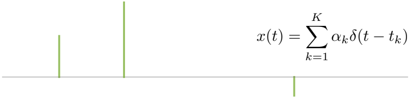
# Sparse Recovery with Finite Rate of Innovation Sampling

Hanjie Pan, Frederike Dümbgen

École Polytechnique Fédérale de Lausanne

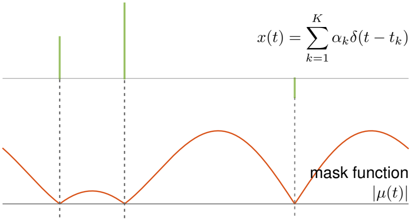
MFSP Application Session
December 17, 2018

# Sensing Sparsity through Annihilation



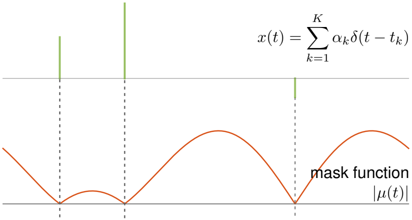
$$x(t) = \sum_{k=1}^{K} \alpha_k \delta(t - t_k)$$

# Sensing Sparsity through Annihilation




$$\boldsymbol{\mu(t)x(t) = 0}$$

# Sensing Sparsity through Annihilation




$$\mu(t)x(t) = 0$$

- Dirac locations: **zero-crossings** of a mask function, e.g.,

$$\mu(t) = \underbrace{\prod_{k=1}^{K} \left(1 - e^{-j\frac{2\pi}{\tau}t_k} e^{j\frac{2\pi}{\tau}t}\right)}_{\text{poly}\left(e^{j\frac{2\pi}{\tau}t}\right)}$$

# A Generalized FRI Sampling Framework

**Goal**: reconstruct an FRI signal that is *consistent* with the given measurements.

[1] H. Pan, T. Blu, and M. Vetterli. "Towards Generalized FRI Sampling with an Application to Source Resolution in Radioastronomy". In: *IEEE Transactions on Signal Processing* 65.4 (2017), pp. 821–835.

# A Generalized FRI Sampling Framework

**Goal**: reconstruct an FRI signal that is *consistent* with the given measurements.

1. What is an FRI signal?

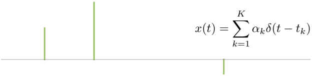
2. What is a consistent reconstruction?

[1] H. Pan, T. Blu, and M. Vetterli. "Towards Generalized FRI Sampling with an Application to Source Resolution in Radioastronomy". In: *IEEE Transactions on Signal Processing* 65.4 (2017), pp. 821–835.

# A Generalized FRI Sampling Framework

**Goal**: reconstruct an FRI signal that is *consistent* with the given measurements.

1. What is an FRI signal?



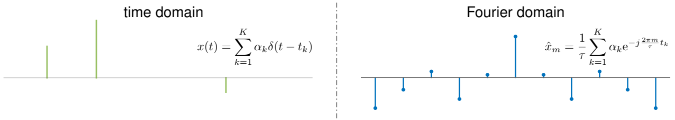
$$x(t) = \sum_{k=1}^{K} \alpha_k \delta(t - t_k)$$

[1] H. Pan, T. Blu, and M. Vetterli. "Towards Generalized FRI Sampling with an Application to Source Resolution in Radioastronomy". In: *IEEE Transactions on Signal Processing* 65.4 (2017), pp. 821–835.

# A Generalized FRI Sampling Framework

**Goal**: reconstruct an FRI signal that is *consistent* with the given measurements.

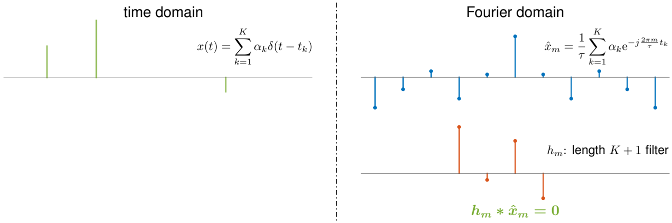
1. What is an FRI signal?




[1] H. Pan, T. Blu, and M. Vetterli. "Towards Generalized FRI Sampling with an Application to Source Resolution in Radioastronomy". In: *IEEE Transactions on Signal Processing* 65.4 (2017), pp. 821–835.

**Goal**: reconstruct an FRI signal that is *consistent* with the given measurements.

1. What is an FRI signal?

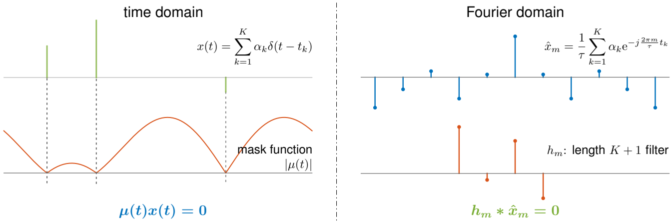



[1] H. Pan, T. Blu, and M. Vetterli. "Towards Generalized FRI Sampling with an Application to Source Resolution in Radioastronomy". In: *IEEE Transactions on Signal Processing* 65.4 (2017), pp. 821–835.

# A Generalized FRI Sampling Framework

**Goal**: reconstruct an FRI signal that is *consistent* with the given measurements.

1. What is an FRI signal?



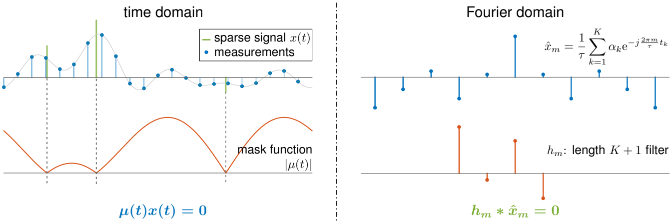


[1] H. Pan, T. Blu, and M. Vetterli. "Towards Generalized FRI Sampling with an Application to Source Resolution in Radioastronomy". In: *IEEE Transactions on Signal Processing* 65.4 (2017), pp. 821–835.

# A Generalized FRI Sampling Framework

**Goal**: reconstruct an FRI signal that is *consistent* with the given measurements.

1 What is an FRI signal?



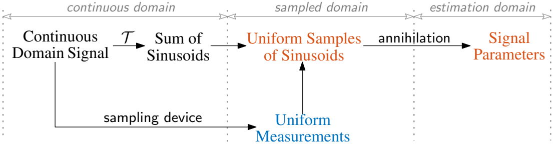


[1] H. Pan, T. Blu, and M. Vetterli. "Towards Generalized FRI Sampling with an Application to Source Resolution in Radioastronomy". In: *IEEE Transactions on Signal Processing* 65.4 (2017), pp. 821–835.

# A Generalized FRI Sampling Framework

**Goal**: reconstruct an FRI signal that is *consistent* with the given measurements.

1. What is an FRI signal?



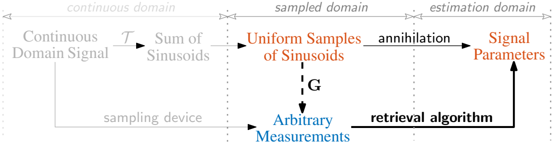


[1] H. Pan, T. Blu, and M. Vetterli. "Towards Generalized FRI Sampling with an Application to Source Resolution in Radioastronomy". In: *IEEE Transactions on Signal Processing* 65.4 (2017), pp. 821–835.

# A Generalized FRI Sampling Framework

**Goal**: reconstruct an FRI signal that is *consistent* with the given measurements.

1. What is an FRI signal?

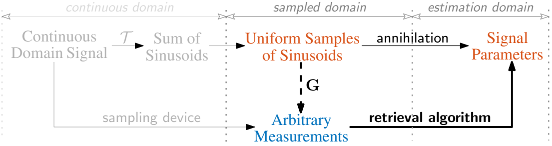



[1] H. Pan, T. Blu, and M. Vetterli. "Towards Generalized FRI Sampling with an Application to Source Resolution in Radioastronomy". In: *IEEE Transactions on Signal Processing* 65.4 (2017), pp. 821–835.

# A Generalized FRI Sampling Framework

**Goal**: reconstruct an FRI signal that is *consistent* with the given measurements.

1. What is an FRI signal?




2. What is a consistent reconstruction?

$$\|\mathbf{y} - \mathbf{y}'\|_2 \leq \varepsilon$$

- $\mathbf{y}$: the given measurements
- $\mathbf{y}'$: the re-synthesized measurements
- $\varepsilon$: noise level

[1] H. Pan, T. Blu, and M. Vetterli. "Towards Generalized FRI Sampling with an Application to Source Resolution in Radioastronomy". In: *IEEE Transactions on Signal Processing* 65.4 (2017), pp. 821–835.

# FRI-based Sparse Recovery

**Problem 1**: Constrained Minimization

$$\min_{\mathbf{h}\in\mathcal{H},\mathbf{b}} \quad \|\mathbf{y}-\mathbf{G}\mathbf{b}\|_2^2$$
$$\text{subject to} \quad \mathbf{b} * \mathbf{h} = \mathbf{0}$$

- $\mathbf{y}$: the given set of measurements
- $\mathbf{b}$: the unknown uniform samples of sinusoids
- $\mathbf{h}$: the annihilating filter coefficients
- $\mathbf{G}$: linear transformation from $\mathbf{b}$ to $\mathbf{y}$

# FRI-based Sparse Recovery

**Problem 2**: Constrained Approximation with Noise Level ($\varepsilon^2$)

$$\begin{aligned} \text{find} \quad & \mathbf{h} \in \mathcal{H}, \mathbf{b} \\ \text{subject to} \quad & \mathbf{b} * \mathbf{h} = \mathbf{0} \\ & \|\mathbf{y} - \mathbf{G}\mathbf{b}\|_2^2 \leq \varepsilon^2 \end{aligned}$$

- $\mathbf{y}$: the given set of measurements
- $\mathbf{b}$: the unknown uniform samples of sinusoids
- $\mathbf{h}$: the annihilating filter coefficients
- $\mathbf{G}$: linear transformation from $\mathbf{b}$ to $\mathbf{y}$

**Problem 1**: Constrained Minimization

$$\min_{\mathbf{h}\in\mathcal{H},\mathbf{b}} \quad \|\mathbf{y}-\mathbf{Gb}\|_2^2$$
$$\text{subject to} \quad \mathbf{b} * \mathbf{h} = \mathbf{0}$$

**any** solution such that $\|\mathbf{y}-\mathbf{Gb}\|_2^2 \leq \varepsilon^2$

- $\mathbf{y}$: the given set of measurements
- $\mathbf{b}$: the unknown uniform samples of sinusoids
- $\mathbf{h}$: the annihilating filter coefficients
- $\mathbf{G}$: linear transformation from $\mathbf{b}$ to $\mathbf{y}$

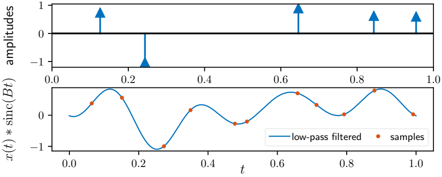
# FRI-based Sparse Recovery

**Problem 1**: Constrained Minimization

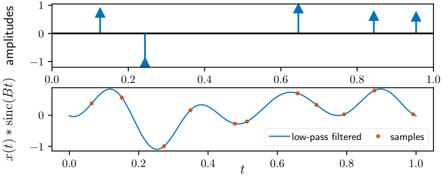
$$\min_{\mathbf{h}\in\mathcal{H},\mathbf{b}} \quad \|\mathbf{y}-\mathbf{G}\mathbf{b}\|_2^2$$
$$\text{subject to} \quad \mathbf{b}*\mathbf{h}=\mathbf{0}$$

- $\mathbf{y}$: the given set of measurements
- $\mathbf{b}$: the unknown uniform samples of sinusoids
- $\mathbf{h}$: the annihilating filter coefficients
- $\mathbf{G}$: linear transformation from $\mathbf{b}$ to $\mathbf{y}$

# Example: Dirac Estimation (Time-domain)



# Example: Dirac Estimation (Time-domain)




1. Uniform sinusoidal samples $\mathbf{b}$ — FS coefficients

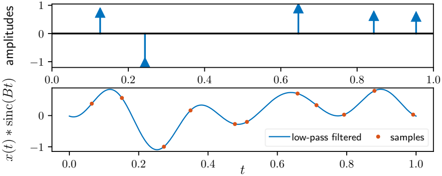
$$\hat{x}_m = \frac{1}{\tau} \sum_{k=1}^{K} \alpha_k \mathrm{e}^{-j\frac{2\pi t_k}{\tau} m}$$

2. Relation with given measurements $\mathbf{G}$ — inverse DFT

$$y_\ell = \frac{1}{B} \sum_{m \in \mathcal{M}} \hat{x}_m \mathrm{e}^{j\frac{2\pi m}{\tau} t'_\ell}$$

$t'_\ell$: sampling locations, $\mathcal{M} = \{m \,|\, |m| \leq \lfloor \tau B/2 \rfloor\}$.

# Example: Dirac Estimation (Time-domain)




1. Uniform sinusoidal samples $\mathbf{b}$ — FS coefficients

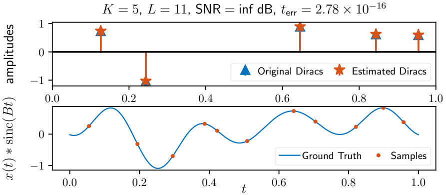
$$\hat{x}_m = \frac{1}{\tau} \sum_{k=1}^{K} \alpha_k \mathrm{e}^{-j \frac{2\pi t_k}{\tau} m}$$

2. Relation with given measurements $\mathbf{G}$ — inverse DFT

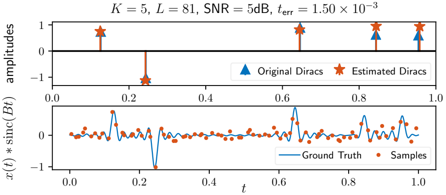
$$y_\ell = \frac{1}{B} \sum_{m \in \mathcal{M}} \hat{x}_m \mathrm{e}^{j \frac{2\pi m}{\tau} t'_\ell}$$

$t'_\ell$: sampling locations, $\mathcal{M} = \{m \,|\, |m| \leq \lfloor \tau B/2 \rfloor\}$.

# Example: Dirac Estimation (Time-domain)




(a) noiseless reconstruction




(b) noisy reconstruction

Figure: Dirac reconstruction from non-uniform time-domain samples.