Banach Algebras

Lecture 1

Spectral Theorem in Finite Dimensions

Definition 1.1. Bunch of things. Let $A =$ square matrix.

- Symmetric: $A = A^T$.
- Orthogonal: $AA^T = A^T A = I$.
- Adjoint: $A^* = \overline{A^T}$.
- Self-adjoint: $A = A^*$.
- Unitary: $AA^* = A^* A = I$.
- Normal: $AA^* = A^* A$.
- Diagonal: $A_{ij} = 0$ whenever $i \neq j$.

Theorem 1.1. *Let A be a normal complex matrix. Then there is a unitary matrix U such that UAU^* is a diagonal matrix. Equivalently, there is an orthogonal basis of eigenvectors for A.*

Example 1.1.

$$A = \begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix},$$
$$A^* = \begin{pmatrix} 1-i & 1+i \\ 1+i & 1-i \end{pmatrix}.$$

Then

$$AA^* = A^* A = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}.$$

A few days later we get

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} & 1 \\ 1 & -1 \end{pmatrix},$$

and

$$UAU^* = \begin{pmatrix} 2 & 0 \\ 0 & 2i \end{pmatrix}.$$

Definition 1.2. A Hilbert space is a complete inner product space.

Proposition 1.1. *If H_1, H_2 are Hilbert spaces with orthonormal bases of the same cardinality, they are isometrically isomorphic.*

Definition 1.3. A Hilbert space is separable if it admits a countable orthonormal basis.

Any infinite dimensional Hilbert space is isometrically isomorphic to $\ell^2(\mathbb{N})$.

Definition 1.4. A bounded operator $A : H \to H$ is compact if the closure of the image of the unit ball in H under A is compact.

Example 1.2.

1. Any finite rank operator is compact.
2. Let $H = \ell^2(\mathbb{N})$. Let $\mathbf{a} = (a_1, a_2, \dots)$ be a sequence of complex numbers. Define $M_{\mathbf{a}}(x_1, x_2, \dots) = (a_1 x_1, a_2 x_2, \dots)$.
   (a) Bounded if $\mathbf{a}$ is bounded.
   (b) Adjoint is $M_{\bar{\mathbf{a}}}$ where $\bar{\mathbf{a}} = (\bar{a}_1, \bar{a}_2, \dots)$.
   (c) Normal cause doesn't matter which way you multiply stuff.
   (d) Self-adjoint if the a_i are real for all i.
   (e) Compact if $a_i \to 0$.

Lecture 2

Theorem 2.1. *Let A be a compact normal operator on a separable infinite dimensional Hilbert space H. Then H contains an orthonormal basis of eigenvectors for A, with eigenvalues tending to* 0.

Eigenvectors for $M_{\mathbf{a}}$ in Example 1.2 — $\{\mathbf{x}_i = (0, \dots, 0, \underbrace{1}_{i^{\text{th}} \text{ slot}}, 0, \dots)\}$ is an orthonormal basis of eigenvectors.

Theorem 2.2. *Let A be a compact normal operator on a separable infinite dimensional Hilbert space. Then there exists a unitary operator $U : H \to \ell^2(\mathbb{N})$ and a vector $\mathbf{a} = (a_1, a_2, \dots)$, $a_i \to 0$, such that $UAU^* = M_{\mathbf{a}}$.*

Proof. Sketch.

1. Pick an orthonormal basis of eigenvectors $\{e_i\}$ with eigenvalues $\{a_i\}$.
2. $U : H \to \ell^2(\mathbb{N})$, with $x \mapsto (\langle x, e_1 \rangle, \langle x, e_2 \rangle, \dots)$.
3. $U^* : \ell^2(\mathbb{N}) \to H$, with $(x_1, x_2, \dots) \mapsto \sum_{i=1}^{\infty} x_i e_i$. □

What about non-compact operators? Is every normal operator unitarily equivalent to a multiplication operator on $\ell^2(\mathbb{N})$?

Example 2.1. Let $H = L^2([0,1])$. For f bounded, define $M_f : L^2([0,1]) \to L^2([0,1])$ with $M_f g = fg$. Let $f_0(x) = x$. What are the eigenvalues of M_{f_0}? We have $M_{f_0} g = \lambda g$ if $xg(x) = \lambda g(x)$ for all $x \in [0,1]$. But then $g(x) = 0$ almost everywhere, so there are no eigenvalues: so M_{f_0} cannot be unitarily equivalent to a multiplication operator on $\ell^2(\mathbb{N})$.

Theorem 2.3. *(Spectral Theorem.)*
Let A be a normal operator on a separable Hilbert space. Then A is unitarily equivalent to a multiplication operator M_f on "$L^2(\Omega)$".

This Ω will be defined later.

Definition 2.1. An algebra over a field $\mathbb{F}$ is a vector space V with a map $V \times V \to \mathbb{F}$ such that (for $a \in \mathbb{F}$, $x, y, z \in V$):

1. $(ax + y)z = a(xz) + yz$.
2. $z(ax + y) = a(zx) + zy$.
3. $(xy)z = x(yz)$.

It is commutative if $xy = yx$, and unital if there exists some $\mathbf{1}$ such that $\mathbf{1}x = x\mathbf{1} = x$ for all x.

Example 2.2. Algebraaas.

1. $\mathbb{F}$.
2. $\mathbb{F}[x]$.
3. Functions $X \to \mathbb{F}$ — X any set, product done pointwise.
4. $n \times n$ matrices over $\mathbb{F}$.
5. All linear operators on a vector space, with composition as the product.
6. Let G be a group. Take a vector space with basis indexed by G, $\{e_g\}$, multiplication on basis $e_g e_h = e_{gh}$.

Definition 2.2. A Banach algebra is an algebra over $\mathbb{C}$ such that the underlying vector space is a Banach space, and $\|x \cdot y\| \leq \|x\|\|y\|$ for all x, y.

Lecture 3

Example 3.1. BANACH Algebraaas.

1. $\mathbb{C}$.
2. Any Banach space, with $ab = 0$ for all a, b.
3. $C(X)$, continuous functions on a compact metric space with the sup norm and pointwise product.
4. $C_b(X)$, bounded continuous functions on a metric space.
5. $C_0(X)$, continuous functions "vanishing at ∞" on some metric space.
6. Disk algebra. Continuous functions on the unit disk which are holomorphic on the interior.
7. For any Banach space E, the space of bounded operators $B(E)$ is a Banach algebra with the operator norm and composition as the product.
8. $M_n(\mathbb{C})$, with matrix product and norm $\|M\| = \sum_{i,j} |M_{ij}|$ (in this case $\|\mathbf{1}\| = \|I_n\| = n$).
9. $\ell^1(\mathbb{Z})$, with $(a * b)_i = \sum_{j \in \mathbb{Z}} a_j b_{i-j}$. This converges as

$$\sum_{j \in \mathbb{Z}} |a_j b_{i-j}| \leq \sum_{j \in \mathbb{Z}} |b_{i-j}||a_j| \leq \sup_k |b_k| \sum_{j \in \mathbb{Z}} |a_j| < \infty$$

since $(a_i), (b_i) \in \ell^1(\mathbb{Z})$. Check condition from Definition 2.2:

$$\begin{aligned} \|a * b\| = \sum_i |(a*b)_i| &= \sum_i \left| \sum_j a_j b_{i-j} \right| \\ &\le \sum_{i,j} |a_j b_{i-j}| \\ &= \sum_j \left(|a_j| \sum_i |b_{i-j}| \right) \\ &= \sum_j |a_j| \|b\| \\ &= \|a\| \|b\|. \end{aligned}$$

10. $L^1(\mathbb{R})$, with $(f*g)(x) = \int_{\mathbb{R}} f(y)g(x-y)\, dy$.

Example 3.2. MAYBE BANACH ALGEBRas.

1. Polynomial functions on $[0,1]$, with sup norm and pointwise product — not complete.
2. $L^1([0,1])$, pointwise product — not closed under this multiplication.
3. $\ell^1(\mathbb{Z})$, pointwise product — should be okay.
4. $C(\mathbb{R})$ has no obvious norm...
5. All bounded functions on $\mathbb{R}$, sup norm, pointwise product — should be okay.

Invertibility and Spectrum

Definition 3.1. A bounded operator $A : E \to E$ is invertible if there exists some bounded operator $B : E \to E$ such that $AB = BA = \mathrm{id}_E$.

Theorem 3.1. *The following are equivalent:*

(1) *A is invertible.*

(2) *For every $x, y \in E$, $Ax = y$ has a unique solution, that is, A is a bijection.*

Proof. (1) $\implies$ (2) is clear, since any invertible map is bijective.
For (2) $\implies$ (1), we need to show that if A is bijective, then A^{-1} is a bounded operator. The graph of A, $\{(x, Ax) : x \in E\}$, is closed in $E \times E$ since A is continuous. Equivalently, $\{(Ay, y)\}$ is closed in $E \times E$, but this is the graph of A^{-1} since A is a bijection, so A^{-1} is bounded. □

Definition 3.2. The spectrum of an operator $\sigma(A)$ is $\{\lambda \in \mathbb{C} : \lambda I - A \text{ is not invertible}\}$.

Lecture 4

Example 4.1. Shifts.
Let $T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be the right unilateral shift, $(x_1, x_2, \dots) \mapsto (0, x_1, x_2, \dots)$, and $S : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be the left shift, $(x_1, x_2, \dots) \mapsto (x_2, x_3, \dots)$. Both fail to be invertible: T is not surjective, and

S is not injective. Note $ST = I$, but $TS(x_1, x_2, x_3, \dots) = (0, x_2, x_3, \dots)$.

Does T have eigenvalues? No — $T\mathbf{x} = \lambda\mathbf{x} \implies 0 = \lambda x_1,\ x_1 = \lambda x_2,$ etc.... If $\lambda = 0$ then $\mathbf{x} = 0$; otherwise $x_1 = 0$ and $\mathbf{x} = 0$ anyway... so no eigenvalues.

The spectrum of ST (when is $I - \lambda I$ not invertible?) is $\sigma(ST) = \{1\}$.

The spectrum of TS is $\sigma(TS) = \{0, 1\}$. Note TS is the projection onto $\{(0, x_2, x_3, \dots)\}$... let P be any projection onto a Hilbert space. Write $I = P + P^\perp$; when is $P - \lambda I$ invertible? We have $P - \lambda I = P - \lambda(P + P^\perp) = (1 - \lambda)P - \lambda P^\perp$. The inverse is given by

$$\frac{1}{1-\lambda}P - \frac{1}{\lambda}P^\perp,$$

which is okay as long as $\lambda \notin \{0, 1\}$. Hence $\sigma(P) \subseteq \{0, 1\}$; we can also check that $0 \in \sigma(P)$ if $P \neq I$ and $1 \in \sigma(P)$ if $P \neq \mathbf{0}$.

If

$$A = \sum_{i=1}^{n} \lambda_i P_i,$$

where the P_i are non-zero projections, $P_i P_j = 0$ for $i \neq j$ and $\sum P_i = I$, then $\sigma(A) = \{\lambda_i\}$.

Fact. $\sigma(AB)\backslash\{0\} = \sigma(BA)\backslash\{0\}$, that is, the spectra are the same if we ignore zero.

This follows from:

Fact. $\mathbf{1} - AB$ is invertible if and only if $\mathbf{1} - BA$ is invertible.

Example 4.2. Spectrum of multiplication map.
Let $\mathbf{a} = (a_1, a_2, \dots) \in \ell^\infty(\mathbb{N})$, and let $M_\mathbf{a} : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ with $(x_1, x_2, \dots) \mapsto (a_1 x_1, a_2 x_2, \dots)$. What is $\sigma(M_\mathbf{a})$? We have $\{a_i\} \subseteq \sigma(M_\mathbf{a})$, since $M_\mathbf{a} - a_i I$ has a non-trivial kernel. Also, for any λ

$$(M_\mathbf{a} - \lambda I)(x_1, x_2, \dots) = \big((a_1 - \lambda)x_1, (a_2 - \lambda)x_2, \dots\big).$$

As long as $\lambda \notin \{a_i\}$, we can try to invert with $M_\mathbf{b}$, where

$$\mathbf{b} = \left(\frac{1}{a_1 - \lambda}, \frac{1}{a_2 - \lambda}, \dots\right).$$

But $M_\mathbf{b}$ is a bounded operator of $\ell^2(\mathbb{N})$ if and only if $\lambda \notin \overline{\{a_i\}}$. It follows that $\sigma(M_\mathbf{a}) = \overline{\{a_i\}}$.

Example 4.3. Construct an operator whose spectrum is $[0, 1]$.

Take any countable dense set in $[0, 1]$, look at the corresponding multiplication operator.

Lecture 5

Definition 5.1. An element x in a unital Banach algebra A is invertible if there is some $y \in A$ such that $xy = yx = 1$. The spectrum $\sigma(x) = \{\lambda \in \mathbb{C} : x - \lambda\mathbf{1} \text{ is invertible}\}$.

Conventions:

1. Always assume $\|\mathbf{1}\| = 1$.

2. Write $x - \lambda$ for $x - \lambda\mathbf{1}$.

Lemma 5.1.

1. *If* $\|x\| < 1$, *then* $\mathbf{1} - x$ *is invertible.*

2. *If* $\|x\| < 1$, *then* $\|(\mathbf{1} - x)^{-1}\| \leq \dfrac{1}{1 - \|x\|}$.

Proof. Let

$$z = \sum_{n=0}^{\infty} x^n.$$

(It converges because $\|x^n\| \leq \|x\|^n$.) Then

$$\begin{aligned}
(\mathbf{1} - x)z &= (\mathbf{1} - x)\left(\sum_{n=0}^{\infty} x^n\right) \\
&= (\mathbf{1} - x) \lim_{N\to\infty} \sum_{n=0}^{N} x^n \\
&= \lim_{N\to\infty} \left((\mathbf{1} - x)\sum_{n=0}^{N} x^n\right) \\
&= \lim_{N\to\infty} (\mathbf{1} - x^{N+1}) \\
&= \mathbf{1}.
\end{aligned}$$

So z is a right inverse; it's also a left inverse. Then

$$\|(\mathbf{1} - x)^{-1}\| = \left\|\sum_{n=0}^{\infty} x^n\right\| \leq \sum_{n=0}^{\infty} \|x\|^n = \frac{1}{1 - \|x\|}.$$

□

Let A^{-1} be the *group* of invertible elements of A.

Theorem 5.1. A^{-1} *is an open set, and* $x \mapsto x^{-1}$ *is a continuous map.*

Proof. If x is invertible, then $x + h = x(\mathbf{1} + x^{-1}h)$, so by the previous lemma, $x + h$ will be invertible if $\|x^{-1}h\| < 1$. So, if $\|h\| < \frac{1}{\|x^{-1}\|}$, then $\|x^{-1}h\| < 1$, and $x + h$ is invertible implies A^{-1} is open. For continuity, use estimate on $\|(\mathbf{1} - x)^{-1}\|$. □

Theorem 5.2. *For any* x, $\sigma(x)$ *is a compact set and* $\sigma(x) \subseteq \{\lambda : |\lambda| \leq \|x\|\}$.

Proof. We first show $\sigma(x)$ is closed. If $\lambda \notin \sigma(x)$, then $x - \lambda_0$ is invertible. If $|\lambda - \lambda_0| < \delta$, then $\|(x - \lambda) - (x - \lambda_0)\| = |\lambda - \lambda_0| < \delta$. Since A^{-1} is open, this means that for δ sufficiently small, λ will be in the "resolvent" $(\mathbb{C}\backslash\sigma(x))$ as well, which implies that the resolvent is open.

Next, we show that $\sigma(x)$ is bounded by $\|x\|$, that is, any λ with $|\lambda| > \|x\|$ is not in $\sigma(x)$. If $|\lambda| > \|x\|$, then $x - \lambda = \lambda(\frac{x}{\lambda} - \mathbf{1})$. Since $\left\|\frac{x}{\lambda}\right\| = \frac{1}{\lambda}\|x\| < 1$, we know that $x - \lambda$ is invertible, that is, $\lambda \notin \sigma(x)$. □

Theorem 5.3. $\sigma(x)$ *is non-empty.*

Proof. Basic idea: if $\sigma(x) = \varnothing$, then $x - \lambda$ is invertible for all $\lambda \in \mathbb{C}$. We want to show that this doesn't make sense. First approach: use complex analysis for functions from $\mathbb{C} \to A$, but we need to show that stuff from complex analysis sensibly extends to such functions.

Second approach: look at $f\left((x-\lambda)^{-1}\right)$ for bounded linear functionals f, and use functional analysis. We'll go with this. Fix x, and suppose for a contradiction that $\sigma(x) = \varnothing$. **Claim:** for any bounded linear functional f on A, $f\left((x-\lambda)^{-1}\right)$ is a bounded, entire function which tends to 0.

Proof of claim. We have, for a fixed λ_0,

$$\begin{aligned}\lim_{\lambda\to\lambda_0} \frac{(x-\lambda)^{-1} - (x-\lambda_0)^{-1}}{\lambda - \lambda_0} &= \lim_{\lambda\to\lambda_0} \frac{(x-\lambda)^{-1}((x-\lambda)-(x-\lambda_0))(x-\lambda_0)^{-1}}{\lambda-\lambda_0} \\ &= \lim_{\lambda\to\lambda_0} (x-\lambda)^{-1}(x-\lambda_0)^{-1} \\ &= (x-\lambda_0)^{-2}.\end{aligned}$$

Thus $f((x-\lambda)^{-1})$ is analytic for all f (exercise).

Similarly, if $\lambda \neq 0$

$$\|(x-\lambda)^{-1}\| = \left\|\lambda^{-1}\left(\frac{x}{\lambda} - 1\right)^{-1}\right\| \leq \frac{1}{|\lambda|}\frac{1}{1-\frac{\|x\|}{|\lambda|}} \to 0$$

as $\lambda \to \infty$. □

But this means that $(x-\lambda)^{-1}$ is 0 (Hahn-Banach) for all λ which is absurd. □

Definition 5.2. The spectral radius is $r(x) = \sup\{|\lambda| : \lambda \in \sigma(x)\}$.

Note $r(x) \leq \|x\|$.

Theorem 5.4.

$$r(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}}.$$

Lecture 6

MIA — see Ben's stuff (or use that anyway if you want something more orderly ☺)

Lecture 7

Proof of Theorem 5.4. (Sketch.)

We show that $r(x) \leq \liminf_{n\to\infty} \|x^n\|^{\frac{1}{n}}$, and $r(x) \geq \limsup_{n\to\infty} \|x^n\|^{\frac{1}{n}}$.

$(r(x) \leq \liminf)$.

If $\lambda \in \sigma(x)$, then $\lambda^n \in \sigma(x^n)$ (see the Spectral Mapping Theorem). Then

$$\begin{gathered}|\lambda^n| \leq \|x^n\|, \text{ and} \\ |\lambda| \leq \|x^n\|^{\frac{1}{n}}.\end{gathered}$$

So $\sigma(x)$ is bounded in absolute value by $\|x^n\|^{\frac{1}{n}}$ for every n, which implies that $r(x) \leq \liminf_{n\to\infty} \|x^n\|^{\frac{1}{n}}$.

$(r(x) \geq \limsup)$.

It suffices to show that for any $\lambda > r(x)$, $\lambda \geq \limsup_{n\to\infty} \|x^n\|^{\frac{1}{n}}$, so suppose $\lambda > r(x)$.

Claim: $\left\{\dfrac{x^n}{\lambda^n}\right\}$ is bounded in norm.

Assume the claim is true. Then

$$\left\|\frac{x^n}{\lambda^n}\right\| = \frac{\|x^n\|}{|\lambda|^n} < M \quad \forall n.$$

Then $\|x^n\| < |\lambda|^n M$, so $\|x^n\|^{\frac{1}{n}} < |\lambda|\ M^{\frac{1}{n}}$. $\limsup$ everything to get

$$\limsup_{n\to\infty} \|x^n\|^{\frac{1}{n}} < |\lambda| \limsup_{n\to\infty} M^{\frac{1}{n}} = |\lambda|.$$

Proof of claim. It suffices to show that $f(x^n/\lambda^n)$ is bounded for every bounded linear functional f. If $f(x^n/\lambda^n)$ is bounded for each $f \in A^*$, that means $\{x^n/\lambda^n\}$ is bounded pointwise as elements of A^{**}.

Take $f \in A^*$. Look at the function $f[(1-zx)^{-1}]$. Assume $r(x) \neq 0$. The domain is

$$\{0\} \cup \left\{z : \frac{1}{z} \notin \sigma(x)\right\},$$

or $|z| < \frac{1}{r(x)}$. On the disk $|z| < \frac{1}{\|x\|}$, we can take a power series for $(1-zx)^{-1}$ to get

$$f((1-zx)^{-1}) = 1 + zf(x) + z^2 f(x^2) + \dots.$$

On the larger disk $|z| < \frac{1}{r(x)}$, $f((1-zx)^{-1})$ should still be analytic, so $f((1-zx)^{-1}) = \sum f(x^n)z^n$. In particular, $f(x^n)z^n$ is bounded when $|z| < \frac{1}{r(x)}$. □

□

Ideals

Definition 7.1. An ideal in a Banach algebra A is a subspace $I \subseteq A$, such that $xi, ix \in I$ for all $x \in A$, $i \in I$.

Given an ideal I in an algebra A, we can take the quotient A/I. Aside: If $B_0 \subseteq B$ is a closed subspace of a Banach space, then B/B_0 is a Banach space with $\|[b]\| = \inf\{\|b + b_0\| : b_0 \in B_0\}$. We would have seen this in functional analysis last semester.

So for a Banach algebra A, if $I \subseteq A$ is a closed ideal, then A/I is a Banach space and an algebra. But is it... a BANACH ALGEBRA?

Check: is it true that $\|[x][y]\| \leq \|[x]\|\ \|[y]\|$ for all x, y?

$$\begin{aligned}
\|[x][y]\| &= \|[xy]\| \\
&= \inf_{i\in I} \|xy + i\| \\
&\leq \|xy + \underbrace{i_1 y + i_2 x + i_1 i_2}_{\in I}\| \quad \text{for any } i_1, i_2 \in I \\
&= \|(x+i_1)(y+i_2)\| \\
&\leq \|x+i_1\|\|y+i_2\|.
\end{aligned}$$

This shows that $\|[x][y]\| \leq \inf_{i_1,i_2} \|x+i_1\|\|y+i_2\| = \|[x]\|\,\|[y]\|$.

Lecture 8

True facts about ideals.

Theorem 8.1. *Let I be a closed ideal in a Banach algebra A. Then*

1. *A/I is a Banach algebra.*
2. *If $T : A \to B$ is a bounded homomorphism then $ker(T)$ is a closed ideal, and there is a unique $\dot{T} : A/ker(T) \to B$ such that $A \xrightarrow{T} B$ is equal to $A \to A/ker(T) \xrightarrow{\dot{T}} B$. Moreover, $\|\dot{T}\| = \|T\|$.*

Definition 8.1. An ideal $I \subseteq A$ is called proper if $I \neq A$. It is maximal if I is proper and there is no ideal lying strictly between I and A.

Theorem 8.2. *Let $I \subseteq A$ be a proper ideal in a unital Banach algebra. Then $\bar{I}$ is a proper ideal.*

Proof. Note that I does not contain any invertible elements, since $I \neq A$. Then for any $x \in I$, by Lemma 5.1, $\|1-x\| \geq 1$. But then $1 \notin \bar{I}$. □

Theorem 8.3. *Let A be a unital Banach algebra. Then every proper ideal is contained in a maximal ideal and every maximal ideal is closed.*

Proof. If I is a maximal ideal, it is proper, so $\bar{I}$ is also a proper ideal. But $I \subseteq \bar{I} \subset A$ implies $I = \bar{I}$, since I is maximal. This implies that I is closed. For the first part, let I be a proper ideal of A. Let $X = \{J : J \text{ is a proper ideal containing } I\}$ with the partial ordering of inclusion. Any chain has an upper bound; namely, the union, which is a proper ideal (see proof of Theorem 8.2 — 1 is still too far away!). So by Zorn's lemma, X has a maximal element, which must be a maximal ideal. □

Definition 8.2. Let A be a unital, commutative Banach algebra. The Gelfand spectrum $\mathrm{sp}(A)$ is the set of non-zero homomorphisms from A to $\mathbb{C}$.

Theorem 8.4.

1. *Every element in $sp(A)$ is continuous with norm 1.*
2. *$sp(A)$ is in bijective correspondence with the set of maximal ideals of A.*

Proof.

1.
2. Given $\omega \in \mathrm{sp}(A)$, $\ker(\omega)$ is an ideal of codimension 1, so it is maximal (call this ideal I_ω). Conversely, starting with an ideal I of codimension (the dimension of A/I) 1, we can write $A \to A/I \cong \mathbb{C}$ to get an element of $\mathrm{sp}(A)$ (call this ω_I). (Here we have used the true fact that in a commutative algebra, every maximal ideal has codimension 1. We'll explain this later.)

   Claim: $\omega_{I_\omega} = \omega$ and $I_{\omega_I} = I$.

   Proof. $I_{\omega_I} = \ker(\omega_I) = I$, and $\omega_{I_\omega} = \omega_{\ker(\omega)} = (A \to A/\ker(\omega) \to \mathbb{C}) = \omega$, by uniqueness of the map from $A/\ker(\omega) \to \mathbb{C}$. □

It remains to show that the ideals of codimension 1 are exactly the maximal ideals. Suppose $I \subset A$ has codimension 1. Suppose $x \notin I$. Then $[x] \neq 0$ in A/I, so we can write $[x] = \lambda[\mathbf{1}]$ for some $\lambda \neq 0$, so $x = \lambda\mathbf{1} + I$. But then the ideal generated by I and x contains $\lambda\mathbf{1} + I$, so it contains $\lambda\mathbf{1}$ and hence $\mathbf{1}$. Conversely, let $I \subset A$ be a proper ideal, and suppose A/I has dimension greater than 1. Choose $x \in A \backslash I$ such that $[x]$ is not invertible in A/I (Theorem 9.1). Consider the ideal $J = I + Ax$ (that it is an ideal depends on commutativity). Then J is a proper ideal — $\mathbf{1}$ cannot be in J, because if it were, then $\mathbf{1} = i + ax$ for some $i \in I$ and $a \in A$. But then $[a][x] = [x][a] = [\mathbf{1}]$ — but we took x so that $[x]$ was not invertible in A/I. So I is not maximal.

□

Lecture 9

Theorem 9.1. *Every Banach division algebra (unital algebra where every non-zero element is invertible) is* 1*-dimensional.*

Proof. Let A be a unital Banach division algebra, and let $x \in A$. Suppose $\lambda \in \sigma(x)$. Then $x - \lambda$ is not invertible, so $x - \lambda = 0$, which means $x = \lambda\mathbf{1}$. □

Proposition 9.1. *Every 1-dimensional unital Banach algebra is isometrically isomorphic to $\mathbb{C}$, and this isomorphism is unique.*

Proof. Can construct the obvious isomorphism, just have to check it actually is an isomorphism. For uniqueness, let A be a 1-dimensional unital Banach algebra, and let $\phi : A \to \mathbb{C}$ be an isomorphism of complex algebras. Then $\phi(\mathbf{1}) = \phi(\mathbf{1} \cdot \mathbf{1}) = \phi(\mathbf{1}) \cdot \phi(\mathbf{1})$, so $\phi(\mathbf{1}) = 0$ or $\phi(\mathbf{1}) = 1$. If $\phi(\mathbf{1}) = 0$, then ϕ is the zero homomorphism, so it's not an isomorphism — this means that $\phi(\mathbf{1}) = 1$. □

Proposition 9.2. *Let A be a unital Banach algebra, and let $I \subset A$ be a proper ideal. Then A/I is a unital Banach algebra (including $\|\mathbf{1}\| = 1$).*

Proof. Assume we already know A/I is a Banach algebra (see Lecture 7). The element $[\mathbf{1}]$ is a unit for A/I. We need to show that $\|[\mathbf{1}]\|_{A/I} = 1$. We have

$$\begin{aligned} \|[\mathbf{1}]\|_{A/I} &= \inf_{i \in I} \|\mathbf{1} + i\| \\ &\leq \|\mathbf{1} + 0\| \\ &= 1. \end{aligned}$$

For the other inequality, we want to show that for every $i \in I$, $\|\mathbf{1} + i\| \geq 1$. But if $\|1 + i\| < 1$ for any $i \in I$, then i is invertible (Lemma 5.1), contradicting the fact that I is a proper ideal. □

Lecture 10

Proposition 10.1. *$sp(A)$ is non-empty.*

Proof. Since $\{0\}$ is a proper ideal of A, it is contained in a maximal ideal, which is enough by Theorem 8.4 (2). □

Theorem 10.1. *For all $\omega \in sp(A)$, $\|\omega\| = \omega(1) = 1$.*

Proof. (Sketch.)
For $A \xrightarrow{\omega} \mathbb{C}$, consider $A \xrightarrow{\pi} A/I \xrightarrow{\dot{\omega}} \mathbb{C}$. Use the true fact: $\|\omega\| = \|\dot{\omega}\|$ and the fact that a non-zero homomorphism between one-dimensional algebras is pretty much the identity. $\square$

Let B be a Banach space and B^* be the Banach space of bounded linear functionals on B. There's an isometry $B \to B^{**}$ — if $x \in B$, $\rho \in B^*$, define $\hat{x}(\rho) = \rho(x)$.

Definition 10.1. The *weak* topology on B is the coarsest topology which makes every $\rho \in B^*$ continuous. The *weak-** topology on B^* is the topology on B^* which makes every $\rho \in B \subseteq B^**$ continuous.

Theorem 10.2. *(Banach-Alaoglu Theorem.)*
The unit ball of B^* is compact in the weak- topology.*

Theorem 10.3. *Suppose A is a unital, commutative Banach algebra. Then $sp(A)$ is a compact Hausdorff space in the weak-* topology.*

Proof. (Sketch.)
We know that $\mathrm{sp}(A)$ is a subset of the unit ball of A^*, so by Banach-Alaoglu, we just need to show that $\mathrm{sp}(A)$ is weak-*closed (exercise). $\square$

Definition 10.2. The *Gelfand transform* from A to $C(\mathrm{sp}(A))$ is defined by $x \mapsto \hat{x} \in C(\mathrm{sp}(A))$, where $\hat{x}(\omega) = \omega(x)$.

(Note that $\hat{x}$ is continuous by definition of the weak-*topology.)

Theorem 10.4.

1. *The Gelfand transform is a continuous unital homomorphism from A to $C(sp(A))$.*
2. *For any $x \in A$, $\sigma(x) = \hat{x}(sp(A))$.*

Proof. True facts about Gelfand stuff:

- It's a homomorphism (need to show $\hat{x} \cdot \hat{y} = \widehat{xy}$ and $\hat{x} + \hat{y} = \widehat{x+y}$). Indeed, we have $\hat{x}\hat{y}(\omega) = \omega(x)\omega(y) = \omega(xy) = \widehat{xy}(w)$.
- Unital. $\mathbf{1}$ is the constant function $1 \in C(\mathrm{sp}(A))$. For any $\omega \in \mathrm{sp}(A)$, $\hat{\mathbf{1}}(\omega) = \omega(\mathbf{1}) = \mathbf{1}$, so $\hat{\mathbf{1}}$ is the constant function $\mathbf{1}$.
- $\|\hat{x}\| = \sup_{\omega \in \mathrm{sp}(A)} \|\hat{x}(\omega)\| = \sup_{\omega \in \mathrm{sp}(A)} \|\omega(x)\| \leq 1\|x\|$.
- **Claim:** $x \in A$ is invertible iff $\hat{x}$ is nowhere vanishing.

  Proof of claim. If x is invertible then $\hat{x} \cdot \hat{x}^{-1} = \widehat{xx^{-1}} = \hat{\mathbf{1}} = \mathbf{1}$, so $\hat{x}$ is invertible as well, which means that $\hat{x}$ is nowhere vanishing. If $\hat{x}$ is nowhere vanishing then $\hat{x}(\omega) \neq 0$ for all $\omega \in \mathrm{sp}(A)$. Therefore, x is not contained in any maximal ideal. Then x must be invertible (because otherwise xA would be a proper ideal). $\square$

From the claim:

$$\begin{aligned}
\sigma(x) &= \{\lambda : x - \lambda \text{ is not invertible}\} \\
&= \{\lambda : \widehat{x - \lambda} \text{ is somewhere vanishing}\} \\
&= \{\lambda : \hat{x} \text{ is somewhere equal to } \lambda\} \\
&= \{\lambda : \hat{x} \text{ takes the value } \lambda \text{ for some } w \in \mathrm{sp}(A)\}.
\end{aligned}$$

$\square$

Lecture 11

Example 11.1. Let $A = C(X)$, the continuous functions on a compact Hausdorff space, e.g. with $X = [0, 1]$.

Let $Y \subseteq X$. Then the set of functions which vanish on Y is an ideal, say, I_Y. If $Y_1 \subseteq Y_2$, then $I_{Y_2} \subseteq I_{Y_1}$. The largest such possible ideal is $I_{\{x\}}$ for some $x \in X$. Now, $I_{\{x\}}$ is maximal — can see it constructively, or because $I_{\{x\}}$ is the kernel of the homomorphism $\omega_x : f \mapsto f(x)$ (since $C(X), \mathbb{C}$ are commutative, $\mathbb{C}$ is a field and ω_x is clearly surjective).

Theorem 11.1. *Every maximal ideal of $C(X)$ is of the form $I_{\{x\}}$ for some $x \in X$.*

Proof. Let $\omega \in \mathrm{sp}(A)$. Suppose $\omega \neq \omega_x$ for all x. Then

$$\bigcap_{f \in A} \{x \in X : \omega(f) = f(x)\} = \varnothing.$$

Then by compactness, there exist a finite number of functions $\{f_k\}$ such that $\bigcap_f \{x \in X : \omega(f) = f(x)\} = \varnothing$. So, we have a finite set of functions $\{f_k\}$ such that for each $x \in X$, $\omega(f_k) \neq f_k(x)$ for at least one k. Let $g_k = f_k - \omega(f_k)$ for each k. Then $\omega(g_k) = \omega(f_k) - \omega(f_k) = 0$ for all k, and for each x there is some k such that $g_k(x) \neq 0$. Let $g = \sum_k g_k \overline{g_k}$. Then $\omega(g) = 0$, and for each x, $g(x) > 0$. So g is an invertible element of $C(X)$ which is in $\ker(\omega)$, but this contradicts the fact that $\omega \in \mathrm{sp}(A)$. □

Theorem 11.2. *Let $A = C(X)$, where X is a compact Hausdorff space. For each $x \in X$, let $\omega_x : C(X) \to \mathbb{C}$ be the homomorphism sending $f \mapsto f(x)$. Then $x \mapsto \omega_x$ is a homeomorphism from X to sp(A). When X and sp(A) are identified via this homeomorphism, the Gelfand transform is the identity map.*

Bits of proof. The map $x \mapsto \omega_x$ is injective since $C(X)$ separates points, and is surjective by the previous compactness argument. Since $X, \mathrm{sp}(C(X))$ are compact, it suffices to show continuity in one direction. **True fact:** continuous bijection from a compact space to a Hausdorff space is a homeomorphism[1]. Suppose $x_n \to x$ is a convergent net in X. Then for every $f \in C(X)$, $f(x_n) \to f(x)$, so $\omega_{x_n}(f) \to \omega_x(f)$ and $\hat{f}(\omega_{x_n}) \to \hat{f}(\omega_x)$. Therefore, $\omega(x_n) \to \omega_x$ in the weak-*topology. □

Lecture 12

Let's back up a bit and investigate some true facts about general topological junk (which may explain the end of that last proof). Let X be a set, Y a topological space, and a family of functions $\{f_i\}_{i \in I}$. The weak topology of $\{f_i\}$ on X is the coarsest topology that makes all the f_i continuous.

Example 12.1.

- If B is a Banach space, and $X = B$, $Y = C$, $\{f_i\} = B^*$, then we get the weak topology on B.
- If $X = B^*$, $Y = \mathbb{C}$, $\{f_i\} = B \subseteq B^{**}$, then we get the weak-*topology.

True fact: If for some set $\{x_n\}$ and a point x, $f_i(x_n) \to f_i(x)$ for all i, then $x_n \to x$ in the weak topology.

For X a compact Hausdorff space, $X \cong \mathrm{sp}(C(X))$ via $x \mapsto \omega_x$. For $f \in C(X)$, $\hat{f}(\omega_x) = \omega_x(f) = f(x)$. This proves that the Gelfand transform is quite boring in some sense. :(

[1] http://www.proofwiki.org/wiki/Continuous_Bijection_from_Compact_to_Hausdorff_is_Homeomorphism

Lecture 13

Example 13.1. Let X, Y be compact Hausdorff spaces. Show that if $C(X)$ is isometrically isomorphic to $C(Y)$, then X is homeomorphic to Y.

Proof. $X \cong \mathrm{sp}(C(X)) \cong \mathrm{sp}(C(Y)) \cong Y$. $\square$

Example 13.2. Let $A = \ell^1(\mathbb{Z})$ with convolution, $(a * b)_i = \sum_{j\in\mathbb{Z}} a_j b_{i-j}$. Let $e_i \in A$ be the element in A with 1 in the i^{th} position, and 0 elsewhere. Then

$$(e_i * e_j)_k = \sum_{ell} (e_i)_\ell (e_j)_{k-\ell} = \delta_k^{i+j},$$

so that $e_i * e_j = e_{i+j}$. True fact: A is commutative and unital (e_0 acts as identity).

What is the Gelfand spectrum? Every homomorphism to $\mathbb{C}$ is determined by what it does to e_1, since $\omega(e_n) = (\omega(e_1))^n$, for all $n \in \mathbb{Z}$. So the question becomes: for which $\lambda \in \mathbb{C}$ does $\omega(e_1) = \lambda$ extend to a non-zero homomorphism on A? We know from Theorem 8.4 (1) that $\|\omega\| = 1$, so $|\lambda| \le \|\omega\|\|e_1\| = 1$. But we also have $|\lambda^{-1}| \le \|\omega\|\|e_{-1}\| = 1$, which implies that $|\lambda| = 1$. For any $|\lambda| = 1$, $(a_i) \mapsto \sum_i a_i \lambda^i$ is a homomorphism (check). It's an absolutely convergent sequence since $|\lambda| = 1$ and $\sum_i |a_i| < \infty$. Conclusion: Gelfand spectrum is homeomorphic to the circle ($\omega \in \mathrm{sp}(A) \mapsto \omega(e_1)$ is a continuous bijection).

What is the Gelfand transform? Denote the circle from the previous part by π. We have $A \to C(\mathrm{sp}(A)) = C(\pi)$ with $(a_i) \mapsto (\hat{a}_i)$, with

$$(\hat{a}_i)(\omega_\lambda) = \omega_\lambda((a_i)) = \sum_{i\in\mathbb{Z}} a_i \lambda^i.$$

In other words, a sequence $(a_i) \in \ell^1(\mathbb{Z})$ maps to the function $\sum_i a_i z^i \in \mathbb{C}(\pi)$.

Now, $\sum_i a_i z^i = 0$ implies $a_i = 0$ for all i, so the only function in the kernel of the Gelfand transform is 0, and hence it is injective. But it's not surjective — not every continuous function can be written as $\sum_i a_i z^i$ with $\sum_i |a_i| < \infty$.

Definition 13.1. The *Weiner algebra* is the subalgebra of $C(\pi)$ of functions of the form $\sum_i a_i z^i$ with $\sum_i |a_i| < \infty$.

Theorem 13.1. *(Weiner's Theorem.)*
Let f be a nowhere vanishing function in the Weiner algebra. Then $\frac{1}{f}$ is in the Weiner algebra as well.

Proof. Recall that an element in a unital commutative Banach algebra is invertible if its Gelfand transform is non-vanishing ($\sigma(x) = \mathrm{Range}(\hat{x})$). If f is in the Weiner algebra, then $f = (\hat{a}_i)$ for some $(a_i) \in \ell^1(\mathbb{Z})$. If f is non-vanishing, then (a_i) is invertible. Then $(a_i)(a_i)^{-1} = \mathbf{1}$, and $(\hat{a}_i)\widehat{(a_i)^{-1}} = \hat{\mathbf{1}}$. Thus $f \cdot \widehat{(a_i)^{-1}} = 1$, so f is invertible in the Weiner algebra. $\square$