A probability model requires at least two outcomes.
↳ if only one → it' deterministic model.

These outcomes collected in a set called sample space (S)

Example: * Coin Toss

Possible sample spaces

S = { Heads, edges, tails }
or
S = { Heads, Tails }

your decision.

* Triple Coin Toss

All details (Including order)

S = { HHH, HHT, HTH, THH, TTH, THT, HTT, TTT }

or just the number of "Heads"

S = { 0, 1, 2, 3 }

Model must fit problem!!!

* Exam Grade

S = { Pass, fail }
or
S = { 1,0 ; 1,3 ; --- ; 4,0 ; 5,0 }

Sometimes outcomes are combined into an Event.
→ subset of the sample space

German Grades:

Sehr Gut = { 1,0 ; 1,3 }
Gut = { 1,7 ; 2,0 ; 2,3 }
⋮
Mangelhaft = { 5,0 }

Compound Events } Contains more than one outcome

Elementary Event } Contains exactly one outcome

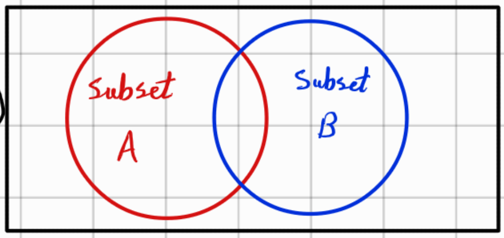
I) Special Events: { } (empty set), also ∅

is called impossible event because at least one outcome must occur.

II) Special Events: S (whole sample space)

is called $\boxed{\text{Certain event}}$ because whatever outcome occures must belong to the sample space.

Sets & Subsets (Samplespace & events)

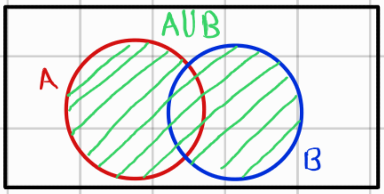


→ set (superset)

* John venn (venn diagrams)

We "calculate" with events (subsets) using the following operations

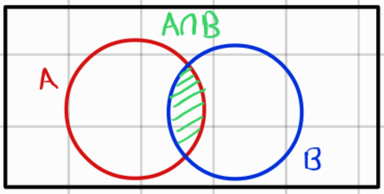
- **Union**: Contains all elements that belonges to A or B.



$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

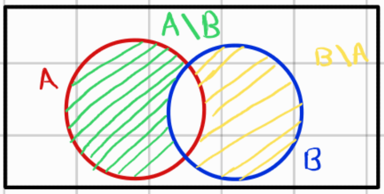
Symbol $\cup$ as in union

- **Intersection**: of A & B contains all elements that belong to A & B



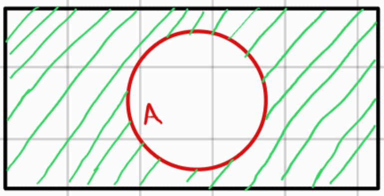
$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

- **Difference**: "A Minus B" contains all elements that belong to A but do not belong to B.



$A \setminus B = A - B = \{x \mid x \in A,\ x \notin B\}$

- **Complement**: of a subset contains all elements from the "superset" that do not belong to the subset.



$\bar{A} = A' = S \setminus A$

$= \{x \in S \mid x \notin A\}$

"Super set" (Sample space)

Two events (subsets) are Disjoint or Mutually exclusive if they do not have a common outcome (elements)

$A \,\&\, B$ Disjoint $\Longleftrightarrow A \cap B = \{\}$

How do we set up a probability model?

(Step 1: Outcomes, Sample space, and events) ✓

$\boxed{\text{Step 2: Assign probabilities}}$

Answer unclear yet!

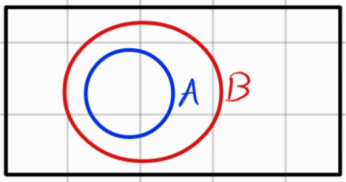
Kolmogorov Axioms

A Function that assigns to each event A of a given sample space, a number $0 \leqslant P(A) \leqslant 1$

Can be considered a $\boxed{\text{probability distribution}}$ if:

(i) $A \leqslant B \implies P(A) \leqslant P(B)$

If the event B contains all the outcomes of event A, then B is at least as likely to occur as A.



(ii) $P(\{\}) = 0$; $P(S) = 1$

(iii) If $A_1, A_2, \cdots$ are Pairwise Disjoint events, $A_i \cap A_j = \{\}$, $i \neq j$,

Then, the probabilities add $P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots$

[Finite sum or infinite sum]

Simple consequences

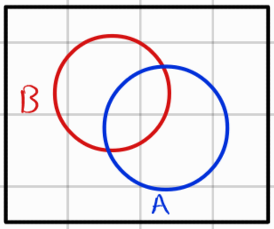
Important rules:

Let A be any event. Then $A \cap \bar{A} = \{\}$ and $A \cup \bar{A} = S$.

Therefore

$$1 = P(S) = P(A \cup \bar{A}) = P(A) + P(\bar{A}) \implies P(\bar{A}) = 1 - P(A)$$

Difference of events



$$P(B) = P(P(B \setminus A) \cup P(A \cap B))$$

$$\implies \boxed{P(B \setminus A) = P(B) - P(A \cap B)}$$

Probability of unions (not necessary disjoint)

Trick: use Disjoint parts A and $B \setminus A$

$$\Longrightarrow P(A \cup B) = P(A \cup (B \setminus A))$$

$$P(A \cup B) = P(A) + P(B \setminus A)$$

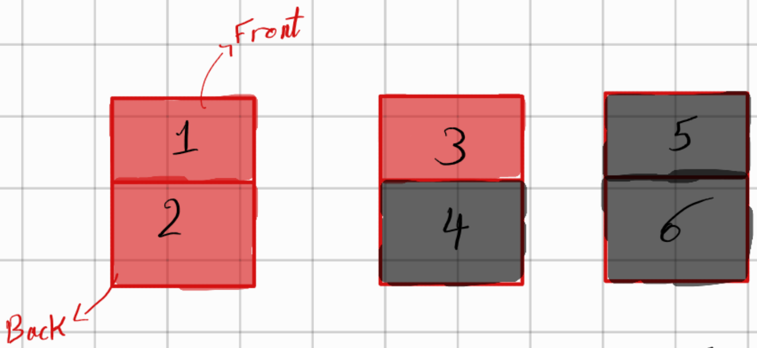
$$\boxed{P(A \cup B) = PA + P(B) - P(A \cap B)}$$

Note: When A & B are disjoint $P(A \cap B) = 0$

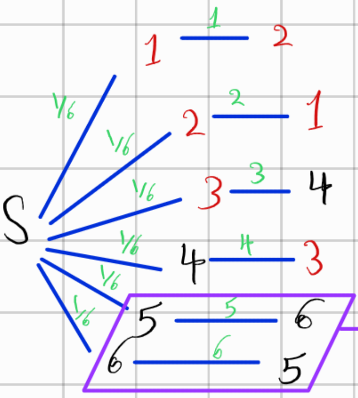
How do we assign concrete probabilities?

- Symmetry Arguments. (Laplace Assumption)
  There is no reason to believe that the outcome of having Head is more likely to occur than having a Tail when Tossing a Fair Coin.

- Relative Frequencies of past data.
- Subjective choice.



Sample space $S = \{1, 2, 3, 4, 5, 6\}$



1. $P(\text{Front} = \text{Black}) = \frac{3}{6} = 0.5 = 50\%$

2. $P(\text{Back} = \text{Black}) = \frac{3}{6} = 0.5 = 50\%$

3. $P(\text{Back} = \text{Black} \mid \text{Front} = \text{Black})$ = the probability of having a Black back card given that the front is Black.

$$3) \longrightarrow P(B = B \mid F = B) = \frac{P(B = B \cap F = B)}{P(F = B)} = \frac{\frac{2}{6}}{\frac{3}{6}} = \frac{4}{6} = \frac{2}{3}$$

$= 66.7\%$

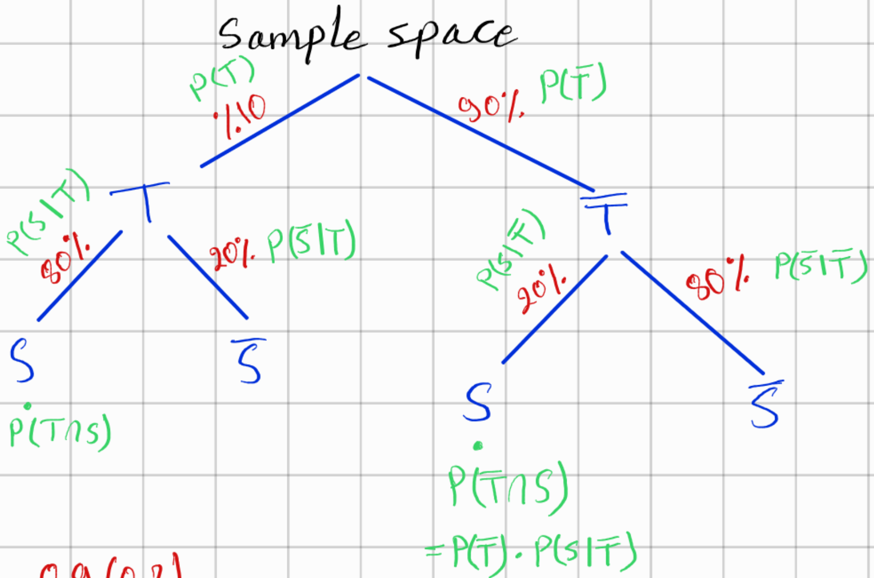
Conditional probability: $\boxed{P(A \mid B) = \frac{P(A \cap B)}{P(B)}}$

Where $P(A)$ & $P(B)$ are unconditional probability

FICTITIOUS EXAMPLE: THE MANAGEMENT OF A COMPANY KNOWS FROM HISTORICAL DATA THAT 10% OF THE EMPLOYEES ARE STEALING FROM THE COMPANY. TO FIND OUT WHO ARE THE THIEVES THEY PERFORM A LIE DETECTOR TEST. ACCORDING TO THE MANUAL THE TEST IS 80% RELIABLE. WHAT IS THE PROBABILITY THAT AN HONEST EMPLOYEE IS FALSELY ACCUSED?

T = "The radomly chosen employee is a thief"

S = "The lie detector test suspects the (randomly chosen) employee to be a thief"



$$P(\bar{T}\mid S) = \frac{P(\bar{T}\cap S)}{P(S)}$$

$$= \frac{P(\bar{T})\cdot P(S\mid\bar{T})}{P(\bar{T}\cap S)+P(T\cap S)} = \frac{0.9\,(0.2)}{0.9(0.2)+0.1\,(0.8)}$$

$$\approx 0.6923 = 69.23\ \%$$

Random variable: Describes a random experiment whose outcomes are real numbers.

Example: Roll a 20-sided die:-

Sample space $S = \{1, 2, ---, 20\}$

X = "The result of rolling a 20-sided die" is a random variable.

Custom: Use Capital letter for R.V. and corresponding small letters for results. ($x_1 = 7, x_2 = 13, \dots$)

Example: Production of Metal Bolts:-

L: "Length of a produced bolt in mm"

Sample space $S = [49, 51]$

All infinitely many values of an interval may be outcomes of a random variable.

Example: Accidents in a company:-

A: "Number of work-related accidents in a year"

Sample space $S = \{0, 1, 2, 3, 4, ------\}$

Infinitely many outcomes, but with gaps between the outcomes (not an interval).

R.V. (choice not alway obvious)

Discrete	Continuous
Gaps between outcome	All numbers of an interval are outcomes
(Uses Finite sum or Infinite series)	(uses integral)
Related to Counting.	Related to measuring
Uses Finite sums, or Infinite series.	Uses integrals

To Describe a discrete R.V completely, we need the outcomes and their probabilities.

This is called the Distribution of the R.V

x	$P(X=x)$	
1	$P(X=1)$	"Success probability"
0	$1-P(X=1)$	"Failure probability"

Example: Continuous R.V.

City of Utopia; Bus arrives every 20 minutes. We arrive "Randomly" at the bus Station. How Long do we have to wait for the next bus?

W = "Waiting time in minutes until the next bus arrives"

Sample space $S = [0, 20]$

Laplace assumption: No reason to believe that any outcome is more likely to happen than the others.

$P(W \leq 5) = 0.25$

$P(5 < W \leq 10) = 0.25$

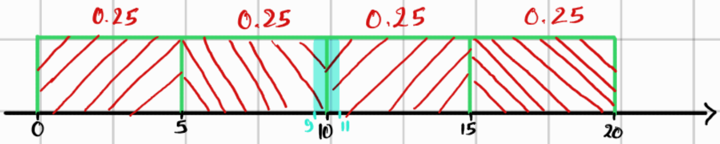
$P(10 < W \leq 15) = 0.25$

$P(15 < W \leq 20) = 0.25$

$\Rightarrow P(W \leq 20) = 1$

$P(W \leq 10) = 0.5$

Grouping & Histogram ➡



$P(9 < W \leq 11) = 0.1$

$P(9.5 < W \leq 10.5) = 0.05$

$P(9.75 < W \leq 10.25) = 0.025$

$P(W = 10) = 0 \Rightarrow$ For Continuous R.V., the probability of an elementry event is zero

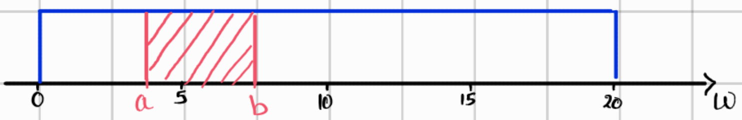
Two ways of visualization for the distribution of a continuous R.V.

1) Probability Density Function (PDF) [Probability Histogram]

2) Cumulative Distribution Function (CDF)

Always consider both.

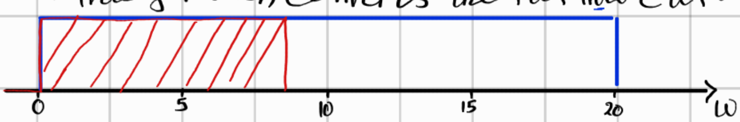
* Probability Density Function (PDF):



The probability is given by the area

$$P(a \leq W \leq b) = \underbrace{\int_a^b \underbrace{f(w)}_{\text{Pdf}}\, dw}_{\text{Probability}}$$

-Integeration converts the Pdf into Cdf: $F(w) = \int_{-\infty}^{w} \underbrace{f(x)}_{\text{Pdf}} dx$



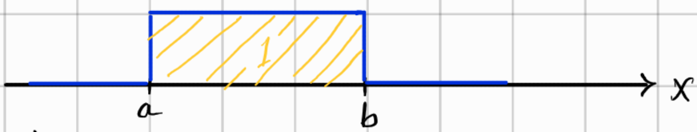
$F(w) = P(W \leq w)$

➡ The Pdf is the derivative of the Cdf:

$$f(w) = F'(w) = \frac{d}{dw} P(W \leq w)$$

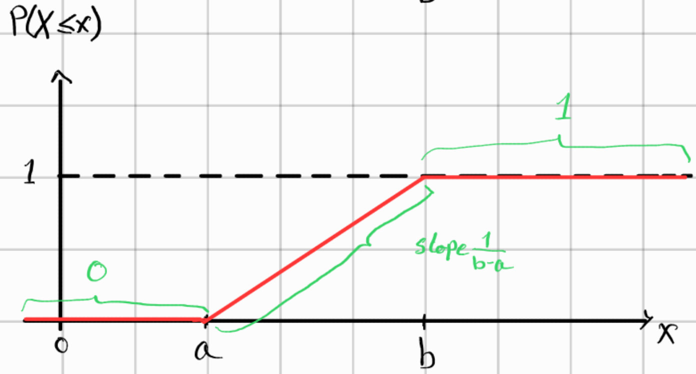
Also $P(a < W \leq b) = \int_a^b f(w)\,dw = \int_{-\infty}^{b} f(w)\,dw - \int_{-\infty}^{a} f(w)\,dw = P(W \leq b) - P(W \leq a) = F(b) - F(a)$

Uniform Continuous Distribution on $[a, b]$



Pdf $f(x) = \begin{cases} \frac{1}{b-a} & , \text{ if } a \leq x \leq b \\ 0 & , \text{ otherwise} \end{cases}$

↓ Integrate ↑ Differentiate



Cdf $F(x) = \begin{cases} 0 & , \text{ if } x < a \\ \frac{x-a}{b-a} & , \text{ if } a \leq x \leq b \\ 1 & , \text{ if } b < x \end{cases}$

* Since $P(X=c) = \int_c^c f(x)\,dx = 0$

➡ $P(a \leq X < b) = P(a \leq X \leq b) = P(a < X \leq b) = P(a < X < b)$

A Pdf $f(x)$ must satisfy:

(i) $f(x) \geq 0$ for all x

(ii) $\int_{-\infty}^{+\infty} f(x)\,dx = 1$

Both properties are directly related to Kolmogorov's Axioms.

Exercise: A Beverage bottle is filled with $(500 + X)$ mL, where X is a random variable. The pdf of X is $f(x) = 0.006(25 - x^2)$ for $-5 \leq x \leq 5$ and $f(x) = 0$ otherwise.

a) Check that f is a Pdf

b) Find the Cdf and sketch both functions.

c) Find the number c such that there is 95% probability of the bottle having a content between $(500 - c)$ mL and $(500 + c)$ mL

a) $f(x) = \begin{cases} 0.006(25 - x^2) & \text{for } -5 \leq x \leq 5 \\ 0 & \text{otherwise} \end{cases}$