# 4.2　换元积分法

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高等数学

# 一、第一类换元积分法（凑微分法）

设 $f(t)$ 具有原函数 $F(t)$，即

$$F'(t)=f(t) \quad , \qquad \int f(t)dt=F(t)+C$$

如果 $t=\varphi(x)$，

则 $dt=d\left[\varphi(x)\right]=\varphi'(x)dx$

$$\int f\left[\varphi(x)\right]\boxed{\cdot\varphi'(x)dx}=\int f\left[\varphi(x)\right]d\left[(\varphi(x))\right]=\int f(t)dt$$

$$=F(t)+C=F\left[\varphi(x)\right]+C$$

例1

(1) $\int e^{\boxed{\frac{1}{3}x}}dx = 3\cdot\int e^{\frac{1}{3}x}d\left(\frac{1}{3}x\right)$

$$\int e^{\boxed{x}}d\boxed{x} = e^{x} + C$$

$= 3\int e^{t}dt \quad = 3e^{t} + C$

$= 3e^{\frac{x}{3}} + C$

(2) $\int \frac{1}{x-3}dx = \int \frac{1}{x-3}d(x-3)$

$$\int \frac{1}{\boxed{x}}d\boxed{x} = \ln|x| + C$$

$= \int \frac{1}{t}dt \quad = \ln|t| + C$

$= \ln|x-3| + C$

$$dx = \frac{1}{k} d\left(kx+1\right)$$

$$(3) \quad \int (2x+1)^{10} dx = \frac{1}{2} \int (2x+1)^{10} d\left(2x+1\right)$$

$$= \frac{1}{2} \cdot \frac{1}{11} \left(2x+1\right)^{11} + C$$

$$= \frac{1}{22} (2x+1)^{11} + C$$

$$\int \boxed{x}^{10}\, d\boxed{x} = \frac{1}{11} x^{11} + C$$

例2 (1) $\int \frac{x}{1+x^2}dx$

$$= \int \frac{1}{1+x^2} \cdot \boxed{x\,dx}$$

$$xdx = d\left(\frac{1}{2}x^2\right)$$

$$= \int \frac{1}{1+x^2} d\left(\frac{1}{2}x^2\right)$$

$$= \frac{1}{2}\int \frac{1}{1+x^2} d\left(x^2\right)$$

$$\int \frac{1}{\boxed{x}} d\boxed{x} = \ln|x| + C$$

$$= \frac{1}{2}\int \frac{1}{1+x^2} d\left(x^2+1\right)$$

$$\int \frac{1}{t} dt = \ln|t| + C$$

$$= \frac{1}{2}\ln\left|1+x^2\right| + C \quad = \frac{1}{2}\ln\left(1+x^2\right) + C$$

$$(2)\quad \int x^3 \cdot e^{x^4}\,dx$$

$$= \int e^{x^4} \cdot \boxed{x^3\,dx}$$

$$\boxed{x^3 dx = d\left(\frac{1}{4}x^4\right)}$$

$$= \int e^{x^4}\,d\left(\frac{1}{4}x^4\right)$$

$$= \frac{1}{4}\int e^{x^4}\,d\left(x^4\right)$$

$$\boxed{\int e^t\,dt = e^t + C}$$

$$= \frac{1}{4}e^{x^4} + C$$

例3

$$\int \sin\frac{1}{x}\cdot\boxed{\frac{1}{x^2}dx}$$

$$\boxed{\frac{1}{x^2}dx = d\left(-\frac{1}{x}\right)}$$

$$= \int \sin\frac{1}{x}\, d\left(-\frac{1}{x}\right)$$

$$= -\int \sin\frac{1}{x}\, d\left(\frac{1}{x}\right)$$

$$\boxed{\int \sin t\, dt = -\cos t + C}$$

$$= -\left(-\cos\frac{1}{x}\right) + C \ = \cos\frac{1}{x} + C$$

**例4** (1) $\int \frac{\sin\sqrt{x}}{\sqrt{x}}dx$

$$=\int \sin\sqrt{x}\cdot\boxed{\frac{1}{\sqrt{x}}dx}$$

$$\boxed{\frac{1}{\sqrt{x}}dx=d\left(2\sqrt{x}\right)}$$

$$=\int \sin\sqrt{x}d\left(2\sqrt{x}\right)$$

$$=2\int \sin\sqrt{x}d\left(\sqrt{x}\right)$$

$$\boxed{\int \sin t\,dt=-\cos t+C}$$

$$=2\cdot\left(-\cos\sqrt{x}\right)+C$$

$$=-2\cos\sqrt{x}+C$$

$$\frac{1}{\sqrt{x}}dx = d\left(2\sqrt{x}\right)$$

(2) $$\int \frac{1}{\sqrt{x}\cdot(1+x)}dx = \int \frac{1}{(1+x)}\cdot \boxed{\frac{1}{\sqrt{x}}dx}$$

$$= \int \frac{1}{(1+x)} d\left(2\sqrt{x}\right)$$

$$= 2\int \frac{1}{(1+\boxed{x})} d\left(\sqrt{x}\right)$$

$$\boxed{\int \frac{1}{1+t^2}dt = \arctan t + C}$$

$$= 2\int \frac{1}{1+\left(\sqrt{x}\right)^2} d\left(\sqrt{x}\right)$$

$$= 2\arctan\sqrt{x} + C$$

例5

$$\int \frac{e^x}{e^x+2}dx$$

$$e^x dx = d\left(e^x + C\right)$$

$$= \int \frac{1}{e^x+2} \cdot \boxed{e^x\, dx}$$

$$= \int \frac{1}{e^x+2} d\left(e^x + 2\right)$$

$$\int \frac{1}{t} dt = \ln|t| + C$$

$$= \ln\left|e^x + 2\right| + C$$

$$= \ln\left(e^x + 2\right) + C$$

例6 $\int \frac{1-2\ln x}{x}dx$

$= \int (1-2\ln x)\cdot \boxed{\frac{1}{x}dx}$

$\boxed{\frac{1}{x}dx = d(\ln x)}$

$= \int (1-2\ln x)d(\ln x)$

令 $\ln x = t$

$= \int (1-2t)dt$

$= t - t^2 + C$

$= \ln x - \ln^2 x + C$

**例7** (1) $\int \tan x\,dx$

$$= \int \frac{\sin x}{\cos x} dx$$

$$= \int \frac{1}{\cos x} \cdot \boxed{\sin x\,dx}$$

$$\boxed{\sin x dx = d\left(-\cos x\right)}$$

$$= \int \frac{1}{\cos x} d\left(-\cos x\right)$$

$$= -\int \frac{1}{\cos x} d\left(\cos x\right)$$

$$\boxed{\int \frac{1}{t} dt = \ln\left|t\right| + C}$$

$$= -\ln\left|\cos x\right| + C$$

$$(2)\quad \int \frac{\tan x}{\cos^2 x}dx = \int \tan x \cdot \frac{1}{\cos^2 x}dx$$

$$= \int \tan x \cdot \boxed{\sec^2 x\, dx}$$

$$\boxed{\sec^2 x dx = d(\tan x)}$$

$$= \int \tan x\, d(\tan x)$$

$$\boxed{\int t\, dt = \frac{1}{2}t^2 + C}$$

$$= \frac{1}{2}\tan^2 x + C$$

$$(3)\quad \int \sin^2 x \cdot \cos^3 x dx$$

$$= \int \sin^2 x \cdot \cos^2 x \cdot \boxed{\cos x \, dx}$$

$$\boxed{\cos x dx = d(\sin x)}$$

$$= \int \sin^2 x \cdot \cos^2 x \, d(\sin x)$$

$$= \int \sin^2 x \cdot (1 - \sin^2 x) d(\sin x)$$

$$= \int (\sin^2 x - \sin^4 x) d(\sin x)$$

$$= \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C$$

$$\int (t^2 - t^4) dt = \frac{1}{3}t^3 - \frac{1}{5}t^5 + C$$

(4) $\int \sec x dx = \int \frac{\sec x}{1} dx$

$$= \int \frac{\sec x \cdot (\sec x + \tan x)}{\sec x + \tan x} dx$$

$$= \int \frac{\sec^2 x + \sec x \cdot \tan x}{\sec x + \tan x} dx$$

$$\boxed{\int \frac{1}{t} dt = \ln|t| + C}$$

$$= \int \frac{1}{\sec x + \tan x} d(\sec x + \tan x)$$

$$= \ln|\sec x + \tan x| + C$$

例8

(1) $$\int \frac{1}{x^2+3x+2}dx = \int \frac{1}{(x+1)(x+2)}dx$$

$$= \int \left( \frac{1}{x+1} - \frac{1}{x+2} \right) dx$$

$$\boxed{\int \frac{1}{t}dt = \ln|t| + C}$$

$$= \int \frac{1}{x+1}d(x+1) - \int \frac{1}{x+2}d(x+2)$$

$$= \ln|x+1| - \ln|x+2| + C$$

$$= \ln\left|\frac{x+1}{x+2}\right| + C$$

$$(2)\quad \int \frac{1}{x^2+4x+5}\,dx = \int \frac{1}{(x^2+4x+4)+1}\,dx$$

$$= \int \frac{1}{1+(x+2)^2}\,dx$$

$$\boxed{\int \frac{1}{1+t^2}\,dt = \arctan t + C}$$

$$= \int \frac{1}{1+(x+2)^2}\,d(x+2)$$

$$= \arctan(x+2) + C$$

$$(3)\quad \int \frac{2x+3}{x^2+3x+2}\,dx = \int \frac{1}{x^2+3x+2}\cdot \boxed{(2x+3)\,dx}$$

$$\boxed{\int \frac{1}{t}\,dt = \ln|t| + C}$$

$$= \int \frac{1}{x^2+3x+2}\,d\left(x^2+3x+2\right)$$

$$= \ln\left|x^2+3x+2\right| + C$$

## 练习

$$(9)\quad \int (3x-1)^2 dx = \frac{1}{3}\int (3x-1)^2 d(3x-1)$$

$$= \frac{1}{3} \cdot \frac{1}{3}(3x-1)^3 + C$$

$$= \frac{1}{9}(3x-1)^3 + C$$

$$\boxed{\int t^2\, dt = \frac{1}{3}t^3 + C}$$

**方法二：**

$$(9)\quad \int(3x-1)^2dx = \int(9x^2-6x+1)dx$$

$$= 9\cdot\frac{1}{3}x^3 - 6\cdot\frac{1}{2}x^2 + x + C$$

$$= 3x^3 - 3x^2 + x + C$$

(10) $\int xe^{x^2}dx = \int e^{x^2} \boxed{\cdot xdx}$

$$= \int e^{x^2} d\left(\frac{1}{2}x^2\right)$$

$$= \frac{1}{2}\int e^{x^2} d\left(x^2\right)$$

$$\boxed{\int e^t\,dt = e^t + C}$$

$$= \frac{1}{2}e^{x^2} + C$$

$$(11)\quad \int \cos x \cdot \sin^3 x\,dx = \int \sin^3 x \boxed{\cdot \cos x\,dx}$$

$$= \int \sin^3 x\,d(\sin x)$$

$$= \frac{1}{4}\sin^4 x + C$$

$$\boxed{\int t^3\,dt = \frac{1}{4}t^4 + C}$$

$$(12)\quad \int \frac{2x-3}{x^2-3x+4}dx = \int \frac{1}{x^2-3x+4} \cdot \boxed{(2x-3)\,dx}$$

$$= \int \frac{1}{x^2-3x+4} d\left( x^2-3x+4 \right)$$

$$= \ln\left|x^2-3x+4\right| + C$$

$$\boxed{\int \frac{1}{t}dt = \ln|t| + C}$$

(13) $$\int \frac{1}{x^2+x-2}dx = \int \frac{1}{(x+2)(x-1)}dx$$

$$= \frac{1}{3}\int \left(\frac{1}{x-1} - \frac{1}{x+2}\right)dx$$

$$\boxed{\int \frac{1}{t}dt = \ln|t| + C}$$

$$= \frac{1}{3}\left[\int \frac{1}{x-1}d(x-1) - \int \frac{1}{x+2}d(x+2)\right]$$

$$= \frac{1}{3}\left(\ln|x-1| - \ln|x+2|\right) + C$$

$$= \frac{1}{3}\ln\left|\frac{x-1}{x+2}\right| + C$$

(14) $\int \frac{1}{1+9x^2}dx = \int \frac{1}{1+(3x)^2}dx$

$$= \frac{1}{3}\int \frac{1}{1+(3x)^2}d(3x)$$

$$= \frac{1}{3}\arctan 3x + C$$

$$\boxed{\int \frac{1}{1+t^2}dt = \arctan t + C}$$

$$(15)\quad \int \frac{e^x}{1+e^x}\,dx = \int \frac{1}{1+e^x}\cdot \boxed{e^x\,dx}$$

$$= \int \frac{1}{1+e^x}\,d\left(e^x+1\right)$$

$$= \ln\left|e^x+1\right| + C$$

$$= \ln\left(e^x+1\right) + C$$

$$\boxed{\int \frac{1}{t}\,dt = \ln|t| + C}$$

$$(16)\quad \int \frac{1}{e^x + e^{-x}}dx = \int \frac{1}{e^x + \frac{1}{e^x}}dx$$

$$= \int \frac{1}{\frac{(e^x)^2 + 1}{e^x}}dx = \int \frac{e^x}{1+(e^x)^2}dx$$

$$= \int \frac{1}{1+(e^x)^2} \boxed{\cdot e^x dx} = \int \frac{1}{1+(e^x)^2}d(e^x)$$

$$= \arctan e^x + C$$

$$\boxed{\int \frac{1}{1+t^2}dt = \arctan t + C}$$

# 二、第二类换元积分法

讨论 $\int \frac{\sqrt{x-1}}{x} dx$

**例1** 求 $\int \frac{1}{2+\sqrt{x-1}} dx$

**解：** 令 $\sqrt{x-1}=t$，得到 $x-1=t^2$，即 $x=t^2+1$，

$$d(x)=d\left(t^2+1\right) \Rightarrow dx = 2t\,dt$$

$$\text{原积分} = \int \frac{1}{2+t} 2t\,dt = 2\int \frac{t}{2+t} dt$$

$$= 2\int \frac{t+2-2}{t+2} dt = 2\int \left(1-\frac{2}{t+2}\right) dt$$

$$= 2\int 1\,dt - 2\int \frac{2}{t+2}\,dt$$

$$= 2\int 1\,dt - 4\int \frac{1}{t+2}\,d(t+2)$$

$$= 2t - 4\ln|t+2| + C$$

$$\boxed{t = \sqrt{x-1}}$$

$$= 2\sqrt{x-1} - 4\ln\left(\sqrt{x-1}+2\right) + C$$

# 练习

(17) $\int \frac{\sqrt{x}}{1+\sqrt{x}} dx$

**解:** 令 $\sqrt{x} = t$ ， 得到 $x = t^2$ ，

$$d(x) = d(t^2) \Rightarrow dx = 2t\,dt$$

$$\text{原积分} = \int \frac{t}{1+t} 2tdt = 2\int \frac{t^2}{1+t} dt$$

$$= 2\int \frac{t^2-1+1}{1+t} dt = 2\int \left( t-1 + \frac{1}{t+1} \right) dt$$

$$\boxed{2\int\left(t-1+\frac{1}{t+1}\right)dt}$$

$$= 2\left(\frac{1}{2}t^2 - t + \ln|t+1|\right) + C$$

$$\boxed{t=\sqrt{x}}$$

$$= t^2 - 2t + 2\ln|t+1| + C$$

$$= x - 2\sqrt{x} + 2\ln\left(\sqrt{x}+1\right) + C$$

# 练习

(18) $\int \frac{1}{1+\sqrt[3]{x+2}}dx$

**解：** 令 $\sqrt[3]{x+2}=t$，得到 $x+2=t^3$，即 $x=t^3-2$，

$d(x)=d\left(t^3-2\right) \quad \Rightarrow \quad dx = 3t^2dt$

原积分 $= \int \frac{1}{1+t}3t^2dt = 3\int \frac{t^2}{1+t}dt$

$= 3\int \frac{t^2-1+1}{1+t}dt = 3\int \left(t-1+\frac{1}{t+1}\right)dt$

$$3\int\left(t-1+\frac{1}{t+1}\right)dt$$

$$=3\left(\frac{1}{2}t^2-t+\ln|t+1|\right)+C$$

$$=\frac{3}{2}t^2-3t+3\ln|t+1|+C$$

$$t=\sqrt[3]{x+2}$$

$$=\frac{3}{2}(x+2)^{\frac{2}{3}}-3\cdot\sqrt[3]{x+2}+3\ln\left|\sqrt[3]{x+2}+1\right|+C$$

**例2** 求 $\int \frac{1}{\left(1+\sqrt[3]{x}\right)\cdot\sqrt{x}}dx$

**解：** 令 $\sqrt[6]{x}=t$ ，得到 $\boxed{x=t^6}$，$\sqrt{x}=t^3$，$\sqrt[3]{x}=t^2$

$$d\left(x\right)=d\left(t^6\right) \Rightarrow dx = 6t^5dt$$

$$\text{原积分} = \int \frac{1}{\left(1+t^2\right)t^3}6t^5dt$$

$$=6\int \frac{t^2}{1+t^2}dt$$

$$= 6\int \frac{t^2 + 1 - 1}{1 + t^2} dt$$

$$= 6\int \left( 1 - \frac{1}{1 + t^2} \right) dt$$

$$= 6( t - \arctan t ) + C$$

$$\boxed{t = \sqrt[6]{x}}$$

$$= 6\left( \sqrt[6]{x} - \arctan \sqrt[6]{x} \right) + C$$

# 练习

(19) $\int \frac{1}{\sqrt{x}+\sqrt[4]{x}}dx$

**解:** 令 $\sqrt[4]{x}=t$ ，得到 $\boxed{x=t^4}$，$\sqrt{x}=t^2$

$d(x)=d(t^4) \quad \Longrightarrow \quad dx = 4t^3dt$

原积分 $=\int \frac{1}{t^2+t}4t^3dt$

$=4\int \frac{t^3}{t(t+1)}dt$

$$= 4\int \frac{t^2}{t+1}\,dt$$

$$= 4\int \frac{t^2-1+1}{t+1}\,dt$$

$$= 4\int \left( t-1 + \frac{1}{t+1} \right) dt$$

$$\boxed{\sqrt[4]{x} = t,\ \sqrt{x} = t^2}$$

$$= 4\left( \frac{1}{2}t^2 - t + \ln\left|t+1\right| \right) + C$$

$$= 2\sqrt{x} - 4\cdot\sqrt[4]{x} + 4\ln\left(\sqrt[4]{x}+1\right) + C$$

(20) $\int \frac{1}{\sqrt{x}+\sqrt[3]{x}} dx$

**解：** 令 $\sqrt[6]{x}=t$ ， 得到 $\boxed{x=t^6}$， $\sqrt{x}=t^3$， $\sqrt[3]{x}=t^2$

$$d(x)=d\left(t^6\right) \quad \Rightarrow \quad dx = 6t^5 dt$$

$$\text{原积分} = \int \frac{1}{t^3+t^2} 6t^5 dt$$

$$= 6\int \frac{t^5}{t^2(t+1)} dt$$

$$= 6\int \frac{t^3}{t+1}\,dt$$

$$= 6\int \frac{t^3+1-1}{t+1}\,dt$$

$$t^3+1=(t+1)(t^2-t+1)$$

$$= 6\int \left( t^2 - t + 1 - \frac{1}{t+1} \right) dt$$

$$= 6\left( \frac{1}{3}t^3 - \frac{1}{2}t^2 + t - \ln|t+1| \right) + C$$

$$= 2t^3 - 3t^2 + 6t - 6\ln|t+1| + C$$

$$= 2\sqrt{x} - 3\cdot\sqrt[3]{x} + 6\cdot\sqrt[6]{x} - 6\ln\left(\sqrt[6]{x}+1\right) + C$$

$$\sqrt[6]{x} = t\,,\ \sqrt{x} = t^3\,,\ \sqrt[3]{x} = t^2$$

目的：消去根式

利用三角恒等式：

$\sin^2 t + \cos^2 t = 1$

$dx = \cos t dt$

**例3**　求 $\int \sqrt{1-x^2}\,dx$

**解：**　令 $x = \sin t$，$\left(0 < t < \dfrac{\pi}{2}\right)$

$$\text{原积分} = \int \sqrt{1-\sin^2 t} \cdot \cos t dt = \int \cos t \cdot \cos t dt$$

$$= \int \cos^2 t dt = \int \frac{1+\cos 2t}{2} dt = \frac{1}{2}\int (1+\cos 2t) dt$$

$$= \frac{1}{2}\int 1 dt + \frac{1}{2}\cdot\frac{1}{2}\int \cos \boxed{2t} d(2t)$$

$$= \frac{1}{2} t + \frac{1}{4} \sin 2t + C$$

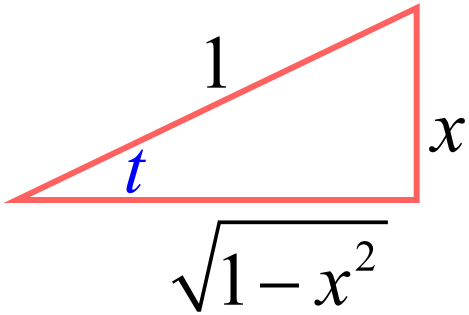
（其中 $t = \arcsin x$）

$$= \frac{1}{2} t + \frac{1}{4} \sin 2t + C$$

$$\because \sin t = x$$

$$\therefore t = \arcsin x$$

$$= \frac{1}{2} t + \frac{1}{2} \sin t \cdot \cos t + C$$




$$= \frac{1}{2} \arcsin x + \frac{1}{2} x \cdot \sqrt{1-x^2} + C$$

**例4** $\displaystyle\int \frac{dx}{\sqrt{x^2+1}}$ $\quad \tan^2 t + 1 = \sec^2 t$

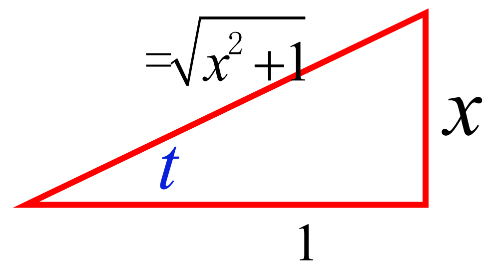
**解：** 令 $x = \tan t$, $\quad t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\quad dx = \sec^2 t dt$

$$\text{原式} = \int \frac{\sec^2 t}{\sec t} dt = \int \sec t dt$$

$$= \ln\left|\sec t + \tan t\right| + C$$

$\because \tan t = x$

$$= \ln\left|\sqrt{x^2+1} + x\right| + C$$



**例5** $\int \frac{dx}{\sqrt{x^2-1}}$