Combinatorics

Problem Set 1

1) Ignore the condition that the two sets of people are disjoint, because you can just end the people who are in both to make the sets disjoint. The number of non-empty subsets of the 10 people is $2^{10} - 1 = 1023$, but the possible set of age sums for non-empty subsets is $\{1, 2, \ldots, 600\}$ (this is a very loose upper bound, because we can assume all ages are distinct — or else just choose singleton sets — but this works so why bother). By the pigeonhole principle, since $1023 > \mathbf{1} \times 600$, there is at least one age sum (pigeonhole) containing more than **1** subset of people (pigeons)... which is what we wanted all along.

   With 9 people, $2^9 - 1 = 511$. Be a bit smarter and use distinct ages to show that the possible set of age sums for non-empty subsets is $\{1, 2, \ldots, 52 + 53 + \cdots + 60\} = \{1, 2, \ldots, 504\}$ — it still works.

2) Let $f : \{1, \ldots, m\} \to \{1, \ldots, m-1\}$ be defined by $f(i) = a^i \bmod m$ ($a^i \not\equiv 0 \bmod m$ cause they're coprime). By pigeonhole, there are distinct $s, t \in \{1, \ldots, m\}$ such that $f(s) = f(t)$. Assume $s > t$. So $a^s \equiv a^t \bmod m$. Since $\gcd(a, m) = 1$, this implies that $a^{s-t} \equiv 1 \bmod m$.

3) "Direct" approach: look for contiguous blocks of stuff — let x_i be the sum of the games played over the first i days; then if $x_j - x_i = 21$ for some $j > i$ then we're done.

   The total number of games over the 77 days can be at most $11 \times 12 = 132$. Let x_i be the number of games played on days $1, 2, \ldots, i$ inclusive, for $i = 1, \ldots, 77$. We want to use pigeonhole, but it's not immediately able to tell us $x_j - x_i = 21$ — it's better for getting some kind of equality. So let $y_i = x_i + 21$ for $i = 1, \ldots, 77$. But we are most interested in y_i when $y_i \leq 132$, or equivalently, $x_i \leq 111$. This will be definitely true for $i = 1, \ldots, 63$, since $x_{63} \leq 12 \times 9 = 108$.

   Consider the $63 + 77 = 140$ values $\{x_1, \ldots, x_{77}, y_1, \ldots, y_{63}\}$ which lie in the range $\{1, \ldots, 132\}$. By the pigeonhole principle, since $140 > 132$, there is a value $v \in \{1, \ldots, 132\}$ such that at least two of the elements of $\{x_1, \ldots, x_{77}, y_1, \ldots, y_{63}\}$ equal v. Since $x_i < x_{i+1}$ for all $i = 1, \ldots, 76$, the x_i are all distinct, which also implies that the y_i are all distinct. It must be that some $x_j = y_i$ for some i, j. That is, $x_j = x_i + 21$, which is what we wanted all along.

   Alt: Use Example 1.2 from lectures, in any 21 days, there is a consecutive subsequence adding up to a multiple of 21, but there are at most $12 \times 3 = 36$ games, which means it must be 21.

$6\frac{1}{2}$) (a) Define

$$S = \{(A, b) : A \subseteq \{1, \ldots, n\}, b \in A\}.$$

   Calculating the size of S by first counting over A gives

$$|S| = \sum_{\substack{A \subseteq \{1,\ldots,n\} \\ |A| = k}} k = k\binom{n}{k}.$$

Summing over b instead gives

$$\begin{aligned}
|S| &= \sum_{b\in\{1,\ldots,n\}} |\{A\subseteq\{1,\ldots,n\} : b\in A, |A|=k\}| \\
&= \sum_{b\in\{1,\ldots,n\}} \binom{n-1}{k-1} \\
&= \frac{n(n-1)!}{(k-1)!(n-1-(k-1))!} \\
&= \frac{n!}{(k-1)!(n-k)!} = (n-k+1)\binom{n}{k-1}.
\end{aligned}$$

Problem Set 2

3) Let $n = R(s-1,t)+R(s,t-1)-1$. Colour the edges of K_n red or blue arbitrarily. Let x be a vertex. The degree of x is $n-1 = R(s-1,t)+R(s,t-1)-2$. By the proof of Erdős-Szekeres upper bound (Lemma 5.2), if x is incident with $\geq R(s-1,t)$ red edges *or* $\geq R(s,t-1)$ blue edges, then all is well. So, suppose that x is incident with precisely$R(s-1,t)-1$ red edges *and* $R(s,t-1)-1$ blue edges (both these numbers are odd). In fact, we can assume that this holds for all vertices in K_n. Note, n is odd. Consider the subgraph of K_n consisting of just the red edges. The sum of the degrees of this subgraph is odd, as it is the sum of an odd number of odd numbers. This contradicts the handshaking lemma, completing the proof.

4) a) Suppose that ij and jk are red, where $i<j<k$. Then

$$k-i = (k-j)+(j-i) \equiv 2 \mod 3,$$

which shows that edge ik is coloured blue. Next we show that there is no blue K_t.

Induction: $t=3$, K_5 is fine. Assume it's okay for t. Consider K_{3t-1}. To make a blue K_{t+1} without a blue K_t on the vertices $1,\ldots,3t-4$, we must include two new vertices and they must be $3t-1$ and $3t-3$, or else we have a red edge. If i is in the blue $K_{t+!}$, then

$$\begin{aligned}
3t-1-i &\not\equiv 1 \mod 3, \text{ and} \\
3t-3-i &\not\equiv 1 \mod 3.
\end{aligned}$$

Hence the only possibility is $i\equiv 0 \mod 3$. There are at most $t-2$ choices for i, which, together with $3t-1$ and $3t-3$ only give t vertices, not $t+1$.

b) Next apply Question 3 for an upper bound,

$$\begin{aligned}
R(3,4) &\leq R(2,4)+R(3,3)-1 \quad \text{if both } R(2,4) \text{ and } R(3,3) \text{ are even} \\
&= 4+6-1 \\
&= 9.
\end{aligned}$$

5) a) Let $n = R(p_1, R(p_2,\ldots,p_t))$, and consider an arbitrary colouring of the edges of K_n with t distinct colours. Recolour all the edges coloured $2,\ldots,t$ by a new colour, say, 0. If there is a K_{p_1} in this new colouring, then we are done. Otherwise, by choice of n, there is a copy of K_{n_0} coloured 0 where $n_0 = R(p_2,\ldots,p_t)$. Reinstate the original colours on these edges (that is, with colours $2,\ldots,t$). Hence by choice of n_0, there is a K_{p_i} coloured with colour i for at least one $i\in\{2,\ldots,t\}$.

b) When $t = 2$, $R(p_1, p_2)$ is finite, by the Erdős-Szekeres Theorem. Assume that $t \geq 3$ and that k-colour Ramsey numbers are finite for all $k \leq t-1$. Then $R(p_1, p_2, \ldots, p_t) \leq R(p_1, \underbrace{R(p_2, \ldots, p_t)}_{\text{finite, by induction hypothesis}})$ by (a), which is finite by the base case.

6) Write $r(3;t)$ to denote $R(\underbrace{3, 3, \ldots, 3, 3}_{t})$.

a) Let $n = t(r(3;t-1) - 1) + 2$, and colour the edges of K_n with t colours arbitrarily. Let x be any vertex. Then x is incident with $n - 1 = t(r(3;t-1) - 1) + 1$ edges. By the pigeonhole principle, there exists a colour i such that x is incident with at least $r(3;t-1)$ edges coloured i. Let S be a set of $r(3;t-1)$ neighbours of x along edges coloured i. If any edge between two elements of S is coloured i, then we have a triangle (with x) coloured i. Otherwise, the edges of S are coloured with precisely $t-1$ colours. Since S has $r(3;t-1)$ vertices, there must be a monochromatic triangle in S.

8) From lectures, $S(t) \leq r(3;t) - 1 \leq 3t!$.

9) a) From lectures, we need $p_0 \geq S(3) + 1$. Now $S(3) \leq r(3;3) - 1 \leq 16$, by question 7. Therefore we may take $p_0 = 17$

b) By (a), such a p must satisfy $p < 17$. The subgroup $H = \{x^3 : x \in \mathbb{Z}_p^*\}$ has index $\gcd(3, p-1)$ in $\mathbb{Z}_p^*$. Maybe things go wrong if H is *small*, that is, when the index is *large*. Now $\gcd(3, p-1) = 3$, and this holds for $p = 7$, say. In $\mathbb{Z}_7^*$, $\{x^3 : x \in \mathbb{Z}_7^*\} = \{1, 6\}$. Furthermore, $x^3 + y^3 \in \{0, 2, 5\}$ for all $x, y \in \mathbb{Z}_7^*$. These are not equal to z^3 for any $z \in \mathbb{Z}_7^*$. Therefore $x^3 + y^3 = z^3$ has no solution in $\mathbb{Z}_7^*$.

c) Is it true that for all $p < p_0$, p prime, there is no solution to $x^3 + y^3 = z^3$ in Z_p^*? No: for example, $1^3 + 1^3 = 2^3$ in $\mathbb{Z}_3^*$.