trajectory_utils: Mathematical background

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1. Cartpoles

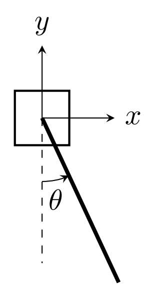

Figure 1: Cart–pole system. The cart is actuated by some means and is constrained to move along $\pm x$. Objectives are to swing the pole up to vertical - and/or stabilize it there. The physical cart is actuated by a belt drive, and the pole is a steel rod.

1.1 Equations of motion

By inspection, figure 1 offers:

$$x_r(t) = rsin\theta(t), r \in [0, L] \quad (1)$$
$$y_r(t) = -rcos\theta(t), r \in [0, L] \quad (2)$$
$$\dot{x_r}(t) = rcos\theta(t)\dot{\theta}(t) + \dot{x}(t), r \in [0, L] \quad (3)$$
$$\dot{y_r}(t) = rsin\theta(t)\dot{\theta}(t), r \in [0, L] \quad (4)$$

where r parameterizes the location of infinitesimal mass along the pole. It is straightforward to compute the Lagrangian $\mathcal{L} = T - V$ - the first step in deriving the equations of motion.

The kinetic energy T can be written as a sum of the cart kinetic energy and an integral over the pole:

$$T = \int_{r=0}^{r=L} (\dot{x}_r^2 + \dot{y}_r^2)\rho dr + \frac{1}{2} m_c \dot{x}^2 \tag{5}$$

where ρ is the pole mass per unit length. A bit of algebra yields:

$$T = m_p \frac{L^2}{6} \dot{\theta}^2 + \frac{1}{2} m_p L \cos\theta \dot{\theta} \dot{x} + \frac{1}{2} (m_p + m_c) \dot{x}^2 \tag{6}$$

where we have used the fact that $\rho L = m_p$.

The (gravitational) potential energy of the cart does not change, so we are left with an integral over the pole:

$$V = g \int_{r=0}^{r=L} y_r \rho dr \tag{7}$$

or, equivalently, the potential energy of a mass concentrated at the pole's center of mass:

$$V = -\frac{g m_p L}{2} \cos\theta \tag{8}$$

Euler-Lagrange now yields two equations:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = F_x \tag{9}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0 \tag{10}$$

The first (force) equation yields:

$$\frac{1}{2} m_p L \left(\cos\theta \ddot{\theta} - \sin\theta \dot{\theta}^2\right) + (m_p + m_c) \ddot{x} = F_x \tag{11}$$

and the second (torque) equation yields:

$$\frac{2}{3} L \ddot{\theta} + \cos\theta \ddot{x} + g \sin\theta = 0 \tag{12}$$

or, conveniently:

$$\ddot{\theta} = -\frac{3}{2L} (\cos\theta \ddot{x} + g \sin\theta) \tag{13}$$

This equation alone can be used to establish an ODE for the cartpole system with the "velocity" servo actuation model:

$$\dot{v} = \frac{v_c - v}{\tau} \tag{14}$$

or:

$$\ddot{x} = \frac{v_c - \dot{x}}{\tau} \tag{15}$$

where v_c is the velocity control. The state ODE for this system is below.

For cart force actuation, we can decouple the above (coupled) equations in $\ddot{\theta}$ and $\ddot{x}$ by thrashing about with some algebra, yielding:

$$\ddot{\theta} = -\frac{6F_x \cos\theta + 6g\sin\theta\,(m_c + m_p) + 3m_p L \sin\theta\cos\theta\dot{\theta}^2}{dL} \tag{16}$$

$$\ddot{x} = \frac{4F_x \cos\theta + 3g m_p \sin\theta\cos\theta + 2m_p L \sin\theta\dot{\theta}^2}{d} \tag{17}$$

where the denominator factor d is:

$$d = 4m_c + m_p\left(1 + 3\sin^2\theta\right) \tag{18}$$

Below we assume the following state space:

$$s = \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} \tag{19}$$

1.2 Cart force control

The above equations yield the following first order cartpole system, assuming force control:

$$\dot{s} = \begin{bmatrix} \dot{x} \\ \dot{\theta} \\ \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \dot{\theta} \\ \frac{4F_x \cos\theta + 3g m_p \sin\theta\cos\theta + 2m_p L\sin\theta\dot{\theta}^2}{d} \\ -\frac{6F_x\cos\theta + 6g\sin\theta(m_c+m_p) + 3m_p L\sin\theta\cos\theta\dot{\theta}^2}{dL} \end{bmatrix} \tag{20}$$

1.3 Cart velocity servo control

For the velocity servo control, we have the simpler (force-free) equations:

$$\dot{s} = \begin{bmatrix} \dot{x} \\ \dot{\theta} \\ \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \dot{\theta} \\ \frac{v_c - \dot{x}}{\tau} \\ -\frac{3}{2L}\left(\frac{v_c - \dot{x}}{\tau}\cos\theta + g\sin\theta\right) \end{bmatrix} \tag{21}$$

1.4 The `cvxpy` experience

2. Differential Drive Control trajectories

2.1 Signed Distance Function (SDF) for obstacle avoidance

2.2 Differentiable SDF in pyTorch

2.3 The `cvxpy` experience

References