Measure and Integration

Lecture 1

Fix a partition of the interval $[a,b] \subset \mathbb{R}$, $\rho : a = x_0 < x_1 < \cdots < x_n = b$. Let $m_k = \min f(x)$ and $M_k = \max f(x)$, where the min and max are taken over $x \in [x_{k-1}, x_k]$. Let $d = \max \Delta x_k$, where $\Delta x_k = x_k - x_{k-1}$. The *oscillation* is given by $\omega_k = M_k - m_k$; then a function is Riemann integrable if

$$\lim_{d\to 0} \sum_{k=1}^{n} \omega_k \Delta x_k = 0.$$

Denote the Riemann integrable functions over the interval $[a,b]$ by $R[a,b]$.

Lemma 1.1. $C[a,b] \subseteq R[a,b]$.

Proof. If $f \in C[a,b]$, it is uniformly continuous, so for any $\epsilon > 0$, choose d small enough so that

$$\sum_{k=1}^{n} \omega_k \Delta x_k < \epsilon \sum_{k=1}^{n} \Delta x_k = \epsilon(b-a) \to 0.$$

□

Some funky examples:

$$f(x) = \begin{cases} \dfrac{1}{q} & x = \frac{p}{q} \in \mathbb{Q}; \gcd(p.q) = 1 \\ 0 & x \in \mathbb{R} - \mathbb{Q} \end{cases}$$

Show f is continuous on all $x \in \mathbb{R} - \mathbb{Q}$.

Lecture 2

Dirichlet function:

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} - \mathbb{Q} \end{cases}$$

A general function of this type is based on indicator functions: for subsets $A_1, \ldots, A_n, \cdots \subseteq X$, where $A_j \cap A_k = \varnothing$ for $j \neq k$, then

$$f(x) = \sum_{k=1}^{\infty} a_k \chi_A(x),$$

where the $a_k \in \mathbb{C}$, defines a **simple function** $f : X \to \mathbb{C}$.

Let $X =$ set, $2^X = \{A \subseteq X\}$.

Definition 2.1. A measure is some $m : 2^X \to \mathbb{R}^+$ with:

1. Additivity: $m(A \cup B) = m(A) + m(B)$ when $A \cap B = \varnothing$.
2. If $A \subseteq B$, then $m(A) \leq m(B)$.

For f a simple function, we can define the integral with respect to this measure

$$\int_X f(x)\ dm \sim \sum_{k=1}^{\infty} a_k m(A_k).$$

Brave people can try integrate the Dirichlet function with

$$\int_{\mathbb{R}} f(x)\ dm \sim \sum_{r\in\mathbb{Q}} m(\{r\}).$$

Example 2.1. Let $X = [0, 1]$.

1. Boring measure $m \equiv 0$. Then $\int f(x)\ dm \sim 0$.
2. Set

$$m(A) = \begin{cases} 1 & \text{if } \frac{1}{2} \in A \\ 0 & \text{if } \frac{1}{2} \notin A. \end{cases}$$

   (This kind of measure is called a *point mass* measure.) With respect to this measure, we can find the integral of a simple function. Let $k_0 \in \mathbb{Z}^+$ with $\frac{1}{2} \in k_0$.

$$\int_X f(x)\ dm \sim \sum_{k=1}^{\infty} a_k m(A_k) = a_{k_0} m(A_{k_0}) = a_{k_0} = f\left(\frac{1}{2}\right).$$

   If no such k_0 exists, it is still consistent — $f(\frac{1}{2})$ must be zero.

3. Let $x_1, \ldots, x_n \in X$, $b_1, \ldots, b_n \in \mathbb{R}^+$ and $A \subseteq X$. Let

$$m(A) = \sum_{k: x_k \in A} b_k.$$

   Then

$$\int_X f(x)\ dm \sim \sum_{k=1}^{n} b_k f(x_k).$$

Theorem 2.1. *(Vitali's Theorem.)*
There is no non-trivial additive measure $m : 2^{\mathbb{R}} \mapsto \mathbb{R}^+$ such that

$$m(A) = m(A + x)$$

where $A \subseteq \mathbb{R}$, $x \in \mathbb{R}$ and $A + x = \{y + x : y \in A\}$.

Proof. Suppose m is a non-trivial translation invariant measure as above. Define equivalence relation on $[0, 1]$ given by $x \sim y \iff x - y \in \mathbb{Q}$. Define a Vitali set $V \subseteq [0, 1]$ by choosing one class representative from each equivalence class. We claim that for any non-zero $r \in \mathbb{Q}$, $V \cap V + r = \varnothing$. To see this, suppose $x \in V \cap (V + r)$. Then $x \in V + r$ implies $x = y + r$ for some $y \in V$. This means that $x \sim y$ and $\bar{x} = \bar{y}$, but by the definition of V this implies that $x = y$ and hence $r = 0$, a contradiction.

Also,

$$[0, 1] \subseteq \bigcup_{r\in\mathbb{Q}\cap[-1,1]} (V + r) \subseteq [-1, 2].$$

To see this, for each $x \in [0,1]$, there is some $x_1 \in \bar{x}$ such that $x_1 \in V$. Then $x - x_1 = r \in \mathbb{Q}$, so $x = x_1 + r \in V + r$, that is, $x \in \bigcup_{r \in \mathbb{Q} \cap [-1,1]} (V + r)$. Taking the measure of everything gives

$$m([0,1]) \leq \sum_{r \in \mathbb{Q} \cap [-1,1]} m(V + r) \leq m([-1,2]).$$

Since $m(V+r) = m(V)$ for all r, the sum is an infinite sum of a fixed non-negative real number. But it is also bounded above by a fixed number, $m([-1,2])$, so we must have $m(V) = 0$. This implies that $m([0,1]) = 0$, and hence $m \equiv 0$, a contradiction. $\square$

Lecture 3

Some sets are too freaky, want to restrict stuff. Let $S \subseteq 2^X$.

Definition 3.1. S is a semi-ring if:

1. $S \neq \varnothing$.
2. For any $A, B \in S$, $A \cap B \in S$.
3. For any $A, B \in S$, $A \backslash B = \bigsqcup_{k=1}^{n} C_k$ with $C_k \in S$.

Example 3.1. Semi-rings.

1. $X = \mathbb{R}, S = \{[a,b) : a \leq b\}$.
2. $X = \mathbb{R}^2, S = \{[a,b) \times [c,d) : a \leq b, c \leq d\}$.

Definition 3.2. S is a ring of subsets if:

1. $S \neq \varnothing$.
2. For any $A, B \in S$, $A \cup B \in S$.
3. For any $A, B \in S$, $A \backslash B \in S$.
4. For any $A, B \in S$, $A \cap B \in S$.

A ring S is called an *algebra* if $X \in S$. A ring (resp. algebra) S is called a σ-ring (resp. σ-algebra) if it is also closed under countably many unions/intersections.

Example 3.2.

1. $R = \{\varnothing\}$ is a σ-ring; $R = \{\varnothing, X\}$ is a σ-algebra.
2. $R = 2^X$ is a σ-algebra.

Lecture 4

The stuff before lets us define the measure for semi-rings in a hopefully nicer way:

Definition 4.1. Let S be a semi-ring of subsets of X. A measure is some $m : S \to \mathbb{R}^+$ with:

1. $m(A \sqcup B) = m(A) + m(B)$, for $A, B \in S$ and $A \sqcup B \in S$.
2. $A_1, A_2, \ldots, A_n \in S \implies m\left(\bigsqcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} m(A_k)$ when $\bigsqcup_{k=1}^{m} A_k \in S$ for all $m \leq n$.

It is σ-additive if (2) works for $n = \infty$. Semi-rings make things easy, but they don't allow for very much, so we look to extend measures to larger structures in a sane way.

Let $S \subseteq 2^X$.

Definition 4.2. The minimal ring enveloping S is defined as

$$R(S) = \bigcap_{\substack{S \subseteq R, \\ R \text{ a ring}}} R.$$

The minimal σ-ring enveloping S is (similarly)

$$R_\sigma(S) = \bigcap_{\substack{S \subseteq R_\sigma, \\ R_\sigma \text{ a } \sigma\text{-ring}}} R_\sigma.$$

Proposition 4.1. *Let S be a semi-ring. Then*

$$R(S) = \left\{ \bigsqcup_{k=1}^{n} A_k : A_k \in S \right\}.$$

Proof. Denote the right hand side by R_0. We just need to show that R_0 is a ring. Suppose $A, B \in R_0$. Write

$$A = \bigsqcup_{k=1}^{n} A_k, \qquad B = \bigsqcup_{s=1}^{m} B_s,$$

where the $A_k, B_s \in S$. Then (exercise, or youtube):

$$A \backslash B = \bigsqcup_{k=1}^{N} C_k, \qquad A \cup B = \bigsqcup_{s=1}^{M} D_s,$$

where the $C_k, D_s \in S$. □

Lemma 4.1. *Suppose $m : S \to \mathbb{R}^+$ is a measure. This extends to a measure $\widetilde{m} : R(S) \to \mathbb{R}^+$, where $\widetilde{m}(A) = m(A)$ for all $A \in S$. Also, $\widetilde{m}$ is σ-additive if m is σ-additive.*

Proof. See video, hardest part is σ-additivity. □

For S a semi-ring, what about $R_\sigma(S)$? Can we say

$$R_\sigma(S) = \left\{ \bigsqcup_{n=1}^{\infty} A_n : A_n \in S \right\} := R_{\sigma,0}?$$

No — take the semi-ring of half open intervals, $S = \{[a, b)\}$. Then $[0, 1] \notin R_{\sigma,0}$, for if $[0, 1] = \bigcup_{n=1}^{\infty} [a_n, b_n)$, then there is some n such that $1 \in [a_n, b_n)$. This means there is some $\varepsilon > 0$ with $[1, 1 + \varepsilon] \subseteq [a_n, b_n)$ and hence $[1, 1 + \varepsilon] \subseteq [0, 1]$, a contradiction. On the other hand, that $[0, 1] = [0, 2) \backslash \bigcup_{n=1}^{\infty} [1 + \frac{1}{n}, 2)$ shows it must be in $R_{\sigma,0}$ if it were to be the minimal σ-ring enveloping S, which is kinda sucky.

Lecture 5

Let $S = \{[a, b)\}$, then $R_\sigma(S)$ is the Borel σ-algebra. (It is an algebra because $\mathbb{R} = \bigsqcup_{n\in\mathbb{Z}}[n, n+1)$.) Last time, we saw a botched attempt at describing some sort of structure on $R_\sigma(S)$. Let's try again:

$$R_\sigma(S) = \bigcup_{n=0}^{\infty} R_{\sigma,n},$$

where $R_{\sigma,0} = S$, and

$$R_{\sigma,n} = \left\{ \bigcup_{k=1}^{\infty} A_k, A \cap B, A \backslash B;\ A_k, A, B \in R_{\sigma,n-1} \right\}.$$

Then $|R_\sigma(S)| = 2^{\aleph_0}$. But we see it's not that great — for example, the Cantor set C has measure zero but cardinality $2^{\aleph_0}$. So $|P(C)| > 2^{\aleph_0}$, but this implies we can choose a subset that should definitely be measurable (with measure zero) but is not in the Borel σ-algebra. (I may have missed the point of this bit, not sure.)

Some properties of measures:

Proposition 5.1. *Let $R =$ ring, and $m : R \to \mathbb{R}^+$ be a measure. Then:*

1. $m(\varnothing) = 0$.
2. *If $A, B \in R$ and $A \subseteq B$, then $m(B\backslash A) = m(B) - m(A)$. (Hence $m(A) \leq m(B)$.)*
3. $m(A \cup B) = m(A) + m(B) - m(A \cap B)$.
4. *If m is σ-additive:*
$$m\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} m(A_k).$$

Proof.

1. $m(\varnothing) = m(\varnothing \sqcup \varnothing) = 2m(\varnothing)$.
2. $B = A \sqcup (B\backslash A)$, so $m(B) = m(A) + m(B\backslash A)$.
3. Follows from $A \cup B = A \sqcup (B\backslash(A \cap B))$.

□

Lecture 6

MIA

Lecture 7

Extended "Measure"/Outer "Measure"

Take a measure m on a semi-ring S, and let $A \subseteq X$ be a subset of the enormous set. Define the external "measure" by

$$m^*(A) = \inf \sum_{n=1}^{\infty} m(A_n),$$

where $A \subseteq \bigcup_n A_n$, $A_n \in S$. (It is not a 'proper' measure. We'll eventually limit our choice of subsets of X so that m^* is actually a measure.) Properties:

1. $A \subseteq B \subseteq X \implies m^*(A) \subseteq m^*(B)$.
2. Semi-additivity:
$$m^* \left(\bigcup_{k=1}^{\infty} A_k \right) \leq \sum_{k=1}^{\infty} m^*(A_k).$$
3. Whacked up triangle inequality:
$$|m^*(A) - m^*(B)| \leq m^*(A \triangle B).$$

Proof.

2. For finitely many only — check brains or youtube for countably infinite. We want $m^*(A \cup B) \leq m^*(A) + m^*(B)$. Fix $\varepsilon > 0$. Take coverings $\{A_n\}$ and $\{B_n\}$ from the semi-ring for A and B respectively, such that $\sum m(A_n) < m^*(A) + \varepsilon$ and $\sum m(B_n) < m^*(B) + \varepsilon$. Then
$$A \cup B \subseteq \left(\bigcup_{n=1}^{\infty} A_n \right) \cup \left(\bigcup_{n=1}^{\infty} B_n \right),$$
so
$$m^*(A \cup B) \leq \sum_{n=1}^{\infty} m(A_n) + \sum_{n=1}^{\infty} m(B_n) < m^*(A) + m^*(B) + 2\varepsilon.$$

$\square$

Proposition 7.1.

1. *$X \in R(S) \implies m^*(A) < \infty \ \forall A \subseteq X$.*
2. *If m is σ-additive, then $m^*(B) = \widetilde{m}(B)$ for all $B \in R(S)$.*

Proof.

2. Suppose m is σ-additive. Write
$$B = \bigsqcup_{k=1}^{n} B_k,$$
where each $B_k \in S$. Then
$$m^*(B) \leq \sum_{k=1}^{n} m(B_k) = \widetilde{m}(B).$$

Fix an $\varepsilon > 0$, and choose a covering $\{A_n\}$ from S such that

$$\sum_{n=1}^{\infty} m(A_n) < m^*(B) + \varepsilon.$$

Write

$$B = \bigcup_{n=1}^{\infty} B \cap A_n,$$

Then by semi-additivity

$$\widetilde{m}(B) \leq \sum_{n=1}^{\infty} \widetilde{m}(B \cap A_n).$$

But $\widetilde{m}(B \cap A_n) \leq \widetilde{m}(A_n) = m(A_n)$, so

$$\widetilde{m}(B) \leq \sum_{n=1}^{\infty} m(A_n) < m^*(B) + \varepsilon.$$

□

Theorem 7.1. *Suppose m is σ-additive and $X \in R(S)$. Let*

$$\mathcal{F} = \{A \subseteq X : \forall \varepsilon > 0, \exists A' \in R(S) : m^*(A \triangle A') < \varepsilon\}.$$

Then $\mathcal{F}$ is a σ-algebra and m^* is a σ-additive measure in $\mathcal{F} \subseteq 2^X$.

There's heaps of junk to prove here.

Lecture 8

Proof of Theorem 7.1. $X \in \mathcal{F}$ is clear (take " X' "$= X$). Suppose $A, B \in \mathcal{F}$. Closure under union: fix an $\varepsilon > 0$, and take $A', B' \in R(S)$ such that

$$m^*(A \triangle A') < \varepsilon, \text{and}$$
$$m^*(B \triangle B') < \varepsilon.$$

Now, $(A \cup B) \triangle \underbrace{(A' \cup B')}_{\in R(S)} \subseteq (A \triangle A') \cup (B \triangle B')$. So

$$m^*((A \cup B) \triangle (A' \cup B')) \leq m^*(A \triangle A') + m^*(B \triangle B') < 2\varepsilon.$$

Closure under set difference: show $(A \backslash B) \triangle (A' \backslash B') \subseteq (A \triangle A') \cup (B \triangle B')$, and use the same argument as before.

Closure under countable union: suppose $A_n \in \mathcal{F}$ for $n = 1, \ldots, \infty$. Let $A = \bigcup A_n$. Fix an $\varepsilon > 0$. For each n, choose $A'_n \in R(S)$ such that

$$m(A_n \triangle A'_n) < \frac{\varepsilon}{2^n}.$$

Let

$$A' = \bigcup_{n=1}^{\infty} A'_n.$$

Then

$$m^*(A \triangle A') \le \sum_{n=1}^{\infty} m^*(A_n \triangle A'_n) < \varepsilon.$$

But this isn't enough because A' is not necessarily in $R(S)$. Now,

$$A \triangle A' \subseteq \bigcup_{n=1}^{\infty} (A_n \triangle A'_n)$$

and

$$A' \subseteq A \cup \left[\bigcup_{n=1}^{\infty} (A_n \triangle A'_n) \right].$$

Observe that

$$\sum_{n=1}^{\infty} \widetilde{m}(A'_n) < \infty.$$

Why? We have

$$\begin{aligned}
\sum_{n=1}^{N} \widetilde{m}(A'_n) &= \widetilde{m}\left(\bigcup_{n=1}^{N} A'_n \right) \\
&= m^*\left(\bigcup_{n=1}^{N} A'_n \right) \\
&\le m^*\left(\underbrace{\bigcup_{n=1}^{\infty} A'_n}_{A'} \right) \\
&\le m^*(A) + \sum_{n=1}^{\infty} m^*(A_n \triangle A'_n) \\
&\le m^*(A) + \varepsilon \\
&\le m^*(A) + 1.
\end{aligned}$$

Now, how do we fix the A'? Choose $N_\varepsilon \ge 1$ such that

$$\sum_{n=N_\varepsilon+1}^{\infty} \widetilde{m}(A'_n) < \varepsilon.$$

Let

$$A'' = \bigcup_{n=1}^{N_\varepsilon} A'_n.$$

Then

$$A \triangle A'' \subseteq \left[\bigcup_{n=1}^{\infty} (A_n \triangle A'_n) \right] \cup \left[\bigcup_{n=N_\varepsilon+1}^{\infty} A'_n \right].$$

So

$$\begin{aligned}
m^*(\text{LHS}) &\le \sum_{n=1}^{\infty} m^*(A_n \triangle A'_n) + \sum_{n=N_\varepsilon+1}^{\infty} m^*(A'_n) \\
&\le 2\varepsilon.
\end{aligned}$$

We'll still need to show that it's a proper measure!

Lecture 9

Continuing on with the proof from last time.

Proof. We want to show that for $A, B \in \mathcal{F}$ with $A \cap B = \varnothing$, $m^*(A \sqcup B) = m^*(A) + m^*(B)$. Semi-additivity gives us "$\leq$", so we'll only need to prove "$\geq$". Fix $\varepsilon > 0$, and take $A', B' \in R(S)$ such that

$$m^*(A \triangle A') < \varepsilon \quad \text{and} \quad m^*(B \triangle B') < \varepsilon.$$

Now $A \subseteq A' \cup (A \triangle A')$ and $B \subseteq B' \cup (B \triangle B')$. Thus

$$m^*(A) \leq m^*(A') + \varepsilon \quad \text{and} \quad m^*(B) \leq m^*(B') + \varepsilon.$$

Adding these gives

$$m^*(A) + m^*(B) \leq \widetilde{m}^(A') + \widetilde{m}(B') + 2\varepsilon,$$

since m^* and $\widetilde{m}$ coincide on $R(S)$. Then

$$m^*(A) + m^*(B) \leq \widetilde{m}(A' \cup B') + \widetilde{m}(A' \cap B') + 2\varepsilon.$$

Now,

$$\begin{aligned} A' \cup B' &\subseteq (A \sqcup B) \cup (A \triangle A') \cup (B \triangle B'), \text{ and} \\ A' \cap B' &\subseteq (\underbrace{A \cap B}_{\varnothing}) \cup (A \triangle A') \cup (B \triangle B'). \end{aligned}$$

So

$$\begin{aligned} \widetilde{m}(A' \cup B') &= m^*(A' \cup B') \leq m^*(A \sqcup B) + 2\varepsilon, \text{ and} \\ \widetilde{m}(A' \cap B') &= m^*(A' \cap B') \leq 2\varepsilon. \end{aligned}$$

Thus

$$m^*(A) + m^*(B) \leq m^*(A \sqcup B) + 6\varepsilon.$$

What about for countable disjoint unions? For a measure on a ring, additivity with semi-additivity implies σ-additivity. $\square$

If m is a σ-additive measure on a semi-ring S and $X \in R(S)$, then $(X; S, m) \mapsto (\mathcal{F}, m^*)$ is a finite Lebesgue extension.

If we relax the restriction that $X \in R(S)$ to just that

$$X = \bigsqcup_{n=1}^{\infty} X_n,$$

where $X_n \in S$, then we call it a σ-finite extension. In this case, define new semi-rings

$$S_n = \{A \cap X_n : A \in S\} \subseteq S.$$

Then restrict $m : S_n \to \mathbb{R}^+$, to get a finite Lebesgue extension

$$(X_n; S_n, m) \mapsto (X_n; \mathcal{F}_n, m_n^*).$$

LET'S KEEP GOING, define

$$\mathcal{F} = \{A \subseteq X : A \cap X_n \in \mathcal{F}_n\},$$
$$\mathcal{F}_0 = \left\{ A \in \mathcal{F} : \sum_{n=1}^{\infty} m_n^*(A \cap X_n) < \infty \right\}.$$

Then let $\mu : \mathcal{F}_0 \to \mathbb{R}^+$, with $\mu(A) = \sum_{n=1}^{\infty} m_n^*(A \cap X_n)$.

Theorem 9.1.

1. *$\mathcal{F}$ is a σ-algebra.*

2. *$\mathcal{F}_0$ is a ring.*

3. *μ is σ-additive.*

3.1. *If $A_n \in \mathcal{F}_0$, and $A_i \cap A_j = \varnothing$ for $i \neq j$, and $\sum \mu(A_n) < \infty$, then $A = \bigsqcup A_n \in \mathcal{F}_0$, and*

$$\mu(A) = \sum_{n=1}^{\infty} \mu(A_n).$$

3.2. *If $\sum \mu(A_n) = \infty$ then $A \notin \mathcal{F}_0$.*

Lecture 10

From now on, we'll call $(X; \mathcal{F}, m)$ a *measure space*, where $\mathcal{F}$ is a σ-algebra and $m : \mathcal{F}_0 \to \mathbb{R}^+$, where $\mathcal{F}_0$ is a ring and m is σ-additive.

Definition 10.1. A function $f : X \to \mathbb{R}$ is measurable if

$$\{f < c\} := f^{-1}((-\infty, c)) = \{x \in X : f(x) < c\} \in \mathcal{F} \text{ for all } c \in \mathbb{R}.$$

Lemma 10.1. *The following are equivalent:*

1. *$\{f > c\} \in \mathcal{F}$ for all $c \in \mathbb{R}$.*

2. *$\{f \geq c\} \in \mathcal{F}$ for all $c \in \mathbb{R}$.*

3. *$\{f < c\} \in \mathcal{F}$ for all $c \in \mathbb{R}$.*

4. *$\{f \leq c\} \in \mathcal{F}$ for all $c \in \mathbb{R}$.*

5. *$f^{-1}(B) \in \mathcal{F}$ for all $B \in \mathcal{B}(\mathbb{R})$.*

Proof. Based on closure conditions on $\mathcal{F}$.
(1) $\implies$ (2) — $\{f \geq c\} = \bigcap_n \{f > c - \frac{1}{n}\}$.
(2) $\implies$ (3) — $\{f < c\} = X \backslash \{f \geq c\}$.
(3) $\implies$ (4) — $\{f \leq c\} = \bigcap_n \{f < c + \frac{1}{n}\}$.
(5) $\implies$ (1) — $\{f > c\} = f^{-1}((c, \infty))$, note $(c, \infty) \in \mathcal{B}(\mathbb{R})$.

(4) $\implies$ (5) — we want to show (2,3) $\implies$ (5).

Recall $\mathcal{B}(\mathbb{R}) = \bigcup_n \mathcal{B}_n(\mathbb{R})$, where $\mathcal{B}_1(\mathbb{R}) = \{[a,b) : a \le b\}$ and

$$\mathcal{B}_n(\mathbb{R}) = \left\{ \bigcup_{k=1}^{\infty} A_k, A \cap B, A \backslash B : A_k, A, B \in \mathcal{B}_{n-1}(\mathbb{R}) \right\}.$$

Do it with induction: based on $f^{-1}([a,b)) = \{f \ge a\} \cap \{f \ge b\}$.

$\square$