directional derivative $\hat{=}$ The derivative of function $z=u(x,y)$ along direction $\vec{l}$

denoted as $\frac{\partial u}{\partial \vec{l}}$ or $D_{\vec{l}}\, u$, where $\vec{l} = (\cos\alpha, \sin\alpha)$

$\Delta x = \rho\cos\alpha$
$\Delta y = \rho\sin\alpha$

since u is differentiable, $u(x_0+\Delta x, y_0+\Delta y) - u(x,y) = u_x(x_0,y_0)\Delta x + u_y(x_0,y_0)\Delta y + o(\rho)$

Therefor $\frac{\partial u}{\partial \vec{l}} = \lim_{\rho\to 0^+} \frac{u(x+\Delta x, y+\Delta y) - u(x,y)}{\rho}$

$= \lim_{\rho\to 0^+} \frac{u(x+\rho\cos\alpha, y+\rho\sin\alpha) - u(x,y)}{\rho}$

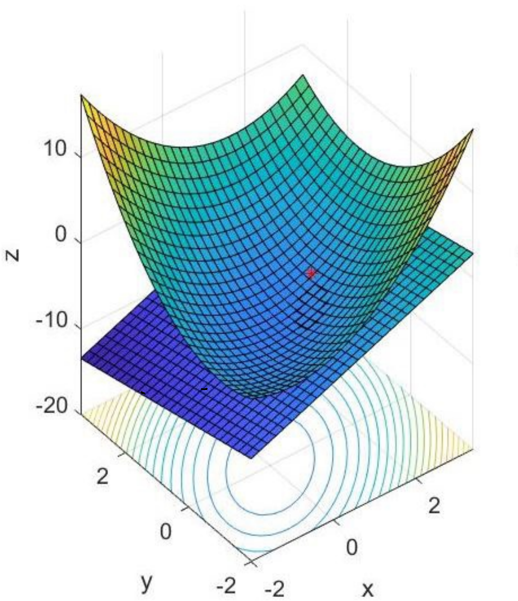
$= \lim_{\rho\to 0^+} u_x(x_0,y_0)\cdot\cos\alpha + u_y(x_0,y_0)\sin\alpha.$

Not important !
Gradient $\nabla u = \text{grad}\, u = u_x\vec{i} + u_y\vec{j} = \begin{pmatrix} u_x \\ u_y \end{pmatrix}$

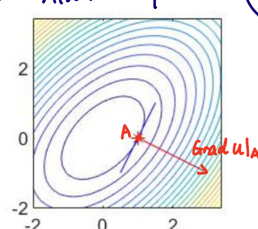
$\frac{\partial u}{\partial \vec{l}} = \nabla u \cdot \vec{l}$

dot product.

The direction of greatest increase of u is the direction as the gradient vector.



Also not important!



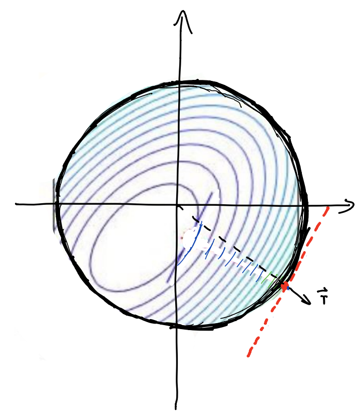
In ex. 2 f assignment 1.

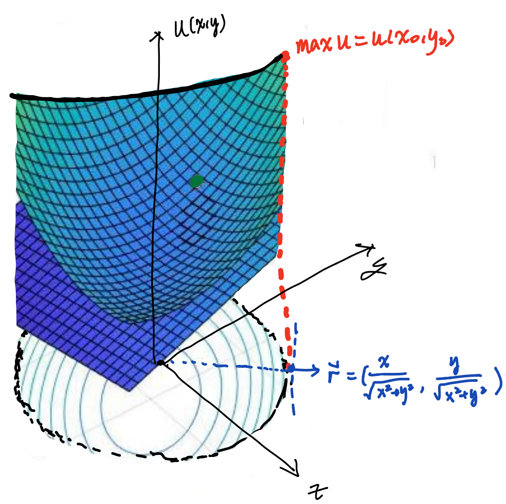
If we attain max u on the boundary $C=\{x^2+y^2=a^2\}$, suppose the max. point (x_0, y_0)

Then we must have $\left.\frac{\partial u}{\partial \vec{r}}\right|_{(x_0,y_0)} \geq 0 \Leftrightarrow \left.\frac{\partial u}{\partial \vec{r}}\right|_{(x_0,y_0)} = \left.\nabla u \cdot \vec{r}\right|_{(x_0,y_0)} = \left.\frac{xu_x + yu_y}{\sqrt{x^2+y^2}}\right|_{(x_0,y_0)} \geq 0$

$$\max u = u(x_0,y_0) = -(xu_x + yu_y)\Big|_{(x_0,y_0)} \leq 0$$

- similar to $\min u \geq 0$





In ex. 3 of assignment 1.

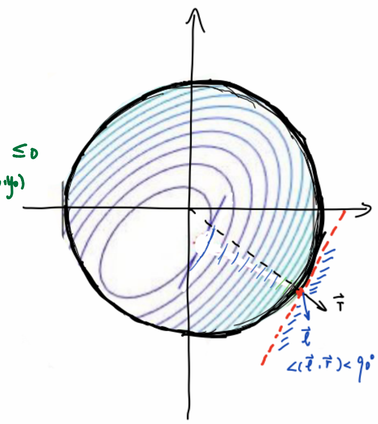
If we attain max u on the boundary $C=\{x^2+y^2=a^2\}$, suppose the max. point (x_0, y_0)

Under the condition: $\vec{r}\cdot\vec{\ell} = a(x,y)x + b(x,y)y < 0$, which means the angle between $\vec{r}$ and $\vec{l}$ is less than $\frac{\pi}{2}$. where $\vec{\ell} = \left(\frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}\right)$

Therefor, the derivation of u along $\vec{\ell}$ at max. point (x_0, y_0): $\left.\frac{\partial u}{\partial \vec{\ell}}\right|_{(x_0,y_0)} \geq 0$

$$\Leftrightarrow \left.\frac{\partial u}{\partial \vec{l}}\right|_{(x_0,y_0)} = \left.\frac{a(x,y)u_x + b(x,y)u_y}{\sqrt{a^2+b^2}}\right|_{(x_0,y_0)} \geq 0$$

$$\max u = u(x_0,y_0) = -\left.\left[a(x,y)u_x + b(x,y)u_y\right]\right|_{(x_0,y_0)} \leq 0$$



• similar to $\min u \geq 0$