Banach Algebras

Lecture 1

Spectral Theorem in Finite Dimensions

Definition 1.1. Bunch of things. Let $A =$ square matrix.

- Symmetric: $A = A^T$.
- Orthogonal: $AA^T = A^T A = I$.
- Adjoint: $A^* = \overline{A^T}$.
- Self-adjoint: $A = A^*$.
- Unitary: $AA^* = A^* A = I$.
- Normal: $AA^* = A^* A$.
- Diagonal: $A_{ij} = 0$ whenever $i \neq j$.

Theorem 1.1. *Let A be a normal complex matrix. Then there is a unitary matrix U such that UAU^* is a diagonal matrix. Equivalently, there is an orthogonal basis of eigenvectors for A.*

Example 1.1.

$$A = \begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix},$$
$$A^* = \begin{pmatrix} 1-i & 1+i \\ 1+i & 1-i \end{pmatrix}.$$

Then

$$AA^* = A^* A = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}.$$

A few days later we get

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} & 1 \\ 1 & -1 \end{pmatrix},$$

and

$$UAU^* = \begin{pmatrix} 2 & 0 \\ 0 & 2i \end{pmatrix}.$$

Definition 1.2. A Hilbert space is a complete inner product space.

Proposition 1.1. *If H_1, H_2 are Hilbert spaces with orthonormal bases of the same cardinality, they are isometrically isomorphic.*

Definition 1.3. A Hilbert space is separable if it admits a countable orthonormal basis.

Any infinite dimensional Hilbert space is isometrically isomorphic to $\ell^2(\mathbb{N})$.

Definition 1.4. A bounded operator $A : H \to H$ is compact if the closure of the image of the unit ball in H under A is compact.

Example 1.2.

1. Any finite rank operator is compact.

2. Let $H = \ell^2(\mathbb{N})$. Let $\mathbf{a} = (a_1, a_2, \dots)$ be a sequence of complex numbers. Define $M_{\mathbf{a}}(x_1, x_2, \dots) = (a_1 x_1, a_2 x_2, \dots)$.

   (a) Bounded if $\mathbf{a}$ is bounded.

   (b) Adjoint is $M_{\bar{\mathbf{a}}}$ where $\bar{\mathbf{a}} = (\bar{a}_1, \bar{a}_2, \dots)$.

   (c) Normal cause doesn't matter which way you multiply stuff.

   (d) Self-adjoint if the a_i are real for all i.

   (e) Compact if $a_i \to 0$.

Lecture 2

Theorem 2.1. *Let A be a compact normal operator on a separable infinite dimensional Hilbert space H. Then H contains an orthonormal basis of eigenvectors for A, with eigenvalues tending to* 0.

Eigenvectors for $M_{\mathbf{a}}$ in Example 1.2 — $\{\mathbf{x}_i = (0, \dots, 0, \underbrace{1}_{i^{\text{th}} \text{ slot}}, 0, \dots)\}$ is an orthonormal basis of eigenvectors.

Theorem 2.2. *Let A be a compact normal operator on a separable infinite dimensional Hilbert space. Then there exists a unitary operator $U : H \to \ell^2(\mathbb{N})$ and a vector $\mathbf{a} = (a_1, a_2, \dots)$, $a_i \to 0$, such that $UAU^* = M_{\mathbf{a}}$.*

Proof. Sketch.

1. Pick an orthonormal basis of eigenvectors $\{e_i\}$ with eigenvalues $\{a_i\}$.

2. $U : H \to \ell^2(\mathbb{N})$, with $x \mapsto (\langle x, e_1\rangle, \langle x, e_2\rangle, \dots)$.

3. $U^* : \ell^2(\mathbb{N}) \to H$, with $(x_1, x_2, \dots) \mapsto \sum_{i=1}^{\infty} x_i e_i$. □

What about non-compact operators? Is every normal operator unitarily equivalent to a multiplication operator on $\ell^2(\mathbb{N})$?

Example 2.1. Let $H = L^2([0,1])$. For f bounded, define $M_f : L^2([0,1]) \to L^2([0,1])$ with $M_f g = fg$. Let $f_0(x) = x$. What are the eigenvalues of M_{f_0}? We have $M_{f_0} g = \lambda g$ if $xg(x) = \lambda g(x)$ for all $x \in [0,1]$. But then $g(x) = 0$ almost everywhere, so there are no eigenvalues: so M_{f_0} cannot be unitarily equivalent to a multiplication operator on $\ell^2(\mathbb{N})$.

Theorem 2.3. *(Spectral Theorem.)*
Let A be a normal operator on a separable Hilbert space. Then A is unitarily equivalent to a multiplication operator M_f on "$L^2(\Omega)$".

This Ω will be defined later.

Definition 2.1. An algebra over a field $\mathbb{F}$ is a vector space V with a map $V \times V \to \mathbb{F}$ such that (for $a \in \mathbb{F}$, $x, y, z \in V$):

1. $(ax + y)z = a(xz) + yz$.
2. $z(ax + y) = a(zx) + zy$.
3. $(xy)z = x(yz)$.

It is commutative if $xy = yx$, and unital if there exists some $\mathbf{1}$ such that $\mathbf{1}x = x\mathbf{1} = x$ for all x.

Example 2.2. Algebraaas.

1. $\mathbb{F}$.
2. $\mathbb{F}[x]$.
3. Functions $X \to \mathbb{F}$ — X any set, product done pointwise.
4. $n \times n$ matrices over $\mathbb{F}$.
5. All linear operators on a vector space, with composition as the product.
6. Let G be a group. Take a vector space with basis indexed by G, $\{e_g\}$, multiplication on basis $e_g e_h = e_{gh}$.

Definition 2.2. A Banach algebra is an algebra over $\mathbb{C}$ such that the underlying vector space is a Banach space, and $\|x \cdot y\| \leq \|x\|\|y\|$ for all x, y.

Lecture 3

Example 3.1. BANACH Algebraaas.

1. $\mathbb{C}$.
2. Any Banach space, with $ab = 0$ for all a, b.
3. $C(X)$, continuous functions on a compact metric space with the sup norm and pointwise product.
4. $C_b(X)$, bounded continuous functions on a metric space.
5. $C_0(X)$, continuous functions "vanishing at ∞" on some metric space.
6. Disk algebra. Continuous functions on the unit disk which are holomorphic on the interior.
7. For any Banach space E, the space of bounded operators $B(E)$ is a Banach algebra with the operator norm and composition as the product.
8. $M_n(\mathbb{C})$, with matrix product and norm $\|M\| = \sum_{i,j} |M_{ij}|$ (in this case $\|\mathbf{1}\| = \|I_n\| = n$).
9. $\ell^1(\mathbb{Z})$, with $(a * b)_i = \sum_{j \in \mathbb{Z}} a_j b_{i-j}$. This converges as

$$\sum_{j \in \mathbb{Z}} |a_j b_{i-j}| \leq \sum_{j \in \mathbb{Z}} |b_{i-j}||a_j| \leq \sup_k |b_k| \sum_{j \in \mathbb{Z}} |a_j| < \infty$$

since $(a_i), (b_i) \in \ell^1(\mathbb{Z})$. Check condition from Definition 2.2:

$$\begin{aligned}
\|a * b\| = \sum_i |(a * b)_i| &= \sum_i \left| \sum_j a_j b_{i-j} \right| \\
&\leq \sum_{i,j} |a_j b_{i-j}| \\
&= \sum_j \left(|a_j| \sum_i |b_{i-j}| \right) \\
&= \sum_j |a_j| \|b\| \\
&= \|a\| \|b\|.
\end{aligned}$$

10. $L^1(\mathbb{R})$, with $(f * g)(x) = \int_{\mathbb{R}} f(y) g(x - y)\, dy$.

Example 3.2. MAYBE BANACH ALGEBRas.

1. Polynomial functions on $[0, 1]$, with sup norm and pointwise product — not complete.
2. $L^1([0, 1])$, pointwise product — not closed under this multiplication.
3. $\ell^1(\mathbb{Z})$, pointwise product — should be okay.
4. $C(\mathbb{R})$ has no obvious norm...
5. All bounded functions on $\mathbb{R}$, sup norm, pointwise product — should be okay.

Invertibility and Spectrum

Definition 3.1. A bounded operator $A : E \to E$ is invertible if there exists some bounded operator $B : E \to E$ such that $AB = BA = \mathrm{id}_E$.

Theorem 3.1. *The following are equivalent:*

(1) *A is invertible.*

(2) *For every $x, y \in E$, $Ax = y$ has a unique solution, that is, A is a bijection.*

Proof. (1) $\implies$ (2) is clear, since any invertible map is bijective.
For (2) $\implies$ (1), we need to show that if A is bijective, then A^{-1} is a bounded operator. The graph of A, $\{(x, Ax) : x \in E\}$, is closed in $E \times E$ since A is continuous. Equivalently, $\{(Ay, y)\}$ is closed in $E \times E$, but this is the graph of A^{-1} since A is a bijection, so A^{-1} is bounded. $\square$

Definition 3.2. The spectrum of an operator $\sigma(A)$ is $\{\lambda \in \mathbb{C} : \lambda I - A \text{ is not invertible}\}$.

Lecture 4

Example 4.1. Shifts.
Let $T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be the right unilateral shift, $(x_1, x_2, \dots) \mapsto (0, x_1, x_2, \dots)$, and $S : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be the left shift, $(x_1, x_2, \dots) \mapsto (x_2, x_3, \dots)$. Both fail to be invertible: T is not surjective, and

S is not injective. Note $ST = I$, but $TS(x_1, x_2, x_3, \dots) = (0, x_2, x_3, \dots)$.

Does T have eigenvalues? No — $T\mathbf{x} = \lambda\mathbf{x} \implies 0 = \lambda x_1,\ x_1 = \lambda x_2,$ etc.... If $\lambda = 0$ then $\mathbf{x} = 0$; otherwise $x_1 = 0$ and $\mathbf{x} = 0$ anyway... so no eigenvalues.

The spectrum of ST (when is $I - \lambda I$ not invertible?) is $\sigma(ST) = \{1\}$.

The spectrum of TS is $\sigma(TS) = \{0, 1\}$. Note TS is the projection onto $\{(0, x_2, x_3, \dots)\}$... let P be any projection onto a Hilbert space. Write $I = P + P^\perp$; when is $P - \lambda I$ invertible? We have $P - \lambda I = P - \lambda(P + P^\perp) = (1 - \lambda)P - \lambda P^\perp$. The inverse is given by

$$\frac{1}{1-\lambda}P - \frac{1}{\lambda}P^\perp,$$

which is okay as long as $\lambda \notin \{0, 1\}$. Hence $\sigma(P) \subseteq \{0, 1\}$; we can also check that $0 \in \sigma(P)$ if $P \neq I$ and $1 \in \sigma(P)$ if $P \neq \mathbf{0}$.

If

$$A = \sum_{i=1}^n \lambda_i P_i,$$

where the P_i are non-zero projections, $P_i P_j = 0$ for $i \neq j$ and $\sum P_i = I$, then $\sigma(A) = \{\lambda_i\}$.

Fact. $\sigma(AB)\backslash\{0\} = \sigma(BA)\backslash\{0\}$, that is, the spectra are the same if we ignore zero.

This follows from:

Fact. $\mathbf{1} - AB$ is invertible if and only if $\mathbf{1} - BA$ is invertible.

Example 4.2. Spectrum of multiplication map.
Let $\mathbf{a} = (a_1, a_2, \dots) \in \ell^\infty(\mathbb{N})$, and let $M_\mathbf{a} : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ with $(x_1, x_2, \dots) \mapsto (a_1 x_1, a_2 x_2, \dots)$. What is $\sigma(M_\mathbf{a})$? We have $\{a_i\} \subseteq \sigma(M_\mathbf{a})$, since $M_\mathbf{a} - a_i I$ has a non-trivial kernel. Also, for any λ

$$(M_\mathbf{a} - \lambda I)(x_1, x_2, \dots) = \big((a_1 - \lambda)x_1, (a_2 - \lambda)x_2, \dots\big).$$

As long as $\lambda \notin \{a_i\}$, we can try to invert with $M_\mathbf{b}$, where

$$\mathbf{b} = \left(\frac{1}{a_1 - \lambda}, \frac{1}{a_2 - \lambda}, \dots\right).$$

But $M_\mathbf{b}$ is a bounded operator of $\ell^2(\mathbb{N})$ if and only if $\lambda \notin \overline{\{a_i\}}$. It follows that $\sigma(M_\mathbf{a}) = \overline{\{a_i\}}$.

Example 4.3. Construct an operator whose spectrum is $[0, 1]$.

Take any countable dense set in $[0, 1]$, look at the corresponding multiplication operator.

Lecture 5

Definition 5.1. An element x in a unital Banach algebra A is invertible if there is some $y \in A$ such that $xy = yx = 1$. The spectrum $\sigma(x) = \{\lambda \in \mathbb{C} : x - \lambda\mathbf{1} \text{ is invertible}\}$.

Conventions:

1. Always assume $\|\mathbf{1}\| = 1$.
2. Write $x - \lambda$ for $x - \lambda\mathbf{1}$.

Lemma 5.1.

1. *If* $\|x\| < 1$*, then* $\mathbf{1} - x$ *is invertible.*
2. *If* $\|x\| < 1$*, then* $\|(\mathbf{1} - x)^{-1}\| \leq \dfrac{1}{1 - \|x\|}$.

Proof. Let

$$z = \sum_{n=0}^{\infty} x^n.$$

(It converges because $\|x^n\| \leq \|x\|^n$.) Then

$$\begin{aligned}
(\mathbf{1} - x)z &= (\mathbf{1} - x)\left(\sum_{n=0}^{\infty} x^n\right) \\
&= (\mathbf{1} - x) \lim_{N\to\infty} \sum_{n=0}^{N} x^n \\
&= \lim_{N\to\infty} \left((\mathbf{1} - x) \sum_{n=0}^{N} x^n\right) \\
&= \lim_{N\to\infty} (\mathbf{1} - x^{N+1}) \\
&= \mathbf{1}.
\end{aligned}$$

So z is a right inverse; it's also a left inverse. Then

$$\|(\mathbf{1} - x)^{-1}\| = \left\|\sum_{n=0}^{\infty} x^n\right\| \leq \sum_{n=0}^{\infty} \|x\|^n = \frac{1}{1 - \|x\|}.$$

□

Let A^{-1} be the *group* of invertible elements of A.

Theorem 5.1. A^{-1} *is an open set, and* $x \mapsto x^{-1}$ *is a continuous map.*

Proof. If x is invertible, then $x + h = x(\mathbf{1} + x^{-1}h)$, so by the previous lemma, $x + h$ will be invertible if $\|x^{-1}h\| < 1$. So, if $\|h\| < \frac{1}{\|x^{-1}\|}$, then $\|x^{-1}h\| < 1$, and $x + h$ is invertible implies A^{-1} is open. For continuity, use estimate on $\|(\mathbf{1} - x)^{-1}\|$. □

Theorem 5.2. *For any* x*,* $\sigma(x)$ *is a compact set and* $\sigma(x) \subseteq \{\lambda : |\lambda| \leq \|x\|\}$.

Proof. We first show $\sigma(x)$ is closed. If $\lambda \notin \sigma(x)$, then $x - \lambda_0$ is invertible. If $|\lambda - \lambda_0| < \delta$, then $\|(x - \lambda) - (x - \lambda_0)\| = |\lambda - \lambda_0| < \delta$. Since A^{-1} is open, this means that for δ sufficiently small, λ will be in the "resolvent" $(\mathbb{C}\backslash\sigma(x))$ as well, which implies that the resolvent is open.

Next, we show that $\sigma(x)$ is bounded by $\|x\|$, that is, any λ with $|\lambda| > \|x\|$ is not in $\sigma(x)$. If $|\lambda| > \|x\|$, then $x - \lambda = \lambda(\frac{x}{\lambda} - \mathbf{1})$. Since $\left\|\frac{x}{\lambda}\right\| = \frac{1}{\lambda}\|x\| < 1$, we know that $x - \lambda$ is invertible, that is, $\lambda \notin \sigma(x)$. □

Theorem 5.3. $\sigma(x)$ *is non-empty.*

Proof. Basic idea: if $\sigma(x) = \varnothing$, then $x - \lambda$ is invertible for all $\lambda \in \mathbb{C}$. We want to show that this doesn't make sense. First approach: use complex analysis for functions from $\mathbb{C} \to A$, but we need to show that stuff from complex analysis sensibly extends to such functions.

Second approach: look at $f\left((x-\lambda)^{-1}\right)$ for bounded linear functionals f, and use functional analysis. We'll go with this. Fix x, and suppose for a contradiction that $\sigma(x) = \varnothing$. **Claim:** for any bounded linear functional f on A, $f\left((x-\lambda)^{-1}\right)$ is a bounded, entire function which tends to 0.

Proof of claim. We have, for a fixed λ_0,

$$\begin{aligned}\lim_{\lambda \to \lambda_0} \frac{(x-\lambda)^{-1} - (x-\lambda_0)^{-1}}{\lambda - \lambda_0} &= \lim_{\lambda \to \lambda_0} \frac{(x-\lambda)^{-1}((x-\lambda)-(x-\lambda_0))(x-\lambda_0)^{-1}}{\lambda - \lambda_0} \\ &= \lim_{\lambda \to \lambda_0} (x-\lambda)^{-1}(x-\lambda_0)^{-1} \\ &= (x-\lambda_0)^{-2}.\end{aligned}$$

Thus $f((x-\lambda)^{-1})$ is analytic for all f (exercise).

Similarly, if $\lambda \neq 0$

$$\|(x-\lambda)^{-1}\| = \left\| \lambda^{-1} \left(\frac{x}{\lambda} - 1\right)^{-1} \right\| \leq \frac{1}{|\lambda|} \frac{1}{1 - \frac{\|x\|}{|\lambda|}} \to 0$$

as $\lambda \to \infty$. □

But this means that $(x-\lambda)^{-1}$ is 0 (Hahn-Banach) for all λ which is absurd. □

Definition 5.2. The spectral radius is $r(x) = \sup\{|\lambda| : \lambda \in \sigma(x)\}$.

Note $r(x) \leq \|x\|$.

Theorem 5.4.

$$r(x) = \lim_{n \to \infty} \|x^n\|^{\frac{1}{n}}.$$

Lecture 6

MIA — see Ben's stuff (or use that anyway if you want something more orderly ☺)

Lecture 7

Proof of Theorem 5.4. (Sketch.)

We show that $r(x) \leq \liminf_{n \to \infty} \|x^n\|^{\frac{1}{n}}$, and $r(x) \geq \limsup_{n \to \infty} \|x^n\|^{\frac{1}{n}}$.

($r(x) \leq \liminf$).

If $\lambda \in \sigma(x)$, then $\lambda^n \in \sigma(x^n)$ (see the Spectral Mapping Theorem). Then

$$\begin{aligned}|\lambda^n| &\leq \|x^n\|, \text{ and} \\ |\lambda| &\leq \|x^n\|^{\frac{1}{n}}.\end{aligned}$$

So $\sigma(x)$ is bounded in absolute value by $\|x^n\|^{\frac{1}{n}}$ for every n, which implies that $r(x) \leq \liminf_{n\to\infty} \|x^n\|^{\frac{1}{n}}$.

$(r(x) \geq \limsup)$.

It suffices to show that for any $\lambda > r(x)$, $\lambda \geq \limsup_{n\to\infty} \|x^n\|^{\frac{1}{n}}$, so suppose $\lambda > r(x)$.

Claim: $\left\{\dfrac{x^n}{\lambda^n}\right\}$ is bounded in norm.

Assume the claim is true. Then
$$\left\|\frac{x^n}{\lambda^n}\right\| = \frac{\|x^n\|}{|\lambda|^n} < M \quad \forall n.$$
Then $\|x^n\| < |\lambda|^n M$, so $\|x^n\|^{\frac{1}{n}} < |\lambda|\ M^{\frac{1}{n}}$. $\limsup$ everything to get
$$\limsup_{n\to\infty} \|x^n\|^{\frac{1}{n}} < |\lambda| \limsup_{n\to\infty} M^{\frac{1}{n}} = |\lambda|.$$
Proof of claim. It suffices to show that $f(x^n/\lambda^n)$ is bounded for every bounded linear functional f. If $f(x^n/\lambda^n)$ is bounded for each $f \in A^*$, that means $\{x^n/\lambda^n\}$ is bounded pointwise as elements of A^{**}.

Take $f \in A^*$. Look at the function $f[(1-zx)^{-1}]$. Assume $r(x) \neq 0$. The domain is
$$\{0\} \cup \left\{z : \frac{1}{z} \notin \sigma(x)\right\},$$
or $|z| < \frac{1}{r(x)}$. On the disk $|z| < \frac{1}{\|x\|}$, we can take a power series for $(1-zx)^{-1}$ to get
$$f((1-zx)^{-1}) = 1 + zf(x) + z^2 f(x^2) + \dots.$$
On the larger disk $|z| < \frac{1}{r(x)}$, $f((1-zx)^{-1})$ should still be analytic, so $f((1-zx)^{-1}) = \sum f(x^n)z^n$. In particular, $f(x^n)z^n$ is bounded when $|z| < \frac{1}{r(x)}$. □

□

Ideals

Definition 7.1. An ideal in a Banach algebra A is a subspace $I \subseteq A$, such that $xi, ix \in I$ for all $x \in A$, $i \in I$.

Given an ideal I in an algebra A, we can take the quotient A/I. Aside: If $B_0 \subseteq B$ is a closed subspace of a Banach space, then B/B_0 is a Banach space with $\|[b]\| = \inf\{\|b + b_0\| : b_0 \in B_0\}$. We would have seen this in functional analysis last semester.

So for a Banach algebra A, if $I \subseteq A$ is a closed ideal, then A/I is a Banach space and an algebra. But is it... a BANACH ALGEBRA?

Check: is it true that $\|[x][y]\| \leq \|[x]\|\ \|[y]\|$ for all x, y?
$$\begin{aligned}
\|[x][y]\| &= \|[xy]\| \\
&= \inf_{i\in I} \|xy + i\| \\
&\leq \|xy + \underbrace{i_1 y + i_2 x + i_1 i_2}_{\in I}\| \quad \text{for any } i_1, i_2 \in I \\
&= \|(x + i_1)(y + i_2)\| \\
&\leq \|x + i_1\|\|y + i_2\|.
\end{aligned}$$

This shows that $\|[x][y]\| \leq \inf_{i_1,i_2} \|x+i_1\|\|y+i_2\| = \|[x]\|\ \|[y]\|$.

Lecture 8

True facts about ideals.

Theorem 8.1. *Let I be a closed ideal in a Banach algebra A. Then*

1. *A/I is a Banach algebra.*

2. *If $T : A \to B$ is a bounded homomorphism then $ker(T)$ is a closed ideal, and there is a unique $\dot{T} : A/ker(T) \to B$ such that $A \xrightarrow{T} B$ is equal to $A \to A/ker(T) \xrightarrow{\dot{T}} B$. Moreover, $\|\dot{T}\| = \|T\|$.*

Definition 8.1. An ideal $I \subseteq A$ is called proper if $I \neq A$. It is maximal if I is proper and there is no ideal lying strictly between I and A.

Theorem 8.2. *Let $I \subseteq A$ be a proper ideal in a unital Banach algebra. Then $\bar{I}$ is a proper ideal.*

Proof. Note that I does not contain any invertible elements, since $I \neq A$. Then for any $x \in I$, by Lemma 5.1, $\|1-x\| \geq 1$. But then $1 \notin \bar{I}$. □

Theorem 8.3. *Let A be a unital Banach algebra. Then every proper ideal is contained in a maximal ideal and every maximal ideal is closed.*

Proof. If I is a maximal ideal, it is proper, so $\bar{I}$ is also a proper ideal. But $I \subseteq \bar{I} \subset A$ implies $I = \bar{I}$, since I is maximal. This implies that I is closed. For the first part, let I be a proper ideal of A. Let $X = \{J : J \text{ is a proper ideal containing } I\}$ with the partial ordering of inclusion. Any chain has an upper bound; namely, the union, which is a proper ideal (see proof of Theorem 8.2 — 1 is still too far away!). So by Zorn's lemma, X has a maximal element, which must be a maximal ideal. □

Definition 8.2. Let A be a unital, commutative Banach algebra. The Gelfand spectrum $\mathrm{sp}(A)$ is the set of non-zero homomorphisms from A to $\mathbb{C}$.

Theorem 8.4.

1. *Every element in $sp(A)$ is continuous with norm 1.*

2. *$sp(A)$ is in bijective correspondence with the set of maximal ideals of A.*

Proof.

1.

2. Given $\omega \in \mathrm{sp}(A)$, $\ker(\omega)$ is an ideal of codimension 1, so it is maximal (call this ideal I_ω). Conversely, starting with an ideal I of codimension (the dimension of A/I) 1, we can write $A \to A/I \cong \mathbb{C}$ to get an element of $\mathrm{sp}(A)$ (call this ω_I). (Here we have used the true fact that in a commutative algebra, every maximal ideal has codimension 1. We'll explain this later.)

   Claim: $\omega_{I_\omega} = \omega$ and $I_{\omega_I} = I$.

   Proof. $I_{\omega_I} = \ker(\omega_I) = I$, and $\omega_{I_\omega} = \omega_{\ker(\omega)} = (A \to A/\ker(\omega) \to \mathbb{C}) = \omega$, by uniqueness of the map from $A/\ker(\omega) \to \mathbb{C}$. □

It remains to show that the ideals of codimension 1 are exactly the maximal ideals. Suppose $I \subset A$ has codimension 1. Suppose $x \notin I$. Then $[x] \neq 0$ in A/I, so we can write $[x] = \lambda[\mathbf{1}]$ for some $\lambda \neq 0$, so $x = \lambda\mathbf{1} + I$. But then the ideal generated by I and x contains $\lambda\mathbf{1} + I$, so it contains $\lambda\mathbf{1}$ and hence $\mathbf{1}$. Conversely, let $I \subset A$ be a proper ideal, and suppose A/I has dimension greater than 1. Choose $x \in A \backslash I$ such that $[x]$ is not invertible in A/I (Theorem 9.1). Consider the ideal $J = I + Ax$ (that it is an ideal depends on commutativity). Then J is a proper ideal — $\mathbf{1}$ cannot be in J, because if it were, then $\mathbf{1} = i + ax$ for some $i \in I$ and $a \in A$. But then $[a][x] = [x][a] = [\mathbf{1}]$ — but we took x so that $[x]$ was not invertible in A/I. So I is not maximal.

□

Lecture 9

Theorem 9.1. *Every Banach division algebra (unital algebra where every non-zero element is invertible) is* 1*-dimensional.*

Proof. Let A be a unital Banach division algebra, and let $x \in A$. Suppose $\lambda \in \sigma(x)$. Then $x - \lambda$ is not invertible, so $x - \lambda = 0$, which means $x = \lambda\mathbf{1}$. □

Proposition 9.1. *Every 1-dimensional unital Banach algebra is isometrically isomorphic to $\mathbb{C}$, and this isomorphism is unique.*

Proof. Can construct the obvious isomorphism, just have to check it actually is an isomorphism. For uniqueness, let A be a 1-dimensional unital Banach algebra, and let $\phi : A \to \mathbb{C}$ be an isomorphism of complex algebras. Then $\phi(\mathbf{1}) = \phi(\mathbf{1} \cdot \mathbf{1}) = \phi(\mathbf{1}) \cdot \phi(\mathbf{1})$, so $\phi(\mathbf{1}) = 0$ or $\phi(\mathbf{1}) = 1$. If $\phi(\mathbf{1}) = 0$, then ϕ is the zero homomorphism, so it's not an isomorphism — this means that $\phi(\mathbf{1}) = 1$. □

Proposition 9.2. *Let A be a unital Banach algebra, and let $I \subset A$ be a proper ideal. Then A/I is a unital Banach algebra (including $\|\mathbf{1}\| = 1$).*

Proof. Assume we already know A/I is a Banach algebra (see Lecture 7). The element $[\mathbf{1}]$ is a unit for A/I. We need to show that $\|[\mathbf{1}]\|_{A/I} = 1$. We have

$$\begin{aligned} \|[\mathbf{1}]\|_{A/I} &= \inf_{i \in I} \|\mathbf{1} + i\| \\ &\leq \|\mathbf{1} + 0\| \\ &= 1. \end{aligned}$$

For the other inequality, we want to show that for every $i \in I$, $\|\mathbf{1} + i\| \geq 1$. But if $\|1 + i\| < 1$ for any $i \in I$, then i is invertible (Lemma 5.1), contradicting the fact that I is a proper ideal. □

Lecture 10

Proposition 10.1. *$sp(A)$ is non-empty.*

Proof. Since $\{0\}$ is a proper ideal of A, it is contained in a maximal ideal, which is enough by Theorem 8.4 (2). □

Theorem 10.1. *For all $\omega \in sp(A)$, $\|\omega\| = \omega(1) = 1$.*

Proof. (Sketch.)
For $A \xrightarrow{\omega} \mathbb{C}$, consider $A \xrightarrow{\pi} A/I \xrightarrow{\dot{\omega}} \mathbb{C}$. Use the true fact: $\|\omega\| = \|\dot{\omega}\|$ and the fact that a non-zero homomorphism between one-dimensional algebras is pretty much the identity. □

Let B be a Banach space and B^* be the Banach space of bounded linear functionals on B. There's an isometry $B \to B^{**}$ — if $x \in B$, $\rho \in B^*$, define $\hat{x}(\rho) = \rho(x)$.

Definition 10.1. The *weak* topology on B is the coarsest topology which makes every $\rho \in B^*$ continuous. The *weak-** topology on B^* is the topology on B^* which makes every $\rho \in B \subseteq B^* *$ continuous.

Theorem 10.2. *(Banach-Alaoglu Theorem.)*
The unit ball of B^* is compact in the weak-* topology.

Theorem 10.3. *Suppose A is a unital, commutative Banach algebra. Then $sp(A)$ is a compact Hausdorff space in the weak-* topology.*

Proof. (Sketch.)
We know that $\mathrm{sp}(A)$ is a subset of the unit ball of A^*, so by Banach-Alaoglu, we just need to show that $\mathrm{sp}(A)$ is weak-*closed (exercise). □

Definition 10.2. The *Gelfand transform* from A to $C(\mathrm{sp}(A))$ is defined by $x \mapsto \hat{x} \in C(\mathrm{sp}(A))$, where $\hat{x}(\omega) = \omega(x)$.

(Note that $\hat{x}$ is continuous by definition of the weak-*topology.)

Theorem 10.4.

1. *The Gelfand transform is a continuous unital homomorphism from A to $C(sp(A))$.*
2. *For any $x \in A$, $\sigma(x) = \hat{x}(sp(A))$.*

Proof. True facts about Gelfand stuff:

- It's a homomorphism (need to show $\hat{x} \cdot \hat{y} = \widehat{xy}$ and $\hat{x} + \hat{y} = \widehat{x+y}$). Indeed, we have $\hat{x}\hat{y}(\omega) = \omega(x)\omega(y) = \omega(xy) = \widehat{xy}(w)$.
- Unital. $\mathbf{1}$ is the constant function $1 \in C(\mathrm{sp}(A))$. For any $\omega \in \mathrm{sp}(A)$, $\hat{\mathbf{1}}(\omega) = \omega(\mathbf{1}) = \mathbf{1}$, so $\hat{\mathbf{1}}$ is the constant function $\mathbf{1}$.
- $\|\hat{x}\| = \sup_{\omega \in \mathrm{sp}(A)} \|\hat{x}(\omega)\| = \sup_{\omega \in \mathrm{sp}(A)} \|\omega(x)\| \leq 1\|x\|$.
- **Claim:** $x \in A$ is invertible iff $\hat{x}$ is nowhere vanishing.

  Proof of claim. If x is invertible then $\hat{x} \cdot \hat{x}^{-1} = \widehat{xx^{-1}} = \hat{\mathbf{1}} = \mathbf{1}$, so $\hat{x}$ is invertible as well, which means that $\hat{x}$ is nowhere vanishing. If $\hat{x}$ is nowhere vanishing then $\hat{x}(\omega) \neq 0$ for all $\omega \in \mathrm{sp}(A)$. Therefore, x is not contained in any maximal ideal. Then x must be invertible (because otherwise xA would be a proper ideal). □

From the claim:

$$\begin{aligned}
\sigma(x) &= \{\lambda : x - \lambda \text{ is not invertible}\} \\
&= \{\lambda : \widehat{x - \lambda} \text{ is somewhere vanishing}\} \\
&= \{\lambda : \hat{x} \text{ is somewhere equal to } \lambda\} \\
&= \{\lambda : \hat{x} \text{ takes the value } \lambda \text{ for some } \omega \in \mathrm{sp}(A)\}.
\end{aligned}$$

□

Lecture 11

Example 11.1. Let $A = C(X)$, the continuous functions on a compact Hausdorff space, e.g. with $X = [0, 1]$.

Let $Y \subseteq X$. Then the set of functions which vanish on Y is an ideal, say, I_Y. If $Y_1 \subseteq Y_2$, then $I_{Y_2} \subseteq I_{Y_1}$. The largest such possible ideal is $I_{\{x\}}$ for some $x \in X$. Now, $I_{\{x\}}$ is maximal — can see it constructively, or because $I_{\{x\}}$ is the kernel of the homomorphism $\omega_x : f \mapsto f(x)$ (since $C(X), \mathbb{C}$ are commutative, $\mathbb{C}$ is a field and ω_x is clearly surjective).

Theorem 11.1. *Every maximal ideal of $C(X)$ is of the form $I_{\{x\}}$ for some $x \in X$.*

Proof. Let $\omega \in \mathrm{sp}(A)$. Suppose $\omega \neq \omega_x$ for all x. Then

$$\bigcap_{f \in A} \{x \in X : \omega(f) = f(x)\} = \varnothing.$$

Then by compactness, there exist a finite number of functions $\{f_k\}$ such that $\bigcap_f \{x \in X : \omega(f) = f(x)\} = \varnothing$. So, we have a finite set of functions $\{f_k\}$ such that for each $x \in X$, $\omega(f_k) \neq f_k(x)$ for at least one k. Let $g_k = f_k - \omega(f_k)$ for each k. Then $\omega(g_k) = \omega(f_k) - \omega(f_k) = 0$ for all k, and for each x there is some k such that $g_k(x) \neq 0$. Let $g = \sum_k g_k \overline{g_k}$. Then $\omega(g) = 0$, and for each x, $g(x) > 0$. So g is an invertible element of $C(X)$ which is in $\ker(\omega)$, but this contradicts the fact that $\omega \in \mathrm{sp}(A)$. $\square$

Theorem 11.2. *Let $A = C(X)$, where X is a compact Hausdorff space. For each $x \in X$, let $\omega_x : C(X) \to \mathbb{C}$ be the homomorphism sending $f \mapsto f(x)$. Then $x \mapsto \omega_x$ is a homeomorphism from X to sp(A). When X and sp(A) are identified via this homeomorphism, the Gelfand transform is the identity map.*

Bits of proof. The map $x \mapsto \omega_x$ is injective since $C(X)$ separates points, and is surjective by the previous compactness argument. Since $X, \mathrm{sp}(C(X))$ are compact, it suffices to show continuity in one direction. **True fact:** continuous bijection from a compact space to a Hausdorff space is a homeomorphism[1]. Suppose $x_n \to x$ is a convergent net in X. Then for every $f \in C(X)$, $f(x_n) \to f(x)$, so $\omega_{x_n}(f) \to \omega_x(f)$ and $\hat{f}(\omega_{x_n}) \to \hat{f}(\omega_x)$. Therefore, $\omega(x_n) \to \omega_x$ in the weak-*topology. $\square$

Lecture 12

Let's back up a bit and investigate some true facts about general topological junk (which may explain the end of that last proof). Let X be a set, Y a topological space, and a family of functions $\{f_i\}_{i \in I}$. The weak topology of $\{f_i\}$ on X is the coarsest topology that makes all the f_i continuous.

Example 12.1.

- If B is a Banach space, and $X = B$, $Y = C$, $\{f_i\} = B^*$, then we get the weak topology on B.
- If $X = B^*$, $Y = \mathbb{C}$, $\{f_i\} = B \subseteq B^{**}$, then we get the weak-*topology.

True fact: If for some set $\{x_n\}$ and a point x, $f_i(x_n) \to f_i(x)$ for all i, then $x_n \to x$ in the weak topology.

For X a compact Hausdorff space, $X \cong \mathrm{sp}(C(X))$ via $x \mapsto \omega_x$. For $f \in C(X)$, $\hat{f}(\omega_x) = \omega_x(f) = f(x)$. This proves that the Gelfand transform is quite boring in some sense. :(

[1] http://www.proofwiki.org/wiki/Continuous_Bijection_from_Compact_to_Hausdorff_is_Homeomorphism

Lecture 13

Example 13.1. Let X, Y be compact Hausdorff spaces. Show that if $C(X)$ is isometrically isomorphic to $C(Y)$, then X is homeomorphic to Y.

Proof. $X \cong \mathrm{sp}(C(X)) \cong \mathrm{sp}(C(Y)) \cong Y$. $\square$

Example 13.2. Let $A = \ell^1(\mathbb{Z})$ with convolution, $(a * b)_i = \sum_{j\in\mathbb{Z}} a_j b_{i-j}$. Let $e_i \in A$ be the element in A with 1 in the i^{th} position, and 0 elsewhere. Then

$$(e_i * e_j)_k = \sum_{ell} (e_i)_\ell (e_j)_{k-\ell} = \delta_k^{i+j},$$

so that $e_i * e_j = e_{i+j}$. True fact: A is commutative and unital (e_0 acts as identity).

What is the Gelfand spectrum? Every homomorphism to $\mathbb{C}$ is determined by what it does to e_1, since $\omega(e_n) = (\omega(e_1))^n$, for all $n \in \mathbb{Z}$. So the question becomes: for which $\lambda \in \mathbb{C}$ does $\omega(e_1) = \lambda$ extend to a non-zero homomorphism on A? We know from Theorem 8.4 (1) that $\|\omega\| = 1$, so $|\lambda| \leq \|\omega\|\|e_1\| = 1$. But we also have $|\lambda^{-1}| \leq \|\omega\|\|e_{-1}\| = 1$, which implies that $|\lambda| = 1$. For any $|\lambda| = 1$, $(a_i) \mapsto \sum_i a_i \lambda^i$ is a homomorphism (check). It's an absolutely convergent sequence since $|\lambda| = 1$ and $\sum_i |a_i| < \infty$. Conclusion: Gelfand spectrum is homeomorphic to the circle ($\omega \in \mathrm{sp}(A) \mapsto \omega(e_1)$ is a continuous bijection).

What is the Gelfand transform? Denote the circle from the previous part by $\mathbb{T}$. We have $A \to C(\mathrm{sp}(A)) = C(\mathbb{T})$ with $(a_i) \mapsto (\hat{a}_i)$, with

$$(\hat{a}_i)(\omega_\lambda) = \omega_\lambda((a_i)) = \sum_{i\in\mathbb{Z}} a_i \lambda^i.$$

In other words, a sequence $(a_i) \in \ell^1(\mathbb{Z})$ maps to the function $\sum_i a_i z^i \in \mathbb{C}(\mathbb{T})$.

Now, $\sum_i a_i z^i = 0$ implies $a_i = 0$ for all i, so the only function in the kernel of the Gelfand transform is 0, and hence it is injective. But it's not surjective — not every continuous function can be written as $\sum_i a_i z^i$ with $\sum_i |a_i| < \infty$.

Definition 13.1. The *Weiner algebra* is the subalgebra of $C(\mathbb{T})$ of functions of the form $\sum_i a_i z^i$ with $\sum_i |a_i| < \infty$.

Theorem 13.1. *(Weiner's Theorem.)*
Let f be a nowhere vanishing function in the Weiner algebra. Then $\frac{1}{f}$ is in the Weiner algebra as well.

Proof. Recall that an element in a unital commutative Banach algebra is invertible if its Gelfand transform is non-vanishing ($\sigma(x) = \mathrm{Range}(\hat{x})$). If f is in the Weiner algebra, then $f = (\hat{a}_i)$ for some $(a_i) \in \ell^1(\mathbb{Z})$. If f is non-vanishing, then (a_i) is invertible. Then $(a_i)(a_i)^{-1} = \mathbf{1}$, and $(\hat{a}_i)\widehat{(a_i)^{-1}} = \hat{\mathbf{1}}$. Thus $f \cdot \widehat{(a_i)^{-1}} = 1$, so f is invertible in the Weiner algebra. $\square$

Lecture 14

Aside: true facts about Fourier series. $C(\mathbb{T}) \subseteq L^2(\mathbb{T})$. For $f \leftrightarrow \sum_i a_i z^i$ (where $a_i = (2\pi)^{-1/2} \int_{\mathbb{T}} f(z) z^{-1}\, dz$). Convergence to f?

- in $L^2(\mathbb{T})$ — automatic
- a.e. — true
- pointwise — false in general
- uniformly — false, but true if f is continuous and piecewise smooth
- absolutely — can be false even for piecewise smooth (only maybe a true fact)

Example 14.1. Let $A = C(\mathbb{T})$, $f : \mathbb{T} \to \mathbb{T}$ where $\zeta(z) = z$, and let B be the Banach subalgebra of A generated by $\mathbf{1}$ and ζ. What are $\sigma_A(\zeta)$ and $\sigma_B(\zeta)$? $\sigma_A(\zeta) = \hat{\zeta}(\mathrm{sp}(A)) = \zeta(\mathbb{T}) = \mathbb{T}$. Also, $\sigma_B(\zeta) = \hat{\zeta}(\mathrm{sp}(B))$, but what is $\mathrm{sp}(B)$?

- Any $\omega \in \mathrm{sp}(B)$ is determined by $\omega(\zeta)$ (for a polynomial $p(z)$, $\omega(p(z)) = p(\omega(\zeta))$).
- $|\omega(\zeta)| \leq \|w\|\|\zeta\| = 1 \cdot 1 = 1$.
- For $|\lambda| < 1$, define $\omega_\lambda(p(z)) = p(\lambda)$. Then

  $$|\omega_\lambda(p(z))| = |p(\lambda)| \leq \sup_{|z| \leq 1i} |p(z)| \leq \sup_{z \in \mathbb{T}} |p(z)| = \|p(z)\|,$$

  by the maximum modulus principle.
- Since ω_λ is a bounded homomorphism on polynomials, it can be extended to a homomorphism on B.
- $\mathrm{sp}(B) \cong \mathbb{D}$, the closed unit disk. $\hat{\zeta} : \omega_\lambda \mapsto \lambda$.
- $\widehat{p(z)}(\omega_\lambda) = \omega_\lambda(p(z)) = p(\lambda)$.

So $\sigma_B(\zeta) = \hat{\zeta}(\mathrm{sp}(B)) = \hat{\zeta}(\mathbb{D}) = \mathbb{D}$.

Lecture 15

Continuing on from last time. (Recall $A = C(\mathbb{T})$, $B = \overline{\{p(z)\}} \subseteq A$. We have an embedding $A = C(\mathbb{T}) \subseteq L^\infty(\mathbb{T}) \subseteq B(L^2(\mathbb{T}))$. Take a basis $\{\frac{1}{\sqrt{2\pi}} z^n\}_{n \in \mathbb{Z}}$ for $L^2(\mathbb{T})$, and let $H^2(\mathbb{Z}) = \mathrm{span}\{z^n\}_{n \in \mathbb{N}} \subset L^2(\mathbb{T})$. Then $B \subseteq B(H^2(\mathbb{Z}))$. How does ζ act? M_ζ on $H^2(\mathbb{Z})$) is unitarily equivalent to the rightward shift in $\ell^2(\mathbb{N})$.

So we have $B \subset C(\mathbb{T})$ and $B \subset B(\ell^2(\mathbb{N}))$, but you get weird junk happening, like $\sigma_B(\zeta) = \mathbb{D} = \sigma_{B(\ell^2(\mathbb{N}))}(\zeta)$, but $\sigma_{C(\mathbb{T})}(\zeta) = \mathbb{T}$. So it's pretty hard to tell what will happen when you move to a larger algebra; C^* algebras will make things nicer.

What are the maximal ideals of $C_0(\mathbb{R})$? Three questions:

- What are the codimension 1 ideals/complex homomorphisms?
- What are the closed maximal ideals?
- What are all the maximal ideals?

We've seen that these are pretty much the same in a unital algebra.

Exercise: Let $L^1(\mathbb{R})$ with convolution $(f * g)(x) = \int_{\mathbb{R}} f(y)g(x-y)\,dy$. Should check that:

- It's well defined almost everywhere.
- $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$.
- It's commutative.
- It's non-unital.

What are the complex homomorphisms? For $t \in \mathbb{R}$, let $\omega_t : L^1(\mathbb{R}) \to \mathbb{C}$ be given by $\omega_t(f) = \int_{\mathbb{R}} f(x)e^{-itx}\,dx$, $f \in L^1(\mathbb{R})$. Then ω_t is a homomorphism, and all homomorphisms to $\mathbb{C}$ are of this form.

What about Gelfand theory? Let $A^+ = L^1(\mathbb{R}) \oplus \mathbb{C}\delta$, where δ is a formal unit. Then $\mathrm{sp}(A^+) = \mathbb{R} \cup \{\infty\}$; also, $\omega_\infty(\delta) = 1$ and $\omega_\infty(f) = 0$.

Let $f \in L^1(\mathbb{R}) \subseteq A^+$. What is $\hat{f}$? It's pretty much a Fourier transform; $\hat{f}(\omega_t) = \omega_t(f) = \int_{\mathbb{R}} f(x)e^{-itx}\,dx$.

Spectral Permanence

If $\mathbf{1}_A \in B \subseteq A$, then $\sigma_A(x) \subseteq \sigma_B(x)$ for all $x \in B$. On the other hand $\sigma_b(x)$ could be bigger. Recall that the boundary ∂S of a subset $S \subset X$, where X is a topological space, is $\overline{S} \cap \overline{X \backslash S}$.

Theorem 15.1. *Let $\mathbf{1}_A \in B \subseteq A$, where A, B are Banach algebras, $x \in B$. Then $\partial\sigma_B(x) \subseteq \sigma_A(x)$.*

Proof. Want to show $\lambda \in \partial\sigma_B(x) \implies \lambda \in \sigma_A(x)$. Suppose $\lambda \in \partial\sigma_B(x)$, that is, $\lambda \in \overline{\sigma_B(x)} \cap \overline{\mathbb{C}\backslash\sigma_B(x)}$. So $\lambda \in \sigma_B(x)$ and there exist some $\lambda_n \notin \sigma_B(x)$ such that $\lambda_n \to \lambda$. So $x - \lambda$ is no invertible in B, but the $x - \lambda_n$ are invertible in B.

Now suppose for a contradiction that $x - \lambda \in A^{-1}$ (that is, it's invertible in A). Then since $\lambda_n \to \lambda$, we have $x - \lambda_n \to x - \lambda$. But then $(x-\lambda_n)^{-1} \to (x-\lambda)^{-1}$ (taking inverses is a continuous map). Since $(x-\lambda_n)^{-1} \in B$, this implies $x - \lambda \in B$ as well, which is a contradiction. □

Lecture 16

$\mathbb{C}\backslash\sigma_A(x)$ is an open set, can decompose into countably many connected components, where one component is unbounded, and the rest are bounded (we call these "holes"). Here's a corollary of Theorem 15.1:

Corollary. *Let Y be a hole of $\sigma_A(x)$. Then either $Y \subseteq \sigma_B(x)$, or $Y \cap \sigma_B(x) = \varnothing$.*

That is, every hole gets completely filled or not touched at all.

Proof. (Sketch.)
Suppose $Y \cap \sigma_B(x) \neq \varnothing$, but $Y \cap \sigma_B(x) \neq Y$. Then the boundary of $Y \cap \sigma_B(x)$ in Y is nonempty (otherwise $Y \cap \sigma_B(x)$ and $(\mathbb{C}\backslash Y) \cap \sigma_B(x)$ would be a decomposition of Y into closed and open subsets, contradicting connectedness of Y). Then one (a smart "one") can get a boundary point in $\sigma_B(x)$ which is not in $\sigma_A(x)$. □

So with the example $B = \overline{\{p(x)\}} \subset C(\mathbb{T}) = A$ before, $\sigma_A(\zeta) = \mathbb{T}$. If C is any subalgebra of A, then $\sigma_C(\zeta)$ is either $\mathbb{T}$ or $\mathbb{D}$.

Analytic Functional Calculus

If $x \in A$, where A is a unital Banach algebra, we can form $p(x)$ for any polynomial and some power series, e.g. $\sum_n \frac{x^n}{n!}$. Recall Cauchy's integral formula,

$$f(a) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z-a}\, dz,$$

where Γ is a simple closed curve, and a is in the interior of Γ, and f is analytic on the interior of Γ. Let Γ be a simple closed curve contained in $\mathbb{C}\backslash\sigma(x)$. Define

$$f(x) = \frac{1}{2\pi i} \int_\Gamma f(z)(z-x)^{-1}\, dz.$$

- Integrals of functions from $\mathbb{C}$ to A can be defined using Riemann sums.
- Makes sense as long as $z \notin \sigma(x)$.

Theorem 16.1. *If $\sigma(x)$ is contained in the interior of Γ, then*

1. *$f_\Gamma(x)$ doesn't depend on Γ.*
2. *$f(x)$ only depends on values of f in some open set containing $\sigma(x)$.*

We can define $A(\sigma(x))$, the algebra of locally analytic functions on $\sigma(x)$, as the algebra of all functions which are analytic on some open set containing $\sigma(x)$ modulo the following equivalence: $f \sim g$ if f and g are the same on some open set containing x.

Theorem 16.2. *Fix x in a unital Banach algebra. The analytic functional calculus $f \mapsto f(x)$ is a homomorphism from $A(\sigma(x)) \to A$. Moreover, for every power series $f(z) = \sum a_i z^i$ which converges on a disk centered at 0 and containing $\sigma(x)$, $f(x) = \sum a_i x^i$.*

Fun things to do. Take Ω to be a σ-finite measure space. Then $L^\infty(\Omega)$, with pointwise operations and essential supremum norm is a unital commutative Banach algebra. If $f \in L^\infty(\Omega)$, what is $\sigma(f)$?

An easier question: when is f invertible? f is invertible if and only if there exists some $g \in L^\infty(\Omega)$ such that $fg = 1$ almost everywhere, which is true if $g = \frac{1}{f}$ is defined almost everywhere and bounded, or equivalently, there exists some ε such that $|f| > \varepsilon$ almost everywhere.

So what is $\sigma(f)$? We have $\lambda \in \sigma(f)$ if and only if $f - \lambda$ is not invertible, or, for every $\varepsilon > 0$, $\mu(\{x : |f(x) - \lambda| < \varepsilon\}) > 0$. This set is called the essential range of range (points that the function gets arbitrarily close to on sets of positive measure).

Lecture 17

Recall that a (complex) Hilbert space is a complex vector space H with a map $\langle\ ,\ \rangle : H \times H \to \mathbb{C}$ such that

1. $\langle\ ,\ \rangle$ is linear in the first component.
2. $\langle x, y\rangle = \overline{\langle y, x\rangle}$ for all $x, y \in H$.
3. $x \mapsto \sqrt{\langle x, x\rangle}$ is a complete norm.

Denote the space of bounded operators from $H \to H$ by $B(H)$. For any $T \in B(H)$, there exists some $T^* \in B(H)$ such that $\langle Tx, y \rangle = \langle x, T^*y \rangle$ for all $x, y \in H$. True facts about the adjoint (here let $U \in B(H)$ as well):

1. $T^{**} = T$.

2. $(\lambda T + U)^* = \bar{\lambda} T^* + U^*$ (where $\lambda \in \mathbb{C}$).

3. $(TU)^* = U^* T^*$.

Definition 17.1. An involution on a complex algebra A is a map from A to A satisfying (1)-(3) above.

Lemma 17.1. *If $T \in B(H)$, then $\|T\| = \|T^*\|$.*

Proof. We have

$$\sup_{\|x\| \leq 1} \|Ax\| = \|A\| = \sup_{\|x\|, \|y\| \leq 1} |\langle Ax, y \rangle|.$$

(Take $y = Ax/\|Ax\|$ if $\|Ax\| \neq 0$ for $\leq$, Cauchy-Schwarz for $\geq$.) But

$$\|A^*\| = \sup_{\|x\|, \|y\| \leq 1} |\langle A^*x, y \rangle| = \sup_{\|x\|, \|y\| \leq 1} |\langle Ay, x \rangle| = \|A\|.$$

$\square$

Proposition 17.1. *If $T \in B(H)$, then $\|T^*T\| = \|T\|^2$.*

Proof. First, we have

$$\|T^*T\| \leq \|T^*\| \|T\| = \|T\|^2.$$

Also,

$$\begin{aligned} \|T\|^2 &= \sup_{\|x\| \leq 1} \langle Tx, Tx \rangle \\ &= \sup_{\|x\| \leq 1} \langle T^*Tx, x \rangle \\ &\leq \sup_{\|x\| \leq 1} \|T^*Tx\| \|x\| \\ &\leq \|T^*T\|, \end{aligned}$$

where we have used Cauchy-Schwarz. $\square$

Definition 17.2. A C^*-algebra is a Banach algebra with an involution * such that $\|x^*x\| = \|x\|^2$ for all x.

We say x is self-adjoint if $x^* = x$, normal if $x^*x = xx^*$ and unitary if $x^*x = 1 = xx^*$ (if we're in a unital algebra). If A, B are two algebras with involutions, can define an x-homomorphism $\rho : A \to B$ as a homomorphism of algebras which preserves the involutions, that is, $\rho(x^{*_A}) = (\rho(x))^{*_B}$.

Example 17.1.

1. $\mathbb{C}$ as an algebra over itself, where * is complex conjugate.

2. $B(H)$, where H is a Hilbert space and * is the adjoint.

3. $C_0(X)$, where X is a locally compact Hausdorff space, where * is pointwise conjugation.
4. $L^\infty(\Omega)$, where Ω is a measure space, again * is pointwise conjugation.
5. $C^*(T)$, where $T \in B(H)$ or $T \in A$, a C^*-algebra.
6. $C^*(T, 1)$.

Example 17.2.

1. What is $C^*(T)$, $C^*(1, T)$? In general, $C^*(\text{blah})$ is made of "words" in blah. If $T = T^* = T^2$ (we call this a projection), $T \neq 0, 1$, $T \in B(H)$, then $C^*(T) = \{\lambda T\}$, that is, scalar multiples of T. Also, $C^*(1, T) = \{\lambda T + \mu 1\} = \{\lambda T + \mu(1 - T)\}$. (The last one is sometimes helpful to use.)
2. Let T be the right shift on $\ell^2(\mathbb{N})$. What is $C^*(T)$?Let $\{e_i\}$ be the standard basis for $\ell^2(\mathbb{N})$. Let f_{ij} be the operator that takes e_j to e_i and kills all other e_k. Then $C^*(T) \supseteq \{f_{ij}\}$ (Exercise). Turns out to be the sum of certain continuous multiplications operators and compact operators of some sort; it's a bit weird. Maybe get to it later.
3. What is $C^*(1, T)$ when $T = M_{\{\frac{1}{n}\}} \in B(\ell^2(\mathbb{N}))$? Show that $C^*(1, T) = M_{(a_n)}$, where (a_n) is a convergent sequence.

Lecture 18

Proposition 18.1. *If A is a unital C^*-algebra, then $\mathbf{1}^* = \mathbf{1}$ and $\|\mathbf{1}\| = 1$.*

Proof. We have $\mathbf{1}^* x = (x^* \mathbf{1})^* = (x^*)^* = x = x\mathbf{1}^*$ for all $x \in A$, which implies that $\mathbf{1}^* = \mathbf{1}$. Then

$$\|\mathbf{1}\|^2 = \|\mathbf{1}^*\mathbf{1}\| = \|\mathbf{1}\| \qquad \text{(by Proposition 17.1)},$$

so $\|\mathbf{1}\| \in \{0, 1\}$, but since A is unital, $\|\mathbf{1}\| \neq 0$. ☐

Proposition 18.2. *If A is a C^*-algebra then $\|x^*\| = \|x\|$, and if x is normal, then $\|x^2\| = \|x\|^2$.*

Proof. We have $\|x\|^2 = \|x^*x\| \leq \|x\|\|x^*\|$. This gives $\|x\| \leq \|x^*\|$. Similarly, by considering $\|x^*\|^2$, we have $\|x^*\| \leq \|x\|$, so $\|x\| = \|x^*\|$. If x is normal, then

$$\begin{aligned}
\|x^2\|^2 &= \|(x^2)^* x^2\| && \text{by } C^*\text{-property} \\
&= \|(x^*x)(x^*x)\| && \text{by normality} \\
&= \|x^*x\|^2 && \text{by } C^*\text{-property} \\
&= \left(\|x\|^2\right)^2 && \text{by } C^*\text{-property.}
\end{aligned}$$

☐

Proposition 18.3. *Any element in a C^*-algebra can be written uniquely as $x = y + iz$ with $y = y^*$ and $z = z^*$.*

Proof. Let $y = \frac{x+x^*}{2}$ and $z = \frac{x-x^*}{2i} = -\frac{i}{2}(x - x^*)$ Then

$$y + iz = \frac{1}{2}(x + x^*) + i\left(-\frac{i}{2}(x - x^*)\right) = x.$$

☐

Lecture 19

Theorem 19.1. *(Stone-Weierstrass Theorem.)*
Suppose X is a compact Hausdorff space, and let $A \subseteq C_{real}(X)$ be a (real) subalgebra which contains $\mathbf{1}$ and separates points: for every $x, y \in X$, there exists some $f \in A$ such that $f(x) \neq f(y)$. Then A is uniformly dense in $C_{real}(X)$.

Example 19.1. Any $f \in C([a, b])$ can be uniformly approximated by polynomials.

Theorem 19.2. *(Stone-Weierstrass, complex version.)*
If $A \subseteq C(X)$ is a *-algebra containing $\mathbf{1}$ and separating points, then A is uniformly dense in $C(X)$.

Example 19.2. In $C([0, 1])$, (complex) polynomials are uniformly dense. On $C(\mathbb{T})$, is $\{p(z)\}$ dense? No: it is not a *-algebra. If we allow negative powers it would be (since the conjugate is the inverse).

Lemma 19.1. *Let A be a unital commutative C^*-algebra. The Gelfand transform from A to $C(sp(A))$ is an isometry onto a subalgebra of $C(sp(A))$ which contains $\mathbf{1}$ and separates points.*

Proof. "Contains $\mathbf{1}$ and separates points" always true: for any $\omega \in \operatorname{sp}(A)$, $\hat{\mathbf{1}}(\omega) = \omega(\mathbf{1}) = 1$, so $\hat{\mathbf{1}} = 1 \in C(\operatorname{sp}(A))$. For any distinct $\omega_1, \omega_2 \in \operatorname{sp}(A)$, there exists $x \in A$ such that $\omega_1(x) \neq \omega_2(x)$. Then $\hat{x}(\omega_1) = \omega_1(x) \neq \hat{x}(\omega_2)$.

Isometry follows from $\|x\|^2 = \|x^2\|$ for normal x, since A is commutative. □

What is missing for Gelfand transform to be surjective? If the image is a *-algebra, then the image is dense in $C(\operatorname{sp}(A))$ (S-W), so it is surjective (dense isometry).

Definition 19.1. The exponential map is given by

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

for $x \in A$.

If $xy = yx$, then $e^{x+y} = e^x e^y$:

$$\sum_{n=0}^{\infty} \frac{(x+y)^n}{n!} = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{y^n}{n!}\right).$$

Fix a commutative unital C^*-algebra A.

Lemma 19.2. *The Gelfand transform is a *-homomorphism from A to $C(sp(A))$, that is, $\widehat{x^*} = \overline{\hat{x}}$.*

Proof. The statement is equivalent to $\omega(x^*) = \overline{\omega(x)}$ for all $\omega \in \operatorname{sp}(A)$.

Assume x is self-adjoint, and let $u_t = e^{itx} = \sum_n \frac{(itx)^n}{n!} = \sum_n \frac{(it)^n x^n}{n!}$. Then

$$u_t^* = \sum_{n=0}^{\infty} \frac{(-it)^n}{n!} x^n = u_{-t},$$

and $u_t u_t^* = u_t^* u_t = e^{itx - itx} = \mathbf{1}$ (since itx and $-itx$ commute). Then $\|u_t\|^2 = \|u_t u_t^*\| = \|\mathbf{1}\| = 1$, so $\|u_t\| = 1$ for all t.

Claim: For any $\omega \in \operatorname{sp}(A)$, $\omega(x) \in \mathbb{R}$.

Proof of claim. For any $t \in \mathbb{R}$,

$$\omega(u_t) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!}\omega(x)^n = e^{it\omega(x)}.$$

So $e^{itx\omega(x)} = |\omega(u_t)| \leq 1$. This implies that $\omega(x)$ is real, since if it had an imaginary part, the t could be scaled to break that inequality. $\square$

So we have shown that $\omega(x) \in \mathbb{R}$ for all $x = x^*$ and $\omega \in \mathrm{sp}(A)$. This implies $\omega(x^*) = \overline{\omega(x)}$ for all x: decompose $x = y + iz$ where y, z are self-adjoint. $\square$

Theorem 19.3. *Let A be a unital commutative C^*-algebra. The Gelfand transform is an isometric *-isomorphism from A onto $C(sp(A))$.*

Proof. We proved that it is an isometry onto a subalgebra of $C(\mathrm{sp}(A))$ containing $\mathbf{1}$ and separating points. Since t is a *-homomorphism, the image is a *-algebra. By S-W, the image is dense in $C(\mathrm{sp}(A))$. But since it is an isometry, the image is complete. Therefore the image is all of $C(\mathrm{sp}(A))$. $\square$

Theorem 19.4. *The category of unital commutative C^*-algebras (with continuous *-homomorphisms) is equivalent to the category of compact Hausdorff spaces and continuous maps.*

If $X \xrightarrow{f} Y$, then for any $g \in C(Y)$, we have $C(X) \xleftarrow{f^*} C(Y)$: $f^*(g) = g \circ f$. Similarly, for $\phi : A \to B$ a continuous *-homomorphism between unital commutative C^*-algebras, we can take $sp(A) \xleftarrow{\phi^*} sp(B)$ with $\phi^*(\omega) = \omega \circ \phi$. Applying the * in both these cases commutes with composition.

Lecture 20

Proposition 20.1. *Suppose V and W are Banach spaces and W is finite-dimensional. Then $T : V \to W$ is continuous if and only if $\ker(T)$ is closed.*

Why? If $\ker(T)$ is closed, then $T : V/\ker(T) \to W$ is a linear map between two finite dimensional Banach spaces.

Exercise: If x is a normal element in a unital C^*-algebra A, then $\|x\| = r(x)$ (the spectral radius).

Gelfand transform is an isometry from $C^*(\mathbf{1}, x)$ to $C(\mathrm{sp}(C^*(\mathbf{1}, x)))$. So $\|x\| = \sigma_{C^*(\mathbf{1},x)}(x)$; what about $\sigma_A(x)$? See the next couple of theorems, everything is okay.

Theorem 20.1. *Let A be a unital C^*-algebra, $x = x^* \in A$. Then $\sigma(x) = \mathbb{R}$.*

Proof. Let $B = C^*(1, x) \subseteq A$. Then $\sigma_B(x)$ is real (since the Gelfand transform of a self-adjoint element is a real function.) So $\sigma_A(x) \subseteq \sigma_B(x) \subseteq \mathbb{R}$. $\square$

Theorem 20.2. *Let $B \subseteq A$ be unital C^*-algebras (where $\mathbf{1}_A \in B$ — we call this kind of inclusion a unital inclusion). For any $x \in B$, $\sigma_B(x) = \sigma_A(x)$.*

Proof. It suffices to show that if $x \in B$ is invertible in A, then $x^{-1} \in B$. Suppose $x \in B$ is invertible in A. Then xx^* is also invertible in A. Why? We have $xx^{-1} = x^{-1}x = \mathbf{1}$, so $(x^*)(x^{-1})^* = (x^{-1})^*(x^*) = \mathbf{1}^* = \mathbf{1}$. Then $(xx^*)(x^{-1})^*(x^{-1}) = \mathbf{1}$, and similarly for the other way.

Then xx^* is invertible in B, since xx^* is self-adjoint, so $\sigma(xx^*)$ is real and hence has no holes. (The spectrum can't get any bigger.)

It follows that x is invertible in B, since $x \underbrace{x^*}_{\in B} \underbrace{(xx^*)^{-1}}_{\in B} = \mathbf{1}$ (multiplying both sides by x^{-1} on the left shows that $x^{-1} \in B$). □

Theorem 20.3. *Let T be a normal bounded operator on a Hilbert space. Then there is an isometric *-isomorphism from $C^*(\mathbf{1}, T)$ to $C(\sigma(T))$ which sends T to $\zeta : z \mapsto z$.*

Proof. Let $A = C^*(\mathbf{1}, T)$. Consider $\hat{T} : \mathrm{sp}(A) \to \mathbb{C}$, sending $\omega \mapsto \omega(T)$. We claim that $\hat{T}$ is a homeomorphism.

Proof of claim. First, $\hat{T}$ is injective since $\hat{T}(\omega_1) = \hat{T}(\omega_2) \iff \omega_1(T) = \omega_2(T) \iff \omega_1 = \omega_2$. It is continuous by definition ($\hat{x}$ is always continuous in the weak-* topology). So $\hat{T}$ is a continuous bijection from $\mathrm{sp}(A)$ to a subset of $\mathbb{C}$; hence, it is a homeomorphism from $\mathrm{sp}(A)$ to $\sigma(T)$. □

Then...

- Gelfand transform:
$$C^*(\mathbf{1}, T) \leftrightarrow C(\mathrm{sp}(C^*(\mathbf{1}, T))).$$
- $\mathrm{sp}(C^*(\mathbf{1}, T)) \leftrightarrow \sigma(T)$.

Putting this together, we have an isometric *-isomorphism from $C^*(\mathbf{1}, T) \to C(\sigma(T))$, $\hat{T}(\omega_\lambda) = \lambda$.

(First we go from $C^*(\mathbf{1}, T) \to C(\mathrm{sp}(C^*(\mathbf{1}, T)))$ via $T \mapsto \hat{T}$, then to $C(\sigma(T))$ via $\hat{T} \mapsto \hat{T}'$, where $\hat{T}'(\lambda) = \hat{T}(\omega_\lambda) = \omega(T) = \lambda$.) □

Under the above map: $T \leftrightarrow z$, $T^2 \leftrightarrow z^2$ and $T^* \leftrightarrow \bar{z}$. Polynomials $p(T, T^*) \leftrightarrow p(z, \bar{z})$...

Lecture 21

Let (Ω, μ) be a measure space. Then $L^2(\Omega)$ (complex) is a Hilbert space and for any $f \in L^\infty(\Omega)$, we can define $M_f \in B(L^2(\Omega))$ by
$$M_f(g) = f \cdot g \qquad \forall g \in L^2(\Omega).$$

True facts:

1. $\sigma(M_f)$ is the essential range of f : $\{\lambda : \mu(\{x : |f(x) - \lambda| < \varepsilon\}) > 0 \ \forall \varepsilon > 0\}$.
2. If μ is σ-finite, then $f \mapsto M_f$ is an isometric *-isomorphism from $L^\infty(\Omega)$ to $B(L^2(\Omega))$.

Definition 21.1. A bounded operator T on a Hilbert space H is diagonalizable if it is unitarily equivalent to a multiplication operator, that is, if there exists a measure space $\Omega, \mu)$ and a unitary operator $U : H \to L^2(\Omega)$ and a function $f \in L^\infty(\Omega)$ such that $T = U^* M_f U$.

Theorem 21.1. *(Spectral Theorem.)*
Let T be a normal bounded operator on a separable Hilbert space. Then T is diagonalizable.

What about the converse? Also true. We can show $(M_f)^* = M_{\bar{f}}$ and $M_{f \cdot g} = M_f M_g$, so $M_f (M_f)^* = M_{f\bar{f}} = M_{\bar{f}f} = (M_f)^* M_f$, and normality is preserved under unitary equivalence.

Sketch of Spectral Theorem. Recall the following construction:

$$\begin{array}{ccc} C([0,1]) \subset L^\infty([0,1]) \subset B(L^2([0,1])) \\ f \quad \mapsto \quad M_f. \end{array}$$

We already have a way to think about T as a continuous function ($\zeta : z \to z$ on $\sigma(T)$); $T \leftrightarrow \zeta \in C(\sigma(T))$. Want to write

$$C(\sigma(T)) \subset L^\infty(\sigma(T)) \subset B(L^2(\sigma(T))).$$

Idea of proof: show that T is unitarily equivalent to $M_\zeta \in B(L^2(\sigma(T)))$. We need a measure on $\sigma(T)$. We use the following:

Theorem 21.2. *(Riesz-Markov):*
Let X be a compact Hausdorff space and let ρ be a linear functional on $C(X)$ such that $\rho(f) \geq 0$ whenever $f(x) \geq 0$ for all $x \in X$. Then there is a unique regular Borel measure μ on X such that $\rho(f) = \int f \, d\mu$ for all $f \in C(X)$.

Here the Borel σ-algebra is the smallest σ-algebra containing all open sets, and a regular measure is one where $\mu(Y) = \sup\{\mu(Z) : Z \subseteq Y,\ Z \text{ compact}\}$ for all $Y \subseteq X$. $\square$

Lecture 22

Definition 22.1. Let S be a set of operators on a Hilbert space H. A vector $\xi \in H$ is cyclic for S if $\overline{S\xi} = H$.

(So keep applying S and you get everything.)

Lemma 22.1. *Let T be a normal bounded operator on a separable Hilbert space H, and suppose that H contains a cyclic vector for $C^*(\mathbf{1}, T)$. Then there is a regular Borel measure μ on $\sigma(T)$ such that T is unitarily equivalent to $M_z \in B(L^2(\sigma(T), \mu))$.*

Proof. Consider $\rho : C(\sigma(T)) \to \mathbb{C}$, given by $f \mapsto \langle f(T)\xi, \xi\rangle$. (Use the identification $C(\sigma(T)) \cong C^*(\mathbf{1}, T)$.)

We claim that ρ is a positive linear functional. To see this:

$$\begin{aligned} \rho(af+g) &= \langle (af+g)(T)\xi, \xi\rangle \\ &= \langle [af(T) + g(T)]\xi, \xi\rangle \\ &= a\langle f(T)\xi, \xi\rangle + \langle g(T)\xi, \xi\rangle \\ &= a\rho(f) + \rho(g). \end{aligned}$$

If $f \geq 0$, then

$$\begin{aligned} \rho(f) &= \langle f(T)\xi, \xi\rangle \\ &= \left\langle \left(\sqrt{f}\right)^2 (T)\xi, \xi \right\rangle \\ &= \left\langle \sqrt{f}(T)\sqrt{f}(T)\xi, \xi \right\rangle \\ &= \left\langle \sqrt{f}(T)\xi, \left(\sqrt{f}(T)\right)^* \xi \right\rangle \\ &= \left\langle \sqrt{f}(T)\xi, \overline{\sqrt{f}}(T)\xi \right\rangle \\ &= \left\langle \sqrt{f}(T)\xi, \sqrt{f}(T)\xi \right\rangle \\ &\geq 0. \end{aligned}$$

By Riesz-Markov, there exists some μ on $\sigma(T)$ such that

$$\int f \; d\mu = \rho(f) = \langle f(T)\xi, \xi\rangle$$

for all $f \in C(\sigma(T))$.

Next, we want a unitary operator from $L^2(\sigma(T), \mu) \to H$. Define $U_0 : C(\sigma(T)) \to H$ with $f \mapsto f(T)\xi$.

We claim that for any $f, g \in C(\sigma(T))$, $\langle f, g\rangle_{L^2(\sigma(T),\mu)} = \langle U_0 f, U_0 g\rangle_H$: we have

$$\begin{aligned}
\langle U_0 f, U_0 g\rangle_H &= \langle f(T)\xi, g(T)\xi\rangle_H \\
&= \langle (g(T)^* f(t)\xi, \xi\rangle_H \\
&= \langle \bar{g}(T) f(t)\xi, \xi\rangle_H \\
&= \langle (\bar{g}f)(T)\xi, \xi\rangle_H \\
&= \int f\bar{g} \; d\mu \\
&= \langle f, g\rangle_{L^2(\sigma(T),\mu)}.
\end{aligned}$$

True facts:

- $C(\sigma(T))$ is dense in $L^2(\sigma(T))$ (analysis result).
- The image of U_0 is $C^*(\mathbf{1}, T)\xi$, which is dense in H, since ξ is cyclic for $C^*(\mathbf{1}, T)$.

Therefore, U_0 extends to a unitary operator U from $L^2(\sigma(T), \mu)$ onto H.

Claim: $TU = UM_z$. Why? For $f \in C(\sigma(T)) \subseteq L^2(\sigma(T), \mu)$,

$$TUf = (Tf(T))\xi = (zf)(T)\xi = Uzf = UM_z f.$$

Since $C(\sigma(T))$ is dense in $L^2(\sigma(T), \mu)$, $TU = UM_z$.

For general T (normal bounded operator on a separable H), the basic idea is to write $H = \bigoplus_i H_i$ and $T = \bigoplus_i T_i$, such that each H_i is invariant for $C^*(\mathbf{1}_{H_i}, T_i)$ and contains a cyclic vector for $C^*(1_{H_i}, T_i)$. We gotta make some bold claims.

- If H_i, $i \in I$ is a family of Hilbert spaces, can form $\bigoplus_{i\in I} H_i$. If $T_i \in B(H_i)$, can form $\bigoplus_i T_i \in B(\bigoplus_i H_i)$ if $\|T_i\| \leq K$ for all i.
- A subspace W of H is invariant for a set of operators S if $T\xi \in W$ for all $\xi \in W, T \in S$.
- If W is invariant for a *-algebra of operators A, then $\xi \in W$ is cyclic for A if $\overline{A\xi} = W$.

  Lemma.

  1. Let H be separable. If $T \in B(H)$ is the direct sum of $T_i \in B(H_i)$, where $H = \bigoplus_i H_i$, $i \in \mathbb{N}$, and if T_i is diagonalizable on a σ-finite measure space for each i, then T is diagonalizable on σ-finite space as well.
  2. If A is a *-algebra of operators on H, then H decomposes as a direct sum $H = \bigoplus_i H_i$, where each H_i is invariant for A, and contains a cyclic vector for A.

Proof. (sorta)

1. – Suffices to consider the case that each T_i is already diagonal, i.e. $T_i = M_{f_i} \in B(L^2(\Omega_i, \mu_i))$ with μ_i σ-finite.
   – Let $H = L^2(\bigsqcup_i \Omega_i, \oplus_i \mu_i)$ and then $\bigoplus_i T_i = M_f$, where $f|_{\Omega_i} = f_i$.

$\square$

What we actually need: if T is a normal bounded operator on H, then H breaks up as $H = \bigoplus_i H_i$, where each H_i is invariant under $C^*(\mathbf{1}_i, T_i)$, with a cyclic vector for $C^*(\mathbf{1}_i, T_i)$. $\square$

Lecture 23

Theorem 23.1. *(Spectral Theorem.)*
Let T be a normal bounded operator on a separable Hilbert space. Then there is a σ-finite measure space (Ω, μ) and a function $f \in L^\infty(\Omega, \mu)$ such that T is unitarily equivalent to $M_f \in B(L^2(\Omega, \mu))$.

Proof. By the previous lemma part (2), we can write $H = \bigoplus_i H_i$, where each H_i is invariant for $C^*(\mathbf{1}, T)$ with a cyclic vector for $C^*(\mathbf{1}, T)$. Let $T_i = T|_{H_i}$. Then $T = \bigoplus_{i \in \mathbb{N}} T_i$ and each H_i is invariant for $C^*(\mathbf{1}, T_i)$ with a cyclic vector for $C^*(\mathbf{1}, T_i)$. Each T_i is diagonalizable. Since $T = \bigoplus_i T_i$, T is diagonalizable as well. $\square$

Where does the measure in the proof of the spectral theorem live?

- If T has a cyclic vector, measure space is $\sigma(T)$.
- In general, the measure space is $\bigcup_i \sigma(T_i) \subseteq \bigcup_i \sigma(T)$ (each $\sigma(T_i) \subseteq \sigma(T)$, so the union is contained in a bunch of copies of $\sigma(T)$).

Example 23.1. Look at $I = \ell^2(\mathbb{N})$ (already diagonal). Then $\sigma(I) = \{1\}$. There is no way to have $L^2(\sigma(I)) \cong \ell^2(\mathbb{N})$ for any measure on $\sigma(I)$. If we do the construction in the proof above, every subspace is invariant:

$$\ell^2(\mathbb{N}) = \ell^2(\{1\}) \oplus \ell^2(\{2\}) \oplus \dots$$

and the measure space we end up with is $\{1\} \cup \{1\} \cup \dots \cong \mathbb{N}$.

Lecture 24

Example 24.1. Let $H = \ell^2(\mathbb{Z})$, and let $T \in B(\ell^2(\mathbb{Z}))$ defined by $(T\xi)_n = \xi_{n+1}$, where $\xi \in \ell^2(\mathbb{Z})$. Now, $(\dots, 0, 1, 0, \dots)$ is a cyclic vector for $C^*(\mathbf{1}, T)$. Let $H = L^2(\mathbb{T})$ with Lebesgue measure, with orthonormal basis $\{\frac{z^n}{\sqrt{2\pi}}\}_{n \in \mathbb{Z}}$. We want to take $z^n \mapsto z^{n-1}$. This is just multiplication by $\frac{1}{z}$. Just remains to show T is unitarily equivalent to $M_{\frac{1}{z}}$ (identify $(\dots, 0, \underbrace{1}_{n^{\text{th}}}, 0, \dots) \mapsto \frac{z^n}{\sqrt{2\pi}}$). What is $\sigma(T)$?

$\sigma(T) = \text{ess range}(f) = \mathbb{T}$.

Example 24.2. (Non-example.)
Let $T \in B(\ell^2(\mathbb{N}))$, and $(T\xi)_{n+1} = \xi_n$. It is not a normal operator, so it is not diagonalizable. But we can do the following. Let $H^2(\mathbb{T}) \subseteq L^2(\mathbb{T})$ be the subspace generated by z^n, $n \geq 0$. Let $M_z \in B(H^2(\mathbb{T}))$ be the multiplication operator of z. Then T is unitarily equivalent to M_z.

Example 24.3. Compact normal operators.

Lemma. Let T be a normal compact operator on a separable Hilbert space. Then every $0 \neq \lambda \in \sigma(T)$ is isolated, and has a finite dimensional eigenspace.

Proof. Second part: If $\lambda \neq 0$ has an infinite dimensional eigenspace, then there exists an orthonormal sequence $e_1, e_2, \ldots,$ such that $Te_i = \lambda e_i$, which contradicts the compactness of T.

It suffices to consider the case when $T = M_f \in B(L^2(\Omega, \mu))$. Then as before, $\sigma(T) = \text{ess range}(f)$. Suppose $0 \neq \lambda \in \sigma(T)$ which is not isolated, i.e. $\lambda_n \to \lambda$, where $\lambda_n \in \sigma(T)$. Take $\varepsilon_n \to 0$ such that the disks $|\lambda_n - z| < \varepsilon_n$ are disjoint. Then there exist sets of finite positive measure E_n such that $|f(E_n) - \lambda_n| < \varepsilon_n$ for all n. Then for n large,

$$\|(M_f - \lambda)\chi_{E_n}\| \leq |f - \lambda| \mu(E_n) \leq \varepsilon$$

(for n large). Here we've used $|f - \lambda| \leq |f - \lambda_n| + |\lambda_n - \lambda|$, etc.

Summary: if $\lambda_n \to \lambda \neq 0$ in the essential range of f, we can find an orthonormal sequence of vectors in L^2 which are "very close" to being eigenvectors with eigenvalue λ, so their images under M_f are bounded apart, which contradicts compactness. □

Theorem 24.1. *Let T be a compact normal operator on H. Then H admits an orthonormal basis of eigenvectors for T. Moreover,*

$$T = \sum \lambda_n E_n,$$

where the λ_n are the eigenvalues, the E_n are the corresponding eigenspaces, and convergence is in norm.

Proof. Assume $\sigma(T)$ is an infinite set. Consider $z \in C(\sigma(T))$. Since every $0 \neq \lambda \in \sigma(T)$ is isolated, $\chi_{\{\lambda\}}$ is continuous. Claim: $z = \sum_{i=1}^{\infty} \lambda_i \chi_{\{\lambda_i\}}$ in sup norm.

To see this, use some mad wizardry:

$$\left\| z - \sum_{i=1}^{N} \lambda_i \chi_{\{\lambda_i\}} \right\| = \sup_{i>N} |\lambda_i| \to 0$$

as $N \to \infty$.

From the claim,

$$T = \sum_{i=1}^{\infty} \lambda_i \chi_{\{\lambda_i\}}(T).$$

Now, the $\chi_{\{\lambda_i\}}$ are non-zero, mutually orthogonal projections. Exercise: show that the $\chi_{\{\lambda_i\}}$ $(\lambda_i \neq 0)$ are the projections onto the eigenspaces of λ_i. □

Lecture 25

Some true facts that might be useful.

- Commutative C^*-algebra $\leftrightarrow$ compact Hausdorff space
- Positive linear functional ($\rho(f) \geq 0$ for $f \geq 0$) $\leftrightarrow$ regular Borel measure.

Hint for assignment problem: self-adjoint elements in a C^*-algebra generates commutative C^*-algebras, so all spectrum and norm properties can be described by looking at $C(X)$.

Topologies from seminorms

Definition 25.1. A seminorm ρ on a complex vector space V is a function $\rho : V \to \mathbb{R}$ such that:

- $\rho(\lambda v) = |\lambda|\rho(v)$.
- $\rho(v_1 + v_2) \le \rho(v_1) + \rho(v_2)$.

Definition 25.2. A family $\{\rho_i\}$ of seminorms on V is *separating* if $\rho_i(v) = 0$ for all i implies $v = 0$.

Definition 25.3. Given a separating family of seminorms $\{\rho_i\}$ on V, the initial topology on V is the coarsest topology which makes all the functions $v \mapsto \rho_i(v - v_0)$ continuous.

- This makes V into a locally convex topological vector space.
- A net x_n converges to x if and only if $\rho_i(x_n - x) \to 0$ for all i.

(We already know that $x_n \to x$ if $\rho_i(x_n - x_0) \to \rho_i(x - x_0)$ for all i, x_0.)

Topologies on $B(H)$:

1. Operator norm topology ("semi"norm $T \mapsto \|T\|$).
2. Weak topology (makes all bounded linear functionals continuous).
3. Strong operator topology (S.O.T.) — seminorms $\{T \mapsto \|T\xi\|, \xi \in H\}$.
4. Weak operator topology (W.O.T.) — seminorms $\{T \mapsto |\langle T\xi, \eta\rangle|, \xi, \eta \in H\}$.

We have

$$\text{W.O.T.} \subset \text{S.O.T.} \subset \text{Norm topology}.$$

Why? Look at convergence of nets. In W.O.T., $T_n \to T$ iff

$$|\langle (T_n - T)\xi, \eta\rangle| \to 0$$

for all ξ, η, and $|\langle (T_n - T)\rangle| \le \|T_n - T\|\|\xi\|\|\eta\|$. In S.O.T., $T_n \to T$ iff

$$\|(T_n - T)\xi\| \to 0$$

for all $\xi \in H$, and $\|(T_n - T)\xi\| \le \|T_n - T\|\|\xi\|$. In norm topology, $T_n \to T$ iff $\|T_n - T\| \to 0$.

Is * continuous? It is norm continuous, *not* continuous in S.O.T., but continuous in W.O.T.

Proof that * is W.O.T. continuous: If $T_n \xrightarrow{\text{W.O.T.}} T$, then $|\langle (T_n - T)\xi, \eta\rangle| \to 0$ for all $\xi, \eta \in H$. Then $|\langle \xi, (T_n^* - T^*)\eta\rangle| \to 0$ for all $\xi, \eta \in H$, or $\||\langle (T_n^* - T^*)\eta, \xi\rangle| \to 0$ for all $\xi, \eta \in H$. So $T_n^* \xrightarrow{\text{W.O.T.}} T$.

Proof that * is not S.O.T. continuous: Let S be the right shift on $\ell^2(\mathbb{N})$, and look at the sequence S^n. Exercise: S^n has no strong limit. What about $(S^n)^* = (S^*)^n$? Another exercise: $(S^n)^* \xrightarrow{\text{S.O.T.}} 0$.

In fact, in the norm limit, both S^n and $(S^n)^*$ has no limit. In S.O.T., S^n has no limit while $(S^n)^*$ goes to zero. In W.O.T., they both go to zero.

Lecture 26

Definition 26.1. Let $S \subseteq B(H)$ be a set of bounded operators. The commutant is given by

$$S' = \{T \in B(H) : TR = RT \text{ for all } R \in S\}.$$

Proposition 26.1. *With the above notation,*

1. $S \subseteq S''$.
2. S' *is W.O.T. closed.*
3. *If* $S = S^*$, *then* $(S')^* = S'$.

Proof.

1. If $T \in S$, $R \in S'$, then $TR = RT$ by definition of S', so $T \in S''$ by definition of S''.
2. Let $T_n \in S'$ be a net, and suppose $T_n \xrightarrow{\text{W.O.T.}} T$ (that is, $\langle T_n\xi, \eta\rangle \to \langle T\xi, \eta\rangle$ for all $\xi, \eta \in H$). We want to show that $T \in S'$ as well. Let $R \in S$ and $\xi, \eta \in H$. Then

$$\langle RT_n\xi, \eta\rangle = \langle T_n\xi, R^*\eta\rangle.$$

   The LHS equals $\langle T_nR\xi, \eta\rangle$ which goes to $\langle TR\xi, \eta\rangle$, while the RHS goes to $\langle T\xi, R^*\eta\rangle = \langle RT\xi, \eta\rangle$.
3. Exercise.

□

Let A be a $*$-algebra of bounded operators on H containing I. Then

$$A \subseteq \bar{A}^{\text{W.O.T.}} \subseteq A''.$$

Theorem 26.1. *(von Neumann's Double Commutant Theorem.)*
Let A be a $*$-algebra of bounded operators on H containing I. The following are equivalent:

1. $A = A''$.
2. A *is W.O.T. closed.*
3. A *is S.O.T. closed.*

Definition 26.2. A von Neumann algebra is a $*$-algebra of bounded operators on a Hilbert space which is closed in the W.O.T.

True fact: Any vN algebra is a C^*-algebra.

Example 26.1.

1. $\mathbb{C}I$.
2. $B(H)$.

3. Let Ω be a σ-finite measure space, so that $L^\infty(\Omega) \subseteq B(L^2(\Omega))$. Then $L^\infty(\Omega)$ is a vN algebra: let $A = L^\infty(\Omega) \subseteq B(L^2(\Omega))$, then $A \subseteq A'$, since A is commutative. Claim: $A' \subseteq A$ (this would imply $A = A' = A''$). To see this, let $T \in A'$, so that $TM_f = M_fT$ for all $f \in L^\infty(\Omega)$. Assume Ω is a finite measure space. Let $h = T(1)$. Then $h \in L^\infty(\Omega)$ (exercise: follows from boundedness of T).

   Let $g \in L^\infty(\Omega) \subseteq L^2(\Omega)$. Then
   $$T(g) = TM_g(1) = M_gT(1) = M_gh = gh = hg = M_h(g).$$
   Since L^∞ is $\|\cdot\|_2$-dense in L^2, $T = M_h$.

Exercise: $C([0,1]) \subseteq L^\infty([0,1])$ (here $C([0,1])$ is equipped with sup norm; L^∞, with essential sup). They're both C^*-algebras, and L^∞ is also a vN algebra.

Theorem 26.2. *Every commutative vN algebra on a separable Hilbert space is isometrically $*$-isomorphic to $L^\infty(\Omega) \subseteq B(L^2(\Omega))$ for some σ-finite Ω.*

So we have identifications between commutative C^*-algebras and compact Hausdorff spaces, and between commutative vN algebras and σ-finite measure spaces.

Lecture 27

Theorem 27.1. *(Spectral Theorem [V1]).*

1. *Let T be a normal bounded operator on a separable H such that $C^*(1,T)$ has a cyclic vector in H. Then there is a measure μ on $\sigma(T)$ such that T is unitarily equivalent to $M_z \in B(L^2(\sigma(T),\mu))$.*

2. *Let T be a normal bounded operator on a separable H. Then there is a (finite or infinite) sequence of measures (μ_n) on $\sigma(T)$ such that T is unitarily equivalent to $M_z \in B(L^2(\Omega,\mu))$, where $(\Omega,\mu) = \bigcup(\sigma(T),\mu_n)$. Here Ω is a disjoint union of a bunch of copies of $\sigma(T)$.*

Definition 27.1. Let X be a compact Hausdorff space. The algebra of bounded Borel functions $B_b(X)$ is a C^*-algebra with supremum norm.

Theorem 27.2. *(Spectral Theorem [V2]).*
Let T be a normal bounded operator on a separable Hilbert space H. Then there is a unique *$*$-homomorphism π from $B_b(\sigma(T))$ to $B(H)$ such that:*

1. *π agrees with the continuous functional calculus on $C(\sigma(T)) \subseteq B_b(\sigma(T))$.*

2. *If f_n is a uniformly bounded sequence of Borel functions converging pointwise to 0, then $\pi(f_n) \xrightarrow{S.O.T.} 0$.*

Proof. Problem Sheet 5. □

Say $\Omega = \sigma(T) \cup \sigma(T) \cup \sigma(T)$ (3 copies), $\mu = \mu_1 + \mu_2 + \mu_3$. Then
$$f \in C(\sigma(T)) \quad \to \quad M_f \in B(L^2(\Omega,\mu)).$$

Is it true that $\|M_f\| = \|f\|_\infty$? It's kinda easier to see $\|M_f\| \leq \|f\|_\infty$, for $\geq$: if $M = |f(z_0)| = \|f\|_\infty$ then there is some $U \ni z_0$ such that $|f(z) - M| < \varepsilon$ for $z \in U$, then $\|M_f\chi_U\|_2$ is "almost" $M\|\chi_U\|$ (this is related to Q2(ii) in PS5).

To look at uniqueness (see Q3) — the idea is to show that $\langle \pi(f)\xi, \xi\rangle$ is determined by (1) and (2) in Theorem 27.2.

Theorem 27.3. *(Spectral Theorem [V2']). How many of these are there?!*
Let T be a normal bounded operator on a separable Hilbert space. Then there is a finite measure μ on $\sigma(T)$ and an isometric $*$-isomorphism from $L^\infty(\sigma(T), \mu)$ to the vN algebra generated by T which extends the continuous functional calculus.

Example 27.1. Remember C^*-algebra $\leftrightarrow C([0,1])$, vN algebra $\leftrightarrow L^\infty([0,1])$, one contained in the other. Projection ($p = p^* = p^2$) in a C^*-algebra: if $p \in B(H)$, $p = p^* = p^2$ is equivalent to saying p projects onto a closed subspace of H. Prove that the only projections on $C([0,1])$ are $0, 1$.

Proof. Let $f \in C([0,1])$ be a projection. Then $f = f^2$, so $f(x) = (f(x))^2$ for all $x \in [0,1]$, so $f(x) \in \{0,1\}$ for all x. Since f is continuous and $[0,1]$ is connected, this implies that $f \equiv 0$ or $f \equiv 1$. $\square$

In $L^\infty([0,1])$, projections are χ_E for all measurable E.

Lecture 28

Projections in $B(H)$ correspond to closed subspaces; in L^∞, to measurable sets modulo sets of measure zero.

True facts about projections in $B(H)$:

1. There is a partial order given by $p_1 \leq p_2 \iff p_1 H \subseteq p_2 H \iff p_1 p_2 = p_1$.
2. They form a lattice: every 2 elements have a supremum (write $\vee$) and infimum (write $\wedge$); $p_1 \wedge p_2$ is a projection onto $p_1 H \cap p_2 H$, and $p_1 \vee p_2$ is a projection onto $p_1 H + p_2 H$.
3. Complete: $\bigwedge_{i \in I} p_i$ is a projection onto $\bigcap_{i \in I} p_i H$, and $\bigvee_{i \in I} p_i$ is a projection onto $\overline{\sum_{i \in I} p_i H}$.

Lemma 28.1. *Let (p_n) be a sequence of mutually orthogonal projections. Then $\bigvee_n p_n = \sum_{n=1}^\infty p_n$ (convergence in S.O.T.).*

Proof. We want to compare $\bigvee_n p_n \xi$ to $\sum_{n=1}^N p_n \xi$ for vectors $\xi \in H$. If $\xi \in (\bigvee_n p_n)^\perp$ then $\bigvee_n p_n \xi = \sum_{n=1}^N p_n \xi = 0$. If $\xi \in (\bigvee_n P_n) H$, then $\sum_{n=1}^N p_n \xi \to \bigvee_n p_n \xi$ as $N \to \infty$. $\square$

In fact, for any vN algebra A, the projections in A form a complete lattice. But *not* for C^*-algebras.

Example 28.1. If K is the C^*-algebra of compact operators, K contains all finite rank projections, but no infinite rank projections.

Definition 28.1. A spectral measure on a Hilbert space H is a function $P : \{\text{Borel sets in } C\} \to \{\text{projections in } B(H)\}$ such that:

1. $P(\varnothing) = 0$.
2. $P(\mathbb{C}) = I$.
3. $P(\bigcup E_n) = \sum_n P(E_n)$ (convergence in S.O.T.) if E_n is a sequence of mutually disjoint Borel sets.

Lemma 28.2. *Let P be a spectral measure. Then:*

1. *If $E_1 \subseteq E_2$, then $P(E_1) \leq P(E_2)$.*

2. *If $E_1 \cap E_2 = \varnothing$, then $P(E_1) \perp P(E_2)$.*

3. *$P(E_1 \cap E_2) = P(E_1)P(E_2)$.*

Proof.

1. $P(E_2) = P(E_1 \cup (E_2 \backslash E_1)) = P(E_1) + P(E_2 \backslash E_1)$.

2. $P(E_1) = P(E_1 \backslash E_2) + P(E_1 \cap E_2) = P(E_1 \backslash E_2) \leq P(\mathbb{C} \backslash E_2) = P(\mathbb{C}) - P(E_2) = I - P(E_2)$.

3. $$\begin{aligned} P(E_1) &= P(E_1 \backslash E_2) + P(E_1 \cap E_2), \\ P(E_2) &= P(E_2 \backslash E_1) + P(E_1 \cap E_2), \\ P(E_1)P(E_2) &= 0 + 0 + 0 + P(E_1 \cap E_2). \end{aligned}$$

□

Let $f = \sum_{i=1}^{n} c_i \chi_{E_i} \in B_b(\mathbb{C})$. We can define

$$\int f \; dP = \sum_{i=1}^{n} c_i P(E_i) \in B(H).$$

Theorem 28.1. *Given a spectral measure (supported on a compact set $X \subseteq \mathbb{C}$) P, for every $f \in B_b(X)$, there exists some $T_f \in B(H)$ such that*

$$\langle T_f \xi, \xi \rangle = \int f \; dP,$$

where $\mu_\xi(E) = \langle P(E)\xi, \xi \rangle$ is the measure assigned to ξ.

Theorem 28.2. *Spectraslfldakjeapal Theorem.*
Let T be a normal bounded operator on a separable H. Then there is a spectral measure P supported on $\sigma(T)$ such that

$$\int z \; dP = T.$$