

# Measure and Integration

## Lecture 1

Fix a partition of the interval  $[a, b] \subset \mathbb{R}$ ,  $\rho : a = x_0 < x_1 < \dots < x_n = b$ . Let  $m_k = \min f(x)$  and  $M_k = \max f(x)$ , where the min and max are taken over  $x \in [x_{k-1}, x_k]$ . Let  $d = \max \Delta x_k$ , where  $\Delta x_k = x_k - x_{k-1}$ . The *oscillation* is given by  $\omega_k = M_k - m_k$ ; then a function is Riemann integrable if

$$\lim_{d \rightarrow 0} \sum_{k=1}^n \omega_k \Delta x_k = 0.$$

Denote the Riemann integrable functions over the interval  $[a, b]$  by  $R[a, b]$ .

**Lemma 1.1.**  $C[a, b] \subseteq R[a, b]$ .

*Proof.* If  $f \in C[a, b]$ , it is uniformly continuous, so for any  $\epsilon > 0$ , choose  $d$  small enough so that

$$\sum_{k=1}^n \omega_k \Delta x_k < \epsilon \sum_{k=1}^n \Delta x_k = \epsilon(b - a) \rightarrow 0.$$

□

Some funky examples:

$$f(x) = \begin{cases} \frac{1}{q} & x = \frac{p}{q} \in \mathbb{Q}; \gcd(p, q) = 1 \\ 0 & x \in \mathbb{R} - \mathbb{Q} \end{cases}$$

Show  $f$  is continuous on all  $x \in \mathbb{R} - \mathbb{Q}$ .

## Lecture 2

Dirichlet function:

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} - \mathbb{Q} \end{cases}$$

A general function of this type is based on indicator functions: for subsets  $A_1, \dots, A_n, \dots \subseteq X$ , where  $A_j \cap A_k = \emptyset$  for  $j \neq k$ , then

$$f(x) = \sum_{k=1}^{\infty} a_k \chi_{A_k}(x),$$

where the  $a_k \in \mathbb{C}$ , defines a **simple function**  $f : X \rightarrow \mathbb{C}$ .

Let  $X = \text{set}$ ,  $2^X = \{A \subseteq X\}$ .

**Definition 2.1.** A measure is some  $m : 2^X \rightarrow \mathbb{R}^+$  with:

1. Additivity:  $m(A \cup B) = m(A) + m(B)$  when  $A \cap B = \emptyset$ .
2. If  $A \subseteq B$ , then  $m(A) \leq m(B)$ .

For  $f$  a simple function, we can define the integral with respect to this measure

$$\int_X f(x) \, dm \sim \sum_{k=1}^{\infty} a_k m(A_k).$$

Brave people can try integrate the Dirichlet function with

$$\int_{\mathbb{R}} f(x) \, dm \sim \sum_{r \in \mathbb{Q}} m(\{r\}).$$

**Example 2.1.** Let  $X = [0, 1]$ .

1. Boring measure  $m \equiv 0$ . Then  $\int f(x) \, dm \sim 0$ .

2. Set

$$m(A) = \begin{cases} 1 & \text{if } \frac{1}{2} \in A \\ 0 & \text{if } \frac{1}{2} \notin A. \end{cases}$$

(This kind of measure is called a *point mass* measure.) With respect to this measure, we can find the integral of a simple function. Let  $k_0 \in \mathbb{Z}^+$  with  $\frac{1}{2} \in A_{k_0}$ .

$$\int_X f(x) \, dm \sim \sum_{k=1}^{\infty} a_k m(A_k) = a_{k_0} m(A_{k_0}) = a_{k_0} = f\left(\frac{1}{2}\right).$$

If no such  $k_0$  exists, it is still consistent —  $f(\frac{1}{2})$  must be zero.

3. Let  $x_1, \dots, x_n \in X$ ,  $b_1, \dots, b_n \in \mathbb{R}^+$  and  $A \subseteq X$ . Let

$$m(A) = \sum_{k: x_k \in A} b_k.$$

Then

$$\int_X f(x) \, dm \sim \sum_{k=1}^n b_k f(x_k).$$

**Theorem 2.1.** (*Vitali's Theorem.*)

There is no non-trivial additive measure  $m : 2^{\mathbb{R}} \mapsto \mathbb{R}^+$  such that

$$m(A) = m(A + x)$$

where  $A \subseteq \mathbb{R}$ ,  $x \in \mathbb{R}$  and  $A + x = \{y + x : y \in A\}$ .

*Proof.* Suppose  $m$  is a non-trivial translation invariant measure as above. Define equivalence relation on  $[0, 1]$  given by  $x \sim y \iff x - y \in \mathbb{Q}$ . Define a Vitali set  $V \subseteq [0, 1]$  by choosing one class representative from each equivalence class. We claim that for any non-zero  $r \in \mathbb{Q}$ ,  $V \cap V + r = \emptyset$ . To see this, suppose  $x \in V \cap (V + r)$ . Then  $x \in V + r$  implies  $x = y + r$  for some  $y \in V$ . This means that  $x \sim y$  and  $\bar{x} = \bar{y}$ , but by the definition of  $V$  this implies that  $x = y$  and hence  $r = 0$ , a contradiction.

Also,

$$[0, 1] \subseteq \bigcup_{r \in \mathbb{Q} \cap [-1, 1]} (V + r) \subseteq [-1, 2].$$

To see this, for each  $x \in [0, 1]$ , there is some  $x_1 \in \bar{x}$  such that  $x_1 \in V$ . Then  $x - x_1 = r \in \mathbb{Q}$ , so  $x = x_1 + r \in V + r$ , that is,  $x \in \bigcup_{r \in \mathbb{Q} \cap [-1, 1]} (V + r)$ . Taking the measure of everything gives

$$m([0, 1]) \leq \sum_{r \in \mathbb{Q} \cap [-1, 1]} m(V + r) \leq m([-1, 2]).$$

Since  $m(V + r) = m(V)$  for all  $r$ , the sum is an infinite sum of a fixed non-negative real number. But it is also bounded above by a fixed number,  $m([-1, 2])$ , so we must have  $m(V) = 0$ . This implies that  $m([0, 1]) = 0$ , and hence  $m \equiv 0$ , a contradiction.  $\square$

## Lecture 3

Some sets are too freaky, want to restrict stuff. Let  $S \subseteq 2^X$ .

**Definition 3.1.**  $S$  is a semi-ring if:

1.  $S \neq \emptyset$ .
2. For any  $A, B \in S$ ,  $A \cap B \in S$ .
3. For any  $A, B \in S$ ,  $A \setminus B = \bigsqcup_{k=1}^n C_k$  with  $C_k \in S$ .

**Example 3.1.** Semi-rings.

1.  $X = \mathbb{R}, S = \{[a, b) : a \leq b\}$ .
2.  $X = \mathbb{R}^2, S = \{[a, b) \times [c, d) : a \leq b, c \leq d\}$ .

**Definition 3.2.**  $S$  is a ring of subsets if:

1.  $S \neq \emptyset$ .
2. For any  $A, B \in S$ ,  $A \cup B \in S$ .
3. For any  $A, B \in S$ ,  $A \setminus B \in S$ .
4. For any  $A, B \in S$ ,  $A \cap B \in S$ .

A ring  $S$  is called an *algebra* if  $X \in S$ . A ring (resp. algebra)  $S$  is called a  $\sigma$ -ring (resp.  $\sigma$ -algebra) if it is also closed under countably many unions/intersections.

**Example 3.2.**

1.  $R = \{\emptyset\}$  is a  $\sigma$ -ring;  $R = \{\emptyset, X\}$  is a  $\sigma$ -algebra.
2.  $R = 2^X$  is a  $\sigma$ -algebra.

## Lecture 4

The stuff before lets us define the measure for semi-rings in a hopefully nicer way:

**Definition 4.1.** Let  $S$  be a semi-ring of subsets of  $X$ . A measure is some  $m : S \rightarrow \mathbb{R}^+$  with:

1.  $m(A \sqcup B) = m(A) + m(B)$ , for  $A, B \in S$  and  $A \sqcup B \in S$ .
2.  $A_1, A_2, \dots, A_n \in S \implies m\left(\bigsqcup_{k=1}^n A_k\right) = \sum_{k=1}^n m(A_k)$  when  $\bigsqcup_{k=1}^m A_k \in S$  for all  $m \leq n$ .

It is  $\sigma$ -additive if (2) works for  $n = \infty$ . Semi-rings make things easy, but they don't allow for very much, so we look to extend measures to larger structures in a sane way.

Let  $S \subseteq 2^X$ .

**Definition 4.2.** The minimal ring enveloping  $S$  is defined as

$$R(S) = \bigcap_{\substack{S \subseteq R, \\ R \text{ a ring}}} R.$$

The minimal  $\sigma$ -ring enveloping  $S$  is (similarly)

$$R_\sigma(S) = \bigcap_{\substack{S \subseteq R_\sigma, \\ R_\sigma \text{ a } \sigma\text{-ring}}} R_\sigma.$$

**Proposition 4.1.** Let  $S$  be a semi-ring. Then

$$R(S) = \left\{ \bigsqcup_{k=1}^n A_k : A_k \in S \right\}.$$

*Proof.* Denote the right hand side by  $R_0$ . We just need to show that  $R_0$  is a ring. Suppose  $A, B \in R_0$ . Write

$$A = \bigsqcup_{k=1}^n A_k, \quad B = \bigsqcup_{s=1}^m B_s,$$

where the  $A_k, B_s \in S$ . Then (exercise, or youtube):

$$A \setminus B = \bigsqcup_{k=1}^N C_k, \quad A \cup B = \bigsqcup_{s=1}^M D_s,$$

where the  $C_k, D_s \in S$ . □

**Lemma 4.1.** Suppose  $m : S \rightarrow \mathbb{R}^+$  is a measure. This extends to a measure  $\tilde{m} : R(S) \rightarrow \mathbb{R}^+$ , where  $\tilde{m}(A) = m(A)$  for all  $A \in S$ . Also,  $\tilde{m}$  is  $\sigma$ -additive if  $m$  is  $\sigma$ -additive.

*Proof.* See video, hardest part is  $\sigma$ -additivity. □

For  $S$  a semi-ring, what about  $R_\sigma(S)$ ? Can we say

$$R_\sigma(S) = \left\{ \bigsqcup_{n=1}^{\infty} A_n : A_n \in S \right\} := R_{\sigma,0}?$$

No — take the semi-ring of half open intervals,  $S = \{[a, b)\}$ . Then  $[0, 1] \notin R_{\sigma,0}$ , for if  $[0, 1] = \bigcup_{n=1}^{\infty} [a_n, b_n)$ , then there is some  $n$  such that  $1 \in [a_n, b_n)$ . This means there is some  $\varepsilon > 0$  with  $[1, 1 + \varepsilon] \subseteq [a_n, b_n)$  and hence  $[1, 1 + \varepsilon] \subseteq [0, 1]$ , a contradiction. On the other hand, that  $[0, 1] = [0, 2) \setminus \bigcup_{n=1}^{\infty} [1 + \frac{1}{n}, 2)$  shows it must be in  $R_{\sigma,0}$  if it were to be the minimal  $\sigma$ -ring enveloping  $S$ , which is kinda sucky.

## Lecture 5

Let  $S = \{[a, b]\}$ , then  $R_\sigma(S)$  is the Borel  $\sigma$ -algebra. (It is an algebra because  $\mathbb{R} = \bigsqcup_{n \in \mathbb{Z}} [n, n+1)$ .) Last time, we saw a botched attempt at describing some sort of structure on  $R_\sigma(S)$ . Let's try again:

$$R_\sigma(S) = \bigcup_{n=0}^{\infty} R_{\sigma,n},$$

where  $R_{\sigma,0} = S$ , and

$$R_{\sigma,n} = \left\{ \bigcup_{k=1}^{\infty} A_k, A \cap B, A \setminus B; A_k, A, B \in R_{\sigma,n-1} \right\}.$$

Then  $|R_\sigma(S)| = 2^{\aleph_0}$ . But we see it's not that great — for example, the Cantor set  $C$  has measure zero but cardinality  $2^{\aleph_0}$ . So  $|P(C)| > 2^{\aleph_0}$ , but this implies we can choose a subset that should definitely be measurable (with measure zero) but is not in the Borel  $\sigma$ -algebra. (I may have missed the point of this bit, not sure.)

Some properties of measures:

**Proposition 5.1.** *Let  $R = \text{ring}$ , and  $m : R \rightarrow \mathbb{R}^+$  be a measure. Then:*

1.  $m(\emptyset) = 0$ .
2. If  $A, B \in R$  and  $A \subseteq B$ , then  $m(B \setminus A) = m(B) - m(A)$ . (Hence  $m(A) \leq m(B)$ .)
3.  $m(A \cup B) = m(A) + m(B) - m(A \cap B)$ .
4. If  $m$  is  $\sigma$ -additive:

$$m\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} m(A_k).$$

*Proof.*

1.  $m(\emptyset) = m(\emptyset \sqcup \emptyset) = 2m(\emptyset)$ .
2.  $B = A \sqcup (B \setminus A)$ , so  $m(B) = m(A) + m(B \setminus A)$ .
3. Follows from  $A \cup B = A \sqcup (B \setminus (A \cap B))$ .

□

## Lecture 6

MIA

# Lecture 7

## Extended “Measure”/Outer “Measure”

Take a measure  $m$  on a semi-ring  $S$ , and let  $A \subseteq X$  be a subset of the enormous set. Define the external “measure” by

$$m^*(A) = \inf \sum_{n=1}^{\infty} m(A_n),$$

where  $A \subseteq \bigcup_n A_n$ ,  $A_n \in S$ . (It is not a ‘proper’ measure. We’ll eventually limit our choice of subsets of  $X$  so that  $m^*$  is actually a measure.) Properties:

1.  $A \subseteq B \subseteq X \implies m^*(A) \leq m^*(B)$ .

2. Semi-additivity:

$$m^*\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} m^*(A_k).$$

3. Whacked up triangle inequality:

$$|m^*(A) - m^*(B)| \leq m^*(A \Delta B).$$

*Proof.*

2. For finitely many only — check brains or youtube for countably infinite. We want  $m^*(A \cup B) \leq m^*(A) + m^*(B)$ . Fix  $\varepsilon > 0$ . Take coverings  $\{A_n\}$  and  $\{B_n\}$  from the semi-ring for  $A$  and  $B$  respectively, such that  $\sum m(A_n) < m^*(A) + \varepsilon$  and  $\sum m(B_n) < m^*(B) + \varepsilon$ . Then

$$A \cup B \subseteq \left(\bigcup_{n=1}^{\infty} A_n\right) \cup \left(\bigcup_{n=1}^{\infty} B_n\right),$$

so

$$m^*(A \cup B) \leq \sum_{n=1}^{\infty} m(A_n) + \sum_{n=1}^{\infty} m(B_n) < m^*(A) + m^*(B) + 2\varepsilon.$$

□

### Proposition 7.1.

1.  $X \in R(S) \implies m^*(A) < \infty \forall A \subseteq X$ .

2. If  $m$  is  $\sigma$ -additive, then  $m^*(B) = \tilde{m}(B)$  for all  $B \in R(S)$ .

*Proof.*

2. Suppose  $m$  is  $\sigma$ -additive. Write

$$B = \bigsqcup_{k=1}^n B_k,$$

where each  $B_k \in S$ . Then

$$m^*(B) \leq \sum_{k=1}^n m(B_k) = \tilde{m}(B).$$

Fix an  $\varepsilon > 0$ , and choose a covering  $\{A_n\}$  from  $S$  such that

$$\sum_{n=1}^{\infty} m(A_n) < m^*(B) + \varepsilon.$$

Write

$$B = \bigcup_{n=1}^{\infty} B \cap A_n,$$

Then by semi-additivity

$$\tilde{m}(B) \leq \sum_{n=1}^{\infty} \tilde{m}(B \cap A_n).$$

But  $\tilde{m}(B \cap A_n) \leq \tilde{m}(A_n) = m(A_n)$ , so

$$\tilde{m}(B) \leq \sum_{n=1}^{\infty} m(A_n) < m^*(B) + \varepsilon.$$

□

**Theorem 7.1.** Suppose  $m$  is  $\sigma$ -additive and  $X \in R(S)$ . Let

$$\mathcal{F} = \{A \subseteq X : \forall \varepsilon > 0, \exists A' \in R(S) : m^*(A \Delta A') < \varepsilon\}.$$

Then  $\mathcal{F}$  is a  $\sigma$ -algebra and  $m^*$  is a  $\sigma$ -additive measure in  $\mathcal{F} \subseteq 2^X$ .

There's heaps of junk to prove here.

## Lecture 8

*Proof of Theorem 7.1.*  $X \in \mathcal{F}$  is clear (take " $X'$ " =  $X$ ). Suppose  $A, B \in \mathcal{F}$ . Closure under union: fix an  $\varepsilon > 0$ , and take  $A', B' \in R(S)$  such that

$$\begin{aligned} m^*(A \Delta A') &< \varepsilon, \text{ and} \\ m^*(B \Delta B') &< \varepsilon. \end{aligned}$$

Now,  $(A \cup B) \Delta \underbrace{(A' \cup B')}_{\in R(S)} \subseteq (A \Delta A') \cup (B \Delta B')$ . So

$$m^*((A \cup B) \Delta (A' \cup B')) \leq m^*(A \Delta A') + m^*(B \Delta B') < 2\varepsilon.$$

Closure under set difference: show  $(A \setminus B) \Delta (A' \setminus B') \subseteq (A \Delta A') \cup (B \Delta B')$ , and use the same argument as before.

Closure under countable union: suppose  $A_n \in \mathcal{F}$  for  $n = 1, \dots, \infty$ . Let  $A = \bigcup A_n$ . Fix an  $\varepsilon > 0$ . For each  $n$ , choose  $A'_n \in R(S)$  such that

$$m(A_n \Delta A'_n) < \frac{\varepsilon}{2^n}.$$

Let

$$A' = \bigcup_{n=1}^{\infty} A'_n.$$

Then

$$m^*(A \triangle A') \leq \sum_{n=1}^{\infty} m^*(A_n \triangle A'_n) < \varepsilon.$$

But this isn't enough because  $A'$  is not necessarily in  $R(S)$ . Now,

$$A \triangle A' \subseteq \bigcup_{n=1}^{\infty} (A_n \triangle A'_n)$$

and

$$A' \subseteq A \cup \left[ \bigcup_{n=1}^{\infty} (A_n \triangle A'_n) \right].$$

Observe that

$$\sum_{n=1}^{\infty} \tilde{m}(A'_n) < \infty.$$

Why? We have

$$\begin{aligned} \sum_{n=1}^N \tilde{m}(A'_n) &= \tilde{m} \left( \bigcup_{n=1}^N A'_n \right) \\ &= m^* \left( \bigcup_{n=1}^N A'_n \right) \\ &\leq m^* \left( \underbrace{\bigcup_{n=1}^{\infty} A'_n}_{A'} \right) \\ &\leq m^*(A) + \sum_{n=1}^{\infty} m^*(A_n \triangle A'_n) \\ &\leq m^*(A) + \varepsilon \\ &\leq m^*(A) + 1. \end{aligned}$$

Now, how do we fix the  $A'$ ? Choose  $N_\varepsilon \geq 1$  such that

$$\sum_{n=N_\varepsilon+1}^{\infty} \tilde{m}(A'_n) < \varepsilon.$$

Let

$$A'' = \bigcup_{n=1}^{N_\varepsilon} A'_n.$$

Then

$$A \triangle A'' \subseteq \left[ \bigcup_{n=1}^{\infty} (A_n \triangle A'_n) \right] \cup \left[ \bigcup_{n=N_\varepsilon+1}^{\infty} A'_n \right].$$

So

$$\begin{aligned} m^*(\text{LHS}) &\leq \sum_{n=1}^{\infty} m^*(A_n \triangle A'_n) + \sum_{n=N_\varepsilon+1}^{\infty} m^*(A'_n) \\ &\leq 2\varepsilon. \end{aligned}$$



We'll still need to show that it's a proper measure!

## Lecture 9

Continuing on with the proof from last time.

*Proof.* We want to show that for  $A, B \in \mathcal{F}$  with  $A \cap B = \emptyset$ ,  $m^*(A \sqcup B) = m^*(A) + m^*(B)$ . Semi-additivity gives us " $\leq$ ", so we'll only need to prove " $\geq$ ". Fix  $\varepsilon > 0$ , and take  $A', B' \in R(S)$  such that

$$m^*(A \Delta A') < \varepsilon \quad \text{and} \quad m^*(B \Delta B') < \varepsilon.$$

Now  $A \subseteq A' \cup (A \Delta A')$  and  $B \subseteq B' \cup (B \Delta B')$ . Thus

$$m^*(A) \leq m^*(A') + \varepsilon \quad \text{and} \quad m^*(B) \leq m^*(B') + \varepsilon.$$

Adding these gives

$$m^*(A) + m^*(B) \leq \tilde{m}(A') + \tilde{m}(B') + 2\varepsilon,$$

since  $m^*$  and  $\tilde{m}$  coincide on  $R(S)$ . Then

$$m^*(A) + m^*(B) \leq \tilde{m}(A' \cup B') + \tilde{m}(A' \cap B') + 2\varepsilon.$$

Now,

$$\begin{aligned} A' \cup B' &\subseteq (A \sqcup B) \cup (A \Delta A') \cup (B \Delta B'), \text{ and} \\ A' \cap B' &\subseteq \underbrace{(A \cap B)}_{\emptyset} \cup (A \Delta A') \cup (B \Delta B'). \end{aligned}$$

So

$$\begin{aligned} \tilde{m}(A' \cup B') &= m^*(A' \cup B') \leq m^*(A \sqcup B) + 2\varepsilon, \text{ and} \\ \tilde{m}(A' \cap B') &= m^*(A' \cap B') \leq 2\varepsilon. \end{aligned}$$

Thus

$$m^*(A) + m^*(B) \leq m^*(A \sqcup B) + 6\varepsilon.$$

What about for countable disjoint unions? For a measure on a ring, additivity with semi-additivity implies  $\sigma$ -additivity.  $\square$

If  $m$  is a  $\sigma$ -additive measure on a semi-ring  $S$  and  $X \in R(S)$ , then  $(X; S, m) \mapsto (\mathcal{F}, m^*)$  is a finite Lebesgue extension.

If we relax the restriction that  $X \in R(S)$  to just that

$$X = \bigsqcup_{n=1}^{\infty} X_n,$$

where  $X_n \in S$ , then we call it a  $\sigma$ -finite extension. In this case, define new semi-rings

$$S_n = \{A \cap X_n : A \in S\} \subseteq S.$$

Then restrict  $m : S_n \rightarrow \mathbb{R}^+$ , to get a finite Lebesgue extension

$$(X_n; S_n, m) \mapsto (X_n; \mathcal{F}_n, m_n^*).$$

LET'S KEEP GOING, define

$$\mathcal{F} = \{A \subseteq X : A \cap X_n \in \mathcal{F}_n\},$$

$$\mathcal{F}_0 = \left\{ A \in \mathcal{F} : \sum_{n=1}^{\infty} m_n^*(A \cap X_n) < \infty \right\}.$$

Then let  $\mu : \mathcal{F}_0 \rightarrow \mathbb{R}^+$ , with  $\mu(A) = \sum_{n=1}^{\infty} m_n^*(A \cap X_n)$ .

**Theorem 9.1.**

1.  $\mathcal{F}$  is a  $\sigma$ -algebra.

2.  $\mathcal{F}_0$  is a ring.

3.  $\mu$  is  $\sigma$ -additive.

3.1. If  $A_n \in \mathcal{F}_0$ , and  $A_i \cap A_j = \emptyset$  for  $i \neq j$ , and  $\sum \mu(A_n) < \infty$ , then  $A = \bigsqcup A_n \in \mathcal{F}_0$ , and

$$\mu(A) = \sum_{n=1}^{\infty} \mu(A_n).$$

3.2. If  $\sum \mu(A_n) = \infty$  then  $A \notin \mathcal{F}_0$ .