Measure and Integration

Lecture 1

Fix a partition of the interval $[a, b] \subset \mathbb{R}$, $\rho : a = x_0 < x_1 < \cdots < x_n = b$. Let $m_k = \min f(x)$ and $M_k = \max f(x)$, where the min and max are taken over $x \in [x_{k-1}, x_k]$. Let $d = \max \Delta x_k$, where $\Delta x_k = x_k - x_{k-1}$. The *oscillation* is given by $\omega_k = M_k - m_k$; then a function is Riemann integrable if

$$\lim_{d \to 0} \sum_{k=1}^{n} \omega_k \Delta x_k = 0.$$

Denote the Riemann integrable functions over the interval $[a, b]$ by $R[a, b]$.

Lemma 1.1. $C[a, b] \subseteq R[a, b]$.

Proof. If $f \in C[a, b]$, it is uniformly continuous, so for any $\epsilon > 0$, choose d small enough so that

$$\sum_{k=1}^{n} \omega_k \Delta x_k < \epsilon \sum_{k=1}^{n} \Delta x_k = \epsilon(b - a) \to 0.$$

$\square$

Some funky examples:

$$f(x) = \begin{cases} \dfrac{1}{q} & x = \frac{p}{q} \in \mathbb{Q}; \gcd(p.q) = 1 \\ 0 & x \in \mathbb{R} - \mathbb{Q} \end{cases}$$

Show f is continuous on all $x \in \mathbb{R} - \mathbb{Q}$.

Lecture 2

Dirichlet function:

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} - \mathbb{Q} \end{cases}$$

A general function of this type is based on indicator functions: for subsets $A_1, \ldots, A_n, \cdots \subseteq X$, where $A_j \cap A_k = \varnothing$ for $j \neq k$, then

$$f(x) = \sum_{k=1}^{\infty} a_k \chi_A(x),$$

where the $a_k \in \mathbb{C}$, defines a **simple function** $f : X \to \mathbb{C}$.

Let $X =$ set, $2^X = \{A \subseteq X\}$.

Definition 2.1. A measure is some $m : 2^X \to \mathbb{R}^+$ with:

1. Additivity: $m(A \cup B) = m(A) + m(B)$ when $A \cap B = \varnothing$.
2. If $A \subseteq B$, then $m(A) \leq m(B)$.

For f a simple function, we can define the integral with respect to this measure

$$\int_X f(x)\ dm \sim \sum_{k=1}^{\infty} a_k m(A_k).$$

Brave people can try integrate the Dirichlet function with

$$\int_{\mathbb{R}} f(x)\ dm \sim \sum_{r\in\mathbb{Q}} m(\{r\}).$$

Example 2.1. Let $X = [0,1]$.

1. Boring measure $m \equiv 0$. Then $\int f(x)\ dm \sim 0$.

2. Set

$$m(A) = \begin{cases} 1 & \text{if } \frac{1}{2} \in A \\ 0 & \text{if } \frac{1}{2} \notin A. \end{cases}$$

   (This kind of measure is called a *point mass* measure.) With respect to this measure, we can find the integral of a simple function. Let $k_0 \in \mathbb{Z}^+$ with $\frac{1}{2} \in k_0$.

$$\int_X f(x)\ dm \sim \sum_{k=1}^{\infty} a_k m(A_k) = a_{k_0} m(A_{k_0}) = a_{k_0} = f\left(\frac{1}{2}\right).$$

   If no such k_0 exists, it is still consistent — $f(\frac{1}{2})$ must be zero.

3. Let $x_1, \ldots, x_n \in X$, $b_1, \ldots, b_n \in \mathbb{R}^+$ and $A \subseteq X$. Let

$$m(A) = \sum_{k:x_k\in A} b_k.$$

   Then

$$\int_X f(x)\ dm \sim \sum_{k=1}^{n} b_k f(x_k).$$

Theorem 2.1. *(Vitali's Theorem.)*
There is no non-trivial additive measure $m : 2^{\mathbb{R}} \mapsto \mathbb{R}^+$ such that

$$m(A) = m(A + x)$$

where $A \subseteq \mathbb{R}$, $x \in \mathbb{R}$ and $A + x = \{y + x : y \in A\}$.

Proof. Suppose m is a non-trivial translation invariant measure as above. Define equivalence relation on $[0,1]$ given by $x \sim y \iff x - y \in \mathbb{Q}$. Define a Vitali set $V \subseteq [0,1]$ by choosing one class representative from each equivalence class. We claim that for any non-zero $r \in \mathbb{Q}$, $V \cap V + r = \varnothing$. To see this, suppose $x \in V \cap (V + r)$. Then $x \in V + r$ implies $x = y + r$ for some $y \in V$. This means that $x \sim y$ and $\bar{x} = \bar{y}$, but by the definition of V this implies that $x = y$ and hence $r = 0$, a contradiction.

Also,

$$[0,1] \subseteq \bigcup_{r\in\mathbb{Q}\cap[-1,1]} (V + r) \subseteq [-1,2].$$

To see this, for each $x \in [0,1]$, there is some $x_1 \in \bar{x}$ such that $x_1 \in V$. Then $x - x_1 = r \in \mathbb{Q}$, so $x = x_1 + r \in V + r$, that is, $x \in \bigcup_{r \in \mathbb{Q} \cap [-1,1]} (V + r)$. Taking the measure of everything gives

$$m([0,1]) \leq \sum_{r \in \mathbb{Q} \cap [-1,1]} m(V + r) \leq m([-1,2]).$$

Since $m(V+r) = m(V)$ for all r, the sum is an infinite sum of a fixed non-negative real number. But it is also bounded above by a fixed number, $m([-1,2])$, so we must have $m(V) = 0$. This implies that $m([0,1]) = 0$, and hence $m \equiv 0$, a contradiction. $\square$

Lecture 3

Some sets are too freaky, want to restrict stuff. Let $S \subseteq 2^X$.

Definition 3.1. S is a semi-ring if:

1. $S \neq \varnothing$.
2. For any $A, B \in S$, $A \cap B \in S$.
3. For any $A, B \in S$, $A \backslash B = \bigsqcup_{k=1}^{n} C_k$ with $C_k \in S$.

Example 3.1. Semi-rings.

1. $X = \mathbb{R}, S = \{[a,b) : a \leq b\}$.
2. $X = \mathbb{R}^2, S = \{[a,b) \times [c,d) : a \leq b, c \leq d\}$.

Definition 3.2. S is a ring of subsets if:

1. $S \neq \varnothing$.
2. For any $A, B \in S$, $A \cup B \in S$.
3. For any $A, B \in S$, $A \backslash B \in S$.
4. For any $A, B \in S$, $A \cap B \in S$.

A ring S is called an *algebra* if $X \in S$. A ring (resp. algebra) S is called a σ-ring (resp. σ-algebra) if it is also closed under countably many unions/intersections.

Example 3.2.

1. $R = \{\varnothing\}$ is a σ-ring; $R = \{\varnothing, X\}$ is a σ-algebra.
2. $R = 2^X$ is a σ-algebra.

Lecture 4

The stuff before lets us define the measure for semi-rings in a hopefully nicer way:

Definition 4.1. Let S be a semi-ring of subsets of X. A measure is some $m : S \to \mathbb{R}^+$ with:

1. $m(A \sqcup B) = m(A) + m(B)$, for $A, B \in S$ and $A \sqcup B \in S$.
2. $A_1, A_2, \ldots, A_n \in S \implies m\left(\bigsqcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} m(A_k)$ when $\bigsqcup_{k=1}^{m} A_k \in S$ for all $m \leq n$.

It is σ-additive if (2) works for $n = \infty$. Semi-rings make things easy, but they don't allow for very much, so we look to extend measures to larger structures in a sane way.

Let $S \subseteq 2^X$.

Definition 4.2. The minimal ring enveloping S is defined as

$$R(S) = \bigcap_{\substack{S \subseteq R, \\ R \text{ a ring}}} R.$$

The minimal σ-ring enveloping S is (similarly)

$$R_\sigma(S) = \bigcap_{\substack{S \subseteq R_\sigma, \\ R_\sigma \text{ a } \sigma\text{-ring}}} R_\sigma.$$

Proposition 4.1. *Let S be a semi-ring. Then*

$$R(S) = \left\{ \bigsqcup_{k=1}^{n} A_k : A_k \in S \right\}.$$

Proof. Denote the right hand side by R_0. We just need to show that R_0 is a ring. Suppose $A, B \in R_0$. Write

$$A = \bigsqcup_{k=1}^{n} A_k, \qquad B = \bigsqcup_{s=1}^{m} B_s,$$

where the $A_k, B_s \in S$. Then (exercise, or youtube):

$$A \backslash B = \bigsqcup_{k=1}^{N} C_k, \qquad A \cup B = \bigsqcup_{s=1}^{M} D_s,$$

where the $C_k, D_s \in S$. □

Lemma 4.1. *Suppose $m : S \to \mathbb{R}^+$ is a measure. This extends to a measure $\widetilde{m} : R(S) \to \mathbb{R}^+$, where $\widetilde{m}(A) = m(A)$ for all $A \in S$. Also, $\widetilde{m}$ is σ-additive if m is σ-additive.*

Proof. See video, hardest part is σ-additivity. □

For S a semi-ring, what about $R_\sigma(S)$? Can we say

$$R_\sigma(S) = \left\{ \bigsqcup_{n=1}^{\infty} A_n : A_n \in S \right\} := R_{\sigma,0}?$$

No — take the semi-ring of half open intervals, $S = \{[a, b)\}$. Then $[0, 1] \notin R_{\sigma,0}$, for if $[0, 1] = \bigcup_{n=1}^{\infty} [a_n, b_n)$, then there is some n such that $1 \in [a_n, b_n)$. This means there is some $\varepsilon > 0$ with $[1, 1 + \varepsilon] \subseteq [a_n, b_n)$ and hence $[1, 1 + \varepsilon] \subseteq [0, 1]$, a contradiction. On the other hand, that $[0, 1] = [0, 2) \backslash \bigcup_{n=1}^{\infty} [1 + \frac{1}{n}, 2)$ shows it must be in $R_{\sigma,0}$ if it were to be the minimal σ-ring enveloping S, which is kinda sucky.

Lecture 5

Let $S = \{[a, b)\}$, then $R_\sigma(S)$ is the Borel σ-algebra. (It is an algebra because $\mathbb{R} = \bigsqcup_{n \in \mathbb{Z}} [n, n+1)$.) Last time, we saw a botched attempt at describing some sort of structure on $R_\sigma(S)$. Let's try again:

$$R_\sigma(S) = \bigcup_{n=0}^{\infty} R_{\sigma,n},$$

where $R_{\sigma,0} = S$, and

$$R_{\sigma,n} = \left\{ \bigcup_{k=1}^{\infty} A_k, A \cap B, A \backslash B;\ A_k, A, B \in R_{\sigma,n-1} \right\}.$$

Then $|R_\sigma(S)| = 2^{\aleph_0}$. But we see it's not that great — for example, the Cantor set C has measure zero but cardinality $2^{\aleph_0}$. So $|P(C)| > 2^{\aleph_0}$, but this implies we can choose a subset that should definitely be measurable (with measure zero) but is not in the Borel σ-algebra. (I may have missed the point of this bit, not sure.)

Some properties of measures:

Proposition 5.1. *Let $R =$ ring, and $m : R \to \mathbb{R}^+$ be a measure. Then:*

1. $m(\varnothing) = 0$.
2. *If $A, B \in R$ and $A \subseteq B$, then $m(B \backslash A) = m(B) - m(A)$. (Hence $m(A) \leq m(B)$.)*
3. $m(A \cup B) = m(A) + m(B) - m(A \cap B)$.
4. *If m is σ-additive:*
$$m\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} m(A_k).$$

Proof.

1. $m(\varnothing) = m(\varnothing \sqcup \varnothing) = 2m(\varnothing)$.
2. $B = A \sqcup (B \backslash A)$, so $m(B) = m(A) + m(B \backslash A)$.
3. Follows from $A \cup B = A \sqcup (B \backslash (A \cap B))$.

□

Lecture 6

MIA

Lecture 7

Extended "Measure"/Outer "Measure"

Take a measure m on a semi-ring S, and let $A \subseteq X$ be a subset of the enormous set. Define the external "measure" by

$$m^*(A) = \inf \sum_{n=1}^{\infty} m(A_n),$$

where $A \subseteq \bigcup_n A_n$, $A_n \in S$. (It is not a 'proper' measure. We'll eventually limit our choice of subsets of X so that m^* is actually a measure.) Properties:

1. $A \subseteq B \subseteq X \implies m^*(A) \subseteq m^*(B)$.
2. Semi-additivity:
$$m^*\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} m^*(A_k).$$
3. Whacked up triangle inequality:
$$|m^*(A) - m^*(B)| \leq m^*(A \triangle B).$$

Proof.

2. For finitely many only — check brains or youtube for countably infinite. We want $m^*(A \cup B) \leq m^*(A) + m^*(B)$. Fix $\varepsilon > 0$. Take coverings $\{A_n\}$ and $\{B_n\}$ from the semi-ring for A and B respectively, such that $\sum m(A_n) < m^*(A) + \varepsilon$ and $\sum m(B_n) < m^*(B) + \varepsilon$. Then
$$A \cup B \subseteq \left(\bigcup_{n=1}^{\infty} A_n\right) \cup \left(\bigcup_{n=1}^{\infty} B_n\right),$$
so
$$m^*(A \cup B) \leq \sum_{n=1}^{\infty} m(A_n) + \sum_{n=1}^{\infty} m(B_n) < m^*(A) + m^*(B) + 2\varepsilon.$$

□

Proposition 7.1.

1. *$X \in R(S) \implies m^*(A) < \infty \ \forall A \subseteq X$.*
2. *If m is σ-additive, then $m^*(B) = \widetilde{m}(B)$ for all $B \in R(S)$.*

Proof.

2. Suppose m is σ-additive. Write
$$B = \bigsqcup_{k=1}^{n} B_k,$$
where each $B_k \in S$. Then
$$m^*(B) \leq \sum_{k=1}^{n} m(B_k) = \widetilde{m}(B).$$

Fix an $\varepsilon > 0$, and choose a covering $\{A_n\}$ from S such that

$$\sum_{n=1}^{\infty} m(A_n) < m^*(B) + \varepsilon.$$

Write

$$B = \bigcup_{n=1}^{\infty} B \cap A_n,$$

Then by semi-additivity

$$\widetilde{m}(B) \leq \sum_{n=1}^{\infty} \widetilde{m}(B \cap A_n).$$

But $\widetilde{m}(B \cap A_n) \leq \widetilde{m}(A_n) = m(A_n)$, so

$$\widetilde{m}(B) \leq \sum_{n=1}^{\infty} m(A_n) < m^*(B) + \varepsilon.$$

□

Theorem 7.1. *Suppose m is σ-additive and $X \in R(S)$. Let*

$$\mathcal{F} = \{A \subseteq X : \forall \varepsilon > 0, \exists A' \in R(S) : m^*(A \triangle A') < \varepsilon\}.$$

Then $\mathcal{F}$ is a σ-algebra and m^* is a σ-additive measure in $\mathcal{F} \subseteq 2^X$.

There's heaps of junk to prove here.

Lecture 8

Proof of Theorem 7.1. $X \in \mathcal{F}$ is clear (take " X' "$= X$). Suppose $A, B \in \mathcal{F}$. Closure under union: fix an $\varepsilon > 0$, and take $A', B' \in R(S)$ such that

$$m^*(A \triangle A') < \varepsilon, \text{and}$$
$$m^*(B \triangle B') < \varepsilon.$$

Now, $(A \cup B) \triangle \underbrace{(A' \cup B')}_{\in R(S)} \subseteq (A \triangle A') \cup (B \triangle B')$. So

$$m^*((A \cup B) \triangle (A' \cup B')) \leq m^*(A \triangle A') + m^*(B \triangle B') < 2\varepsilon.$$

Closure under set difference: show $(A \backslash B) \triangle (A' \backslash B') \subseteq (A \triangle A') \cup (B \triangle B')$, and use the same argument as before.

Closure under countable union: suppose $A_n \in \mathcal{F}$ for $n = 1, \ldots, \infty$. Let $A = \bigcup A_n$. Fix an $\varepsilon > 0$. For each n, choose $A'_n \in R(S)$ such that

$$m(A_n \triangle A'_n) < \frac{\varepsilon}{2^n}.$$

Let

$$A' = \bigcup_{n=1}^{\infty} A'_n.$$

Then

$$m^*(A \triangle A') \le \sum_{n=1}^{\infty} m^*(A_n \triangle A'_n) < \varepsilon.$$

But this isn't enough because A' is not necessarily in $R(S)$. Now,

$$A \triangle A' \subseteq \bigcup_{n=1}^{\infty} (A_n \triangle A'_n)$$

and

$$A' \subseteq A \cup \left[\bigcup_{n=1}^{\infty} (A_n \triangle A'_n) \right].$$

Observe that

$$\sum_{n=1}^{\infty} \widetilde{m}(A'_n) < \infty.$$

Why? We have

$$\begin{aligned}
\sum_{n=1}^{N} \widetilde{m}(A'_n) &= \widetilde{m}\left(\bigcup_{n=1}^{N} A'_n \right) \\
&= m^*\left(\bigcup_{n=1}^{N} A'_n \right) \\
&\le m^*\left(\underbrace{\bigcup_{n=1}^{\infty} A'_n}_{A'} \right) \\
&\le m^*(A) + \sum_{n=1}^{\infty} m^*(A_n \triangle A'_n) \\
&\le m^*(A) + \varepsilon \\
&\le m^*(A) + 1.
\end{aligned}$$

Now, how do we fix the A'? Choose $N_\varepsilon \ge 1$ such that

$$\sum_{n=N_\varepsilon+1}^{\infty} \widetilde{m}(A'_n) < \varepsilon.$$

Let

$$A'' = \bigcup_{n=1}^{N_\varepsilon} A'_n.$$

Then

$$A \triangle A'' \subseteq \left[\bigcup_{n=1}^{\infty} (A_n \triangle A'_n) \right] \cup \left[\bigcup_{n=N_\varepsilon+1}^{\infty} A'_n \right].$$

So

$$\begin{aligned}
m^*(\text{LHS}) &\le \sum_{n=1}^{\infty} m^*(A_n \triangle A'_n) + \sum_{n=N_\varepsilon+1}^{\infty} m^*(A'_n) \\
&\le 2\varepsilon.
\end{aligned}$$

We'll still need to show that it's a proper measure!

Lecture 9

Continuing on with the proof from last time.

Proof. We want to show that for $A, B \in \mathcal{F}$ with $A \cap B = \varnothing$, $m^*(A \sqcup B) = m^*(A) + m^*(B)$. Semi-additivity gives us "$\leq$", so we'll only need to prove "$\geq$". Fix $\varepsilon > 0$, and take $A', B' \in R(S)$ such that

$$m^*(A \triangle A') < \varepsilon \quad \text{and} \quad m^*(B \triangle B') < \varepsilon.$$

Now $A \subseteq A' \cup (A \triangle A')$ and $B \subseteq B' \cup (B \triangle B')$. Thus

$$m^*(A) \leq m^*(A') + \varepsilon \quad \text{and} \quad m^*(B) \leq m^*(B') + \varepsilon.$$

Adding these gives

$$m^*(A) + m^*(B) \leq \widetilde{m}^(A') + \widetilde{m}(B') + 2\varepsilon,$$

since m^* and $\widetilde{m}$ coincide on $R(S)$. Then

$$m^*(A) + m^*(B) \leq \widetilde{m}(A' \cup B') + \widetilde{m}(A' \cap B') + 2\varepsilon.$$

Now,

$$\begin{aligned} A' \cup B' &\subseteq (A \sqcup B) \cup (A \triangle A') \cup (B \triangle B'), \text{ and} \\ A' \cap B' &\subseteq (\underbrace{A \cap B}_{\varnothing}) \cup (A \triangle A') \cup (B \triangle B'). \end{aligned}$$

So

$$\begin{aligned} \widetilde{m}(A' \cup B') &= m^*(A' \cup B') \leq m^*(A \sqcup B) + 2\varepsilon, \text{ and} \\ \widetilde{m}(A' \cap B') &= m^*(A' \cap B') \leq 2\varepsilon. \end{aligned}$$

Thus

$$m^*(A) + m^*(B) \leq m^*(A \sqcup B) + 6\varepsilon.$$

What about for countable disjoint unions? For a measure on a ring, additivity with semi-additivity implies σ-additivity. $\square$

If m is a σ-additive measure on a semi-ring S and $X \in R(S)$, then $(X; S, m) \mapsto (\mathcal{F}, m^*)$ is a finite Lebesgue extension.

If we relax the restriction that $X \in R(S)$ to just that

$$X = \bigsqcup_{n=1}^{\infty} X_n,$$

where $X_n \in S$, then we call it a σ-finite extension. In this case, define new semi-rings

$$S_n = \{A \cap X_n : A \in S\} \subseteq S.$$

Then restrict $m : S_n \to \mathbb{R}^+$, to get a finite Lebesgue extension

$$(X_n; S_n, m) \mapsto (X_n; \mathcal{F}_n, m_n^*).$$

LET'S KEEP GOING, define

$$\mathcal{F} = \{A \subseteq X : A \cap X_n \in \mathcal{F}_n\},$$
$$\mathcal{F}_0 = \left\{ A \in \mathcal{F} : \sum_{n=1}^{\infty} m_n^*(A \cap X_n) < \infty \right\}.$$

Then let $\mu : \mathcal{F}_0 \to \mathbb{R}^+$, with $\mu(A) = \sum_{n=1}^{\infty} m_n^*(A \cap X_n)$.

Theorem 9.1.

1. *$\mathcal{F}$ is a σ-algebra.*
2. *$\mathcal{F}_0$ is a ring.*
3. *μ is σ-additive.*

3.1. *If $A_n \in \mathcal{F}_0$, and $A_i \cap A_j = \varnothing$ for $i \neq j$, and $\sum \mu(A_n) < \infty$, then $A = \bigsqcup A_n \in \mathcal{F}_0$, and*

$$\mu(A) = \sum_{n=1}^{\infty} \mu(A_n).$$

3.2. *If $\sum \mu(A_n) = \infty$ then $A \notin \mathcal{F}_0$.*

Lecture 10

From now on, we'll call $(X; \mathcal{F}, m)$ a *measure space*, where $\mathcal{F}$ is a σ-algebra and $m : \mathcal{F}_0 \to \mathbb{R}^+$, where $\mathcal{F}_0$ is a ring and m is σ-additive.

Definition 10.1. A function $f : X \to \mathbb{R}$ is measurable (we say $f \in \mathbb{L}^0(X; \mathcal{F}, m)$) if

$$\{f < c\} := f^{-1}((-\infty, c)) = \{x \in X : f(x) < c\} \in \mathcal{F} \text{ for all } c \in \mathbb{R}.$$

Lemma 10.1. *The following are equivalent:*

1. *$\{f > c\} \in \mathcal{F}$ for all $c \in \mathbb{R}$.*
2. *$\{f \geq c\} \in \mathcal{F}$ for all $c \in \mathbb{R}$.*
3. *$\{f < c\} \in \mathcal{F}$ for all $c \in \mathbb{R}$.*
4. *$\{f \leq c\} \in \mathcal{F}$ for all $c \in \mathbb{R}$.*
5. *$f^{-1}(B) \in \mathcal{F}$ for all $B \in \mathcal{B}(\mathbb{R})$.*

Proof. Based on closure conditions on $\mathcal{F}$.
(1) $\implies$ (2) — $\{f \geq c\} = \bigcap_n \{f > c - \frac{1}{n}\}$.
(2) $\implies$ (3) — $\{f < c\} = X \setminus \{f \geq c\}$.
(3) $\implies$ (4) — $\{f \leq c\} = \bigcap_n \{f < c + \frac{1}{n}\}$.
(5) $\implies$ (1) — $\{f > c\} = f^{-1}((c, \infty))$, note $(c, \infty) \in \mathcal{B}(\mathbb{R})$.

(4) $\implies$ (5) — we want to show (2,3) $\implies$ (5).

Let $\mathcal{R}$ be the ring defined by $\mathcal{R} = \{A \subseteq \mathbb{R} : f^{-1}(A) \in \mathcal{F}\}$. Check: if $A, B \in \mathcal{R}$, then $f^{-1}(A), f^{-1}(B) \in \mathcal{F}$, so $f^{-1}(A) \backslash f^{-1}(B) \in \mathcal{F}$ and $f^{-1}(A) \cup f^{-1}(B) \in \mathcal{F}$. It follows that $A \backslash B \in \mathcal{R}$ and $A \cup B \in \mathcal{R}$. If $A_n \in \mathcal{R}$ and $f^{-1}(A_n) \in \mathcal{F}$, then $f^{-1}(\bigcup A_n) = \bigcup f^{-1}(A_n) \in \mathcal{F}$. So R is a σ-ring.

Now, since $f^{-1}([a,b)) = \{f \geq a\} \cap \{f \geq b\}$, we have $S = \{[a,b)\} \subseteq \mathcal{R}$. But $\mathcal{B}(\mathbb{R})$ is the minimal σ-ring enveloping S, so $S \subseteq \mathcal{B}(\mathbb{R}) \subseteq \mathcal{R}$. □

Lecture 11

There's some connections with algebras we've seen before.

True facts: For $f, g \in \mathbb{L}^0$:

1. $f + g \in \mathbb{L}^0$.
2. $\lambda f \in \mathbb{L}^0$ for all $\lambda \in \mathbb{R}$.
3. $f \cdot g \in \mathbb{L}^0$.
4. $f_n \in \mathbb{L}^0$, $f(x) = \lim_{n \to \infty} f_n(x)\ \forall x \implies f \in \mathbb{L}^0$.

Proof. (Partial.)

1. We show $\{f + g < c\} = \bigcup_{r \in \mathbb{Q}} \{f < r\} \cap \{g < c - r\}$. The $\supseteq$ direction is easy, for $\subseteq$, suppose $x \in$ LHS. Then $f(x) < c - g(x)$, and by the density of $\mathbb{Q}$, choose some $r \in \mathbb{Q}$ such that
$$f(x) < r < c - g(x).$$
Then $x \in \{f < r\}$ *and* $x \in \{g < c - r\}$, so $x \in$ RHS. It follows that $\{f + g < c\} \in \mathcal{F}$.
2. If $\lambda = 0$, then $\lambda f \equiv 0 \in \mathcal{F}$. If $\lambda > 0$, then $\{\lambda f < c\} = \{f < \frac{c}{\lambda}\} \in \mathcal{F}$ since f is measurable. If $\lambda < 0$, then $\{\lambda f < c\} = \{f > \frac{c}{\lambda}\} \in \mathcal{F}$, again since f is measurable (see Lemma 10.1).
4. We show $\{f > c\} = \liminf \{f_n > c\}$. If $x \in$ LHS, then $\lim f_n(x) > c$, so there exists an N such that for all $n \geq N$, $f_n(x) > c$. This implies that $x \in$ RHS; reversing this shows the other direction.

□

Convergence

yayayay everyone loves convergence. Suppose $f_n, f \in \mathbb{L}^0(X, \mathcal{F}, m)$. Types of convergence IN ORDER OF INCREASING WEAKNESS:

- Uniform: $f_n \rightrightarrows f$, means $\displaystyle\lim_{n \to \infty} \sup_{x \in X} |f_n(x) - f(x)| = 0$.
- Pointwise: $f_n \to f$, means $\displaystyle\lim_{n \to \infty} |f_n(x) - f(x)| = 0$ for all $x \in X$.
- Almost everywhere: $f_n \xrightarrow{\text{a.e.}} f$, means $m(X \backslash A) = 0$, where $A = \{x \in X : \lim_{n \to \infty} f_n(x) = f(x)\}$.
- Measure topology: $f_n \xrightarrow{\text{m}} f$, means $\displaystyle\lim_{n \to \infty} m\{x \in X : |f_n(x) - f(x)| \geq \varepsilon\} = 0$ for all $\varepsilon > 0$.

Lecture 12

We prove that the converges from last lecture actually appear in order of weakness.

Proof. Uniform $\implies$ pointwise: $|f_n(x) - f(x)| \leq \sup_{x' \in X} |f_n(x') - f(x')|$ for all $x \in X$.

Pointwise $\implies$ almost everywhere: $D := \{x \in X : f_n(x) \not\to f(x)\} = \varnothing$.

Almost everywhere $\implies$ measure: Fix $\varepsilon > 0$, and let

$$A_n(\varepsilon) = \{x \in X : |f_n(x) - f(x)| \geq \varepsilon\},$$

and

$$B_k(\varepsilon) = \bigcup_{n=k}^{\infty} A_n(\varepsilon).$$

Let $B(\varepsilon) = \bigcap_k B_k(\varepsilon)$. He wrote some set inclusion up, not sure what he was trying to do there — but I guess since f_n converges a.e., $m(B(\varepsilon)) = 0$, so $\lim_k m(B_k) = 0$ and we can find some N for which $m(B_N(\varepsilon)) < \varepsilon$. But $B_N(\varepsilon)$ contains the set $\{x \in X : \forall n > N, |f_n(x) - f(x)| \geq \varepsilon\}$, and we're done. $\square$

Example 12.1. For a sequence that converges almost everywhere but not pointwise, take a pointwise convergent sequence and change one point of the limit function. Function that converges pointwise but not uniformly:

$$f_n(x) = \frac{nx}{n^2 + x^2} \to 0 = f(x)$$

pointwise, but $\sup_x |f_n(x) - f(x)| = \frac{1}{2}$.

Hardest one is a function that converges with respect to $\xrightarrow{\text{m}}$ but not $\xrightarrow{\text{a.e.}}$. Let

$$A_{nk} = \left[\frac{k-1}{n}, \frac{k}{n}\right],$$

where $n = 1, 2, 3, \ldots$ and $k = 1, 2, \ldots, n$. Let $f_{nk} = \chi_{A_{nk}}$, relabel as f_s where

$$s = \frac{n(n-1)}{2} + k.$$

Then, $f_s \xrightarrow{\text{m}} f = 0$. To see this, note that

$$\{|f_s - f| \geq \varepsilon\} = \{f_s \geq \varepsilon\} = \begin{cases} \varnothing & \text{if } \varepsilon > 1 \\ A_{nk} & \text{if } \varepsilon \leq 1 \end{cases}.$$

So $m(\{f_s \geq \varepsilon\}) \leq \frac{1}{n} \to 0$. But $f_s \not\to f$ *anywhere*, so we're done.

Lecture 13

Theorem 13.1. *If $f_n, f \in \mathbb{L}^0(X; \mathcal{F}, m)$ where $m(X) < \infty$, and $f_n \xrightarrow{a.e.} f$, then for all $\delta > 0$, there is some $E \in \mathcal{F}$ such that $m(X \backslash E) < \delta$ and $f_n \rightrightarrows f$ on E.*

Proof. Let $A_n(\varepsilon) = \{|f - f_n| \geq \varepsilon\}$, and

$$C_n(\varepsilon) = \bigcup_{k=n}^{\infty} A_k(\varepsilon).$$

Note $C_1(\varepsilon) \supseteq C_2(\varepsilon) \supseteq \dots$. Let $C(\varepsilon) = \bigcap_n C_n(\varepsilon)$. If $x \in C(\varepsilon)$, then for all n, there exists some $k \geq n$, such that

$$x \in A_k(\varepsilon) \iff |f_k(x) - f(x)| \geq \varepsilon.$$

But for all $\varepsilon > 0$, $m(C(\varepsilon)) = 0 = \lim_n m(C_k(\varepsilon))$. In particular, for $\varepsilon = \frac{1}{k}$, there exists some n_k such that $m(C_{n_k}(\frac{1}{k})) < \frac{d}{2^k}$. Let

$$E = X \backslash \bigcup_{k=1}^{\infty} C_{n_k}\left(\frac{1}{k}\right).$$

Then

$$m(X \backslash E) < \sum_{k=1}^{\infty} \frac{\delta}{2^k} = \delta.$$

Fix $\varepsilon > 0$, and choose k such that $\frac{1}{k} < \varepsilon$. Then if $x \in E$, $x \notin C_{n_k}(\frac{1}{k})$, which implies that $x \notin A_n(\frac{1}{k})$ for all $n \geq n_k$. But

$$\sup_{x \in E} |f_n(x) - f(x)| < \varepsilon \quad \iff \quad |f_n(x) - f(x)| < \frac{1}{k} < \varepsilon \ \forall x \in E,$$

and we're done. □

Theorem 13.2. *If $f_n \xrightarrow{m} f$, then there exists some subsequence n_k such that $f_{n_k} \xrightarrow{a.e.} f$.*

Proof. Again, let

$$A_n(\varepsilon) = \{|f_n - f| \geq \varepsilon\}.$$

Then $\lim_n m(A_n(\varepsilon)) = 0$, so there exists n_k such that $m(A_{n_k}(\frac{1}{k})) < 2^{-k}$. Let

$$B_n = \bigcup_{k=n}^{\infty} A_{n_k}\left(\frac{1}{k}\right),$$

and $B = \bigcap_n B_n$. Then

$$m(B_n) \leq \sum_{k=n}^{\infty} 2^{-k} = 2^{-n+1} \to 0.$$

So $m(B) = 0$. But $\{x \in X : f_{n_k}(x) \not\to f(x)\} \subseteq B$ (check that if $x \notin B$, then $f_{n_k}(x) \to f(x)$.)

□

There's another half-proved theorem, might include it tomorrow.

Lecture 14

Theorem 14.1. *(Lusin's Theorem.)*
Suppose $f \in \mathbb{L}^0[a,b]$. Then for all $\delta > 0$, there exists some $E \in \mathcal{F}[a,b]$ such that $f \in C(E)$ and $m(E^c) < \delta$.

Integration

Suppose $(X;\mathcal{F},m)$ is a finite measure space. Let f be a simple function,

$$f=\sum_{n=1}^{\infty}a_n\chi_{A_n},$$

$a_n\in\mathbb{R}$, $A_n\in\mathcal{F}$, $X=\bigsqcup_n A_n$. We say f is integrable and write $f\in\mathbb{L}^1=\mathbb{L}^1(X;\mathcal{F},m)$ if

$$\sum_{n=1}^{\infty}|a_n|m(A_n)<\infty.$$

The integral is

$$\int_X f\ dm:=\sum_{n=1}^{\infty}a_nm(A_n)..$$

Suppose $f\in\mathbb{L}^0$. Then $f\in\mathbb{L}^1$ if and only if there exist simple $f_n\in\mathbb{L}^1$ such that $f_n\rightrightarrows f$ and

$$\limsup_{n\to\infty}\int_X|f_n|\ dm<\infty,$$

and we say

$$\int_X f\ dm:=\lim_{n\to\infty}\int_X f_n\ dm.$$

Suppose f,g are simple functions in $\mathbb{L}^1$. True facts:

0. $f\in\mathbb{L}^1\iff|f|\in\mathbb{L}^1$.
1. $\alpha f+\beta g\in\mathbb{L}^1$, the integral is linear.
2. $f\in\mathbb{L}^1$ and $|g|\le f\implies g\in\mathbb{L}^1$ and $|\int_X g\ dm|\le\int_X f\ dm$.
3. $A:=\sup_x|f(x)|<\infty\implies f\in\mathbb{L}^1$, and $\int_X|f|\ dm\le A\cdot m(X)$.

Lecture 15

We've been writing $f\in\mathbb{L}^0$ for measurable functions, and $f\in\mathbb{L}^1$ for integrable functions. Let's have a look at some nasty things.

Let $f_n=\chi_{[0,\frac{1}{n}]}$. Then $f_n\xrightarrow{\text{a.e.}}0$, $f_n\xrightarrow{\text{a.e.}}\chi_0$ and $f_n\xrightarrow{\text{a.e.}}\sum_{r\in\mathbb{Q}}\chi_r$. This is quite distressing, so we introduce an equivalence relation given by $f\sim g\iff m\{f\neq g\}=0$. True facts:

1. If f is measurable and $f\sim g$, then g is also measurable.
2. If $f,g\in C(\mathbb{R})$ and $f\sim g$, then $f=g$.
3. If $f_n\xrightarrow{\text{m}}f$ and $f_n\xrightarrow{\text{m}}g$, then $f\sim g$.

Proof.

2. If $f(x_0)\neq g(x_0)$ then $|f(x_0)-g(x_0)|>0$. Since $|f-g|\in C(\mathbb{R})$, there is some $\delta>0$ such that $|f(x)-g(x)|>0$ for all $x\in(x_0,\delta,x_0+\delta)$, contradicting the fact that $f\sim g$.

3. We have $\{f \neq g\} = \{|f-g|>0\} = \bigcup_{k=1}^{\infty} \underbrace{\left\{|f-g| > \frac{1}{k}\right\}}_{A_k}$. Note

$$A_k \subseteq \left\{|f-f_n| > \frac{1}{2k}\right\} \cup \left\{|g-f_n| > \frac{1}{2k}\right\},$$

since if $x \notin$ RHS, then $|f(x)-f_n(x)| \leq \frac{1}{2k}$ and $|g(x)-f_n(x)| \leq \frac{1}{2k}$, and the triangle inequality implies $x \notin A_k$. Note that this is independent of n. Since both terms in the union limit to zero by assumption, $m(A_k) = 0$ for each k. Thus $m(\{f+g\}) = 0$.

□

Denote the equivalence class of f by $\bar{f}$, and write $\mathbb{L}^0 = \{\bar{f} : f \text{ is measurable}\}$. This avoids the nasty stuff before: suppose $\bar{f}_n \in \mathbb{L}^0$. If $f_n \xrightarrow{\text{m}} f$, $f_n' \in \bar{f}_n$ and $f_n' \xrightarrow{\text{m}} f'$, then $f' \in \bar{f}$. So we do not think of functions in $\mathbb{L}^0$ as individual functions, but classes of functions that only disagree on a set of measure zero. We'll write $\mathbb{L}^1$ as

$$\mathbb{L}^1 = \{\bar{f} \in \mathbb{L}^0 : \exists f' \in \bar{f}, f' \text{ Lebesgue integrable}\}.$$

The $\mathbb{L}^1$ norm is given by

$$\|\bar{f}\|_1 = \int_X |f'| \, dm.$$

For all $f'' \in \bar{f}$, f'' is Lebesgue integrable and $\int_X |f''| \, dm = \int_X |f'| \, dm.$

Lemma 15.1. *If $f \sim g$ and f is integrable, then g is integrable and $\int_X f \, dm = \int_X g \, dm$.*

Proof. There exist simple f_n such that $f_n \rightrightarrows f$ and $\limsup_n \int_X |f_n| \, dm < \infty$. Let $A = \{f = g\}$, so that $m(X \backslash A) = 0$. So $f_n \chi_A \rightrightarrows g\chi_A$. But

$$\lim n \to \infty \int |f_n \chi_A| \, dm \leq \limsup_{n\to\infty} \int_X |f_n| \, dm < \infty.$$

For $g\chi_{X\backslash A}$, take $g_n \rightrightarrows g\chi_{X\backslash A}$, where

$$g_n = \sum_{k\in\mathbb{Z}} \frac{k}{n} \chi_{\{\frac{k-1}{n} < g\chi_{X\backslash A} \leq \frac{k}{n}\}}.$$

(In fact, this uniform approximation works for *any* measurable function.) Denote the subscript of the indicator on the RHS by $A_{k,n}$. Now,

$$0 = \int_X g_n \, dm = \sum_{k\in\mathbb{Z}} \frac{k}{n} m(A_{k,n}).$$

□

True facts about things in $\mathbb{L}^1$:

1. $\bar{f} \in \mathbb{L}^1 \iff \overline{|f|} \in \mathbb{L}^1$.
2. $\bar{f}, \bar{g} \in \mathbb{L}^1 \implies \alpha\bar{f} + \beta\bar{g} \in \mathbb{L}^1$, and

$$\|\alpha\bar{f} + \beta\bar{g}\|_1 \leq |\alpha|\|\bar{f}\|_1 + |\beta|\|barg\|_1.$$

3. $\bar{f} \in \mathbb{L}^1, |g| \leq f \implies \bar{g} \in \mathbb{L}^1$, and
$$\left| \int_X g \ dm \right| \leq \|\bar{f}\|_1.$$

We also write $\mathbb{L}^\infty = \{\bar{f} \in \mathbb{L}^0 : \exists f' \in \bar{f} : \sup_x |f'(x)| < \infty\}$, that is, to be in $\mathbb{L}^\infty$ we require only *one* class representative to be bounded. The $\mathbb{L}^\infty$ norm is given by

$$\|\bar{f}\|_\infty = \inf_{f' \in \bar{f}} \sup_{x \in X} |f'(x)|.$$

4. $\mathbb{L}^\infty \subseteq \mathbb{L}^1 \iff \forall \bar{f} \in \mathbb{L}^\infty, \bar{f} \in \mathbb{L}^1$ and $\|\bar{f}\|_1 \leq \|\bar{f}\|_\infty \cdot m(X)$.

5. If $\bar{f}_n \in \mathbb{L}^1$ is a Cauchy sequence, that is,
$$\lim_{n,k \to \infty} \|\bar{f}_n - \bar{f}_k\|_1 = 0,$$
then there exists some $\bar{f} \in \mathbb{L}^1$ such that $\lim_n \|\bar{f}_n - \bar{f}\|_1 = 0$.