PARTIAL DIFFERENTIAL EQUATIONS

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MAT4022 Notebook

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Contents

Acknowledgments

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CUHK(SZ)

Chapter 1

Week1

1.1. Tuesday

1.1.1. Introduction and Examples

$u(x,y)$ a smooth function

$u_x = \frac{\partial u}{\partial x}$

$u_y = \frac{\partial u}{\partial y}$

$u_{xx} = \frac{\partial^2 u}{\partial x \partial x}$

$u_{xy} = \frac{\partial^2 u}{\partial y \partial x}$

In this course, as u is smooth, we don't fuss about difference between u_{xy} and u_{yx}.

Definition 1.1 A PDE is a relation for $F(x, y, u, u_x, y_y, u_{xy}, \dots) = 0$ linear equation. If F is linear in u, u_x, u_y, u_{xx}, $\cdots$ (Not necessarily in x, y)

Example 1.1 Laplace equation: $u_{xx} + u_{xy} = 0$

$(x^2 + y^2)u_{xx} + e^{xy}u_{yy} = 0$

In the second case, it is still linear as it is a linear equation for u_{xx} and u_{yy}.

Other examples

Cauchy-Riemann.

$$\begin{cases} u_x - v_y = 0 \\ u_y + v_x = 0 \end{cases} \quad \text{1st order system}$$

$$\begin{cases} u_{xx} - v_{yx} = 0 \\ u_{yy} + v_{yx} \end{cases} \to u_{xx} + y_{yy} = 0 \text{ harmonic function } (2^{\text{st}} \text{ order}).$$

Order: highest order that partial derivatives are taken.

Notation for laplace operator:

$$\Delta u = u_{xx} + u_{yy},\ \Delta u = u_{xx},\ \Delta u = u_{x_1 x_1} + \cdots + u_{x_n x_n}$$

■ **Example 1.2** Wave equation. $u_{tt} = c^2 \Delta u$, $c > 0$ wave speed a constant, t: time.

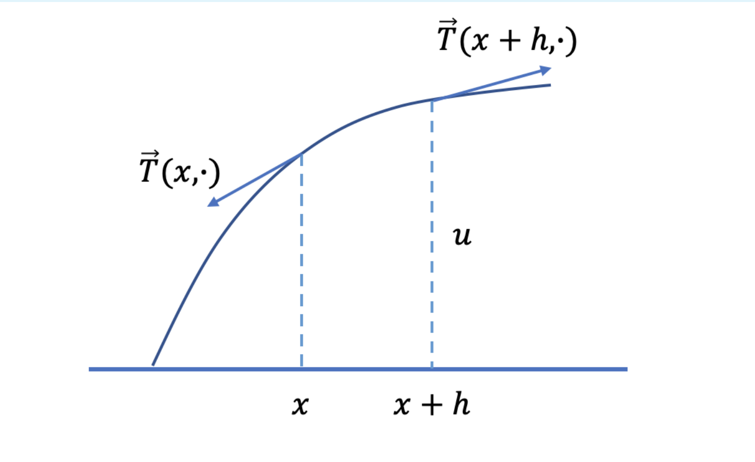
$n = 1$ $u_t t = c^2 u_{xx}$: vibration of a string

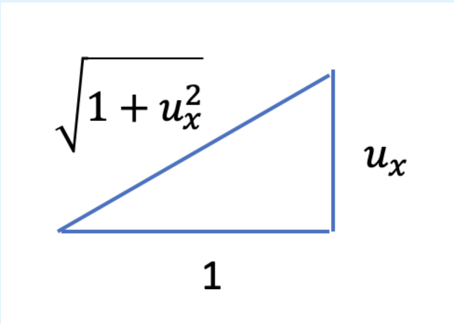
$n = 2$ $u_{tt} = c^2(u_{xx} + u_{yy})$: water wave

$n = 3$ $u_{tt} = c^2(u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3})$: sound wave

n: space dimension

n=1 A string, flexible, elastic, homogeneous with density ρ





When we magnefy part of the string and let h to be small enough. The horizontal parts of forces of $\vec{T}(x,\cdot)$ and $\vec{T}(x+h,\cdot)$ are equal to each other. Vertical forces is equal to am according to Newton's second law. Therefore, we have the following.

$$\begin{cases} \dfrac{|T|(x+h,\cdot)}{\sqrt{1+u_x^2(x+h,\cdot)}} = \dfrac{|T|(x,\cdot)}{\sqrt{1+u_x^2(x,\cdot)}} \\ \dfrac{|T|u_x}{\sqrt{1+u_x^2}}(x+h,\cdot) - \dfrac{|T|u_x}{\sqrt{1+u_x^2}}(x,\cdot) = h\rho u_{tt} \end{cases}$$

When h is small enough,

$$\frac{1}{\sqrt{1+u_x^2}} = (1+u_x^2)^{-\frac{1}{2}} = 1 - \frac{1}{2}u_x^2 + \cdots \approx 1$$

$$|T|(x+h,\cdot) = |T|(x,\cdot)$$

$$\rho u_{tt} = |T| \cdot [u_x(x+h,\cdot) - u_x(x,\cdot)]\frac{1}{h} \to |T|u_{xx}$$

$$u_{tt} = \frac{|T|}{\rho}u_{xx},\ c = \sqrt{\frac{|T|}{\rho}}$$

■

■ **Example 1.3** [Heat equation] $u_t = \Delta u$, u: temprature
$H(t) =$ total amount of heat in $\Omega \subset \mathbb{R}^3$

$$= \iiint_{\Omega} c\rho u \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

$$\iint_{\partial\Omega} \kappa \triangle u\nu \,\mathrm{d}s = \frac{\mathrm{d}H}{\mathrm{d}t} = \iiint_{\Omega} c\rho u \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

κ is heat conduction constant. ν is unit outward normal.

$$\vec{w} = \triangle u = (u_x, u_y, u_z)$$

$$\mathrm{dir}\vec{w} = u_{xx} + u_{yy} + u_{zz} = \Delta u$$

By divergent theorem:

$$\int\int\int_{\Omega} c\rho u_t = \int\int\int_{\Omega} \kappa \Delta u$$

As this is the case for all Ω we can tell $c\rho u_t = \kappa \Delta u$

$$u_t = \frac{\kappa}{c\rho} \Delta u$$

■

Theorem 1.1 — Divergent theorem. $\vec{w} = (w_1, w_2, w_3)$: vector field

$$\int\int\int_{\Omega} \mathrm{dir}\vec{w}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int\int_{\partial\Omega} \vec{w}\nu\,\mathrm{d}s$$

where $\mathrm{dir}\vec{w} = \frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial y} + \frac{\partial w_3}{\partial z}$

1.2. Thursday

1.2.1. Examples

Minimal surface equation.

Given a curve Γ in $\mathbb{R}^3$ to find a surface spanning Γ and has the smallest possible area.

$$(1+u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1+u_x^2)u_{yy} = 0$$

This is a 2^{nd} order linear in u_{xx}, u_{yy} equation. It is called quasi-linear equation.

Now, before we take a look at solving some partial differential equations. Let's review a simple ode case.

In ODE,

$$v' + cv = 0$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = v' = -cv$$

$$\frac{\mathrm{d}v}{v} = -c\,\mathrm{d}t$$

At the moment, just ignore $|v|$

$$\ln v = -ct + \alpha$$

$$v(t) = e^{-ct}e^{\alpha} = v(0)e^{-ct}, t = 0\ v(0) = e^{\alpha}$$

In PDE,

$$u_y + cu_x = 0\ c\text{: constant}$$

This is called transport equation. Later on, we will talk about this name.

$$\frac{\mathrm{d}}{\mathrm{d}t}u(x(t),y(t)) = u_x\frac{\mathrm{d}x}{\mathrm{d}t} + u_y\frac{\mathrm{d}y}{\mathrm{d}t} = cu_x + u_y = 0$$

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = c \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = 1 \end{cases} \Rightarrow \begin{cases} x = ct + X(0) = ct + \xi \\ y = t + y(0) = t \end{cases}$$

$$\Rightarrow \xi = x - ct \qquad \xi = x - cy$$

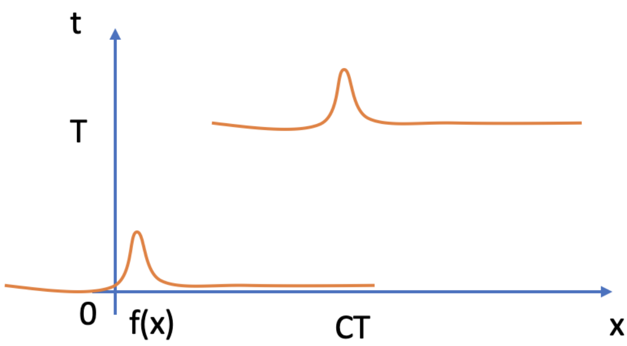
On Γ_ξ, $\frac{\mathrm{d}t}{\mathrm{d}u}(x(t),y(t)) \equiv 0\ \forall t \Rightarrow$, $u(x(t),y(t)) \equiv$ constant on Γ_ξ.

$$u(x,y) = f(x - cy)$$

As convention, let t denote y, transport equation is $\begin{cases} u_t + cu_x = 0 \\ u(x,0) = f(x) \end{cases} \Rightarrow u(x,t) = f(x - ct)$.

The following is a graph used to illustrate why it is called transport equation:

$$u(cT,T) = f(cT - cT) = f(0) = u(0,0)$$



■ **Example 1.4**

$$\begin{cases} xu_x + yu_y = \alpha u \\ u(x,1) = f(x) \end{cases}$$

Γ : characteristic curve $\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = x \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = y \end{cases} \Rightarrow \begin{cases} x = x_0 e^t = se^t \\ y = y_0 e^t = 1e^t \rightarrow t = \ln y \end{cases}$.

Initial curve $\begin{cases} x = s \\ y = 1 \end{cases}$.

$$s = \frac{x}{e^t} = \frac{x}{y}$$

On Γ, we have $\frac{\mathrm{d}u}{\mathrm{d}t} = u_x \frac{\mathrm{d}x}{\mathrm{d}t} + u_y \frac{\mathrm{d}y}{\mathrm{d}t} = xu_x + yu_y = \alpha u \Rightarrow u = ce^{\alpha t} = f(s)e^{\alpha}$ About the last equal sign of above equation, it is because $u(x_0, y_0) = u(s, 1) = c$ when t=0.

$$u(x,y) = f(\frac{x}{y})y^{\alpha}$$

Next lecture will discuss quasi-linear equation.

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u)$$

Chapter 2

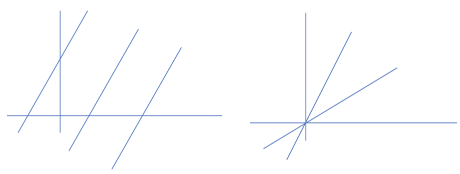
week2

2.1. Tuesday

2.1.1. Quasi-linear Equations

Review: last week we have learn how to solve the following PDE and get the solution along characteristic curves.

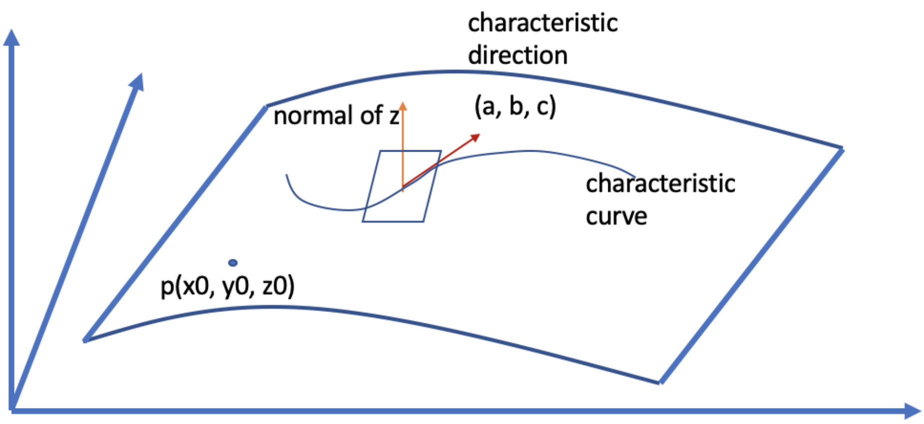
$$u_y + cu_x = 0 \qquad xu_x + yu_y = u$$



$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u)$$

This is what we called quasi-linear equation. At this time, I have to quote our textbook p9 since I cannot explain what professor said better than it does.

We represent the function $u(x,y)$ by a surface $z=u(x,y)$ in xyz-space. Surfaces corresponding to solutions of a P.D.E. are called *integral surfaces* of the P.D.E. The prescribed functions $a(x,y,z), b(x,y,z), c(x,y,z)$ define a field of vectors in xyz-space (or in a portion Ω of that space). Obviously only the direction of the vector, the *characteristic direction*, matters for the P.D.E. (4.2). Since $(u_x, u_y, -1)$ constitute direction numbers of the normal of the surface $z=u(x,y)$, we see that (4.2) is just the condition that the normal of an integral surface at any point is perpendicular to the direction of the vector (a,b,c) corresponding to that point. Thus integral surfaces are surfaces that at each point are tangent to the characteristic direction.



Next we are going to show that the integral surface, that is the solution u we want, is the union of characteristic curves.

Theorem 2.1 If a characteristic curve $\mathscr{C}$ has one point on a integral surface $z=u(x,y)$ then $\mathscr{C}$ lies entirely on the surface.

Proof.

$$z_0 = u(x_0, y_0)$$

$$\begin{cases} \dfrac{dx}{dt} = a(x(t), y(t), u(x_0, y_0)) \\ \dfrac{dy}{dt} = b(x(t), y(t), u(x_0, y_0)) \\ x(0) = x_0 \\ y(0) = y_0 \end{cases}$$

This is a system of odes, there is a solution $(x(t),y(t))$ when t is around 0.

This gives us a curve $(x(t),y(t),u(x(t),y(t)))$ lies on the integral surface.

Claim: It's a characteristic curve.

$$z(t) = u(x(t),y(t))$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = u_x \frac{\mathrm{d}x}{\mathrm{d}t} + u_y \frac{\mathrm{d}y}{\mathrm{d}t} = au_x + bu_y = c$$

■

2.1.2. Cauchy Problem

"We now have a simple description for the general solution u of the quasilinear equation. To have a better insight into the structure of the manifold of solutions it is desirable to have a definite method of generating solutions in terms of a prescribed set F of functions, called "data"."(John, 1982, p.11)

"Finding the function $u(x,y)$ for given data $x = f(s)$, $y = g(s)$, $u = h(s)$ constitutes the *Cauchy problem* for quasi-linear equation." (John, 1982, p.9)

The following φ,ψ,ρ are f,g,h correspondingly.

To solve the Cauchy problem, it suffices to require that $(\psi',\varphi')\&(a,b)$ are linearly independent.

The

$$\begin{cases} \dfrac{\mathrm{d}\bar{X}}{\mathrm{d}t} = a(\bar{X},\bar{Y},Z) \\ \dfrac{\mathrm{d}\bar{Y}}{\mathrm{d}t} = b(\bar{X},\bar{Y},Z) \\ \dfrac{\mathrm{d}\bar{z}}{\mathrm{d}t} = c(\bar{X},\bar{Y},Z) \\ \bar{X}(s,0) = \varphi(s) \\ \bar{Y}(s,0) = \psi(s) \\ \bar{Z}(s,0) = \rho(s) \end{cases}$$

Existence and uniqueness theorem for ODE.

$\Rightarrow$ For every s, $\exists$ one solution $(\bar{X}(s,t),\bar{Y}(s,t),z(s,t))$. In order to make $u(x,y) =$

$z(s(x,y),t(x,y))$ is the case. We need implicit function theorem which makes us able to write s,t as a function of (x,y)

$$\begin{vmatrix} \frac{\partial \bar{X}}{\partial s} & \frac{\partial \bar{X}}{\partial t} \\ \frac{\partial \bar{Y}}{\partial s} & \frac{\partial \bar{Y}}{\partial t} \end{vmatrix} = \begin{vmatrix} \varphi' & a \\ \psi' & b \end{vmatrix} \neq 0$$

■ **Example 2.1**

$$u_y + uu_x = 0, \quad u(x,0) = f(x)$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = z, \ \frac{\mathrm{d}y}{\mathrm{d}t} = 1, \ \frac{\mathrm{d}z}{\mathrm{d}t} = 0$$

z is constant in t. $z(s,0) = f(s) \Rightarrow z(s,t) = f(s)$.

$$\begin{cases} x(s,0) = s \ \Rightarrow \ x(s,t) = f(s)t + s \\ y(s,0) = 0 \ \Rightarrow \ y(s,t) = t \end{cases} \rightarrow \ s = x - f(s)t = x - f(s)y = x - zy$$

$$z(s,0) = f(s) \Rightarrow z(s,t) = f(s) = f(x - zy)$$

$$u(x,y) = f(x - uy)$$

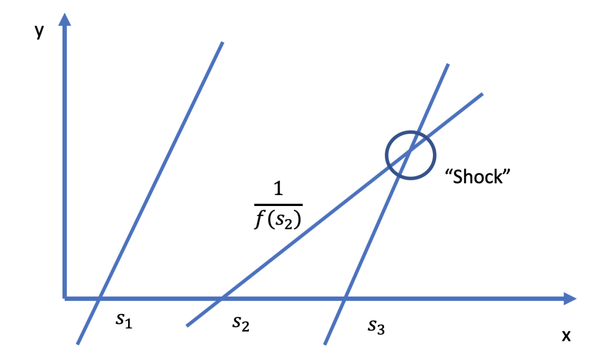
Characteristic curve

$$f(s_2) > f(s_3)$$

$$\frac{1}{f(s_2)} < \frac{1}{s_3}$$

Why it is called a “Shock”?

The reason is that: for same (x,y) z should be the same. However, from the graph we can see there are two z corresponds to the same point marked by the circle. Therefore, the solution does't exist.


y
$\frac{1}{f(s_2)}$
"Shock"
s_1
s_2
s_3
x

■

2.2. Thursday

2.2.1. 2^{nd} order Quasi-linear

$$au_{xx} + 2bu_{xy} + cu_{yy} = d$$

a, b, c, d are functions of (x, y, u, u_x, u_y)

Initial curve $f(s)$, $g(s)$. Initial value $u = h(s), u_x = \varphi(s), u_y = \psi(s)$.

$$u(f(s), g(s)) = h(s)$$

$$u_x f' + u_y g' = h'$$

$$\varphi f' + \psi g' = h'$$

Assume the above equation is satisfied.

$$u_x(f(s), g(s)) = \varphi(s) \qquad u_y(f(s), g(s)) = \psi(s)$$

$$u_{xx} f' + u_{xy} g' = \varphi' \qquad u_{yx} f' + u_{yy} g' = \psi'$$

$$\begin{pmatrix} a & 2b & c \\ f' & g' & 0 \\ 0 & f' & g' \end{pmatrix} \begin{pmatrix} u_{xx} \\ u_{xy} \\ u_{yy} \end{pmatrix} = \begin{pmatrix} d \\ \varphi' \\ \psi' \end{pmatrix}$$

$$0 = \begin{vmatrix} a & 2b & c \\ f' & g' & 0 \\ 0 & f' & g' \end{vmatrix} = ag'^2 - 2bf'g' + cf'^2$$

Personal understanding (may not be correct):

This is called characteristic curve for when determinant is equal to zero above system will not have a solution. This situation will on happen when the initial curve $(f, g, h)$is tangent to a charateristic curve at every point. Therefore, $ag'^2 - 2bf'g' + cf'^2$.

$$\begin{cases} x = f(s) \\ y = g(s) \end{cases} \quad \begin{cases} \dfrac{dx}{ds} = f' \\ \dfrac{dy}{ds} = g' \end{cases} \Rightarrow \frac{dy}{dx} = \frac{g'}{f'}$$

$$a(\frac{dy}{dx})^2 - 2b\frac{dy}{dx} + c = 0$$

$$\frac{dy}{dx} = \frac{2b \pm \sqrt{2b^2 - 4ac}}{2a}$$

- $b^2 - ac > 0$: Hyperbolic
- $b^2 - ac = 0$: Parabolic
- $b^2 - ac < 0$: Elliptic

Example 2.2

- $u_{yy} - c^2 u_{xx} = 0$ $A = -c^2, B = 0, C = 1 \Rightarrow$ $B^2 - AC = C^2$
- heat equation $u_y = c^2 = u_{xx} = 0$ $A = -C^2, B = 0, C = 0$ $\Rightarrow$ $B^2 - AC = 0$
- laplace equation $u_{yy} + u_{xx} = 0$ $A = 1 = C, B = 0$ $\Rightarrow$ $B^2 - AC = -1 < 0$

Wave equation

$$u_{yy} - c^2 u_{xx} = 0$$

$$\frac{dy}{dx} = \frac{1}{-c^2}(\pm c) = \mp\frac{1}{c}$$

$$y = \mp\frac{1}{c}x + \text{const}$$

$$y \pm \frac{1}{c}x = \text{const}$$

$$\xi = y + \frac{1}{c}x$$

$$\eta = y - \frac{1}{c}x$$

$$v(\xi,\eta) = u(x(\xi,\eta), y(\xi,\eta))$$

$$u(x,y) = v(\xi(x,y), \eta(x,y))$$

$$u_x = v_\xi \xi_x + v_\eta \eta_x = v_\xi \frac{1}{c} - v_\eta \frac{1}{c}$$

$$u_{xx} = v_{\xi\xi} \frac{1}{c^2} + v_{\xi\eta}(-\frac{1}{c^2}) - v_{\eta\xi} \frac{1}{c^2} + v_{\eta\eta} \frac{1}{c^2}$$

$$u_y = v_\xi \xi_y + v_\eta \eta_y = v_\xi + v_\eta$$

$$u_{yy} = v_{\xi\xi} + 2v_{\xi\eta} + v_{yy}$$

$$0 = u_{yy} - c^2 u_{xx} = v_{\xi\xi} + 2v_{\xi\eta} + v_{yy} - v_{\xi\xi} + 2v_{\xi\eta} - v_{\eta\eta} = 4v_{\xi\eta}$$

$$v_{\eta\xi} = 0 \;\Rightarrow\; (v_\xi)_\eta = 0 \;\Rightarrow\; v_\xi = f'(\xi)$$

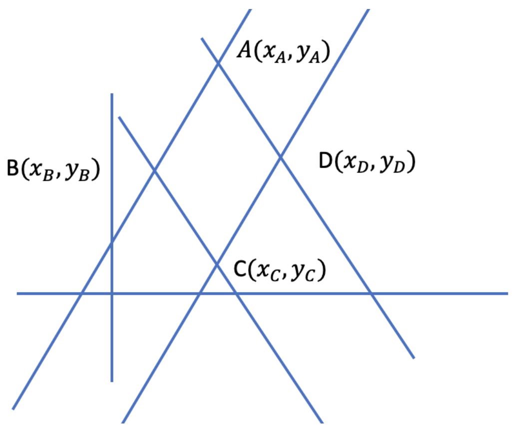
$$v = f(\xi) = \text{const in } \xi$$

$$v(\xi,\eta) = f(\xi) + g(\eta) = u(x,y) = f(y + \frac{1}{c}x) + g(y - \frac{1}{c}x)$$

An observation:

$$u(A) + u(C) = f(y_A + \frac{1}{c}x_A) + g(\underline{y_A - \frac{1}{c}x_A}) + f(y_C \frac{1}{c}x_C) + g(y_C - \frac{1}{c}x_C)$$

$$u(B) + u(D) = f(y_B + \frac{1}{c}x_B) + g(\underline{y_B - \frac{1}{c}x_B}) + f(y_D \frac{1}{c}x_D) + g(y_D - \frac{1}{c}x_D)$$



Take $c = 1$ for simplicity

$$\begin{cases} u_{yy} - u_{xx} = 0 \\ u(x,0) = \varphi(x) \\ u_y(x,0) = \psi(x) \end{cases}$$

$$u(x,y) = f(y+x) + g(y-x)$$

$$\varphi(x) = u(x,0) = f(x) + g(-x)$$

$$\psi(x) = u_y(x,0) = f'(x) + g'(-x)$$

$$\varphi(x) = f(x) + g(-x) \ \Rightarrow\ \varphi'(x) = f'(x) - g'(-x)$$

$$\begin{cases} \varphi(x) = f(x) = g(-x) \\ \psi(x) = f'(x) = g'(x) \end{cases} \Rightarrow \psi'(x) = f'(x) - g'(-x)$$

$$\frac{\varphi'(x) + \psi(x)}{2} = f'(x) \Rightarrow\ f(x) = \frac{1}{2}\psi(x) + \frac{1}{2}\int_0^x \psi(s)\,\mathrm{d}s + \mathrm{const}\alpha$$

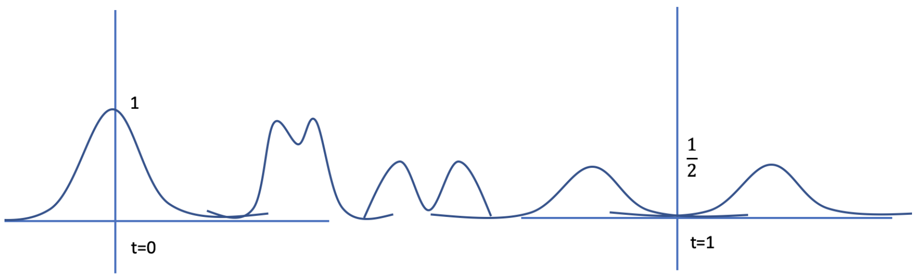
$$\frac{-\varphi'(x) + \psi(x)}{2} = g'(-x) \ \Rightarrow\ g(-x) = \frac{1}{2}\varphi(x) - \frac{1}{2}\int_0^x \psi(s)\,\mathrm{d}s + \mathrm{const}\beta$$

$$\begin{aligned} u(x,y) &= f(x+y) + g(y-x) \\ &= \frac{1}{2}\varphi(x+y) + \frac{1}{2}\int_0^{x+y} \psi(s)\,\mathrm{d}s + \alpha + \frac{1}{2}\varphi(x-y) - \frac{1}{2}\int_0^{x-y} \psi(s)\,\mathrm{d}s + \beta \\ &= \frac{1}{2}[\varphi(x+y) + \varphi(x-y)] + \frac{1}{2}\int_{x-y}^{x+y} \psi(s)\,\mathrm{d}s + \alpha + \beta \end{aligned}$$

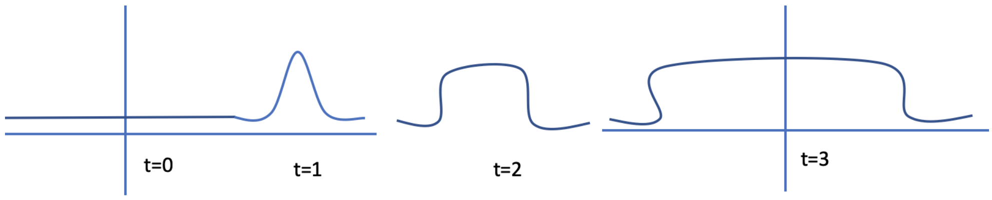
$$u(x,y) = \frac{1}{2}[\varphi(x+y) + \varphi(x-y)] + \frac{1}{2}\int_{x-y}^{x+y} \psi(s)\,\mathrm{d}s$$

$$\begin{cases} u_{yy} - u_{xx} = 0 \\ u(x,0) = \varphi(x)\text{–initial displacement} \\ u_y(x,0) = \psi(x)\text{–initial velocity} \end{cases}$$

case(i) $\psi \equiv 0$ $t = y = 1$ $u(x,1) = \frac{1}{2}\varphi(x+1) + \frac{1}{2}\varphi(x-1)$



case(ii)$\psi = 0$



Chapter 3

week3

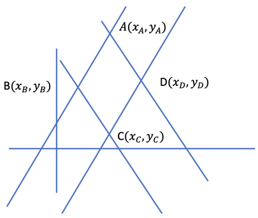
3.1. Tuesday

3.1.1. Initial-boundary Value Problem

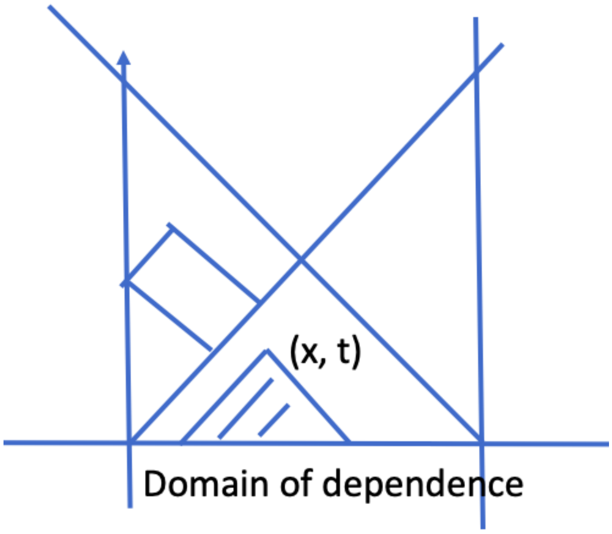
Review:

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) = \varphi(x) \\ u_t(x,0) = \psi(x) \\ u(x,t) = \dfrac{1}{2}[\psi(x+ct) + \psi(x-ct)] + \dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct} \psi(\xi)\,\mathrm{d}\xi \end{cases}$$

General solution: $u(x,t) = f(x+ct) + g(x-ct)$



$$\begin{cases} u_{tt} = c^2 u_{xx} \quad x \in (0,L),\ t \geq 0 \\ u(x,0) = \varphi(x) \quad x \in (0,L),\ t \geq 0 \\ u_t(x,0) = \psi(x) \quad x \in (0,L),\ t \geq 0 \\ u(0,t) = u(L,t) = 0 \quad t \geq 0 \end{cases}$$



Assume

$$u(x,t) = X(x)T(t)$$

$$u_{tt} = XT''$$

$$u_{xx} = X''T$$

$$XT'' = c^2X''T$$

$$\frac{T''}{c^2T} = \frac{X''}{X} = \lambda \text{ a constant}$$

$$X'' = \lambda X \quad \text{in } (0, L)$$

Case (1): $\lambda = 0$

$$X'' = 0$$

$$X(x) = Ax + B$$

$$X(0) = B = X(L) = AL = 0$$

Case (2): $0 < \lambda = \alpha^2$

$$\frac{X''}{X} = \alpha^2$$

$$X(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$$

$$X(0) = c_1 + c_2 = 0$$

$$X(L) = c_1 e^{\alpha L} + c_2 e^{-\alpha L} = 0$$

$$c_1 e^{2\alpha L} + c_2 = 0$$

$$c_1 (e^{2\alpha L} - 1) = 0$$

Similiarly, $c_2 = 0$

Case (3): $\lambda = -\beta^2 < 0$

$$\begin{cases} X'' + \beta^2 X = 0 \\ X(0) = X(L) = 0 \end{cases}$$

$$X(x) = c_1 \cos \beta x + c_2 \sin \beta x$$

$$X(0) = 0 = c_1$$

$$X(L) = 0 = c_2 \sin(\beta L)$$

Want: $\beta L = n\pi$, n=1, 2, 3,... $\beta = \frac{n\pi}{L}$, n=1, 2, 3, ...

$$T'' = -\beta^2 c^2 T$$

$$T(t) = b_1 \cos(\beta c t) + b_2 \sin(\beta c t) = b_1 \cos(\frac{n\pi}{L} c t) + b_2 \sin(\frac{n\pi}{L} c t)$$

As for any n, u_n is a solution for ode, then for any linear combination of it, if it converges, is still a solution. We have:

$$u(x,t) = \sum_{n=1}^{\infty} [b_n \cos(\frac{n\pi}{L} c t) + c_n \sin(\frac{n\pi}{L} c t)] sin(\frac{n\pi}{L} x)$$

Wish List:

(i) Converge for all x, t. In particular, it converges to $\varphi(x)$ when $t = 0$

(ii)differentiable and converge to $\psi(x)$ when $t = 0$

When $t = 0$, $\psi(x) = u(x,0) = \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi}{L} x)$. This series can converge to φ, as when

$L = \pi$, $\{\sin x, \sin 2x, \sin 3x, \cdots, \sin nx, \cdots\}$ is a basis for the continuous function space with inner product $< f, g >= \frac{1}{L}\int_0^L fg(x)\,\mathrm{d}x$.

Chapter 4

week3

4.1. Thursday

4.1.1. Fourier Series

On $[-\pi, \pi]$ want to approximate $f(x)$ by $a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots + a_k \cos kx + b_k \sin kx + \ldots$.

Want to show $\{1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots, \cos kx, \sin kx, \ldots\}$ is a family of orthoganol functions s.t. they for a basis of continuous function space.

Inner product:

$$< f, g >= \int_{-\pi}^{\pi} f(x)g(x)\,\mathrm{d}x$$

The following is aim to show it is a orthoganol family.

$$< 1, \cos kx >= \int_{-\pi}^{\pi} 1 \cdot \cos kx\,\mathrm{d}x = \frac{1}{k} \sin kx|_{-\pi}^{\pi} = 0$$

$$< 1, \sin lx >= \int_{-\pi}^{\pi} 1 \cdot \sin lx\,\mathrm{d}x = -\frac{1}{L} \cos lx|_{-\pi}^{\pi} = 0$$

$$\begin{aligned}
< \cos kx, \cos lx > &= \int_{-\pi}^{\pi}] \cos kx \cos lx\,\mathrm{d}x \\
&= \frac{1}{2}\int_{-\pi}^{\pi} [\cos(k+l)x + \cos(k-l)x]\,\mathrm{d}x \\
&= \frac{1}{2}\int_{-\pi}^{\pi} \cos(k-l)x\,\mathrm{d}x \\
&= \begin{cases} 0 \text{ if } k \neq l \\ \pi \text{ if } k = l \neq 0 \end{cases}
\end{aligned}$$

Next we are going to find all their coefficient.

Assume $f(x) = a_0 + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$.

$$< f, 1 >= \int_{-\pi}^{\pi} f(x)\,\mathrm{d}x \Rightarrow a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x$$

$$< a_0 + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx), 1 >=< a_0, 1 > + \sum_{k=1}^{\infty} < a_k \cos kx + b_k \sin kx, 1 >=< a_0, 1 >= 2\pi a_0$$

$$< f, \cos x >=< a_0, \cos x > + \sum_{k=1}^{\infty}[< a_k \cos kx, \cos x > + < b_k \sin kx, \cos x >] =< a_1 \cos x, \cos x >= a_1\pi$$

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos x\,\mathrm{d}x$$

$$b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin x\,\mathrm{d}x$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,\mathrm{d}x$$

$a_0 + \sum_{k=1}^{N}(a_k \cos kx + b_k \sin kx) \to f(x)$ for as $N \to \infty$ $\{1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots, \cos kx, \sin kx, \ldots\}$ form a complete basis for function space.

$$* \begin{cases} X'' + \lambda X = 0 \text{ in } (a,b) \\ a_1 X(a) + a_2 X'(a) = 0 \\ \beta_1 X(b) + \beta_2 X'(b) = 0 \end{cases}$$

$\alpha_1, \alpha_2, \beta_1, \beta_2$ are constants. $\alpha_1^2 + \alpha_2^2 > 0, \beta_1^2 + \beta_2^2 > 0$

Theorem 4.1 (1) The eigenvalues of (*) form an increasing sequence $\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_k < \cdots \to \infty$.

(2)For each eigenvalue λ_k the corresponding eigenspace is 1-dimensional. Denote eigen function by X_k.

(3) X_k has exactly $k-1$ zeros in (a,b).

(4) $\{X_1, X_2, \ldots, x_k, \ldots\}$ is a family of orthogonal $f(n)$ i.e. $\int_a^b x_k x_l = 0$ if $k \neq l$.

(5)If $\varphi(x)$ is $C^2[a,b]$ then its eigen function expansion $\sum_{k=1}^{\infty} \varphi_k X_k$ converges to $\varphi(x)$ uniformly on $[a,b]$ when $\varphi_k = \frac{\int_a^b \varphi(x) X_k(x)\,\mathrm{d}x}{\int_a^b X_k^2(x)\,\mathrm{d}x}$

(6)If φ is only square intergrable i.e. if $\int_a^b \varphi^2(x)\,\mathrm{d}x < \infty$, then $\int_a^b [\varphi(x) - \sum_{k=1}^{N} \varphi_k X_k(x)]^2 \mathrm{d}x] \to 0$ as $N \to \infty$.

(1) $\begin{cases} \alpha_1 = 1, \alpha_2 = 0 \\ \beta_1 = 1, \beta_2 = 0 \end{cases}$ Dirichlet

(2) $\begin{cases} \alpha_1 = 0, \alpha_2 = 1 \\ \beta_1 = 0, \beta_2 = 1 \end{cases}$ Neumann.

The following is going to show when two solutions of $x'' + \lambda x = 0$ are different. x_k is orthorganol to x_l.

$$\begin{cases} x_k'' + \lambda_k x_k = 0 \\ x_l'' + \lambda_l x_l = 0 \end{cases}$$

Multiply x_l and x_k correspondingly to two equations and then take the integral. We have

$$0 = \int_a^b (x_k'' x_l - x_l'' x_k) + \int_a^b (\lambda_k - \lambda_l) x_k x_l = (x_k' - x_l' - x_l' x_k)|_a^b + \int_a^b (\lambda_k - \lambda_l) x_k x_l$$

$$= x_k'(b) x_l(b) - x_l'(b) x_k(b) - x_k'(a) x_l(a) + x_l'(a) x_k(a) + \int_a^b (\lambda_k - \lambda_l) x_k x_l = 0$$

$$\int_a^b x_k x_l = 0$$

Sturm-Liouville

$$\frac{\mathrm{d}}{\mathrm{d}x}(p(x)\frac{\mathrm{d}u}{\mathrm{d}x}) + 1(x)u = 0$$

Chapter 5

Week4

5.1. Tuesday

5.1.1. Inhomogeneous Wave Equation

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) \qquad x \in [o, L],\ t \geq 0 \\ u(x,0) = \varphi(x), u_t(x,0) = \psi(x),\ x \in [0, L] \\ u(0,t) = h(t),\ u(L,t) = k(t) \end{cases}$$

Assume $h(0) = k(0) = 0$.Write

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin\frac{n\pi x}{L},\ u_n(t) = \frac{2}{L}\int_0^L u(x,t) \sin\frac{n\pi x}{L}\,\mathrm{d}x$$

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin\frac{n\pi x}{L},\ f_n(t) = \frac{2}{L}\int_0^L f(x,t) \sin\frac{n\pi x}{L}\,\mathrm{d}x$$

$$u_{tt}(x,t) = \sum_{n=1}^{\infty} V_n(t) \sin\frac{n\pi x}{L}, \quad V_n(t) = \frac{2}{L}\int_0^L u_{tt}(x,t) \sin\frac{n\pi x}{L}\,\mathrm{d}x = \frac{\mathrm{d}^2 u_n}{\mathrm{d}t}$$

$$u_{xx}(x,t) = \sum_{n=1}^{\infty} w_n(t) \sin\frac{n\pi x}{L}, \quad w_n(t) = \frac{2}{L}\int_0^L u_{xx}(x,t) \sin\frac{n\pi x}{L}\,\mathrm{d}x$$

$$
\begin{aligned}
w_n(t) &= \frac{2}{L}\int_0^L u_{xx}(x,t)\sin\frac{n\pi x}{L}\,\mathrm{d}x \\
&= \frac{2}{L}[u_x \sin\frac{n\pi x}{L}|_0^L - \frac{n\pi}{L}\int_0^L u_x\cos\frac{n\pi x}{L}\,\mathrm{d}x] \\
&= -\frac{2n\pi}{L^2}[n\cos\frac{n\pi x}{L}|_0^L + \frac{n\pi}{L}\int_0^L u\sin\frac{n\pi x}{L}\,\mathrm{d}x] \\
&= -\frac{2n\pi}{L^2}[u(L,t)(-1)^n - u(0,t) + \frac{n\pi}{L}\int_0^L u\sin\frac{n\pi x}{L}\,\mathrm{d}x] \\
&= -\frac{2n\pi}{L^2}[(-1)^n k(t) - h(t) + \frac{n\pi}{2}u_n(t)] \\
&= \frac{2n\pi}{L^2}[h(t)-(-1)^n k(t)] - (\frac{n\pi}{L})^2 u_n(t)
\end{aligned}
$$

$$
v_n(t) - c^2 w_n(t) = \frac{2}{L}\int_0^L (u_{tt}-c^2u_{xx})\sin\frac{n\pi x}{L}\,\mathrm{d}x = \frac{2}{L}\int_0^L f(x,t)\sin\frac{n\pi x}{L}\,\mathrm{d}x = f_n(t)
$$

$$
\frac{\mathrm{d}^2 u_n}{\mathrm{d}t^2} + c^2\lambda_n u_n(t) - \frac{2n\pi}{L^2}[h(t)-(-1)^n k(t)]c^2 = f_n(t),\ n \geq 1
$$

Solve this ODE of u_n, we get the solution.

$$
\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) \\ u(0,t = u(L,t) = 0 \end{cases}
$$

To solve

$$
\begin{cases} U_{tt} - c^2 U_{xx} = g(x,t) \\ U(0,L) = h(t), U(L,t) = k(t) \end{cases}
$$

Set $\rho(x,t) = \frac{(L-x)}{L}h(t) + \frac{x}{L}k(t) \ \Rightarrow\ \rho(0,t) = h(t),\ \rho(L,t) = k(t)$

$$
u(x,t) = U(x,t) - \rho(x,t)
$$

$$
u_{tt} - c^2 u_{xx} = U_{tt} - c^2 U_{xx} - \rho_{tt} + c^2\rho_{xx} = g(x,t) - \frac{L-x}{L}h_{tt} - \frac{x}{L}k_{tt} = f(x,t)
$$

Set $U(x,t) = u(x,t) + \frac{L-x}{L}h(t) + \frac{x}{L}k(t)$.

To solve the given inhomogeneous problem

$$
\begin{cases} U_{tt} - c^2 U_{xx} = g(x,t) \\ U(0,L) = h(t), U(L,t) = k(t) \end{cases}
$$

Set $f(x,t) = g(x,t) - \frac{L-x}{L} h_{tt}(t) - \frac{x}{L} k_{tt}(t)$ and we know the solution for $\begin{cases} u_t t - c^u_{xx} = f(x,t) \\ u(0,t) = 0,\ u(L,t) = 0 \end{cases}$

Then $U(x,t) = u(x,t) + \frac{L-x}{L} h(t) + \frac{x}{L} k(t)$ solves. Solve $\begin{cases} v_{tt} - c^2 v_{xx} = 0 \\ v(x,0) = \varphi(x), v_t(x,0) = \psi(x) \end{cases} \Rightarrow$

$U + v$

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) \quad x \in \mathbb{R}, t \geq 0 \\ u(x,0) = \varphi(x) \\ u_r(x,0) = \psi(x) \end{cases}$$

$$u(x,t) = \frac{1}{2}(\varphi(x+ct) + \varphi(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,\mathrm{d}y + \frac{1}{2c}\int\int_D f(y,s)\,\mathrm{d}y\,\mathrm{d}s$$

Duhamel's Principle

$$\begin{cases} \frac{\mathrm{d}^2 v}{\mathrm{d}t^2} + A^2 v = f(t) \\ v(0) = a \\ v'(0) = b \end{cases}$$

Let $S(t) = \frac{1}{A}\sin At \Rightarrow v(t) = S'(t)a + S(t)b + \int_0^t S(t-s)f(s)\,\mathrm{d}s$

Pluge S in the system of equations. We get the solution. (Remind what we do in ODE with variation of parameter. This two method is the same in essence. In addition, what we do in PDE is similiar.)

Chapter 6

week4

6.1. Thursday

6.1.1.

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) \\ u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x) \end{cases}$$

$$\begin{cases} v_{tt} - c^2 v_{xx} = 0 \\ v(x,0) = \varphi(x),\ v_t(x,0) = \psi(x) \end{cases}$$

$\tilde{u} = u - v$

$$\Rightarrow \begin{cases} \tilde{u}_{tt} - c^2 \tilde{u}_{xx} = f(x,t) \\ \tilde{u}(x,0) = 0,\ \tilde{u}_t(x,0) = 0 \end{cases}$$

$$\begin{aligned} \tilde{u}(x,t) &= \frac{1}{2c} \int\int_D f(y,s)\,\mathrm{d}y\,\mathrm{d}s \\ &= \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,\mathrm{d}y\,\mathrm{d}s \\ &= \int_0^t \Big(\frac{1}{2c} \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,\mathrm{d}y\Big)\,\mathrm{d}s \end{aligned}$$

For each fixed $0 \le s \le t$

$$\begin{cases} U_{tt} - c^2 U_{xx} = 0 \\ U(x,s) = 0,\ U_t(x,s) = f(x,s) \end{cases} \text{from s to t}$$

i.e. $U(x,t,s)$ is a solution for

$$\begin{cases} U_{tt} - c^2 U_{xx} = 0 \\ U(x,s,s) = 0,\ U_t(x,s,s) = f(x,s) \end{cases}$$

$$\tilde{u}(x,t) = \int_0^t U(x,t,s)\,\mathrm{d}s$$

$$U(x,t,s) = \frac{1}{2c}\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,\mathrm{d}y$$

Check the answer was right

$$\tilde{u}_t = U(x,t,t) + \int_0^t U_t(x,t,s)\,\mathrm{d}s$$

$$\begin{aligned} \tilde{u}_{tt} &= U_t(x,t,t) + \int_0^t U_{tt}(x,t,s)\,\mathrm{d}s \\ &= f(x,t) + \int_0^t c^2 U_{xx}(x,t,s)\,\mathrm{d}s \\ &= f(x,t) + c^2 \tilde{u}_{xx}(x,t) \end{aligned}$$

$$\tilde{u}_{tt} - c^2 \tilde{u_{xx}} = f(x,t)$$

Go through initial boundary problem once more.

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t),\ 0 \le x \le L,\ t \ge 0 \\ u(x,0) = \varphi(x), u_t(x,0) = \psi(x) \\ u(0,t) = h(t),\ u(L,t) = k(t) \end{cases}$$

$$\rho(x,t) = \frac{L-x}{L}h(t) + \frac{x}{L}k(t)$$

$$\tilde{u}(x,t) - \rho(x,t)$$

$$\begin{aligned} &\tilde{u}_{tt} - c^2 \tilde{u_{xx}} \\ =&u_{tt} - \rho_{tt} - c^2 u_{xx} + c^2 \rho_{xx} \\ =&f(x,t) - [\rho_{tt} - c^2 \rho_{xx}] \\ \equiv&\tilde{f}(x,t) \end{aligned}$$

%%%%%

$$\tilde{u}_{tt} - c^2\tilde{u}_{xx} = \tilde{f}(x,t)$$

$$\tilde{u}(x,0) = \varphi(x) - [\frac{L-x}{L}h(0) + \frac{x}{L}k(0)] \ \equiv \tilde{\varphi(x)}$$

$$\tilde{u}_r(x,0) = \psi(x) - [\frac{L-x}{L}h'(0) + \frac{x}{L}k'(0)] \ \equiv \tilde{\psi}(x)$$

%%%%%

$$\tilde{u}(0,t) = u(0,t) - \rho(0,t) = h(t) - h(t) = 0$$

$$\tilde{u}(L,t) = u(L,t) - \rho(L,t) = k(t) - k(t) = 0$$

$$\begin{cases} \tilde{u}_{tt} - c^2\tilde{u}_{xx} = \tilde{f}(x,t) \\ \tilde{u}(x,0) = \tilde{\varphi}(x) \ \tilde{u}_t(x,0) = \tilde{\psi}(x) \\ \tilde{u}(0,t) = 0 = \tilde{u}(L,t) \end{cases}$$

$$\begin{cases} v_{tt} - c^2 v_{xx} = 0 \\ v(x,0) = \tilde{\varphi(x)}, \ v_t(x,0) = \tilde{\psi}(x) \\ v(0,t) = 0 = v(L,t) \end{cases}$$

$u^* = \tilde{u} - v$

$$\Rightarrow \begin{cases} u^*_{tt} - c^2 u^*_{xx} = \tilde{f}(x,t) \\ u^*(x,0) = 0, \ u^*_t(x,0) = 0 \\ u^*(0,t) = 0 = u^*(L,t) \end{cases}$$

%%%%%

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(x,0) = \varphi(x), u_t(x,0) = \psi(x) \end{cases}$$

$\Rightarrow$

$$u(x,t) = \frac{1}{2}[\varphi(x+ct) + \varphi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,\mathrm{d}y$$

no “smoothing” effect

%%%%%

$$(*)\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) \\ u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x) \\ u(0,t) = h(t), u(L,t) = k(t) \end{cases}$$

Fact: The solution is unique. Suppose both u_1, u_2 are solutions to (*), Set $v = u_1 - u_2$

$$\Rightarrow \begin{cases} v_{tt} - c^2 v_{xx} = 0 \\ v(x,0) = 0,\ v_t(x,0) = 0 \Rightarrow v \equiv 0 \\ v(0,t) = 0,\ v(L,t) = 0 \end{cases}$$

Conservation Law

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, x \in (0,L), t \geq 0 \\ u(0,t) = 0 = u(L,t) \end{cases}$$

Energy

$$E(t) = \frac{1}{2}\int_0^L [u_t^2(x,t) + c^2 u_x^2(x,t)]\,\mathrm{d}x$$

$$E'(t) = \int_0^L [u_t u_{tt} + c^2 u_x u_{xt}]\,\mathrm{d}x = c^2(u_t u_x)(L,t) - c^2(u_t u_x)(0,t) = 0$$

$$u_t(L,t) = 0,\ u_t(0,t) = 0$$

$$E' \equiv 0$$

$$E(t) = E(0) = 0$$

$$E(0) = \frac{1}{2}\int_0^L [v_t^2(x,0) + c^2 v_x^2(x,0)]\,\mathrm{d}x$$

$$v_t \equiv 0,\ v_x \equiv 0$$

Chapter 7

Week5

7.1. Tuesday

7.1.1. Heat Equation

We will restrict to one dimension.

$$\begin{cases} u_t = c^2 u_{xx} \\ u(x,0) = \varphi(x) \\ u(0) = u(L) = 0 \quad \text{(Dirichlet B.C.)} \end{cases}$$

Dirichlet means keep temperature on the boundary to be zero. Neumann B.C. ($u_x(0) = u_x(L) = 0$) means the tube is insulated.

$$u(x,t) = X(x)T(t)$$

$$u_t = XT'$$

$$u_{xx} = X''T$$

$$XT' = c^2 X''T$$

$$\frac{T'}{c^2 T} = \frac{X''}{X} = -\lambda \text{ (a constant)}$$

$$X'' + \lambda X = 0$$

$$u(0,t) = X(0)T(t) = 0$$

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(L) = 0 \end{cases}$$

(Previous calculation $\rightarrow$)

$$\lambda = (\frac{n\pi}{L})^2,\ X(x) = \sin\frac{n\pi x}{L},\ n = 1,2\dots$$

$$T' = c^2 T[-\frac{n\pi}{L}]^2 = -(c\frac{n\pi}{L})^2 T$$

$$T(t) = e^{-(\frac{cn\pi}{L})^2 t}$$

$$u(x,t) = cost.e^{-(\frac{cn\pi}{L})^2 t}\sin(\frac{n\pi x}{L})$$

$$u(x,t) = \sum \alpha_n e^{-(\frac{cn\pi}{L})^2 t}\sin(\frac{n\pi x}{L})$$

$$u(x,0) = \varphi(x) = \sum_{n=1}^{\infty}\alpha_n \sin\frac{n\pi x}{L}$$

$$\Rightarrow \alpha_n = \varphi_n = \text{ the Fourier coeff. of } \varphi = \frac{2}{L}\int_0^L \varphi(x)\sin\frac{n\pi x}{L}\,\mathrm{d}x$$

$$u(x,t) = \sum_{n=1}^{\infty}\varphi_n \sin\frac{n\pi x}{L} e^{-(\frac{cn\pi}{L})^2 t}$$

Suppose φ is bounded on [0,L] , $|\varphi| \le M$

$$\Rightarrow |\varphi_n| \le \frac{2}{L}\int_0^L |\varphi||\sin\frac{n\pi x}{L}|\,\mathrm{d}x \le \frac{2}{L}\int_0^L M = 2M$$

$\sum_{n=1}^{\infty} e^{-\beta n^2}$ is convergent, by ratio test, implies the convergence of u.

Smoothing effect:

Suppose the initial value φ is bounded. Then as soon as t becomes positive, the solution becomes C^∞ smooth.

Rk: φ could be even discontinuous and the conclusion still holds.

$$u_N(x,t) = \sum_{n=1}^{N}\varphi_n \sin\frac{n\pi x}{L} e^{-(\frac{cnt}{L})^2 t} \rightarrow u(x,t)$$

A lemma to show the limit is a smooth function: Suppose that f_n converges to f on [a,b] and $f_n' \rightarrow g$ uniformly on [a,b]. Then $f' = g$.

Proof.

$$f_n(x) = f_n(a) = \int_a^x f_n'(y)\,\mathrm{d}y, n \to \infty$$

$$f(x) = f(a) + \int_a^x g(y)\,\mathrm{d}y$$

%%% back to the proof t is fixed

$$\frac{\mathrm{d}u_N(x,t)}{\mathrm{d}x} = \sum_{n=1}^{N} \varphi_n \cos\frac{n\pi x}{L} e^{-(\frac{cnt}{L})^2 t} (\frac{n\pi}{L})$$

The derivative converges to g uniformly by ratio test. The derivative is infinitely differentiable. Therefore, smoothing effect is true.

Theorem 7.1 Uniqueness:

$$\begin{cases} u_t = c^2 u_{xx} \\ u(x,0) = \varphi(x) \\ u(0) = u(L) = 0 \end{cases}$$

The solution is unique.

Proof. Suppose we have 2 solutions u_1 and u_2. Set $u = u_1 - u_2$. Then u satisfies:

$$u_t = c^2 u_{xx} in (0,L), u(x,0) = 0, u(0,t) = u(L,t) = 0$$

$$E(t) = \int_0^L u^2(x,t)\,\mathrm{d}x$$

$$E'(t) = \int_0^L u u_t\,\mathrm{d}x = \int_0^L u u_{xx}\,\mathrm{d}x = -\int_0^L u_x u_x + u u_x|_0^L = -\int_0^L u_x^2 \leq 0$$

E(t) is non-increasing. i.e. $E(t) \leq E(0)\ \forall t \geq 0$ $E(0) = \frac{1}{2}\int_0^L u^2(x,0)\,\mathrm{d}x = 0$

$$\Rightarrow E(t) = 0 = \frac{1}{2} u^2(x,t)\,\mathrm{d}x$$

$$u(x,t) \equiv 0 in x, and t$$

$$i.e. u_1(x,t) \equiv u_2(x,t)$$

■

Backwards Uniqueness

Q: Suppose that u is a solution for

$$\begin{cases} u_t = u_{xx}, x \in [0,L] \\ u(x,0) = \varphi(x), x \in [0,L] \end{cases} u(0,t) = u(L,t) = 0, t \geq 0$$

Suppose $u(x,T) \equiv 0 \ \forall x \in [0,L]$ for some $T > 0$. Then, what can you conclude about the initial value $\varphi(x)$? Yes is 0

Well-posedness. Uniqueness, existence, dependence.

7.2. Thursday

7.2.1. Fourier Transformation

$$f \in C_0^\infty(\mathbb{R}), f(x)$$

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi} f(x) \,\mathrm{d}x$$

Fourier transform of f.

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Use fourier transform to change PDE to ODE.

$$\begin{aligned}
\widehat{f'}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi} f'(x) \,\mathrm{d}x \\
&= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\frac{\mathrm{d}}{\mathrm{d}x} e^{-ix\xi}) f(x) \,\mathrm{d}x \\
&= i\xi \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi} f(x) \,\mathrm{d}x = i\xi \hat{f}(\xi)
\end{aligned}$$

$$u_t = u_{xx}, \quad u(x,0) = \varphi(x) on \mathbb{R}$$

$$\hat{u}_t = \hat{u_{xx}} = i\xi \hat{u_x} = (i\xi)^2 \hat{u} = -\xi^2 \hat{u}$$

$$\hat{u}_t(\xi,t) = -\xi^2 \hat{u}(\xi,t)$$

$$\hat{u}(\xi,t) = C(\xi) e^{-\xi^2 t}$$

$$\hat{u}(\xi,0) = \hat{\varphi}(\xi) = C(\xi)$$

$$\hat{u}(\xi,t) = \hat{\varphi}(\xi) e^{-\xi^2 t}$$

Inverse Fourier Transform

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\xi} \hat{f}(\xi) \,\mathrm{d}\xi$$

$$u(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ix\xi}\hat{\varphi}(\xi)e^{-\xi^2t}\,\mathrm{d}\xi=\frac{1}{2\pi}\int_{-\infty}^{\infty}(\int_{-\infty}^{\infty}e^{ix\xi-iy\xi-\xi^2t}\,\mathrm{d}\xi)\varphi(y)\,\mathrm{d}y$$

where $\hat{\varphi}(\xi)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-iy\xi}\varphi(y)\,\mathrm{d}y$

$$K(x,y,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i(x-y)\xi-\xi^2t}\,\mathrm{d}\xi=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-t(\xi-\frac{i(x-y)}{2t})^2}e^{-\frac{(x-y)^2}{4t}}\,\mathrm{d}\xi$$

where $i(x-y)\xi-\xi^2t=-t[\xi^2-i\frac{x-y}{t}\xi+(\frac{i(x-y)}{2t})^2)]-\frac{(x-y)^2}{4t}$. We have

$$\begin{aligned}K(x,y,t)&=\frac{1}{2\pi}e^{-\frac{(x-y)^2}{4t}}\int_{-\infty}^{\infty}e^{-t[\xi-i\frac{(x-y)}{2t}]^2}\,\mathrm{d}\xi \quad t>0\\&=\frac{1}{2\pi}e^{-\frac{(x-y)^2}{4t}}\int_{-\infty}^{\infty}e^{-[\sqrt{t}\xi-\frac{i(x-y)}{2\sqrt{t}}]^2}\,\mathrm{d}\xi\end{aligned}$$

Let $\eta=\sqrt{t}\xi-\frac{i(x-y)}{2\sqrt{t}}$ and $\mathrm{d}\eta=\sqrt{t}\,\mathrm{d}\xi$. Above is equal to

$$\frac{1}{2\pi}e^{-\frac{(x-y)^2}{4t}}(\int_{-\infty}^{\infty}e^{-\eta^2}\,\mathrm{d}\eta)\frac{1}{\sqrt{t}}=\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}$$

$$K(x,y,t)=\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}\ \text{Heat Kernel}$$

$$u(x,t)=\frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty}e^{-\frac{(x-y)^2}{4t}}\varphi(y)\,\mathrm{d}y=\int_{-\infty}^{\infty}K(x,y,t)\varphi(y)\,\mathrm{d}y$$

Theorem 7.2 Let φ be a bounded continuous function on $\mathbb{R}$. Then

$$u(x,t)=\int_{-\infty}^{\infty}K(x,y,t)\varphi(y)\,\mathrm{d}y=\frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty}e^{-\frac{(x-y)^2}{4t}}\varphi(y)\,\mathrm{d}y,\ t>0$$

is a solution for $u_t=u_{xx}$ with $\lim_{x\to x_0\ t\to 0}u(x,t)=\varphi(x_0)$ for any $x_o\in\mathbb{R}$.

Proof. Basic properties of $K: t>0$

(1) K is C^∞ $\forall x,y\in\mathbb{R},t>0$

(2) $(K_t-K_{xx})(x,y,t)=0,\ t>o$

(3) $K(x,y,t)>0$

(4) $\int_{-\infty}^{\infty}K(x,y,t)\,\mathrm{d}y=1\ \forall x\in\mathbb{R},t>0$

(5) For any $\epsilon>0$, $\int K(x,y,t)\,\mathrm{d}y\to 0$ uniformly in x, as $t\to 0$ $|x-y|>\epsilon$ ■

R 1.$u \in C^\infty$ as soon as t becomes positive.

2. Speed of propogation of " heat" transfer is ∞.

3. $\inf_{x\in\mathbb{R}} \varphi(x) \le u(x,t) \le \sup_{x\in\mathbb{R}} \varphi(x)$ (Max principle.)

4. Soltion is NOT unique.

List of Properties:function $\rightarrow$ after transformation

- $f'(x) \rightarrow i\xi\hat{f}(\xi)$

- $xf(x) \rightarrow i\hat{f}'(\xi)$

- $f(x-a) \rightarrow e^{-ia\xi}\hat{f}(\xi)$

- $e^{iax} \rightarrow \hat{f}(\xi - a)$

- $f(ax) \rightarrow \frac{1}{|a|}\hat{f}(\frac{\xi}{a})$, $(a \neq 0)$

Definition 7.1 A distribution is a linear functional $f[\phi]$ defined for all $\phi \in C_0^\infty(\Omega)$, which is continuous in the following sense:

Let $\{\phi_k\}$ a sequence of functions in $C_0^\infty(\Omega)$ s.t.

(i) Support $\phi_l \subset K << \Omega, \forall l$ (K: independent of l)

(ii) $D^\alpha \phi_l \rightarrow 0$ uniformly in $x \in \Omega$ for each α.

Then $\lim_{l\rightarrow\infty} f[\phi_l] = 0$ ■

R Support is a mathematical concept.

■ **Example 7.1** 1 f is a functional. $f[\phi] = \int_\Omega f\phi \quad \phi \in C_0^\infty(\Omega)$

2. Dirac δ-function, x is fixed in Ω

$$\delta_x[\phi] = \int_\Omega \delta_x \phi = \phi(x)$$

List: function $\to$ Fourier transformation

$\delta(x) \to \frac{1}{2\pi}$

$e^{-a|x|} \to \frac{1}{\sqrt{2\pi}} \frac{2a}{a^2+\xi^2}$ $a > 0$

$1 \to \sqrt{2\pi}\delta(\xi)$

$e^{-\frac{x^2}{2}} \to e^{-\frac{\xi^2}{2}}$

$H(a - |x|) \to \frac{1}{2\pi} \frac{2}{\xi} \sin(a\xi), a > 0$

R $\hat{f}'$ and $\widehat{f'}$ are different. Be careful.