f'(s) \, \mathrm{d}s = \int_0^t -a(s)f(s) \, \mathrm{d}s$$

Then, absolute both sides.

$$\begin{aligned}|f(t)| &= |\int_0^t (-a(s))f(s)\,\mathrm{d}s| \\ &\le \int_0^t |a(s)f(s)|\,\mathrm{d}s \\ &\le C\int_0^t |f(s)|\,\mathrm{d}s \qquad ,\forall |t| < T\end{aligned}$$

Differentiate $Z(t)$,

$$Z'(t) = |f(t)| \le C\int_0^t |f(s)|\,\mathrm{d}s = CZ(t)$$

Move right-hand side to the left and then multiply e^{-Ct}.

$$e^{-Ct}(Z' - CZ) \le 0$$

$$(e^{-Ct}Z)' \le 0$$

Integrate both sides from 0 to t,

$$e^{-Ct}Z(t) - e^{-C0}Z(0) \le 0$$

holds for $t \ge 0$

$$\Rightarrow Z(t) \le 0$$

Now, we have proven the $Z(t) \equiv 0$ for $t \ge 0$.

For $t < 0$ the situation is similiar. Observe $Z(t) \le 0$ (by definition). We want to show $Z(t) \ge 0$.With the same argument we have

$$(e^{-Ct}Z)' \le 0$$

Integrate both side from $-t$ to 0.

$$e^{-C_0}Z(0) - e^{-C_{-t}}Z(-t) \le 0$$

Therefore,

$$e^{-C_{-t}}Z(-t) \geq 0$$

Therefore, $Z \equiv 0$, $f \equiv 0$ for both $t \leq 0$ and $t \geq 0$. ■

R Inside this proof, I said the case $t < 0$ but I write $-t$ during the procedure.

1.2.2. Separable equations

Why. this topic come into hand? It is because sometimes we deal with more general DE which might be **non-linear**. (The form y in $y' + a(t)y = 0$ changes to a complicated $f(y)$ form.) We still want to use this sort of way to solve the equation, namely find the formula of $y(t)$.

$$y'(t) + a(t)y = 0$$

$$\frac{y'}{y} = -a(t)$$

$$ln|y| = -\int a(t)\,\mathrm{d}t + c$$

$$|y| = e^{-\int a(t)\,\mathrm{d}t}e^{c}$$

Definition 1.2 [Separable equation]

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{g(t)}{f(y)}$$

$$\int f(y)\,\mathrm{d}y = \int g(t)\,\mathrm{d}t$$

$$F(y) = G(t) + c$$

Then we might be able to use implicit function theorem to get $y(t)$ (locally).

■ **Example 1.2**

$$\begin{cases} y' = 1 + y^2 \\ y(0) = 0 \end{cases}$$

$$\int \frac{\mathrm{d}y}{1+y^2} = \int \mathrm{d}t$$

$$\tan^{-1} y = t + c$$

$$y = \tan(t+c)$$

$$0 = y(0) = \tan(0+c) = \tan c$$

$$\Rightarrow c = k\pi \qquad \text{k:integer}$$

Is this contradict to the uniqueness we just proved? ■

R

[1] The difference between linear and non-linear DE is that the domain of solution we got for l.DE is the whole domain, while the domain of solution for non-linear DE mightbe a small neighbourhood depend on initial value.

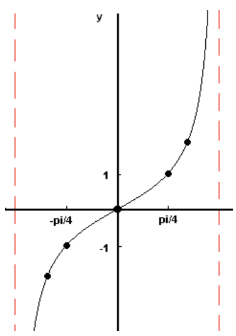


Figure 1.1: Graph of solution

There is "nothing" outside $\frac{\pi}{2}$. In fact, there might be some graph satisfied the equation. However, it is non-sense because we can not specify the value of $y(t)$ outside $\frac{\pi}{2}$.

[2] See definition of *direction field* in text book.

■ **Example 1.3**

$$\begin{cases} y' = 1 + y^2 \\ y(0) = 1 \end{cases}$$

$\Rightarrow \quad \ldots y = \tan(t + c) \qquad 1 = y(0) = \tan c \Rightarrow c = \frac{\pi}{4} + k\pi$

$$y = \tan(t + \frac{\pi}{4})$$

[Difference equation of the example 1.2]

$$\frac{y(t_2) - y(t_1)}{t_2 - t_1} = 1 + y^2(t_1)$$

Difference equation can be used to approximate the behaviour of the solution of DE. Take $t_2 = t_1 + 1$, $y(0) = 0, y(1) = 1, y(2) = 3, y(3) = 13, y(4) = 183, \ldots$, we roughly guess the behaviour of $y(t)$. Although, this is not quite accurate.

1.2.3. Population method

$P(t)$: total population at time t.

$r(t, p)$: growth rate (birthrate-death rate)

Malthus theory: $\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}t} = r$ is a constant. (The differentiation of population by time $\frac{\mathrm{d}p}{\mathrm{d}t}$ is rp)

$$lnp = rt + c$$

$$p(t) = e^c e^{rt} = \tilde{c}e^{rt}$$

$$p(t) = p(0)e^{rt}$$

This is imposible considering the limited nature resource.

Logistic equation(Verhulst 1837)

$$\begin{cases} p' = p(a - bp) \\ \quad p(0) = p_0 \end{cases}$$

$$\int \frac{\mathrm{d}p}{p(a-bp)} = \int \mathrm{d}t = t + c$$

To integrate left-hand side, we need a little trick:

$$\frac{1}{p(a-bp)} = \frac{A}{p} + \frac{B}{a-bp}$$

$$\begin{aligned} \frac{1}{p(a-bp)} &= \frac{A}{p} + \frac{B}{a-bp} \\ &= \frac{aA + (B-bA)p}{p(a-bp)} \end{aligned}$$

$\Rightarrow A = \frac{1}{a}$, $B = \frac{b}{a}$

$$\begin{aligned} \int \frac{\mathrm{d}p}{p(a-bp)} &= \frac{1}{a}\int \frac{1}{p}\mathrm{d}p + \frac{b}{a}\int \frac{\mathrm{d}p}{a-bp} = t + c \\ &= \frac{1}{a} lnp - \frac{1}{a} ln|a-bp| = \frac{1}{a} ln \frac{p}{|a-bp|} \end{aligned}$$

$$\boxed{\frac{p}{|a-bp|} = \tilde{c}e^{at}, \tilde{c} > 0}$$

Chapter 2

Week2

2.1. Monday

2.1.1. Logistic equations

$$\frac{\mathrm{d}p}{\mathrm{d}t} = p(a - bp)$$

where a is intrinsic equation, $\frac{a}{b}$ is carrying capacity, and p_0: the initial population $= p(0)$.

$$\int \frac{\mathrm{d}p}{p(a-bp)} = \int \mathrm{d}t$$

$$\cdots$$

$$\frac{p}{|a-bp|} = e^{at+c}$$

In order to get away from absolute value we need to separate the initial value into different cases.

1. $p_0 = \frac{a}{b}$ $\left.\begin{aligned}\frac{\mathrm{d}p}{\mathrm{d}t} = p(a-bp)\\ p(0) = p_0\end{aligned}\right\}. \Rightarrow P(t) \equiv P_0$ By uniqueness.
2. $P_0 < \frac{a}{b}$ Then we can get away from absolute value. However, here comes the question;

   Why p_t never reaches $\frac{a}{b}$? (Uniqueness)

$$\frac{p}{a-bp} = \tilde{c}e^{at}, \tilde{c} = \frac{P_0}{a-bP_0}(= e^c)$$

$$
\begin{aligned}
P =& (a-bp)\tilde{c}e^{at} \\
=& a\tilde{c}e^{at} - b\tilde{c}pe^{at} \\
p(1+b\tilde{c}e^{at}) =& a\tilde{c}e^{at} \\
p =& \frac{a\tilde{c}e^{at}}{1+b\tilde{c}e^{at}}
\end{aligned}
$$

Move the upper $\tilde{c}e^{at}$ down and pluge $\tilde{c}$ in;

$$p = \frac{a}{b + \frac{a-bP_0}{P_0}e^{-at}}$$

3. $P_0 > \frac{a}{b}$

$$\frac{P}{-(a-bP)} = \frac{P_0}{-(a-bP_0)}e^{at}$$

$$\ldots \boxed{P(t) = \frac{a}{b+\frac{a-bp_0}{p_0}e^{-at}} \to \frac{a}{b}} \text{ as } t \to \infty.$$

Question: $\begin{cases} \dfrac{\mathrm{d}p}{\mathrm{d}t} = p(a-bp) \\ p(0) = p_0 \end{cases}$ At time s, $P(s) = P_s$, for P is a dummy variable, we can write a another linear DE;

$$\begin{cases} \dfrac{\mathrm{d}q}{\mathrm{d}t} = q(a-bq) \\ q(0) = p_s \end{cases}$$

For the solution of those two linear DEs, is $P(t) = q(t-s)$?

Proof. From above disscussion, we know that the solution of these two DE is:

$$P(t) = \frac{a}{b+\frac{a-bP_0}{P_0}e^{-at}}$$

$$q(t-s) = \frac{a}{b+\frac{a-bq_0}{q_0}e^{-a(t-s)}}$$

With the second equation of second DEs; $q(0) = p_s$

$$q(t-s) = \frac{a}{b+\frac{a-bP_s}{P_s}e^{-a(t-s)}} \ldots (1)$$

$$P_s = P(s) = \frac{a}{b + \frac{a-bP_0}{P_0} e^{-a(s)}}$$

Substitude P_s into (1), we get;

$$q(t-s) = \frac{a}{b + \frac{a-bP_0}{P_0} e^{-at}} = P(t)$$

■

R This question shows that the solution of the linear differential equations is unique regardless of time (when the time start).

■ **Example 2.1** A commercial fish population is estimated to have carrying capacity 10,000kg of certain kind of fish. Suppose the annual growth of the total fish population is governed by $\frac{dp}{dt} = p(1 - \frac{p}{10{,}000})$ and initially there are 2,000kg fish.

(1) What is the fish population after 1 year?Suppose, after waiting for a certain period of time, the owner of the fishery decide to harvest 2,400kg of annually at a constant rate.

(2) What is the differential equations governing the fish population now?

(3) Draw the slope field of the equation in (2).

(4) What is the minimal waiting period you would recommand to the owner (before harvesting)?

Answer:

(1)

$$\frac{dp}{dt} = p(1 - \frac{p}{10000})$$

$$\int \frac{dp}{p(1-\frac{p}{10000})} = \int dt$$

$$\frac{1}{p(1-\frac{p}{10000})} = \frac{A}{P} + \frac{B}{1-\frac{P}{1000}} = \frac{A(1-\frac{p}{10000}) + BP}{p(1-\frac{p}{10000})} = \frac{A + (B - \frac{A}{10000}P)}{p(1-\frac{p}{10000})}$$

Then substitude $A = 1$, $B = \frac{1}{10000}$ to the second equation;

$$\rightarrow \int (\frac{1}{p} + \frac{\frac{1}{10000}}{1 - \frac{p}{10000}}) \, \mathrm{d}p = t + c$$

$$lnp - ln(1 - \frac{p}{10000}) = t + c$$

$$ln \frac{p}{1 - \frac{p}{10000}} = t + c$$

$$\frac{p}{1 - \frac{p}{10000}} = e^t e^c \qquad \ldots (1)$$

(1) is the equation shows relation between fish population and time,now substitude the initial value into it, it means when $t = 0$;

$$\frac{2000}{1 - \frac{2000}{10000}} = e^c = 2500$$

Substitude it back to (1),

$$p = (1 - \frac{p}{10000}) 2500 e^t$$

We can get the formula of fish population.

$$p = \frac{2500e^t}{1 + \frac{1}{4}e^t}$$

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{p}{1 - \frac{p}{10000}} - 2400$$

$$p = \frac{2500e^t}{1 + \frac{1}{4}e^t}$$

(2)

$$\frac{\mathrm{d}p}{\mathrm{d}t} = p(1 - \frac{p}{10000}) - 2400$$

(3)

$$\begin{aligned}\frac{\mathrm{d}p}{\mathrm{d}t} &= p - \frac{p^2}{10000} - 2400 \\ &= \frac{-1}{10000}(p^2 - 10000p + 24000000) \\ &= \frac{-1}{10000}(p - 4000)(p - 6000)\end{aligned}$$

Now we can draw the slope field:

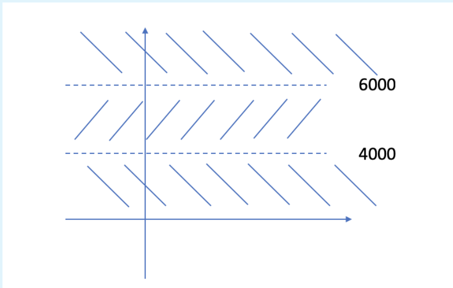


Figure 2.1: Slope field

(4) One year. From calculation, after one year the population of fish is around 4046 ($P(1)$) which is greater than 4000. This implies the number of fish wouldn't decline as time goes by as we can see from slope field.

2.2. Monday-tutorial

Newton's Cooling Law. The rate of change of the temperature $T(t)$ of a body immersed in a medium of constant temperature A is proportional to $A - T$; i.e.

$$\frac{\mathrm{d}T}{\mathrm{d}t} = k(A - T), \qquad k \text{ a constant}$$

$$\begin{cases} \dfrac{\mathrm{d}T}{\mathrm{d}t} = k(A - T) \\ T(0) = T_0 \end{cases}$$

$$\int \mathrm{d}TA - T = \int k\,\mathrm{d}t$$

$$-ln|A - T| = kt + c$$

$$|A - T| = e^{-c}e^{-kt}$$

$$\text{At time } t = 0, \qquad |A - T_0| = e^{-c}$$

$$|A - T| = |A - T_0|e^{-kt}$$

1. $T_0 = A$: $\Rightarrow \quad T(t) \equiv A \qquad \forall t \geq 0$
2. $T_0 > A$: $\Rightarrow \quad T(t) > A$ (Never reach A, the same reason illustrated in the lecture. In addition, this is only true when it is first order linear DE.)

$$T - A = (T_0 - A)e^{-kt} \to 0 \qquad \text{as } t \to \infty$$

   Therefore, $T(t) \to A$ as $t \to \infty$.
3. $T_0 < A$: $\Rightarrow \quad T(t) < A \qquad \forall t \geq 0$

$$A - T(t) = (A - T_0)e^{-kt} \to 0 \qquad \text{as } t \to \infty$$

■ **Example 2.2** temperature of room A, temperature of coffe T_c, amount of coffee 1, temperature of cream T_m, amount of cream $r(\ll 1)$, Boy: adds cream at $t = 0$, Girl adds cream at $t = 10$. Question: whose cream is cooler when they drink at $t = 10$? **Answer:generally girl's**

Proof. First, let's calculate the temperature of boy's coffee at the beginning:

$$T_b(0) = \frac{T_c + rT_m}{1 + r}$$

After tem minutes;

$$T_b(10) = (T_b(0) - A)e^{-10k} + A$$

Temperature of girl's coffee at the beginning $T_g(0) = T_c$ and after ten minutes:

$$T_g(10) = (T_c - A)e^{-10k} + A$$

After girl added cream, the temperature of coffee is:

$$T'_g(10) = \frac{T_g(10) + rT_m}{1 + r}$$

Now let's see which one is cooler;

$$\begin{aligned}
T_b(10) - T'_g(10) &= (\frac{T_c + rT_m}{1 + r} - A)e^{-10k} + A - \frac{(T_c - A)e^{-10k} + A + rT_m}{1 + r} \\
&= \frac{r(A - T_m)}{1 + r} + e^{-10k}\frac{r(T_m - A)}{1 + r} \\
&= \frac{r(A - T_m)}{1 + r}(1 - e^{-10k})
\end{aligned}$$

(1)$T_m = A \Rightarrow T_b(10) = T'_g(10)$

(2)$T_m < A \Rightarrow T_b(10) > T'_g(10)$

(3)$T_m > A \Rightarrow T_b(10) < T'_g(10)$

For most cases, the temperature of cream should be less than the temperature of the room, hence, girl's coffee is cooler than boy's. ■

2.3. Wednesday

2.3.1. Application

Mixture problem. There are two things that you need to bear in mind:

$\frac{dy}{dt}$ = input rate - output rate, carry the units.

■ **Example 2.3** A 120-gal tank initially contain 90kg salt dissoved in 90-gal of water. Brine containing 2kg/gal of salt flows into the tank at the water. Brine containing 2kg/gal of salt flows into the tank at the rate of 4gal/min, & the well-stirned mixture flows out at the rate of 3 gal/min. How much salt does the tank contain when it is full?

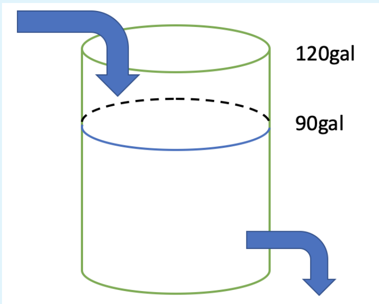


Figure 2.2: First

Set $y(t)$ = the amount of salt at time t,
$V(t)$ =the amount of brine at time t= 90 +t(gal)(t:minute)

$$\begin{aligned} y'(t) &= 2\text{kg/gal} \cdot 4\text{gal/min} - \frac{y(t)}{V(t)}\text{kg/gal} \cdot 3\text{gal/min} \\ &= 8\text{kg/min} - 3\frac{y(t)}{V(t)}\text{kg/min} \end{aligned}$$

$$y' = 8 - 3\frac{y}{90+t}$$

$$y' + \frac{3}{90+t}y = 8$$

By intergrating factor,

$$(e^{\int \frac{3}{90+t}\,dt} \cdot y)' = 8e^{\int \frac{3}{90+t}\,dt}$$

Try to simplify $e^{\int \frac{3}{90+t}\,\mathrm{d}t}$;

$$\int \frac{3}{90+t}\,\mathrm{d}t = 3ln|90+t| + \tilde{c}$$
$$= 3ln(90+t) + \tilde{c}$$

Then,

$$e^{\int \frac{3}{90+t}\,\mathrm{d}t} = (90+t)^3 \cdot c$$

Integrate both sides,

$$(90+t)^3 \cdot c \cdot y = 8\int (90+t)^3 \cdot c\,\mathrm{d}t$$

$$(90+t)^3 y = 2\cdot(90+t)^4 + C$$

$$y = 2(90+t) + \frac{c}{(90+t)^3}$$

$$y(0) = 90 = 180 + \frac{c}{90^3} \quad \Rightarrow \quad c = -90^4$$

$$y = 2(90+t) - \frac{90^4}{(90+t)^3}$$

At $t = 30$ the tank is full & $y(30) = 240 - \frac{90^3}{120^3}\cdot 90$ ■

■ **Example 2.4** [Persuit problem] In a naval excercise, a distroyler D is hunting a submarine S. Suppose D at $(9,0)$ detects S at $(0,0)$ & at the same time S detects D. Assuming that S will dive immediately & depart at full speed, 15mile/hr in a straight course of unknown direction. What path should the destroyer D follows to be certain of passing directly over the submarine S. If the speed of D is 30mile/hr at all time of the pursuit?

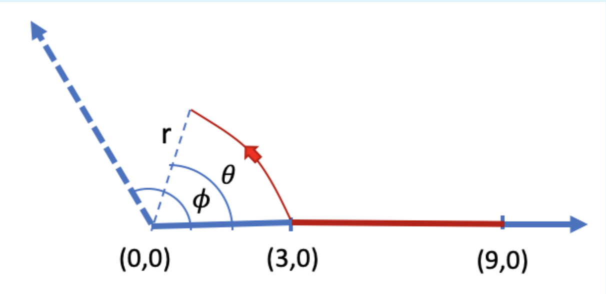


Figure 2.3: The path of destroyler in red.

Distance D has travelled$=6+\int_0^{\phi}\sqrt{(r(\theta))^2+(r'(\theta))^2}\,\mathrm{d}\theta$.

Distance S has travelled$=r(\phi)$.

For the speed of D is twice than that of S, with the same period of time:

$$6+\int_0^{\phi}\sqrt{(r(\theta))^2+(r'(\theta))^2}\,\mathrm{d}\theta=2r(\phi)$$

This isn't a form that can be dealt with. Differentiate both side,

$$\sqrt{(r(\phi))^2+(r'(\phi))^2}=2r'(\phi)$$

$$(r(\phi))^2+(r'(\phi))^2=4(r'(\phi))^2$$

$$r'(\phi)=\pm\frac{1}{\sqrt{3}}r(\phi)$$

W.L.O.G pick plus sign $\frac{r'}{r}=\frac{1}{\sqrt{3}}$

$$lnr=\frac{1}{\sqrt{3}}\phi+\tilde{c}$$

$$r=e^{\frac{\phi}{\sqrt{3}}}\cdot c$$

$$r(0)=3=c$$

$$r(\phi)=3e^{\frac{\phi}{\sqrt{3}}}$$

Given the direction S take ϕ, if D want to be above it, they would reach each other at $(\phi,3e^{\frac{\phi}{\sqrt{3}}})$. This implies that if D follows the path of $r(\theta)=3e^{\frac{\theta}{\sqrt{3}}}$, D can catch S whichever direction S take. ■

R There is a link about basic idea of how to compute the length of a curve: http://tutorial.math.lamar.edu/Classes/CalcII/ArcLength.aspx. Check it if you are interested.

Orthoganal trajectories. Given a family of curve $f(x,y,c)=0$. To find its orthoganal trajectories, the slope of the graph is needed. Differentiate F by x;

$$F_x + F_y y_x = 0$$

Slope is $y_x = -\frac{F_x}{F_y}$. Then the slope of the graph that is orthoganal to the original one is

$$y_x = \frac{F_y}{F_x} = \frac{\frac{\partial F}{\partial y}(x,y,c)}{\frac{\partial F}{\partial x}(x,y,c)}$$

■ **Example 2.5** $y = cx^2$ c:parameter. To find another family of curves which is orthogonal to $y = cx^2$ whenever they intersect with each other.

$y = cx^2$, $\frac{\mathrm{d}f}{\mathrm{d}x} = 2cx$ O.T. $\rightarrow$ $\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{-1}{2cx} = \frac{-1}{2\frac{y}{x^2}x} = -\frac{x}{2y}$ (A separable equation)

$2y\,\mathrm{d}y + x\,\mathrm{d}x = 0 \quad \rightarrow \quad y^2 + \frac{1}{2}x^2 = c$ ■

Chapter 3

Week3

3.1. Monday

3.1.1. Uniqueness of first ODE (Include non-linear)

Question statement:. Consider

$$\begin{cases} y' = f(t,y) \\ f(t_0) = y_0. \end{cases}$$

There exists a unique function y that satisfies those equations near (t_0, y_0)

Before the proof begin, there is something need to be clarified. First f and f_y are continuous function on $\mathcal{R} : [t_0 - a, t_0 + a] \times [y_0 - b, y_0 + b]$ (a set of point that constitutes a rectangular area). Second $M = \sup_R |f(t,y)|$ $L = \sup_R |f_y(t,y)|$ (f_y means $\frac{\partial f}{\partial y}$).

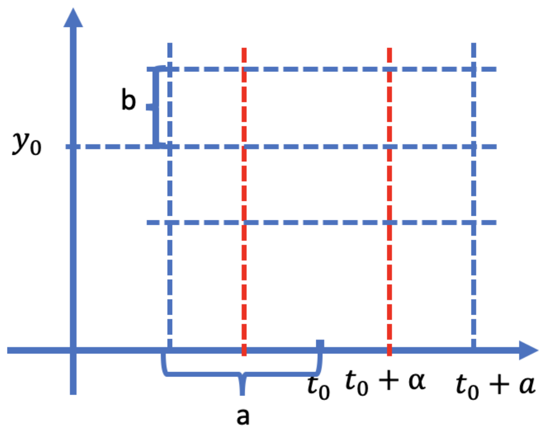


Figure 3.1: Domain

Proof. Existence: $\exists\alpha > 0$, s.t. ODE has a solution on $(t_0 - \alpha, t_0 + \alpha)$. There are two methods mentioned during lecutre. The first is the one using contraction mapping. The other is **Picard iteration**:(set $t_0 = 0$) Observe that:

$$y(t) - y(0) = \int_0^t f(s, y(s))\,\mathrm{d}s$$

$$y(t) = y(0) + \int_0^t f(s, y(s))\,\mathrm{d}s$$

This equation has the same solution as the ODE. (Why is that? Think about it.) We are looking to this integral equation which are continuous. Approximating solutions

$$y_0(t) = y_0 \quad \forall t$$

$$y_1(t) = y_0 + \int_0^t f(s, y(s))\,\mathrm{d}s$$

$$\dots$$

$$y_{n+1}(t) = y_0 + \int_0^t f(s, y_n(s))\,\mathrm{d}s$$

$$\dots$$

$$\{y_0, y_1, y_2, \dots, y_n, \dots\}$$

If the sequence of those functions converges uniformly, say it converges to g. Then we can find out that g is the function we are looking for. ($\lim_{n+1\to\infty} y_{n+1}(t) = y_0 + \lim_{n\to\infty} \int_0^t f(s, y_n(s))\,\mathrm{d}s$). W.T.S. uniform convergnece:

$$y_n(t) = y_0(t) + (y_1(t) - y_0(t)) + \cdots + (y_n(t) - y_{n-1}(t))$$

Without loss of generality, consider $t > 0$

$$|y_1(t) - y_0(t)| \leq \int_0^t |f(s, y_0(s))|\,\mathrm{d}t \leq M \int_0^t \mathrm{d}s = Mt \quad \dots(1)$$

$$\begin{aligned}
|y_2 - y_1(t)| &\le \int_0^t |f(s,y_1(s) - f(s,y_0(s))|\,\mathrm{d}s \\
&= \int_0^t |f_y(s,\xi(s))||y_1(s) - y_0(s)|\,\mathrm{d}s \qquad \text{mean value theorem} \\
&\le LM\int_0^t s\,\mathrm{d}s = \frac{LMt^2}{2} \qquad \text{with (1)}
\end{aligned}$$

$$\begin{aligned}
|y_3(t) - y_2(t)| &\le \int_0^t |f(s,y_2(s)) - f(s,y_1(s))|\,\mathrm{d}s \\
&= \int_0^t |f_y(s,\xi_1(s))||y_2(s) - y_1(s)|\,\mathrm{d}s \\
&\le \frac{L^2M}{2}\int_0^t s^2\,\mathrm{d}s = \frac{L^2Mt^3}{3\cdot 2\cdot 1}
\end{aligned}$$

The same computation implies $|y_n(t) - y_{n-1}(t)| \le \frac{L^{n-1}M}{n!}t^n$.

Claim $\{y_0, y_1, y_2, \dots, y_n, \dots\}$ is a Cauchy sequence i.e. for $|t| \le \alpha$ $n > m$

$$\begin{aligned}
|y_n(t) - y_m(t)| &\le |y_n(t) - y_{n-1}(t)| + |y_{n-1}(t) - y_{n-2}(t)| + \cdots + |y_{m+1}(t) - y_m(t)| \\
&\le \frac{L^{n-1}M}{n!}t^n + \frac{L^{n-2}M}{(n-1)!}t^{n-1} + \cdots + \frac{L^mM}{(m+1)!}t^{m+1} \\
&\le \frac{M}{L}[e^{Lt} - (1 + Lt + \cdots + \frac{(Lt)^m}{m!})] \qquad \text{Taylor expansion} \\
&\le \frac{M}{L}[e^{L\alpha} - (1 + L\alpha + \cdots + \frac{(L\alpha)^m}{m!})] \to 0 \text{ as } m \to \infty
\end{aligned}$$

Before moving to prove the uniqueness of the solution, let's have a look at why the solution can only be gotten within a small neighbourhood ($t < |\alpha|$). The leak appear when we use mean value theorem. $\xi_n(s)$, the partial y direvative of f, which is between $y_{n+1}(s)$ and $y_n(s)$ need to be inside the domain. This is the same as every $y_n(s)$ needs to be inside domain.(Shown by induction)

$$|y_1(t) - y_0(t)| \le \int_0^t |f(s,y_0(s))|\,\mathrm{d}s \le M\int_0^t \mathrm{d}s = Mt \le M\alpha \le b$$

$$|y_{n+1} - y_0| \le \int_0^t |f(s,y_n(s))|\,\mathrm{d}s \le M\int_0^t \mathrm{d}s = Mt \le M\alpha < b$$

$$y_{n+1} = y_0 + \int_0^t f(s,y_n(s))\,\mathrm{d}s$$

Therefore, every y_n lies in $|t_0| < \alpha$.

Uniqueness

Suppose the ODE has two sols y_1 and y_2

$$y_1(t) = y_0 + \int_0^t f(s, y_1(s))\,\mathrm{d}s$$

$$y_2(t) = y_0 + \int_0^t f(s, y_2(s))\,\mathrm{d}s$$

$$\begin{aligned} |(y_1 - y_2)(t)| &\le \int_0^t |f(s, y_1(s)) - f(s, y_2(s))|, (t > 0) \\ &\le \int_0^t |f_y(s, \xi(s))||y_1(s) - y_2(s)|\,\mathrm{d}s \end{aligned}$$

$$|(y_1 - y_2)(t)| \le L \int_0^t |y_1(s) - y_2(s)|\,\mathrm{d}s \qquad |t| \le \alpha \qquad \ldots(2)$$

Set $z(t) = \int_0^t |y_1(t) - y_2(t)|\,\mathrm{d}s \Rightarrow z'(t) = |y_1(t) - y_2(t)| \le Lz(t)$ by (2)

$$z'(t) - Lz(t) \le 0$$

$$(e^{-Lt} z(t))' \le 0$$

$$e^{-Lt} z(t) \le 0$$

$$z(t) \le 0 \quad \Rightarrow |y_1(s) - y_2(s)| = 0$$

■

■ **Example 3.1**

$$y' = \frac{3}{2} y^{\frac{1}{3}}$$

$y \equiv 0$ and $y = t^{\frac{3}{2}}$ are solutions of above. Why this is the case? We just proved uniqueness of ODE.

The reason is that it doesn't fit the assumption that f_y is continuous. ■

3.2. Wednesday

3.2.1. Condition for existence and uniqueness of first ODE (Include non-linear)

Consider $\begin{cases} y' = f(t,y) \\ y(t_0) = y_0. \end{cases}$ Recall that when f and $\frac{\partial f}{\partial y}$ are continuous on a nbhd of (t_0, y_0), there exists a unique solution to (IVP). There comes a question: what are the least conditions required for existence of a solution?

It only require f is continuous.

Theorem 3.1 Suppose that $f(t,y)$ is continuous on $\mathcal{R} = [t_0 - a, t_0 + a] \times [y_0 - b, y_0 + b]$ $a, b > 0$. Then IVP has a solution $y = y(t)$ in $(t_0 - \alpha, t_0 + \alpha)$ where $\alpha = min\{a, \frac{b}{m}\}$ $M = max_{\mathcal{R}}|f(t,y)|$

Proof outline:

1. $\{y_0, y_1, \ldots, y_n, \ldots\}$ is an equicontinuous family.
2. Arzela-Ascoli theorem implies there exists a convergent subsequence and the limit is our sol.

Manual Script. y_n is in $[y_0 - b, y_0 + b]$ which means it is bounded uniformly. In addition, for any ε there exists $\delta = \frac{\varepsilon}{M}$ such that when $|t_2 - t_1| < \delta$, $y_n(t_1) - y_n(t_2) = \int_{t_1}^{t_2} f(s, y_{n-1}(s))\, \mathrm{d}s \leq M \int_{t_1}^{t_2} 1\, \mathrm{d}s = M(t_2 - t_1) = \varepsilon$. Therefore, $\{y_0, y_1, \ldots, y_n, \ldots\}$ is indeed an equicontinuous. Then by Arzela-Ascoli theorem there exists a uniformly convergent subsequence. ■

Rather nothing more than 1. is proved, if you are interested in the complete prove, check *Theory of ordinary differential equations* by Coddington & Levinson The link of the book. Though the construct of sequence of approximating functions is slightly different, the main idea is the same.

Condition for uniqueness to hold. Replace "$\frac{\partial f}{\partial y}$" being continuous on $\mathcal{R}$ "by" $|f(s,y_1) - f(s,y_2)| \leq L|y_1 - y_2|$ (Lipshitz continuity)

R

- Intuition of uniqueness

  Consider a simple case $\begin{cases} y' = f(t,y) \\ y(0) = 0. \end{cases}$

  $y \equiv 0$ where $f(0) = 0$, $f(y) \geq 0$, $f(y) \leq Cy$, for $y \geq 0$ small.(y increases in $t > 0$ small).

  Suppose $\exists$ sol$\neq 0$, $t \geq 0$

$$
\begin{aligned}
0 \leq y(t) = \int_0^t f(y(s))\,\mathrm{d}s &\leq C\int_0^t y(s)\,\mathrm{d}s \\
&\leq Cy(t)\int_0^t 1\,\mathrm{d}s \\
&= Cty(t) \\
&\Rightarrow \quad 1 \leq Ct
\end{aligned}
$$

  For C is a constant, when $t < \frac{1}{2C}$, the above inequality doesn't hold. Therefore, a contradiction appears. This implies that the only solution must be identically 0.

  $f(y) \leq Cy$ is because of $f'(0)$. The derivative exists implies that $f(y)$ can be expressed linearly. Therefore, the derivative at initial value point leads to the uniqueness of the solution. (Personal interpretation of professor's words, check that.)

- Picard iteration

  This is an example shows how picard iteration is working.

■ **Example 3.2** $\begin{cases} y' = 1 + y^2 \\ y(0) = 0. \end{cases} \quad \Rightarrow y = \tan t$

$$\begin{aligned}
y_0(t) &\equiv 0 \\
y_1(t) &= \textstyle\int_0^t f(s, y_0)\,\mathrm{d}s &&= \textstyle\int_0^t 1\,\mathrm{d}s = t \\
y_2(t) &= \textstyle\int_0^t f(s, y_1)\,\mathrm{d}s &&= \textstyle\int_0^t 1 + s^2\,\mathrm{d}s = t + \frac{1}{3}t^3 \\
y_3(t) &= \textstyle\int_0^t (1 + y_2^2)\,\mathrm{d}s &&= \textstyle\int_0^t 1 + s^2 + \frac{2}{3}s^4 + \frac{1}{9}s^6\,\mathrm{d}s \\
& &&= t + \tfrac{1}{3}t^3 + \tfrac{2}{15}t^5 + \tfrac{1}{63}t^7 \\
y_4 &= \dots
\end{aligned}$$

$$tant = t + \frac{1}{3}t^3 + \frac{2}{15}t^5 + \frac{17}{315}t^7 + \frac{62}{2835}t^9 = \dots$$

It's easy to observe that the approximating function is meaningful at any point. While, the solution is only meaningful at$(-\frac{\pi}{2}, \frac{\pi}{2})$.(Shown before, check note.)

PS: Just show you how complicated the taylor expansion of a common function can be.

$$\tan t = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+2}(2^{2n+2} - 1)B_{2n+2}t^{2n+1}}{(2n+2)}$$

where B_n is a Bernoulli number defined by $\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!}x^n$ ■

3.2.2. Exact Equations

"We can solve all differential equations which are, or can be put, in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(t, y) = 0$$

for some function $\phi(t, y)$ "(Martin, p58). If we want such ϕ exists, it has to satisfies $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$. (That s why the definition requires this.)

Definition 3.1 The differential equation

$$M(t,y)+N(t,y)\frac{\mathrm{d}y}{\mathrm{d}t}=0$$

is said to be exact if $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial t}$.

Theorem 3.2 Given $M(t,y)$, $N(t,y)\in C'(\bar{\mathcal{R}})$ where $\bar{\mathcal{R}}=[a,b]\times[c,d]$. Then $\exists\ \phi$ s.t. $M=\phi_t$ $N=\phi_y$ if and only if $M_y=N_t$

Proof. ($\Rightarrow$) $M=\phi_t$ and $N=\phi_y \Rightarrow\ M_y=\phi_{ty}=\phi_{yt}=N_t$ ($\Leftarrow$) let $\phi(t,y)=\int M(t,y)\,\mathrm{d}t+\int[N(t,y)-\int\frac{\partial M}{\partial y}\,\mathrm{d}t]\,\mathrm{d}y$ (If you are wondering why there is such a come-from-nowhere function, see textbook.)

$$\phi_t=M(t,y)+\underbrace{\int N_t\,\mathrm{d}y-\int M_y\,\mathrm{d}y}_{=0}$$

$$\phi_y=\int M_y\,\mathrm{d}t+N-\int M_y\,\mathrm{d}t=N$$

■

Integrating factor might be helpful to find $\phi(t,y)$, need

$$(\mu M)_y=(\mu N)_t$$

$$\mu M_y+\mu_y M=\mu_t N+\mu N_t$$

This is a partial differential equation. (We want μ) It seems we have dig a hole for ourselves. However, if μ is a function only concern one of t and y, life would be easier. If $\mu=\mu(t)$ $\mu(M_y-N_t)=\mu_t N$, $\frac{\mu_t}{\mu}=\frac{M_y-N_t}{N}$. When $\mu=\mu(y)$, the situation is similiar. (There is a example 6 in p65 in textbook.)

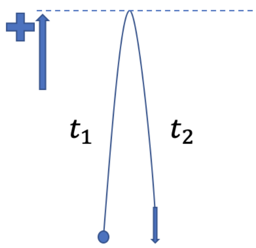
Chapter 4

Week4

4.1. Monday

4.1.1. Which part of motion last longer?

Now we consider throwing a ball straight upward. At first, it will go up. Then because of gravity, it will goes down. If air friction is considered, which part of motion experience more time?



In order to make content complete, first, let's consider the procedure without any friction.

$$mv' = -mg$$

$$v(t) = -gt + v_0$$

,where v_0 is a initial velocity.

$$v(T_{up}) = v_0 - gT_{up}$$

$$T_{up} = \frac{v_0}{g},\ T_{down} = \frac{v_0}{g}$$

Now let's consider the case where air friction exists. By recent discovery, relationship between air friction and the speed of the objection is; $F = kv^p$ where p $(1 \leq p \leq 2)$ is a constant related to the speed, i.e. when the speed is high, p is aproximately 2; otherwise, p is more close to 1 and k is a constant related to the shape of the object.

$$F = -kv|v|$$

(This hold nomatter p is equal to 1 or 2 where v is a vector.) To make our life easier, take $p = 1$,

$$mv' = -mg - kv$$

$$v' = -g - \frac{k}{m}v$$

Let $\rho = \frac{k}{m}$

$$v' + \rho v = -g$$

$$e^{\rho t}(v' + \rho v) = -ge^{\rho t}$$

Take integral on both side of the equation,

$$e^{\rho t}v(t) - v(0) = -\int_0^t ge^{\rho s}\,\mathrm{d}s = -\frac{g}{\rho}(e^{\rho t} - 1)$$

$$e^{\rho t}v(t) = v_0 + \frac{g}{\rho} - \frac{g}{\rho}e^{\rho t}$$

$$v(t) = e^{-\rho t}(v_0 + \frac{g}{\rho}) - \frac{g}{\rho}$$

$$v(T_{up}) = 0 = e^{-\rho T_{up}}(v_0 + \frac{g}{\rho}) - \frac{g}{\rho}$$

$$v_0 + \frac{g}{\rho} = \frac{g}{\rho}e^{\rho T_{up}}$$

$$\boxed{e^{\rho T_{up}} = \frac{v_0 + \frac{g}{\rho}}{\frac{g}{\rho}}}$$

Max height $H = \int_0^{T_{up}} v(t)\,\mathrm{d}t$

$$T_{down} = T_{total} - T_{up}$$

T_{total} is given by $h(T_{total}) = 0$, where $h(t) = \int_0^t v(s)\,\mathrm{d}s$

$$h(t) = \int_0^t [e^{\rho t}(v_0 + \frac{g}{\rho}) - \frac{g}{\rho}]\,\mathrm{d}s = -\frac{1}{\rho}(e^{-\rho t} - 1)(v_0 + \frac{g}{\rho}) - \frac{g}{\rho}t$$

$h(total) = h(0)$ is given by $\frac{1}{\rho}(e^{-\rho T_{total}} - 1)(v_0 + \frac{g}{\rho}) = \frac{g}{\rho}T_{total}$

Question: Can you see $T_{up} < T_{down}$?

Answer: Interesting.

$k = 0$(no air)

$T_{up} = \frac{v_0}{g}$

$$\begin{aligned} H &= \int_0^{T_{up}} (-gt + v_0)\,\mathrm{d}t \\ &= (-\frac{g}{2}t^2 + v_0 t)|_0^{\frac{v_0}{g}} \\ &= \frac{v_0^2}{g} - \frac{g}{2}(\frac{v_0}{g})^2 = \frac{v_0^2}{2g} \end{aligned}$$

$$T_{total} = 2\frac{v_0}{g}$$

$k > 0 \quad T_{up} = \frac{1}{\rho}ln(1 + \frac{g}{\rho}v_0) \quad H = h_0 + \frac{v_0}{\rho} - \frac{g}{\rho}T_{up}$

Question: please examine when $\rho \to 0$ everything goes to the right quantity.

■ **Example 4.1** A bolt shot straight upward with initial velocity 49m/sec from a cross bow at ground level. With air resistance take into account. Assuming the constant $\rho = 0.04$ (and $\rho = 0.2$ for comparison). Compute T_{up} and T_{down}.

$P = 0.04$ $v_0 = 49$, $g = 9.8m/sec^2$

$$T_{up} = \frac{1}{0.04}ln(1 + 0.04 \cdot 5) = \frac{1}{0.04}ln(1.2) \approx 4.56sec$$

$$T_{total} \approx 9.41sec$$

$$T_{down} = 9.41 - 4.56 = 4.85sec$$

$p = 0.2$

$$T_{up} = 3.47$$

$$T_{total} = 8sec$$

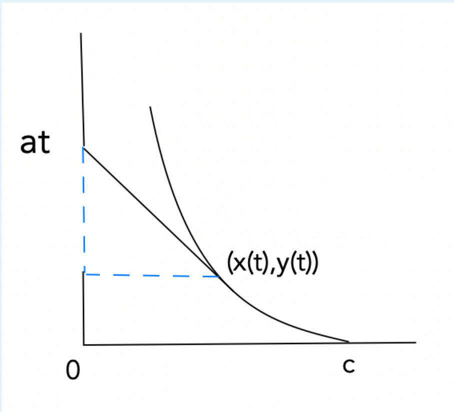
$$T_{down} = 4.53sec$$

Indeed, upper time is shorter than downward time.

4.2. Wednesday

4.2.1. Three Examples

■ **Example 4.2** [Chase problem] A rabit starts at (0,0) and runs up the y-axis with speed a toward its burrow. A dog at the same time running with speed b at (c,0) and persuit the rabbit. What is the path of the dog?



From the figure, we can get that;

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{at - y(t)}{x(t)}$$

By the way, all the following " $'$ " will represent $\frac{\mathrm{d}}{\mathrm{d}x}$.

Rewrite it a little bit;

$$xy' = y - at.$$

Differentiate it with respect to x,

$$y' + xy'' = y' - \frac{a\,\mathrm{d}t}{\mathrm{d}x}.$$

Now, it seems there is a need to know more about $\frac{\mathrm{d}t}{\mathrm{d}x}$.

By calculus, we know the formula to compute the length of a curve:

$$S(x) = \int_x^c \sqrt{1 + y'^2}\,\mathrm{d}x$$

The speed of dog is the derivative of length it goes by time.

$$b = \frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\mathrm{d}s}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = -\sqrt{1 + y'^2}\frac{\mathrm{d}x}{\mathrm{d}t} \qquad \cdots (1)$$

By (1), we know what is $\frac{\mathrm{d}x}{\mathrm{d}t}$. Pluge that in, we get.

$$xy'' = \frac{a\sqrt{1 + y'^2}}{b}$$

Set $z = y'$, $r = \frac{a}{b}$, we rewrite the above equation.

$$xz' = r\sqrt{1 + z^2}$$

God bless us that this is a first order linear differential equation we can solve.

By the way we solve seperable equation,

$$\int \frac{\mathrm{d}z}{\sqrt{1 + z^2}} = \int \frac{r}{x}\,\mathrm{d}x$$

With ingenious, skillful, and some how tedious substitution, i.e. $z = tan\theta$, $\mathrm{d}z = \sec^2\theta\,\mathrm{d}\theta$, the computation of left hand side can be managed. By the way, with this replacement, $1 + z^2 = 1 + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} = \sec^2\theta$

$$\int \frac{\mathrm{d}z}{\sqrt{1 + z^2}} = \int \frac{\sec^2\theta}{\sec}\,\mathrm{d}\theta = \int \frac{1}{\cos\theta}\,\mathrm{d}\theta = \int \frac{\cos\theta}{\cos^2\theta}\,\mathrm{d}\theta = \int \frac{\mathrm{d}\sin\theta}{1 - \sin^2\theta}$$

Again, with unspeakable thoughts, let $u = \sin\theta$, the above

$$= \int \frac{1}{1 - u^2}\,\mathrm{d}u = \int (\frac{\frac{1}{2}}{1 - u} + \frac{\frac{1}{2}}{1 + u})\,\mathrm{d}u$$

$$= \frac{1}{2} ln|\frac{1+u}{1-u}| + C = ln|\sec\theta + \tan\theta| + C$$

This shows you why the last equal sign is the case.

$$\frac{1+\sin\theta}{1-\sin\theta}\frac{1+\sin\theta}{1+\sin\theta} = (\frac{1+\sin\theta}{\cos\theta})^2 = (\sec\theta + \tan\theta)^2$$

Now, with those details, we know why the seperable equation looks like this;

$$ln|z + \sqrt{1+z^2}| = rln|x| + C.$$

$$|z + \sqrt{1+z^2}| = |x|^r \tilde{c}$$

That is

$$y' + \sqrt{1+y'^2} = \tilde{c}x^r.$$

By observation,

$$y(s) = 0 = y'(c).$$

Therefore, $\tilde{c} = (\frac{1}{c})^r$.

Now, let's take square and then integrate y' to get what we want.

$$y' + \sqrt{1+y'^2} = (\frac{x}{c})^r$$

$$1 + y'^2 = (\frac{x}{c})^{2r} - 2(\frac{x}{c})^r y' + y'^2$$

$$\boxed{y' = \frac{1}{2}(\frac{x}{c})^r - \frac{1}{2}(\frac{x}{c})^{-r}}$$

r is the rate of rabbit's speed verses dog's.

When $r = 1$,

$$y = \frac{1}{2}\frac{1}{2}(\frac{x}{c})^2 \cdot c - \boxed{\frac{c}{2} lnx} + C.$$

When $r \neq 1$,

$$y = \frac{1}{2}\frac{1}{c^r}\frac{1}{r+1}x^{r+1} - \frac{1}{2}c^r\frac{1}{1-r}x^{1-r} + C$$

■ **Example 4.3** Four bugs sit at the corners of a square table of side a. At the same time, they all begin to walk at the same speed, each moving steadily toward the bug to its right. Describe the path of the bugs. In addition, compute the length of the path.

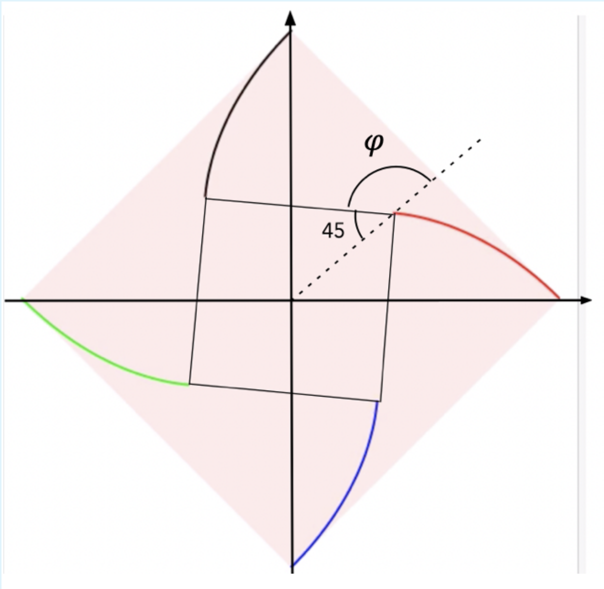
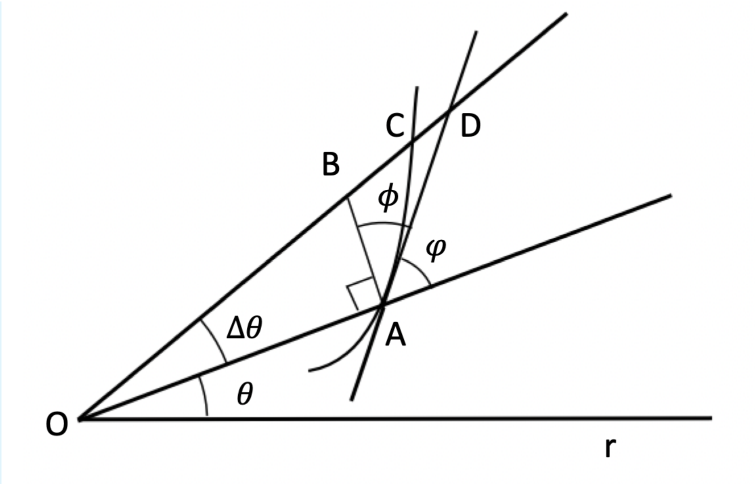


Figure 4.1: Pictures with further edition from; https://demonstrations.wolfram.com/FourBugProblem/

The ode knowledge used in this example is pretty trivial, that is, we use polar cordinate to illustrate the path with relation between r (the length from the bug to origin) and θ. However, before that, the relation between r and θ need to be derived, i.e. the direvative of r over θ. (Why is this? Frankly, I don't know. Maybe, it is by experience.)

Now, let's have a look at detailed graph of the path in order to compute the derivative.



In this graph, AC is the path that bug goes, AD is tangent line of AC at A, (r,θ) is the polar cordinate. We know from figure 7.1 (see above) $\varphi = 135^\circ$. $\tan\varphi = \cot\phi$. As $\Delta\theta$ is pretty small, $\cot\phi = \frac{AB}{BD}$. (Somehow fishy as I see. Please figure it out, if this interpretation need improvement.) In addition, with the same reason, BC ($r(\theta+\Delta\theta) - r(\theta) = r'(\theta)\Delta\theta$) is used to represent the length of BD and the length of arc, OA$\times\Delta\theta$, is used as length of AB.

Anyway, we get;

$$\tan\frac{3\pi}{4} = -1 = \cot\phi = \frac{r\Delta\theta}{r'(\theta)\Delta\theta} = \frac{r}{r'}.$$

Here comes ode;

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = -r$$

$$\frac{\mathrm{d}r}{r} = -\mathrm{d}\theta$$

$$ln|r| = -\theta + c$$

$$r = e^{-\theta}\tilde{c}$$

Let's take the path of red bug, the one on the right in figure7.1, as a example.

Initially, the red by is at $(0, \frac{a}{\sqrt{2}})$. With this, we get $\tilde{c} = \frac{a}{\sqrt{2}}$.

Talk about the length, formula of polar cordination from Calculus comes again.

As $r \to 0$, $\theta \to \infty$,

$$
\begin{aligned}
s &= \int_0^\infty \sqrt{r'^2(\theta) + r^2(\theta)}\,\mathrm{d}\theta \\
&= \int_0^\infty \sqrt{a^2 e^{-2\theta}}\,\mathrm{d}\theta \\
&= \int_0^\infty a e^{-\theta}\,\mathrm{d}\theta = a(-e^{-\theta}) \mid_0^\infty = a
\end{aligned}
$$

■

■ **Example 4.4** It has been observed that a mothball of radius $\frac{1}{2}$ inch evaperates to leave a ball of radius $\frac{1}{4}$ inch in 6 months. Find the radius as function of time. After how many months will it disappear althogether?

As physics tell us, the rate mothball evaperate (the rate is volumn change) is proportion to its surface area.

That is to say

$$(\frac{4}{3}\pi r^3)' = k 4\pi r$$

$$4\pi r^2 r' = k 4\pi r^2$$

$$r' = k$$

$$r = kt + c$$

Now, with the condition provided in the problem, let's figure out what is k and c.

$$r(0) = 0 + c = \frac{1}{2}$$

$$r(6) = \frac{1}{4} = 6k + \frac{1}{2}$$

$$6k = \frac{1}{4}$$

$$r = -\frac{1}{24}t + \frac{1}{2}$$

$$r(T) = 0 = -\frac{T}{24} + \frac{1}{2}$$

$$T = 12$$

Chapter 5

Week5

5.1. Wednesday

5.1.1. Second order differential equation

$$F(t,y,y',y'')=0$$

$$y''=f(t,y,y'')$$

$$(*)\begin{cases} y''+P(t)y'+q(t)y=0 \\ y(t_0)=y_0 \\ y'(t_0)=y_0'. \end{cases}$$

Theorem 5.1 Let $P(t)$, $q(t)$ be continuous on (α,β) and $t_0 \in (\alpha,\beta)$. Then there exists a unique solution for the IVP (*) on the entire interval (α,β).

R Remember previous example $y'=1+y^2$, the solution is $y=\tan x$ which makes the above theorem dubious(p, q are continuous on entire $\mathbb{R}$, while the solution is only on $(-\frac{\pi}{2},\frac{\pi}{2})$). However, the theorem requires DEs to be linear($y'=1+y^2$ isn't linear). (PS: linear means the variable only appears with power of 1.)

In addition, this theorem will be proved latter.

Definition 5.1 Define $L[y] = y'' + P(t)y' + q(t)y$

The set of all solutions to $L[y] = 0$ is denoted by $ker(L) = \{y|L[y] = 0\}$

Proposition 5.1

1. Ker(L) is a vector space of dimension 2.

2. $\{y_1, y_2\}$ is a basis for ker(L) where

$$\begin{cases} y_1'' + P(t)y_1' + q(t)y_1 = 0 \\ y_1(t_0) = 1, y_1'(t_0) = 0. \end{cases} \qquad \begin{cases} y_2'' + P(t)y_2' + q(t)y_2 = 0 \\ y_2(t_0) = 0, y_2'(t_0) = 1. \end{cases}$$

Proof. In order to show it's a basis, there are two things that we need to show. First, any solution y to (*) can be written as a linear combination of y_1, y_2. Assume it is $y = c_1y_1 + c_2y_2$. What we need to do now is checking whether it satisfies the equation and finding out what c_1 and c_2 are exactly.

$$\begin{aligned} L[c_1y_1 + c_2y_2] &= (c_1y_1'' + c_2y_2'') + p(t)(c_1y_1' + c_2y_2') + q(t)(c_1y_1 + c_2y_2) \\ &= c_1y_1'' + p(t)c_1y_1' + q(t)c_1y_1 + c_2y_2'' + p(t)c_2y_2' + q(t)c_2y_2 \\ &= c_1L[y_1] + c_2L[y_2] = 0 \end{aligned}$$

Therefore, it satisfies the first equation. Now, let's find out the value of coefficient.

$$[c_1y_1 + c_2y_2](t_0) = c_1y_1(t_0) + c_2y_2(t_0) = c_1 = y(t_0)$$

$$[c_1y_1 + c_2y_2]'(t_0) = c_1y_1'(t_0) + c_2y_2'(t_0) = c_2 = y'(t_0)$$

With $c_1 = y(t_0)$, $c_2 = y'(t_0)$, y is a solution of (*). In addition, with uniqueness, $y \equiv z = c_1y_1 + c_2y_2$.

Second, we need to show they are linearly independent.

Claim $\{y_1, y_2\}$ are linearly independent, i.e. if there exists constants c_1, c_2 s.t. $c_1y_1 + c_2y_2 = 0$ $\forall t \in (\alpha, \beta)$ then $c_1 = c_2 = 0$.

Let's see why this is the case. $z = c_1y_1 + c_2y_2$ is a solution for (*).

$$z(t_0) = 0 = c_1y_1(t_0) + c_2y_2(t_0) = c_1 + 0$$

$$z'(t_0) = c_1 y_1'(t_0) + c_2 y_2'(t_0) = c_2 = 0$$

■

$L[y] = y'' + P(t)y' + q(t)y = 0$

$\begin{cases} y(t_0) = y_0 \\ y'(t_0) = Y_0'. \end{cases}$ Suppose y_1, y_2 are solutions i.e. y_1,$y_2 \in$ ker(L).

Definition 5.2 [Wronskian]$W(t) = y_1(t)y_2'(t) - y_2(t)y_1'(t)$

Lemma:

(i)$W' + p(t)W = 0$

(ii)W is either $\equiv 0$ or never 0.

Proof.

$$W = y_1 y_2' - y_2 y_1'$$

$$\begin{aligned} W' &= y_1' y_2' + y_1 y_2'' - y_2' y_1' - y_2 y_1'' \\ &= y_1(-py_2' - qy_2) + y_2(py_1' + qy_1) \\ &= -py_1 y_2' - y_1 q y_2 + y_2 p y_1' + y_2 q y_1 = p(y_1' y_2 - y_1 y_2') = -PW \end{aligned}$$

This is $W' + PW = 0$.

$$(e^{\int P(t)\,\mathrm{d}t} W)' = 0$$

$\Rightarrow \qquad e^{\int P(t)\,\mathrm{d}t} W \equiv constant.$

Therefore, $W(t) = W(t_0)(e^{-\int_{t_0}^{t} P(s)\,\mathrm{d}s})$ ■

Proposition 5.2 For any two solutions y_1, y_2 such that y_1, y_2 are linearly dependent.

$\iff W[y_1,y_2] \equiv 0$

Proof. ($\Rightarrow$)y_1, y_2 l.dep

$\Rightarrow \exists$ c_1, c_2 s.t. $c_1 y_1 + c_2 y_2 = 0$ (not both 0; definition of linearly dependent.)

Suppose $c_1 \neq 0$, $y_1 = -\frac{c_2}{c_1} y_2 = \delta y_2$

$W[y_1, y_2] = y_1 y_2' - y_2 y_1' = \delta y_2 y_2' - y_2 \delta y_2' = 0 \qquad (y_1 = \delta y_2)$

$(\Leftarrow)$

This direction we need to consider two different cases.

(i) $\quad y_1(t), y_2(t)$ are never 0 in (α, β)

$\Rightarrow \quad \frac{y_2'}{y_2} = \frac{y_1'}{y_1} \quad \Rightarrow ln|\frac{y_1}{y_2}| = constant$

$|\frac{y_1}{y_2}| = Constant$, i.e. $\frac{y_1}{y_2} = constant$

(ii)If $y_1(E), y_2(E)$ =0 for some $E \in (\alpha, \beta)$, assume $y_1(E) = 0$.

If $y_1'(E) = 0 \Rightarrow y_1 \equiv 0$.

If $y_1'(E) \neq 0 \quad y_1(E) \cdot y_2'(E) = y_2(E) \cdot y_1'(E)$

$y_2(E) = \frac{y_2'(E)}{y_1'(E)} y_1(e) = \delta y_1(E)$

$$\begin{cases} y_2, \delta y_1 \textit{ are solutions never 0} \\ \qquad y_2(E) = \delta y_1(E) \\ \qquad y_2'(E) = \delta y_1'(E). \end{cases}$$

View δy_1 as a solution to IVP. By uniqueness, $y_2 \equiv \delta y_1$ ■

As we showed before, $z' = t^2 + z^2$ (1) doesn't have a solution that can be expressed by primary functions. Let $z = -\frac{y'}{y}$.

$$z' = -\frac{y''}{y} + \frac{y'^2}{y^2} = t^2 + \frac{y'^2}{y^2}$$

From right-hand side equation, we get

$$y'' + t^2 y = 0 \cdots (2)$$

Because (1) doesn't have a solution, then (2) equation doesn't have either.

Chapter 6

Week6

6.1. Monday

6.1.1. 2^{nd} Order linear equation with constant coefficients

■ **Example 6.1**

$$y'' - 3y' + 2y = 0$$

Define operator: $Dy \triangleq y'$, $\quad D^2y \triangleq y''$

Then $D^2y - 3Dy + 2y = 0$,

our operator $L \triangleq D^2 - 3D + 2$, $\quad$ then $L[y] = 0 \quad (D^2 - 3D + 2)y = 0$.

$(D-2)(D-1)y = 0 \quad$ then $Dz - 2z = 0$

i.e. $z' - 2z = 0$ $\mu = e^{\int -2\mathrm{d}t} = e^{-2t} \quad (e^{-2t}z)' = 0$

$e^{-2t}z = c_1$ Therefore, $z = c_1e^{2t}$

Moreover, $(D-1)y = z = c_1e^{2t}$

Means, $y' - y = c_1e^{2t}$

$(e^{-ty}y)' = c_1e^{2t}e^{-t} = c_1e^t$

$e^{-t}y = c_1e^t + c_2$

$y = c_1e^{2t} + c_2e^t$

$\{e^{2t}, e^t\}$ is linearly independent. (Then this is a basis spanning the set of all solutions.)

There are two ways to prove independence; use definition, and Wronskian.

(1) Suppose $c_1e^{2t} + c_2e^t = 0$

$$\begin{cases} c_1 + c_2 = 0 & (t=0) \\ c_1 e^2 + c_2 e = 0 & (t=1). \end{cases}$$ Solve these equations, we get $c_1 = c_2 = 0$.

(2)$W[e^{2t}, e^t] = y_1 y_2' - y_2 y_1' = e^{2t} e^t - e^t 2e^{2t} = -e^{3t} \neq 0$

In general, when a, b, c are constants.

$$ay'' + by' + cy = 0$$

$$aD^2 + bD + C = 0$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

(i)$b^2 - 4ac > 0 \Rightarrow 2$ real roots r_1, r_2

(ii)$b^2 - 4ac < 0 \; -\frac{b}{2a} \pm i\frac{\sqrt{4ac-b^2}}{2a}$

(iii)$b^2 - 4ac = 0$ double root

■ **Example 6.2**

$$4y'' + 4y' + 5y = 0$$

$$4r^2 + 4r + 5 = 0 \quad \Rightarrow \quad r = \frac{-4 \pm \sqrt{16 - 80}}{8} = -\frac{1}{2} \pm i$$

$$e^{(-\frac{1}{2}+i)t} = e^{-\frac{1}{2}t} e^{it} = e^{-\frac{1}{2}t}(\cos t + i \sin t)$$

$$e^{-\frac{1}{2}t} \cos t \pm i e^{-\frac{1}{2}t} \sin t$$

$$y = u(t) + iv(t)$$

$$0 = L[t] = L[u + iv] = L[u] + iL[v]$$

$\Longleftrightarrow$ By complex number definition of zero.

$$L[u] = 0 \quad \& \quad L[v] = 0$$

This implies $\begin{cases} u = e^{-\frac{1}{2}t}\cos t \\ v = e^{-\frac{1}{2}t}\sin t. \end{cases}$ both are solutions.

Just take a simple check to see whether this is true or not. Take v as an example.

$$5v = 5e^{-\frac{1}{2}t}\sin t$$

$$4v' = -2e^{-\frac{1}{2}t}\sin t + 4e^{-\frac{1}{2}t}\cos t$$

$$4v'' = e^{-\frac{1}{2}t}\sin t - 4e^{-\frac{1}{2}t}\cos t - 4e^{-\frac{1}{2}t}\sin t$$

Therefore,

$$L[v] = 4v'' + 4v' + 5v = 0.$$

This implies v is a solution. In addition, these two solutions are linearly independent. $\{e^{-\frac{1}{2}t}\sin t, e^{-\frac{1}{2}t}\cos t\}$ are l. indep.

$c_1 e^{-\frac{1}{2}t}\sin t + c_2 e^{-\frac{1}{2}t}\cos t = 0 \ t = 0 \Rightarrow c_2 = 0$

■

■ Example 6.3

$$y'' + 4y' + 4y = 0$$

$$r^2 + 4r + 4 = 0$$

$$(r+2)^2 = 0 \qquad \Rightarrow r = -2, -2$$

$\Rightarrow$ we have at least one solution e^{-2t}

Q; How to find 2^{nd} solution?

$$4y = 4C(t)e^{-2t}$$

(Variation of parameters)

$$4y' = 4C'e^{-2t} - 8ce^{-2t}$$

$$y'' = c''e^{-2t} - 4c'e^{-2t} + 4ce^{-2t}$$

Sum them up

$$L[Cy] = c''e^{-2t} = 0 \qquad \Rightarrow c'' = 0 \quad c' = \alpha \qquad \alpha \text{ is a constant}$$

$$c = \alpha t + c_1 \qquad c = t$$

$$y_2 = te^{-2t}$$

We just take $c_1 = 0$ and $\alpha = 1$ for simplicity. The reason is that other coefficients just contribute to an additional general solution term which is redundant. ■

■ **Example 6.4**

$$(1-t^2)y'' + 2ty' - 2y = 0$$

$t = \pm 1$, $1 - t^2 = 0$, singularity for a solution. (PS: Roughly speaking, singularity means the coefficient highest order is zero. However, specific details of singularity is somehow exotic. You can check wikipidia yourself. Or later on section 2.8, the solution is analytic at $t = t_0$ requires the $\frac{Q(x)}{P(x)}$ and $\frac{R(x)}{P(x)}$ have convergent taylor series expansions at $t = t_0$. For instance, in this example, the fraction doesn't make any sense at $t = 1$ at all.)

Lateron in 2.8, we will know how to factor out this singularity. At present, let's just focus on $-1 < t < 1$. Observe:

$y_1 = t$ is a solution.

$y = c(t)t$

$y' = c't + c$

$y'' = c''t + 2c'$

$$\begin{aligned}(1-t^2)y'' + 2ty' - 2y &= (1-t^2)(c''t + 2c') + 2t(c't + c) - 2ct \\ &= t(1-t^2)c'' + [2(1-t^2) + 2t^2]c' \\ &= t(1-t^2)c'' + 2c' = b\end{aligned}$$

$g = c' \Rightarrow t(1-t^2)g' + 2g = 0$

$t \neq 0$

$$\int \frac{\mathrm{d}g}{g} = -\int \frac{2}{t(1-t^2)}\,\mathrm{d}t$$

$$ln|g| = [-2ln|t| + ln|1-t^2|] = ln|\frac{1-t^2}{t^2} + C|$$

$$|g| = |\frac{1-t^2}{t^2}|\tilde{c}, \quad \tilde{c} > 0$$

$\Rightarrow g = \bar{c}\frac{1-t^2}{t^2}$, $\bar{c} \in \mathbb{R}$

$c' = \bar{c}(\frac{1}{t^2} - 1) \Rightarrow c = \frac{1}{t} + t \ (\bar{c} = -1)$

$$y_2 = (\frac{1}{t} + t)t = 1 + t^2$$

$\{t, t^2+1\}$ is l.dep. ■

6.2. wednesday

6.2.1. Nonhomogenous equation

$$L[y] = y'' + p(t)y' + q(t)y = g(t) \qquad \cdots (N)$$

$$y'' + p(t)y' + q(t)y = 0 \qquad \cdots (H)$$

Suppose we have found total linearly independent solutions for the homogeneous equation y_1, y_2.

$$y = c_1y_1 + c_2y_2$$

Like we have done before, a resonable guess form of the nonhomogeneous solution may take the form of:

$$y_p = u_1(t)y_1 + u_2(t)y_2$$

$$y' = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'$$

Furthermore, if we again compute y'', there will be **8** terms which will make things less interesting. Let's make $u_1'y_1 + u_2'y_2 = 0$

$$\begin{aligned} y'' &= u_1'y_1' && +u_1y_1'' + u_2'y_2' + u_2y_2'' \\ py' &= && pu_1y_1' \qquad\qquad + pu_2y_2' \\ qy &= && qu_1y_1 \qquad\qquad + qu_2y_2 \end{aligned}$$

Add them up together, we will get

$u_1'y_1' + u_2'y_2' = g$

Now, in order to get one solution, all we need to do is to solve;

$$\begin{cases} u_1'y_1 + u_2'y_2 = 0 \\ u_1'y_1' + u_2'y_2' = g. \end{cases}$$

By cramer's rule (check MAT2040),

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ g & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = -\frac{gy_2}{W[y_1, y_2]}$$

In summary, we have find one particular solution y_p for (N).

Claim: any solution $z(t)$ for (N) must be of the form $z = y_p + c_1y_1 + c_2y_2$ for some c_1, c_2.

Proof. For any given solution $z(t)$ for (N),

$$\begin{aligned}(z - y_p)'' + p(t)(z - y)' + q(t)(z - y_p) &= z'' + p(t)z' + q(t)z - (y_p'' + p(t)y_p' + q(t)y_p) \\ &= q(t) - q(t) = 0\end{aligned}$$

Therefore, $z - y_p$ is a solution of (H). ■

However, as professor said,

Life is, fortunately or unfortunately, not that simple.

Even a simplest nonhomogeneous equation can be a disaster to calculate.

■ **Example 6.5**

$$y'' + y' + y = t$$

$$y'' + y' + y = 0$$

$$r'' + r' + 1 = 0$$

$$r = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

$$y_1 = e^{-\frac{1}{2}t}\cos(\frac{\sqrt{3}}{2}) \qquad y_2 = e^{-\frac{1}{2}t}\sin(\frac{\sqrt{3}}{2})$$

With the formula above and "a little bit" calculation,

$$u_1' = \frac{-t^2 e^{-\frac{1}{2}t}\sin(\frac{\sqrt{3}}{2}t)2e^t}{\sqrt{3}}$$

$$u_1 = \int t^2 e^{\frac{1}{2}t}\sin(\frac{\sqrt{3}}{2}t)\,\mathrm{d}t$$

With similiar procedure, we can get u_2 which means y_p is known which is the particular solution we want.

Anyway, you get the idea of how messy the computation can be. ■

R It's much more easier to compute the solution of above example by reasonable guessing. See next section.

6.2.2. Undetermined coefficient question

Sometimes, it's better to guess the form of the solution than calculating them in a formal way. It's ok to have a reasonable guess.

For example,

$$y'' + y' + y = t^2 - t$$

$$y_p = At^2 + Bt + C$$

$$y' = 2At + B$$

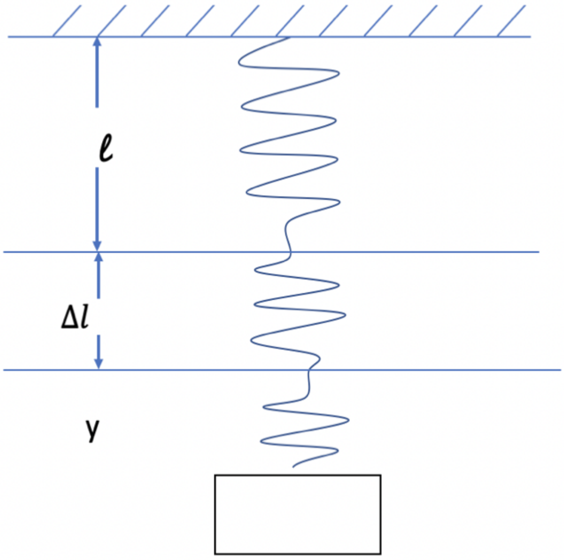
$$y'' = 2A$$

$$y'' + y' + y = 2A + 2At + B + At^2 + Bt + C = At^2 + (B + 2A)t + 2A + B + C$$

$$A = 1, \quad B = -2, \qquad C = 0$$

There are kinds of $g(t)$ that can use this method to solve. For instance, $t^k e^{at}$, $t^k\cos(wt)$, $t^k\sin(wt)$, $t6ke^{at}\cos(wt)$, ...

6.2.3. Mechanical Vibrations



The original length of spring is l. After placing block on it, it increase δl. With a force place on it, it, again, increase y. W: be weight mg. R is restoring force. $R = -(\Delta l + y)k$. Damping force $D = cy'$. External force F

$$my'' = mg - k(\Delta l + y) - cy' + F$$

$$mg = k\Delta l$$

$$my'' + cy' + ky = F$$

Let's see the case without damping force, i.e. $c = 0$.

$$y'' + \frac{k}{m}y = \frac{F}{M}$$

$$y'' + \frac{k}{m}y = \frac{F}{m}$$

By physics, w_0 is the frequency.

$$w_0^2 = \frac{k}{m}$$

Free vibration($F=0$),

$$y'' + w_0^2 y = 0$$

$$r^2 + w_0^2 = 0$$

$$r = \pm i w_0$$

$$\begin{aligned}
y &= a\cos(w_0 t) + b\sin(w_0 t)\\
&= \sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}}\cos w_0 t + \frac{b}{\sqrt{a^2+b^2}}\sin w_0 t\right)\\
&= \sqrt{a^2+b^2}(\cos\delta\cos w_0 t + \sin\delta\sin w_0 t) \quad \text{Auxiliary angle method}\\
&= \sqrt{a^2+b^2}\cos(w_0 t - \delta)
\end{aligned}$$

(2)Damped free vibration ($v \neq 0$)

$$wy'' + cy' + ky = 0$$

$$mr^2 + cr + k = 0$$

$$r = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

- $c^2 - 4mk > 0$:(over damped) 2 real roots both <0 $y = c_1 e^{r_1 t} + c_2 e^{r_2 t} \quad \to 0$ not $\to \infty$
- $c^2 - 4mk < 0$ 2 complex conjugate roots

$$r = -\frac{c}{2m} \pm i\frac{\sqrt{4mk - c^2}}{2m}$$

$$\begin{aligned}
y &= c_1 e^{-\frac{c}{2m}t}\cos\left(\frac{\sqrt{4mk-c^2}}{2m}t\right) + c_2 e^{-\frac{c}{2m}t}\sin\left(\frac{\sqrt{4mk-c^2}}{2m}t\right)\\
&= e^{-\frac{c}{2m}t}\left[c_1\cos\left(\frac{\sqrt{4mk-c^2}}{2m}t\right) + c_2\sin\left(\frac{\sqrt{4mk-c^2}}{2m}t\right)\right]
\end{aligned}$$

- $c^2 = 4mk \quad c_1 e^{-\frac{c}{2m}t} + c_2 t e^{-\frac{c}{2m}t}$

R Those only hold when y is small.

Chapter 7

Week7

7.1. Monday

7.1.1. The Tacoma Bridge disaster

$$my'' + cy' + ky = F$$

Forced free vibration, $c = 0$,

$$y'' + \frac{k}{m}y = \frac{F}{m}$$

$w_0^2 = \frac{k}{m}$

$$y'' + w_0^2 y = \frac{F}{m}$$

Characteristic equation is$r^2 + w_0^2 = 0$, which means $r = \pm i w_0$

Therefore, the general solution is $y = c_1 \cos(w_0 t) + c_2 \sin(w_0 t)$.

By physics,$F = F_0 \cos(wt)$

$$y^2 + w_0^2 y = \frac{F_0}{m}\cos(wt)$$

1. $w \neq w_0$ Guess the form of particular solution is $y_\delta = A\cos wt$

$$y'_\delta = -Aw\sin wt$$

$$y''_\delta = -Aw^2\cos wt$$

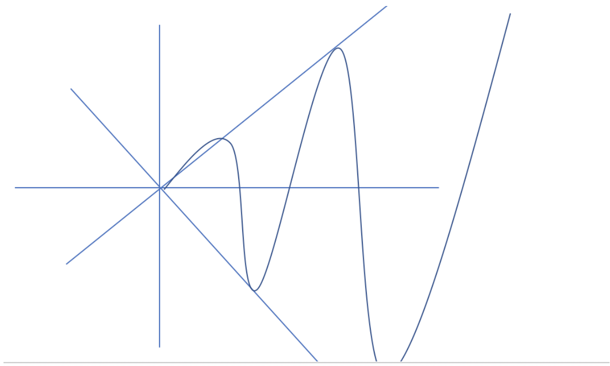
$$y''_\delta + w_0^2 y_\delta = A(w_0^2 - w^2)\cos wt$$

This implies $A = \frac{F_0}{m(w_0^2 - w^2)}$ The general solution is

$$y = c_1 \cos(w_0 t) + c_2 \sin(w_0 t) + \frac{F_0}{m(w_0^2 - w^2)} \cos(wt)$$

2. $w = w_0$ Want to solve $y'' + w_0^2 y = \cos(w_0 t)$ either we guess the solution form or use variation of variable to get the solution.

$$y = c_1 cos(w_0 t) + c_2 \sin(w_0 t) + t \frac{F_0 \sin(w_0 t)}{2 m w_0}$$



R

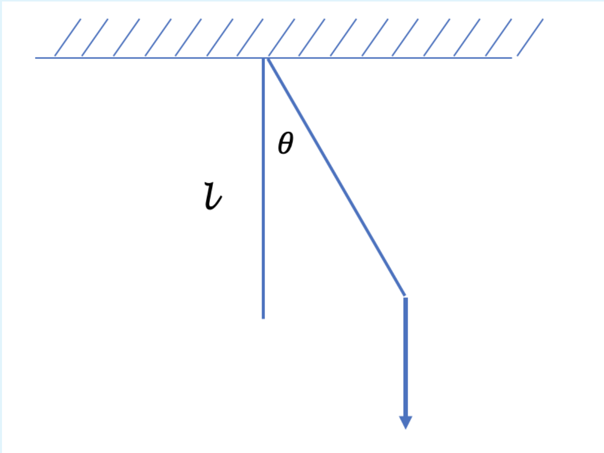
1. Again, the model is suitable when y is small as it's the maximum amplitude can only be the width of the bridge.
2. Why we separate into two case?

   When $w = w_0$, the solution in first case doesn't hold.

   If we pick some part in first term of general solution $c_1 \cos(w_0 t)$ and let $w \to w_0$, we can see that

$$\frac{F_0}{m(w^2 - w_0^2)} \cos wt - \frac{F_0}{m(w^2 - w_0^2)} \cos w_0 t = \frac{F_0}{m(w^2 - w_0^2)} (\cos wt - \cos w_0 t)$$

   By L'hospital's rule, it is equal to $-\frac{F_0}{2mw} t \sin wt \to -\frac{F_0}{2mw_0} t \sin w_0 t$ which is equal to the specific solution of the equation in second case.



■ **Example 7.1** The world largest damper is modelized as a pendulum with period 7sec. The aim is to find the length of the cable to hung the mass.

Because of energy conservation,

$$\frac{1}{2}ml^2(\frac{\mathrm{d}\theta}{\mathrm{d}t})^2 + mgl(1-\cos\theta) \equiv constant$$

Differentiate θ,

$$ml^2\theta''\theta' + mgl\theta'\sin\theta = 0$$

$$\theta'' + \frac{g}{l}\sin\theta = 0$$

$$\theta'' + \frac{g}{l}\theta = 0$$

By physics, $w^2 = \frac{g}{l}$, period=$2\pi\sqrt{\frac{l}{g}} = 7$. We get l=12.18m. ■

■ **Example 7.2** It is known that 1kg mass moves a spring $\frac{49}{320}$m. If the mass is pulled down with an additional $\frac{1}{4}$m and released. Find amplitude, period, and frequency of the motion neglecting air resistance.

$$my'' + cy' + ky = 0$$

m=1kg, c=0, $mg = k\Delta l \ \Rightarrow 1kg \cdot 9.8m/sec^2 = k\frac{49}{320}m$, k=64

$$\begin{cases} y'' + 64y = 0 \\ y(0) = \dfrac{1}{4} \\ y'(0) = 0. \end{cases}$$

$$y = c_1 \cos(8t) + c_2 \sin(8t)$$

$$y(0) = c_1 = \frac{1}{4}$$

$$y'(0) = 0 = -(\sin 8t)8 + 8c_2 \cos 8t$$

$$c_2 = 0$$

$$y = \frac{1}{4}\cos(8t)$$

Therefore, amplitude=0.25m and period=0.25π

7.2. Wednesday

7.3. Series solutions

Introduction. As we have already seen before, $y'' + p(t)y' + q(t)y = 0$, when $p(t)$ and $q(t)$ are constant, is solvable. Now we begin to wondering about the case when $p(t)$ and $q(t)$ are not constant. Like the following one.

■ Example 7.3

$$y'' - 2ty' - 2y = 0$$

Obviously, first guess the solution as polynomial fails. What about infinite series?

$$y(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n + \cdots = \sum_{n=0}^{\infty} a_n t^n$$

$$y'(t) = \sum_{n=0}^{\infty} n a_n t^{n-1}$$

$$y''(t) = \sum_{n=0}^{\infty} n(n-1) a_n t^{n-2} = \sum_{m=0}^{\infty} (m+1)(m+2) a_{m+2} t^m$$

$$y'' - 2ty' - 2y = \sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} - 2na_n - 2a_n] t^n = 0$$

$$\Rightarrow (n+1)(n+2)a_{n+2} = 2(n+1)a_n$$

$$a_{n+2} = \frac{2}{n+2} a_n, \quad n = 0,1,2,\ldots$$

As we have shown before, the second order differential equation can be spanned by two linearly independent solutions. We only need to choose two linearly independent solutions. Take $a_0 = 1$, $a_1 = 0$ or $a_0 = 0$, $a_1 = 1$. Let's take the first case as an example.

$$a_{n+2} = \frac{2}{n+2} a_n = \frac{2}{n+2}\frac{2}{n} a_{n-2} = \frac{2}{n+2}\frac{2}{n}\frac{2}{n-2} a_{n-4} = \ldots$$

In first case, we only focus on even subscript terms as odd subscript terms are all zeros.

Take $n = 2k$, we get $a_{2k+2} = \frac{1}{(k+1)!}a_0$.

$$y = \sum_{k=0}^{\infty} \frac{1}{k!} t^{2k}$$

With calculus knowledge we "easily" see that $y = e^{t^2}$

Now the following will show how to use variation of parameter to find the other solution.

(By the way, Prof. Ni said variation of parameter will definitely be in midterm.)

$$y_1 = e^{t^2}$$

$$y_2 = v y_1$$

$$y_2' = v' y_1 + v y_1'$$

$$y_2'' = v'' y_1 + 2v' y_1'$$

$$y_2'' + p(t) y_2' + q(t) y_2 = 0$$

$$\Rightarrow v'' y_1 + v'(p y_1 + 2y_1') = 0$$

Pluge y_1 in,

$$v'' + 2tv' = 0$$

$u = v'$,

$$u' + 2tu = 0$$

$$(e^{\int 2t \, \mathrm{d}t} u)' = 0$$

$$e^{t^2} u = c \;\Rightarrow\; v' = ce^{-t^2}$$

$$v = \int e^{-t^2} \, \mathrm{d}t$$

$$y_2 = e^{t^2} \int e^{-t^2} \, \mathrm{d}t$$

■

Theorem 7.1 Suppose $\frac{Q(t)}{P(t)}$, $\frac{R(t)}{P(t)}$ have convergent taylor series expansions at t_o for $|t-t_0|<\rho$ Then every solutions of the equation $P(t)y''+q(t)y'+R(t)y=0$ is analytic at t_0 and the solutions of convergnece for series at t_0 is at least ρ.

R This theorem only has restriction to the quiotient but not the individual $Q(t)$, $R(t)$, $P(t)$. Therefore, this theorem is more powerful.

Example 7.4

$$y''+\frac{3t}{1+t^2}y'+\frac{1}{1+t^2}y=0$$

$$y=a_0+a_1t+a_2t^2+\cdots=\sum_{n=0}^{\infty}a_nt^n$$

$$y'=\sum_{n=0}^{\infty}na_nt^{n-1}$$

$$y''=\sum_{n=0}^{\infty}n(n-1)a_nt^{n-2}$$

$$3ty'=\sum_{n=0}^{\infty}3na_nt^n$$

$$(1+t^2)y''=\sum_{n=0}^{\infty}(n)(n-1)a_nt^{n-2}+\sum_{n=0}^{\infty}n(n-1)a_nt^n$$

Pluge those terms in

$$(1+t^2)y''+3ty'+y=0$$

We get

$$\sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}t^m+\sum_{n=0}^{\infty}[n(n-1)a_n+3na_n+a_n]t^n=0$$

$$0=\sum_{n=0}^{\infty}t^n\{(n+2)(n+1)a_{n+2}+[n^2+2n+1]a_n\}$$

Therefore, $a_{n+2}=-\frac{n+1}{n+2}a_n$ ■

R It's much more easy to compute with both side of the equation multiplies $1+t^2$.

■ **Example 7.5** [Initial value problem]

$$(t^2-2t)y''+5(t-1)y'+3y=0$$

$y(1)=7$, $y'(1)=3$ ■

Change the variable with $s=(t-1)$, we get

$$(s^2-1)y''+5sy'+3y=0$$

which is a initial value problem with initial value at 0. As

$$\begin{aligned}(t^2-2t)&=[(t-1)+1]^2-2[(t-1)+1]\\&=(t-1)^2+2(t-1)+1-2(t-1)-2\\&=(t-1)^2-1\end{aligned}$$

Chapter 8

Week8

8.1. Monday

8.1.1. Euler's Equation

Recall previous knowledge,

$$t^2y'' + \alpha ty' + \beta y = 0 \qquad t > 0$$

$y = t^r \Rightarrow \qquad r(r-1) + \alpha r + \beta = 0$

$r^2 + (\alpha - 1)r + \beta = 0$

$r_1, r_2 = \frac{-(\alpha-1) \pm \sqrt{(\alpha-1)-4\beta}}{2}$

1. $(\alpha - 1)^2 - 4\beta > 0$ two real roots, $y = c_1 t^{r_1} + c_2 t^{r_2}$
2. $(\alpha - 1)^2 - 4\beta = 0$ $r_1 = r_2 = r$, $y = c_1 t^r + c_2 t^r lnt$
3. $(\alpha - 1)^2 - 4\beta < 0$ $r = \lambda \pm i\mu$, $y = t^\lambda [c_1 \cos(\mu lnt) + c_2 \sin(\mu lnt)]$

Some motivations of Euler's Equation: Bessel's equation of order $\frac{1}{2}$

$$t^2y'' + ty' + (t^2 - \frac{1}{4})y = 0$$

There doesn't exist a solution with the form of $y = \sum_{n=0}^{\infty} a_n t^n$. When $t = 0$ it's a first order differential equation, the solution of this equation is only at one point $t = 0$ which doesn't make much sense. We call this a singular point. We need to do something to factor out singularity.

Generalization.

$$L[y] \equiv t^2 y'' + t[p_0 + p_1 t + p_2 t^2 + \dots]y' + [q_0 + q_1 t + q_2 t^2 + \dots]y = 0$$

Method of Frobenius,

$$y = \sum_{n=0}^{\infty} a_n t^{n+r}$$

R r can be $\mathbb{R}$.

In order to avoid ambiguity, $a_0 \neq 0$. Otherwise, it becomes $y = \sum_{n=1}^{\infty} a_n t^{n+r}$ which are the same as $y = \sum_{n=0}^{\infty} b_n t^{n+r'}$. As r and $r' = r+1$ need to be determined, we have $y = \sum_{n=1}^{\infty} a_n t^{n+r}$ with the first term equals to zero, and $y = \sum_{n=0}^{\infty} b_n t^{n+r'}$ to represent the same thing. Simply put, $a_0 \neq 0$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n t^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n t^{n+r-2}$$

$$\begin{aligned}
L[y] &= \sum_{0}^{\infty}(n+r)(n+r-1)a_n t^{n+r} + (\sum_{m=0}^{\infty} p_m t^m)(\sum_{0}^{\infty}(n+r)a_n t^{n+r}) + \sum_{m=0}^{\infty} q_n t^m \sum_{0}^{\infty} a_n t^{n+r} \\
&= r(r-1)a_0 + p_0 r a_0 + q_0 a_0 + [(1+r)r a_1 + p_0(1+r)a_1 + p_1 r a_0 + q_0 a_1 q_1 a_0]t + \dots \\
&= [r(r-1) + p_0 r + q_0]a_0 + \underline{[\{(1+r)r + p_0(1+r) + q_0\}a_1 + \{p_1 r + q_1\}a_0]t} + \dots \\
&\quad + \underline{[(k+r)(k+r-1)a_k + \sum_{m=0}^{k} p_m(k-m+r)a_{k-m} + \sum_{n=0}^{k} q_m a_{k-m}]t^k} + \dots
\end{aligned}$$

In order to have a solution $L[y] = 0$, Indicial equation

$$F(r) = r(r-1) + p_0 r + q_0 = 0$$

$$F(r+1)a_1 = -[p_1 r + q_1]a_0$$

retrieved from first underline.

Let's rewrite second underline a little bit.

$$= [(k+r)(k+r-1)a_k + \sum_{0}^{k-l} p_{k-l}(l+r)a_l + p_0 a_k(k+r) + \sum_{n=1}^{k} q_{k-l}a_l + q_0 a_k]t^k$$

$$F(r+k)a_k = -[\sum_0^{k-1}\{p_{k-l}(l+1)+q_{k-l}\}a_l]$$

It is clear that all a_n can be solved recursively.

If there exists two real roots $r_1 \geq r_2$, then

- $r_1 > r_2$ and $r_1 - r_2$ is not a positive integer. We will have two solutions.
- $r_1 > r_2$ and $r_1 - r_2$ is a positive ineger then, you need to check textbook for more information as this will not be tested in the final.
- $r_1 = r_2$ check §2.8.3 for more information.

■ **Example 8.1**

$$t^2y'' + ty' + (-\frac{1}{4} + t^2)y = 0$$

$p_0 = 1,\ p_1 = p_2 = \cdots = 0;\ q = -\frac{1}{4},\ q_1 = 0,\ q_2 = 1,\ q_3 = 0$

Look at $L[y] \equiv t^2y'' + t[p_0 + p_1t + p_2t^2 + \dots]y' + [q_0 + q_1t + q_2t^2 + \dots]y = 0$. You will know how we get all those stuffs.

$r(r-1) + r - \frac{1}{4} = 0,\ r = \pm\frac{1}{2}$

First look at the first case $r = r_1 = \frac{1}{2}$

$F(q+\frac{1}{2})a_1 = 0 \Rightarrow a_1 = 0 \dots (1)$ (We just pluge those stuffs in $F(r+1)a_1 = -[p_1r + q_1]a_0$.

$F(k+r)a_k = k(k+1)a_k = -a_{k-2} \quad \Rightarrow ak = -\frac{1}{(k+1)ka_{k-2}} \dots (2)$ (As $F(k+r) = (r+k)^2 - \frac{1}{4} = k(k+1))$

With (1) and (2), we get $a_3 = a_5 = \cdots = 0$

$$a_2 = -\frac{1}{3!}a_0$$

$$a_4 = -\frac{1}{5\cdot 4}a_2 = \frac{1}{5!}a_0$$

$$a_{2n} = \frac{(-1)^n}{(2n+1)!}a_0$$

$$y_1 = a_0t^{\frac{1}{2}}(1 - \frac{1}{3!}t^2 + \frac{1}{5!}t^2 - \dots)$$

$$y_1 = a_0t^{-\frac{1}{2}}\sin t$$

$r_2 = -\frac{1}{2}$

$$F(1+r_2)a_1 = F(\frac{1}{2})a_1 = 0$$

It's lucky both sides are equal to zero, else we cannot solve the second solution in this way.

$$F(k+r_2) = k(k-1)a_k = -q_2 a_{k-2} = -a_{k-2}$$

$$a_k = -\frac{1}{(k-1)k}a_{k-2}$$

$$a_2 = -\frac{1}{2}a_0$$

$$a_4 = -\frac{a_4}{4 \cdot 3}a_2 = \frac{1}{4!}a_0$$

$$a_6 = -\frac{a_4}{6 \cdot 5} = -\frac{1}{6!}a_0$$

$$y_2 = t^{-\frac{1}{2}}a_0[1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 + \dots] = a_0 t^{-\frac{1}{2}} \cos t$$

Therefore, the general solutions is $y = \frac{1}{\sqrt{t}}(c_1 \cos t + c_2 \sin t)$ ■

Chapter 9

Week9

9.1. Monday

9.1.1. Laplace Transformation

Definition 9.1 Let f be a piecewise continuous function on $(0,\infty)$ and is of "exponential" order, i.e., $|f(t)| \le Me^{ct}$, where M, C are two constants. Then the **Laplace transformation of f, denoted by $\mathscr{L}\{f\}$, is given by**

$$F(s) = \mathscr{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$$

, for $s > c$

R $\mathscr{L}$ is an operator on f. F is a function of s.

Properties:

- $\mathscr{L}$ is linear.
- $\mathscr{L}\{f'\}(s) = s\mathscr{L}\{f\} - f(0)$ (Often used)
- $\mathscr{L}\{f''\}(s) = s^2F(s) - sf(0) - f'(0)$
- $\mathscr{L}\{e^{at}f(t)\}(s) = F(s-a)$

(1) $\mathscr{L}$ is an operator on f.

$$\begin{aligned}\mathscr{L}\{c_1 f_1 + c_2 f_2\}(s) &= \int_0^\infty e^{-st}[c_1 f_1(t) + c_2 f_2(t)]\,\mathrm{d}t \\ &= c_1 \int_0^\infty e^{-st} f_1(t)\,\mathrm{d}t + c_2 \int_0^\infty e^{-st} f_2(t)\,\mathrm{d}t \\ &= c_1 \mathscr{L}\{f_1\}(s) + c_2 \mathscr{L}\{f_2\}(s)\end{aligned}$$

(2)

$$\begin{aligned}\mathscr{L}\{f'\}(s) &= \int_0^\infty e^{-st} f'(t)\,\mathrm{d}t \\ &= \lim_{A\to\infty} \int_0^A e^{-st}\,\mathrm{d}f(t) \\ &= \lim_{A\to\infty} (e^{-st} f(t) \mid_{t=0}^{A} - \int_0^A (-s) f(t) e^{-st}\,\mathrm{d}t) \quad \text{Differentiation by part} \\ &= \lim_{A\to\infty} e^{-sA} f(A) - f(0) + s \lim_{A\to\infty} \int_0^A f(t) e^{-st}\,\mathrm{d}t \\ &= -f(0) + s\mathscr{L}\{f\}(s)\end{aligned}$$

(3)

$$\begin{aligned}\mathscr{L}\{f''\}(s) &= s\mathscr{L}\{f'\} - f'(0) \\ &= s[s\mathscr{L}\{f\} - f(0)] - f'(0) \\ &= s^2\mathscr{L}\{f\} - sf(0) - f'(0) \\ &= s^2 F(s) - sf(0) - f'(0)\end{aligned}$$

(4)

$$\begin{aligned}\mathscr{L}\{e^{at} f\}(s) &= \int_0^\infty e^{-st} e^{at} f(t)\,\mathrm{d}t \\ &= \int_0^\infty e^{-(s-a)t} f(t)\,\mathrm{d}t \\ &= \mathscr{L}\{f\}(s-a) = F(s-a)\end{aligned}$$

Use laplace transformation to solve nonhomogeneous differential equations

$$ay'' + by' + cy = f(t) \ldots (*) \qquad \text{a, b, c are constants}$$

$Y(s) = \mathscr{L}\{y\}(s) \qquad F(s) = \mathscr{L}\{f\}(s)$

Apply laplace transformation to the both sides of the (*).

$$a\mathscr{L}\{y''\} + b\mathscr{L}\{y'\} + c\mathscr{L}\{y\} = \mathscr{L}\{f\}(s)$$

$$a[s^2Y(s) - sy(0) - y'(0)] + b[sY(s) - y(0)] + cY(s) = F(s)$$

$$Y(s)(as^2 + bs + c) - ay(0)s - [ay'(0) + by(0)] = F(s)$$

$$Y(s) = \frac{ay(0)s + [ay'(0) + by(0)] + F(s)}{as^2 + bs + c}$$

Apply inverse of the laplace transformation to get the solution of nonhomogeneous solution of (*).

R The laplace transformation should be a bijection map to make the process rigorous.

■ **Example 9.1** $\begin{cases} y'' - 3y' + 2y = e^{3t} \\ y(0) = 1, y'(0) = 0 \end{cases}$

$$Y(s) = \frac{s - 3 + \mathscr{L}\{e^{3t}\}(s)}{s^2 - 3s + 2}$$

$$\begin{aligned} \mathscr{L}\{e^{3t}\}(s) &= \int_0^\infty e^{-st}e^{3t}\,\mathrm{d}t \\ &= \int_0^\infty e^{(3-s)t}\,|_{t=0}^\infty \\ &= -\frac{1}{3-s} = \frac{1}{s-3} \end{aligned}$$

$$\begin{aligned} Y(s) =& \frac{s-3}{(s-1)(s-2)} + \frac{1}{(s-1)(s-2)(s-3)} \\ &= \frac{2}{s-1} + \frac{-1}{s-2} + \frac{\frac{1}{2}}{s-1} + \frac{-1}{s-2} + \frac{\frac{1}{2}}{s-3} \\ &= \frac{5}{2}\frac{1}{s-1} - 2\frac{1}{s-2} + \frac{1}{2}\frac{1}{s-3} \end{aligned}$$

We want to find g s.t. $\mathscr{L}\{g\} = \frac{1}{s-1}$

We can verify $\mathscr{L}\{e^{at}\} = \frac{1}{s-a}$

$$\int_0^\infty e^{-st}e^{at}\,\mathrm{d}t = \frac{1}{s-a}$$

Then $y(t) = \frac{5}{2}e^{1-t} - 2e^{2t} + \frac{1}{2}e^{3t}$ ■

Uniqueness of laplace transformation.

Proof. Suppose f, g are two piecewise continuous functions. If $\mathscr{L}\{f\}\mathscr{L}\{g\}$ for large s, then $f \equiv g$.

Sketch

Theorem 9.1 — **Weirstrass Approximation Theorem.** Any continuous function on a interval $[a, b]$ is the uniform limit of a sequence of polynomials. (i.e $\exists$ a sequence of polynomials $\{P_n\}$. s.t. $\max_{a\le t\le b}|P_n(t) - f(t)| \to 0$, as $n \to \infty$)

Let $h = f - g$. Then we have $\mathscr{L}\{f-g\}(s) = 0$ for s large enough.

Claim:$h \equiv 0$

For s large, we have $\int_0^\infty e^{-st}h(t)\,\mathrm{d}t = 0$

$$\begin{aligned}
&\int_0^\infty e^{-nt}e^{-ct}h(t)\,\mathrm{d}t = 0 \qquad s = n+1+c \text{ Just for construction.}\\
=&\int_0^\infty e^{-nt}(e^{-ct}h(t))e^{-t}\,\mathrm{d}t = 0\\
=&\int_1^0 x^n(e^{-ct}h(t)) - \mathrm{d}x
\end{aligned}$$

$x = e^{-t}$, $\mathrm{d}x = -e^{-t}\,\mathrm{d}t$

$$\int_0^1 x^n q(x)\,\mathrm{d}x = 0 \quad \forall n$$

Just sum up some of above equations.

$$\begin{aligned}
&\Rightarrow \int_0^1 P_m q(x)\,\mathrm{d}x = 0 \qquad \text{any polynomial} P_m\\
&\Rightarrow \int_0^1 q^2(x)\,\mathrm{d}x = 0\ldots(2)\\
&\Rightarrow q \equiv 0\\
&h(t) = 0
\end{aligned}$$

To prove (2), by theorem, pick a seq $P_m \to q$

$$\begin{aligned} |\int P_m q - \int q^2| &= |\int (P_m - q) q| \\ &\leq \int |P_m - q| \cdot |q| \to 0 \end{aligned}$$

In addition, the it indeed $\to 0$ is because q is also $\to 0$

$q(x) = e^{-ct} h(t)$, $h = f - g$ is bounded by a exponential function. Therefore, $c \to 0$, $q(x) \to 0$

■

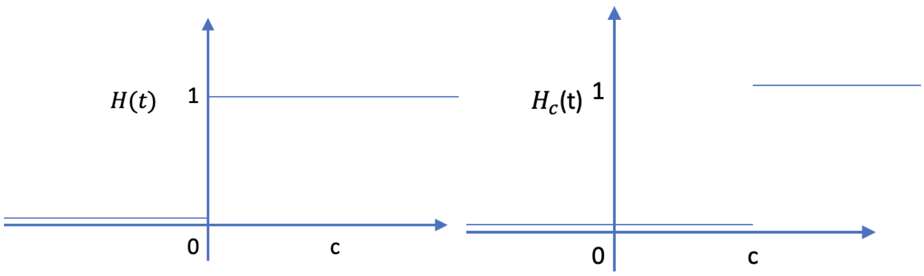
9.2. Wednesday

9.2.1. Laplace Transformation

$$\mathscr{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$$

$$|f(t) \le Me^{ct}| \qquad \text{for t large, } s > c$$

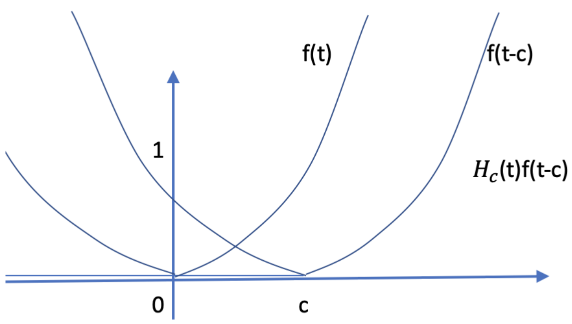
Heariside function $H(t)$



lemma.

$$\mathscr{L}\{H_c(t)f(t-c)\}(s) = e^{-cs}F(s)$$

$(F(s) = \mathscr{L}\{f\}(s))$



Proof.

$$\begin{aligned}
&\mathscr{L}\{H_c(t)f(t-c)\}(s) \\
&= \int_0^\infty e^{-st} H_c(t) f(t-c)\,\mathrm{d}t \\
&= \int_c^\infty e^{-st} f(t-c)\,\mathrm{d}t
\end{aligned}$$

$\bar{t} = t - c$

$$= \int_0^\infty e^{-s(\bar{t}+c)} f(\bar{t})\, \mathrm{d}\bar{t}$$

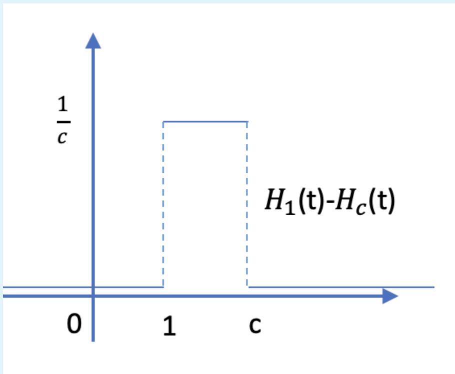
$$= \int_0^\infty e^{-s\bar{t}} f(\bar{t})\, \mathrm{d}\bar{t} e^{-sc}$$

$$= \mathscr{L}\{f\}(s) e^{-sc}$$

■

■ **Example 9.2**

$$y'' - 3y' + 2y = \begin{cases} \frac{1}{c}, & 1 < t < 1 + c \\ 0, \text{otherwise} \end{cases}$$



$$Y = \mathscr{L}\{y\} \qquad y(0) = y'(0) = 0$$

$$\mathscr{L}\{y'' - 3y' + 2y\}(s)$$

$$= s^2 Y(s) - 3sY(s) + 2Y(s)$$

$$= Y(s)(s^2 - 3s + 2) \ldots (1)$$

$$\mathscr{L}\{\frac{1}{c}[H_1(t) - H_{1+c}(t)]\}(s) = \frac{1}{c}(\mathscr{L}\{H_1(t)\}(s) - \mathscr{L}\{H_{1+c}(t)\}(s))$$
$$= \frac{1}{c}(\frac{e^{-s}}{s} - \frac{e^{-(1+c)s}}{s}) \ldots (2)$$

(1)=(2), as we take laplace transformation on both side of the given equation in this example.

$$\begin{aligned}
\therefore Y(s) &= \frac{\frac{1}{c}e^{-s}(1-e^{-cs})}{(s-1)(s-2)s} \\
&= \frac{1}{c}e^{-s}(1-e^{-cs})[\frac{1}{2s} - \frac{1}{s-1} + \frac{1}{2}\frac{1}{s-2}]\ldots(3) \\
&= \frac{1}{c}e^{-s}(1-e^{-cs})\mathscr{L}\{\frac{1}{2} - e^{t} + \frac{1}{2}e^{2t}\}\ldots(4) \\
&= \frac{1}{c}\mathscr{L}\{H_1(t)[\frac{1}{2} - e^{t-1} + \frac{1}{2}e^{2(t-1)}] - H_{1+c}[\frac{1}{2} - e^{t-1-c} + \frac{1}{2}e^{2(t-1-c)}]\}\ldots(5)
\end{aligned}$$

From (3) to (4), we do reverse laplace transformation, such as $\mathscr{L}\{1\}(s) = \int_0^\infty e^{-st}1\,\mathrm{d}t = \cdots = \frac{1}{s}$ and $\mathscr{L}\{H_1(t)\}(s) = e^{-s}\mathscr{L}\{1\}(s)$ (lemma above).

Procedure from (4) to (5) is due to lemma above.

Remember $Y(s) = \mathscr{L}\{y\}$

$$\begin{aligned}
y(t) &= \frac{1}{c}\{H_1(t)[\frac{1}{2} - e^{t-1} + \frac{1}{2}e^{2(t-1)}] - H_{1+c}[\frac{1}{2} - e^{t-1-c} + \frac{1}{2}e^{2(t-1-c)}]\} \\
&= \frac{1}{c}\begin{cases} 0, \ t \le 1 \\ \frac{1}{2} - e^{t-1} + \frac{1}{2}e^{2(t-1)}, \ 1 < t < 1+c \\ -e^{t-1} + \frac{1}{2}e^{2(t-1)} + e^{t-1-c} - \frac{1}{2}e^{2(t-1-c)}, \ t \ge 1+c \end{cases}
\end{aligned}$$

As $c \to 0$, $\frac{1}{2}e^{2(t-1)}\frac{1-e^{-2c}}{c} - e^{t-1}\frac{1-e^{-c}}{c} \to e^{2(t-1)} - e^{t-1}$ (by L'hospital's rule).

■

There is another way of doing this problem.

Definition 9.2 [“Delta” function: δ] For any smooth function φ with compact support on $\mathbb{R}$

$$\int_{-\infty}^{\infty} \delta(t)\varphi(t)\,\mathrm{d}t = \varphi(0)$$

■

■ **Example 9.3** [Example of delta function: “Approximation to identity”] A sequence of functions $\{\eta_\varepsilon\}$, $\eta(x)$ smooth supported inside $(-1,1)$, $\eta(-x) = \eta(x) \ge 0$ and $\int_{-1}^{1}\eta(x)\,\mathrm{d}x = 1$

$$\eta(x) = \begin{cases} ce^{-\frac{1}{1-x^2}} & |x| \le 1 \\ 0 & |x| \ge 1 \end{cases}$$

c is chosen such that $\int_{-1}^{1} \eta = 1$.

$$\eta_\varepsilon = \frac{1}{\varepsilon}\eta(\frac{x}{\varepsilon})$$

Want to show $\lim_{\varepsilon\to 0} \eta_\varepsilon = \delta$, i.e. $\lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} \eta_\varepsilon(x)\varphi(x)\,\mathrm{d}x = \varphi(0)$. ■

Proof.

$$\int_{-\infty}^{\infty} \eta_\varepsilon(x)\,\mathrm{d}x = 1$$

$$\begin{aligned} |\int_{-\infty}^{\infty} \eta_\varepsilon(x)\varphi(x)\,\mathrm{d}x - \varphi(0)| &= |\int_{-\infty}^{\infty} \eta_\varepsilon(x)\varphi(x)\,\mathrm{d}x - \int_{-\infty}^{\infty} \eta_\varepsilon(x)\varphi(0)\,\mathrm{d}x| \\ &\le \int_{-\infty}^{\infty} |\eta_\varepsilon(x)[\varphi(x) - \varphi(0)]|\,\mathrm{d}x = \int_{-\varepsilon}^{\varepsilon} \eta_\varepsilon(x)|\varphi(x) - \varphi(0)|\,\mathrm{d}x \\ &\le \int_{-\varepsilon}^{\varepsilon} \eta_\varepsilon(x)\,\mathrm{d}x |\max_{|x|\le\varepsilon} |\varphi(x) - \varphi(0)|| \to 0 \end{aligned}$$

■

R Delta function isn't a function. As $\delta = \lim_{\varepsilon\to 0} \eta_\varepsilon$, we can see delta function doesn't satisfy the definition of function at point 0.

In addition, the integral of delta function is equal to 1.

■ **Example 9.4**

$$y'' - 3y' + 2y = \begin{cases} \frac{1}{c}, & 1 < t < 1 + c \\ 0, & \text{othrwise} \end{cases}$$

$$y(0) = y'(0) = 0$$

When $c \to 0$, this question is the same as solve

$$y'' - 3y' + 2y = \delta(t-1)$$

$$Y(s)(s^2 - 3s + 2) = \mathscr{L}\{\delta(t-1)\}(s) = \int_0^\infty e^{-st}\delta(t-1)\,\mathrm{d}t$$

$$
\begin{aligned}
\therefore Y(s) &= \frac{e^{-s}}{(s-1)(s-2)} \\
&= e^{-s}(\frac{-1}{s-1} + \frac{1}{s-2}) \\
&= \mathscr{L}\{H_1(t)e^{t-1} + H_1(t)e^{2(t-1)}\}(s) \\
&= \begin{cases} 0,\ t \leq 1 \\ e^{2(t-1)} - e^{ct-1},\ t \geq 1 \end{cases}
\end{aligned}
$$

Chapter 10

Week10

10.1. Monday

10.1.1. Convolution

April 27. (Saturday) 13pm-16pm ChengDao 103; Addition lecture. Next week's lectures are cancelled.

Definition 10.1 [Convolution]

$$(f * g)(t) := \int_0^t f(t-u)g(u)\,\mathrm{d}u$$

Property.

$$\mathscr{L}\{f * g(t)\}(s) = \mathscr{L}\{g\}(s) \cdot \mathscr{L}\{f\}(s)$$

Proof.

$$\begin{aligned}
\mathscr{L}\{f * g(t)\}(s) &= \int_0^\infty e^{-st} f * g(t)\,\mathrm{d}t \\
&= \int_0^\infty e^{-st}[\int_0^t f(t-u)g(u)]\,\mathrm{d}t \\
&= \int_0^\infty \int_0^t (e^{-st} f(t-u)g(u))\,\mathrm{d}u\,\mathrm{d}t \\
&= \int_0^\infty \int_u^\infty (e^{-st} f(t-u)g(u))\,\mathrm{d}t\,\mathrm{d}u \text{ change order of integration}
\end{aligned}$$

Take a look at inner integration.

Take $v = t - u$, $t = v + u$

$$\begin{aligned}
\int_u^\infty e^{-st} f(t-u) g(u) \, \mathrm{d}t &= \int_0^\infty e^{-s(u+v)} f(v) g(u) \, \mathrm{d}v \\
&= \int_0^\infty e^{-su} e^{-sv} f(v) g(u) \, \mathrm{d}v \\
&= e^{-su} g(u) \int_0^\infty e^{-sv} f(v) \, \mathrm{d}v
\end{aligned}$$

Pluge back into the previous equation.

$$\begin{aligned}
\mathscr{L}\{f * g(t)\}(s) &= \int_0^\infty \int_u^\infty (e^{-st} f(t-u) g(u)) \, \mathrm{d}t \, \mathrm{d}u \\
&= \int_0^\infty e^{-su} g(u) \int_0^\infty e^{-sv} f(v) \, \mathrm{d}v \, \mathrm{d}u \\
&= \int_0^\infty e^{-su} g(u) \, \mathrm{d}u \int_0^\infty e^{-sv} f(v) \, \mathrm{d}v \\
&= \mathscr{L}\{g\}(s) \times \mathscr{L}\{f\}(s)
\end{aligned}$$

■

With this property bear in mind, we are able to solve some of problems.

■ **Example 10.1**

$$\begin{cases} ay'' + by' + cy = 0 \\ y(0) = y_0, \ y'(0) = y'_0 \end{cases}$$

At time $t = t_0$ give the particle an impulse force I_0. The first equation becomes

$$ay'' + by' + cy = I_0 \delta(t - t_0)$$

$$\mathscr{L}\{ay'' + by' + cy\} = \mathscr{L}\{I_0 \delta(t - t_0)\} = \int_0^\infty e^{-st} I_0 \delta(t - t_0) \, \mathrm{d}t = e^{-st_0} I_0$$

$$\mathscr{L}\{y'\} = sY(s) - Y_0$$

$$\mathscr{L}\{y''\} = s^2 Y(s) - sY_0 - y'_0$$

$$a(s^2 Y - sy_0 - y'_0) + b(sY(s) - Y_0) + cY = e^{-st_0} I_0$$

$$(as^2+bs+c)Y(s) = y_0(as+b) + ay_0' + I_0 e^{-st_0}$$

$$Y(s) = \frac{y_0(as+b)}{as^2+bs+c} + \frac{ay_0'}{as^2+bs+c} + \frac{I_0 e^{-st_0}}{as^2+bs+c} \dots (1)$$

There are two linearly independent solutions y_1, y_2 where $y_1(0) = 1$, $y_1'(0) = 0$; $y_2(0) = 0$, $y_2'(0) = 1$

$$y(t) = y_0 y_1(t) + y_0' y_2(t) + y_3(t)$$

We get the answer by solving out y_1, y_2, y_3 with the help of reverse laplace transformation. Especially, y_3 needs the property of convolution.

$$\mathscr{L}\{y_1\}(s) = \frac{as+b}{as^2+bs+c}$$

$$\mathscr{L}\{y_2\}(s) = \frac{a}{as^2+bs+c} \Rightarrow \mathscr{L}\{\frac{y_2}{a}\} = \frac{1}{as^2+bs+c}$$

$$\mathscr{L}^{-1}\{\frac{I_0 e^{-st_0}}{as^2+bs+c}\} = \mathscr{L}^{-1}\{I_0 e^{-st_0} \cdot \frac{1}{as^2+bs+c}\} = y_3 \dots (2)$$

$$I_0 e^{-st_0} \cdot \frac{1}{as^2+bs+c} = \mathscr{L}\{I_0 \delta(t-t_0)\} \cdot \mathscr{L}\{\frac{y_2}{a}\} = \mathscr{L}\{I_0 \delta(t-t_0) * \frac{y_2}{a}\}$$

Substitude back to (2), we get

$$y_3 = I_0 \delta(t-t_0) \times \frac{y_2}{a}$$

$$y_3 = \int_0^t I_0 \delta(t-t_0-u) \frac{y_2}{a}(u)\, \mathrm{d}u = \begin{cases} 0, \ t < t_0 \\ I_0 \frac{y_2}{a}(t-t_0), \ t \geq t_0 \end{cases}$$

■

10.1.2. Systems of Linear Equations(First order)

Every order differential equation can be viewed as a system of first order differential equations. Take second order as an example.

$$y'' + p(t)y' + q(t)y = f(t)$$

$$\text{Let } x_1 = y \qquad x_1' = x_2$$

$$x_2 = y' \qquad x_2' = -qx_1 - px_2 + f$$

$$\begin{aligned} x_2' &= y'' \\ &= -py' - qy + f \\ &= -px_2 - qx_1 + f \end{aligned}$$

Define $\mathbb{X} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix} \mathbb{X} + \bar{f}$$

$$\mathbb{X}' = A\mathbb{X} + b$$

Look at the general case.

$$y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_1(t)y' + a_0(t)y = f(t)$$

$$\begin{matrix} x_1 = y \\ x_2 = y \\ \vdots \\ x_n = y \end{matrix} \qquad \begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix} = \begin{pmatrix} 0\,1\,0\,0 \ldots 0\,0\,0 \\ 0\,0\,1\,0 \ldots 0\,0\,0 \\ \ldots \quad \ldots \\ \ldots \quad \ldots \\ 0\,0\,0\,0 \ldots 0\,0\,1 \\ -a_0 - a_1 \cdots - a_{n-1} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ f \end{pmatrix}$$

$$\mathbb{X}' = A\mathbb{X} + b$$

$$\mathbb{X}' = A\mathbb{X}$$

$$\mathbb{X} = \mathbb{C}e^{At}$$

$$e^A = I + A + \frac{A^2}{2} + \cdots + \frac{A^n}{n!} + \cdots$$

If A is diagonal, there is a great property that

$$A = \begin{pmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_n \end{pmatrix} \quad \Rightarrow \quad A^k = \begin{pmatrix} a_1^k & & & 0 \\ & a_2^k & & \\ & & \ddots & \\ 0 & & & a_n^k \end{pmatrix}$$

General system

$$\begin{cases} x_1' = a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + g_1(t) \\ x_2' = a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + g_2(t) \\ \qquad \ldots \\ x_n' = a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + g_n(t) \end{cases}$$

$$\mathbb{X}'(t) = A\mathbb{X}(t) + \vec{g}(t) \ldots (*)$$

$$A = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \ldots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \ldots & a_{2n}(t) \\ & & \ldots & \\ a_{n1}(t) & a_{n2}(t) & \ldots & a_{nn}(t) \end{pmatrix}$$

Theorem 10.1 (Existence/Uniqueness) There exists a unique solution to the IVP of (*) with $\mathbb{X}(0) = \begin{pmatrix} x_1^0 \\ \vdots \\ x_m^0 \end{pmatrix}$ on entire $\mathbb{R}$

Theorem 10.2 Let V be the set of all solutions to $\mathbb{X}' = A\mathbb{X}$. Then V is a vector space of dimension n spanned by $\{\mathbb{X}_1(t), \ldots, \mathbb{X}_n(t)\}$ where $\mathbb{X}_j(0) = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$. The 1 is in the j^{th} entry.

10.2. Wednesday

10.2.1. System of 1^{th} order linear differential equations

Notification: Next week's lectures are cancelled together with tutorials on monday and tuesday. However, the wednesday's tutorial will be held as usually. In addition, when lecture is resumed, you are suppose to have read the whole chapter 3.

$$\begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix} = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \dots & a_{2n}(t) \\ & \dots & & \\ a_{n1}(t) & a_{n2}(t) & \dots & a_{nn}(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{pmatrix}$$

$$\mathbb{X}' = A\mathbb{X} + \vec{g}$$

$$\vec{g} \equiv 0 \begin{cases} \mathbb{X}' = A\mathbb{X} \\ \mathbb{X}(0) = \mathbb{X}_0 \end{cases}$$

Set of solutions: vector space of dimension n.

Theorem 10.3 Suppose $\mathbb{X}_1, \dots, \mathbb{X}_k(t)$ are linearly independent for any t $\iff$ $\mathbb{X}_1(t_0), \dots, \mathbb{X}_k(t_0)$ are linearly independent for some t_0.

Definition 10.2 $\mathbb{X}_1, \dots, \mathbb{X}_n$ are solutions for $\mathbb{X}' = A\mathbb{X}$

$$[\mathbb{X}_1, \dots, \mathbb{X}_n] = \begin{vmatrix} | & | & | \\ x_1 & x_2 & x_3 \\ | & | & | \end{vmatrix}$$

Lemma. $\mathbb{X}_1, \dots, \mathbb{X}_n$ are linearly independent $\Rightarrow W[\mathbb{X}_1, \dots, \mathbb{X}_n]$ is never 0.

Proof.

$$W' = [a_{11}(t) + \cdots + a_{mm}(t)]W$$

■

First have a look at the case when $n = 2$

$$\mathbb{X}' = A\mathbb{X}$$

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Two solution $\mathbb{X}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ and $\bar{\mathbb{X}}(t) = \begin{pmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{pmatrix}$

$$W[\mathbb{X}, \bar{\mathbb{X}}] = \begin{vmatrix} x_1(t) & \bar{x}_1(t) \\ x_2(t) & \bar{x}_2(t) \end{vmatrix}$$

$$W' = \begin{vmatrix} x_1'(t) & \bar{x}_1(t) \\ x_2'(t) & \bar{x}_2(t) \end{vmatrix} + \begin{vmatrix} x_1(t) & \bar{x}_1'(t) \\ x_2(t) & \bar{x}_2'(t) \end{vmatrix}$$

This is the differentiation of a determinant instead of a matrix.

Also,

$$\begin{aligned} W' &= \begin{vmatrix} x_1' & \bar{x}_1' \\ x_2 & \bar{x}_2' \end{vmatrix} + \begin{vmatrix} x_1 & \bar{x}_1 \\ x_2' & \bar{x}_2' \end{vmatrix} \\ &= \begin{vmatrix} a_{11}x_1 + a_{12}x_2 & a_{11}\bar{x}_1 + a_{12}\bar{x}_2 \\ x_2 & \bar{x}_2 \end{vmatrix} + \begin{vmatrix} x_1 & \bar{x}_1 \\ a_{21}x_1 + a_{22}x_2 & a_{21}\bar{x}_1 + a_{22}\bar{x}_2 \end{vmatrix} \end{aligned}$$

Take a look at the first determinant.

$$\begin{vmatrix} a_{11}x_1 + a_{12}x_2 & a_{11}\bar{x}_1 + a_{12}\bar{x}_2 \\ x_2 & \bar{x}_2 \end{vmatrix} = \begin{vmatrix} a_{11}x_1 & a_{11}\bar{x}_1 \\ x_2 & \bar{x}_2 \end{vmatrix} + \begin{vmatrix} a_{12}x_2 & a_{12}\bar{x}_2 \\ x_2 & \bar{x}_2 \end{vmatrix}$$
$$= a_{11}\begin{vmatrix} x_1 & \bar{x}_1 \\ x_2 & \bar{x}_2 \end{vmatrix} + a_{12}\begin{vmatrix} x_2 & \bar{x}_2 \\ x_2 & \bar{x}_2 \end{vmatrix} = a_{11}W$$

$$W' = [a_{11}(t) + \cdots + a_{nn}(t)]W$$

$$W(t) = e^{\int [a_n(t) + \cdots + a_{nn}(t)]\,\mathrm{d}t} c$$

■ **Example 10.2**

$$\begin{cases} \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{cases}$$

$$\mathbb{X}' = A\mathbb{X}$$

$$\mathbb{X} = e^{At}\mathbb{X}(0)$$

$$\begin{aligned} e^A &= I + A + \frac{A^2}{2!} + \cdots + \frac{A^n}{n!} + \dots \\ &= I + Q^{-1}DQ + Q^{-1}D^2Q + \cdots + Q^{-1}D^nQ + \dots \\ &= Q^{-1}(I + D + D^2 + \cdots + D^n + \dots)Q \\ &= Q^{-1}e^DQ = Q^{-1}\begin{pmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{pmatrix} Q \end{aligned}$$

■

$$|A - \lambda I| = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} - \begin{pmatrix} e^{\lambda} & 0 \\ 0 & e^{\lambda_2} \end{pmatrix} = |Q^{-1}DQ - Q^{-1}\lambda IQ|$$
$$= |Q^{-1}(D - \lambda I)Q| + |Q^{-1}||D - \lambda I||Q|$$

$$0 = \begin{vmatrix} 1-\lambda & 3 \\ 4 & 3-\lambda \end{vmatrix} = (\lambda-1)(\lambda-3) - 8 = (\lambda-5)(\lambda+1)$$

Therefore, $\lambda = -1{,}5$ eigenvalues.

$$Q = \begin{pmatrix} | & | \\ v & w \\ | & | \end{pmatrix}$$

$\lambda_1 = -1 \ (A - \lambda_1 I) = 0$

$$0 = \begin{pmatrix} 1+1 & 2 \\ 4 & 3+1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 2v_1 + 2v_2 \\ 4v_1 + 4v_2 \end{pmatrix} \Rightarrow \text{ eigenvector } v_1 = -v_2, \ v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\lambda_2 = 5 \ (A - \lambda_2 I)W = 0$

$$0 = \begin{pmatrix} 1-5 & 2 \\ 4 & 3+1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} -4 & 2 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \Rightarrow w_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\delta = \{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \} \qquad \beta = \{v, w\}$$

$Av = -v, (= \lambda, v) = \lambda_1 v + 0w$

$Aw = 5w, (= \lambda_2 w) = 0v + \lambda_2 w$

$A = [A]_\delta^\delta \qquad [A]_\beta^\beta = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}$

$$[A]_\delta^\delta = [I]_\beta^\delta [A]_\beta^\beta [I]_\delta^\beta \to \begin{pmatrix} | & | \\ v & w \\ | & | \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$$

$$\mathbb{X}(t) = c_1 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{5t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\mathbb{X}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 \\ -c_1 + 2c_2 \end{pmatrix}$$

Therefore, $c_1 = \frac{1}{3}$ $c_2 = \frac{2}{3}$

Chapter 11

Week11

11.1. Monday

11.1.1. The eigenvalue-eigenvector method of finding solutions

To solve the problem, we seperate the situation of the following equation.

$$\dot{\mathbb{X}} = A\mathbb{X}$$

where $A : n \times n$ matrix and $\mathbb{X} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

i A is diagonable

ii A has real eigenvalues but not diagonable

iii A has 2 complex eigen values

■ **Example 11.1**

$$\dot{\mathbb{X}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbb{X}$$

Need to find two linearly independent solutions or e^{At}.

$$0 = |A - \lambda I| = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1$$

$$\lambda = \pm i$$

For $\lambda = i$, $A - iI = \begin{pmatrix} -i & 1 \\ 1 & -i \end{pmatrix}$

$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, $(A - iI)v = 0$

In this equation, v and 0 are vectors with corresponding size.

$$0 = \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -iv_1 - v_2 \\ v_1 - iv_2 \end{pmatrix}$$

$$v = \begin{pmatrix} i \\ 1 \end{pmatrix}$$

Similiarly when $\lambda = -i$, $w = \bar{v}$

Therefore, should be something like $e^{At} = \begin{pmatrix} & \\ & \end{pmatrix} \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \begin{pmatrix} & \\ & \end{pmatrix}$

$\beta = \{ \begin{pmatrix} i \\ 1 \end{pmatrix}, \begin{pmatrix} -i \\ 1 \end{pmatrix} \}$

$$\begin{aligned} A = [A]_\delta &= [I]_\beta^\delta [A]_\beta [I]_\delta^\beta \\ &= \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} it & 0 \\ 0 & -it \end{pmatrix} \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}^{-1} \end{aligned}$$

$$\mathbb{X} = e^{At} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$(i) = e^{\lambda t} v = e^{it} \begin{pmatrix} i \\ 1 \end{pmatrix}$$

$$(ii) = e^{-\lambda t} \bar{v} = e^{-it} \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

$$\begin{aligned} e^{it} \begin{pmatrix} i \\ 1 \end{pmatrix} &= (\cos t + i \sin t)[\begin{pmatrix} 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \end{pmatrix}] \\ &= [\cos t \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \sin t \begin{pmatrix} 1 \\ 0 \end{pmatrix}] + i[\sin t \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \cos t \begin{pmatrix} 1 \\ 0 \end{pmatrix}] \\ &= r(t) + is(t) \end{aligned}$$

We have got two solutions $r(t)$ and $s(t)$. Let $t = 0$, we can see they are indeed linearly independent. ▪

The underlying reason how we got two solution in previous example. Let $\mathbb{X} = r(t) + is(t)$ be a solution the equation $\dot{\mathbb{X}} = A\mathbb{X}$. Differentiate the solution. We get

$$\dot{\mathbb{X}} = \dot{r}(t) + i\dot{s}(t) = A(r(t) + is(t))$$

This implies $\dot{r}(t) = Ar(t)$ and $\dot{s}(t) = As(t)$ which means they are the solution of the equation $\mathbb{X} = A\dot{\mathbb{X}}$.

After we have seen the example of diagonalizable A, know take a look at the second case.

▪ **Example 11.2**

$$\mathbb{X} = A\dot{\mathbb{X}}, A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) + 1 = (\lambda - 2)^2$$

$$0 = (A - 2I)v = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -v_1 - v_2 \\ v_1 + v_2 \end{pmatrix}$$

$\lambda = 2,2$

$$v_1 = -v_2 \qquad v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

A isn't diagonalizable.

Generalized eigenvector w $Aw = \lambda w + v$

$\beta = \{v, w\}$ and $\delta = \{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \}$

$$A = [A]_\delta = [I]_\beta^\delta [A]_\beta [I]_\delta^\beta$$
$$= \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}^{-1}$$

The $[A]_\beta$ is got from the facts that $Av = 2v + 0 \cdot w$ and $Aw = 2w + v$ (definition of w).

The $[I]_\beta^\delta$ can be calculated with v known. (See following.)

$$(A - \lambda I)w = v$$

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Choose w to be $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$

As $[I]_\beta^\delta = \begin{matrix} | & | \\ v & w \\ | & | \end{matrix}$, we get the target form of $[A]_\beta^\delta$.

With the expression of A,

$$e^{At} = \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix} e^{\begin{pmatrix} 2t & t \\ 0 & 2t \end{pmatrix}} \begin{pmatrix} 2t & t \\ 0 & 2t \end{pmatrix}^{-1}$$

$$e^{Dt} = I + Dt + \frac{(Dt)^2}{2!} + \cdots + \frac{(Dt)^n}{n!} + \cdots$$

Decomposite $Dt = \begin{pmatrix} 2t & t \\ 0 & 2t \end{pmatrix} = \begin{pmatrix} 2t & 0 \\ 0 & 2t \end{pmatrix} + \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}$. Let the first term to be I and second term to be J.

$$(I+J)^2 = I^2 + IJ + JI + J^2 = I^2 + 2IJ + J^2 = I + \begin{pmatrix} 0 & 2t^2 \\ 0 & 0 \end{pmatrix}$$

As I is a diagonal matrix, the second equal sign holds.

Similiarly

$$\begin{aligned} (I+J)^n = I^n + nI^{n-1}J + \ldots &= \begin{pmatrix} (2t)^n & 0 \\ 0 & (2t)^n \end{pmatrix} + n \begin{pmatrix} (2t)^{n-1} & 0 \\ 0 & (2t)^{n-1} \end{pmatrix} \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} (2t)^n & nt(2t)^{n-1} \\ 0 & (2t)^n \end{pmatrix} \end{aligned}$$

Higher order terms of J are zero, for instance $J^2 = 0$. This is the reason why the first equal sign holds.

Now, we are able to simplify e^{Dt}.

$$\begin{aligned} e^{Dt} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + Dt + \frac{(Dt)^2}{2!} + \cdots + \frac{(Dt)^n}{n!} + \cdots \\ &= \begin{pmatrix} 1 + 2t + \frac{(2t)^2}{2!} + \cdots & t(1 + 2t + \cdots + \frac{n(2t)^n - 1}{n!} + \cdots) \\ 0 & 1 + 2t + \frac{(2t)^2}{2!} + \cdots \end{pmatrix} \\ &= \begin{pmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{pmatrix} \end{aligned}$$

Definition 11.1 [Foundamental matrix]

$$\dot{\mathbb{X}} = A\mathbb{X}$$

Suppose we have n linearly independent solutions $\mathbb{X}_1(t), \mathbb{X}_2, \ldots, \mathbb{X}_n(t)$. Then, the matrix

$$\underset{\sim}{\mathbb{X}}(t) = \begin{pmatrix} | & | & & | \\ \mathbb{X}_1(t) & \mathbb{X}_2(t) & \ldots & x_n(t) \\ | & | & & | \end{pmatrix}$$

is called fundamental matrix.

R e^{At} is a fundamental matrix and $e^{At} = \underset{\sim}{\mathbb{X}}(t)\underset{\sim}{\mathbb{X}}^{-1}(0)$, where $\underset{\sim}{\mathbb{X}}(t)$ is any fundamental matrix.

Variation of parameters

$$\dot{\mathbb{X}} = A\mathbb{X} + \vec{f}(t)$$

Let $\underset{\sim}{\mathbb{X}}(t)$ be a fundamental matrix for $\mathbb{X} = A\mathbb{X}$. To solve $\dot{\mathbb{X}} = A\mathbb{X} + \vec{f}(t)$, we set $\mathbb{X}(t) =$

$\bar{\underline{X}}(t)\vec{c}(t)$

$$\begin{aligned}\dot{\mathbb{X}}(t) &= \dot{\bar{\underline{X}}}(t)\vec{c}(t) + \bar{\underline{X}}(t)\dot{\vec{c}}(t)\\ &= A\bar{\underline{X}}(t)\vec{c}(t) + \bar{\underline{X}}(t)\dot{\vec{c}}(t)\\ &= A\mathbb{X}(t) + \vec{f}(t)\end{aligned}$$

To solve

$$\bar{\underline{X}}(t)\vec{c}(t) = \vec{f}(t)$$

$$\vec{c} = \bar{\underline{X}}(t)^{-1}\vec{f}(t)$$

$$\therefore \quad \vec{c}(t) = \int \bar{\underline{X}}^{-1}(t) f(t)\,\mathrm{d}t$$

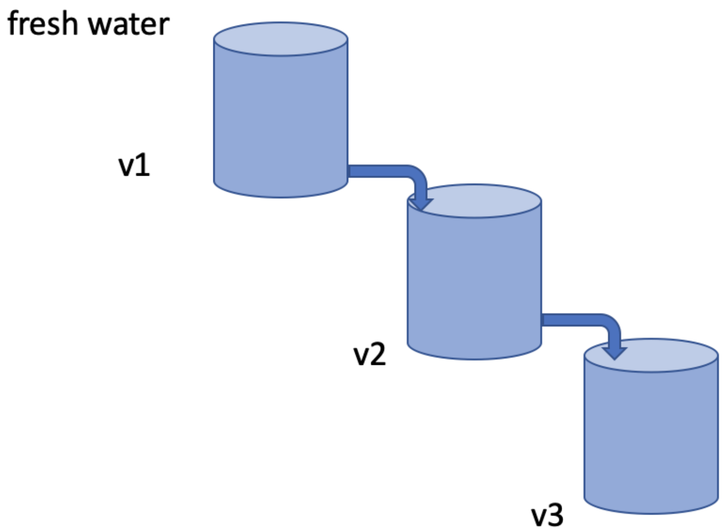
A particular solution is

$$\mathbb{X}(t) = \bar{\underline{X}}(t)\int \bar{\underline{X}}^{-1}(t)\vec{f}(t)\,\mathrm{d}t$$

11.2. Wednesday

11.2.1. Applications of system of differential equations

Fresh water flows into a full container V_1 with a constant flow rate r L/min. Let the amount of salt in tank i be noted as $x_i(t)$.



$$x_1'(t) = -k_1 x_1$$

$$x_2'(t) = k_1 x_1 - k_2 x_2$$

$$x_3'(t) = k_2 x_2 - k_3 x_3$$

where $k_1 = \frac{r}{v_1}$, $k_2 = \frac{r}{v_2}$, $k_3 = \frac{r}{v_3}$

■ **Example 11.3** Conditions:

$$V_1 = 20L \quad V_2 = 40L \quad V_3 = 50L \quad r = 10L/min$$

$$x_1(0) = 15kg \quad \& \quad x_2(0) = 0 \quad \& \quad x_3(0) = 0$$

Find the expression of all x_i with respect to t.

$$\begin{cases} x_1' = -\frac{1}{2}x_1 \\ x_2' = \frac{1}{2}x_1 - \frac{1}{4}x_2 \\ x_3' = \frac{1}{4}x_2 - \frac{1}{5}x_3 \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{4} & -\frac{1}{5} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$0 = |A - \lambda I| = \begin{vmatrix} -\frac{1}{2} - \lambda & 0 & 0 \\ \frac{1}{2} & -\frac{1}{4} - \lambda & 0 \\ 0 & \frac{1}{4} & -\frac{1}{5}\lambda \end{vmatrix}$$

$$\lambda = -\frac{1}{2}, -\frac{1}{4}, -\frac{1}{5}$$

When $\lambda = -\frac{1}{2}$

$$0 = (A - \frac{1}{2}I)v_1 = \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & -\frac{3}{10} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2}a + \frac{1}{4}b \\ \frac{1}{4} + \frac{3}{10}c \end{pmatrix}$$

$$v = \begin{pmatrix} -3 \\ 6 \\ -5 \end{pmatrix}$$

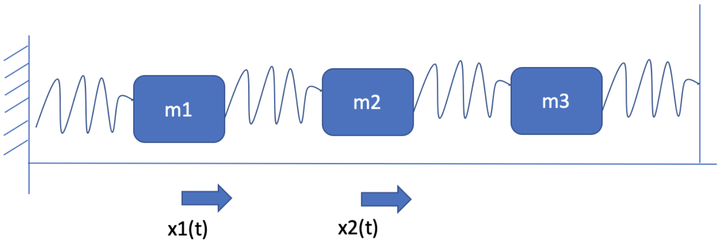
With exactly the same procedure, we can get the other to eigenvector with respect to $\lambda = -\frac{1}{4}, \lambda = -\frac{1}{5}$.

$$v_2 = \begin{pmatrix} 0 \\ 1 \\ -5 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\mathbb{X}(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 + c_3 e^{\lambda_3 t} v_3$$

$$\begin{pmatrix} 15 \\ 0 \\ 0 \end{pmatrix} = \mathbb{X}(0) = c_1 \begin{pmatrix} -3 \\ 6 \\ 5 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix} + c_3 e^{-\frac{1}{5}t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$c_1 = -5 \qquad c_2 = 30 \qquad c_3 = 125$$



Three mases slide without friction. Four spring with spring constant k_1,k_2,k_3,k_4. $x_i(t)$ is the displacement of m_i from its equilibrium equation. First spring is stratcheed by x_1. Second is stratched by $x_2 - x_1$ and the third is stratched by $x_3 - x_2$. The fourth spring is stratched by $-x_4$

With

$$my'' + cy' + ky = 0$$

derived previously, we have

$$m_1 x_1'' = -k_1 x_1 + k_2(x_2 - x_1)$$

$$m_2 x_2'' = -k_2(x_2 - x_1) + k_3(x_3 - x_2)$$

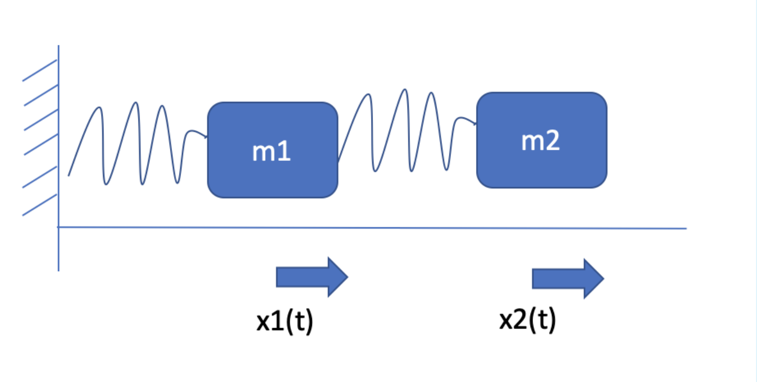
$$m_3 x_3'' == -k_3(x_3 - x_2) - k_4 x_3$$

. That is

$$\begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}'' = \begin{pmatrix} -(k_1 + k_2) & k_2 & 0 \\ k_2 & -(k_2 + k_3) & k_3 \\ 0 & k_3 & -(k_3 + k_4) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

The matrix is tri-diagonal.

■ **Example 11.4** Consider a simpler case.



$$k_1 = 100, k_2 = 50, m_1 = 2, m_2 = 1$$

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}'' = \begin{pmatrix} -150 & 50 \\ 50 & -50 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

There are two methods to solve this question.

First method is change this question to a 4×4 first order system. Convert the question to something we are familiar with; $\dot{\mathbb{X}} = A\mathbb{X}$

Let

$$y_1 = x_1, y_2 = x_1', y_3 = x_2, y_4 = x_2'$$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}' = \begin{pmatrix} y_2 \\ x_1'' \\ y_4 \\ x_2'' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -75 & 0 & 25 & 0 \\ 0 & 0 & 0 & 1 \\ 50 & 0 & -50 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$$

$$0 = |A - \lambda I|$$

We can get four solutions.

Second method

$$\mathbb{X}'' = A\mathbb{X}, \mathbb{X} = e^{\alpha t}v, \mathbb{X}' = \alpha e^{\alpha t}v, \mathbb{X}'' = \alpha^2 e^{\alpha t}v$$

$$0 = |A - \lambda I| = \begin{vmatrix} -75-\lambda & 25 \\ 50 & -50 \end{vmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= (\lambda + 75)(lambda + 50) - 1250$$

$$= (\lambda + 100)(\lambda + 25)$$

$$\alpha^2 = -25, -100$$

Let's have a look at the first solution with $\alpha = 5i$

$$0 = \begin{pmatrix} -75+25 & 25 \\ 50 & -50+25 \end{pmatrix} \begin{pmatrix} c_1 \\ v_2 \end{pmatrix}$$

$$v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\mathbb{X}(t) = e^{\alpha t} v = [\cos 5t + i \sin 5t] \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Similiarly, when $\alpha = 10i$

$$\mathbb{X}(t) = [\cos(10t) + i \sin(10t)] \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

With real parts and imagine parts, there are four solutions. ■

Chapter 12

Week12

12.1. Monday

12.1.1. Qualitative Theory

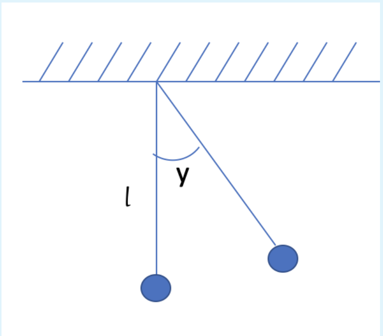
Stability or "equilibria":

$$\dot{\mathbb{X}} = f(t, \mathbb{X})$$

If a solution $\mathbb{X}(t)$ is independent of t, i.e. $\dot{\mathbb{X}} \equiv 0$, then we call it equilibrium station.

Thus, $\dot{\mathbb{X}} = f(t, \mathbb{X}) = 0$ or $f(t, \xi)$ since $\mathbb{X}(t) = \xi$ for some constant ξ.

Autonomous system. $\dot{\mathbb{X}} = f(\mathbb{X})$ $\quad$ f is independent of t.



■ **Example 12.1** tangential component:

$$F = -mg - \sin y = ma$$

i.e

$$m(ly)'' = -mg\sin y$$

or

$$y'' + \frac{g}{l}\sin y = 0$$

Let $\mathbb{X} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, where $x_1 = y$, $x_2 = y'$

$$\mathbb{X}' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} y' \\ y'' \end{pmatrix} = \begin{pmatrix} y' \\ -\frac{g}{l}\sin y \end{pmatrix} = \begin{pmatrix} x_2 \\ -\frac{g}{l}\sin x_1 \end{pmatrix}$$

Equilibira point: $\mathbb{X}' = 0$, i.e.

$$\begin{pmatrix} x_2 \\ -\frac{g}{l}\sin x_1 \end{pmatrix} \quad \Rightarrow \quad \mathbb{X} = \begin{pmatrix} k\pi \\ 0 \end{pmatrix} \quad k \in \mathbb{Z}$$

■

The equilibium solution ξ is stable means:

"If $\mathbb{X}(0)$ is close to the equilibrium solution $\mathbb{X}(t) = \xi$, then $\mathbb{X}(t)$ stays close to the equilibrium solution ξ for all $t > 0$"

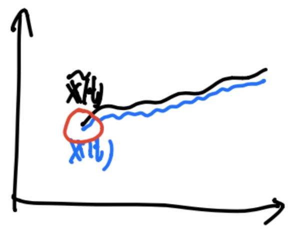
"For any given $\varepsilon > 0, \exists\, \delta > 0$ s.t. for any $||\mathbb{X}(0) - \xi|| < \delta$, we have $||\mathbb{X}(t) - \xi|| < \varepsilon \quad \forall t > 0$"

■ **Example 12.2** In the last example $(0,0)$ is stable since for every $\varepsilon > 0$, $||\mathbb{X}(t) - (0,0)|| < \varepsilon$ for all t, if $||\mathbb{X}(0) - (0,0)|| < \delta$

$(\pi, 0)$ is unstable because there exists $\varepsilon_0 > 0$ s.t. $\forall \delta > 0$, we have $||x(t) - (x,0)|| > \varepsilon_0$ for some $t > 0$, with $||\mathbb{X}(0) - (x,0)|| < \delta$. To show, pick $\varepsilon = \frac{\pi}{2}$, then $\forall \delta > 0$, $||x(t_0) - (\pi,0)|| = \pi > \frac{\pi}{2} = \varepsilon$

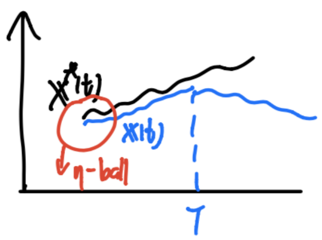
■

Let $\tilde{\mathbb{X}}(t)$ be a solution, then we say $\tilde{\mathbb{X}}(t)$ is stable if $\forall \varepsilon > 0, \exists \delta > 0$, s.t. a solution $\mathbb{X}(t)$ with $|\mathbb{X}(0) - \tilde{\mathbb{X}}(0)| < \delta$ satisfies $|\mathbb{X}(t) - \tilde{\mathbb{X}}(t)| < \varepsilon$ for all $t > 0$.



Continuous dependence on initial value:

For any $T > 0$ given and for any $\varepsilon > 0$, there exists a $\eta > 0$ s.t. $||\mathbb{X}(t) - \mathbb{X}^*(t)|| < \varepsilon$ if $||\mathbb{X}(0) - \mathbb{X}^*(0)|| < \eta$ for all $t < T$



Stability of linear system:

Consider $\dot{X} = AX$ where A is a matrix with constant coeffcient.

(i) If A has an eigenvalue with positive real root, then all solutions are countable.

(ii) If all eigenvalue of A have negative real part, then all soluions are stable.

(iii)Suppose $\lambda_1 = i\delta_1, \lambda_2 = i\delta_2, ..., \lambda_k = i\delta_n$ and $\lambda_{k+1}, ..., \lambda_n$ are eigenvalues with negative real parts. Let $\lambda_j = i\delta_j, j = 1, ...,$ all solutions are mustable.

Proof. (i) Assume A has an eigenvalue $\lambda = \alpha + i\beta$, *where* $\alpha > 0$.

Case 1: $\beta = 0$. Let v be the corresponding eigenvalue of λ. $X = e^{\alpha t}$ is a solution. $||X(t)|| = ||e^{\lambda t}v|| = e^{\alpha t} \Rightarrow as\ t \to \infty$.

Then $||\delta X(t)|| = \delta e^{\alpha t}||v|| \to \infty\ as\ t \to \infty$, Thus $||\delta X(0) - 0|| = ||\delta X(0)|| = \delta||v||$, which implies solution is not stable. ■

12.2. Wednesday

12.2.1. Stability of Linear System

Theorem 12.1 $\dot{\mathbb{X}} = A\mathbb{X}$, where A is a constant matrix

(i) If A has an eigenvalue with positive real part, then all solutions are unstable.

(ii) If all eigenvalues of A have negative real parts, then all solutions are stable.

(iii) Suppose $\lambda_1 = i\delta_1, \ldots, \lambda_k = i\delta_k$, and $\lambda_{k+1}, \ldots, \lambda_n$ have negative real parts. Assume multiplicity of λ_j is m_j, $j = 1, \ldots, k$. Then all solutions are stable if A has m_j linearly indepedent eigenvectors corresponding to λ_j $j = 1, \ldots, k$. Otherwise, all solutions are unstable.

Lemma:. $\mathbb{X} \equiv 0$ is a solution. Suppose $\tilde{\mathbb{X}} \equiv 0$ is stable, then $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $||\mathbb{X}(t) - 0|| < \varepsilon$ if $||\mathbb{X}(0) - 0|| < \delta$ for all $t > 0$. We want to show that $\mathbb{X}^*$ is also stable, where $\mathbb{X}^*(t)$ is an arbitrary solution. i.e. We want to show that $\forall \varepsilon > 0, \exists \delta > 0$, s.t. $||\mathbb{X}(t) - \mathbb{X}^*(t)|| < \varepsilon$ if $||\mathbb{X}(0) - \mathbb{X}^*(0)|| < \delta$ for all $t > 0$.

■ **Example 12.3**

$$\dot{\mathbb{X}} = \begin{pmatrix} 2 & -3 & 0 \\ 0 & -6 & -2 \\ -6 & 0 & -3 \end{pmatrix} \mathbb{X}$$

$$0 = |A - \lambda I| = \begin{vmatrix} 2-\lambda & -3 & 0 \\ 0 & -6-\lambda & -2 \\ -6 & 0 & -3-\lambda \end{vmatrix} = -\lambda^2(\lambda + 7)$$

$$\Rightarrow \lambda_1 = 0 \lambda_2 = 0 \lambda_3 = -7$$

When $\lambda = 0$,

$$Av = \begin{pmatrix} 2 & -3 & 0 \\ 0 & -6 & -2 \\ -6 & 0 & -3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2a - 3b \\ -6b - 2c \\ -6a - 3c \end{pmatrix} = \underline{0} \Rightarrow v = \begin{pmatrix} 3 \\ 2 \\ -6 \end{pmatrix}$$

Since v is the only eigenvector of 0, we know all solutions are unstable. ■

When f is arbitrary (f is not linear), can we still understand the solutions near an equilibrium solution $\mathbb{X}_0$; $f(\mathbb{X}_0) = 0$?

Idea: near the point x_0, write

$$f(x_0) = A(x - x_0) + o(||x - x_0||^2)$$

Chapter 13

4.27

13.0.1. Stability of Equilibrium Solutions

$$\dot{\mathbb{X}} = (\mathbb{X} - \mathbb{X}_0)' \qquad \mathbb{Y} = \mathbb{X} - \mathbb{X}_0$$

$$(\mathbb{X} - \mathbb{X}_0)' = A(\mathbb{X} - \mathbb{X}_0) + o(||x - x_0||)$$

$$\dot{\mathbb{Y}} = A\mathbb{Y} + p(||\mathbb{Y}||)$$

Definition 13.1 A solution $\phi(t)$ of a system is said to be stable if $\forall \varepsilon > 0 \exists \delta > 0$ s.t.

$$||\varphi(0) - \phi(0)|| < \delta \Rightarrow ||\varphi(t) - \phi(t)|| < \varepsilon \quad \forall t > 0$$

Definition 13.2 A solution $\phi(t)$ is said to be asymptotically stable if $\exists \delta > 0$ s.t.

$$||\varphi(0) - \phi(0)|| < \delta \Rightarrow ||\varphi(t) - \phi(t)|| \to 0 \quad ast \to \infty$$

R A symptotically stability $\Rightarrow$ Stability

A general system

$$(*) \quad \dot{\mathbb{X}} = \vec{f}(\mathbb{X}) = \begin{pmatrix} f_1(\mathbb{X}) \\ \vdots \\ f_n(\mathbb{X}) \end{pmatrix}$$

Let x_0 be an equilibrium solution to (*), i.e. $\vec{f}(\mathbb{X}_0) = 0$. Want to understand the stability properties of x_0 $\vec{f}(\mathbb{X}_0) = 0$.

$$\begin{pmatrix} f_1(\mathbb{X}) \\ f_2(\mathbb{X}) \end{pmatrix} = \begin{pmatrix} f_1(\mathbb{X}_0) + \Delta f_1(\mathbb{X}_0)(\mathbb{X} - \mathbb{X}_0) = O(||\mathbb{X} - \mathbb{X}_0||^2) \\ f_2(\mathbb{X}_0) + \dots \end{pmatrix}$$

$$\vec{f}(\mathbb{X}) = \vec{f}(\mathbb{X}_0) + \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}_{\mathbb{X}_0} \begin{pmatrix} x_1 - x_{0,1} \\ \vdots \\ x_n - x_{0,n} \end{pmatrix} + O(||\mathbb{X} - \mathbb{X}_0||^2)$$

$$(\mathbb{X} - \mathbb{X}_0)' = \dot{\mathbb{X}} = \vec{f}(\mathbb{X}) = A(\mathbb{X} - \mathbb{X}_0) + o(||\mathbb{X} - \mathbb{X}_0||)$$

$$\dot{\mathbb{Y}} = A\mathbb{Y} + o(||\mathbb{Y}||)$$

$$\dot{\mathbb{Y}} = A\mathbb{Y} + g(\mathbb{Y}), \text{where} g(\mathbb{Y}) o(||\mathbb{Y}||) \quad \text{near} \mathbb{Y} = 0$$

Theorem 13.1 (1) The equilibrium solution $\mathbb{Y} = 0$ is unstable if A has (at least) one eigenvalue with positive real part.

(2)$\mathbb{Y} = 0$ is a sympototically stable if all eigenvalues of A have negative real parts.

(3)Otherwise, the stability of $\mathbb{Y} = 0$ cannot be determined from the linear system

■ **Example 13.1**

$$\dot{\mathbb{X}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbb{X} \pm \begin{pmatrix} x_1(x_1^2 + x_2^2) \\ x_2(x_1^2 + x_2^2) \end{pmatrix}, \quad \mathbb{X} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$\frac{(x_1||X||^2, x_2||X||^2)}{||X||} = (x_1||X||, x_2||X||) \to 0$ as $x \to 0$

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0$$

$\lambda = \pm i$

$$\frac{d}{dt}||X||^2(t) = (x_1^2 + x_2^2) = 2x_1\dot{x_1} + 2x_2\dot{x_2}$$
$$= 2x_1x_1(x_1^2 + x_2^2) + 2x_2x_2(x_1^2 + x_2^2) = 2(x_1^2 + x_2^2)^2 > 0$$

i.e. $||x||^2$ is increasing in t along a solution curve ■

■ **Example 13.2** $\begin{cases} \frac{dx}{dt} = 1 - xy \\ \frac{dy}{dt} = x - y^3 \end{cases}$ Equilibrium solutions $\begin{cases} 1 - xy = 0 \\ x - y^3 = 0 \end{cases}$ $\therefore (1,1)(-1,-1)$

$(1,1)$

$$1 - xy = 1 - [(x-1)+1][(y-1)+1] = -(x-1) - (y-1) - (x-1)(y-1)$$

$$x - y^3 = (x-1) + 1 - [(y-1)+1]^3 = (x-1) - 3(y-1) - 3(y-1)^2 - (y-1)^3$$

$$\begin{cases} (x-1)' = -(x-1) - (y-1) - (x-1)(y-1) \\ (y-1)' = (x-1) - 3(y-1) - 3(y-1)^2 - (y-1)^3 \end{cases}$$

$$\begin{pmatrix} x-1 \\ y-1 \end{pmatrix}' = \begin{pmatrix} -1 & -1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} x-1 \\ y-1 \end{pmatrix} + \begin{pmatrix} -(x-1)(y-1) \\ -3(y-1)^2 - (y-1)^3 \end{pmatrix}$$

$$0 = |A - \lambda I| = \begin{vmatrix} -1-\lambda & -1 \\ 1 & -3-\lambda \end{vmatrix} = (\lambda+1)(\lambda+3) + 1 = \lambda^2 + 4\lambda + 4$$

$\lambda = -2, -2 \therefore (0,0)$ is asymptotically stable.

Another method:

$$\begin{cases} \dot{x} = 1 - xy = f_1(x,y) \\ \dot{y} = x - y^3 = f_2(x,y) \end{cases}$$

$$\begin{pmatrix} x-1 \\ y-1 \end{pmatrix}' = A \begin{pmatrix} x-1 \\ y-1 \end{pmatrix} + g(||x-y||)$$

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}_{(1,1)} = \begin{pmatrix} -y & -x \\ 1 & -3y^2 \end{pmatrix}_{(1,1)} = \begin{pmatrix} -1 & -1 \\ 1 & -3 \end{pmatrix}$$

13.0.2. Phases Plane

2×2 autonomous system

$$\begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{cases}$$

$$\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)}$$

Example 13.3 $\begin{cases} \dot{x} = y^2 \\ \dot{y} = x^2 \end{cases}$

$$\frac{dy}{dx} = \frac{x^2}{y^2}$$

$$y^2 dy = x^2 dx$$

$$y^3 = x^3 + C$$

Example 13.4 (1) $\begin{cases} \dot{x} = y \\ \dot{y} = -x \end{cases}$ $\frac{dy}{dx} = -\frac{x}{y} \Rightarrow ydy = -xdx \Rightarrow (x^2 + y^2) = C$ Solutions are circles.

(2) $\begin{cases} \dot{x} = y(1 + x^2 + y^2) \\ \dot{y} = -x(1 + x^2 + y^2) \end{cases}$ $\frac{dy}{dx} = -\frac{x}{y}$

(3) $\begin{cases} \dot{x} = y(1 - x^2 - y^2) \\ \dot{y} = -x(1 - x^2 - y^2) \end{cases}$ What are the differences between phase plane of (2) and that of (3)?

13.0.3. Qualitative Theory of Orbits

Autonomous system $(n \times n)$

$$\dot{\mathbb{X}} = \vec{f}(\mathbb{X}) = \begin{pmatrix} f_1(x_1, \ldots, x_n) \\ \ldots \\ f_n(x_1, \ldots, x_n) \end{pmatrix}$$

Orbit is also called trajectory or solution curves.

Proposition 13.1 Suppose $f_1, f_2, \ldots, f_n$ are C Then for every point $\mathbb{X}_0 \in \mathbb{R}^n$, $\exists$ a unique orbit through $\mathbb{X}_0$

Proposition 13.2 Let $\phi(t)$ be a solution. If $\phi(t_0 + T) = \phi(t_0)$ for some $t_0, T > 0$, then $\phi(t + T) = \phi(t) \forall t > 0$ (i.e. ϕ must be periodic.)

Proof. Def $\varphi(t) = \phi(t + T), t > 0$

$$\Rightarrow \varphi(t_0) = \phi(t_0) \& \varphi \text{is a solution}$$

$$\Rightarrow \varphi(t) \equiv \phi(t) \qquad \forall t \text{ by Prop1}$$

$$(\dot{\varphi} = \frac{d}{dt}\phi(t + T) = \vec{f}(\phi(t + T)) = \vec{f}(\varphi(t)))$$

■

13.0.4. Phase Portraits for linear systems

■ **Example 13.5**

$$\dot{\mathbb{X}} = \begin{pmatrix} 1 & -3 \\ -3 & 1 \end{pmatrix} \mathbb{X}$$

$$0 = |A - \lambda I| = \begin{vmatrix} 1-\lambda & -3 \\ -3 & 1-\lambda \end{vmatrix} = (\lambda - 1)^2 - 9 = (\lambda - 4)(\lambda + 2)$$

$$\lambda = -2, 4$$

$$\lambda = -2 \qquad 0 = (A + 2I) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 3 & -3 \\ -3 & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda = 4 \qquad 0 = (A - 4I) \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} -3 & -3 \\ -3 & -3 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

$$w = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\mathbb{X}(t) = c_1 e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

■

$$\dot{\mathbb{X}} = A\mathbb{X} + o(||X||), (0,0) : equilibrium A : 2 \times 2$$

Let λ_1, λ_2 be 2 eigenvalues of A. (1) $0 > \lambda_1 > \lambda_2$ stable node

(2) $\lambda_2 = \lambda_1 < 0$: stable or spiral

(3)$\lambda_2 < 0 < \lambda_1$: unstable saddle

(4)$0 < \lambda_2 = \lambda_1$: unstable node or spiral

(5)$0 < \lambda_2 < \lambda_1$: unstable node

(6)$\lambda_1, \lambda_2 = \alpha \pm i\beta, \alpha < 0$: stable spiral

(7)$lambda_1, \lambda_2 = \alpha \pm i\beta, \alpha > 0$:unstable spiral out or center

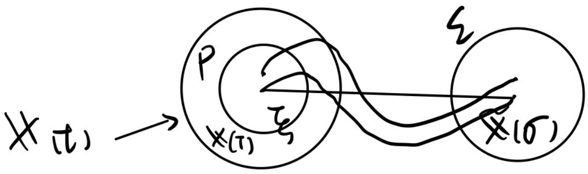
Chapter 14

Week13

14.1. Monday

$$\dot{\mathbb{X}} = f(\mathbb{X})$$

Proposition 14.1 Suppose a solution $\mathbb{X}(t) \to \xi$ as $t \to \infty$. Then $f(\xi) = 0$



Proof. Suppose Not, i.e. suppose $f(\xi) \neq 0$

Consider IVP $\begin{cases} \dot{\tilde{\mathbb{X}}} = f(\tilde{\mathbb{X}}) \\ \tilde{\mathbb{X}}(0) = \xi \end{cases}$

$$f(\xi) \neq 0 \Rightarrow \tilde{\mathbb{X}}(\delta) \neq \xi \text{ for some } \delta > 0$$

Let $\varepsilon = ||\xi - \tilde{\mathbb{X}}(\delta)|| > 0$ as $\mathbb{X}(t) \to \xi$ as $t \to \infty \Rightarrow \exists T$ s.t. $||\mathbb{X}(t) - \xi|| < \frac{\varepsilon}{3}\ \forall t > T$

$\exists \rho > 0$ s.t. $\forall ||\mathbb{X}^*(0) - \xi|| < \rho \Rightarrow ||\mathbb{X}^*(\delta) - \tilde{\mathbb{X}}(\delta)|| < \frac{\varepsilon}{3}$ when $\mathbb{X}^*$ is a solution for $\dot{\mathbb{X}} = f(\mathbb{X})$

(Continuous dependence on IV)

$\mathbb{X}(t) \to \xi \Rightarrow \exists S > 0$ s.t. $||\mathbb{X}(t) - \xi|| < min\{\rho, \frac{\varepsilon}{3}\}$

$\forall t \geq \max(S, T) = \bar{T}$

$$||\mathbb{X}(\bar{T} + \delta) - \tilde{\mathbb{X}}(\delta)|| < \frac{\varepsilon}{3} \qquad \bar{T} + \delta \geq T$$

On the other hand $||\mathbb{X}(\bar{T}+\delta)-\xi|| < \frac{\varepsilon}{3}$ ■

■ **Example 14.1**

$$\begin{cases} \dot{x}(t) = ax - bxy - ex^2 = x(a - by - ex) \\ \dot{y} = -cy + dxy - fy^2 = y(-c + dx - fy) \end{cases}$$

where a,b,c,d,e,f are positive constant and $\frac{a}{e} < \frac{c}{d}, x > 0, y > 0$.

$$x = 0 \text{ or } a - by - ex = 0$$

$$y = 0 \text{ or } -c + dx - fy = 0$$

Conclusion: all solutions with $x(0) > 0, y(0) > 0$ will converge to $(\frac{a}{e}, 0)$ as t $\to \infty$

At A $\begin{cases} \dot{x} = x(a - by - ex) = 0 \\ \dot{y}(-c + dx - fy) < 0 \end{cases}$, at B $\begin{cases} \dot{x} < 0 \\ \dot{y} = 0 \end{cases}$

- All solutions start in iii will enter ii at a later time
- All solutions start in ii will enter i at a later time or converge to $(\frac{a}{e}, 0)$
- All solutions in i will converge to $(\frac{a}{e}, 0)$

Question what if $\frac{a}{e} > \frac{c}{d}$ ■

14.1.1. Poincare-Bandixon Theorem

Consider $\dot{\mathbb{X}} = f(\mathbb{X})$, $n = 2$ Suppose $\mathbb{R}$ is a bounded closed subset in $\mathbb{R}^2$ which contains no equilibrium of $f(\mathbb{X})$. If $\exists a$ solution $\mathbb{X}(t)$ which is contained entirely in R, then either $\mathbb{X}(t)$ is periodic or its orbit spirals into a simple closed curve.(i.e. the orbit of a periodic solution.) In particular, $\mathbb{R}$ contains a periodic solution.

■ **Example 14.2**

$$\ddot{z} + \dot{z}(z^2 + 2\dot{z}^2 - 1) + z = 0$$

Let $\dot{z} = y$, $\dot{y} = \ddot{z} = -[\dot{z}(z^2 + 2\dot{z}^2 - 1) + z]$.

$$\begin{cases} \dot{z} = y \\ \dot{y} = y(1 - z^2 - 2y^2) - z \end{cases}$$

$$\frac{d}{dt}(z^2 + y^2) = 2z\dot{z} + 2y\dot{y} = 2y^2(1 - z^2 - 2y^2)$$

$$z^2 + y^2 = \frac{1}{4} \text{ on } z^2 + y^2 = \frac{1}{4}, \frac{d}{dt}(z^2 + y^2) \geq 0$$

$$z^2 + y^2 = 4 \text{ on } z^2 + y^2 = 4, \frac{d}{dt}(z^2 + y^2) \leq 0$$

P-B theorem $\Rightarrow \exists$ periodic orbit in $[\frac{2}{4} \leq y^2 + z^2 \leq 4]$. The annular area is chosen arbitrarily.

You can choose any circle you like only if it satisfies the condition of the theorem. ■

Chapter 15

Week14

15.1. Monday

15.1.1. applications

$$\begin{cases} \dot{x} = ax - bxy = x(a - by) \\ \dot{y} = -cy + \tilde{d}xy = y(-c + \tilde{d}x) \end{cases}$$

$x : preyer\ y : predators\ a,b,c,d,$ are constants >0.

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\dot{y}}{\dot{x}} = \frac{y(-c + \tilde{d}x)}{x(a - by)}$$

$$(a - by)\frac{dy}{y} = \frac{-c + \tilde{d}x}{x}dx$$

$$(alny - by) = -clnx + \tilde{d}x + \bar{c}$$

$$\frac{y^a}{e^{by}} = \frac{e^{\tilde{d}x}}{x^c}K$$

$$\frac{x^c y^a}{e^{by}e^{\tilde{d}x}} = K$$

Definition 15.1 [α-level curve]

$$\{(x,y)|\varphi(x,y) = \alpha\}$$

lemma: The level curves of the function $\varphi(x,y) = \frac{x^c}{e^{\tilde{d}x}}\frac{y^a}{e^{by}}(\equiv g(x)h(y))$ simple closed curves.

Proof.

$$\nabla\varepsilon = < \frac{\partial\varepsilon}{\partial x}, \frac{\partial\varepsilon}{\partial y} >$$

$$\varphi_x = (\frac{cx^{c-1}}{e^{\tilde{d}x}} - \tilde{d}\frac{x^c}{e^{\tilde{d}x}})h(y) = 0$$

$$\varepsilon_y = g(x)(\frac{ay^{a-1}}{e^{by}} - b\frac{y^a}{e^{by}}) = 0$$

Therefore, the critical points of φ: $\nabla\varphi(x,y) = 0$ is:

$$(\frac{c}{\tilde{d}}, \frac{a}{b})$$

The area $\varphi > \alpha$ defined by a closed curve.

Assume this isn't the case, there are two closed region $\varphi > \alpha$. Then, there will be two local maximum in the first quadrant. This is contradict to the statement that $(\frac{c}{\tilde{d}}, \frac{a}{b})$ is the only local maximum in the first quadrant.

■

L. et $(x(t), y(t))$ be a periodic solution with period T. Then

$$\bar{x} \equiv \int_0^T x(t)\,\mathrm{d}t = \frac{c}{\tilde{d}}$$

$$\bar{y} \equiv \int_0^T y(t)\,\mathrm{d}t = \frac{a}{b}$$

Proof.

$$\frac{\dot{x}}{x} = a - by$$

$$ln(x(t))|_0^T = \int_0^T (a - by)\,\mathrm{d}t$$

$$0 = aT - b\int_0^T y(t)\,\mathrm{d}t$$

$$\frac{a}{b} = \frac{1}{T}\int_0^T y(t)\,\mathrm{d}t$$

$\blacksquare$

Let's see why, after the world war, the amount of preyer captured increased and the predator's fishery amount decrease.

During the war, they satisfies the equation.

$$\begin{cases} \dot{x} = x(a - by) \\ \dot{y} = y(-c + \tilde{d}y) \end{cases}$$

The average amount is given by above lamma. While, after the war, equations change to

$$\begin{cases} \dot{x} = x(a - \epsilon - by) \\ \dot{y} = y(-(c + \epsilon) + \tilde{d}y) \end{cases}$$

$$\bar{x} = \frac{c}{\tilde{d}} \Rightarrow \frac{c+\epsilon}{\tilde{d}} \qquad \bar{y} = \frac{a}{b} \Rightarrow \frac{a-c}{b}$$

Logistic equation

$$u_t = au(1 - \frac{u}{k})$$

where a is growth rate, k is carrying capacity.

$$\begin{cases} u_t = au(1 - \frac{u}{k_1} - \frac{v}{L_1}) \\ v_t = cv(1 - \frac{v}{k_2} - \frac{u}{l_2}) \end{cases}$$

$$\begin{cases} u_t = u(a_1 - b_1 u - c_1 v) & u = 0 \text{ or } a_1 - b_1 u - c_1 v = 0 \\ v_t = v(a_2 - b_2 u - c_2 v) & v = 0 \text{ or } a_2 - b_2 u - c_2 v \end{cases}$$

$$\frac{b_1}{b_2} > \frac{a_1}{a_2} > \frac{c_1}{c_2} : \text{ weak competition}$$

$$\frac{b_1}{b_2} < \frac{a_1}{a_2} < \frac{c_1}{c_2} : \text{ strong competition}$$

$$\begin{cases} a_1 - b_1 u_* - c_1 v_* = 0 \\ a_2 - b_2 v_* - c_2 v_* = 0 \end{cases}$$

[Lemma] The equilibrium (v_*, v_*) is

(i) unstable: if competition is trong (i.e. if $\frac{b_1}{b_2} < \frac{a_1}{a_2} < \frac{c_1}{c_2}$)

(ii) asymptotically stable: if the competition is weak.

Pf: linearlize the system at (u_*, v_*).

15.2. Wednesday

15.2.1. Epidemiology

Population N fixed (ignoring birth, death, ...)

S(t): the number of individuals in the susceptible class.

I(t): the number of individuals in the infected class.

R(t): the number of individuals in the removed class.

Rate of change of $S(t)$ is proportional to the product of $S(t)\&I(t)$.

Rate of change of $R(t)$ is proportional to hte size of $I(t)$.

$$\begin{cases} \dot{s} = -rsI \\ \dot{I} = rsI - \delta I \\ \dot{R} = \delta I \end{cases}$$

$$\frac{dI}{ds} = \frac{\dot{I}}{\dot{s}} = \frac{rsI - \delta I}{-rsI} = -1 + \frac{\delta}{r}\frac{1}{S}$$

$$I(t) - I(0) = -(s - s_0) + \frac{\delta}{r} ln \frac{s(t)}{s_0}$$

$$I(t) = I_0 + s_0 - s + \rho ln \frac{s}{s_0}, \qquad \rho = \frac{\delta}{r}$$

as $s \to 0, I \to -\infty$ $0 = -1 + \rho\frac{1}{s}$, $s = \rho$