Banach Algebras

Lecture 1

Spectral Theorem in Finite Dimensions

Definition 1.1. Bunch of things. Let $A =$ square matrix.

- Symmetric: $A = A^T$.
- Orthogonal: $AA^T = A^T A = I$.
- Adjoint: $A^* = \overline{A^T}$.
- Self-adjoint: $A = A^*$.
- Unitary: $AA^* = A^*A = I$.
- Normal: $AA^* = A^*A$.
- Diagonal: $A_{ij} = 0$ whenever $i \neq j$.

Theorem 1.1. *Let A be a normal complex matrix. Then there is a unitary matrix U such that UAU^* is a diagonal matrix. Equivalently, there is an orthogonal basis of eigenvectors for A.*

Example 1.1.

$$A = \begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix},$$
$$A^* = \begin{pmatrix} 1-i & 1+i \\ 1+i & 1-i \end{pmatrix}.$$

Then

$$AA^* = A^*A = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}.$$

A few days later we get

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} & 1 \\ 1 & -1 \end{pmatrix},$$

and

$$UAU^* = \begin{pmatrix} 2 & 0 \\ 0 & 2i \end{pmatrix}.$$

Definition 1.2. A Hilbert space is a complete inner product space.

Proposition 1.1. *If H_1, H_2 are Hilbert spaces with orthonormal bases of the same cardinality, they are isometrically isomorphic.*

Definition 1.3. A Hilbert space is separable if it admits a countable orthonormal basis.

Any infinite dimensional Hilbert space is isometrically isomorphic to $\ell^2(\mathbb{N})$.

Definition 1.4. A bounded operator $A : H \to H$ is compact if the closure of the image of the unit ball in H under A is compact.

Example 1.2.

1. Any finite rank operator is compact.

2. Let $H = \ell^2(\mathbb{N})$. Let $\mathbf{a} = (a_1, a_2, \dots)$ be a sequence of complex numbers. Define $M_{\mathbf{a}}(x_1, x_2, \dots) = (a_1x_1, a_2x_2, \dots)$.

   (a) Bounded if $\mathbf{a}$ is bounded.

   (b) Adjoint is $M_{\bar{\mathbf{a}}}$ where $\bar{\mathbf{a}} = (\bar{a}_1, \bar{a}_2, \dots)$.

   (c) Normal cause doesn't matter which way you multiply stuff.

   (d) Self-adjoint if the a_i are real for all i.

   (e) Compact if $a_i \to 0$.

Lecture 2

Theorem 2.1. *Let A be a compact normal operator on a separable infinite dimensional Hilbert space H. Then H contains an orthonormal basis of eigenvectors for A, with eigenvalues tending to 0.*

Eigenvectors for $M_{\mathbf{a}}$ in Example 1.2 — $\{\mathbf{x}_i = (0, \dots, 0, \underbrace{1}_{i^{\text{th}} \text{ slot}}, 0, \dots)\}$ is an orthonormal basis of eigenvectors.

Theorem 2.2. *Let A be a compact normal operator on a separable infinite dimensional Hilbert space. Then there exists a unitary operator $U : H \to \ell^2(\mathbb{N})$ and a vector $\mathbf{a} = (a_1, a_2, \dots)$, $a_i \to 0$, such that $UAU^* = M_{\mathbf{a}}$.*

Proof. Sketch.

1. Pick an orthonormal basis of eigenvectors $\{e_i\}$ with eigenvalues $\{a_i\}$.

2. $U : H \to \ell^2(\mathbb{N})$, with $x \mapsto (\langle x, e_1\rangle, \langle x, e_2\rangle, \dots)$.

3. $U^* : \ell^2(\mathbb{N}) \to H$, with $(x_1, x_2, \dots) \mapsto \displaystyle\sum_{i=1}^{\infty} x_i e_i$. □

What about non-compact operators? Is every normal operator unitarily equivalent to a multiplication operator on $\ell^2(\mathbb{N})$?

Example 2.1. Let $H = L^2([0, 1])$. For f bounded, define $M_f : L^2([0, 1]) \to L^2([0, 1])$ with $M_f g = fg$. Let $f_0(x) = x$. What are the eigenvalues of M_{f_0}? We have $M_{f_0}g = \lambda g$ if $xg(x) = \lambda g(x)$ for all $x \in [0, 1]$. But then $g(x) = 0$ almost everywhere, so there are no eigenvalues: so M_{f_0} cannot be unitarily equivalent to a multiplication operator on $\ell^2(\mathbb{N})$.

Theorem 2.3. *(Spectral Theorem.)*
Let A be a normal operator on a separable Hilbert space. Then A is unitarily equivalent to a multiplication operator M_f on "$L^2(\Omega)$".

This Ω will be defined later.

Definition 2.1. An algebra over a field $\mathbb{F}$ is a vector space V with a map $V \times V \to \mathbb{F}$ such that (for $a \in \mathbb{F}$, $x, y, z \in V$):

1. $(ax + y)z = a(xz) + yz$.
2. $z(ax + y) = a(zx) + zy$.
3. $(xy)z = x(yz)$.

It is commutative if $xy = yx$, and unital if there exists some $\mathbf{1}$ such that $\mathbf{1}x = x\mathbf{1} = x$ for all x.

Example 2.2. Algebraaas.

1. $\mathbb{F}$.
2. $\mathbb{F}[x]$.
3. Functions $X \to \mathbb{F}$ — X any set, product done pointwise.
4. $n \times n$ matrices over $\mathbb{F}$.
5. All linear operators on a vector space, with composition as the product.
6. Let G be a group. Take a vector space with basis indexed by G, $\{e_g\}$, multiplication on basis $e_g e_h = e_{gh}$.

Definition 2.2. A Banach algebra is an algebra over $\mathbb{C}$ such that the underlying vector space is a Banach space, and $\|x \cdot y\| \leq \|x\|\|y\|$ for all x, y.

Lecture 3

Example 3.1. BANACH Algebraaas.

1. $\mathbb{C}$.
2. Any Banach space, with $ab = 0$ for all a, b.
3. $C(X)$, continuous functions on a compact metric space with the sup norm and pointwise product.
4. $C_b(X)$, bounded continuous functions on a metric space.
5. $C_0(X)$, continuous functions "vanishing at ∞" on some metric space.
6. Disk algebra. Continuous functions on the unit disk which are holomorphic on the interior.
7. For any Banach space E, the space of bounded operators $B(E)$ is a Banach algebra with the operator norm and composition as the product.
8. $M_n(\mathbb{C})$, with matrix product and norm $\|M\| = \sum_{i,j} |M_{ij}|$ (in this case $\|\mathbf{1}\| = \|I_n\| = n$).
9. $\ell^1(\mathbb{Z})$, with $(a * b)_i = \sum_{j \in \mathbb{Z}} a_j b_{i-j}$. This converges as

$$\sum_{j \in \mathbb{Z}} |a_j b_{i-j}| \leq \sum_{j \in \mathbb{Z}} |b_{i-j}||a_j| \leq \sup_k |b_k| \sum_{j \in \mathbb{Z}} |a_j| < \infty$$

since $(a_i), (b_i) \in \ell^1(\mathbb{Z})$. Check condition from Definition 2.2:

$$\begin{aligned}
\|a * b\| = \sum_i |(a * b)_i| &= \sum_i \left| \sum_j a_j b_{i-j} \right| \\
&\leq \sum_{i,j} |a_j b_{i-j}| \\
&= \sum_j \left(|a_j| \sum_i |b_{i-j}| \right) \\
&= \sum_j |a_j| \|b\| \\
&= \|a\| \|b\|.
\end{aligned}$$

10. $L^1(\mathbb{R})$, with $(f * g)(x) = \int_{\mathbb{R}} f(y)g(x-y)\,dy$.

Example 3.2. MAYBE BANACH ALGEBRas.

1. Polynomial functions on $[0,1]$, with sup norm and pointwise product — not complete.
2. $L^1([0,1])$, pointwise product — not closed under this multiplication.
3. $\ell^1(\mathbb{Z})$, pointwise product — should be okay.
4. $C(\mathbb{R})$ has no obvious norm...
5. All bounded functions on $\mathbb{R}$, sup norm, pointwise product — should be okay.

Invertibility and Spectrum

Definition 3.1. A bounded operator $A : E \to E$ is invertible if there exists some bounded operator $B : E \to E$ such that $AB = BA = \mathrm{id}_E$.

Theorem 3.1. *The following are equivalent:*

(1) *A is invertible.*

(2) *For every $x, y \in E$, $Ax = y$ has a unique solution, that is, A is a bijection.*

Proof. (1) $\implies$ (2) is clear, since any invertible map is bijective.
For (2) $\implies$ (1), we need to show that if A is bijective, then A^{-1} is a bounded operator. The graph of A, $\{(x, Ax) : x \in E\}$, is closed in $E \times E$ since A is continuous. Equivalently, $\{(Ay, y)\}$ is closed in $E \times E$, but this is the graph of A^{-1} since A is a bijection, so A^{-1} is bounded. $\square$

Definition 3.2. The spectrum of an operator $\sigma(A)$ is $\{\lambda \in \mathbb{C} : \lambda I - A \text{ is not invertible}\}$.

Lecture 4

Example 4.1. Shifts.
Let $T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be the right unilateral shift, $(x_1, x_2, \dots) \mapsto (0, x_1, x_2, \dots)$, and $S : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be the left shift, $(x_1, x_2, \dots) \mapsto (x_2, x_3, \dots)$. Both fail to be invertible: T is not surjective, and

S is not injective. Note $ST = I$, but $TS(x_1, x_2, x_3, \dots) = (0, x_2, x_3, \dots)$.

Does T have eigenvalues? No — $T\mathbf{x} = \lambda\mathbf{x} \implies 0 = \lambda x_1,\ x_1 = \lambda x_2,$ etc.... If $\lambda = 0$ then $\mathbf{x} = 0$; otherwise $x_1 = 0$ and $\mathbf{x} = 0$ anyway... so no eigenvalues.

The spectrum of ST (when is $I - \lambda I$ not invertible?) is $\sigma(ST) = \{1\}$.

The spectrum of TS is $\sigma(TS) = \{0, 1\}$. Note TS is the projection onto $\{(0, x_2, x_3, \dots)\}$... let P be any projection onto a Hilbert space. Write $I = P + P^\perp$; when is $P - \lambda I$ invertible? We have $P - \lambda I = P - \lambda(P + P^\perp) = (1 - \lambda)P - \lambda P^\perp$. The inverse is given by

$$\frac{1}{1 - \lambda}P - \frac{1}{\lambda}P^\perp,$$

which is okay as long as $\lambda \notin \{0, 1\}$. Hence $\sigma(P) \subseteq \{0, 1\}$; we can also check that $0 \in \sigma(P)$ if $P \neq I$ and $1 \in \sigma(P)$ if $P \neq \mathbf{0}$.

If

$$A = \sum_{i=1}^{n} \lambda_i P_i,$$

where the P_i are non-zero projections, $P_i P_j = 0$ for $i \neq j$ and $\sum P_i = I$, then $\sigma(A) = \{\lambda_i\}$.

Fact. $\sigma(AB) \backslash \{0\} = \sigma(BA) \backslash \{0\}$, that is, the spectra are the same if we ignore zero.

This follows from:

Fact. $\mathbf{1} - AB$ is invertible if and only if $\mathbf{1} - BA$ is invertible.

Example 4.2. Spectrum of multiplication map.
Let $\mathbf{a} = (a_1, a_2, \dots) \in \ell^\infty(\mathbb{N})$, and let $M_\mathbf{a} : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ with $(x_1, x_2, \dots) \mapsto (a_1 x_1, a_2 x_2, \dots)$. What is $\sigma(M_\mathbf{a})$? We have $\{a_i\} \subseteq \sigma(M_\mathbf{a})$, since $M_\mathbf{a} - a_i I$ has a non-trivial kernel. Also, for any λ

$$(M_\mathbf{a} - \lambda I)(x_1, x_2, \dots) = \big((a_1 - \lambda)x_1, (a_2 - \lambda)x_2, \dots\big).$$

As long as $\lambda \notin \{a_i\}$, we can try to invert with $M_\mathbf{b}$, where

$$\mathbf{b} = \left(\frac{1}{a_1 - \lambda}, \frac{1}{a_2 - \lambda}, \dots\right).$$

But $M_\mathbf{b}$ is a bounded operator of $\ell^2(\mathbb{N})$ if and only if $\lambda \notin \overline{\{a_i\}}$. It follows that $\sigma(M_\mathbf{a}) = \overline{\{a_i\}}$.

Example 4.3. Construct an operator whose spectrum is $[0, 1]$.

Take any countable dense set in $[0, 1]$, look at the corresponding multiplication operator.

Lecture 5

Definition 5.1. An element x in a unital Banach algebra A is invertible if there is some $y \in A$ such that $xy = yx = 1$. The spectrum $\sigma(x) = \{\lambda \in \mathbb{C} : x - \lambda\mathbf{1}$ is invertible$\}$.

Conventions:

1. Always assume $\|\mathbf{1}\| = 1$.

2. Write $x - \lambda$ for $x - \lambda\mathbf{1}$.

Lemma 5.1.

1. *If $\|x\| < 1$, then $\mathbf{1} - x$ is invertible.*

2. *If $\|x\| < 1$, then $\|(\mathbf{1} - x)^{-1}\| \leq \dfrac{1}{1 - \|x\|}$.*

Proof. Let

$$z = \sum_{n=0}^{\infty} x^n.$$

(It converges because $\|x^n\| \leq \|x\|^n$.) Then

$$\begin{aligned}
(\mathbf{1} - x)z &= (\mathbf{1} - x)\left(\sum_{n=0}^{\infty} x^n\right) \\
&= (\mathbf{1} - x) \lim_{N\to\infty} \sum_{n=0}^{N} x^n \\
&= \lim_{N\to\infty} \left((\mathbf{1} - x)\sum_{n=0}^{N} x^n\right) \\
&= \lim_{N\to\infty} (\mathbf{1} - x^{N+1}) \\
&= \mathbf{1}.
\end{aligned}$$

So z is a right inverse; it's also a left inverse. Then

$$\|(\mathbf{1} - x)^{-1}\| = \left\|\sum_{n=0}^{\infty} x^n\right\| \leq \sum_{n=0}^{\infty} \|x\|^n = \frac{1}{1 - \|x\|}.$$

□

Let A^{-1} be the *group* of invertible elements of A.

Theorem 5.1. *A^{-1} is an open set, and $x \mapsto x^{-1}$ is a continuous map.*

Proof. If x is invertible, then $x + h = x(\mathbf{1} + x^{-1}h)$, so by the previous lemma, $x + h$ will be invertible if $\|x^{-1}h\| < 1$. So, if $\|h\| < \frac{1}{\|x^{-1}\|}$, then $\|x^{-1}h\| < 1$, and $x + h$ is invertible implies A^{-1} is open. For continuity, use estimate on $\|(\mathbf{1} - x)^{-1}\|$. □

Theorem 5.2. *For any x, $\sigma(x)$ is a compact set and $\sigma(x) \subseteq \{\lambda : |\lambda| \leq \|x\|\}$.*

Proof. We first show $\sigma(x)$ is closed. If $\lambda \notin \sigma(x)$, then $x - \lambda_0$ is invertible. If $|\lambda - \lambda_0| < \delta$, then $\|(x - \lambda) - (x - \lambda_0)\| = |\lambda - \lambda_0| < \delta$. Since A^{-1} is open, this means that for δ sufficiently small, λ will be in the "resolvent" ($\mathbb{C}\backslash\sigma(x)$) as well, which implies that the resolvent is open.

Next, we show that $\sigma(x)$ is bounded by $\|x\|$, that is, any λ with $|\lambda| > \|x\|$ is not in $\sigma(x)$. If $|\lambda| > \|x\|$, then $x - \lambda = \lambda(\frac{x}{\lambda} - \mathbf{1})$. Since $\left\|\frac{x}{\lambda}\right\| = \frac{1}{\lambda}\|x\| < 1$, we know that $x - \lambda$ is invertible, that is, $\lambda \notin \sigma(x)$. □

Theorem 5.3. *$\sigma(x)$ is non-empty.*

Proof. Basic idea: if $\sigma(x) = \varnothing$, then $x - \lambda$ is invertible for all $\lambda \in \mathbb{C}$. We want to show that this doesn't make sense. First approach: use complex analysis for functions from $\mathbb{C} \to A$, but we need to show that stuff from complex analysis sensibly extends to such functions.

Second approach: look at $f\left((x-\lambda)^{-1}\right)$ for bounded linear functionals f, and use functional analysis. We'll go with this. Fix x, and suppose for a contradiction that $\sigma(x) = \varnothing$. **Claim:** for any bounded linear functional f on A, $f\left((x-\lambda)^{-1}\right)$ is a bounded, entire function which tends to 0.

Proof of claim. We have, for a fixed λ_0,

$$\begin{aligned}\lim_{\lambda \to \lambda_0} \frac{(x-\lambda)^{-1} - (x-\lambda_0)^{-1}}{\lambda - \lambda_0} &= \lim_{\lambda \to \lambda_0} \frac{(x-\lambda)^{-1}((x-\lambda) - (x-\lambda_0))(x-\lambda_0)^{-1}}{\lambda - \lambda_0} \\ &= \lim_{\lambda \to \lambda_0} (x-\lambda)^{-1}(x-\lambda_0)^{-1} \\ &= (x-\lambda_0)^{-2}.\end{aligned}$$

Thus $f((x-\lambda)^{-1})$ is analytic for all f (exercise).

Similarly, if $\lambda \neq 0$

$$\|(x-\lambda)^{-1}\| = \left\| \lambda^{-1} \left(\frac{x}{\lambda} - 1 \right)^{-1} \right\| \leq \frac{1}{|\lambda|} \frac{1}{1 - \frac{\|x\|}{|\lambda|}} \to 0$$

as $\lambda \to \infty$. □

But this means that $(x-\lambda)^{-1}$ is 0 (Hahn-Banach) for all λ which is absurd. □

Definition 5.2. The spectral radius is $r(x) = \sup\{|\lambda| : \lambda \in \sigma(x)\}$.

Note $r(x) \leq \|x\|$.

Theorem 5.4.

$$r(x) = \lim_{n \to \infty} \|x^n\|^{\frac{1}{n}}.$$