

# Combinatorics

## Problem Set 1

- 1) Ignore the condition that the two sets of people are disjoint, because you can just end the people who are in both to make the sets disjoint. The number of non-empty subsets of the 10 people is  $2^{10} - 1 = 1023$ , but the possible set of age sums for non-empty subsets is  $\{1, 2, \dots, 600\}$  (this is a very loose upper bound, because we can assume all ages are distinct — or else just choose singleton sets — but this works so why bother). By the pigeonhole principle, since  $1023 > 1 \times 600$ , there is at least one age sum (pigeonhole) containing more than 1 subset of people (pigeons)... which is what we wanted all along.

With 9 people,  $2^9 - 1 = 511$ . Be a bit smarter and use distinct ages to show that the possible set of age sums for non-empty subsets is  $\{1, 2, \dots, 52 + 53 + \dots + 60\} = \{1, 2, \dots, 504\}$  — it still works.

- 2) Let  $f : \{1, \dots, m\} \rightarrow \{1, \dots, m-1\}$  be defined by  $f(i) = a^i \bmod m$  ( $a^i \not\equiv 0 \bmod m$  cause they're coprime). By pigeonhole, there are distinct  $s, t \in \{1, \dots, m\}$  such that  $f(s) = f(t)$ . Assume  $s > t$ . So  $a^s \equiv a^t \bmod m$ . Since  $\gcd(a, m) = 1$ , this implies that  $a^{s-t} \equiv 1 \bmod m$ .
- 3) “Direct” approach: look for contiguous blocks of stuff — let  $x_i$  be the sum of the games played over the first  $i$  days; then if  $x_j - x_i = 21$  for some  $j > i$  then we're done.

The total number of games over the 77 days can be at most  $11 \times 12 = 132$ . Let  $x_i$  be the number of games played on days  $1, 2, \dots, i$  inclusive, for  $i = 1, \dots, 77$ . We want to use pigeonhole, but it's not immediately able to tell us  $x_j - x_i = 21$  — it's better for getting some kind of equality. So let  $y_i = x_i + 21$  for  $i = 1, \dots, 77$ . But we are most interested in  $y_i$  when  $y_i \leq 132$ , or equivalently,  $x_i \leq 111$ . This will be definitely true for  $i = 1, \dots, 63$ , since  $x_{63} \leq 12 \times 9 = 108$ .

Consider the  $63 + 77 = 140$  values  $\{x_1, \dots, x_{77}, y_1, \dots, y_{63}\}$  which lie in the range  $\{1, \dots, 132\}$ . By the pigeonhole principle, since  $140 > 132$ , there is a value  $v \in \{1, \dots, 132\}$  such that at least two of the elements of  $\{x_1, \dots, x_{77}, y_1, \dots, y_{63}\}$  equal  $v$ . Since  $x_i < x_{i+1}$  for all  $i = 1, \dots, 76$ , the  $x_i$  are all distinct, which also implies that the  $y_i$  are all distinct. It must be that some  $x_j = y_i$  for some  $i, j$ . That is,  $x_j = x_i + 21$ , which is what we wanted all along.

Alt: Use Example 1.2 from lectures, in any 21 days, there is a consecutive subsequence adding up to a multiple of 21, but there are at most  $12 \times 3 = 36$  games, which means it must be 21.

- $6\frac{1}{2}$ ) (a) Define

$$S = \{(A, b) : A \subseteq \{1, \dots, n\}, b \in A\}.$$

Calculating the size of  $S$  by first counting over  $A$  gives

$$|S| = \sum_{\substack{A \subseteq \{1, \dots, n\} \\ |A|=k}} k = k \binom{n}{k}.$$

Summing over  $b$  instead gives

$$\begin{aligned}
|S| &= \sum_{b \in \{1, \dots, n\}} |\{A \subseteq \{1, \dots, n\} : b \in A, |A| = k\}| \\
&= \sum_{b \in \{1, \dots, n\}} \binom{n-1}{k-1} \\
&= \frac{n(n-1)!}{(k-1)!(n-1-(k-1))!} \\
&= \frac{n!}{(k-1)!(n-k)!} = (n-k+1) \binom{n}{k-1}.
\end{aligned}$$

## Problem Set 2

3) Let  $n = R(s-1, t) + R(s, t-1) - 1$ . Colour the edges of  $K_n$  red or blue arbitrarily. Let  $x$  be a vertex. The degree of  $x$  is  $n-1 = R(s-1, t) + R(s, t-1) - 2$ . By the proof of Erdős-Szekeres upper bound (Lemma 5.2), if  $x$  is incident with  $\geq R(s-1, t)$  red edges *or*  $\geq R(s, t-1)$  blue edges, then all is well. So, suppose that  $x$  is incident with precisely  $R(s-1, t) - 1$  red edges *and*  $R(s, t-1) - 1$  blue edges (both these numbers are odd). In fact, we can assume that this holds for all vertices in  $K_n$ . Note,  $n$  is odd. Consider the subgraph of  $K_n$  consisting of just the red edges. The sum of the degrees of this subgraph is odd, as it is the sum of an odd number of odd numbers. This contradicts the handshaking lemma, completing the proof.

4) a) Suppose that  $ij$  and  $jk$  are red, where  $i < j < k$ . Then

$$k - i = (k - j) + (j - i) \equiv 2 \pmod{3},$$

which shows that edge  $ik$  is coloured blue. Next we show that there is no blue  $K_t$ .

Induction:  $t = 3$ ,  $K_5$  is fine. Assume it's okay for  $t$ . Consider  $K_{3t-1}$ . To make a blue  $K_{t+1}$  without a blue  $K_t$  on the vertices  $1, \dots, 3t-4$ , we must include two new vertices and they must be  $3t-1$  and  $3t-3$ , or else we have a red edge. If  $i$  is in the blue  $K_{t+1}$ , then

$$\begin{aligned}
3t-1-i &\not\equiv 1 \pmod{3}, \text{ and} \\
3t-3-i &\not\equiv 1 \pmod{3}.
\end{aligned}$$

Hence the only possibility is  $i \equiv 0 \pmod{3}$ . There are at most  $t-2$  choices for  $i$ , which, together with  $3t-1$  and  $3t-3$  only give  $t$  vertices, not  $t+1$ .

b) Next apply Question 3 for an upper bound,

$$\begin{aligned}
R(3, 4) &\leq R(2, 4) + R(3, 3) - 1 \quad \text{if both } R(2, 4) \text{ and } R(3, 3) \text{ are even} \\
&= 4 + 6 - 1 \\
&= 9.
\end{aligned}$$

5) a) Let  $n = R(p_1, R(p_2, \dots, p_t))$ , and consider an arbitrary colouring of the edges of  $K_n$  with  $t$  distinct colours. Recolour all the edges coloured  $2, \dots, t$  by a new colour, say, 0. If there is a  $K_{p_1}$  in this new colouring, then we are done. Otherwise, by choice of  $n$ , there is a copy of  $K_{n_0}$  coloured 0 where  $n_0 = R(p_2, \dots, p_t)$ . Reinststate the original colours on these edges (that is, with colours  $2, \dots, t$ ). Hence by choice of  $n_0$ , there is a  $K_{p_i}$  coloured with colour  $i$  for at least one  $i \in \{2, \dots, t\}$ .

- b) When  $t = 2$ ,  $R(p_1, p_2)$  is finite, by the Erdős-Szekeres Theorem. Assume that  $t \geq 3$  and that  $k$ -colour Ramsey numbers are finite for all  $k \leq t - 1$ . Then  $R(p_1, p_2, \dots, p_t) \leq R(p_1, \underbrace{R(p_2, \dots, p_t)}_{\text{finite, by induction hypothesis}})$  by (a), which is finite by the base case.

6) Write  $r(3; t)$  to denote  $R(\underbrace{3, 3, \dots, 3}_t)$ .

- a) Let  $n = t(r(3; t - 1) - 1) + 2$ , and colour the edges of  $K_n$  with  $t$  colours arbitrarily. Let  $x$  be any vertex. Then  $x$  is incident with  $n - 1 = t(r(3; t - 1) - 1) + 1$  edges. By the pigeonhole principle, there exists a colour  $i$  such that  $x$  is incident with at least  $r(3; t - 1)$  edges coloured  $i$ . Let  $S$  be a set of  $r(3; t - 1)$  neighbours of  $x$  along edges coloured  $i$ . If any edge between two elements of  $S$  is coloured  $i$ , then we have a triangle (with  $x$ ) coloured  $i$ . Otherwise, the edges of  $S$  are coloured with precisely  $t - 1$  colours. Since  $S$  has  $r(3; t - 1)$  vertices, there must be a monochromatic triangle in  $S$ .

8) From lectures,  $S(t) \leq r(3; t) - 1 \leq 3t!$ .

- 9) a) From lectures, we need  $p_0 \geq S(3) + 1$ . Now  $S(3) \leq r(3; 3) - 1 \leq 16$ , by question 7. Therefore we may take  $p_0 = 17$ .
- b) By (a), such a  $p$  must satisfy  $p < 17$ . The subgroup  $H = \{x^3 : x \in \mathbb{Z}_p^*\}$  has index  $\gcd(3, p - 1)$  in  $\mathbb{Z}_p^*$ . Maybe things go wrong if  $H$  is *small*, that is, when the index is *large*. Now  $\gcd(3, p - 1) = 3$ , and this holds for  $p = 7$ , say. In  $\mathbb{Z}_7^*$ ,  $\{x^3 : x \in \mathbb{Z}_7^*\} = \{1, 6\}$ . Furthermore,  $x^3 + y^3 \in \{0, 2, 5\}$  for all  $x, y \in \mathbb{Z}_7^*$ . These are not equal to  $z^3$  for any  $z \in \mathbb{Z}_7^*$ . Therefore  $x^3 + y^3 = z^3$  has no solution in  $\mathbb{Z}_7^*$ .
- c) Is it true that for all  $p < p_0$ ,  $p$  prime, there is no solution to  $x^3 + y^3 = z^3$  in  $\mathbb{Z}_p^*$ ? No: for example,  $1^3 + 1^3 = 2^3$  in  $\mathbb{Z}_3^*$ .

## Problem Set 3

- 2) Let  $A_1$  be all orderings which contain ABLE;  $A_2$ , HYPNOTIC;  $A_3$ , RUG. We want  $|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}|$ , which we calculate using inclusion-exclusion. For  $A_1$ , we have 12 symbols

$$ABLE, C, G, H, I, N, O, P, R, T, U, Y$$

to permute, so  $|A_1| = 12!$ . Similarly,  $|A_2| = 8!$  and  $|A_3| = 13!$ . Next,  $A_1 \cap A_2$  consists of all words which are permutations of the 5 symbols

$$ABLE, HYPNOTIC, R, U, G,$$

so  $|A_1 \cap A_2| = 5!$ . Similarly,  $|A_1 \cap A_3| = 10!$  and  $|A_2 \cap A_3| = 6!$ . Finally,  $|A_1 \cap A_2 \cap A_3| = 3!$ . There are  $15!$  total orderings of the 15 letters in total, so

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| = 15! - (12! + 8! + 13!) + (5! + 10! + 6!) - 3!.$$

(Leave answer as is in exams and stuff.)

- 3) Let  $Q_n = \#$  permutations  $\sigma \in S_n$  such that  $\sigma(i + 1) \neq \sigma(i) + 1$  for  $i = 1, \dots, n - 1$ .

a)  $Q_1$  obvious,  $Q_2$  we can only have  $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ ,  $Q_3$

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}.$$

b) Let  $S = S_n$ , the set of all  $n!$  permutations of  $\{1, \dots, n\}$ . Define  $A_i = \{\sigma \in S_n : \sigma(j) = i, \sigma(j+1) = i+1 \text{ for some } j \in \{1, \dots, n-1\}\}$ , for  $i = 1, \dots, n-1$ . Then  $Q_n = |\overline{A_1} \cap \dots \cap \overline{A_{n-1}}|$ ; use inclusion-exclusion. For  $A_1$ , we are permuting symbols

$$12, 3, 4, \dots, n,$$

so  $|A_1| = (n-1)!$ . Similarly  $|A_j| = (n-1)!$  for all  $j = 1, \dots, n-1$ .

Next, consider  $A_1 \cap A_2$ . Here we permute the symbols

$$123, 4, \dots, n,$$

and hence  $|A_1 \cap A_2| = (n-2)!$ . What about  $A_1 \cap A_3$ ? Here we permute

$$12, 34, 5, \dots, n,$$

but it's still  $(n-2)!$ . Hence  $|A_{i_1} \cap A_{i_2}| = (n-2)!$  for all  $i_1 < i_2$ . It can be shown by induction that

$$|A_{i_1} \cap \dots \cap A_{i_k}| = (n-k)!$$

for all  $1 \leq i_1 < \dots < i_k \leq n-1$ . Since there are  $\binom{n-1}{k}$  such intersections, for  $k = 1, \dots, n-1$ , the result follows by inclusion-exclusion.

c) We have

$$Q_n = (n-1)! \sum_{k=0}^{n-1} (-1)^k \frac{n-k}{k!}.$$

From lectures,

$$D_n = n! \sum_{k=0}^n \frac{(-1)^k}{k!},$$

so

$$\begin{aligned}
D_n + D_{n+1} &= n! \left( \sum_{k=0}^n \frac{(-1)^k}{k!} \right) + (n-1)! \left( \sum_{i=0}^{n-1} \frac{(-1)^i}{i!} \right) \\
&= n! + n! \left( \sum_{i=0}^{n-1} \frac{(-1)^{i+1}}{(i+1)!} \right) + (n-1)! \left( \sum_{i=0}^{n-1} \frac{(-1)^i}{i!} \right) \\
&= n! + \sum_{i=0}^{n-1} \left( (-1)^{i+1} \frac{n!}{(i+1)!} + \frac{(-1)^i (n-1)!}{i!} \right) \\
&= n! + \sum_{i=0}^{n-1} \frac{(-1)^i (n-1)!}{(i+1)!} (-n+i+1) \\
&= n! + \sum_{i=0}^{n-1} \frac{(-1)^{i+1} (n-1)!}{(i+1)!} (n-i-1) \\
&= n! + \sum_{i=0}^{n-2} \frac{(-1)^{i+1} (n-1)!}{(i+1)!} (n-(i+1)) \\
&= n! + \sum_{j=1}^{n-1} \frac{(-1)^j (n-1)!}{j!} (n-j) \\
&= (n-1)! \sum_{j=0}^{n-1} (-1)^j \frac{n-j}{j!} \\
&= Q_n.
\end{aligned}$$

- 4) Write  $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ , where  $p_1 < \cdots < p_r$  are prime and  $\alpha_i \in \mathbb{Z}^+$ . Let  $A_i = \{k \in \{1, \dots, n\} : p_i \mid k\}$  for  $i = 1, \dots, r$ . Then  $\phi(n) = |\overline{A_1} \cap \cdots \cap \overline{A_r}|$ , calculate using inclusion-exclusion. Now  $|A_i| = \frac{n}{p_i}$ ; similarly,  $|A_{i_1} \cap A_{i_2}| = \frac{n}{p_{i_1} p_{i_2}}$  for  $i_1 < i_2$ , and in general,

$$|A_{i_1} \cap \cdots \cap A_{i_k}| = \frac{n}{p_{i_1} \cdots p_{i_k}}.$$

Hence

$$\begin{aligned}
\phi(n) &= |\overline{A_1} \cap \cdots \cap \overline{A_r}| = n - \left( \frac{n}{p_1} + \cdots + \frac{n}{p_r} \right) \\
&\quad + \left( \frac{n}{p_1 p_2} + \cdots + \frac{n}{p_{r-1} p_r} \right) \\
&\quad + \cdots + (-1)^r \frac{n}{p_1 \cdots p_r}.
\end{aligned}$$

Can check that this is equal to

$$\begin{aligned}
\phi(n) &= n \prod_{i=1}^r \left( 1 - \frac{1}{p_i} \right) \\
&= \cdots = n \prod_{\substack{p \mid n, \\ p \text{ prime}}} \left( 1 - \frac{1}{p} \right).
\end{aligned}$$

5)  $\Pi_4 = \{(4), (3, 1), (2, 2), (2, 1, 1), (1, 1, 1, 1)\}$ .

a) For all  $\lambda_1 \in \Pi_4$ , we can obtain  $\lambda_1$  from itself by partitioning 0 parts; hence  $\leq$  is reflexive.

Suppose that  $\lambda_1 \leq \lambda_2$  and  $\lambda_2 \leq \lambda_1$ , for  $\lambda_i \in \Pi_4$ . Then  $\lambda_1$  has at least as many parts as  $\lambda_2$ , since  $\lambda_1 \leq \lambda_2$ . But also  $\lambda_2$  has at least as many parts as  $\lambda_1$  as well. Hence  $\lambda_1$  and  $\lambda_2$  have the same number of parts, and one is a refinement of the other. Hence  $\lambda_1 = \lambda_2$ , so  $\leq$  is antisymmetric.

Suppose that  $\lambda_1 \leq \lambda_2$  and  $\lambda_2 \leq \lambda_3$ , for  $\lambda_i \in \Pi_4$ . Then  $\lambda_1$  is a refinement of a refinement of  $\lambda_3$ , so it's a refinement of  $\lambda_3$ . Thus  $\leq$  is transitive. Hasse diagram has  $(4); (3, 1), (2, 2); (2, 1, 1); (1, 1, 1, 1)$ .

b) There are two:

$$(1, 1, 1, 1) \leq^* (2, 1, 1) \leq^* (2, 2) \leq^* (3, 1) \leq^* (4),$$

or

$$(1, 1, 1, 1) \leq^* (2, 1, 1) \leq^* (3, 1) \leq^* (2, 2) \leq^* (4).$$

c) Choose the second. We have

$$A_\zeta = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(Here the rows and columns are indexed according to the linear extension we chose.)

d) Invert  $A_\zeta$  to find  $\mu$ , we get

$$A_\mu = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Hence

$$\mu(\lambda_1, \lambda_2) = \begin{cases} 1 & \text{if } \lambda_1 = \lambda_2 \\ 0 & \text{if } \lambda_1 \not\leq \lambda_2 \\ -1 & \text{if } \lambda_1 = (1, 1, 1, 1) \text{ and } \lambda_2 = (2, 1, 1), \text{ or} \\ & \lambda_1 = (2, 1, 1) \text{ and } \lambda_2 = (3, 1), \text{ or} \\ & \lambda_1 = (3, 1) \text{ and } \lambda_2 = (4), \text{ or} \\ & \lambda_1 = (2, 2) \text{ and } \lambda_2 = (4). \end{cases}$$

e) By Möbius inversion,

$$F(\lambda_2) = \sum_{\lambda_1 \leq \lambda_2} \mu(\lambda_1, \lambda_2) G(\lambda_1).$$

So

$$\begin{aligned} F(1, 1, 1, 1) &= G(1, 1, 1, 1) = 4, \\ F(2, 1, 1) &= G(1, 1, 1, 1) \cdot -1 + G(2, 1, 1) \cdot 1 \\ &= -4 + 3 \\ &= -1. \end{aligned}$$

Similarly,

$$\begin{aligned} F(3, 1) &= G(1, 1, 1, 1) \cdot 0 + G(2, 1, 1) \cdot -1 + G(3, 1) \cdot 1 \\ &= 2 - 3 \\ &= -1, \end{aligned}$$

$$F(2, 2) = -1,$$

$$\begin{aligned} F(4) &= G(1, 1, 1, 1) \cdot 0 + G(2, 1, 1) \cdot 0 + G(3, 1) \cdot -1 + G(2, 2) \cdot -1 + G(4) \cdot 1 \\ &= -2 - 2 + 1 \\ &= -3. \end{aligned}$$

Alternatively,  $(4 \ 3 \ 2 \ 2 \ 1) = F(A_\zeta)$ , so

$$\begin{aligned} F &= (4 \ 3 \ 2 \ 2 \ 1)A_\mu \\ &= (4 \ 3 \ 2 \ 2 \ 1) \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \\ &= (4 \ -1 \ -1 \ -1 \ -3). \end{aligned}$$

This is wrong.

6) Let  $N_1, \leq$  be the linear order on  $N = \{1, \dots, n\}$ .

a) We have

$$A_\zeta = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 0 & \ddots & 1 & 1 \\ 0 & \dots & \ddots & 1 \\ 0 & \dots & 0 & 1 \end{pmatrix}.$$

We get that  $A_\mu$  is the matrix with 1 on the diagonal,  $-1$  on the entries above the diagonal, 0 elsewhere.

b) By Möbius inversion,

$$\begin{aligned} F(m) &= \sum_{k=1}^m \mu(k, m)G(k) \\ &= \begin{cases} G(1) & \text{if } m = 1, \\ G(m) - G(m-1) & \text{if } m \geq 2. \end{cases} \end{aligned}$$

7) If  $a = b$  then  $\mu(a, b) = \mu(a, a) = 1 = \mu(1, \frac{b}{a})$  by properties of a Möbius function. Now suppose that  $b \neq a, a \mid b$ . By induction, assume that  $\mu(a, b) = \mu(1, \frac{c}{a})$  for all  $c$  where  $a \mid c, c \mid b, c \neq b$ . Then

$$\begin{aligned} \mu(a, b) &= - \sum_{\substack{c \in \mathbb{N}, \\ a \mid c, c \mid b, c \neq b}} \mu(a, c) \\ &= - \sum_{\substack{c \in \mathbb{N}, \\ a \mid c, c \mid b, c \neq b}} \mu\left(1, \frac{c}{a}\right) \quad \text{by inductive hypothesis.} \end{aligned}$$

In the above sum, the values of  $c$  range over  $c = ak$  where  $1 \mid k, k \mid \frac{b}{a}, k \neq \frac{b}{a}$ . So

$$\begin{aligned}\mu(a, b) &= - \sum_{\substack{k \in \{1, \dots, \frac{b}{a}\}, \\ 1 \mid k, k \mid \frac{b}{a}, k \neq \frac{b}{a}}} \mu(1, k) \\ &= \mu\left(1, \frac{b}{a}\right),\end{aligned}$$

by recursive definition of  $\mu$  for divisibility poset on  $\{1, \dots, \frac{b}{a}\}$ .

- 8) a)  $k \in B_n^d$  iff  $\gcd(k, n) = d$  iff  $\gcd(\frac{k}{d}, \frac{n}{d}) = 1$  for  $k, n$  multiples of  $d$ . There are  $\phi(\frac{n}{d})$  integers  $i \in \{1, \dots, \frac{n}{d}\}$  which are coprime to  $\frac{n}{d}$ , and setting  $k = di$  shows that  $|B_n^d| = \phi(\frac{n}{d})$ .

b) We have

$$n = |\{1, \dots, n\}| = \sum_{d \mid n} |B_n^d|,$$

as the  $B_n^d$  are disjoint and  $|B_n^d| = 0$  if  $d \nmid n$ . So

$$n = \sum_{d \mid n} \phi\left(\frac{n}{d}\right) = \sum_{d \mid n} \phi(d) = G(n).$$

c) Using Möbius inversion, it follows that

$$\begin{aligned}\phi(n) &= \sum_{d \mid n} \mu(d, n) G(d) \\ &= \sum_{d \mid n} \mu\left(1, \frac{n}{d}\right) G(d) \quad \text{by Q7} \\ &= \sum_{d \mid n} \mu\left(\frac{n}{d}\right) d \\ &= \sum_{d \mid n} \mu(d) \frac{n}{d}.\end{aligned}$$

d) Recall

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^k & \text{if } n \text{ is the product of } k \text{ distinct primes,} \\ 0 & \text{otherwise.} \end{cases}$$

Write  $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ , where  $p_1 < p_2 < \cdots < p_r$  are distinct primes, and  $\alpha_i \in \mathbb{Z}^+$ . Then  $\mu(d) = 0$  if  $d \neq 1$ , unless  $d = p_1^{\beta_1} \cdots p_r^{\beta_r}$  where  $\beta_i \in \{0, 1\}$ . Hence

$$\begin{aligned}\phi(n) &= n - \left(\frac{n}{p_1} + \frac{n}{p_2} + \cdots + \frac{n}{p_r}\right) + (-1)^2 \left(\frac{n}{p_1 p_2} + \cdots + \frac{n}{p_{r-1} p_r}\right) \\ &\quad + \cdots + (-1)^r \frac{n}{p_1 p_2 \cdots p_r} \\ &= n \prod_{i=1}^r \left(1 - \frac{1}{p_i}\right) \quad \text{as in Q4} \\ &= n \prod_{p \mid n} \left(1 - \frac{1}{p}\right).\end{aligned}$$



## Problem Set 4

- 2) Let  $C \xleftrightarrow{ogf} (c_n)$ . Now  $c_0 = 0, c_1 = 10, c_2 = 90$ . More generally,  $c_n = 9 \cdot 10^{n-1}$  for  $n \geq 2$ , so  $c_n = 10c_{n-1}$  for  $n \geq 3$ . Hence

$$\sum_{n=3}^{\infty} c_n x^n = 10 \sum_{n=3}^{\infty} 1 - c_{n-1} x^n.$$

So

$$C(x) - (c_0 + c_1 x + c_2 x^2) = 10x \sum_{n=3}^{\infty} c_{n-1} x^{n-1} = 10x(C(x) - (c_0 + c_1 x)).$$

That is,

$$C(x) - (10x + 90x^2) = 10xC(x) - 100x^2.$$

Rearranging gives

$$(1 - 10x)C(x) = 10x(1 - x),$$

so

$$C(x) = \frac{10x(1-x)}{1-10x}.$$

- 3) Let  $F \xleftrightarrow{ogf} (F_n)_{n=0}^{\infty}$ . The ogf for the LHS is  $\frac{F(x)}{1-x}$ , as it is a partial sum. The ogf for the RHS is

$$\begin{aligned} \sum_{n=0}^{\infty} F_{n+2} x^n - \sum_{n=0}^{\infty} x^n &= \frac{1}{x^2} \left( \sum_{n=0}^{\infty} F_{n+2} x^{n+2} \right) - \frac{1}{1-x} \\ &= \frac{F(x) - x}{x^2} - \frac{1}{1-x}. \end{aligned}$$

Now, these two ogfs are equal if and only if (cross multiplying):

$$\begin{aligned} x^2 F(x) + x^2 &= (1-x)F(x) - x + x^2 \\ \iff (1-x-x^2)F(x) &= x, \\ \iff F(x) &= \frac{x}{1-x-x^2}. \end{aligned}$$

But this is the ogf of the Fibonacci numbers. Hence the Fibonacci numbers satisfy the given recurrence.

- 4) We have

$$\begin{aligned} B(x) &= \sum_{n=0}^{\infty} b_n x^n \\ &= \sum_{n=0}^{\infty} a_{2n+3} x^n \\ &= \frac{1}{x^{3/2}} \sum_{n=0}^{\infty} a_{2n+3} (\sqrt{x})^{2n+3} \\ &= \frac{1}{x^{3/2}} \left( \sum_{s=0}^{\infty} a_{2s+1} (\sqrt{x})^{2s+1} - a_1 \sqrt{x} \right) \\ &= \frac{A(\sqrt{x}) - A(-\sqrt{x}) - 2a_1 \sqrt{x}}{2x^{3/2}}. \end{aligned}$$

5) (a) Here  $k$  is the number of parts. We have  $p_n = [x^n]G(x)$ , and we want to write this as

$$\begin{aligned} p_n &= [x^n] \sum_{k=1}^{\infty} y(x)^k \\ &= \sum_{k=1}^{\infty} \underbrace{[x^n] y(x)^k}_{\# \text{ of ordered partitions of } n \text{ with } k \text{ parts}}. \end{aligned}$$

So  $y(x)$  should be the ogf for the number of ordered partitions of  $n$  with 1 parts, i.e.

$$y(x) = x + x^2 + \dots = \frac{x}{1-x} \xleftrightarrow{\text{ogf}} (1)_{n=1}^{\infty}.$$

So

$$G(x) = \sum_{k=1}^{\infty} \left( \frac{x}{1-x} \right)^k.$$

Note:  $[x^n](x + x^2 + \dots)^k$  gives all terms  $x^{z_1} x^{z_2} \dots x^{z_k}$ . Also, no relabelling  $\leadsto$  use ogf.

(b) From (a),

$$A(x) = \sum_{k=1}^{\infty} \left( \frac{x}{1-x} \right)^{2k},$$

and

$$B(x) = \sum_{k=1}^{\infty} \left( \frac{x}{1-x} \right)^{2k-1}.$$

(c) From (b),

$$\begin{aligned} A(x) - B(x) &= \sum_{k=1}^{\infty} \left( \frac{x}{1-x} - 1 \right) \left( \frac{x}{1-x} \right)^{2k-1} \\ &= \sum_{k=1}^{\infty} \frac{2x-1}{1-x} \cdot \frac{1-x}{x} \left( \frac{x^2}{(1-x)^2} \right)^k \\ &= \frac{2x-1}{x} \left( \frac{1}{1 - \frac{x^2}{(1-x)^2}} - 1 \right) \\ &= \frac{2x-1}{x} \left( \frac{(1-x)^2}{1-2x} - 1 \right) \\ &= -\frac{(1-x)^2}{x} - \frac{2x-1}{x} \\ &= -\frac{1-2x+x^2+2x-1}{x} \\ &= -x. \end{aligned}$$

Hence

$$\begin{aligned} a_n - b_n &= [x^n](A(x) - B(x)) \\ &= [x^n](-x) \\ &= \begin{cases} -1 & \text{if } n = 1, \\ 0 & \text{if } n \geq 2. \end{cases} \end{aligned}$$

6) (a) First calculate

$$\begin{aligned}
[x^n](1-4x)^{\frac{1}{2}} &= [x^n] \sum_{s=0}^{\infty} \binom{\frac{1}{2}}{s} (-4x)^s \\
&= \binom{\frac{1}{2}}{n} (-4)^n \\
&= \frac{\left(-\frac{1}{2}\right) \left(-\frac{3}{2}\right) \cdots \left(\frac{-(2n-3)}{2}\right) (-4)^n}{n!} \\
&= (-1)^{2n-1} \frac{(2n-3)(2n-5) \cdots 3 \cdot 1}{2^n \cdot n!} 4^n \\
&= -\frac{2^n (2n-3)(2n-5) \cdots 3 \cdot 1}{n!} \\
&= -2 \frac{(2n-2)!}{n!(n-1)!} \quad (\text{use the } 2^n \text{ to fill in the gaps, fixing stuff up in the denominator}) \\
&= -\frac{2}{n} \binom{2n-2}{n-1}.
\end{aligned}$$

Therefore

$$[x^n](1-\sqrt{1-4x}) = \begin{cases} 1-1=0 & \text{if } n=0, \\ \frac{2}{n} \binom{2n-2}{n-1} & \text{if } n \geq 1. \end{cases}$$

(b) From (a), we have

$$1 - \sqrt{1-4x} = \sum_{n=1}^{\infty} \frac{2}{n} \binom{2n-2}{n-1} x^n,$$

so

$$\begin{aligned}
\frac{1}{2x}(1-\sqrt{1-4x}) &= \sum_{n=1}^{\infty} \frac{1}{n} \binom{2n-2}{n-1} x^{n-1} \\
&= \sum_{s=0}^{\infty} \frac{1}{s+1} \binom{2s}{s} x^s,
\end{aligned}$$

as required.

(c) We multiply by  $x^n$  and sum. Let

$$f_n = \sum_k \binom{n+k}{m+2k} \binom{2k}{k} \frac{(-1)^k}{k+1},$$

for  $n \in \mathbb{N}$ . Define  $F \xleftrightarrow{ogf} (f_n)_{n=0}^\infty$ . Then

$$\begin{aligned}
F(x) &= \sum_{n=0}^{\infty} x^n \sum_k \binom{n+k}{m+2k} \binom{2k}{k} \frac{(-1)^k}{k+1} \\
&= \sum_k \binom{2k}{k} \frac{(-1)^k}{k+1} x^k \sum_{n=0}^{\infty} x^{n+k} \binom{n+k}{m+2k} \\
&= \sum_k \binom{2k}{k} \frac{(-1)^k}{k+1} x^{-k} \frac{x^{m+2k}}{(1-x)^{m+2k+1}} \\
&= \frac{x^m}{(1-x)^{m-1}} \sum_k \frac{1}{k+1} \binom{2k}{k} \left( \frac{-x}{(1-x)^2} \right)^k \\
&= \frac{x^m}{(1-x)^{m+1}} \cdot \frac{(1-x)^2}{-2x} \left( 1 - \sqrt{1 + \frac{4x}{(1-x)^2}} \right) \\
&= -\frac{x^{m-1}}{2(1-x)^{m-1}} \left( 1 - \sqrt{\frac{1+2x+x^2}{(1-x)^2}} \right) \\
&= -\frac{x^{m-1}}{2(1-x)^{m-1}} \left( 1 - \frac{1+x}{1-x} \right) \\
&= -\frac{x^{m-1}(-2x)}{2(1-x)^{m-1}(1-x)} \\
&= \frac{x^m}{(1-x)^m}.
\end{aligned}$$

Therefore

$$\begin{aligned}
f_n &= [x^n] \frac{x^m}{(1-x)^m} \\
&= [x^{n-1}] \frac{x^{m-1}}{(1-x)^m} \\
&= \binom{n-1}{m-1}.
\end{aligned}$$

We have proved

$$\sum_k \binom{n+k}{m+2k} \binom{2k}{k} \frac{(-1)^k}{k+1} = \binom{n-1}{m-1}.$$

## Problem Set 5

- 2) Suppose there are  $n_i$  occurrences of  $i$  for all  $i \in \{\alpha, \beta, \gamma, \delta, \varepsilon\}$ . Without caring for restrictions, this gives

$$\begin{aligned}
&\binom{n}{n_\alpha} \binom{n-n_\alpha}{n_\beta} \binom{n-n_\alpha-n_\beta}{n_\gamma} \binom{n-n_\alpha-n_\beta-n_\gamma}{n_\delta} \binom{n-n_\alpha-n_\beta-n_\gamma-n_\delta}{n_\varepsilon} \\
&= \frac{n!}{n_\alpha! n_\beta! n_\gamma! n_\delta! n_\varepsilon!} \\
&= \binom{n}{n_\alpha, n_\beta, n_\gamma, n_\delta, n_\varepsilon} \quad (\text{multinomial coefficient}).
\end{aligned}$$

(Just write out factorials.) Let  $S(x) \xleftrightarrow{egf} (s_n)$ . Then

$$S(x) = \sum_{n \geq 0} \sum_{\text{those given conditions on } n_i} \binom{n}{n_\alpha, n_\beta, n_\gamma, n_\delta, n_\varepsilon}.$$

This is a product of 5 exponential generating functions. Let  $A, B, C, D, E$  be the exponential generating functions for the legal values of  $n_\alpha, n_\beta, n_\gamma, n_\delta, n_\varepsilon$  respectively. Then,

$$\begin{aligned} A(x) &= \frac{1}{2}(e^x + e^{-x}), \\ B(x) &= \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} \xleftrightarrow{egf} (0, 0, 0, 1, 1, 1, 0, \dots), \\ C(x) &= e^x \xleftrightarrow{egf} (1)_{n \geq 0}, \\ D(x) &= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!}, \\ E(x) &= e^x - \left(1 + x + \frac{x^2}{2} + \frac{x^3}{3}\right). \end{aligned}$$

Therefore  $S(x) = A(x)B(x)C(x)D(x)E(x)$ .

3) (a) Consider the deck

$$(1), (1 \ 2 \ 3), \dots, (1 \ 2 \ \dots \ 2n+1).$$

The deck enumerator is

$$D(x) = \frac{1}{2}(e^x - e^{-x}) = \sinh(x).$$

Therefore  $(b_n) \xleftrightarrow{egf} e^{D(x)}$  by the exponential formula.

(b) For  $n \geq 1$ , we have

$$nb_n = \sum_k \binom{n}{k} k d_k b_{n-k}$$

where

$$d_k = \begin{cases} 0 & \text{if } k \text{ even,} \\ 1 & \text{if } k \text{ odd.} \end{cases}$$

That is,

$$nb_n = \sum_{s=0}^{\infty} \binom{n}{2s+1} (2s+1) b_{n-2s-1}$$

for  $n \geq 1$ .

4) Suppose  $H(x) = e^{D(x)}$ ,  $D(x) \xleftrightarrow{egf} (a_n)_{n=0}^{\infty}$ ,  $H(x) \xleftrightarrow{egf} (h_n)_{n=0}^{\infty}$ , where  $d_0 = 0$ .

(a) We have  $h_0 = H(0) = e^{D(0)} = e^{d_0} = e^0 = 1$ .

(b) Recall

$$nh_n = \sum_k \binom{n}{k} k a_k h_{n-k}, \tag{1}$$

where  $n \geq 1$ . Suppose that  $(h_n)_{n \geq 0}$  is known and we want to find  $(a_n)_{n \geq 0}$ . Rearranging (1) gives

$$na_n = nh_n - \sum_{k=1}^{n-1} \binom{n}{k} ka_k h_{n-k},$$

for  $n \geq 1$ . From  $h_1$  we calculate  $a_1$  ( $a_1 = h_1$ ). From  $h_1, h_2$  and  $a_1$  we calculate  $a_2$ . From  $h_1, \dots, h_n, a_1, \dots, a_{n-1}$  we calculate  $a_n$ .

- 5) (a) The LHS is the number of permutations of  $\{1, \dots, n\}$ . The summand on the RHS is zero unless  $k \in \{0, \dots, n\}$ . For such  $k$ , the expression  $\binom{n}{k} d_{n-k}$  counts the number of permutations of  $\{1, \dots, n\}$  with  $k$  fixed points — choose the  $k$  fixed points with  $\binom{n}{k}$ , then count the number of ways to complete the permutation without any other fixed points with  $d_{n-k}$ . Summing over  $k = 0, \dots, n$  counts each permutation of  $\{1, \dots, n\}$  exactly once.
- (b) Multiply (a) by  $\frac{x^n}{n!}$  and sum, giving

$$\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} \sum_k \binom{n}{k} d_{n-k} \frac{x^n}{n!}.$$

Now,

$$\text{LHS} = \frac{1}{1-x},$$

and the RHS is the exponential generating function of

$$\sum_k \binom{n}{k} 1 \cdot d_{n-k},$$

which is  $e^x F(x)$  (see Proposition 16.1).

- (c) From the generating function, we have

$$e^{-x} = (1-x)F(x).$$

Differentiating gives

$$-e^{-x} = (1-x)F'(x) - F(x),$$

so

$$F'(x) = xF'(x) + F(x) - e^{-x}.$$

The LHS is the exponential generating function for  $(d_{n+1})_{n=0}^{\infty}$ . The RHS is the exponential generating function for  $(nd_n + d_n + (-1)^{n+1})_{n=0}^{\infty}$  (see Lemma 16.1). Equating coefficients of  $\frac{x^n}{n!}$  gives the result.

- (d) Let

$$\begin{aligned} f_n &= \text{number of permutations of } \{1, \dots, n\} \text{ with exactly one fixed point} \\ &= nd_{n-1}. \end{aligned}$$

Part (c) gives

$$f_n - d_n = nd_{n-1} - d_n = -(-1)^{(n-1)+1} = (-1)^{n-1} \in \{-1, 1\}.$$

Also  $f_n - d_n = 1$  iff  $n$  is odd.

- 6) (a) There are  $\binom{n}{2}$  possible edges, for each one decide whether we want it in or not.  
 (b) A hand of weight  $n$  is a graph with  $n$  vertices. Every graph on  $n$  vertices can be found in this way, from a unique hand. Hence  $h_n = 2^{\binom{n}{2}}$ . So

$$H(x) = \sum 2^{\binom{n}{2}} \frac{x^n}{n!},$$

not useful analytically.

- (c) From Q4,

$$\begin{aligned} ng_n &= nh_n - \sum_{k=1}^{n-1} kg_k h_{n-k} \\ &= n2^{\binom{n}{2}} - \sum_{k=1}^{n-1} \binom{n}{k} kg_k 2^{\binom{n-k}{2}}. \end{aligned}$$

We can use this to recursively calculate  $g_n$  for small  $n$ .