1. Pólya Counting

Texts:

- **van Lint and Wilson**
- Keller and Trotter
- Brualdi

Pólya counting is a method for counting colourings in the presence of symmetry. It can also be used to count unlabelled objects.

Example: Consider a square. We will colour the vertices of the square black or white. If the square is fixed in space there are 16 possibilities. But, if rotations and reflections are allowed, then some of these become equivalent. We now want to count/list colourings up to symmetry. There are only 6.

Group theory Revision: definition of a group, left cosets, Lagrange's theorem. group actions: orbit $G \cdot s$, stabilizer $\mathrm{stab}_G s = \{g : g \cdot s = s\}$, $\mathrm{fix}_S g = \{s : g \cdot s = s\}$, set of orbits $G \backslash S$

Example: The group of symmetries of the square acts on the set of vertex labels $\{1, 2, 3, 4\}$ (not yet on colourings) as follows: rotation corresponds to (1234) or $(13)(24)$ or (1432), reflections correspond to blah (there are 4). These are the only symmetries of the square. There is a single orbit $G \backslash S$. $\mathrm{stab}_G 1 = \{(), (24)\}$, $\mathrm{fix}_S (24) = \{1, 3\}$.

The group of symmetries of the square is the dihedral group with order 8. In general, the group of symmetries of a regular n-gon is the dihedral group D_{2n} of order $2n$, generated by a rotation by $2\pi/n$ and a reflection.

Now suppose G is a group which acts on a finite set A, and let B be a finite set of colours. Then G acts on the set B^A of functions from A to B as follows: given an element $g \in G$ and a $f \in B^A$, define $g \cdot f \in B^A$ by $(g \cdot f)(a) = f\left(g^{-1} \cdot a\right)$ for all $a \in A$. We will check that this defines a G-action.

Proof: Clearly $1 \cdot f = f$ for all $f \in B^A$, where 1 denotes the identity of G. Next, let $g, h \in G$ and $f \in B^A$. Then

$$\begin{aligned}
(gh \cdot f)(a) &= f\left((gh)^{-1} \cdot a\right) \\
&= f\left(\left(h^{-1} g^{-1}\right) \cdot a\right) \\
&= f\left(h^{-1} \cdot \left(g^{-1} \cdot a\right)\right) \\
&= f\left(h^{-1} \cdot \left(g^{-1} \cdot a\right)\right)
\end{aligned}$$

Also,

$$\begin{aligned}
(g \cdot (h \cdot f))(a) &= (h \cdot f)\left(g^{-1} \cdot a\right) \\
&= f\left(h^{-1} \cdot g^{-1} \cdot a\right)
\end{aligned}$$

so $(gh) \cdot f = g \cdot (h \cdot f)$ for all $g, h \in G$, $f \in B^A$, as required. $\square$

We wish to count orbits of the action of G on B^A, which is the same as counting inequivalent B-colourings of A (under the action of G).

Typical question: How many inequivalent colourings of faces of a cube are there, with k colours, up to rotation?

1.1. **Burnside's Lemma.** First we prove

Proposition: Let G be a finite group which acts on the finite set S. Then, for all $s \in S$,

$$\sum_{\hat{s} \in G \cdot s} |\mathrm{stab}_G(\hat{s})| = |G|.$$

Proof (combinatorial): Write $\mathrm{stab}_G(s) = \{g_1, g_2, \ldots, g_k\}$ (g_i distinct) for some $k \in \mathbb{Z}^+$. For any $s' \in G \cdot s$, define $T(s, s') = \{g \in G : g \cdot s = s'\}$. Note, $T(s, s) = \{g \in G : g \cdot s = s\} = \mathrm{stab}_G(s)$. Fix $s' \in G \cdot s$ and choose $g \in T(s, s')$. Then $gg_i \in T(s, s')$ for $i = 1, \ldots, k$. If $gg_i = gg_j$ then $g_i = g_j$, so $i = j$. Hence, the function $\mathrm{stab}_G(s) \to T(s, s') : g_i \mapsto gg_i$ is injective. Furthermore, if $g' \in T(s, s')$ then $g^{-1}g' \in \mathrm{stab}_G(s)$, so $g^{-1}g' = g_j$ for some $j \in \{1, \ldots, k\}$. Hence $g' = gg_j$, which shows the above map is onto. Hence the map is a bijection and so

$$|\mathrm{stab}_G(s)| = |T(s, s')| \tag{1.1}$$

[Note, $s' \in G \cdot s$ was arbitrary.]

Also, for all $s' \in G \cdot s$ we have

$$T(s', s) = \left\{g^{-1} : g \in T(s, s')\right\}. \tag{1.2}$$

Hence $|\mathrm{stab}_G(s')| = |T(s', s)| = |T(s', s)| = |\mathrm{stab}_G(s)|$ by (1.1) and (1.2).

Therefore

$$\sum_{\hat{s} \in G \cdot s} |\mathrm{stab}_G(\hat{s})| = \sum_{\hat{s} \in G \cdot s} |T(s, \hat{s})|$$

by (1.1) and each element of G appears in precisely one set $T(s, \hat{s})$. Therefore the right hand side of the above expression equals $|G|$, as required. □

Lemma (Burnside's Lemma): let G be a finite group which acts on the finite set S, and let $N = |G \backslash S|$ be the number of G-orbits of S. Then $N = \frac{1}{|G|} \sum_{g \in G} |\mathrm{fix}_S(g)|$, the average number of fixed points.

Proof: Let $\mathcal{A} = \{(g, s) : g \in G, s \in S \text{ and } g \cdot s = s\}$. Apply double counting. Counting by g first gives $|\mathcal{A}| = \sum_{g \in G} |\mathrm{fix}_S(g)|$, and counting by s gives

$$|\mathcal{A}| = \sum_{s \in S} |\mathrm{stab}_G(s)| = \sum_{U \in G \backslash S} \sum_{\hat{s} \in U} |\mathrm{stab}_G(\hat{s})| = \sum_{U \in G \backslash S} |G| = N|G|.$$

□

Specialize to our setting:

Theorem: Let A, B be finite sets and let G be a finite group which acts on A. For $i \in \mathbb{Z}^+$, let $c_i(G)$ be the number of elements of G which have exacty i cycles in their disjoint cycle decomposition, considered as permutations of A. Then, the number of orbits of G on B^A is

$$\frac{1}{|G|}\sum_{i=1}^{\infty} c_i(G)\,|B|^i = \#\text{ of inequivalent } B\text{-colourings unde the action of } G.$$

Proof: By Burnside's Lemma, the answer is

$$\frac{1}{|G|}\sum_{g\in G}\Psi(g),$$

where $\Psi(g)$ is the number of colourings $f \in B^A$ which are fixed by g. Suppose that g has i cycles in its cycle decomposition (as a permutation of A). Then $f \in B^A$ is fixed by g iff f is a constant on each of these i cycles. Hence there are exactly $|B|^i$ colourings $f \in B^A$ fixed by g, and the result follows. $\square$

Recall, symmetries of the square, acting on vertex labels. $|B| = 2$. () fixes 2^4 colourings (all of them). (1234) fixes 2^1colourings, etc

By the theorem, the number of inequivalent 2-colourings of the vertices of a square is $\frac{1}{8}\left(1\times 2^4 + 2\times 2^3 + 3\times 2^2 + 2\times 2^1\right) = 6$