



# 第六讲 无穷小的比较

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- 内容概要
  - 一、无穷小的比较
  - 二、等价无穷小替换
  - 三、小结 思考题



# 一、无穷小的比较

例如, 当 $x \rightarrow 0$ 时,  $x, x^2, \sin x, x^2 \sin \frac{1}{x}$  都是无穷小.

观察各极限

$$\lim_{x \rightarrow 0} \frac{x^2}{3x} = 0,$$

$x^2$ 比 $3x$ 要快得多;

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1,$$

$\sin x$ 与 $x$ 大致相同;

$$\lim_{x \rightarrow 0} \frac{x^2 \sin \frac{1}{x}}{x^2} = \lim_{x \rightarrow 0} \sin \frac{1}{x} \text{ 不存在. 不可比.}$$

极限不同, 反映了趋向于零的“快慢”程度不同.

定义：设 $\alpha, \beta$ 是同一过程中的两个无穷小,且 $\alpha \neq 0$ .

(1) 如果  $\lim \frac{\beta}{\alpha} = 0$ , 就说 $\beta$ 是比 $\alpha$ 高阶的无穷小,

记作  $\beta = o(\alpha)$ ;

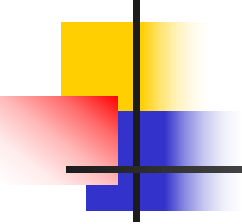
(2) 如果  $\lim \frac{\beta}{\alpha} = C (C \neq 0)$ , 就说 $\beta$ 与 $\alpha$ 是同阶的无穷小;

特殊地 如果  $\lim \frac{\beta}{\alpha} = 1$ , 则称 $\beta$ 与 $\alpha$ 是等价的无穷小;

记作  $\alpha \sim \beta$ ;

(3) 如果  $\lim \frac{\beta}{\alpha^k} = C (C \neq 0, k > 0)$ , 就说 $\beta$ 是 $\alpha$ 的 $k$ 阶的无穷小.

例1


$$(1) \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

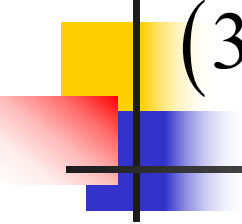
(当  $x \rightarrow 0$  时,  $\sin x \sim x$ )

$$(2) \lim_{x \rightarrow 0} \frac{\arcsin x}{x} = 1 \quad (\text{当 } x \rightarrow 0 \text{ 时, } \arcsin x \sim x)$$

令  $\arcsin x = t$ , 则  $\sin t = x$ ,

当  $x \rightarrow 0$  时,  $t \rightarrow 0$

$$\text{原式} = \lim_{t \rightarrow 0} \frac{t}{\sin t} = 1$$


$$(3) \quad \lim_{x \rightarrow 0} \frac{\tan x}{x} = 1$$

(当  $x \rightarrow 0$  时,  $\tan x \sim x$ )

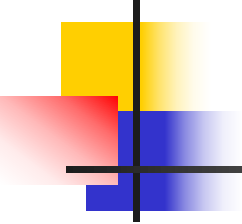
$$(4) \quad \lim_{x \rightarrow 0} \frac{\arctan x}{x} = 1$$

(当  $x \rightarrow 0$  时,  $\arctan x \sim x$ )

$$(5) \quad \lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = \lim_{x \rightarrow 0} \frac{1}{x} \cdot \ln(1+x) = \lim_{x \rightarrow 0} \ln(1+x)^{\frac{1}{x}}$$

$$= \ln \left[ \lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} \right] = \ln e = 1$$

(当  $x \rightarrow 0$  时,  $\ln(1+x) \sim x$ )


$$(6) \quad \lim_{x \rightarrow 0} \frac{e^x - 1}{x}$$

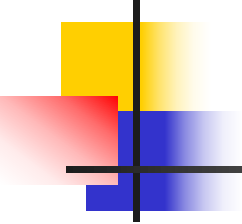
$$\text{令 } e^x - 1 = t, \quad \text{则 } e^x = 1 + t \Rightarrow x = \ln(1 + t)$$

$$\text{当 } x \rightarrow 0 \text{ 时,} \quad t \rightarrow 0$$

$$\text{原式} = \lim_{t \rightarrow 0} \frac{t}{\ln(1 + t)} = 1$$

$$(5) \quad \lim_{x \rightarrow 0} \frac{\ln(1 + x)}{x} = 1$$

$$(\text{当 } x \rightarrow 0 \text{ 时, } e^x - 1 \sim x)$$

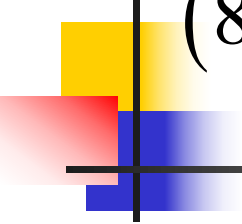


$$(7) \quad \lim_{x \rightarrow 0} \frac{1 - \cos x}{\frac{1}{2}x^2} = \lim_{x \rightarrow 0} \frac{1 - \left(1 - 2\sin^2 \frac{x}{2}\right)}{\frac{x^2}{2}}$$

$$= \lim_{x \rightarrow 0} \frac{2\sin^2 \frac{x}{2}}{\frac{x^2}{2}} = \lim_{x \rightarrow 0} \frac{\sin^2 \frac{x}{2}}{\frac{1}{2} \cdot \frac{x^2}{2}}$$

$$\left( \begin{array}{l} \text{当 } x \rightarrow 0 \text{ 时,} \\ 1 - \cos x \sim \frac{1}{2}x^2 \end{array} \right)$$

$$= \lim_{x \rightarrow 0} \frac{\sin^2 \frac{x}{2}}{\frac{x^2}{4}} = \lim_{x \rightarrow 0} \left( \frac{\sin \frac{x}{2}}{\frac{x}{2}} \right)^2 = 1$$


$$(8) \quad \lim_{x \rightarrow 0} \frac{(1+x)^\alpha - 1}{\alpha x} \quad \left( \text{当 } x \rightarrow 0 \text{ 时, } (1+x)^\alpha - 1 \sim \alpha x \right)$$

$$\text{令 } (1+x)^\alpha - 1 = t, \quad \text{当 } x \rightarrow 0 \text{ 时, } t \rightarrow 0$$

$$\text{且 } (1+x)^\alpha = t+1, \quad \ln \left[ (1+x)^\alpha \right] = \ln(t+1)$$

$$\alpha \ln(1+x) = \ln(t+1), \quad \ln(1+x) = \frac{1}{\alpha} \ln(t+1)$$

$$\lim_{x \rightarrow 0} \frac{(1+x)^\alpha - 1}{\alpha x} = \lim_{x \rightarrow 0} \frac{(1+x)^\alpha - 1}{\alpha \ln(1+x)}$$

$$\begin{array}{l} \text{当 } x \rightarrow 0 \text{ 时} \\ \ln(1+x) \sim x \end{array}$$

$$= \lim_{t \rightarrow 0} \frac{t}{\alpha \frac{1}{\alpha} \ln(1+t)} = \lim_{t \rightarrow 0} \frac{t}{\ln(1+t)} = 1$$





## 常用等价无穷小:

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当 $x \rightarrow 0$ 时,

$$\sin x \sim x, \quad \arcsin x \sim x,$$

$$\tan x \sim x, \quad \arctan x \sim x,$$

$$\ln(1+x) \sim x, \quad e^x - 1 \sim x, \quad 1 - \cos x \sim \frac{1}{2}x^2.$$

$$(1+x)^\alpha - 1 \sim \alpha x \quad (\alpha \neq 0)$$



## 推广成一般形式

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当  $\square \rightarrow 0$  时,

$$\sin \square \sim \square,$$

$$\arcsin \square \sim \square,$$

$$\tan \square \sim \square,$$

$$\arctan \square \sim \square,$$

$$\ln(1 + \square) \sim \square,$$

$$e^{\square} - 1 \sim \square,$$

$$1 - \cos \square \sim \frac{1}{2} \square^2,$$

$$(1 + \square)^{\alpha} - 1 \sim \alpha \square \quad (\alpha \neq 0).$$



## 二、等价无穷小替换

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定理(等价无穷小替换定理)

设  $\alpha \sim \alpha'$ ,  $\beta \sim \beta'$  且  $\lim \frac{\beta'}{\alpha'}$  存在, 则  $\lim \frac{\beta}{\alpha} = \lim \frac{\beta'}{\alpha'}$ .

证 
$$\begin{aligned}\lim \frac{\beta}{\alpha} &= \lim \left( \frac{\beta}{\beta'} \cdot \frac{\beta'}{\alpha'} \cdot \frac{\alpha'}{\alpha} \right) \\ &= \lim \frac{\beta}{\beta'} \cdot \lim \frac{\beta'}{\alpha'} \cdot \lim \frac{\alpha'}{\alpha} = \lim \frac{\beta'}{\alpha'}.\end{aligned}$$

例2

$$\lim_{x \rightarrow 0} \frac{\ln(1+x^2)(e^x-1)}{(1-\cos x)\sin 4x} = \lim_{x \rightarrow 0} \frac{x^2 \cdot x}{\frac{1}{2}x^2 \cdot 4x} = \frac{1}{2}$$

思考:  $x \rightarrow 0$ 时,

$$\ln(1+x^2) \sim x^2$$

$$e^x - 1 \sim x$$

$$1 - \cos x \sim \frac{1}{2}x^2$$

$$\sin 4x \sim 4x$$

当  $\square \rightarrow 0$  时

$$\sin \square \sim \square, \quad \arcsin \square \sim \square, \quad \tan \square \sim \square, \quad \arctan \square \sim \square,$$

$$\ln(1+\square) \sim \square, \quad e^\square - 1 \sim \square, \quad 1 - \cos \square \sim \frac{1}{2}\square^2,$$

$$(1+\square)^\alpha - 1 \sim \alpha\square \quad (\alpha \neq 0).$$

例3

$$\lim_{x \rightarrow 0} \frac{\sqrt{1+x^2} - 1}{x^2}$$

解：

方法一（有理化分子）

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sqrt{1+x^2} - 1}{x^2} &= \lim_{x \rightarrow 0} \frac{\sqrt{1+x^2} - 1}{x^2} \times \frac{\sqrt{1+x^2} + 1}{\sqrt{1+x^2} + 1} \\ &= \lim_{x \rightarrow 0} \frac{(1+x^2) - 1}{x^2 (\sqrt{1+x^2} + 1)} = \lim_{x \rightarrow 0} \frac{x^2}{x^2 (\sqrt{1+x^2} + 1)} \\ &= \lim_{x \rightarrow 0} \frac{1}{\sqrt{1+x^2} + 1} = \frac{1}{2} \end{aligned}$$



方法二（等价无穷小替换）

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$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sqrt{1+x^2} - 1}{x^2} &= \lim_{x \rightarrow 0} \frac{(1 + \boxed{x^2})^{\frac{1}{2}} - 1}{x^2} \\ &= \lim_{x \rightarrow 0} \frac{\frac{1}{2}x^2}{x^2} = \frac{1}{2}\end{aligned}$$

当  $\square \rightarrow 0$  时,  $(1 + \square)^\alpha - 1 \sim \alpha \square \quad (\alpha \neq 0).$

例4

$$\lim_{x \rightarrow 3} \frac{\sin(x^2 - 9)}{x - 3}$$

解：

方法一（构造第一类重要极限）

$$\begin{aligned} \lim_{x \rightarrow 3} \frac{\sin(x^2 - 9)}{x - 3} &= \lim_{x \rightarrow 3} \frac{\sin(x^2 - 9)}{(x - 3)(x + 3)} \cdot (x + 3) \\ &= \lim_{x \rightarrow 3} \left[ \frac{\sin(x^2 - 9)}{x^2 - 9} \times (x + 3) \right] \\ &= 1 \times 6 = 6 \end{aligned}$$



方法二（等价无穷小替换）


当  $\square \rightarrow 0$  时,  $\sin \square \sim \square$

$$\lim_{x \rightarrow 3} \frac{\sin(x^2 - 9)}{x - 3} = \lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3}$$

$$= \lim_{x \rightarrow 3} (x + 3)$$

$$= 6$$





例5 求  $\lim_{x \rightarrow 0} \frac{\tan^2 2x}{1 - \cos x}$ .

当  $\square \rightarrow 0$  时

$$\tan \square \sim \square, \quad 1 - \cos \square \sim \frac{1}{2} \square^2$$

解：当  $x \rightarrow 0$  时， $1 - \cos x \sim \frac{1}{2}x^2$ ， $\tan 2x \sim 2x$ .

$$\lim_{x \rightarrow 0} \frac{\tan^2 2x}{1 - \cos x} = \lim_{x \rightarrow 0} \frac{(\tan 2x)^2}{1 - \cos x} = \lim_{x \rightarrow 0} \frac{(2x)^2}{\frac{1}{2}x^2} = 8$$

注：只对因子整体代换；

对于代数和中各无穷小不能分别替换。

例6

$$\text{求 } \lim_{x \rightarrow 0} \frac{\tan x - \sin x}{\sin^3 2x}.$$

$$\tan x = \frac{\sin x}{\cos x}$$

$$\sin x = \tan x \cdot \cos x$$

解 当  $x \rightarrow 0$  时,  $\tan x \sim x$ ,  $\sin x \sim x$ .

$$\text{原式} \times \lim_{x \rightarrow 0} \frac{x - x}{(2x)^3} = 0. \quad \text{错}$$

解:

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\tan x - \sin x}{\sin^3 2x} &= \lim_{x \rightarrow 0} \frac{\tan x - \tan x \cdot \cos x}{\sin^3 2x} \\ &= \lim_{x \rightarrow 0} \frac{\tan x \cdot (1 - \cos x)}{\sin^3 2x} = \lim_{x \rightarrow 0} \frac{x \cdot \frac{1}{2} x^2}{(2x)^3} = \frac{1}{16} \end{aligned}$$



例7

$$\lim_{x \rightarrow a} \frac{e^x - e^a}{x - a}$$

当  $\square \rightarrow 0$  时,  $e^{\square} - 1 \sim \square$

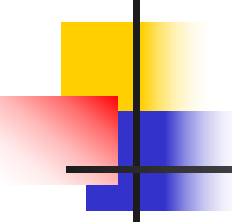
解:

$$\text{原式} = \lim_{x \rightarrow a} \frac{e^a \cdot (e^{x-a} - 1)}{x - a}$$

$$= e^a \cdot \lim_{x \rightarrow a} \frac{e^{\boxed{x-a}} - 1}{x - a}$$

$$= e^a \cdot \lim_{x \rightarrow a} \frac{x - a}{x - a}$$

$$= e^a \cdot 1 = e^a$$



例8  $\lim_{x \rightarrow a} \frac{\ln x - \ln a}{x - a}$

解:  $\lim_{x \rightarrow a} \frac{\ln x - \ln a}{x - a} = \lim_{x \rightarrow a} \frac{\ln \frac{x}{a}}{\frac{x}{a} - 1} = \lim_{x \rightarrow a} \frac{\ln \frac{x}{a}}{a \cdot \left( \frac{x}{a} - 1 \right)}$

令  $\frac{x}{a} - 1 = t$ , 则  $\frac{x}{a} = t + 1$ ,

当  $x \rightarrow a$  时,  $t \rightarrow 0$

当  $\square \rightarrow 0$  时,  
 $\ln(1 + \square) \sim \square$

原式  $= \lim_{t \rightarrow 0} \frac{\ln(t+1)}{a \cdot t} = \lim_{t \rightarrow 0} \frac{t}{a \cdot t} = \frac{1}{a}$



## 练习题

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$$(1) \quad \lim_{x \rightarrow 0} \frac{\tan 3x}{\sin 2x}$$

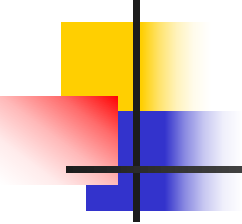
$$(2) \quad \lim_{x \rightarrow 0} \frac{\arcsin x^n}{(\sin x)^m}$$

$$(3) \quad \lim_{x \rightarrow 0} \frac{\ln(1+2x)}{x}$$

$$(4) \quad \lim_{x \rightarrow 0} \frac{\sqrt{1+x \sin x} - 1}{x \cdot \arctan x}$$

$$(5) \quad \lim_{n \rightarrow \infty} 2^n \sin \frac{x}{2^n}$$

$$(6) \quad \lim_{x \rightarrow 0} \frac{(1+ax)^{\frac{1}{n}} - 1}{x}$$



当  $\square \rightarrow 0$  时

$\sin \square \sim \square$ ,  $\arcsin \square \sim \square$ ,  $\tan \square \sim \square$ ,  $\arctan \square \sim \square$ ,

$\ln(1 + \square) \sim \square$ ,  $e^{\square} - 1 \sim \square$ ,  $1 - \cos \square \sim \frac{1}{2} \square^2$ ,

$(1 + \square)^{\alpha} - 1 \sim \alpha \square \quad (\alpha \neq 0)$ .

解:

$$(1) \quad \lim_{x \rightarrow 0} \frac{\tan 3x}{\sin 2x} = \lim_{x \rightarrow 0} \frac{3x}{2x} = \frac{3}{2}$$

$$(2) \quad \lim_{x \rightarrow 0} \frac{\arcsin x^n}{(\sin x)^m} = \lim_{x \rightarrow 0} \frac{x^n}{(x)^m} = \lim_{x \rightarrow 0} \frac{x^n}{x^m} \begin{cases} 0, m < n \\ 1, m = n \\ \infty, m > n \end{cases}$$

当  $\square \rightarrow 0$  时

$\sin \square \sim \square$ ,  $\arcsin \square \sim \square$ ,  $\tan \square \sim \square$ ,  $\arctan \square \sim \square$ ,

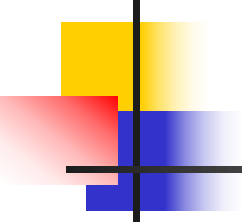
$\ln(1 + \square) \sim \square$ ,  $e^{\square} - 1 \sim \square$ ,  $1 - \cos \square \sim \frac{1}{2}\square^2$ ,

$(1 + \square)^{\alpha} - 1 \sim \alpha \square$  ( $\alpha \neq 0$ ).

$$(3) \quad \lim_{x \rightarrow 0} \frac{\ln(1 + \boxed{2x})}{x} = \lim_{x \rightarrow 0} \frac{2x}{x} = 2$$

$$(4) \quad \lim_{x \rightarrow 0} \frac{\sqrt{1 + x \sin x} - 1}{x \cdot \arctan x} = \lim_{x \rightarrow 0} \frac{(1 + \boxed{x \sin x})^{\frac{1}{2}} - 1}{x \cdot \arctan x}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{1}{2} x \sin x}{x \cdot x} = \frac{1}{2} \lim_{x \rightarrow 0} \frac{\sin x}{x} = \frac{1}{2}$$



当  $\square \rightarrow 0$  时

$\sin \square \sim \square$ ,  $\arcsin \square \sim \square$ ,  $\tan \square \sim \square$ ,  $\arctan \square \sim \square$ ,

$\ln(1 + \square) \sim \square$ ,  $e^{\square} - 1 \sim \square$ ,  $1 - \cos \square \sim \frac{1}{2}\square^2$ ,

$(1 + \square)^{\alpha} - 1 \sim \alpha \square$  ( $\alpha \neq 0$ ).

$$(5) \quad \lim_{n \rightarrow \infty} 2^n \sin \frac{x}{2^n} = \lim_{n \rightarrow \infty} 2^n \cdot \frac{x}{2^n} = \lim_{n \rightarrow \infty} x = x$$

$$(6) \quad \lim_{x \rightarrow 0} \frac{(1 + ax)^{\frac{1}{n}} - 1}{x} = \lim_{x \rightarrow 0} \frac{\frac{1}{n} \cdot ax}{x} = \lim_{x \rightarrow 0} \frac{a}{n} = \frac{a}{n}$$





## 三、小结

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### 1. 无穷小的比较:

反映了同一过程中, 两无穷小趋于零的速度快慢, 但并不是所有的无穷小都可进行比较.  
高(低)阶无穷小; 等价无穷小; 无穷小的阶.

### 2. 等价无穷小的替换:

求极限的又一种方法, 注意适用条件.

## 思考题

任何两个无穷小量都可以比较吗？

## 思考题解答

不能. 例当  $x \rightarrow +\infty$  时

$f(x) = \frac{1}{x}$ ,  $g(x) = \frac{\sin x}{x}$  都是无穷小量

但  $\lim_{x \rightarrow +\infty} \frac{g(x)}{f(x)} = \lim_{x \rightarrow +\infty} \sin x$  不存在且不为无穷大

故当  $x \rightarrow +\infty$  时  $f(x)$  和  $g(x)$  不能比较.

# 练习题

## 一、填空题：

$$1、\lim_{x \rightarrow 0} \frac{\tan 3x}{\sin 2x} = \underline{\hspace{2cm}}.$$

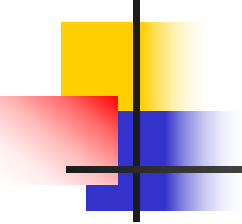
$$2、\lim_{x \rightarrow 0} \frac{\arcsin x^n}{(\sin x)^m} = \underline{\hspace{2cm}}.$$

$$3、\lim_{x \rightarrow 0} \frac{\ln(1+2x)}{x} = \underline{\hspace{2cm}}.$$

$$4、\lim_{x \rightarrow 0} \frac{\sqrt{1+x \sin x} - 1}{x^2 \arctan x} = \underline{\hspace{2cm}}.$$

$$5、\lim_{n \rightarrow \infty} 2^n \sin \frac{x}{2^n} = \underline{\hspace{2cm}}.$$

$$6、\lim_{x \rightarrow 0} \frac{(1+ax)^{\frac{1}{n}} - 1}{x} = \underline{\hspace{2cm}}.$$



7、当  $x \rightarrow 0$  时，无穷小  $1 - \cos x$  与  $mx^n$  等价，则  
 $m = \underline{\hspace{2cm}}, n \underline{\hspace{2cm}}.$

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二、求下列各极限：

1、 $\lim_{x \rightarrow 0} \frac{\tan x - \sin x}{\sin^3 x};$

2、 $\lim_{\alpha \rightarrow \beta} \frac{e^\alpha - e^\beta}{\alpha - \beta};$

3、 $\lim_{x \rightarrow 0} \frac{\sin \alpha x - \sin \beta x}{x};$

4、 $\lim_{x \rightarrow a} \frac{\tan x - \tan a}{x - a};$



## 练习题答案

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一、 1、  $\frac{3}{2}$ ;      2、  $\begin{cases} 0, m < n \\ 1, m = n \\ \infty, m > n \end{cases}$ ;      3、 2;      4、  $\infty$ ;

5、  $x$ ;      6、  $\frac{a}{n}$ ;      7、 3;      8、  $\frac{1}{2}, 2$ .

二、 1、  $\frac{1}{2}$ ;      2、  $e^\beta$ ;      3、  $\alpha - \beta$ ;      4、  $\sec^2 a$ .