

高等数学

4.2 换元积分法

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一、第一类换元积分法（凑微分法）

设 $f(t)$ 具有原函数 $F(t)$ ，即

$$F'(t) = f(t) \quad , \quad \int f(t) dt = F(t) + C$$

如果 $t = \varphi(x)$ ，

则 $dt = d[\varphi(x)] = \varphi'(x) dx$

$$\begin{aligned} \int f[\varphi(x)] \cdot \varphi'(x) dx &= \int f[\varphi(x)] d[\varphi(x)] = \int f(t) dt \\ &= F(t) + C = F[\varphi(x)] + C \end{aligned}$$

例1

$$\begin{aligned}
 (1) \quad \int e^{\frac{1}{3}x} dx &= 3 \cdot \int e^{\frac{1}{3}x} d\left(\frac{1}{3}x\right) \\
 &= 3 \int e^t dt = 3e^t + C \\
 &= 3e^{\frac{x}{3}} + C
 \end{aligned}$$

$$\int e^x dx = e^x + C$$

$$\begin{aligned}
 (2) \quad \int \frac{1}{x-3} dx &= \int \frac{1}{x-3} d(x-3) \\
 &= \int \frac{1}{t} dt = \ln|t| + C \\
 &= \ln|x-3| + C
 \end{aligned}$$

$$\int \frac{1}{x} dx = \ln|x| + C$$

$$dx = \frac{1}{k} d(kx + 1)$$

$$(3) \quad \int (2x + 1)^{10} dx = \frac{1}{2} \int (2x + 1)^{10} d(2x + 1)$$

$$= \frac{1}{2} \cdot \frac{1}{11} (2x + 1)^{11} + C$$

$$= \frac{1}{22} (2x + 1)^{11} + C$$

$$\int x^{10} dx = \frac{1}{11} x^{11} + C$$

例2 (1) $\int \frac{x}{1+x^2} dx$

$$= \int \frac{1}{1+x^2} \cdot x dx$$

$$x dx = d\left(\frac{1}{2} x^2\right)$$

$$= \int \frac{1}{1+x^2} d\left(\frac{1}{2} x^2\right)$$

$$= \frac{1}{2} \int \frac{1}{1+x^2} d(x^2)$$

$$\int \frac{1}{x} dx = \ln|x| + C$$

$$= \frac{1}{2} \int \frac{1}{1+x^2} d(x^2 + 1)$$

$$\int \frac{1}{t} dt = \ln|t| + C$$

$$= \frac{1}{2} \ln|1+x^2| + C = \frac{1}{2} \ln(1+x^2) + C$$

$$(2) \int x^3 \cdot e^{x^4} dx$$

$$= \int e^{x^4} \cdot x^3 dx$$

$$= \int e^{x^4} d\left(\frac{1}{4}x^4\right)$$

$$= \frac{1}{4} \int e^{x^4} d(x^4)$$

$$= \frac{1}{4} e^{x^4} + C$$

$$x^3 dx = d\left(\frac{1}{4}x^4\right)$$

$$\int e^t dt = e^t + C$$

例3

$$\int \sin \frac{1}{x} \frac{1}{x^2} dx$$

$$\frac{1}{x^2} dx = d\left(-\frac{1}{x}\right)$$

$$= \int \sin \frac{1}{x} d\left(-\frac{1}{x}\right)$$

$$= - \int \sin \frac{1}{x} d\left(\frac{1}{x}\right)$$

$$\int \sin t dt = -\cos t + C$$

$$= - \left(-\cos \frac{1}{x}\right) + C = \cos \frac{1}{x} + C$$

例4

$$(1) \int \frac{\sin \sqrt{x}}{\sqrt{x}} dx$$

$$= \int \sin \sqrt{x} \cdot \frac{1}{\sqrt{x}} dx$$

$$= \int \sin \sqrt{x} d(2\sqrt{x})$$

$$= 2 \int \sin \sqrt{x} d(\sqrt{x})$$

$$= 2 \cdot (-\cos \sqrt{x}) + C$$

$$= -2 \cos \sqrt{x} + C$$

$$\frac{1}{\sqrt{x}} dx = d(2\sqrt{x})$$

$$\int \sin t dt = -\cos t + C$$

$$\frac{1}{\sqrt{x}} dx = d(2\sqrt{x})$$

$$(2) \quad \int \frac{1}{\sqrt{x} \cdot (1+x)} dx = \int \frac{1}{(1+x)} \cdot \frac{1}{\sqrt{x}} dx$$

$$= \int \frac{1}{(1+x)} d(2\sqrt{x})$$

$$= 2 \int \frac{1}{(1+\boxed{x})} d(\sqrt{x})$$

$$\int \frac{1}{1+t^2} dt = \arctan t + C$$

$$= 2 \int \frac{1}{1+(\sqrt{x})^2} d(\sqrt{x})$$

$$= 2 \arctan \sqrt{x} + C$$

例5

$$\int \frac{e^x}{e^x + 2} dx$$

$$e^x dx = d(e^x + C)$$

$$= \int \frac{1}{e^x + 2} \cdot e^x dx$$

$$= \int \frac{1}{e^x + 2} d(e^x + 2)$$

$$\int \frac{1}{t} dt = \ln|t| + C$$

$$= \ln|e^x + 2| + C$$

$$= \ln(e^x + 2) + C$$

例6

$$\int \frac{1-2\ln x}{x} dx$$

$$= \int (1-2\ln x) \cdot \frac{1}{x} dx$$

$$\frac{1}{x} dx = d(\ln x)$$

$$= \int (1-2\ln x) d(\ln x)$$

令 $\ln x = t$

$$= \int (1-2t) dt$$

$$= t - t^2 + C$$

$$= \ln x - \ln^2 x + C$$

例7

$$(1) \int \tan x \, dx$$

$$= \int \frac{\sin x}{\cos x} \, dx$$

$$= \int \frac{1}{\cos x} \cdot \sin x \, dx$$

$$= \int \frac{1}{\cos x} d(-\cos x)$$

$$= - \int \frac{1}{\cos x} d(\cos x)$$

$$= - \ln |\cos x| + C$$

$$\sin x \, dx = d(-\cos x)$$

$$\int \frac{1}{t} \, dt = \ln |t| + C$$

$$(2) \quad \int \frac{\tan x}{\cos^2 x} dx = \int \tan x \cdot \frac{1}{\cos^2 x} dx$$

$$= \int \tan x \cdot \sec^2 x dx$$

$$\sec^2 x dx = d(\tan x)$$

$$= \int \tan x d(\tan x)$$

$$\int t dt = \frac{1}{2} t^2 + C$$

$$= \frac{1}{2} \tan^2 x + C$$

$$(3) \quad \int \sin^2 x \cdot \cos^3 x dx$$

$$= \int \sin^2 x \cdot \cos^2 x \cdot \cos x dx$$

$$\cos x dx = d(\sin x)$$

$$= \int \sin^2 x \cdot \cos^2 x d(\sin x)$$

$$= \int \sin^2 x \cdot (1 - \sin^2 x) d(\sin x)$$

$$= \int (\sin^2 x - \sin^4 x) d(\sin x)$$

$$= \frac{1}{3} \sin^3 x - \frac{1}{5} \sin^5 x + C$$

$$\begin{aligned} & \int (t^2 - t^4) dt \\ &= \frac{1}{3} t^3 - \frac{1}{5} t^5 + C \end{aligned}$$

$$\begin{aligned}
 (4) \quad \int \sec x dx &= \int \frac{\sec x}{1} dx \\
 &= \int \frac{\sec x \cdot (\sec x + \tan x)}{\sec x + \tan x} dx
 \end{aligned}$$

$$= \int \frac{\sec^2 x + \cancel{\sec x \cdot \tan x}}{\cancel{\sec x + \tan x}} dx$$

$$= \int \frac{1}{\sec x + \tan x} d(\sec x + \tan x)$$

$$= \ln |\sec x + \tan x| + C$$

$$\int \frac{1}{t} dt = \ln |t| + C$$

例8

$$\begin{aligned}(1) \quad \int \frac{1}{x^2 + 3x + 2} dx &= \int \frac{1}{(x+1)(x+2)} dx \\ &= \int \left(\frac{1}{x+1} - \frac{1}{x+2} \right) dx\end{aligned}$$

$$\int \frac{1}{t} dt = \ln |t| + C$$

$$\begin{aligned}&= \int \frac{1}{x+1} d(x+1) - \int \frac{1}{x+2} d(x+2) \\ &= \ln |x+1| - \ln |x+2| + C \\ &= \ln \left| \frac{x+1}{x+2} \right| + C\end{aligned}$$

$$(2) \quad \int \frac{1}{x^2 + 4x + 5} dx = \int \frac{1}{(x^2 + 4x + 4) + 1} dx$$

$$= \int \frac{1}{1 + (x + 2)^2} dx$$

$$\boxed{\int \frac{1}{1 + t^2} dt = \arctan t + C} = \int \frac{1}{1 + (x + 2)^2} d(x + 2)$$

$$= \arctan(x + 2) + C$$

$$(3) \quad \int \frac{2x+3}{x^2+3x+2} dx = \int \frac{1}{x^2+3x+2} \cdot (2x+3) dx$$

$$\int \frac{1}{t} dt = \ln|t| + C$$

$$= \int \frac{1}{x^2+3x+2} d(x^2+3x+2)$$

$$= \ln|x^2+3x+2| + C$$

练习

$$(9) \quad \int (3x-1)^2 dx = \frac{1}{3} \int (3x-1)^2 d(3x-1)$$

$$= \frac{1}{3} \cdot \frac{1}{3} (3x-1)^3 + C$$

$$= \frac{1}{9} (3x-1)^3 + C$$

$$\int t^2 dt = \frac{1}{3} t^3 + C$$

方法二:

$$\begin{aligned}(9) \quad \int (3x-1)^2 dx &= \int (9x^2 - 6x + 1) dx \\&= 9 \cdot \frac{1}{3} x^3 - 6 \cdot \frac{1}{2} x^2 + x + C \\&= 3x^3 - 3x^2 + x + C\end{aligned}$$

$$(10) \quad \int x e^{x^2} dx = \int e^{x^2} \boxed{x dx}$$

$$= \int e^{x^2} d\left(\frac{1}{2}x^2\right)$$

$$= \frac{1}{2} \int e^{x^2} d(x^2)$$

$$= \frac{1}{2} e^{x^2} + C$$

$$\boxed{\int e^t dt = e^t + C}$$

$$(11) \quad \int \cos x \cdot \sin^3 x \, dx = \int \sin^3 x \cdot \boxed{\cos x \, dx}$$

$$= \int \sin^3 x \, d(\sin x)$$

$$= \frac{1}{4} \sin^4 x + C$$

$$\boxed{\int t^3 \, dt = \frac{1}{4} t^4 + C}$$

$$\begin{aligned}(12) \quad \int \frac{2x-3}{x^2-3x+4} dx &= \int \frac{1}{x^2-3x+4} \cdot (2x-3) dx \\&= \int \frac{1}{x^2-3x+4} d(x^2-3x+4) \\&= \ln|x^2-3x+4| + C\end{aligned}$$

$$\int \frac{1}{t} dt = \ln|t| + C$$

$$\begin{aligned}
 (13) \quad \int \frac{1}{x^2 + x - 2} dx &= \int \frac{1}{(x+2)(x-1)} dx \\
 &= \frac{1}{3} \int \left(\frac{1}{x-1} - \frac{1}{x+2} \right) dx
 \end{aligned}$$

$$\int \frac{1}{t} dt = \ln|t| + C$$

$$= \frac{1}{3} \left[\int \frac{1}{x-1} d(x-1) - \int \frac{1}{x+2} d(x+2) \right]$$

$$= \frac{1}{3} \left(\ln|x-1| - \ln|x+2| \right) + C$$

$$= \frac{1}{3} \ln \left| \frac{x-1}{x+2} \right| + C$$

$$\begin{aligned}(14) \quad \int \frac{1}{1+9x^2} dx &= \int \frac{1}{1+(\color{blue}{3x})^2} dx \\&= \frac{\color{blue}{1}}{\color{blue}{3}} \int \frac{1}{1+(\color{blue}{3x})^2} d(\color{blue}{3x}) \\&= \frac{1}{3} \arctan 3x + C\end{aligned}$$

$$\int \frac{1}{1+t^2} dt = \arctan t + C$$

$$\begin{aligned}(15) \quad \int \frac{e^x}{1+e^x} dx &= \int \frac{1}{1+e^x} \cdot \boxed{e^x dx} \\&= \int \frac{1}{1+e^x} d(\textcolor{blue}{e^x} + \textcolor{red}{1}) \\&= \ln|e^x + 1| + C \\&= \ln(e^x + 1) + C\end{aligned}$$

$$\boxed{\int \frac{1}{t} dt = \ln|t| + C}$$

$$(16) \quad \int \frac{1}{e^x + e^{-x}} dx = \int \frac{1}{e^x + \frac{1}{e^x}} dx$$

$$= \int \frac{1}{\frac{(e^x)^2 + 1}{e^x}} dx = \int \frac{e^x}{1 + (e^x)^2} dx$$

$$= \int \frac{1}{1 + (e^x)^2} \cdot e^x dx = \int \frac{1}{1 + (e^x)^2} d(e^x)$$

$$= \arctan e^x + C$$

$$\int \frac{1}{1 + t^2} dt = \arctan t + C$$