不定积分习题课

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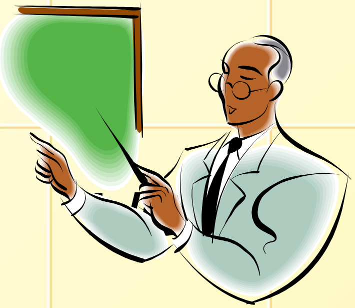

高等数学

例1 (1) $\int \frac{1}{1+x} dx = \int \frac{1}{1+x} d(x+1) = \ln|x+1| + C$

(2) $\int \frac{x}{1+x} dx = \int \frac{x+1-1}{1+x} dx = \int \left(1 - \frac{1}{x+1}\right) dx$

$= x - \ln|x+1| + C$

(3) $\int \frac{1}{1+x^2} dx = \arctan x + C$

$$(4)\quad \int \frac{x}{1+x^2}dx = \int \frac{1}{1+x^2}\cdot \boxed{x\,dx} = \int \frac{1}{1+x^2}d\left(\frac{1}{2}x^2\right)$$

$$= \frac{1}{2}\int \frac{1}{1+x^2}d\left(x^2+1\right)$$

$$= \frac{1}{2}\ln\left|1+x^2\right| + C$$

$$= \frac{1}{2}\ln\left(1+x^2\right) + C$$

$$(5)\quad \int \frac{x^4}{1+x^2}dx = \int \frac{x^4-1+1}{1+x^2}dx$$

$$= \int \left(\frac{x^4-1}{1+x^2} + \frac{1}{1+x^2} \right) dx$$

$$= \int \left(\frac{\left(x^2+1\right)\left(x^2-1\right)}{1+x^2} + \frac{1}{1+x^2} \right) dx$$

$$= \int \left(x^2 - 1 + \frac{1}{1+x^2} \right) dx$$

$$= \frac{1}{3}x^3 - x + \arctan x + C$$

(6) $\int \frac{x}{1+x^4}dx = \int \frac{x}{1+\left(x^2\right)^2}dx$

$$= \int \frac{1}{1+\left(x^2\right)^2}\cdot \boxed{x\,dx}$$

$$= \int \frac{1}{1+\left(x^2\right)^2}d\left(\frac{1}{2}x^2\right)$$

$$= \frac{1}{2}\int \frac{1}{1+\left(x^2\right)^2}d\left(x^2\right)$$

$$= \frac{1}{2}\arctan x^2 + C$$

例2 (1) $\int \cos x\,dx = \sin x + C$

(2) $\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}\,dx$

$$= \frac{1}{2}\int (1+\cos 2x)\,dx$$

$$= \frac{1}{2}\left[\int 1dx + \frac{1}{2}\int \cos 2x\,d(2x)\right]$$

$$= \frac{1}{2}x + \frac{1}{4}\sin 2x + C$$

$$(3)\quad \int \cos^3 x\,dx = \int \cos^2 x \cdot \boxed{\cos x dx}$$

$$= \int \cos^2 x d(\sin x)$$

$$= \int (1-\sin^2 x)\, d(\sin x)$$

$$= \sin x - \frac{1}{3}\sin^3 x + C$$

$$\boxed{\begin{aligned} &\int \left(1-t^2\right) dt \\ &= t - \frac{1}{3}t^3 + C \end{aligned}}$$

例3 (1) $\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx$

$$= \int \frac{1}{\cos x} \boxed{\sin x \, dx}$$

$$= \int \frac{1}{\cos x} d(-\cos x)$$

$$= -\ln|\cos x| + C$$

(2) $\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx = \tan x - x + C$

(3) $\int \tan^3 x\,dx = \int \tan x \cdot \tan^2 x\,dx = \int \tan x \cdot (\sec^2 x - 1)\,dx$

$$= \int \left(\tan x \cdot \sec^2 x - \tan x\right) dx$$

$$= \int \tan x \cdot \boxed{\sec^2 x dx} - \int \boxed{\frac{\sin x}{\cos x} dx}$$

$$= \int \tan x d(\tan x) - \int \frac{1}{\cos x} d(-\cos x)$$

$$= \frac{1}{2}\tan^2 x + \ln|\cos x| + C$$

例4 (1) $$\int \frac{e^x}{1+e^x}dx = \int \frac{1}{1+e^x}\boxed{\cdot e^x dx}$$

$$= \int \frac{1}{1+e^x} d\left(e^x + 1 \right)$$

$$= \ln\left|1+e^x\right| + C$$

$$= \ln\left(1+e^x\right) + C$$

$$(2)\quad \int \frac{1}{1+e^x}dx = \int \frac{1+e^x-e^x}{1+e^x}dx$$

$$= \int \left(1 - \frac{e^x}{1+e^x}\right)dx$$

$$= x - \int \frac{e^x}{1+e^x}dx$$

$$= x - \int \frac{1}{1+e^x}d\left(e^x+1\right)$$

$$= x + \ln\left(1+e^x\right) + C$$

$$(3)\quad \int \frac{1}{e^{x}+e^{-x}}dx = \int \frac{1}{e^{x}+\frac{1}{e^{x}}}dx$$

$$= \int \frac{1}{\frac{\left(e^{x}\right)^{2}+1}{e^{x}}}dx = \int \frac{e^{x}}{1+\left(e^{x}\right)^{2}}dx$$

$$= \int \frac{1}{1+\left(e^{x}\right)^{2}} \boxed{\cdot e^{x}dx} = \int \frac{1}{1+\left(e^{x}\right)^{2}} d\left(e^{x}\right)$$

$$= \arctan e^{x} + C$$

$$\boxed{\int \frac{1}{1+t^{2}}dt = \arctan t + C}$$

(4) $\int \frac{1}{e^x - e^{-x}} dx = \int \frac{1}{e^x - \frac{1}{e^x}} dx$

$$= \int \frac{1}{\frac{(e^x)^2 - 1}{e^x}} dx = \int \frac{e^x}{(e^x)^2 - 1} dx = \int \frac{e^x}{(e^x+1)(e^x-1)} dx$$

$$= \int \frac{1}{(e^x-1)(e^x+1)} d(e^x) = \frac{1}{2} \int \left(\frac{1}{e^x-1} - \frac{1}{e^x+1} \right) d(e^x)$$

$$= \frac{1}{2} \left(\ln|e^x - 1| - \ln|e^x + 1| \right) + C = \frac{1}{2} \ln\left| \frac{e^x-1}{e^x+1} \right| + C$$

例5 (1) $\int \frac{1}{x^2+3x+2}dx = \int \frac{1}{(x+1)(x+2)}dx$

$$= \int \left(\frac{1}{x+1} - \frac{1}{x+2} \right) dx$$

$$\boxed{\int \frac{1}{t}dt = \ln|t| + C}$$

$$= \int \frac{1}{x+1}d(x+1) - \int \frac{1}{x+2}d(x+2)$$

$$= \ln|x+1| - \ln|x+2| + C$$

$$= \ln\left|\frac{x+1}{x+2}\right| + C$$

$$(2)\quad \int \frac{1}{x^2+4x+5}dx = \int \frac{1}{(x^2+4x+4)+1}dx$$

$$= \int \frac{1}{1+(x+2)^2}dx$$

$$\boxed{\int \frac{1}{1+t^2}dt = \arctan t + C}$$

$$= \int \frac{1}{1+(x+2)^2}d(x+2)$$

$$= \arctan(x+2) + C$$

$$(3)\quad \int \frac{2x+3}{x^2+3x+2}dx = \int \frac{1}{x^2+3x+2}\cdot \boxed{(2x+3)\,dx}$$

$$\boxed{\int \frac{1}{t}dt = \ln|t| + C}$$

$$= \int \frac{1}{x^2+3x+2}d\left(x^2+3x+2\right)$$

$$= \ln\left|x^2+3x+2\right| + C$$

(4) $\int \frac{2x+4}{x^2+3x+2}dx$

$$= \int \frac{2x+3+1}{x^2+3x+2}dx$$

例5（3）

例5（1）

$$= \int \frac{2x+3}{x^2+3x+2}dx + \int \frac{1}{x^2+3x+2}dx$$

$$= \ln\left|x^2+3x+2\right| + \ln\left|\frac{x+1}{x+2}\right| + C$$

$$(5)\quad \int \frac{x+1}{x^2+4x+5}dx = \frac{1}{2}\int \frac{2(x+1)}{x^2+4x+5}dx$$

$$= \frac{1}{2}\int \frac{2x+2}{x^2+4x+5}dx = \frac{1}{2}\int \frac{2x+4-2}{x^2+4x+5}dx$$

$$= \frac{1}{2}\int \frac{2x+4}{x^2+4x+5}dx - \frac{1}{2}\int \frac{2}{x^2+4x+5}dx$$

例5（2）

$$= \frac{1}{2}\int \frac{1}{x^2+4x+5}d\left(x^2+4x+5\right) - \int \frac{1}{x^2+4x+5}dx$$

$$= \ln\left|x^2+4x+5\right| - \arctan(x+2) + C$$

例6 $\int \arctan\sqrt{x}\,dx$

解: 令 $\sqrt{x}=t$ ，得到 $\boxed{x=t^2}$，

$$d(x)=d\left(t^2\right) \quad \Rightarrow \quad dx = 2t\ dt$$

$$原积分 = \int \arctan t\ \boxed{\cdot 2tdt}$$

$$= \int \arctan t d\left(\ t^2\right)$$

$$= t^2 \cdot \arctan t - \int t^2 d\left(\arctan t\right)$$

$$= t^2 \cdot \arctan t - \int t^2 \cdot \frac{1}{1+t^2} dt$$

$$= t^2 \cdot \arctan t - \int \frac{t^2 + 1 - 1}{1+t^2} dt$$

$$= t^2 \cdot \arctan t - \int \left(1 - \frac{1}{1+t^2} \right) dt$$

$$= t^2 \cdot \arctan t - (t - \arctan t) + C \qquad \boxed{t = \sqrt{x}}$$

$$= x \cdot \arctan \sqrt{x} - \sqrt{x} + \arctan \sqrt{x} + C$$