4.3　分部积分法

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高等数学

分部积分法

设函数 $u=u(x)$ ， $v=v(x)$ 可导，

导数为 $u'(x)$ 和 $v'(x)$ ，则 $(uv)'=u'v+uv'$.

$$d(uv)=vdu+\boxed{udv} \Longrightarrow udv=d(uv)-vdu$$

等式左右两边同时求积分，

$$\int udv=\int 1d(uv)-\int vdu \Longrightarrow \int udv=uv-\int vdu$$

得到分部积分公式 $\boxed{\int udv=uv-\int vdu}$

例1（1）$\int \underline{x} d\left(\boxed{e^x}\right) = xe^x - \int e^x dx$

$= xe^x - e^x + C$

（2）$\int x \cdot \boxed{e^x dx} = \int xd\left(e^x\right)$

$= xe^x - \int e^x dx$

$= xe^x - e^x + C$

例2（1）$\int x \cdot \boxed{\sin x dx} = \int x d(-\cos x)$

$$= x \cdot (-\cos x) - \int -\cos x dx$$

$$= -x \cdot \cos x + \int \cos x dx$$

$$= -x \cdot \cos x + \sin x + C$$

选择函数移动的优先级顺序

$$e^{x},\quad e^{2x},\quad e^{-x}\cdots\cdots$$

$$\sin x,\ \cos x,\ \sin 2x\cdots\cdots$$

$$x,\quad x^{2}\cdots\cdots$$

（2） $\int x \cdot \boxed{\sin 2x dx} = \frac{1}{2}\int x d\left(-\cos 2x\right)$

$$= -\frac{1}{2}\int x d\left(\cos 2x\right)$$

$$= -\frac{1}{2}\left(x \cdot \cos 2x - \int \cos 2x dx\right)$$

$$= -\frac{1}{2}\left[x \cdot \cos 2x - \frac{1}{2}\int \cos \boxed{2x} d(2x)\right]$$

$$= -\frac{1}{2}x \cdot \cos 2x + \frac{1}{4}\sin 2x + C$$

例3 $\int x\cdot\ln x dx=\int \ln x\cdot \boxed{xdx}$

$$=\int \ln x d\left(\frac{1}{2}x^2\right)$$

$$=\frac{1}{2}x^2\ln x-\int\frac{1}{2}x^2 d(\ln x)$$

$$=\frac{1}{2}x^2\ln x-\int\frac{1}{2}x^2\cdot\frac{1}{x}dx$$

$$=\frac{1}{2}x^2\ln x-\frac{1}{2}\int x\,dx$$

$$=\frac{1}{2}x^2\ln x-\frac{1}{4}x^2+C$$

练习

$$(1)\quad \int x\boxed{\cos x\, dx} = \int x d(\sin x)$$

$$= x\sin x - \int \sin x dx$$

$$= x\sin x - (-\cos x) + C$$

$$= x\sin x + \cos x + C$$

$$(2)\quad \int x\boxed{e^{2x}dx} = \frac{1}{2}\int x d\left(e^{2x}\right)$$

$$=\frac{1}{2}\left[\ xe^{2x} - \frac{1}{2}\int e^{\boxed{2x}} d(2x)\right]$$

$$=\frac{1}{2}xe^{2x} - \frac{1}{4}\int e^{2x} d(2x)$$

$$=\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$$

(3) $\int x^2 \ln x\, dx = \int \ln x \boxed{\cdot x^2\, dx} = \int \ln x d\left(\frac{1}{3}x^3\right)$

$$= \frac{1}{3}x^3 \cdot \ln x - \int \frac{1}{3}x^3 d(\ln x)$$

$$= \frac{1}{3}x^3 \cdot \ln x - \frac{1}{3}\int x^3 \cdot \frac{1}{x} dx$$

$$= \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2 dx$$

$$= \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C$$

$$(4)\quad \int x\ln^2 x\,dx = \int \ln^2 x\cdot \boxed{x\,dx}$$

$$= \int \ln^2 x\,d\left(\frac{1}{2}x^2\right)$$

$$= \frac{1}{2}x^2\cdot\ln^2 x - \int \frac{1}{2}x^2\,d\left(\ln^2 x\right)$$

$$= \frac{1}{2}x^2\ln^2 x - \frac{1}{2}\int x^2\cdot 2\ln x\cdot(\ln x)'\,dx$$

$$= \frac{1}{2}x^2\ln^2 x - \int x^2\cdot\ln x\cdot\frac{1}{x}\,dx$$

$$= \frac{1}{2}x^2 \ln^2 x - \int x \ln x dx$$

$$= \frac{1}{2}x^2 \ln^2 x - \int \ln x d\left(\frac{1}{2}x^2\right)$$

$$= \frac{1}{2}x^2 \ln^2 x - \left[\frac{1}{2}x^2 \cdot \ln x - \int \frac{1}{2}x^2 d(\ln x)\right]$$

$$= \frac{1}{2}x^2 \ln^2 x - \frac{1}{2}x^2 \ln x + \frac{1}{2}\int x^2 \cdot \frac{1}{x} dx$$

$$= \frac{1}{2}x^2 \ln^2 x - \frac{1}{2}x^2 \ln x + \frac{1}{2}\int x dx$$

$$= \frac{1}{2}x^2 \ln^2 x - \frac{1}{2}x^2 \ln x + \frac{1}{4}x^2 + C$$

$$(5)\quad \int \underline{\ln x}\, d\boxed{x} = x\ln x - \int x\, d(\ln x)$$

$$= x\ln x - \int x \cdot \frac{1}{x}\, dx$$

$$= x\ln x - \int 1\, dx$$

$$= x\ln x - x + C$$

$$(6)\quad \int \underline{\arctan x}\, d\boxed{x} = x\cdot\arctan x - \int x\, d(\arctan x)$$

$$= x\cdot\arctan x - \int x\cdot\frac{1}{1+x^2}\,dx$$

$$= x\cdot\arctan x - \int \frac{1}{1+x^2}\boxed{\cdot x\,dx}$$

$$= x\cdot\arctan x - \int \frac{1}{1+x^2}\,d\left(\frac{1}{2}x^2\right)$$

$$= x\cdot\arctan x - \frac{1}{2}\int \frac{1}{1+x^2}\,d\left(x^2+1\right)$$

$$= x\cdot\arctan x - \frac{1}{2}\ln\left(1+x^2\right)+C$$

思考题

(1) $\int \arctan\sqrt{x}\,dx$

解： 令 $\sqrt{x}=t$ ，得到 $\boxed{x=t^2}$，

$$d(x)=d(t^2) \quad \Longrightarrow \quad dx = 2t\,dt$$

$$原积分=\int \arctan t \boxed{\cdot 2tdt}$$

$$=\int \arctan t\,d(\ t^2\ )$$

$$\boxed{\int \arctan t d\left(t^2\right)}$$

$$= t^2 \cdot \arctan t - \int t^2 d\left(\arctan t\right)$$

$$= t^2 \cdot \arctan t - \int t^2 \cdot \frac{1}{1+t^2} dt$$

$$= t^2 \cdot \arctan t - \int \frac{t^2 + 1 - 1}{1+t^2} dt$$

$$= t^2 \cdot \arctan t - \int \left(1 - \frac{1}{1+t^2} \right) dt$$

$$\boxed{t = \sqrt{x}}$$

$$= t^2 \cdot \arctan t - \left(t - \arctan t \right) + C$$

$$= x \cdot \arctan \sqrt{x} - \sqrt{x} + \arctan \sqrt{x} + C$$

$$(2)\quad \int \boxed{e^x} \cdot \sin x\,dx = \int \sin x\,d(\,e^x\,)$$

$$= e^x \sin x - \int e^x\,d(\sin x)$$

$$= e^x \sin x - \int \boxed{e^x} \cdot \cos x\,dx$$

$$= e^x \sin x - \int \cos x\,d(\,e^x\,)$$

$$= e^x \sin x - [\,e^x \cos x - \int e^x\,d(\cos x)\,]$$

$$= e^x \sin x - e^x \cos x + \int e^x(-\sin x)\,dx$$

$$\int e^x \cdot \sin x\,dx = e^x \sin x - e^x \cos x - \int e^x \cdot \sin x\,dx$$

$$2\int e^x \cdot \sin x\,dx = e^x \sin x - e^x \cos x \quad +C$$

$$\int e^x \cdot \sin x\,dx = \frac{1}{2}\left(e^x \sin x - e^x \cos x\right) + C$$