

Measure and Integration

Lecture 1

Fix a partition of the interval $[a, b] \subset \mathbb{R}$, $\rho : a = x_0 < x_1 < \dots < x_n = b$. Let $m_k = \min f(x)$ and $M_k = \max f(x)$, where the min and max are taken over $x \in [x_{k-1}, x_k]$. Let $d = \max \Delta x_k$, where $\Delta x_k = x_k - x_{k-1}$. The *oscillation* is given by $\omega_k = M_k - m_k$; then a function is Riemann integrable if

$$\lim_{d \rightarrow 0} \sum_{k=1}^n \omega_k \Delta x_k = 0.$$

Denote the Riemann integrable functions over the interval $[a, b]$ by $R[a, b]$.

Lemma 1.1. $C[a, b] \subseteq R[a, b]$.

Proof. If $f \in C[a, b]$, it is uniformly continuous, so for any $\epsilon > 0$, choose d small enough so that

$$\sum_{k=1}^n \omega_k \Delta x_k < \epsilon \sum_{k=1}^n \Delta x_k = \epsilon(b - a) \rightarrow 0.$$

□

Some funky examples:

$$f(x) = \begin{cases} \frac{1}{q} & x = \frac{p}{q} \in \mathbb{Q}; \gcd(p, q) = 1 \\ 0 & x \in \mathbb{R} - \mathbb{Q} \end{cases}$$

Show f is continuous on all $x \in \mathbb{R} - \mathbb{Q}$.

Lecture 2

Dirichlet function:

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} - \mathbb{Q} \end{cases}$$

A general function of this type is based on indicator functions: for subsets $A_1, \dots, A_n, \dots \subseteq X$, where $A_j \cap A_k = \emptyset$ for $j \neq k$, then

$$f(x) = \sum_{k=1}^{\infty} a_k \chi_{A_k}(x),$$

where the $a_k \in \mathbb{C}$, defines a **simple function** $f : X \rightarrow \mathbb{C}$.

Let $X = \text{set}$, $2^X = \{A \subseteq X\}$.

Definition 2.1. A measure is some $m : 2^X \rightarrow \mathbb{R}^+$ with:

1. Additivity: $m(A \cup B) = m(A) + m(B)$ when $A \cap B = \emptyset$.
2. If $A \subseteq B$, then $m(A) \leq m(B)$.

For f a simple function, we can define the integral with respect to this measure

$$\int_X f(x) \, dm \sim \sum_{k=1}^{\infty} a_k m(A_k).$$

Brave people can try integrate the Dirichlet function with

$$\int_{\mathbb{R}} f(x) \, dm \sim \sum_{r \in \mathbb{Q}} m(\{r\}).$$

Example 2.1. Let $X = [0, 1]$.

1. Boring measure $m \equiv 0$. Then $\int f(x) \, dm \sim 0$.

2. Set

$$m(A) = \begin{cases} 1 & \text{if } \frac{1}{2} \in A \\ 0 & \text{if } \frac{1}{2} \notin A. \end{cases}$$

(This kind of measure is called a *point mass* measure.) With respect to this measure, we can find the integral of a simple function. Let $k_0 \in \mathbb{Z}^+$ with $\frac{1}{2} \in A_{k_0}$.

$$\int_X f(x) \, dm \sim \sum_{k=1}^{\infty} a_k m(A_k) = a_{k_0} m(A_{k_0}) = a_{k_0} = f\left(\frac{1}{2}\right).$$

If no such k_0 exists, it is still consistent — $f(\frac{1}{2})$ must be zero.

3. Let $x_1, \dots, x_n \in X$, $b_1, \dots, b_n \in \mathbb{R}^+$ and $A \subseteq X$. Let

$$m(A) = \sum_{k: x_k \in A} b_k.$$

Then

$$\int_X f(x) \, dm \sim \sum_{k=1}^n b_k f(x_k).$$

Theorem 2.1. (*Vitali's Theorem.*)

There is no non-trivial additive measure $m : 2^{\mathbb{R}} \mapsto \mathbb{R}^+$ such that

$$m(A) = m(A + x)$$

where $A \subseteq \mathbb{R}$, $x \in \mathbb{R}$ and $A + x = \{y + x : y \in A\}$.

Proof. Suppose m is a non-trivial translation invariant measure as above. Define equivalence relation on $[0, 1]$ given by $x \sim y \iff x - y \in \mathbb{Q}$. Define a Vitali set $V \subseteq [0, 1]$ by choosing one class representative from each equivalence class. We claim that for any non-zero $r \in \mathbb{Q}$, $V \cap V + r = \emptyset$. To see this, suppose $x \in V \cap (V + r)$. Then $x \in V + r$ implies $x = y + r$ for some $y \in V$. This means that $x \sim y$ and $\bar{x} = \bar{y}$, but by the definition of V this implies that $x = y$ and hence $r = 0$, a contradiction.

Also,

$$[0, 1] \subseteq \bigcup_{r \in \mathbb{Q} \cap [-1, 1]} (V + r) \subseteq [-1, 2].$$

To see this, for each $x \in [0, 1]$, there is some $x_1 \in \bar{x}$ such that $x_1 \in V$. Then $x - x_1 = r \in \mathbb{Q}$, so $x = x_1 + r \in V + r$, that is, $x \in \bigcup_{r \in \mathbb{Q} \cap [-1, 1]} (V + r)$. Taking the measure of everything gives

$$m([0, 1]) \leq \sum_{r \in \mathbb{Q} \cap [-1, 1]} m(V + r) \leq m([-1, 2]).$$

Since $m(V + r) = m(V)$ for all r , the sum is an infinite sum of a fixed non-negative real number. But it is also bounded above by a fixed number, $m([-1, 2])$, so we must have $m(V) = 0$. This implies that $m([0, 1]) = 0$, and hence $m \equiv 0$, a contradiction. \square

Lecture 3

Some sets are too freaky, want to restrict stuff. Let $S \subseteq 2^X$.

Definition 3.1. S is a semi-ring if:

1. $S \neq \emptyset$.
2. For any $A, B \in S$, $A \cap B \in S$.
3. For any $A, B \in S$, $A \setminus B = \bigsqcup_{k=1}^n C_k$ with $C_k \in S$.

Example 3.1. Semi-rings.

1. $X = \mathbb{R}, S = \{[a, b) : a \leq b\}$.
2. $X = \mathbb{R}^2, S = \{[a, b) \times [c, d) : a \leq b, c \leq d\}$.

Definition 3.2. S is a ring of subsets if:

1. $S \neq \emptyset$.
2. For any $A, B \in S$, $A \cup B \in S$.
3. For any $A, B \in S$, $A \setminus B \in S$.
4. For any $A, B \in S$, $A \cap B \in S$.

A ring S is called an *algebra* if $X \in S$. A ring (resp. algebra) S is called a σ -ring (resp. σ -algebra) if it is also closed under countably many unions/intersections.

Example 3.2.

1. $R = \{\emptyset\}$ is a σ -ring; $R = \{\emptyset, X\}$ is a σ -algebra.
2. $R = 2^X$ is a σ -algebra.

Lecture 4

The stuff before lets us define the measure for semi-rings in a hopefully nicer way:

Definition 4.1. Let S be a semi-ring of subsets of X . A measure is some $m : S \rightarrow \mathbb{R}^+$ with:

1. $m(A \sqcup B) = m(A) + m(B)$, for $A, B \in S$ and $A \sqcup B \in S$.
2. $A_1, A_2, \dots, A_n \in S \implies m\left(\bigsqcup_{k=1}^n A_k\right) = \sum_{k=1}^n m(A_k)$ when $\bigsqcup_{k=1}^m A_k \in S$ for all $m \leq n$.

It is σ -additive if (2) works for $n = \infty$. Semi-rings make things easy, but they don't allow for very much, so we look to extend measures to larger structures in a sane way.

Let $S \subseteq 2^X$.

Definition 4.2. The minimal ring enveloping S is defined as

$$R(S) = \bigcap_{\substack{S \subseteq R, \\ R \text{ a ring}}} R.$$

The minimal σ -ring enveloping S is (similarly)

$$R_\sigma(S) = \bigcap_{\substack{S \subseteq R_\sigma, \\ R_\sigma \text{ a } \sigma\text{-ring}}} R_\sigma.$$

Proposition 4.1. Let S be a semi-ring. Then

$$R(S) = \left\{ \bigsqcup_{k=1}^n A_k : A_k \in S \right\}.$$

Proof. Denote the right hand side by R_0 . We just need to show that R_0 is a ring. Suppose $A, B \in R_0$. Write

$$A = \bigsqcup_{k=1}^n A_k, \quad B = \bigsqcup_{s=1}^m B_s,$$

where the $A_k, B_s \in S$. Then (exercise, or youtube):

$$A \setminus B = \bigsqcup_{k=1}^N C_k, \quad A \cup B = \bigsqcup_{s=1}^M D_s,$$

where the $C_k, D_s \in S$. □

Lemma 4.1. Suppose $m : S \rightarrow \mathbb{R}^+$ is a measure. This extends to a measure $\tilde{m} : R(S) \rightarrow \mathbb{R}^+$, where $\tilde{m}(A) = m(A)$ for all $A \in S$. Also, \tilde{m} is σ -additive if m is σ -additive.

Proof. See video, hardest part is σ -additivity. □

For S a semi-ring, what about $R_\sigma(S)$? Can we say

$$R_\sigma(S) = \left\{ \bigsqcup_{n=1}^{\infty} A_n : A_n \in S \right\} := R_{\sigma,0}?$$

No — take the semi-ring of half open intervals, $S = \{[a, b)\}$. Then $[0, 1] \notin R_{\sigma,0}$, for if $[0, 1] = \bigcup_{n=1}^{\infty} [a_n, b_n)$, then there is some n such that $1 \in [a_n, b_n)$. This means there is some $\varepsilon > 0$ with $[1, 1 + \varepsilon] \subseteq [a_n, b_n)$ and hence $[1, 1 + \varepsilon] \subseteq [0, 1]$, a contradiction. On the other hand, that $[0, 1] = [0, 2) \setminus \bigcup_{n=1}^{\infty} [1 + \frac{1}{n}, 2)$ shows it must be in $R_{\sigma,0}$ if it were to be the minimal σ -ring enveloping S , which is kinda sucky.

Lecture 5

Let $S = \{[a, b]\}$, then $R_\sigma(S)$ is the Borel σ -algebra. (It is an algebra because $\mathbb{R} = \bigsqcup_{n \in \mathbb{Z}} [n, n+1)$.) Last time, we saw a botched attempt at describing some sort of structure on $R_\sigma(S)$. Let's try again:

$$R_\sigma(S) = \bigcup_{n=0}^{\infty} R_{\sigma,n},$$

where $R_{\sigma,0} = S$, and

$$R_{\sigma,n} = \left\{ \bigcup_{k=1}^{\infty} A_k, A \cap B, A \setminus B; A_k, A, B \in R_{\sigma,n-1} \right\}.$$

Then $|R_\sigma(S)| = 2^{\aleph_0}$. But we see it's not that great — for example, the Cantor set C has measure zero but cardinality 2^{\aleph_0} . So $|P(C)| > 2^{\aleph_0}$, but this implies we can choose a subset that should definitely be measurable (with measure zero) but is not in the Borel σ -algebra. (I may have missed the point of this bit, not sure.)

Some properties of measures:

Proposition 5.1. *Let $R = \text{ring}$, and $m : R \rightarrow \mathbb{R}^+$ be a measure. Then:*

1. $m(\emptyset) = 0$.
2. If $A, B \in R$ and $A \subseteq B$, then $m(B \setminus A) = m(B) - m(A)$. (Hence $m(A) \leq m(B)$.)
3. $m(A \cup B) = m(A) + m(B) - m(A \cap B)$.
4. σ -additive:

$$m\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} m(A_k).$$

Proof.

1. $m(\emptyset) = m(\emptyset \sqcup \emptyset) = 2m(\emptyset)$.
2. $B = A \sqcup (B \setminus A)$, so $m(B) = m(A) + m(B \setminus A)$.
3. ...

□

Lecture 6

MIA

Lecture 7

Extended “Measure”/Outer “Measure”

Take a measure m on a semi-ring S , and let $A \subseteq X$ be a subset of the enormous set. Define the external “measure” by

$$m^*(A) = \inf \sum_{n=1}^{\infty} m(A_n),$$

where $A \subseteq \bigcup_n A_n$, $A_n \in S$. (It is not a ‘proper’ measure. We’ll eventually limit our choice of subsets of X so that m^* is actually a measure.) Properties:

1. $A \subseteq B \subseteq X \implies m^*(A) \leq m^*(B)$.

2. Semi-additivity:

$$m^*\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} m^*(A_k).$$

3. Whacked up triangle inequality:

$$|m^*(A) - m^*(B)| \leq m^*(A \Delta B).$$

Proof.

2. For finitely many only — check brains or youtube for countably infinite. We want $m^*(A \cup B) \leq m^*(A) + m^*(B)$. Fix $\varepsilon > 0$. Take coverings $\{A_n\}$ and $\{B_n\}$ from the semi-ring for A and B respectively, such that $\sum m(A_n) < m^*(A) + \varepsilon$ and $\sum m(B_n) < m^*(B) + \varepsilon$. Then

$$A \cup B \subseteq \left(\bigcup_{n=1}^{\infty} A_n\right) \cup \left(\bigcup_{n=1}^{\infty} B_n\right),$$

so

$$m^*(A \cup B) \leq \sum_{n=1}^{\infty} m(A_n) + \sum_{n=1}^{\infty} m(B_n) < m^*(A) + m^*(B) + 2\varepsilon.$$

□

Proposition 7.1.

1. $X \in R(S) \implies m^*(A) < \infty \forall A \subseteq X$.

2. If m is σ -additive, then $m^*(B) = \tilde{m}(B)$ for all $B \in R(S)$.

Proof.

2. Suppose m is σ -additive. Write

$$B = \bigsqcup_{k=1}^n B_k,$$

where each $B_k \in S$. Then

$$m^*(B) \leq \sum_{k=1}^n m(B_k) = \tilde{m}(B).$$

Fix an $\varepsilon > 0$, and choose a covering $\{A_n\}$ from S such that

$$\sum_{n=1}^{\infty} m(A_n) < m^*(B) + \varepsilon.$$

Write

$$B = \bigcup_{n=1}^{\infty} B \cap A_n,$$

Then by semi-additivity

$$\tilde{m}(B) \leq \sum_{n=1}^{\infty} \tilde{m}(B \cap A_n).$$

But $\tilde{m}(B \cap A_n) \leq \tilde{m}(A_n) = m(A_n)$, so

$$\tilde{m}(B) \leq \sum_{n=1}^{\infty} m(A_n) < m^*(B) + \varepsilon.$$

□

Theorem 7.1. Suppose m is σ -additive and $X \in R(S)$. Let

$$\mathcal{F} = \{A \subseteq X : \forall \varepsilon > 0, \exists A' \in R(S) : m^*(A \Delta A') < \varepsilon\}.$$

Then \mathcal{F} is a σ -algebra and m^* is a σ -additive measure in $\mathcal{F} \subseteq 2^X$.

There's heaps of junk to prove here.

Lecture 8

Proof of Theorem 7.1. $X \in \mathcal{F}$ is clear (take " X' " = X). Suppose $A, B \in \mathcal{F}$. Closure under union: fix an $\varepsilon > 0$, and take $A', B' \in R(S)$ such that

$$\begin{aligned} m^*(A \Delta A') &< \varepsilon, \text{ and} \\ m^*(B \Delta B') &< \varepsilon. \end{aligned}$$

Now, $(A \cup B) \Delta \underbrace{(A' \cup B')}_{\in R(S)} \subseteq (A \Delta A') \cup (B \Delta B')$. So

$$m^*((A \cup B) \Delta (A' \cup B')) \leq m^*(A \Delta A') + m^*(B \Delta B') < 2\varepsilon.$$

Closure under set difference: show $(A \setminus B) \Delta (A' \setminus B') \subseteq (A \Delta A') \cup (B \Delta B')$, and use the same argument as before.

Closure under countable union: suppose $A_n \in \mathcal{F}$ for $n = 1, \dots, \infty$. Let $A = \bigcup A_n$. Fix an $\varepsilon > 0$. For each n , choose $A'_n \in R(S)$ such that

$$m(A_n \Delta A'_n) < \frac{\varepsilon}{2^n}.$$

Let

$$A' = \bigcup_{n=1}^{\infty} A'_n.$$

Then

$$m^*(A \triangle A') \leq \sum_{n=1}^{\infty} m^*(A_n \triangle A'_n) < \varepsilon.$$

But this isn't enough because A' is not necessarily in $R(S)$. Now,

$$A \triangle A' \subseteq \bigcup_{n=1}^{\infty} (A_n \triangle A'_n)$$

and

$$A' \subseteq A \cup \left[\bigcup_{n=1}^{\infty} (A_n \triangle A'_n) \right].$$

Observe that

$$\sum_{n=1}^{\infty} \tilde{m}(A'_n) < \infty.$$

Why? We have

$$\begin{aligned} \sum_{n=1}^N \tilde{m}(A'_n) &= \tilde{m} \left(\bigcup_{n=1}^N A'_n \right) \\ &= m^* \left(\bigcup_{n=1}^N A'_n \right) \\ &\leq m^* \left(\underbrace{\bigcup_{n=1}^{\infty} A'_n}_{A'} \right) \\ &\leq m^*(A) + \sum_{n=1}^{\infty} m^*(A_n \triangle A'_n) \\ &\leq m^*(A) + \varepsilon \\ &\leq m^*(A) + 1. \end{aligned}$$

Now, how do we fix the A' ? Choose $N_\varepsilon \geq 1$ such that

$$\sum_{n=N_\varepsilon+1}^{\infty} \tilde{m}(A'_n) < \varepsilon.$$

Let

$$A'' = \bigcup_{n=1}^{N_\varepsilon} A'_n.$$

Then

$$A \triangle A'' \subseteq \left[\bigcup_{n=1}^{\infty} (A_n \triangle A'_n) \right] \cup \left[\bigcup_{n=N_\varepsilon+1}^{\infty} A'_n \right].$$

So

$$\begin{aligned} m^*(\text{LHS}) &\leq \sum_{n=1}^{\infty} m^*(A_n \triangle A'_n) + \sum_{n=N_\varepsilon+1}^{\infty} m^*(A'_n) \\ &\leq 2\varepsilon. \end{aligned}$$

We'll still need to show that it's a proper measure!