Combinatorics

Problem Set 1

1) Ignore the condition that the two sets of people are disjoint, because you can just end the people who are in both to make the sets disjoint. The number of non-empty subsets of the 10 people is $2^{10} - 1 = 1023$, but the possible set of age sums for non-empty subsets is $\{1, 2, \ldots, 600\}$ (this is a very loose upper bound, because we can assume all ages are distinct — or else just choose singleton sets — but this works so why bother). By the pigeonhole principle, since $1023 > \mathbf{1} \times 600$, there is at least one age sum (pigeonhole) containing more than $\mathbf{1}$ subset of people (pigeons)... which is what we wanted all along.

   With 9 people, $2^9 - 1 = 511$. Be a bit smarter and use distinct ages to show that the possible set of age sums for non-empty subsets is $\{1, 2, \ldots, 52 + 53 + \cdots + 60\} = \{1, 2, \ldots, 504\}$ — it still works.

2) Let $f : \{1, \ldots, m\} \to \{1, \ldots, m-1\}$ be defined by $f(i) = a^i \bmod m$ ($a^i \not\equiv 0 \bmod m$ cause they're coprime). By pigeonhole, there are distinct $s, t \in \{1, \ldots, m\}$ such that $f(s) = f(t)$. Assume $s > t$. So $a^s \equiv a^t \bmod m$. Since $\gcd(a, m) = 1$, this implies that $a^{s-t} \equiv 1 \bmod m$.

3) "Direct" approach: look for contiguous blocks of stuff — let x_i be the sum of the games played over the first i days; then if $x_j - x_i = 21$ for some $j > i$ then we're done.

   The total number of games over the 77 days can be at most $11 \times 12 = 132$. Let x_i be the number of games played on days $1, 2, \ldots, i$ inclusive, for $i = 1, \ldots, 77$. We want to use pigeonhole, but it's not immediately able to tell us $x_j - x_i = 21$ — it's better for getting some kind of equality. So let $y_i = x_i + 21$ for $i = 1, \ldots, 77$. But we are most interested in y_i when $y_i \leq 132$, or equivalently, $x_i \leq 111$. This will be definitely true for $i = 1, \ldots, 63$, since $x_{63} \leq 12 \times 9 = 108$.

   Consider the $63 + 77 = 140$ values $\{x_1, \ldots, x_{77}, y_1, \ldots, y_{63}\}$ which lie in the range $\{1, \ldots, 132\}$. By the pigeonhole principle, since $140 > 132$, there is a value $v \in \{1, \ldots, 132\}$ such that at least two of the elements of $\{x_1, \ldots, x_{77}, y_1, \ldots, y_{63}\}$ equal v. Since $x_i < x_{i+1}$ for all $i = 1, \ldots, 76$, the x_i are all distinct, which also implies that the y_i are all distinct. It must be that some $x_j = y_i$ for some i, j. That is, $x_j = x_i + 21$, which is what we wanted all along.

   Alt: Use Example 1.2 from lectures, in any 21 days, there is a consecutive subsequence adding up to a multiple of 21, but there are at most $12 \times 3 = 36$ games, which means it must be 21.

$6\frac{1}{2}$) (a) Define
$$S = \{(A, b) : A \subseteq \{1, \ldots, n\}, b \in A\}.$$

   Calculating the size of S by first counting over A gives

$$|S| = \sum_{\substack{A \subseteq \{1, \ldots, n\} \\ |A| = k}} k = k \binom{n}{k}.$$

Summing over b instead gives

$$
\begin{aligned}
|S| &= \sum_{b \in \{1,\dots,n\}} |\{A \subseteq \{1,\dots,n\} : b \in A, |A| = k\}| \\
&= \sum_{b \in \{1,\dots,n\}} \binom{n-1}{k-1} \\
&= \frac{n(n-1)!}{(k-1)!(n-1-(k-1))!} \\
&= \frac{n!}{(k-1)!(n-k)!} = (n-k+1)\binom{n}{k-1}.
\end{aligned}
$$

$$
|S| = \sum_{b \in \{1,\dots,n\}} |\{A \subseteq \{1,\dots,n\} : b \in A, |A| = k\}|
$$