Stanford University
Department of Computer Science

CS 000: A Hypothetical Course on Advanced Topics

Homework 1

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Problem 1

(a) Checking for Independence: To determine independence, we solve the equation $c_1\vec{w_1} + c_2\vec{w_2} + c_3\vec{w_3} = 0$. This expands to:

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ c_2 + c_3 &= 0 \\ c_3 &= 0 \end{aligned}$$

From the last equation, we see that $c_3 = 0$. Substituting into the second equation gives $c_2 = 0$, and finally, the first equation gives $c_1 = 0$. Thus, the three vectors are **independent**.

(b) Orthogonality: The three vectors $\vec{w_k}$ are **not orthogonal** since $\langle \vec{w_1}, \vec{w_2} \rangle = 1$, $\langle \vec{w_2}\vec{w_3} \rangle = 2$, and $\langle \vec{w_1}, \vec{w_3} \rangle = 1$. Hence, we apply the Gram-Schmidt orthogonalization process, obtaining the orthogonal vectors $\vec{b_1}, \vec{b_2}, \vec{b_3}$:

$$\vec{b_1} = \vec{w_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{b_2} = \vec{w_2} - P_{\vec{b_1}}\vec{w_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{b_3} = \vec{w_3} - P_{\vec{b_1}}\vec{w_3} - P_{\vec{b_2}}\vec{w_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Problem 2

(a) Checking for Independence: To check independence, solve $c_1\vec{w_1} + c_2\vec{w_2} + c_3\vec{w_3} = 0$, which when expanded gives the following system:

$$\begin{aligned} c_1 + 2c_2 &= 0 \\ c_1 - c_2 + c_3 &= 0 \\ c_1 - c_2 - c_3 &= 0 \end{aligned}$$

From the first equation, $c_1 = -2c_2$. Substituting into the second and third equations yields $c_2 = 0$, and thus $c_3 = 0$. Finally, $c_1 = 0$. Hence, the vectors $\vec{w_k}$ are **independent**.

(b) Orthogonality: The vectors $\vec{w_k}$ are **orthogonal** since:

$$\langle \vec{w_1}, \vec{w_2} \rangle = 0 \quad \langle \vec{w_2}, \vec{w_3} \rangle = 0 \quad \langle \vec{w_1}, \vec{w_3} \rangle = 0$$

(c) Orthonormal Basis: The norms of the vectors are:

$$|\vec{w_1}| = \sqrt{3} \quad |\vec{w_2}| = \sqrt{6} \quad |\vec{w_3}| = \sqrt{2}$$

Thus, the orthonormal basis is:

$$\hat{b_1} = \frac{\vec{w_1}}{|\vec{w_1}|} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\hat{b_2} = \frac{\vec{w_2}}{|\vec{w_2}|} = \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$$

$$\hat{b_3} = \frac{\vec{w_3}}{|\vec{w_3}|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

(d) Transformation Matrix: The coordinate transformation matrix from $\hat{e_i}$ to $\hat{b_k}$ is:

$$M = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$