Q1: $u = X \cdot T$

$u_t = XT'$

$$u_{tt} + au_t = c^2 u_{xx}.$$

$$XT'' + aXT' - c^2 X''T = 0$$

$$\frac{T''}{c^2T} + a\frac{T'}{c^2T} = \frac{X''}{X}$$

$$\frac{X''}{X} = \lambda$$

$$X'' - \lambda X = 0$$

$b > 0$.

$\lambda = b^2 > 0$ $\quad X = e^{\frac{\pm\sqrt{4b^2}}{2}x} = e^{\pm bx}$

$\lambda = 0$ $\quad X = C_1 x + C_0$

$\lambda = -b^2 < 0$ $\quad X = C_1\cos bx + C_2 \sin bx$

$\frac{T''}{c^2T} + \frac{aT'}{c^2T} = \lambda$ $\qquad T'' + aT' - \lambda c^2 T = 0.$

$\lambda > -\frac{a^2}{4c^2}$ $\quad T = C_1\exp\left(\frac{-a+\sqrt{a^2+4\lambda c^2}}{2}t\right) + C_2\exp\left(\frac{-a-\sqrt{a^2+4\lambda c^2}}{2}t\right).$

~~$\lambda = -4c^2$~~

$\lambda = -\frac{a^2}{4c^2}$ $\quad T = C_1 e^{-\frac{a}{2}t} + C_2 t e^{-\frac{a}{2}t}.$

$\lambda < -\frac{a^2}{4c^2}$ $\quad T = e^{-\frac{a}{2}t}\left(C_1\cos\sqrt{\frac{4b^2c^2-a^2}{2}}t + C_2\sin\sqrt{\frac{4b^2c^2-a^2}{2}}t\right)$

$\lambda = -b^2$

The solution which satisfies initial condition is when $\lambda < 0$ $\quad X = C_2 \sin\frac{n\pi}{a}x$

$n \in N^+$ $\quad n < \frac{a}{2\pi c}$

$\lambda > -\frac{a^2}{4c^2}$ $\quad u_n = \sin\frac{n\pi}{a}x \cdot \left(C_1\exp\left(\frac{-a+\sqrt{a^2+4\lambda c^2}}{2}t\right) + C_2\exp\left(\frac{-a-\sqrt{a^2+4\lambda c^2}}{2}t\right)\right)$

$\lambda = -\frac{a^2}{4c^2}$ $\quad u_n = \sin n\pi x \cdot \left(C_1 e^{-\frac{a}{2}t} + C_2 t e^{-\frac{a}{2}t}\right)$, $n = \frac{a}{2\pi c}$

$\lambda < -\frac{a^2}{4c^2}$ $\quad u_n = e^{-\frac{a}{2}t}\sin n\pi x\left(C_1\cos\sqrt{\frac{4b^2c^2-a^2}{2}}t + C_2\sin\sqrt{\frac{4b^2c^2-a^2}{2}}t\right)$

$n > \frac{a}{2\pi c}$

All u_n are solutions to B.C. $\quad u = \sum_{n=1}^{\infty} a_n u_n$

(b). $u_t(x,0) = \sum_{n=1}^{\infty} a_n \sin n\pi x. = f(x).$

$\therefore a_n = \frac{2}{1}\int_0^1 f(x)\sin n\pi x\, dx.$

u is a solution for I.B.C.

(c) $u \to 0$ as $n \to \infty$

(d) If $a > 2c$, it's over dumped.

Q2: (a)
$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) & x \in [0,L] \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \\ u(0,t) = 0, \ u(L,t) = 0 \end{cases}$$

$$\begin{cases} \tilde{u}_{tt} - c^2 \tilde{u}_{xx} = f(x,t) \\ \tilde{u}(x,0) = 0 \quad \tilde{u}_t(x,0) = 0 \\ \tilde{u}(0,t) = 0 \quad \tilde{u}(L,t) = 0 \end{cases}$$

$$\tilde{u} = \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\, dy\, ds$$

$$\begin{cases} v_{tt} - c^2 v_{xx} = 0 \\ v(x,0) = \phi(x), \ v_t(x,0) = \psi(x) \\ v(0,t) = 0, \ v(L,t) = 0 \end{cases}$$

$$v(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(y)\, dy$$

$$u = \tilde{u} + v$$

b)
$$u = \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} k\, dy\, ds + \frac{1}{2c} \int_{x-ct}^{x+ct} l\, dy$$

$$= \int_0^t (t-s) k\, ds + tl.$$

$$= k(ts - \tfrac{1}{2}s^2)\Big|_0^t + tl = \frac{k}{2}t^2 + tl.$$

$$\frac{1}{2ch}\int_0^t\int_{x+h-c(t-s)}^{x+h+c(t-s)} f\,dy\,ds-\frac{1}{2ch}\int_0^t\int_{x-c(t-s)}^{x+c(t-s)} f\,dy\,ds$$

$$=\lim_{h\to0}\frac{1}{2ch}\int_0^t\left[\int_{x-c(t-s)}^{x+h+c(t-s)} f\,dy+\int_{x+h-c(t-s)}^{x-c(t-s)} f\,dy-\int_{x-c(t-s)}^{x+c(t-s)} f\,dy\right]ds$$

$$=\lim_{h\to0}\frac{1}{2ch}\int_0^t\left[\int_{x+c(t-s)}^{x+c(t-s)+h} f\,dy+\int_{x+h-c(t-s)}^{x-c(t-s)} f\,dy\right]ds$$

$$=\lim_{h\to0}\frac{1}{2c}\int_0^t\frac{1}{h}\cdot h\cdot f(x+c(t-s)+h',s)+\frac{1}{h}\cdot h\, f(x-c(t-s)+h'',s)\,ds$$

$$=\lim_{h\to0}\frac{1}{2c}\int_0^t[f(x+c(t-s)+h',s)+f(x-c(t-s)+h'',s)]ds$$

$$=\frac{1}{2c}\int_0^t[f(x+c(t-s),s)+f(x-c(t-s),s)]ds$$

$$u_{xx}=\lim_{h\to0}\frac{1}{2ch}\left\{\int_0^t[f(x+h+c(t-s),s)+f(x+h-c(t-s),s)]ds-\int_0^t[f(x+c(t-s),s)+f(x-c(t-s),s)]ds\right.$$

$$=\lim_{h\to0}\frac{1}{2ch}\int_0^t[f(x+c(t-s)+h,s)-f(x+c(t-s),s)+f(x+h-c(t-s),s)-f(x-c(t-s),s)]ds$$

$$=\lim_{h\to0}\frac{1}{2c}\int_0^t\left[\frac{1}{h}\cdot h\cdot f_x(x+c(t-s)+h',s)+\frac{1}{h}\cdot h\cdot f_x(x-c(t-s)+h'',s)\right]ds$$

$$=\frac{1}{2c}\int_0^t[f_x(x+c(t-s),s)+f_x(x-c(t-s),s)]ds$$

$$u_t=\lim_{h\to0}\left[\frac{1}{2c}\int_0^{t+h}\int_{x-c(t+h-s)}^{x+c(t+h-s)} f(y,s)\,dy\,ds-\frac{1}{2c}\int_0^t\int_{x-c(t-s)}^{x+c(t-s)} f\,dy\,ds.\right]\frac{1}{h}$$

$$=\lim_{h\to0}\frac{1}{2ch}\left\{\int_t^{t+h}\int_{x-c(t+h-s)}^{x+c(t+h-s)} f\,dy\,ds+\int_0^t\left[\int_{x-c(t+h-s)}^{x+c(t+h-s)} f\,dy-\int_{x-c(t-s)}^{x+c(t-s)} f\,dy\right]ds\right\}$$

$$=\lim_{h\to0}\frac{1}{2ch}\left\{\underbrace{\int_t^{t+h}\int_{x-c(t+h-s)}^{x+c(t+h-s)} f\,dy\,ds}_{①}+\int_0^t\left[\int_{x+c(t-s)}^{x+c(t+h-s)} f\,dy+\int_{x-c(t+h-s)}^{x-c(t-s)} f\,dy\right]ds\right\}$$

$$=\lim_{h\to0}\frac{1}{2c}\left\{\frac{1}{h}①+\int_0^t\left[\frac{1}{h}\int_{x+c(t-s)}^{x+c(t+h-s)} f\,dy+\frac{1}{h}\int_{x-c(t+h-s)}^{x-c(t-s)} f\,dy\right]ds\right\}$$

$$=\lim_{h\to0}\frac{1}{2ch}①+\lim_{h\to0}\frac{1}{2c}\int_0^t\left[\frac{1}{h}\cdot ch\, f(x+c(t-s)+h',s)+\frac{1}{h}\cdot ch\, f(x-c(t-s)+h'',s)\right]ds$$

$$=\lim_{h\to0}\frac{1}{2ch}\cdot① +\frac{1}{2}\int_0^t f(x+c(t-s),s)+f(x-c(t-s),s)\,ds.$$

$$\frac{1}{2ch}\cdot\int_t^{t+h}\int_{x-c(t+h-s)}^{x+c(t+h-s)} f\,dy\,ds.$$

$$\lim_{h\to 0}\frac{1}{2ch}\int_t^{t+h}\left[\int_{x+c(t-s)}^{x+c(t+h-s)} f\,dy+\int_{x-c(t-s)}^{x+c(t-s)} f\,dy+\int_{x-c(t+h-s)}^{x-c(t-s)} f\,dy\right]ds.$$

$$=\lim_{h\to 0}\frac{1}{2ch}\int_t^{t+h}\int_{x-c(t-s)}^{x+c(t-s)} f\,dy\,ds.$$

$$=\lim_{h\to 0}\frac{1}{2c}\int_{x-c(t-s)}^{x+c(t-s)}\int_t^{t+h}\frac{1}{h} f\,ds\,dy$$

$$=\frac{1}{2c}\int_{x-c(t-s)}^{x+c(t-s)} f(y,t)\,dy$$

$$u_{tt}=\lim_{h\to 0}\frac{1}{2ch}\left[\int_{x-c(t-s+h)}^{x+c(t-s+h)} f(y,t)\,dy-\int_{x-c(t-s)}^{x+c(t-s)} f\,dy\right]+\frac{1}{2h}\int_0^t\left[f(x+c(t-s+h),s)\right.$$

$$\left.+f(x-c(t+h-s),s)-f(x+c(t-s),s)-f(x-c(t-s),s)\right]ds$$

$$=\lim_{h\to 0}\frac{1}{2ch}\left[\int_{x+c(t-s)}^{x+c(t-s+h)} f\,dy+\int_{x-c(t-s+h)}^{x-c(t-s)} f\,dy\right]+\int_0^t\left[\frac{1}{2h}\cdot ch\, f_x(x+c(t-s)+h',s)\right.$$

$$\left.+\frac{1}{2h}\cdot ch\cdot f_x(x-c(t-s)+h'',s)\right]ds$$

$$=\frac{1}{2}\left[f(x+c(t-s),t)+f(x-c(t-s),t)\right]+\frac{c}{2}\int_0^t\left[f_x(x+c(t-s),s)\right.$$

$$\left.+f_x(x-c(t-s),s)\right]ds \qquad \therefore\ u_{tt}-c^2u_{xx}=f(x,t)$$

(b). $$u(x,t)=\frac{1}{2c}\int_0^t\int_{x-c(t-s)}^{x+c(t-s)} ys\,dy\,ds$$

$$=\frac{1}{2c}\int_0^t \frac{1}{2}sy^2\Big|_{x-c(t-s)}^{x+c(t-s)}ds$$

$$=\frac{1}{2c}\int_0^t \frac{1}{2}s[x+c(t-s)]^2-\frac{1}{2}s[x-c(t-s)]^2ds$$

$$=\frac{1}{2c}\int_0^t\frac{1}{2}s\cdot 4xc(t-s)\,ds=\frac{1}{2c}\int_0^t 2xc(ts-s^2)\,ds$$

$$=\int_0^t x(ts-s^2)\,ds=\frac{1}{2}xts^2-\frac{1}{3}xs^3\Big|_0^t=\frac{1}{2}xt^3-\frac{1}{3}xt^3=\frac{1}{6}xt^3$$

$$E'(t) = \frac{1}{2}\int_0^L \left[2u_t u_{tt} + c^2 2u_x u_{xt}\right]dx$$

$$= \int_0^L \left[u_t u_{tt} + c^2 u_x u_{xt}\right]dx$$

$$= c^2\int_0^L \left[u_t u_{xx} + u_x u_{xt}\right]dx$$

$$= c^2 u_x u_t\Big|_0^L - c^2\int_0^L u_x \cdot u_{tx}\,dx + c^2\int_0^L u_x u_{xt}\,dx$$

$$= c^2 u_x u_t\Big|_0^L.$$ as u is dirichelet.

$$= 0.$$ $\therefore E(t) = E(0)$ is a constant function.

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Q5: (a) $F(t) = \int_0^L V_t^2 + c^2 V_x^2 \, dx.$ $\quad F(0) = 0$

$$F'(t) = \int_0^L 2V_t V_{tt} + 2c^2 V_x V_{xt} \, dx$$

$$= 2c^2 \int_0^L V_{xx} V_t + V_x V_{xt} \, dx.$$

$$= 2c^2 [V_t \cdot V_x] \Big|_0^L$$

$$= 0.$$

$\therefore F(t) \equiv 0 \Rightarrow V_t(x,t) \equiv 0 \Rightarrow V(x,t) = f(x)$

$\qquad\qquad\qquad\quad V_x(x,t) \equiv 0 \Rightarrow V(x,t) = g(t)$ ①

① shows V is a constant independent of x and t. $V \equiv$ Constant.

$V(x,0) = 0 \Rightarrow V \equiv 0$. is the only solution.

(b) Assume there are two solutions ~~satisfies the~~ to the I.B.V. problem

u_1, u_2.

Let $v = u_1 - u_2$.

Then v satisfies $\begin{cases} V_{tt} = c^2 V_{xx} \\ V(0,t) = V(L,t) = 0 \\ V(x,0) = 0, V_t(x,0) = 0 \end{cases}$ by superposition.

~~Which means~~. However By (a) we knows

$V \equiv 0$.

$\therefore u_1 \equiv u_2$.

Q6: (a).

Neumann Condition.

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & 0 < x < l. \\ u(x,0) = \phi(x), \ u_t(x,0) = \psi(x) \\ u_x(0,t) = u_x(l,t) = 0 \end{cases}$$

$$E(t) = \frac{1}{2}\int_0^L (u_t^2 + c^2 u_x^2)\,dx.$$

$$E'(t) = \frac{1}{2}\int_0^L 2u_t u_{tt} + 2c^2 u_x u_{xt}\,dx$$

$$= c^2\int_0^L u_{xx}u_t + u_x u_{xt}\,dx$$

$$= c^2\, u_x \cdot u_t \Big|_0^L = c^2\left[u_x(L,t)u_t(L,t) - u_x(0,t)u_t(0,t)\right].$$

$$= 0.$$

$\therefore$ the conservation of energy still holds.

(b). Assume there are two solution u_1, u_2. Let $v = u_1 - u_2$.

$$\begin{cases} v_{tt} - c^2 v_{xx} = 0 \\ v(x,0) = 0, \ v_t(x,0) = 0 \\ v_x(0,t) = v_x(l,t) = 0 \end{cases}$$

$\cancel{v_t \equiv 0}$ $\quad v \equiv 0 \Rightarrow u_1 = u_2.$