1 Commutative Algebra

- An ideal I is prime if $fg \in I$ means $f \in I$ or $g \in I$
- An element p is prime if $p|fg$ means $p|f$ or $p|g$
- An element p is irreducible if it cannot be factored into two non-invertible elements
- In a UFD, irreducible and prime are equivalent
- A polynomial ring over a UFD is a UFD
- A ring is Noetherian if every ascending chain of ideals is stationary, or equivalently every ideal is finitely generated
- A polynomial ring over a Noetherian ring is Noetherian
- prime ideals are radical
- quotient of a prime ideal is an integral domain

2 Topology

- Zariski topology is where closed sets are zero loci
- An irreducible set is not a proper union of closed subsets
- continuous images of irreducible sets are irreducible
- products of irreducible sets are irreducible
- A Noetherian space is where every decreasing chain of closed subsets is stationary
- In a Noetherian space decomposition into irreducible components is unique
- The dimension of an irreducible space is the length of the longest chain of nonempty irreducible closed subsets minus 1; the dimension of any space is the maximum dimension of an irreducible component
- product topology is weaker than zariski topology on product

3 Abstract Nonsense

- A presheaf of rings is an assignment of rings to open sets, as well as a restriction homomorphism, in such a way that restriction is transitive and reflexive, and the empty set is assigned to the trivial ring.
- A sheaf is a presheaf with the property that a ring element on the global ring can be defined by its restriction to the sets of an open cover
- The stalk of a presheaf is the set of all ring elements of all neighborhoods of P, identifying elements if they agree on some neighborhood of P.
- germs are the elements of a stalk
- the stalk of O_X at P is $O_{X,P}$
- If $f : X \to Y$ is a function, the pull-back $f^*\phi$ of a regular function ϕ on Y is $\phi \circ f$
- A function is a morphism if it pulls back regular functions to regular functions
- A morphism of sheaves F, G is collection of morphisms $f_U : F(U) \to G(U)$ that commute with restriction maps
- a rational map is a morphism from an open subset, under the equivalence of agreeing on an open set. A rational map is dominant if a representative has a dense image; it is birational if it has a rational inverse.

4 Affine Varieties

- Affine variety corresponds to prime ideal
- The coordinate ring $A(X) = k[x_i]_i / I(X)$ is the "polynomials" on X
- The rational functions $K(X)$ are the fraction field of $A(X)$
- The regular functions at P are those rational functions that can be evaluated at P; alternatively they are the functions that have a representation as a quotient of polynomials in $k[x_i]_i$ on some neighborhood of P.
- The regular functions on an open set U are the functions that are regular at each $P \in U$.
- $O(X_f) = A(X)_f := \{g/f^r\} \subseteq K(X)$
- A product of affine varieties in $\mathbb{A}^n$ and $\mathbb{A}^m$ is an affine variety in $\mathbb{A}^{n+m}$ (not product topology)
- An abstract affine variety is an irreducible space and a sheaf of k-valued functions that is isomorphic to a concrete affine variety
- Distinguished open subsets are abstract affine varieties
- Not all open subsets of varieties are varieties, consider $\mathbb{C}^2 \setminus \{(0,0)\}$
- morphisms can be checked on open sets, germs or global sets
- morphisms of affine varieties f correspond to k-algebra homomorphisms f^*

5 Varieties

- A prevariety is an irreducible set with a sheaf of functions that has a finite cover of affine varieties
- Can create a prevariety by gluing two prevarieties along a common open subset: Let f be the isomorphism between open subsets U_1, U_2 of X_1, X_2 and i_1, i_2 be the inclusions into X. The topology is the quotient topology and the sheaf of functions is pairs $(\phi_1, \phi_2) \in O_{X_1}\left(i_1^{-1}(U)\right) \times O_{X_2}\left(i_2^{-1}(U)\right)$ that agree on overlaps
- Same thing works with finite collection of prevarieties, with each pair glued on an open subset, provided isomorphisms are consistent.
- Let $\{V_i\}$ be an affine cover of Y and $\{U_i\}$ be an open cover of X with $f(U_i) \subseteq V_i$ and f a morphism when restricted to each U_i. Then f is a morphism.
- A variety is a prevariety X so that for any prevariety Y and pair of morphisms $Y \to X$, the set where they agree is closed; equivalently, the diagonal is closed.
- an open or closed subprevariety of a variety is a variety.
- A variety is complete if $\pi : X \times Y \to Y$ is closed for every variety Y
- If X is complete then any morphism $X \to Y$ (Y variety) is closed.
- regular functions on complete varieties are constant
- Let X, Y be affine varieties with Y affine. Morphisms $X \to Y$ are in correspondence with homomorphisms $A(Y) \to O(X)$. If the morphism f is $(f_i)_{i=1}^{\dim X}$, f_i regular, this corresponds to $f^* : \bar{x}_i \mapsto f_i$.
- The rational functions on a general variety are equivalence classes of regular functions that agree on open sets
- rational maps from X to Y are equivalent to homomorphisms between $K(Y)$ and $K(X)$.

6 Projective Space

- think of $\mathbb{P}^n$ as $\mathbb{A}^n$ compactified with a point at infinity for every direction
- A projectivity on $\mathbb{P}^n$ is an element of $\mathrm{GL}_{n+1}(k)/k^*$
- A conic is a symmetric bilinear form on k^3, which can be represented as $\varepsilon_1 X^2 + \varepsilon Y^2 + \varepsilon Z^2$ after a projectivity
- conics in $\mathbb{P}^2_{\mathbb{R}}$ are
  - nondegenerate $X^2 + Y^2 - Z^2$
  - empty $X^2 + Y^2 + Z^2$
  - one point $X^2 + Y^2$
  - two lines $X^2 - Y^2$
  - line X^2
  - everything 0
- nondegenerate conics are equivalently $XY = Z^2$, isomorphic to $\mathbb{P}^1$ with the isomorphism $(U : V) \mapsto \left(U^2, UV, V^2\right)$ which can be interpreted as projection
- conics in $\mathbb{P}^2_{\bar{k}}$
  - nondegenerate $X^2 + Y^2 - Z^2$
  - two lines $X^2 - Y^2$
  - line X^2
  - everything 0
- A degree-d homogeneous form F on $\mathbb{P}^n$ corresponds to a (maximum) degree-d polynomial f in $\mathbb{A}^n$. If $n = 1$, the multiplicity of a zero in F is the multiplicity of the corresponding zero in f, or $d - \deg f$ for the point at infinity.
- Bezout's theorem: for an algebraically closed field the number of intersections of projective curves is the product of their degrees, provided they share no irreducible components and multiplicities are counted appropriately
- Easy cases: line or nondegenerate conic vs a nonincluding curve in $\mathbb{P}^2$, inequality to compensate for multiplicities
- 5 points in general position define a unique conic

7 Projective Varieties

- Homogeneous ideals are generated by homogeneous polynomials or equivalently contain each homogeneous part of each member, or equivalently are fixed by the action of k^*.
- A projective algebraic set X in $\mathbb{P}^n$ corresponds to a cone $C(X)$ in $\mathbb{A}^{n+1}$
- The zero set of a homogeneous ideal in $\mathbb{A}^{n+1}$ is the cone of its zero set in $\mathbb{P}^n$ (provided neither are empty)
- the ideal generated by $X \subseteq \mathbb{P}^n$ is the ideal generated by $C(X) \in \mathbb{A}^{n+1}$.
- $\dim X + 1 = \dim C(X)$ (provided nonempty)
- Nullstellensatz still works provided $Z(I)$ is nonempty; $Z(I)$ can only be empty if $I = \langle 1 \rangle$ or $\sqrt{I} = \langle x_0, \ldots, x_n \rangle$.
- Homogeneous coordinate ring is $S(X) = A(C(X))$; **not** polynomial functions

- rational functions are f/g, where $f, g \in S(X)^{(d)}$ have common degree d
- homogeneous functions of the same degree on homogeneous coordinates give a morphism, provided they never all vanish
- projective varieties are varieties
- The Segre embedding is $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{(n+1)(m+1)-1} : ((x_i), (y_i)) \mapsto (x_i y_j)$. It is the zero locus of $z_{i,j} z_{i',j'} - z_{i,j'} z_{i',j}$.
- projective varieties are complete
- A nontrivial projective variety intersects with the zero locus of any homogeneous polynomial
- The Veronese embedding is $\mathbb{P}^n \to \mathbb{P}^{\binom{n+d}{n}-1} : (x = (x_i)) \mapsto (x^I)_I$ (monomials of degree d). It is the zero locus of $z_I z_J - z_K z_L$, with $I + J = K + L$
- The degree of a projective variety is the maximal finite number of intersection points with a linear subvariety that fills the remaining dimensions. The degree of $Z(F)$ is $\deg(F)$
- Let $B = \{(x; \ell) \in \mathbb{A}^n \times \mathbb{P}^{n-1} : \ell \text{ passes through } x\}$, and let π be the projection $(x, \ell) \to x$. Then the closure of $\pi^{-1}(X \setminus \{p\})$ is the blowup of X at p.
- The Grassmannian $G(k, n)$ is the set of $k-1$ dimensional linear subvarieties of $\mathbb{P}^{n-1}$, and can be embedded as a variety in $\mathbb{P}^{\binom{n}{k}-1}$ with the Plücker embedding $\operatorname{span}(v_i)_i \mapsto \bigwedge_i v_i$
- There are 27 projective lines on any smooth cubic surface

8 Dimension

- projective morphisms cannot map surjectively onto varieties with higher dimension
- projection to a point doesn't decrease dimension
- Adding a single polynomial to a zero locus decreases its dimension by 1.
- open subsets of a variety have the same dimension
- all the components of the zero locus in $\mathbb{P}^n$ or $\mathbb{A}^n$ of a single polynomial have dimension $n-1$; intersecting such a zero locus with a variety decreases its dimension by 1.
- If $f : X \to Y$ is a morphism with $\dim(f^{-1}p) \equiv n$ then $\dim X = \dim Y + n$
- dimension of X is transcendence degree of $K(X)$ (maximal number of algebraically independent elements over k)
- dimension of affine X is the Krull dimension (maximal size of a chain of prime ideals) of $A(X)$

9 Smoothness

- A line $\ell = Z(g)$ is tangent to X with ideal $\langle f_i \rangle_i$ at p if the intersection of f and g has multiplicity at least 2
- tangent space is union of all tangent lines
- equivalently, tangent space is zero locus of linear parts of generators (can compute with taylor expansion)
- tangent space is isomorphic to dual of $\mathfrak{m}_p/\mathfrak{m}_p^2$ (zariski tangent), where $\mathfrak{m}_p$ is the maximal ideal $\{f \in O_{X,p}(X) : f(p) = 0\}$ of $O_{X,p}$

- tangent space of X has at least local dimension of X. Smooth if dimension is same, singular otherwise
- compute dimension of tangent space with rank of jacobian
- The set of singularities of a closed set is a closed set
- Hironaka's theorem: for any projective variety V there is a smooth desingularization X which is birational to X, obtained by blowups