Combinatorics

Problem Set 1

1) Ignore the condition that the two sets of people are disjoint, because you can just end the people who are in both to make the sets disjoint. The number of non-empty subsets of the 10 people is $2^{10} - 1 = 1023$, but the possible set of age sums for non-empty subsets is $\{1, 2, \ldots, 600\}$ (this is a very loose upper bound, because we can assume all ages are distinct — or else just choose singleton sets — but this works so why bother). By the pigeonhole principle, since $1023 > \mathbf{1} \times 600$, there is at least one age sum (pigeonhole) containing more than **1** subset of people (pigeons)... which is what we wanted all along.

   With 9 people, $2^9 - 1 = 511$. Be a bit smarter and use distinct ages to show that the possible set of age sums for non-empty subsets is $\{1, 2, \ldots, 52 + 53 + \cdots + 60\} = \{1, 2, \ldots, 504\}$ — it still works.

2) Let $f : \{1, \ldots, m\} \to \{1, \ldots, m-1\}$ be defined by $f(i) = a^i \bmod m$ ($a^i \not\equiv 0 \bmod m$ cause they're coprime). By pigeonhole, there are distinct $s, t \in \{1, \ldots, m\}$ such that $f(s) = f(t)$. Assume $s > t$. So $a^s \equiv a^t \bmod m$. Since $\gcd(a, m) = 1$, this implies that $a^{s-t} \equiv 1 \bmod m$.

3) "Direct" approach: look for contiguous blocks of stuff — let x_i be the sum of the games played over the first i days; then if $x_j - x_i = 21$ for some $j > i$ then we're done.

   The total number of games over the 77 days can be at most $11 \times 12 = 132$. Let x_i be the number of games played on days $1, 2, \ldots, i$ inclusive, for $i = 1, \ldots, 77$. We want to use pigeonhole, but it's not immediately able to tell us $x_j - x_i = 21$ — it's better for getting some kind of equality. So let $y_i = x_i + 21$ for $i = 1, \ldots, 77$. But we are most interested in y_i when $y_i \leq 132$, or equivalently, $x_i \leq 111$. This will be definitely true for $i = 1, \ldots, 63$, since $x_{63} \leq 12 \times 9 = 108$.

   Consider the $63 + 77 = 140$ values $\{x_1, \ldots, x_{77}, y_1, \ldots, y_{63}\}$ which lie in the range $\{1, \ldots, 132\}$. By the pigeonhole principle, since $140 > 132$, there is a value $v \in \{1, \ldots, 132\}$ such that at least two of the elements of $\{x_1, \ldots, x_{77}, y_1, \ldots, y_{63}\}$ equal v. Since $x_i < x_{i+1}$ for all $i = 1, \ldots, 76$, the x_i are all distinct, which also implies that the y_i are all distinct. It must be that some $x_j = y_i$ for some i, j. That is, $x_j = x_i + 21$, which is what we wanted all along.

   Alt: Use Example 1.2 from lectures, in any 21 days, there is a consecutive subsequence adding up to a multiple of 21, but there are at most $12 \times 3 = 36$ games, which means it must be 21.

$6\frac{1}{2}$) (a) Define

$$S = \{(A, b) : A \subseteq \{1, \ldots, n\}, b \in A\}.$$

Calculating the size of S by first counting over A gives

$$|S| = \sum_{\substack{A \subseteq \{1, \ldots, n\} \\ |A| = k}} k = k\binom{n}{k}.$$

Summing over b instead gives

$$\begin{aligned}
|S| &= \sum_{b\in\{1,\ldots,n\}} |\{A \subseteq \{1,\ldots,n\} : b \in A, |A| = k\}| \\
&= \sum_{b\in\{1,\ldots,n\}} \binom{n-1}{k-1} \\
&= \frac{n(n-1)!}{(k-1)!(n-1-(k-1))!} \\
&= \frac{n!}{(k-1)!(n-k)!} = (n-k+1)\binom{n}{k-1}.
\end{aligned}$$

Problem Set 2

3) Let $n = R(s-1,t) + R(s,t-1) - 1$. Colour the edges of K_n red or blue arbitrarily. Let x be a vertex. The degree of x is $n-1 = R(s-1,t) + R(s,t-1) - 2$. By the proof of Erdős-Szekeres upper bound (Lemma 5.2), if x is incident with $\geq R(s-1,t)$ red edges *or* $\geq R(s,t-1)$ blue edges, then all is well. So, suppose that x is incident with precisely$R(s-1,t) - 1$ red edges *and* $R(s,t-1) - 1$ blue edges (both these numbers are odd). In fact, we can assume that this holds for all vertices in K_n. Note, n is odd. Consider the subgraph of K_n consisting of just the red edges. The sum of the degrees of this subgraph is odd, as it is the sum of an odd number of odd numbers. This contradicts the handshaking lemma, completing the proof.

4) a) Suppose that ij and jk are red, where $i < j < k$. Then

$$k - i = (k-j) + (j-i) \equiv 2 \mod 3,$$

which shows that edge ik is coloured blue. Next we show that there is no blue K_t.

Induction: $t = 3$, K_5 is fine. Assume it's okay for t. Consider K_{3t-1}. To make a blue K_{t+1} without a blue K_t on the vertices $1,\ldots,3t-4$, we must include two new vertices and they must be $3t-1$ and $3t-3$, or else we have a red edge. If i is in the blue $K_{t+!}$, then

$$\begin{aligned}
3t - 1 - i &\not\equiv 1 \mod 3, \text{ and} \\
3t - 3 - i &\not\equiv 1 \mod 3.
\end{aligned}$$

Hence the only possibility is $i \equiv 0 \bmod 3$. There are at most $t-2$ choices for i, which, together with $3t-1$ and $3t-3$ only give t vertices, not $t+1$.

b) Next apply Question 3 for an upper bound,

$$\begin{aligned}
R(3,4) &\leq R(2,4) + R(3,3) - 1 \quad \text{if both } R(2,4) \text{ and } R(3,3) \text{ are even} \\
&= 4 + 6 - 1 \\
&= 9.
\end{aligned}$$

5) a) Let $n = R(p_1, R(p_2,\ldots,p_t))$, and consider an arbitrary colouring of the edges of K_n with t distinct colours. Recolour all the edges coloured $2,\ldots,t$ by a new colour, say, 0. If there is a K_{p_1} in this new colouring, then we are done. Otherwise, by choice of n, there is a copy of K_{n_0} coloured 0 where $n_0 = R(p_2,\ldots,p_t)$. Reinstate the original colours on these edges (that is, with colours $2,\ldots,t$). Hence by choice of n_0, there is a K_{p_i} coloured with colour i for at least one $i \in \{2,\ldots,t\}$.

b) When $t = 2$, $R(p_1, p_2)$ is finite, by the Erdős-Szekeres Theorem. Assume that $t \geq 3$ and that k-colour Ramsey numbers are finite for all $k \leq t-1$. Then $R(p_1, p_2, \ldots, p_t) \leq R(p_1, \underbrace{R(p_2, \ldots, p_t)}_{\text{finite, by induction hypothesis}})$ by (a), which is finite by the base case.

6) Write $r(3;t)$ to denote $R(\underbrace{3, 3, \ldots, 3, 3}_{t})$.

a) Let $n = t(r(3;t-1) - 1) + 2$, and colour the edges of K_n with t colours arbitrarily. Let x be any vertex. Then x is incident with $n - 1 = t(r(3;t-1) - 1) + 1$ edges. By the pigeonhole principle, there exists a colour i such that x is incident with at least $r(3;t-1)$ edges coloured i. Let S be a set of $r(3;t-1)$ neighbours of x along edges coloured i. If any edge between two elements of S is coloured i, then we have a triangle (with x) coloured i. Otherwise, the edges of S are coloured with precisely $t-1$ colours. Since S has $r(3;t-1)$ vertices, there must be a monochromatic triangle in S.

8) From lectures, $S(t) \leq r(3;t) - 1 \leq 3t!$.

9) a) From lectures, we need $p_0 \geq S(3) + 1$. Now $S(3) \leq r(3;3) - 1 \leq 16$, by question 7. Therefore we may take $p_0 = 17$

b) By (a), such a p must satisfy $p < 17$. The subgroup $H = \{x^3 : x \in \mathbb{Z}_p^*\}$ has index $\gcd(3, p-1)$ in $\mathbb{Z}_p^*$. Maybe things go wrong if H is *small*, that is, when the index is *large*. Now $\gcd(3, p-1) = 3$, and this holds for $p = 7$, say. In $\mathbb{Z}_7^*$, $\{x^3 : x \in \mathbb{Z}_7^*\} = \{1, 6\}$. Furthermore, $x^3 + y^3 \in \{0, 2, 5\}$ for all $x, y \in \mathbb{Z}_7^*$. These are not equal to z^3 for any $z \in \mathbb{Z}_7^*$. Therefore $x^3 + y^3 = z^3$ has no solution in $\mathbb{Z}_7^*$.

c) Is it true that for all $p < p_0$, p prime, there is no solution to $x^3 + y^3 = z^3$ in Z_p^*? No: for example, $1^3 + 1^3 = 2^3$ in $\mathbb{Z}_3^*$.

Problem Set 3

2) Let A_1 be all orderings which contain ABLE; A_2, HYPNOTIC; A_3, RUG. We want $|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}|$, which we calculate using inclusion-exclusion. For A_1, we have 12 symbols

$$ABLE, C, G, H, I, N, O, P, R, T, U, Y$$

to permute, so $|A_1| = 12!$. Similarly, $|A_2| = 8!$ and $|A_3| = 13!$. Next, $A_1 \cap A_2$ consists of all words which are permutations of the 5 symbols

$$ABLE, HYPNOTIC, R, U, G,$$

so $|A_1 \cap A_2| = 5!$. Similarly, $|A_1 \cap A_3| = 10!$ and $|A_2 \cap A_3| = 6!$. Finally, $|A_1 \cap A_2 \cap A_3| = 3!$. There are 15! total orderings of the 15 letters in total, so

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| = 15! - (12! + 8! + 13!) + (5! + 10! + 6!) - 3!.$$

(Leave answer as is in exams and stuff.)

3) Let $Q_n = \#$ permutations $\sigma \in S_n$ such that $\sigma(i+1) \neq \sigma(i) + 1$ for $i = 1, \ldots, n-1$.

a) Q_1 obvious, Q_2 we can only have $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$, Q_3

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}.$$

b) Let $S = S_n$, the set of all $n!$ permutations of $\{1, \dots, n\}$. Define $A_i = \{\sigma \in S_n : \sigma(j) = i, \sigma(j+1) = i+1 \text{ for some } j \in \{1, \dots, n-1\}\}$, for $i = 1, \dots, n-1\}$. Then $Q_n = |\overline{A_1} \cap \cdots \cap \overline{A_{n-1}}|$; use inclusion-exclusion. For A_1, we are permuting symbols

$$12, 3, 4, \dots, n,$$

so $|A_1| = (n-1)!$. Similarly $|A_j| = (n-1)!$ for all $j = 1, \dots, n-1$.

Next, consider $A_1 \cap A_2$. Here we permute the symbols

$$123, 4, \dots, n,$$

and hence $|A_1 \cap A_2| = (n-2)!$. What about $A_1 \cap A_3$? Here we permute

$$12, 34, 5, \dots, n,$$

but it's still $(n-2)!$. Hence $|A_{i_1} \cap A_{i_2}| = (n-2)!$ for all $i_1 < i_2$. It can be shown by induction that

$$|A_{i_1} \cap \cdots \cap A_{i_k}| = (n-k)!$$

for all $1 \leq i_1 < \cdots < i_k \leq n-1$. Since there are $\binom{n-1}{k}$ such intersections, for $k = 1, \dots, n-1$, the result follows by inclusion-exclusion.

c) We have

$$Q_n = (n-1)! \sum_{k=0}^{n-1} (-1)^k \frac{n-k}{k!}.$$

From lectures,

$$D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!},$$

so

$$\begin{aligned}
D_n + D_{n+1} &= n!\left(\sum_{k=0}^{n}\frac{(-1)^k}{k!}\right) + (n-1)!\left(\sum_{i=0}^{n-1}\frac{(-1)^i}{i!}\right)\\
&= n! + n!\left(\sum_{i=0}^{n-1}\frac{(-1)^{i+1}}{(i+1)!}\right) + (n-1)!\left(\sum_{i=0}^{n-1}\frac{(-1)^i}{i!}\right)\\
&= n! + \sum_{i=0}^{n-1}\left((-1)^{i+1}\frac{n!}{(i+1)!} + \frac{(-1)^i(n-1)!}{i!}\right)\\
&= n! + \sum_{i=0}^{n-1}\frac{(-1)^i(n-1)!}{(i+1)!}(-n+i+1)\\
&= n! + \sum_{i=0}^{n-1}\frac{(-1)^{i+1}(n-1)!}{(i+1)!}(n-i-1)\\
&= n! + \sum_{i=0}^{n-2}\frac{(-1)^{i+1}(n-1)!}{(i+1)!}(n-(i+1))\\
&= n! + \sum_{j=1}^{n-1}\frac{(-1)^j(n-1)!}{j!}(n-j)\\
&= (n-1)!\sum_{j=0}^{n-1}(-1)^j\frac{n-j}{j!}\\
&= Q_n.
\end{aligned}$$

4) Write $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}$, where $p_1 < \cdots < p_r$ are prime and $\alpha_i \in \mathbb{Z}^+$. Let $A_i = \{k \in \{1,\dots,n\} : p_i \mid k\}$ for $i = 1,\dots,r$. Then $\phi(n) = |\overline{A_1} \cap \cdots \cap \overline{A_r}|$, calculate using inclusion-exclusion. Now $|A_i| = \frac{n}{p_i}$; similarly, $|A_{i_1} \cap A_{i_2}| = \frac{n}{p_{i_1}p_{i_2}}$ for $i_1 < i_2$, and in general,

$$|A_{i_1} \cap \cdots \cap A_{i_k}| = \frac{n}{p_{i_1}\cdots p_{i_k}}.$$

Hence

$$\begin{aligned}
\phi(n) = |\overline{A_1} \cap \cdots \cap \overline{A_r}| = n &- \left(\frac{n}{p_1} + \cdots + \frac{n}{p_r}\right)\\
&+ \left(\frac{n}{p_1p_2} + \cdots + \frac{n}{p_{r-1}p_r}\right)\\
&+ \cdots + (-1)^r\frac{n}{p_1\cdots p_r}.
\end{aligned}$$

Can check that this is equal to

$$\begin{aligned}
\phi(n) &= n\prod_{i=1}^{r}\left(1-\frac{1}{p_i}\right)\\
&= \cdots = n\prod_{\substack{p\mid n,\\ p\ \text{prime}}}\left(1-\frac{1}{p}\right).
\end{aligned}$$

5) $\Pi_4 = \{(4), (3,1), (2,2), (2,1,1), (1,1,1,1)\}$.

a) For all $\lambda_1 \in \Pi_4$, we can obtain λ_1 from itself by partitioning 0 parts; hence $\leq$ is reflexive.

Suppose that $\lambda_1 \leq \lambda_2$ and $\lambda_2 \leq \lambda_1$, for $\lambda_i \in \Pi_4$. Then λ_1 has at least as many parts as λ_2, since $\lambda_1 \leq \lambda_2$. But also λ_2 has at least as many parts as λ_1 as well. Hence λ_1 and λ_2 have the same number of parts, and one is a refinement of the other. Hence $\lambda_1 = \lambda_2$, so $\leq$ is antisymmetric.

Suppose that $\lambda_1 \leq \lambda_2$ and $\lambda_2 \leq \lambda_3$, for $\lambda_i \in \Pi_4$. Then λ_1 is a refinement of a refinement of λ_3, so it's a refinement of λ_3. Thus $\leq$ is transitive. Hasse diagram has $(4); (3,1), (2,2); (2,1,1); (1,1,1,1)$.

b) There are two:

$$(1,1,1,1) \leq^* (2,1,1) \leq^* (2,2) \leq^* (3,1) \leq^* (4),$$

or

$$(1,1,1,1) \leq^* (2,1,1) \leq^* (3,1) \leq^* (2,2) \leq^* (4).$$

c) Choose the second. We have

$$A_\zeta = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(Here the rows and columns are indexed according to the linear extension we chose.)

d) Invert A_ζ to find μ, we get

$$A_\mu = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Hence

$$\mu(\lambda_1, \lambda_2) = \begin{cases} 1 & \text{if } \lambda_1 = \lambda_2 \\ 0 & \text{if } \lambda_1 \not\leq \lambda_2 \\ -1 & \text{if } \lambda_1 = (1,1,1,1) \text{ and } \lambda_2 = (2,1,1), \text{ or} \\ & \quad \lambda_1 = (2,1,1) \text{ and } \lambda_2 = (3,1), \text{ or} \\ & \quad \lambda_1 = (3,1) \text{ and } \lambda_2 = (4), \text{ or} \\ & \quad \lambda_1 = (2,2) \text{ and } \lambda_2 = (4). \end{cases}$$

e) By Möbius inversion,

$$F(\lambda_2) = \sum_{\lambda_1 \leq \lambda_2} \mu(\lambda_1, \lambda_2) G(\lambda_1).$$

So

$$\begin{aligned} F(1,1,1,1) &= G(1,1,1,1) = 4, \\ F(2,1,1) &= G(1,1,1,1) \cdot -1 + G(2,1,1) \cdot 1 \\ &= -4 + 3 \\ &= -1. \end{aligned}$$

Similarly,

$$\begin{aligned}
F(3,1) &= G(1,1,1,1)\cdot 0 + G(2,1,1)\cdot -1 + G(3,1)\cdot 1\\
&= 2-3\\
&= -1,\\
F(2,2) &= -1,\\
F(4) &= G(1,1,1,1)\cdot 0 + G(2,1,1)\cdot 0 + G(3,1)\cdot -1 + G(2,2)\cdot -1 + G(4)\cdot 1\\
&= -2-2+1\\
&= -3.
\end{aligned}$$

Alternatively, $(4\ 3\ 2\ 2\ 1) = F(A_\zeta)$, so

$$\begin{aligned}
F &= (4\ 3\ 2\ 2\ 1)A_\mu\\
&= (4\ 3\ 2\ 2\ 1)\begin{pmatrix}1 & -1 & 0 & 0 & 0\\ 0 & 1 & -1 & -1 & 0\\ 0 & 0 & 1 & 0 & -1\\ 0 & 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 0 & 1\end{pmatrix}\\
&= (4 \quad -1 \quad -1 \quad -1 \quad -3).
\end{aligned}$$

This is wrong.

6) Let $N_1, \leq$ be the linear order on $N = \{1, \dots, n\}$.

   a) We have

$$A_\zeta = \begin{pmatrix}1 & 1 & \cdots & 1\\ 0 & \ddots & 1 & 1\\ 0 & \cdots & \ddots & 1\\ 0 & \cdots & 0 & 1\end{pmatrix}.$$

   We get that A_μ is the matrix with 1 on the diagonal, -1 on the entries above the diagonal, 0 elsewhere.

   b) By Möbius inversion,

$$\begin{aligned}
F(m) &= \sum_{k=1}^{m} \mu(k,m)G(k)\\
&= \begin{cases} G(1) & \text{if } m=1,\\ G(m)-G(m-1) & \text{if } m \geq 2.\end{cases}
\end{aligned}$$

7) If $a=b$ then $\mu(a,b) = \mu(a,a) = 1 = \mu(1,\frac{b}{a})$ by properties of a Möbius function. Now suppose that $b \neq a, a \mid b$. By induction, assume that $\mu(a,b) = \mu(1,\frac{c}{a})$ for all c where $a \mid c, c \mid b, c \neq b$. Then

$$\begin{aligned}
\mu(a,b) &= -\sum_{\substack{c\in\mathbb{N},\\ a|c,\ c|b,\ c\neq b}} \mu(a,c)\\
&= -\sum_{\substack{c\in\mathbb{N},\\ a|c,\ c|b,\ c\neq b}} \mu\left(1,\frac{c}{a}\right) \qquad \text{by inductive hypothesis.}
\end{aligned}$$

In the above sum, the values of c range over $c = ak$ where $1 \mid k, k \mid \frac{b}{a}, k \neq \frac{b}{a}$. So

$$\begin{aligned}
\mu(a,b) &= - \sum_{\substack{k \in \{1, \ldots, \frac{b}{a}\}, \\ 1 \mid k,\ k \mid \frac{b}{a},\ k \neq \frac{b}{a}}} \mu(1,k) \\
&= \mu\left(1, \frac{b}{a}\right),
\end{aligned}$$

by recursive definition of μ for divisibility poset on $\{1, \ldots, \frac{b}{a}\}$.

8) a) $k \in B_n^d$ iff $\gcd(k,n) = d$ iff $\gcd(\frac{k}{d}, \frac{n}{d}) = 1$ for k, n multiples of d. There are $\phi(\frac{n}{d})$ integers $i \in \{1, \ldots, \frac{n}{d}\}$ which are coprime to $\frac{n}{d}$, and setting $k = di$ shows that $|B_n^d| = \phi(\frac{n}{d})$.

b) We have

$$n = |\{1, \ldots, n\}| = \sum_{d \mid n} |B_n^d|,$$

as the B_n^d are disjoint and $|B_n^d| = 0$ if $d \mid n$. So

$$n = \sum_{d \mid n} \phi\left(\frac{n}{d}\right) = \sum_{d \mid n} \phi(d) = G(n).$$

c) Using Möbius inversion, it follows that

$$\begin{aligned}
\phi(n) &= \sum_{d \mid n} \mu(d,n) G(d) \\
&= \sum_{d \mid n} \mu\left(1, \frac{n}{d}\right) G(d) \qquad \text{by Q7} \\
&= \sum_{d \mid n} \mu\left(\frac{n}{d}\right) d \\
&= \sum_{d \mid n} \mu(d) \frac{n}{d}.
\end{aligned}$$

d) Recall

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^k & \text{if } n \text{ is the product of } k \text{ distinct primes,} \\ 0 & \text{otherwise.} \end{cases}$$

Write $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, where $p_1 < p_2 < \cdots < p_r$ are distinct primes, and $\alpha_i \in \mathbb{Z}^+$. Then $\mu(d) = 0$ if $d \neq 1$, unless $d = p_1^{\beta_1} \cdots p_r^{\beta_r}$ where $\beta_i \in \{0, 1\}$. Hence

$$\begin{aligned}
\phi(n) &= n - \left(\frac{n}{p_1} + \frac{n}{p_2} + \cdots + \frac{n}{p_r}\right) + (-1)^2 \left(\frac{n}{p_1 p_2} + \cdots + \frac{n}{p_{r-1} p_r}\right) \\
&\quad + \cdots + (-1)^r \frac{n}{p_1 p_2 \cdots p_r} \\
&= n \prod_{i=1}^{r} \left(1 - \frac{1}{p_i}\right) \qquad \text{as in Q4} \\
&= n \prod_{p \mid n} \left(1 - \frac{1}{p}\right).
\end{aligned}$$

Problem Set 4

2) Let $C \overset{ogf}{\longleftrightarrow} (c_n)$. Now $c_0 = 0, c_1 = 10, c_2 = 90$. More generally, $c_n = 9 \cdot 10^{n-1}$ for $n \geq 2$, so $c_n = 10c_{n-1}$ for $n \geq 3$. Hence

$$\sum_{n=3}^{\infty} c_n x^n = 10 \sum_{n=3}^{\infty} 1 - c_{n-1} x^n.$$

So

$$C(x) - (c_0 + c_1 x + c_2 x^2) = 10x \sum_{n=3}^{\infty} c_{n-1} x^{n-1} = 10x(C(x) - (c_0 + c_1 x)).$$

That is,

$$C(x) - (10x + 90x^2) = 10xC(x) - 100x^2.$$

Rearranging gives

$$(1 - 10x)C(x) = 10x(1 - x),$$

so

$$C(x) = \frac{10x(1-x)}{1-10x}.$$

3) Let $F \overset{ogf}{\longleftrightarrow} (F_n)_{n=0}^{\infty}$. The ogf for the LHS is $\frac{F(x)}{1-x}$, as it is a partial sum. The ogf for the RHS is

$$\begin{aligned}
\sum_{n=0}^{\infty} F_{n+2} x^n - \sum_{n=0}^{\infty} x^n &= \frac{1}{x^2} \left(\sum_{n=0}^{\infty} F_{n+2} x^{n+2} \right) - \frac{1}{1-x} \\
&= \frac{F(x) - x}{x^2} - \frac{1}{1-x}.
\end{aligned}$$

Now, these two ogfs are equal if and only if (cross multiplying):

$$\begin{aligned}
& x^2 F(x) + x^2 = (1-x)F(x) - x + x^2 \\
\iff \quad & (1 - x - x^2)F(x) = x, \\
\iff \quad & F(x) = \frac{x}{1 - x - x^2}.
\end{aligned}$$

But this is the ogf of the Fibonacci numbers. Hence the Fibonacci numbers satisfy the given recurrence.

4) We have

$$\begin{aligned}
B(x) &= \sum_{n=0}^{\infty} b_n x^n \\
&= \sum_{n=0}^{\infty} a_{2n+3} x^n \\
&= \frac{1}{x^{3/2}} \sum_{n=0}^{\infty} a_{2n+3} (\sqrt{x})^{2n+3} \\
&= \frac{1}{x^{3/2}} \left(\sum_{s=0}^{\infty} a_{2s+1} (\sqrt{x})^{2s+1} - a_1 \sqrt{x} \right) \\
&= \frac{A(\sqrt{x}) - A(-\sqrt{x}) - 2a_1 \sqrt{x}}{2x^{3/2}}.
\end{aligned}$$

5) (a) Here k is the number of parts. We have $p_n = [x^n]G(x)$, and we want to write this as

$$\begin{aligned} p_n &= [x^n] \sum_{k=1}^{\infty} y(x)^k \\ &= \sum_{k=1}^{\infty} \underbrace{[x^n]y(x)^k}_{\#\text{ of ordered partitions of } n \text{ with } k \text{ parts}} . \end{aligned}$$

So $y(x)$ should be the ogf for the number of ordered partitions of n with 1 parts, i.e.

$$y(x) = x + x^2 + \cdots = \frac{x}{1-x} \overset{ogf}{\longleftrightarrow} (1)_{n=1}^{\infty}.$$

So

$$G(x) = \sum_{k=1}^{\infty} \left(\frac{x}{1-x}\right)^k .$$

Note: $[x^n](x + x^2 + \dots)^k$ gives all terms $x^{z_1}x^{z_2}\cdots x^{z_k}$. Also, no relabelling $\rightsquigarrow$ use ogf.

(b) From (a),

$$A(x) = \sum_{k=1}^{\infty} \left(\frac{x}{1-x}\right)^{2k},$$

and

$$B(x) = \sum_{k=1}^{\infty} \left(\frac{x}{1-x}\right)^{2k-1}.$$

(c) From (b),

$$\begin{aligned} A(x) - B(x) &= \sum_{k=1}^{\infty} \left(\frac{x}{1-x} - 1\right)\left(\frac{x}{1-x}\right)^{2k-1} \\ &= \sum_{k=1}^{\infty} \frac{2x-1}{1-x} \cdot \frac{1-x}{x} \left(\frac{x^2}{(1-x)^2}\right)^k \\ &= \frac{2x-1}{x}\left(\frac{1}{1 - \dfrac{x^2}{(1-x)^2}} - 1\right) \\ &= \frac{2x-1}{x}\left(\frac{(1-x)^2}{1-2x} - 1\right) \\ &= -\frac{(1-x)^2}{x} - \frac{2x-1}{x} \\ &= -\frac{1 - 2x + x^2 + 2x - 1}{x} \\ &= -x. \end{aligned}$$

Hence

$$\begin{aligned} a_n - b_n &= [x^n](A(x) - B(x)) \\ &= [x^n](-x) \\ &= \begin{cases} -1 & \text{if } n = 1, \\ 0 & \text{if } n \geq 2. \end{cases} \end{aligned}$$

6) (a) First calculate

$$\begin{aligned}
[x^n](1-4x)^{\frac{1}{2}} &= [x^n]\sum_{s=0}^{\infty}\binom{\frac{1}{2}}{s}(-4x)^s \\
&= \binom{\frac{1}{2}}{n}(-4)^n \\
&= \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(\frac{-(2n-3)}{2}\right)(-4)^n}{n!} \\
&= (-1)^{2n-1}\frac{(2n-3)(2n-5)\cdots 3\cdot 1}{2^n\cdot n!}4^n \\
&= -\frac{2^n(2n-3)(2n-5)\cdots 3\cdot 1}{n!} \\
&= -2\frac{(2n-2)!}{n!(n-1)!} \qquad \text{(use the } 2^n \text{ to fill in the gaps, fixing stuff up in the denominato} \\
&= -\frac{2}{n}\binom{2n-2}{n-1}.
\end{aligned}$$

Therefore

$$[x^n](1-\sqrt{1-4x}) = \begin{cases} 1-1=0 & \text{if } n=0, \\ \dfrac{2}{n}\dbinom{2n-2}{n-1} & \text{if } n\geq 1. \end{cases}$$

(b) From (a), we have

$$1-\sqrt{1-4x} = \sum_{n=1}^{\infty}\frac{2}{n}\binom{2n-2}{n-1}x^n,$$

so

$$\begin{aligned}
\frac{1}{2x}(1-\sqrt{1-4x}) &= \sum_{n=1}^{\infty}\frac{1}{n}\binom{2n-2}{n-1}x^{n-1} \\
&= \sum_{s=0}^{\infty}\frac{1}{s+1}\binom{2s}{s}x^s,
\end{aligned}$$

as required.

(c) We multiply by x^n and sum. Let

$$f_n = \sum_k \binom{n+k}{m+2k}\binom{2k}{k}\frac{(-1)^k}{k+1},$$

for $n \in \mathbb{N}$. Define $F \overset{ogf}{\longleftrightarrow} (f_n)_{n=0}^{\infty}$. Then

$$\begin{aligned}
F(x) &= \sum_{n=0}^{\infty} x^n \sum_k \binom{n+k}{m+2k}\binom{2k}{k}\frac{(-1)^k}{k+1} \\
&= \sum_k \binom{2k}{k}\frac{(-1)^k}{k+1} x^k \sum_{n=0}^{\infty} x^{n+k}\binom{n+k}{m_2 k} \\
&= \sum_k \binom{2k}{k}\frac{(-1)^k}{k+1} x^{-k}\frac{x^{m+2k}}{(1-x)^{m+2k+1}} \\
&= \frac{x^m}{(1-x)^{m-1}}\sum_k \frac{1}{k+1}\binom{2k}{k}\left(\frac{-x}{(1-x)^2}\right)^k \\
&= \frac{x^m}{(1-x)^{m+1}}\cdot\frac{(1-x)^2}{-2x}\left(1-\sqrt{1+\frac{4x}{(1-x)^2}}\right) \\
&= -\frac{x^{m-1}}{2(1-x)^{m-1}}\left(1-\sqrt{\frac{1+2x+x^2}{(1-x)^2}}\right) \\
&= -\frac{x^{m-1}}{2(1-x)^{m-1}}\left(1-\frac{1+x}{1-x}\right) \\
&= -\frac{x^{m-1}(-2x)}{2(1-x)^{m-1}(1-x)} \\
&= \frac{x^m}{(1-x)^m}.
\end{aligned}$$

Therefore

$$\begin{aligned}
f_n &= [x^n]\frac{x^m}{(1-x)^m} \\
&= [x^{n-1}]\frac{x^{m-1}}{(1-x)^m} \\
&= \binom{n-1}{m-1}.
\end{aligned}$$

We have proved

$$\sum_k \binom{n+k}{m+2k}\binom{2k}{k}\frac{(-1)^k}{k+1} = \binom{n-1}{m-1}.$$

Problem Set 5

2) Suppose there are n_i occurences of i for all $i \in \{\alpha,\beta,\gamma,\delta,\varepsilon\}$. Without caring for restrictions, this gives

$$\begin{aligned}
&\binom{n}{n_\alpha}\binom{n-n_\alpha}{n_\beta}\binom{n-n_\alpha-n_\beta}{n_\gamma}\binom{n-n_\alpha-n_\beta-n_\gamma}{n_\delta}\binom{n-n_\alpha-n_\beta-n_\gamma-n_\delta}{n_\varepsilon} \\
=&\frac{n!}{n_\alpha! n_\beta! n_\gamma! n_\delta! n_\varepsilon!} \\
=&\binom{n}{n_\alpha, n_\beta, n_\gamma, n_\delta, n_\varepsilon} \qquad \text{(multinomial coefficient).}
\end{aligned}$$

(Just write out factorials.) Let $S(x) \overset{egf}{\longleftrightarrow} (s_n)$. Then

$$S(x) = \sum_{n \geq 0} \sum_{\text{those given conditions on } n_i} \binom{n}{n_\alpha, n_\beta, n_\gamma, n_\delta, n_\varepsilon}.$$

This is a product of 5 exponential generating functions. Let A, B, C, D, E be the exponential generating functions for the legal values of $n_\alpha, n_\beta, n_\gamma, n_\delta, n_\varepsilon$ respectively. Then,

$$\begin{aligned}
A(x) &= \frac{1}{2}(e^x + e^{-x}), \\
B(x) &= \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} \overset{egf}{\longleftrightarrow} (0,0,0,1,1,1,0,\dots), \\
C(x) &= e^x \overset{egf}{\longleftrightarrow} (1)_{n \geq 0}, \\
D(x) &= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!}, \\
E(x) &= e^x - \left(1 + x + \frac{x^2}{2} + \frac{x^3}{3}\right).
\end{aligned}$$

Therefore $S(x) = A(x)B(x)C(x)D(x)E(x)$.

3) (a) Consider the deck

$$(1), (1\ 2\ 3), \dots, (1\ 2\ \dots\ 2n+1).$$

The deck enumerator is

$$D(x) = \frac{1}{2}(e^x - e^{-x}) = \sinh(x).$$

Therefore $(b_n) \overset{egf}{\longleftrightarrow} e^{D(x)}$ by the exponential formula.

(b) For $n \geq 1$, we have

$$nb_n = \sum_k \binom{n}{k} k d_k b_{n-k}$$

where

$$d_k = \begin{cases} 0 & \text{if } k \text{ even}, \\ 1 & \text{if } k \text{ odd}. \end{cases}$$

That is,

$$nb_n = \sum_{s=0}^{\infty} \binom{n}{2s+1} (2s+1) b_{n-2s-1}$$

for $n \geq 1$.

4) Suppose $H(x) = e^{D(x)}$, $D(x) \overset{egf}{\longleftrightarrow} (a_n)_{n=0}^{\infty}$, $H(x) \overset{egf}{\longleftrightarrow} (h_n)_{n=0}^{\infty}$, where $d_0 = 0$.

(a) We have $h_0 = H(0) = e^{D(0)} = e^{d_0} = e^0 = 1$.

(b) Recall

$$nh_n = \sum_k \binom{n}{k} k a_k h_{n-k}, \tag{1}$$

where $n \geq 1$. Suppose that $(h_n)_{n\geq 0}$ is known and we want to find $(a_n)_{n\geq 0}$. Rearranging (1) gives

$$na_n = nh_n - \sum_{k=1}^{n-1} \binom{n}{k} ka_k h_{n-k},$$

for $n \geq 1$. From h_1 we calculate a_1 ($a_1 = h_1$). From h_1, h_2 and a_1 we calculate a_2. From $h_1, \ldots, h_n, a_1, \ldots, a_{n-1}$ we calculate a_n.

5) (a) The LHS is the number of permutations of $\{1, \ldots, n\}$. The summand on the RHS is zero unless $k \in \{0, \ldots, n\}$. For such k, the expression $\binom{n}{k} d_{n-k}$ counts the number of permutations of $\{1, \ldots, n\}$ with k fixed points — choose the k fixed points with $\binom{n}{k}$, then count the number of ways to complete the permutation without any other fixed points with d_{n-k}. Summing over $k = 0, \ldots, n$ counts each permutation of $\{1, \ldots, n\}$ exactly once.

(b) Multiply (a) by $\frac{x^n}{n!}$ and sum, giving

$$\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} \sum_{k} \binom{n}{k} d_{n-k} \frac{x^n}{n!}.$$

Now,

$$\text{LHS} = \frac{1}{1-x},$$

and the RHS is the exponential generating function of

$$\sum_{k} \binom{n}{k} 1 \cdot d_{n-k},$$

which is $e^x F(x)$ (see Proposition 16.1).

(c) From the generating function, we have

$$e^{-x} = (1-x)F(x).$$

Differentiating gives

$$-e^{-x} = (1-x)F'(x) - F(x),$$

so

$$F'(x) = xF'(x) + F(x) - e^{-x}.$$

The LHS is the exponential generating function for $(d_{n+1})_{n=0}^{\infty}$. The RHS is the exponential generating function for $(nd_n + d_n + (-1)^{n+1})_{n=0}^{\infty}$ (see Lemma 16.1). Equating coefficients of $\frac{x^n}{n!}$ gives the result.

(d) Let

$$\begin{aligned} f_n &= \text{number of permutations of } \{1, \ldots, n\} \text{ with exactly one fixed point} \\ &= nd_{n-1}. \end{aligned}$$

Part (c) gives

$$f_n - d_n = nd_{n-1} - d_n = -(-1)^{(n-1)+1} = (-1)^{n-1} \in \{-1, 1\}.$$

Also $f_n - d_n = 1$ iff n is odd.

6) (a) There are $\binom{n}{2}$ possible edges, for each one decide whether we want it in or not.

(b) A hand of weight n is a graph with n vertices. Every graph on n vertices can be found in this way, from a unique hand. Hence $h_n = 2^{\binom{n}{2}}$. So

$$H(x) = \sum 2^{\binom{n}{2}} \frac{x^n}{n!},$$

not useful analytically.

(c) From Q4,

$$\begin{aligned} ng_n &= nh_n - \sum_{k=1}^{n-1} kg_k h_{n-k} \\ &= n2^{\binom{n}{2}} - \sum_{k=1}^{n-1} \binom{n}{k} kg_k 2^{\binom{n-k}{2}}. \end{aligned}$$

We can use this to recursively calculate g_n for small n.

7) (a) Paths of length ≥ 1, cycles of length ≥ 3.

(b) Let h_n be the number of graphs on the vertex set $\{1, 2, \ldots, n\}$ such that every vertex has degree 1 or 2 and $h_0 = 1$. (I'll write $1 \to 2 \to 3$ for a path from vertex 1 to 2 to 3, for example, and use the usual way to write cycles.) Consider the deck

$$1 \to 2, 1 \to 2 \to 3, 1 \to 3 \to 2, 2 \to 1 \to 3, (1\ 2\ 3), \ldots$$

of all paths of length ≥ 1 and all cycles of length ≥ 3 with all possible standard labellings. Then

$$\begin{aligned} d_n &= \#\text{ of cards in the deck of weight } n \\ &= \begin{cases} \dfrac{n!}{2} + \dfrac{(n-1)!}{2} & \text{if } n \geq 3, \\ 1 & \text{if } n = 2, \\ 0 & \text{if } n = 0, 1. \end{cases} \end{aligned}$$

This is because the number of paths with n vertices and standard labelling is $\frac{n!}{2}$, and the number of cycles with n vertices and standard labelling is $\frac{n!}{2n}$. The corresponding deck enumerator is

$$\begin{aligned} D(x) = \sum_{n=0}^{\infty} d_n \frac{x^n}{n!} &= \frac{x^2}{2!} + \sum_{n \geq 3} \left(\frac{n!}{2} + \frac{(n-1)!}{2} \right) \frac{x^n}{n!} \\ &= \sum_{n=2}^{\infty} \frac{x^n}{2} + \sum_{n=3}^{\infty} \frac{x^n}{2n} \\ &= \frac{1}{2} \left(\frac{1}{1-x} - 1 - x \right) + \frac{1}{2} \left(-\log(1-x) - x - \frac{x^2}{2} \right) \\ &= \frac{1}{2(1-x)} - \frac{1}{2} \log(1-x) - \frac{1}{2} - \frac{x}{2} - \frac{x}{2} - \frac{x^2}{4} \\ &= \frac{1}{2(1-x)} - \frac{1}{2} \log(1-x) - \frac{1}{2} \left(1 + 2x + \frac{x^2}{2} \right). \end{aligned}$$

Since $d_0 = 0$ we can apply the exponential counting theorem to conclude that $(h_n)_{n=0}^{\infty} \overset{egf}{\longleftrightarrow}$ $H(x)$ where

$$\begin{aligned} H(x) &= e^{D(x)} \\ &= \frac{\exp\left(\dfrac{1}{2(1-x)} - \dfrac{1}{2}\left(1 + 2x + \dfrac{x^2}{2}\right)\right)}{\sqrt{1-x}} \end{aligned}$$

(c) Let g_n be the number of graphs G on $\{1, \dots, n\}$ where every vertex in G has degree 1 or 2 and g has an odd number of components. Let $G(x) \overset{egf}{\longleftrightarrow} (g_n)_{n=0}^{\infty}$. Then

$$\begin{aligned} G(x) &= \frac{1}{2}\left(e^{D(x)} - e^{-D(x)}\right) \\ &= \sinh(D(x)), \end{aligned}$$

where $D(x)$ is as in (b). Why? We have

$$H(x) = e^{D(x)} = \sum_k \frac{D(x)^k}{k!}$$

and k is the number of components. This follows as $D(x)^k$ is the egf for

$$\left(\sum_{\substack{i_1+\cdots+i_k=n,\\ i_j\in\{0,\dots,n\}}} \binom{n}{i_1, \dots, i_k} d_{i_1} d_{i_2} \cdots d_{i_k}\right)_{n=0}^{\infty}$$

by the product rule for egfs. This counts the number of ordered hands of weight n with k cards, the j^{th} card having weight i_j, $j = 1, \dots, k$. Now divide by $k!$ to get rid of this ordering. Therefore

$$\begin{aligned} G(x) &= \sum_{k=)}^{\infty} \frac{D(x)^{2k+1}}{(2k+1)!} \\ &= \frac{1}{2}(e^{D(x)} - e^{-D(x)}) \end{aligned}$$

as claimed.