We aim to simulate the following 1D transport equations

$$\frac{\partial n_s}{\partial \tau} = -\frac{g}{A}\frac{\partial}{\partial \rho}\left[\frac{A}{B^2}\Gamma_s n_r T_r^{3/2}\right] + S_n \tag{1}$$

$$\begin{aligned} \frac{3}{2}\frac{\partial p_s}{\partial \tau} =& -\frac{g}{A}\frac{\partial}{\partial \rho}\left[\frac{A}{B^2}Q_s n_r T_r^{5/2}\right] \\ &+ \frac{g}{B^2} n_r T_r^{5/2}\left[\frac{T_s}{T_r}\frac{3}{2}\left(\frac{a}{L_{T_s}} - \frac{a}{L_n}\right)\Gamma_s - \frac{a}{L_{T_s}}Q_s\right] \\ &+ \nu_\epsilon^{su} n_s\left(T_u - T_s\right) - \frac{H_s}{B^2} n_r T_r^{5/2} + \frac{3}{2}S_p \end{aligned} \tag{2}$$

where Γ_s, Q_s, H_s are averaged turbulent fluxes obtained from an external code. Effects from 3D shaping on stellarator are contained within these averages, and we consider only 1D radial transport. We assume that they are flux functions parametrized by a gradient scale length

$$\kappa_x = \frac{a}{L_x} = -a\nabla \ln x = -\frac{1}{x}\frac{\partial x}{\partial \rho} \tag{3}$$

We assume a two species plasma $s \in (i, e)$ with quasi-neutrality relation $Z_s n_s = Z_r n_r$, where r is a reference species. Let us set $r = i$. The following conventions are adopted from Chapter 7 of the Barnes Trinity thesis: $g = \langle|\nabla\rho|\rangle$ is a geometry parameter related to the Jacobian. $A = A^N$ is the surface area. Both are flux functions. $B = B_j$ is a characteristic toroidal field defined for each flux surface. S_n, S_p are particle and energy sources. Ideal gas law $p = nT$ is assumed for all species. All quantities are normalized, and the normalizations are documented in Barnes 7.3. ν_ϵ^{su} is a collisional energy exchange between species [1]

$$\nu_\epsilon^{su} = \left[\frac{6.88(q_s q_u)^2 \ln\Lambda_{su}}{m_s m_u}\right] n_u \left(\frac{1}{T_s/m_s + T_u/m_s}\right)^{3/2} \tag{4}$$

TRINITY calculates profile evolution for the state vector $\mathbf{y} = \{n, p_i, p_e\}$.

1 Derivation of (n,P) equations

2 Derivation of (n,T) equations

The original Trinity code derives transport in a density-pressure basis. Here we would like to convert to a density-temperature basis. Let us define

$$\frac{\partial n}{\partial \tau} = -\frac{g}{A}\frac{\partial F}{\partial \rho} + S_n \tag{5}$$

$$\frac{3}{2}\frac{\partial p_s}{\partial \tau} = -\frac{g}{A}\frac{\partial I_s}{\partial \rho} + G_s + J_s + K_s + E_s + \frac{3}{2}S_p \tag{6}$$

[1] The factor 6.88 is $\frac{4\pi}{2^{3/2}} \times 1.55$ where

$$\nu_\epsilon^{su} = 1.55\,\nu_{su}\left(\frac{m_u}{m_s}\right)^{1/2}\left(\frac{T_u}{T_s} + \frac{m_u}{m_s}\right)^{-3/2}$$

and the particle collision frequency

$$\nu_{su} = 4\pi n_u\left(\frac{q_u q_s}{m_s}\right)^2\left(\frac{m_s}{2T_s}\right)^{3/2}\ln\Lambda_{su}$$

where

$$F = \frac{A}{B^2} n_i T_i^{3/2} \Gamma_i \tag{7}$$

$$I_s = \frac{A}{B^2} n_i T_i^{5/2} Q_s \tag{8}$$

$$G_s = \frac{g}{B^2} n_s T_i^{3/2} T_s \frac{Z_s}{Z_i} \left(\frac{3}{2} \frac{a}{L_{T_s}} - \frac{a}{L_n} \right) \Gamma_s \tag{9}$$

$$J_s = -\frac{g}{B^2} n_i T_i^{5/2} \left(\frac{a}{L_{T_s}} \right) Q_s \tag{10}$$

$$K_s = -\frac{1}{B^2} n_i T_i^{5/2} H_s \tag{11}$$

$$E_s = -b_{su} n_s^2 \frac{T_s - T_u}{(T_s/m_s + T_u/m_u)^{3/2}} \tag{12}$$

and the stack of constants governing collisions is stored in

$$b_{su} = 6.88 \frac{(q_s q_u)^2}{m_s m_u} \frac{Z_s}{Z_u} \ln \Lambda_{su} \tag{13}$$

2.1 density evolution

Let us start with the density equation 5 which only has one non-linear term. We wish to descretize

$$\frac{\partial F}{\partial \rho} = \frac{F_+ - F_-}{\Delta \rho} \tag{14}$$

where the subscript indicates half step evaulations $x_\pm = \frac{x_j + x_{j\pm1}}{2}$. Then the density evolution is expressed

$$\frac{n^{m+1} - n^m}{\Delta \tau} = \alpha \left[-\frac{g_j}{A_j} \frac{F_+ - F_-}{\Delta \rho} + S_n \right]^{m+1} + (1-\alpha) \left[-\frac{g_j}{A_j} \frac{F_+ - F_-}{\Delta \rho} + S_n \right]^{m} \tag{15}$$

where α is the implicit-explicit time step mixing parameter. [2] The next step is to descritize the time evolution of F by taking a Taylor expansion over the state vector

$$\begin{aligned} F_\pm^{m+1} &\approx F_\pm^m + (\mathbf{y} - \mathbf{y}_0) \left. \frac{\partial F_\pm}{\partial \mathbf{y}} \right|_{\mathbf{y}_0} \\ &= F_\pm^m + \sum_k \left[\left(n_k^{m+1} - n_k^m \right) \frac{\partial F_\pm}{\partial n_k} + \left(T_{i,k}^{m+1} - T_{i,k}^m \right) \frac{\partial F_\pm}{\partial T_{i,k}} + \left(T_{e,k}^{m+1} - T_{e,k}^m \right) \frac{\partial F_\pm}{\partial T_{e,k}} \right] \end{aligned} \tag{16}$$

Here we have adopted the convention that subscripts indicate space index and superscripts indicate time index. Going forward we will suppress m, such that no superscript index indicates "now", and only write $m+1$. The partial derivaties are

$$\frac{\partial F_\pm}{\partial n_k} = F_\pm \left[\frac{1}{2} \frac{\delta_j^k + \delta_{j\pm1}^k}{n_\pm} + \frac{\partial}{\partial n_k} \Gamma_\pm \right] \tag{17}$$

[2] When $\alpha = 0$, the algorithm is full explicit and n^{m+1} is solved from only information available at time m. When $\alpha = 1$, the algorithm is full impicit and n^{m+1} is solved from an inverting a matrix of $m+1$ information. In general implicit is harder to compute (both numerically and algebraicaly), while explicit is less stable. The gain from doing the work of writing an implicit algorithm is leveraging that extra stability to take much larger time steps. More discussion to follow after we derive the implicit matrix.

$$\frac{\partial F_\pm}{\partial T_{i,k}} = F_\pm \left[\frac{3}{4} \frac{\delta_j^k + \delta_{j\pm1}^k}{T_{i,\pm}} + \frac{\partial}{\partial T_{i,k}} \Gamma_\pm \right] \tag{18}$$

$$\frac{\partial F_\pm}{\partial T_{e,k}} = F_\pm \left[\frac{\partial}{\partial T_{e,k}} \Gamma_\pm \right] \tag{19}$$

Plugging these derivatives into 16, and cancelling terms at the m time-step, we find

$$\begin{aligned} \frac{F_\pm^{m+1}}{F_\pm} = \frac{1}{2} \left(\frac{n_j^{m+1} + n_{j\pm1}^{m+1}}{n_\pm} \right) + \frac{3}{4} \left(\frac{T_{i,j}^{m+1} + T_{i,j\pm1}^{m+1}}{T_{i,\pm}} \right) \\ + \sum_k \left(\mathbf{y}_k^{m+1} - \mathbf{y}_k \right) \frac{\partial}{\partial \mathbf{y}_k} \ln \Gamma_\pm \end{aligned} \tag{20}$$

To unpack the sum we take a chain rule using our assumption that the turbulent fluxes depend on the gradient scale lengths $\Gamma = \Gamma(\kappa_n, \kappa_{T_i}, \kappa_{T_e})$ [3]

$$\frac{\partial}{\partial n_k} \ln \Gamma_\pm = \frac{\partial (a/L_n)_\pm}{\partial n_k} \frac{\partial \ln \Gamma_\pm}{\partial (a/L_n)_\pm} \tag{21}$$

The gradient scale length can be discretized

$$\left(\frac{a}{L_n} \right)_\pm = - \left(\frac{\partial}{\partial \rho} \ln n \right)_\pm = \mp \left(\frac{n_{j\pm1} - n_j}{\Delta\rho} \right) \frac{1}{n_\pm} = \mp \frac{2}{\Delta\rho} \left(\frac{n_{j\pm1} - n_j}{n_{j\pm1} + n_j} \right) \tag{22}$$

such that the parametric derivative becomes a simple combinatoric

$$\begin{aligned} \frac{\partial}{\partial n_k} \left(\frac{a}{L_n} \right)_\pm &= \mp \frac{2}{\Delta\rho} \left[\frac{\delta_{j\pm1}^k - \delta_j^k}{n_{j\pm1} + n_j} - \frac{\delta_{j\pm1}^k + \delta_j^k}{(n_{j\pm1} + n_j)^2} (n_{j\pm1} - n_j) \right] \\ &= \mp \frac{2}{\Delta\rho} \left[\frac{\left(\delta_{j\pm1}^k - \delta_j^k\right)(n_{j\pm1} + n_j) - \left(\delta_{j\pm1}^k + \delta_j^k\right)(n_{j\pm1} - n_j)}{(n_{j\pm1} + n_j)^2} \right] \\ &= \mp \frac{1}{\Delta\rho} \left(\frac{\delta_{j\pm1}^k n_j - \delta_j^k n_{j\pm1}}{n_\pm^2} \right) \end{aligned} \tag{23}$$

where we substituted $n_\pm = \frac{n_{j\pm1} + n_j}{2}$ and used the algebraic identity $(a-b)(c+d) - (a+b)(c-d) = 2(ad - bc)$. This shows that

$$\sum_k \left(x_k^{m+1} - x_k \right) \frac{\partial}{\partial x} \left(\frac{a}{L_x} \right)_\pm = \pm \frac{1}{\Delta\rho} \left[\frac{x_j^{m+1}}{x_\pm} \frac{x_{j\pm1}}{x_\pm} - \frac{x_{j\pm1}^{m+1}}{x_\pm} \frac{x_j}{x_\pm} \right] \tag{24}$$

Now the density part of the summation in Eq 20 becomes

$$\begin{aligned} \sum_k \left(n_k^{m+1} - n_k \right) \frac{\partial (a/L_n)_\pm}{\partial n_k} \frac{\partial \ln \Gamma_\pm}{\partial (a/L_n)_\pm} &= \mp \frac{1}{\Delta\rho} \sum_k \frac{\left(n_k^{m+1} - n_k\right)\left(\delta_{j\pm1}^k n_j - \delta_j^k n_{j\pm1}\right)}{n_\pm^2} \frac{\partial \ln \Gamma_\pm}{\partial (a/L_n)_\pm} \\ &= \mp \frac{1}{\Delta\rho} \left[\frac{n_{j\pm1}^{m+1}}{n_\pm} \frac{n_j}{n_\pm} - \frac{n_j^{m+1}}{n_\pm} \frac{n_{j\pm1}}{n_\pm} \right] \frac{\partial \ln \Gamma_\pm}{\partial (a/L_n)_\pm} \end{aligned} \tag{25}$$

Similar algebra produces expressions for each of the ion and electron temperatures

$$\sum_k \left(T_k^{m+1} - T_k \right) \frac{\partial (a/T_n)_\pm}{\partial T_k} \frac{\partial \ln \Gamma_\pm}{\partial (a/L_T)_\pm} = \mp \frac{1}{\Delta\rho} \left[\frac{T_{j\pm1}^{m+1}}{T_\pm} \frac{T_j}{T_\pm} - \frac{T_j^{m+1}}{T_\pm} \frac{T_{j\pm1}}{T_\pm} \right] \frac{\partial \ln \Gamma_\pm}{\partial (a/L_T)_\pm} \tag{26}$$

[3] Is this a good assumption?

Therefore the full expression for 20 becomes

$$\begin{aligned}
\frac{F_{\pm}^{m+1}}{F_{\pm}} = \frac{1}{2}\left(\frac{n_j^{m+1} + n_{j\pm 1}^{m+1}}{n_{\pm}}\right) + \frac{3}{4}\left(\frac{T_{i,j}^{m+1} + T_{i,j\pm 1}^{m+1}}{T_{i,\pm}}\right) \\
\mp \frac{1}{\Delta\rho}\left[\frac{n_{j\pm 1}^{m+1}}{n_{\pm}}\frac{n_j}{n_{\pm}} - \frac{n_j^{m+1}}{n_{\pm}}\frac{n_{j\pm 1}}{n_{\pm}}\right]\frac{\partial \ln \Gamma_{\pm}}{\partial (a/L_n)_{\pm}} \\
\mp \frac{1}{\Delta\rho}\left(\frac{T_{i,j\pm 1}^{m+1}}{T_{i,\pm}}\frac{T_{i,j}}{T_{i,\pm}} - \frac{T_{i,j}^{m+1}}{T_{i,\pm}}\frac{T_{i,j\pm 1}}{T_{i,\pm}}\right)\frac{\partial \ln \Gamma_{\pm}}{\partial (a/L_{T_i})_{\pm}} \\
\mp \frac{1}{\Delta\rho}\left(\frac{T_{e,j\pm 1}^{m+1}}{T_{e,\pm}}\frac{T_{e,j}}{T_{e,\pm}} - \frac{T_{e,j}^{m+1}}{T_{e,\pm}}\frac{T_{e,j\pm 1}}{T_{e,\pm}}\right)\frac{\partial \ln \Gamma_{\pm}}{\partial (a/L_{T_e})_{\pm}}
\end{aligned} \tag{27}$$

Here we may introduce a shorthand

$$D_x Z = \frac{\partial \ln Z}{\partial (a/L_x)} \tag{28}$$

to describe the derivative of a flux Z by a gradient scale length for state variable x. Rearranging the expression by factoring out $m+1$ time step state vectors we find

$$\begin{aligned}
\frac{F_{\pm}^{m+1}}{F_{\pm}} &= \frac{n_{j\pm 1}^{m+1}}{n_{\pm}}\left(\frac{1}{2} \mp \frac{1}{\Delta\rho}\frac{n_j}{n_{\pm}}D_{n_{\pm}}\Gamma_{\pm}\right) + \frac{n_j^{m+1}}{n_{\pm}}\left(\frac{1}{2} \pm \frac{1}{\Delta\rho}\frac{n_{j\pm 1}}{n_{\pm}}D_{n_{\pm}}\Gamma_{\pm}\right) \\
&+ \frac{T_{ij\pm 1}^{m+1}}{T_{i\pm}}\left(\frac{3}{4} \mp \frac{1}{\Delta\rho}\frac{T_{ij}}{T_{i\pm}}D_{T_{i\pm}}\Gamma_{\pm}\right) + \frac{T_{ij}^{m+1}}{T_{i\pm}}\left(\frac{3}{4} \pm \frac{1}{\Delta\rho}\frac{T_{ij\pm 1}}{T_{i\pm}}D_{T_{i\pm}}\Gamma_{\pm}\right) \\
&\mp \frac{T_{ej\pm 1}^{m+1}}{T_{e\pm}}\frac{1}{\Delta\rho}\frac{T_{ej}}{T_{e\pm}}D_{T_{e\pm}}\Gamma_{\pm} \pm \frac{T_{ej}^{m+1}}{T_{e\pm}}\frac{1}{\Delta\rho}\frac{T_{ej\pm 1}}{T_{e\pm}}D_{T_{e\pm}}\Gamma_{\pm}
\end{aligned} \tag{29}$$

Recall that our original problem is solving

$$\left(\frac{\partial F}{\partial \rho}\right)^{m+1} = \frac{F_{+}^{m+1} - F_{-}^{m+1}}{\Delta\rho}$$

Now we find

$$
\begin{aligned}
F_+ - F_- = n_{j+1}^{m+1} & \left[\frac{1}{2}\frac{F_+}{n_+} - \frac{1}{\Delta\rho}\frac{n_j}{n_+^2}F_+ D_n \Gamma_+\right] \\
&+ n_j^{m+1}\left[\frac{1}{2}\left(\frac{F_+}{n_+} - \frac{F_-}{n_-}\right) + \frac{1}{\Delta\rho}\left(\frac{n_{j+1}}{n_+^2}F_+ D_n\Gamma_+ + \frac{n_{j-1}}{n_-^2}F_- D_n \Gamma_-\right)\right] \\
&+ n_{j-1}^{m+1}\left[-\frac{1}{2}\frac{F_-}{n_-} + \frac{1}{\Delta\rho}\frac{n_j}{n_-^2}F_- D_n\Gamma_-\right] \\
&+ {T_i}_{j+1}^{m+1}\left[\frac{3}{4}\frac{F_+}{T_{i+}} - \frac{1}{\Delta\rho}\frac{T_{ij}}{T_{i+}^2}F_+ D_{T_{i+}}\Gamma_+\right] \\
&+ {T_i}_j^{m+1}\left[\frac{3}{4}\left(\frac{F_+}{T_{i+}} - \frac{F_-}{T_{i-}}\right) + \frac{1}{\Delta\rho}\left(\frac{T_{ij+1}}{T_{i+}^2}F_+ D_{T_{i+}}\Gamma_+ + \frac{T_{ij-1}}{T_{i-}^2}F_- D_{T_{i-}}\Gamma_-\right)\right] \\
&+ {T_i}_{j-1}^{m+1}\left[-\frac{3}{4}\frac{F_-}{T_{i-}} + \frac{1}{\Delta\rho}\frac{T_{ij}}{T_{i-}^2}F_- D_{T_{i-}}\Gamma_-\right] \\
&+ {T_e}_{j+1}^{m+1}\left[-\frac{1}{\Delta\rho}\frac{T_{ej}}{T_{e+}^2}F_+ D_{T_{e+}}\Gamma_+\right] \\
&+ {T_e}_j^{m+1}\left[\frac{1}{\Delta\rho}\left(\frac{T_{ej+1}}{T_{e+}^2}F_+ D_{T_{e+}}\Gamma_+ + \frac{T_{ej-1}}{T_{e-}^2}F_- D_{T_{e-}}\Gamma_-\right)\right] \\
&+ {T_e}_{j-1}^{m+1}\left[\frac{1}{\Delta\rho}\frac{T_{ej}}{T_{e-}^2}F_- D_{T_{e-}}\Gamma_-\right]
\end{aligned} \tag{30}
$$

Let us define another shorthand

$$C_1[x;Y,Z] = -\frac{1}{\Delta\rho}\left(\frac{x_j}{x_+^2}Y_+ D_x Z_+\right) \tag{31}$$

$$C_0[x;Y,Z] = \frac{1}{\Delta\rho}\left(\frac{x_{j+1}}{x_+^2}Y_+ D_x Z_+ + \frac{x_{j-1}}{x_-^2}Y_- D_x Z_-\right) \tag{32}$$

$$C_{-1}[x;Y,Z] = \frac{1}{\Delta\rho}\left(\frac{x_j}{x_-^2}Y_- D_x Z_-\right) \tag{33}$$

where $x \in (n, T_i, T_e)$ while $Y = F$ and $Z = \Gamma$ (for heat transport this will be extended to I and Q). [4] Thus we can use 31-33 in 30, and substitute this into our original discretization Eq 15 to write

$$
\begin{aligned}
n_j^{m+1} - \alpha\Delta\tau \sum_{k=-1}^{1} (\psi_k^{nn})_j\, n_{j+k}^{m+1} + \left(\psi_k^{nT_i}\right)_j {T_i}_{j+k}^{m+1} + \left(\psi_k^{nT_e}\right)_j {T_e}_{j+k}^{m+1} \\
= n_j + (1-\alpha)\Delta\tau\left[-\frac{g}{A}\frac{\partial F}{\partial\rho}\right] + \Delta\tau S_n
\end{aligned} \tag{35}
$$

[4] Let us further observe that since $F \propto \Gamma$, we can rewrite

$$F D_x \Gamma = F\frac{\partial \ln\Gamma}{\partial(a/L_x)} = \frac{F}{\Gamma}\frac{\partial\Gamma}{\partial(a/L_x)} \tag{34}$$

where the expression (F/Γ) usually has further simplifications as a product of n and T to thermodynamic powers.

where

$$\psi_1^{nn} = -\frac{g}{A\Delta\rho}\left[C_1[n] + \frac{1}{2}\frac{F_+}{n_+}\right] \tag{36}$$

$$\psi_0^{nn} = -\frac{g}{A\Delta\rho}\left[C_0[n] + \frac{1}{2}\left(\frac{F_+}{n_+} - \frac{F_-}{n_-}\right)\right] \tag{37}$$

$$\psi_{-1}^{nn} = -\frac{g}{A\Delta\rho}\left[C_{-1}[n] - \frac{1}{2}\frac{F_-}{n_-}\right] \tag{38}$$

and

$$\psi_1^{nT_i} = -\frac{g}{A\Delta\rho}\left[C_1[T_i] + \frac{3}{4}\frac{F_+}{T_{i+}}\right] \tag{39}$$

$$\psi_0^{nT_i} = -\frac{g}{A\Delta\rho}\left[C_0[T_i] + \frac{3}{4}\left(\frac{F_+}{T_{i+}} - \frac{F_-}{T_{i-}}\right)\right] \tag{40}$$

$$\psi_{-1}^{nT_i} = -\frac{g}{A\Delta\rho}\left[C_{-1}[T_i] - \frac{3}{4}\frac{F_-}{T_{i-}}\right] \tag{41}$$

with

$$\psi_1^{nT_e} = -\frac{g}{A\Delta\rho}C_1[T_e] \tag{42}$$

$$\psi_0^{nT_e} = -\frac{g}{A\Delta\rho}C_0[T_e] \tag{43}$$

$$\psi_{-1}^{nT_e} = -\frac{g}{A\Delta\rho}C_{-1}[T_e] \tag{44}$$

Each ψ is a radial profile, and each triplet makes a tridiagonal matrix. Together this set of nine ψ^n functions make the top row of a (3×3) block of tridiagonal matrices. These determine the time evolution of density.

2.2 temperature evolution

For energy transport we should open up the derivative

$$\frac{3}{2}\frac{\partial p_s}{\partial \tau} = \frac{3}{2}\left(n_s\frac{\partial T_s}{\partial \tau} + T_s\frac{\partial n_s}{\partial \tau}\right) = -\frac{g}{A}\frac{\partial I_s}{\partial \rho} + G_s + J_s + K_s + E_s + \frac{3}{2}S_{p_s} \tag{45}$$

Now substituting Eq 5 for the density evolution term, we find

$$\frac{\partial T_s}{\partial \tau} = -\frac{g}{A}\frac{Z_s}{Z_i}\left(\frac{2}{3n_i}\frac{\partial I_s}{\partial \rho} - \frac{T_s}{n_i}\frac{\partial F}{\partial \rho}\right) + \frac{2}{3}\frac{Z_s}{Z_i}\left(\frac{G_s + J_s + K_s + E_s}{n_i}\right) + \frac{Z_s}{Z_i}\left(\frac{S_{p_s} - T_s S_n}{n_i}\right) \tag{46}$$

where we've used two-species quasineutrality $Z_s n_s - Z_i n_i = 0$. This motivates defining

$$S_{T_s} = \frac{Z_s}{Z_i}\left(\frac{S_{p_s} - T_s S_n}{n_i}\right) \tag{47}$$

and rescaling the heat fluxes [5] for $X \in (I, G, J, K, E)$

$$X_{T_s} = \frac{2}{3}\frac{Z_s}{Z_i}\frac{X_s}{n_i} \tag{48}$$

such that

$$G_{T_s} = \frac{g}{B^2}\frac{Z_s}{Z_i}T_i^{3/2}T_s\left(\frac{a}{L_{T_s}} - \frac{2}{3}\frac{a}{L_n}\right)\Gamma_s \tag{49}$$

[5] technicially each of these X_{T_s} are transport terms, with units of power, proportional to non-linear turbulent fluxes Γ, Q, H.

$$J_{T_s} = -\frac{g}{B^2}\frac{2}{3}\frac{Z_s}{Z_i}T_i^{5/2}\left(\frac{a}{L_{T_s}}\right)Q_s \tag{50}$$

$$K_{T_s} = -\frac{1}{B^2}\frac{2}{3}\frac{Z_s}{Z_i}T_i^{5/2}H_s \tag{51}$$

$$E_{T_s} = -\frac{2}{3}b_{su}n_s\frac{T_s - T_u}{\left(T_s/m_s + T_u/m_u\right)^{3/2}} \tag{52}$$

We would also like to rescale the particle fluxes

$$F_{T_s} = \frac{A}{B^2}\frac{Z_s}{Z_i}T_i^{3/2}T_s\Gamma_i \tag{53}$$

$$I_{T_s} = \frac{A}{B^2}\frac{2}{3}\frac{Z_s}{Z_i}T_i^{5/2}Q_s \tag{54}$$

so we find (through integration by parts) [6] that

$$\frac{\partial T_s}{\partial \tau} = -\frac{g}{A}\left[\left(\frac{\partial I_{T_s}}{\partial \rho} - \frac{\partial F_{T_s}}{\partial \rho}\right) + \left(I_{T_s} - F_{T_s}\right)\frac{a}{L_n}\right] + \left(G_{T_s} + J_{T_s} + K_{T_s} + E_{T_s}\right) + S_{T_s} \tag{55}$$

But the terms $(I_{T_s} - F_{T_s})\frac{a}{L_n}$ can be merged into G_{T_s} and J_{T_s} to define

$$G'_{T_s} = \frac{g}{B^2}\frac{Z_s}{Z_i}T_i^{3/2}T_s\left[\frac{a}{L_{T_s}} + \left(\delta_{si} - \frac{2}{3}\right)\frac{a}{L_n}\right]\Gamma_s \tag{56}$$

$$J'_{T_s} = -\frac{g}{B^2}\frac{2}{3}\frac{Z_s}{Z_i}T_i^{5/2}\left(\frac{a}{L_{T_s}} + \frac{a}{L_n}\right)Q_s \tag{57}$$

with the following shorthand defined for convenience

$$\kappa_{G,s} = \frac{a}{L_{T_s}} + \left(\delta_{si} - \frac{2}{3}\right)\frac{a}{L_n} = \kappa \tag{58}$$

$$\kappa_{J,s} = \frac{a}{L_{T_s}} + \frac{a}{L_n} = \lambda \tag{59}$$

Now the equation we need solve simplifies into

$$\frac{\partial T_s}{\partial \tau} = -\frac{g}{A}\left(\frac{\partial I_{T_s}}{\partial \rho} - \frac{\partial F_{T_s}}{\partial \rho}\right) + \left(G'_{T_s} + J'_{T_s} + K_{T_s} + E_{T_s}\right) + S_{T_s} \tag{60}$$

Following the algorithm for density evolution, we will discretize the time evolution with an implicit scheme

$$\begin{aligned}\frac{{T_s}^{m+1} - {T_s}^{m}}{\Delta\tau} = \alpha&\left[-\frac{g}{A}\left(\frac{\partial I_{T_s}}{\partial \rho} - \frac{\partial F_{T_s}}{\partial \rho}\right) + \left(G'_{T_s} + J'_{T_s} + K_{T_s} + E_{T_s}\right) + S_{T_s}\right]^{m+1}\\ &+ (1-\alpha)\left[-\frac{g}{A}\left(\frac{\partial I_{T_s}}{\partial \rho} - \frac{\partial F_{T_s}}{\partial \rho}\right) + \left(G'_{T_s} + J'_{T_s} + K_{T_s} + E_{T_s}\right) + S_{T_s}\right]^{m}\end{aligned} \tag{61}$$

and angain handling the flux evolution by making a Taylor expansion against the state variables

$$X_j^{m+1} = X_j + \sum_k \left(y_k^{m+1} - y_k\right)\frac{\partial X_j}{\partial y_k} \tag{62}$$

[6] By putting the $1/n$ between n, P inside the $\frac{\partial}{\partial \rho}$ derivative we find new terms of the form

$$\frac{1}{n}\frac{\partial Y}{\partial \rho} = \frac{\partial}{\partial \rho}\left(\frac{Y}{n}\right) - \frac{Y}{n^2}\frac{\partial n}{\partial \rho}$$

Going forward, we drop the superscript G', J' and the subscript T_s for convenience. Let us take the following partial derivatives

$$\frac{\partial G_j}{\partial n_k} = G_j \left[\frac{\partial}{\partial n_k} \ln \Gamma_j + \frac{1}{\kappa_j} \left(\delta_{si} - \frac{2}{3} \right) \frac{\partial}{\partial n_k} \left(\frac{a}{L_n} \right)_j \right] \tag{63}$$

$$\frac{\partial G_j}{\partial T_{ik}} = G_j \left[\frac{\partial}{\partial T_{ik}} \ln \Gamma_j + \frac{\delta_{si}}{\kappa_j} \frac{\partial}{\partial T_{ik}} \left(\frac{a}{L_{T_s}} \right)_j + \frac{\delta_j^k}{T_{ik}} \left(\frac{3}{2} + \delta_{si} \right) \right] \tag{64}$$

$$\frac{\partial G_j}{\partial T_{ek}} = G_j \left[\frac{\partial}{\partial T_{ek}} \ln \Gamma_j + \frac{\delta_{se}}{\kappa_j} \frac{\partial}{\partial T_{ek}} \left(\frac{a}{L_{T_s}} \right)_j + \frac{\delta_j^k}{T_{ek}} \delta_{se} \right] \tag{65}$$

and

$$\frac{\partial J_j}{\partial n_k} = J_j \left[\frac{\partial}{\partial n_k} \ln Q_j + \frac{1}{\lambda_j} \frac{\partial}{\partial n_k} \left(\frac{a}{L_n} \right)_j \right] \tag{66}$$

$$\frac{\partial J_j}{\partial T_{ik}} = J_j \left[\frac{\partial}{\partial T_{ik}} \ln Q_j + \frac{\delta_{si}}{\lambda_j} \frac{\partial}{\partial T_{ik}} \left(\frac{a}{L_{T_s}} \right)_j + \frac{5}{2} \frac{\delta_j^k}{T_{ij}} \right] \tag{67}$$

$$\frac{\partial J_j}{\partial T_{ek}} = J_j \left[\frac{\partial}{\partial T_{ek}} \ln Q_j + \frac{\delta_{se}}{\lambda_j} \frac{\partial}{\partial T_{ek}} \left(\frac{a}{L_{T_s}} \right)_j \right] \tag{68}$$

and

$$\frac{\partial K_j}{\partial n_k} = K_j \frac{\partial}{\partial n_k} \ln H_j \tag{69}$$

$$\frac{\partial K_j}{\partial T_{ik}} = K_j \left[\frac{\partial}{\partial T_{ik}} \ln H_j + \frac{5}{2} \frac{\delta_j^k}{T_i} \right] \tag{70}$$

$$\frac{\partial K_j}{\partial T_{ek}} = K_j \frac{\partial}{\partial T_{ek}} \ln H_j \tag{71}$$

and

$$\frac{\partial E_j}{\partial n_k} = E_j \frac{\delta_j^k}{n_j} \tag{72}$$

$$\frac{\partial E_j}{\partial T_{ik}} = E_j \left[\frac{\delta_{si} - \delta_{ui}}{T_s - T_u} - \frac{3}{2} \frac{\delta_{si}/m_s + \delta_{ui}/m_u}{T_s/m_s + T_u/m_u} \right] \delta_j^k \tag{73}$$

$$\frac{\partial E_j}{\partial T_{ek}} = E_j \left[\frac{\delta_{se} - \delta_{ue}}{T_s - T_u} - \frac{3}{2} \frac{\delta_{se}/m_s + \delta_{ue}/m_u}{T_s/m_s + T_u/m_u} \right] \delta_j^k \tag{74}$$

We may clarify the last set of identities using binary logic identities $\delta_{ui} = \delta_{se}$ and $\delta_{ue} = \delta_{si}$. Next we again make the assumption that the fluxes Y depend only on gradient scale lengths

$$\frac{\partial}{\partial x_k} \ln Y_j = \frac{\partial (a/L_x)_j}{\partial x_k} \frac{\partial \ln Y_j}{\partial (a/L_x)_j} \tag{75}$$

Notice that in the previous section we took derivatives on $F_\pm$ (half step), while here we evaluate derivatives on Y_j (full step). As a result the expansion takes a different form (compared to Eq 22)

$$\left(\frac{a}{L_n} \right)_j = -\left(\frac{\partial}{\partial \rho} \ln n \right)_j = -\frac{1}{n_j} \left(\frac{n_{j+1} - n_{j-1}}{2\Delta\rho} \right) \tag{76}$$

so

$$\begin{aligned}\frac{\partial}{\partial n_k}\left(\frac{a}{L_n}\right)_j &= -\frac{1}{2\Delta\rho}\left[\frac{\delta^k_{j+1}-\delta^k_{j-1}}{n_j}-\left(\frac{n_{j+1}-n_{j-1}}{n_j}\right)\frac{\delta^k_j}{n_j}\right] \\ &= -\frac{1}{2\Delta\rho}\left[\frac{\delta^k_{j+1}n_j-\delta^k_{j-1}n_j-\delta^k_j n_{j+1}+\delta^k_j n_{j-1}}{n_j^2}\right]\end{aligned} \tag{77}$$

then

$$\begin{aligned}&\sum_k\left(n_k^{m+1}-n_k\right)\frac{\partial(a/L_n)_j}{\partial n_k} \\ &\qquad= -\frac{1}{2\Delta\rho}\sum_k\left(n_k^{m+1}-n_k\right)\left[\frac{\delta^k_{j+1}n_j-\delta^k_{j-1}n_j-\delta^k_j n_{j+1}+\delta^k_j n_{j-1}}{n_j^2}\right] \\ &\qquad= -\frac{1}{2\Delta\rho}\left[\frac{n^{m+1}_{j+1}n_j-n^{m+1}_{j-1}n_j-n^{m+1}_j n_{j+1}+n^{m+1}_j n_{j-1}}{n_j^2}\right] \\ &\qquad= -\frac{1}{2\Delta\rho}\left[\frac{n^{m+1}_{j+1}}{n_j}-\frac{n^{m+1}_{j-1}}{n_j}-\frac{n^{m+1}_j}{n_j}\left(\frac{n_{j+1}}{n_j}-\frac{n_{j-1}}{n_j}\right)\right]\end{aligned} \tag{78}$$

As for the $\pm$ case above, there are identical expressions for T_i and T_e. Let us again adopt the shorthand $D_x Z$ for writing the derivative of the log flux with respect to a gradient scale length. Now we us Eq 63-65 to write

$$\begin{aligned}G_j^{m+1} &= G_j+\sum_k\left(n_k^{m+1}-n_k\right)\frac{\partial G_j}{\partial n_k}+\left(T_{ik}^{m+1}-T_{ik}\right)\frac{\partial G_j}{\partial T_{ik}}+\left(T_{ek}^{m+1}-T_{ek}\right)\frac{\partial G_j}{\partial T_{ek}} \\ &= G_j+G_j\sum_k\left(n_k^{m+1}-n_k\right)\left[D_n\Gamma_j+\frac{1}{\kappa_j}\left(\delta_{si}-\frac{2}{3}\right)\right]\frac{\partial}{\partial n_k}\left(\frac{a}{L_n}\right)_j \\ &\qquad+\left(T_{ik}^{m+1}-T_{ik}\right)\left[\left(D_{T_i}\Gamma_j+\frac{\delta_{si}}{\kappa_j}\right)\frac{\partial}{\partial T_{ik}}\left(\frac{a}{L_{T_i}}\right)_j+\frac{\delta^k_j}{T_{ik}}\left(\frac{3}{2}+\delta_{si}\right)\right] \\ &\qquad+\left(T_{ek}^{m+1}-T_{ek}\right)\left[\left(D_{T_e}\Gamma_j+\frac{\delta_{se}}{\kappa_j}\right)\frac{\partial}{\partial T_{ek}}\left(\frac{a}{L_{T_e}}\right)_j+\frac{\delta^k_j}{T_{ek}}\delta_{se}\right]\end{aligned} \tag{79}$$

We eliminate the summation of k and divide through by G_j to find

$$\begin{aligned}\frac{G_j^{m+1}}{G_j} = 1 &+ \left\{-\frac{1}{2\Delta\rho}\left[\frac{n^{m+1}_{j+1}}{n_j}-\frac{n^{m+1}_{j-1}}{n_j}-\frac{n^{m+1}_j}{n_j}\left(\frac{n_{j+1}}{n_j}-\frac{n_{j-1}}{n_j}\right)\right]\left[D_n\Gamma_j+\frac{1}{\kappa_j}\left(\delta_{si}-\frac{2}{3}\right)\right]\right\} \\ &+\left\{-\frac{1}{2\Delta\rho}\left[\frac{T_{ij+1}^{m+1}}{T_{ij}}-\frac{T_{ij-1}^{m+1}}{T_{ij}}-\frac{T_{ij}^{m+1}}{T_{ij}}\left(\frac{T_{ij+1}}{T_{ij}}-\frac{T_{ij-1}}{T_{ij}}\right)\right]\left(D_{T_i}\Gamma_j+\frac{\delta_{si}}{\kappa_j}\right)+\left(\frac{T_{ij}^{m+1}}{T_{ij}}-1\right)\left(\frac{3}{2}+\delta_{si}\right)\right\} \\ &+\left\{-\frac{1}{2\Delta\rho}\left[\frac{T_{ej+1}^{m+1}}{T_{ej}}-\frac{T_{ej-1}^{m+1}}{T_{ej}}-\frac{T_{ej}^{m+1}}{T_{ej}}\left(\frac{T_{ej+1}}{T_{ej}}-\frac{T_{ej-1}}{T_{ej}}\right)\right]\left(D_{T_e}\Gamma_j+\frac{\delta_{se}}{\kappa_j}\right)+\left(\frac{T_{ej}^{m+1}}{T_{ej}}-1\right)\delta_{se}\right\}\end{aligned} \tag{80}$$

Here the binary logic in δ_{si} or δ_{se} partially cancels with the 1 out front to obtain

$$\begin{aligned}\frac{G_j^{m+1}}{G_j} = &-\frac{3}{2} - \frac{1}{2\Delta\rho}\left\{\left[\frac{n_{j+1}^{m+1}}{n_j} - \frac{n_{j-1}^{m+1}}{n_j} - \frac{n_j^{m+1}}{n_j}\left(\frac{n_{j+1}}{n_j} - \frac{n_{j-1}}{n_j}\right)\right]\left[D_n\Gamma_j + \frac{1}{\kappa_j}\left(\delta_{si} - \frac{2}{3}\right)\right]\right.\\ &+ \left[\frac{T_{ij+1}^{m+1}}{T_{ij}} - \frac{T_{ij-1}^{m+1}}{T_{ij}} - \frac{T_{ij}^{m+1}}{T_{ij}}\left(\frac{T_{ij+1}}{T_{ij}} - \frac{T_{ij-1}}{T_{ij}}\right)\right]\left(D_{T_i}\Gamma_j + \frac{\delta_{si}}{\kappa_j}\right)\\ &+ \left.\left[\frac{T_{ej+1}^{m+1}}{T_{ej}} - \frac{T_{ej-1}^{m+1}}{T_{ej}} - \frac{T_{ej}^{m+1}}{T_{ej}}\left(\frac{T_{ej+1}}{T_{ej}} - \frac{T_{ej-1}}{T_{ej}}\right)\right]\left(D_{T_e}\Gamma_j + \frac{\delta_{se}}{\kappa_j}\right)\right\}\\ &+ \frac{T_{ij}^{m+1}}{T_{ij}}\left(\frac{3}{2} + \delta_{si}\right) + \frac{T_{ej}^{m+1}}{T_{ej}}\delta_{se}\end{aligned} \tag{81}$$

Following a similar path we can write

$$\begin{aligned}\frac{J_j^{m+1}}{J_j} = 1 + \sum_k &\left(n_k^{m+1} - n_k\right)\left[\left(D_nQ_j + \frac{1}{\lambda_s}\right)\frac{\partial}{\partial n_k}\left(\frac{a}{L_n}\right)_j\right]\\ &+ \left(T_{ik}^{m+1} - T_{ik}\right)\left[\left(D_{T_i}Q_j + \frac{\delta_{si}}{\lambda_s}\right)\frac{\partial}{\partial T_{ik}}\left(\frac{a}{L_{T_i}}\right)_j + \frac{5}{2}\frac{\delta_j^k}{T_{ij}}\right]\\ &+ \left(T_{ek}^{m+1} - T_{ek}\right)\left[\left(D_{T_e}Q_j + \frac{\delta_{se}}{\lambda_s}\right)\frac{\partial}{\partial T_{ek}}\left(\frac{a}{L_{T_e}}\right)_j\right]\end{aligned} \tag{82}$$

such that

$$\begin{aligned}\frac{J_j^{m+1}}{J_j} = &-\frac{3}{2} - \frac{1}{2\Delta\rho}\left\{\left[\frac{n_{j+1}^{m+1}}{n_j} - \frac{n_{j-1}^{m+1}}{n_j} - \frac{n_j^{m+1}}{n_j}\left(\frac{n_{j+1}}{n_j} - \frac{n_{j-1}}{n_j}\right)\right]\left(D_nQ_j + \frac{1}{\lambda_s}\right)\right.\\ &+ \left[\frac{T_{ij+1}^{m+1}}{T_{ij}} - \frac{T_{ij-1}^{m+1}}{T_{ij}} - \frac{T_{ij}^{m+1}}{T_{ij}}\left(\frac{T_{ij+1}}{T_{ij}} - \frac{T_{ij-1}}{T_{ij}}\right)\right]\left(D_{T_i}Q_j + \frac{\delta_{si}}{\lambda_s}\right)\\ &+ \left.\left[\frac{T_{ej+1}^{m+1}}{T_{ej}} - \frac{T_{ej-1}^{m+1}}{T_{ej}} - \frac{T_{ej}^{m+1}}{T_{ej}}\left(\frac{T_{ej+1}}{T_{ej}} - \frac{T_{ej-1}}{T_{ej}}\right)\right]\left(D_{T_e}Q_j + \frac{\delta_{se}}{\lambda_s}\right)\right\} + \frac{5}{2}\frac{T_{ij}^{m+1}}{T_{ij}}\end{aligned} \tag{83}$$

For the electromagnetic term

$$\begin{aligned}\frac{K_j^{m+1}}{K_j} = 1 + \sum_k &\left(n_k^{m+1} - n_k\right)\left[D_nH_j\frac{\partial}{\partial n_k}\left(\frac{a}{L_n}\right)_j\right]\\ &+ \left(T_{ik}^{m+1} - T_{ik}\right)\left[D_{T_i}H_j\frac{\partial}{\partial T_{ik}}\left(\frac{a}{L_{T_i}}\right)_j + \frac{5}{2}\frac{\delta_j^k}{T_{ij}}\right]\\ &+ \left(T_{ek}^{m+1} - T_{ek}\right)\left[D_{T_e}H_j\frac{\partial}{\partial T_{ek}}\left(\frac{a}{L_{T_e}}\right)_j\right]\end{aligned} \tag{84}$$

which makes

$$\begin{aligned}\frac{K_j^{m+1}}{K_j} = &-\frac{3}{2} - \frac{1}{2\Delta\rho}\left\{\left[\frac{n_{j+1}^{m+1}}{n_j} - \frac{n_{j-1}^{m+1}}{n_j} - \frac{n_j^{m+1}}{n_j}\left(\frac{n_{j+1}}{n_j} - \frac{n_{j-1}}{n_j}\right)\right]D_nH_j\right.\\ &+ \left[\frac{T_{ij+1}^{m+1}}{T_{ij}} - \frac{T_{ij-1}^{m+1}}{T_{ij}} - \frac{T_{ij}^{m+1}}{T_{ij}}\left(\frac{T_{ij+1}}{T_{ij}} - \frac{T_{ij-1}}{T_{ij}}\right)\right]D_{T_i}H_j\\ &+ \left.\left[\frac{T_{ej+1}^{m+1}}{T_{ej}} - \frac{T_{ej-1}^{m+1}}{T_{ej}} - \frac{T_{ej}^{m+1}}{T_{ej}}\left(\frac{T_{ej+1}}{T_{ej}} - \frac{T_{ej-1}}{T_{ej}}\right)\right]D_{T_e}H_j\right\} + \frac{5}{2}\frac{T_{ij}^{m+1}}{T_{ij}}\end{aligned} \tag{85}$$

Next the collisionality term gives

$$\frac{E_j^{m+1}}{E_j} = 1 + \sum_k \left(n_k^{m+1} - n_k\right) \frac{\delta_j^k}{n_j} + \left({T_i}_k^{m+1} - T_{ik}\right) \left[\frac{\delta_{si} - \delta_{se}}{T_s - T_u} - \frac{3}{2}\frac{\delta_{si}/m_s + \delta_{se}/m_u}{T_s/m_s + T_u/m_u}\right]\delta_j^k + \left({T_e}_k^{m+1} - T_{ek}\right) \left[\frac{\delta_{se} - \delta_{si}}{T_s - T_u} - \frac{3}{2}\frac{\delta_{se}/m_s + \delta_{si}/m_u}{T_s/m_s + T_u/m_u}\right]\delta_j^k \tag{86}$$

for

$$\begin{aligned}\frac{E_j^{m+1}}{E_j} = \frac{n_j^{m+1}}{n_j} &+ {T_i}_j^{m+1}\left[\frac{\delta_{si} - \delta_{se}}{T_s - T_u} - \frac{3}{2}\frac{\delta_{si}/m_s + \delta_{se}/m_u}{T_s/m_s + T_u/m_u}\right] \\ &+ {T_e}_j^{m+1}\left[\frac{\delta_{se} - \delta_{si}}{T_s - T_u} - \frac{3}{2}\frac{\delta_{se}/m_s + \delta_{si}/m_u}{T_s/m_s + T_u/m_u}\right] \\ &- \delta_{si}\left(\frac{T_i - T_e}{T_s - T_u} - \frac{3}{2}\frac{T_i/m_s + T_e/m_u}{T_s/m_s + T_u/m_u}\right) \\ &- \delta_{se}\left(\frac{T_e - T_i}{T_s - T_u} - \frac{3}{2}\frac{T_e/m_s + T_i/m_u}{T_s/m_s + T_u/m_u}\right)\end{aligned} \tag{87}$$

Note that binary logic reduces the last two terms to exactly 1 such that

$$\begin{aligned}\frac{E_j^{m+1}}{E_j} = \frac{n_j^{m+1}}{n_j} &+ \frac{{T_i}_j^{m+1}}{T_{ij}}\left(\frac{\delta_{si} - \delta_{se}}{T_s - T_u} - \frac{3}{2}\frac{\delta_{si}/m_s + \delta_{se}/m_u}{T_s/m_s + T_u/m_u}\right) T_{ij} \\ &+ \frac{{T_e}_j^{m+1}}{T_{ej}}\left(\frac{\delta_{se} - \delta_{si}}{T_s - T_u} - \frac{3}{2}\frac{\delta_{se}/m_s + \delta_{si}/m_u}{T_s/m_s + T_u/m_u}\right) T_{ej} + 1\end{aligned} \tag{88}$$

Putting together Equations (81,83,85,88) we see

$$\begin{aligned}G_j^{m+1} + J_j^{m+1} + K_j^{m+1} + E_j^{m+1} &= E_j - \frac{3}{2}(G_j + J_j + K_j) \\ &+ \frac{n_j^{m+1}}{n_j}\left[E_j - \left(\frac{n_{j+1} - n_{j-1}}{n_j}\right)\mu_{nj}\right] + \frac{n_{j+1}^{m+1}}{n_j}\mu_{nj} - \frac{n_{j-1}^{m+1}}{n_j}\mu_{nj} \\ &+ \frac{{T_i}_j^{m+1}}{T_{ij}}\left[T_{ij}\Delta_i E_j - \left(\frac{T_{ij+1} - T_{ij-1}}{T_{ij}}\right)\mu_{T_i j} + \left(\frac{13}{2} + \delta_{si}\right)\right] + \frac{{T_i}_{j+1}^{m+1}}{T_{ij}}\mu_{T_i j} - \frac{{T_i}_{j-1}^{m+1}}{T_{ij}}\mu_{T_i j} \\ &+ \frac{{T_e}_j^{m+1}}{T_{ej}}\left[T_{ej}\Delta_e E_j - \left(\frac{T_{ej+1} - T_{ej-1}}{T_{ej}}\right)\mu_{T_e j} + \delta_{se}\right] + \frac{{T_e}_{j+1}^{m+1}}{T_{ej}}\mu_{T_e j} - \frac{{T_e}_{j-1}^{m+1}}{T_{ej}}\mu_{T_e j}\end{aligned} \tag{89}$$

where we define the following shorthand for the collections of flux gradients

$$\mu_{nj} = -\frac{1}{2\Delta\rho}\left\{G_j\left[D_n\Gamma_j + \left(\delta_{si} - \frac{2}{3}\right)\frac{1}{\kappa_j}\right] + J_j\left(D_n Q_j + \frac{1}{\lambda_s}\right) + K_j\left(D_n H_j\right)\right\} \tag{90}$$

$$\mu_{T_i j} = -\frac{1}{2\Delta\rho}\left\{G_j\left(D_{T_i}\Gamma_j + \frac{\delta_{si}}{\kappa_j}\right) + J_j\left(D_{T_i}Q_j + \frac{\delta_{si}}{\lambda_s}\right) + K_j\left(D_{T_i}H_j\right)\right\} \tag{91}$$

$$\mu_{T_e j} = -\frac{1}{2\Delta\rho}\left\{G_j\left(D_{T_e}\Gamma_j + \frac{\delta_{se}}{\kappa_j}\right) + J_j\left(D_{T_e}Q_j + \frac{\delta_{se}}{\lambda_s}\right) + K_j\left(D_{T_e}H_j\right)\right\} \tag{92}$$

and for the collisionality terms

$$\Delta_i = \frac{\delta_{si} - \delta_{se}}{T_s - T_u} - \frac{3}{2}\frac{\delta_{si}/m_s + \delta_{se}/m_u}{T_s/m_s + T_u/m_u} \tag{93}$$

$$\Delta_e = \frac{\delta_{se} - \delta_{si}}{T_s - T_u} - \frac{3}{2}\frac{\delta_{se}/m_s + \delta_{si}/m_u}{T_s/m_s + T_u/m_u} \tag{94}$$

Next, let us consider the particle fluxes. Starting from the gradients

$$\frac{\partial I_\pm}{\partial n_k} = I_\pm \left[D_n Q_\pm \frac{\partial}{\partial n_k} \left(\frac{a}{L_n}\right)_\pm \right] \tag{95}$$

$$\frac{\partial I_\pm}{\partial T_{ik}} = I_\pm \left[D_{T_i} Q_\pm \frac{\partial}{\partial T_{ik}} \left(\frac{a}{L_{T_i}}\right)_\pm + \frac{5}{4}\left(\frac{\delta^k_{j\pm1} + \delta^k_j}{T_{i\pm}}\right) \right] \tag{96}$$

$$\frac{\partial I_\pm}{\partial T_{ek}} = I_\pm \left[D_{T_e} Q_\pm \frac{\partial}{\partial T_{ek}} \left(\frac{a}{L_{T_e}}\right)_\pm \right] \tag{97}$$

We plug these into the state-vector Taylor expansion to write

$$\begin{aligned} I^{m+1}_\pm = I_\pm + \sum_k \left(n^{m+1}_k - n_k\right) &\left[D_n Q_\pm \frac{\partial}{\partial n_k} \left(\frac{a}{L_n}\right)_\pm \right] \\ + \left(T^{m+1}_{ik} - T_{ik}\right) &\left[D_{T_i} Q_\pm \frac{\partial}{\partial T_{ik}} \left(\frac{a}{L_{T_i}}\right)_\pm + \frac{5}{4}\left(\frac{\delta^k_{j\pm1} + \delta^k_j}{T_{i\pm}}\right) \right] \\ + \left(T^{m+1}_{ek} - T_{ek}\right) &\left[D_{T_e} Q_\pm \frac{\partial}{\partial T_{ek}} \left(\frac{a}{L_{T_e}}\right)_\pm \right] \end{aligned} \tag{98}$$

Let us recall the identity from Eq (24) to show

$$\begin{aligned} \frac{I^{m+1}_\pm}{I_\pm} = 1 &+ \left(\frac{n^{m+1}_j}{n_\pm}\frac{n_{j+1}}{n_\pm} - \frac{n^{m+1}_{j+1}}{n_\pm}\frac{n_j}{n_\pm}\right)\left(\pm\frac{1}{\Delta\rho} D_n Q_\pm\right) \\ &+ \left(\frac{T^{m+1}_{ij}}{T_{i\pm}}\frac{T_{ij+1}}{T_{i\pm}} - \frac{T^{m+1}_{ij+1}}{T_{i\pm}}\frac{T_{ij}}{T_{i\pm}}\right)\left(\pm\frac{1}{\Delta\rho} D_{T_i} Q_\pm\right) + \frac{5}{4}\left(\frac{T^{m+1}_{ij} + T^{m+1}_{ij\pm1}}{T_{i\pm}}\right) - \frac{5}{2} \\ &+ \left(\frac{T^{m+1}_{ej}}{T_{e\pm}}\frac{T_{ej+1}}{T_{e\pm}} - \frac{T^{m+1}_{ej+1}}{T_{e\pm}}\frac{T_{ej}}{T_{e\pm}}\right)\left(\pm\frac{1}{\Delta\rho} D_{T_e} Q_\pm\right) \end{aligned} \tag{99}$$

Using this we may evaluate

$$\left(\frac{\partial I^{m+1}_{T_s}}{\partial \rho}\right)_j = \frac{I^{m+1}_+ - I^{m+1}_-}{\Delta\rho} \tag{100}$$

by computing the sum

$$\begin{aligned} I^{m+1}_+ - I^{m+1}_- = n^{m+1}_j C_0[n] &+ n^{m+1}_{j+1} C_+[n] + n^{m+1}_{j-1} C_-[n] \\ &+ T^{m+1}_{ij}\left[C_0[T_i] + \frac{5}{4}\left(\frac{I_+}{T_{i+}} - \frac{I_-}{T_{i-}}\right)\right] + T^{m+1}_{ij+1} C_+[T_i] + T^{m+1}_{ij-1} C_-[T_i] \\ &+ T^{m+1}_{ej} C_0[T_e] + T^{m+1}_{ej+1} C_+[T_e] + T^{m+1}_{ej-1} C_-[T_e] \\ &- \frac{3}{2}\left(I_+ - I_-\right) \end{aligned} \tag{101}$$

where $C_\sigma[x] = C_\sigma[x; I, Q]$ (reusing the definintions in 31-33). Then

$$\begin{aligned}\left(\frac{\partial I_{T_s}}{\partial \rho}\right)_j^{m+1} = \frac{1}{\Delta\rho}&\left\{n_j^{m+1} C_0[n] + n_{j+1}^{m+1} C_+[n] + n_{j-1}^{m+1} C_-[n]\right.\\ &+ T_{ij}^{m+1}\left[C_0[T_i] + \frac{5}{4}\left(\frac{I_+}{T_{i+}} - \frac{I_-}{T_{i-}}\right)\right] + T_{ij+1}^{m+1} C_+[T_i] + T_{ij-1}^{m+1} C_-[T_i]\\ &\left.+ T_{ej}^{m+1} C_0[T_e] + T_{ej+1}^{m+1} C_+[T_e] + T_{ej-1}^{m+1} C_-[T_e]\right\}\\ &- \frac{3}{2}\left(\frac{\partial I_{T_s}}{\partial \rho}\right)_j\end{aligned} \tag{102}$$

A similar argument shows

$$\begin{aligned}\left(\frac{\partial F_{T_s}}{\partial \rho}\right)_j^{m+1} = \frac{1}{\Delta\rho}&\left\{n_j^{m+1} C_0[n] + n_{j+1}^{m+1} C_+[n] + n_{j-1}^{m+1} C_-[n]\right.\\ &+ T_{ij}^{m+1}\left[C_0[T_i] + \frac{1}{2}\left(\frac{3}{2} + \delta_{si}\right)\left(\frac{F_+}{T_{i+}} - \frac{F_-}{T_{i-}}\right)\right] + T_{ij+1}^{m+1} C_+[T_i] + T_{ij-1}^{m+1} C_-[T_i]\\ &\left.+ T_{ej}^{m+1}\left[C_0[T_e] + \frac{\delta_{se}}{2}\left(\frac{F_+}{T_{e+}} - \frac{F_-}{T_{e-}}\right)\right] + T_{ej+1}^{m+1} C_+[T_e] + T_{ej-1}^{m+1} C_-[T_e]\right\}\\ &- \left(\frac{1}{2} + \delta_{si}\right)\left(\frac{\partial F_{T_s}}{\partial \rho}\right)_j\end{aligned} \tag{103}$$

where this time $C_\sigma[x] = C_\sigma[x; F, \Gamma]$. To write the difference let us define

$$\Xi_\sigma[y] = C_\sigma[y, I] - C_\sigma[y, F] \tag{104}$$

then

$$\begin{aligned}\left(\frac{\partial I_{T_s}}{\partial \rho} - \frac{\partial F_{T_s}}{\partial \rho}\right)_j^{m+1} = \frac{1}{\Delta\rho}&\left\{n_j^{m+1} \Xi_0[n] + n_{j+1}^{m+1} \Xi_+[n] + n_{j-1}^{m+1} \Xi_-[n]\right.\\ &+ T_{ij}^{m+1}\left[\Xi_0[T_i] + \frac{5}{4}\left(\frac{I_+}{T_{i+}} - \frac{I_-}{T_{i-}}\right) - \frac{1}{2}\left(\frac{3}{2} + \delta_{si}\right)\left(\frac{F_+}{T_{i+}} - \frac{F_-}{T_{i-}}\right)\right]\\ &\qquad + T_{ij+1}^{m+1} \Xi_+[T_i] + T_{ij-1}^{m+1} \Xi_-[T_i]\\ &\left.+ T_{ej}^{m+1}\left[\Xi_0[T_e] - \frac{\delta_{se}}{2}\left(\frac{F_+}{T_{e+}} - \frac{F_-}{T_{e-}}\right)\right] + T_{ej+1}^{m+1} \Xi_+[T_e] + T_{ej-1}^{m+1} \Xi_-[T_e]\right\}\\ &- \left[\frac{3}{2}\left(\frac{\partial I}{\partial \rho}\right)_j - \left(\frac{1}{2} + \delta_{si}\right)\left(\frac{\partial F}{\partial \rho}\right)_j\right]\end{aligned} \tag{105}$$

At last we can substitute (89) and (105) into (61) to write

$$\begin{aligned}&T_{sj}^{m+1} - \alpha\Delta\tau\sum_{k=-1}^{k}\left[\left(\psi_k^{T_s n}\right)_j n_{j+k}^{m+1} + \left(\psi_k^{T_s T_i}\right)_j T_{ij+k}^{m+1} + \left(\psi_k^{T_s T_e}\right)_j T_{ej+k}^{m+1}\right]\\ &\quad = T_j + (1-\alpha)\Delta\tau\left[-\frac{g}{A}\frac{\partial}{\partial\rho}\left(I_{T_s} - F_{T_s}\right) + G + J + K + E\right] + \Delta\tau S_{T_s}\\ &\qquad + \alpha\Delta\tau\left\{E - \frac{3}{2}\left(G + J + K\right) + \frac{g}{A}\left[\frac{3}{2}\frac{\partial I}{\partial\rho} - \left(\frac{1}{2} + \delta_{si}\right)\frac{\partial F}{\partial\rho}\right]\right\}\end{aligned} \tag{106}$$

which simplifies into

$$
\begin{aligned}
{T_s}_j^{m+1} - \alpha\Delta\tau \sum_{k=-1}^{k} \left[\left(\psi_k^{T_s n}\right)_j n_{j+k}^{m+1} + \left(\psi_k^{T_s T_i}\right)_j {T_i}_{j+k}^{m+1} + \left(\psi_k^{T_s T_e}\right)_j {T_e}_{j+k}^{m+1} \right] \\
= T_j + \frac{5}{2}(1-\alpha)\Delta\tau \left[-\frac{g}{A}\frac{\partial}{\partial\rho}\left(I_{T_s} - \frac{3+2\delta_{si}}{5} F_{T_s} \right) + G + J + K \right] + \Delta\tau \left(E + S_{T_s}\right)
\end{aligned}
\tag{107}
$$

with

$$
\psi_0^{T_s n} = -\frac{g}{A\Delta\rho}\Xi_0[n] - \left(\frac{n_{j+1} - n_{j-1}}{n_j}\right)\frac{\mu_{nj}}{n_j} + \frac{E_j}{n_j} \tag{108}
$$

$$
\psi_+^{T_s n} = -\frac{g}{A\Delta\rho}\Xi_+[n] + \frac{\mu_{nj}}{n_j} \tag{109}
$$

$$
\psi_-^{T_s n} = -\frac{g}{A\Delta\rho}\Xi_-[n] - \frac{\mu_{nj}}{n_j} \tag{110}
$$

and

$$
\begin{aligned}
\psi_0^{T_s T_i} = -\frac{g}{A\Delta\rho} \left[\Xi_0[T_i] + \frac{5}{4}\left(\frac{I_+}{T_{i+}} - \frac{I_-}{T_{i-}}\right) - \frac{1}{2}\left(\frac{3}{2} + \delta_{si}\right)\left(\frac{F_+}{T_{i+}} - \frac{F_-}{T_{i-}}\right) \right] \\
- \left(\frac{{T_i}_{j+1} - {T_i}_{j-1}}{T_{ij}}\right)\frac{\mu_{T_i j}}{T_{ij}} + \left(\frac{13}{2} + \delta_{si}\right)\frac{1}{T_{ij}} + \Delta_i E_j
\end{aligned}
\tag{111}
$$

$$
\psi_+^{T_s T_i} = -\frac{g}{A\Delta\rho}\Xi_+[T_i] + \frac{\mu_{T_i j}}{T_{ij}} \tag{112}
$$

$$
\psi_-^{T_s T_i} = -\frac{g}{A\Delta\rho}\Xi_-[T_i] - \frac{\mu_{T_i j}}{T_{ij}} \tag{113}
$$

and

$$
\begin{aligned}
\psi_0^{T_s T_e} = -\frac{g}{A\Delta\rho} \left[\Xi_0[T_e] - \frac{\delta_{se}}{2}\left(\frac{F_+}{T_{e+}} - \frac{F_-}{T_{e-}}\right) \right] \\
- \left(\frac{{T_e}_{j+1} - {T_e}_{j-1}}{T_{ej}}\right)\frac{\mu_{T_e j}}{T_{ej}} + \frac{\delta_{se}}{T_{ej}} + \Delta_e E_j
\end{aligned}
\tag{114}
$$

$$
\psi_+^{T_s T_e} = -\frac{g}{A\Delta\rho}\Xi_+[T_e] + \frac{\mu_{T_e j}}{T_{ej}} \tag{115}
$$

$$
\psi_-^{T_s T_e} = -\frac{g}{A\Delta\rho}\Xi_-[T_e] - \frac{\mu_{T_e j}}{T_{ej}} \tag{116}
$$

Together with the density equations (36-44) these define a (3×3) block matrix of tri-diagonal matrices, where each of the 27 diagonals corresponds to some function of the radial profile. These can also be arranged as a block diagonal matrix with (3×3) blocks on each diagonal block element.

3 Boundary Conditions