# 1 Commutative Algebra

- An ideal I is prime if  $fg \in I$  means  $f \in I$  or  $g \in I$
- An element p is prime if p|fg means p|f or p|g
- An element p is irreducible if it cannot be factored into two non-invertible elements
- In a UFD, irreducible and prime are equivalent
- A polynomial ring over a UFD is a UFD
- A ring is Noetherian if every ascending chain of ideals is stationary, or equivalently every ideal is finitely generated
- A polynomial ring over a Noetherian ring is Noetherian
- prime ideals are radical
- quotient of a prime ideal is an integral domain

### 2 Topology

- Zariski topology is where closed sets are zero loci
- An irreducible set is not a proper union of closed subsets
- continuous images of irreducible sets are irreducible
- products of irreducible sets are irreducible
- A Noetherian space is where every decreasing chain of closed subsets is stationary
- In a Noetherian space decomposition into irreducible components is unique
- The dimension of an irreducible space is the length of the longest chain of nonempty irreducible closed subsets minus 1; the dimension of any space is the maximum dimension of an irreducible component

#### 3 Abstract Nonsense

- A presheaf of rings is an assignment of rings to open sets, as well as a restriction homomorphism, in such a way that restriction is transitive and reflexive, and the empty set is assigned to the trivial ring.
- A sheaf is a presheaf with the property that a ring element on the global ring can be defined by its restriction to the sets of an open cover
- The stalk of a presheaf is the set of all ring elements of all neighborhoods of P, identifying elements if they agree on some neighborhood of P.
- germs are the elements of a stalk
- the stalk of  $O_X$  at P is  $O_{X,P}$
- If  $f: X \to Y$  is a function, the pull-back  $f^*\phi$  of a regular function  $\phi$  on Y is  $\phi \circ f$
- A function is a morphism if it pulls back regular functions to regular functions
- A morphism of sheaves F, G is collection of morphisms  $f_U : F(U) \to G(U)$  that commute with restriction maps

### 4 Affine Varieties

- Affine variety corresponds to prime ideal
- The coordinate ring  $A(X) = k [x_i]_i / I(X)$  is the "polynomials" on X
- The rational functions are the fraction field of A(X)
- The regular functions at P are those rational functions that can be evaluated at P; alternatively they are the functions that have a representation as a quotient of polynomials in  $k[x_i]_i$  on some neighborhood of P.
- The regular functions on an open set U are the functions that are regular at each  $P \in U$ .
- $O(X_f) = A(X)_f := \{g/f^r\} \subseteq K(X)$
- A product of affine varieties is an affine variety
- An abstract affine variety is an irreducible space and a sheaf of k-valued functions that is isomorphic to a concrete affine variety
- Distinguished open subsets are abstract affine varieties
- Not all open subsets of varieties are varieties, consider  $\mathbb{C}^2 \setminus \{(0,0)\}$
- morphisms can be checked on open sets, germs or global sets
- morphisms of affine varieties f correspond to k-algebra homomorphisms  $f^*$

### 5 Varieties

- A prevariety is an irreducible set with a sheaf of functions that has a finite cover of affine varieties
- Can create a prevariety by gluing two prevarieties along a common open subset: Let f be the isomorphism between open subsets  $U_1, U_2$  of  $X_1, X_2$  and  $i_1, i_2$  be the inclusions into X. The topology is the quotient topology and the sheaf of functions is pairs  $(\phi_1, \phi_2) \in O_{X_1}\left(i_1^{-1}(U)\right) \times O_{X_2}\left(i_2^{-1}(U)\right)$  that agree on overlaps
- Same thing works with finite collection of prevarieties, with each pair glued on an open subset, provided isomorphisms are consistent.
- Let  $\{V_i\}$  be an affine cover of Y and  $\{U_i\}$  be an open cover of X with  $f(U_i) \subseteq V_i$  and f a morphism when restricted to each  $U_i$ . Then f is a morphism.
- A variety is a prevariety X so that for any prevariety Y and pair of morphisms  $Y \to X$ , the set where they agree is closed; equivalently, the diagonal is closed.
- an open or closed subprevariety of a variety is a variety.
- A variety is complete if  $\pi: X \times Y \to Y$  is closed for every variety Y
- If X is complete then any morphism  $X \to Y$  (Y variety) is closed.
- regular functions on complete varieties are constant

## 6 Projective Space

- think of  $\mathbb{P}^n$  as  $\mathbb{A}^n$  compactified with a point at infinity for every direction
- A projectivity on  $\mathbb{P}^n$  is an element of  $GL_{n+1}(k)/k^*$
- A conic is a symmetric bilinear form on  $k^3$ , which can be represented as  $\varepsilon_1 X^2 + \varepsilon Y^2 + \varepsilon Z^2$  after a projectivity
- conics in  $\mathbb{P}^2_{\mathbb{R}}$  are
  - nondegenerate  $X^2 + Y^2 Z^2$
  - empty  $X^2 + Y^2 + Z^2$
  - one point  $X^2 + Y^2$
  - two lines  $X^2 Y^2$
  - line  $X^2$
  - everything 0
- nondegenerate conics are equivalently  $XY = Z^2$ , isomorphic to  $\mathbb{P}^1$  with the isomorphism  $(U:V) \mapsto (U^2, UV, V^2)$  which can be interpreted as projection
- conics in  $\mathbb{P}^2_{\bar{k}}$ 
  - nondegenerate  $X^2 + Y^2 Z^2$
  - two lines  $X^2 Y^2$
  - line  $X^2$
  - everything 0
- A degree-d homogeneous form F on  $\mathbb{P}^n$  corresponds to a (maximum) degree-d polynomial f in  $\mathbb{A}^n$ . If n = 1, the multiplicity of a zero in F is the multiplicity of the corresponding zero in f, or  $d \deg f$  for the point at infinity.
- Bezout's theorem: for an algebraically closed field the number of intersections of projective curves is the product of their degrees, provided they share no irreducible components and multiplicities are counted appropriately
- Easy cases: line or nondegenerate conic vs a nonincluding curve in  $\mathbb{P}^2$ , inequality to compensate for multiplicities
- 5 points in general position define a unique conic

# 7 Projective Varieties

- Homogeneous ideals are generated by homogeneous polynomials or equivalently contain each homogeneous part of each member, or equivalently are fixed by the action of  $k^*$ .
- A projective algebraic set X in  $\mathbb{P}^n$  corresponds to a cone C(X) in  $\mathbb{A}^{n+1}$
- The zero set of a homogeneous ideal in  $\mathbb{A}^{n+1}$  is the cone of its zero set in  $\mathbb{P}^n$  (provided neither are empty)
- the ideal generated by  $X \subseteq \mathbb{P}^n$  is the ideal generated by  $C(X) \in \mathbb{A}^{n+1}$ .
- Nullstellensatz still works provided Z(I) is nonempty; Z(I) can only be empty if  $I = \langle 1 \rangle$  or  $\sqrt{I} = \langle x_0, \dots, x_n \rangle$ .
- Homogeneous coordinate ring is S(X) = A(C(X)); **not** polynomial functions
- rational functions are f/g, where  $f,g \in S(X)^{(d)}$  have common degree d

- homogeneous functions of the same degree on homogeneous coordinates give a morphism, provided they never all vanish
- projective varieties are varieties
- The Segre embedding is  $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{(n+1)(m+1)-1} : ((x_i), (y_i)) \mapsto (x_i y_j)$ . It is the zero locus of  $z_{i,j} z_{i',j'} z_{i,j'} z_{i',j}$ .
- $\bullet\,$  projective varieties are complete
- A nontrivial projective variety intersects with the zero locus of any homogeneous polynomial
- The Veronese embedding is  $\mathbb{P}^n \to \mathbb{P}^{\binom{n+d}{n}-1}$ :  $(x=(x_i)) \mapsto (x^I)_I$  (monomials of degree d). It is the zero locus of  $z_I z_J z_K z_L$ , with I+J=K+L