27) $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} \approx -0.45969\ldots$

n	Sumands	Partial sums
1	$-\frac{1}{2}$	$-\frac{1}{2}$
2	$\frac{1}{24}$	$-\frac{11}{24} = -0.45833$
3	$\frac{-1}{720}$	$-0.459722\ldots$
4	$\frac{1}{40320}$	-0.4596974
5	$\frac{-1}{3628800}$	$-0.459697\ldots$
6	$\vdots$	
	0	

2) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6} \approx 0.9855\ldots$

n	Sumands	Partial sums
1	1	1
2	$\frac{-1}{64}$	$\frac{63}{64} = 0.984375$
3	$\frac{1}{729}$	$0.98574\ldots$
4	$\frac{-1}{4096}$	$0.985502\ldots$
5	$\frac{1}{15625}$	$0.985566\ldots$
6	$\frac{-1}{46656}$	$0.985545\ldots$
	$\vdots$	
	0	

$$29)\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n^2}{10^n} = \sum_{n=1}^{\infty}(-1)^{n-1}\cdot b_n \simeq 0.06761\ldots$$

n	Sumands	Partial sums
1	$\frac{1}{10}$	$\frac{1}{10}=0.1$
2	$-\frac{4}{100}$	0.06
3	$\frac{9}{1000}$	0.069
4	$\frac{-16}{10^4}$	0.0674
5	$\frac{25}{10^5}$	0.06765
6	$\frac{-36}{10^6}$	0.067614

$$b_6 = \frac{6^2}{10^6} = 0.000036$$

$$30)\sum_{n=1}^{\infty}\frac{(-1)^n}{3^n\cdot n!} = \sum_{n=1}^{\infty}(-1)^n\cdot b_n \simeq -0.28347\ldots$$

n	Sumands	Partial sums
1	$-\frac{1}{3\cdot 1!}=-\frac{1}{3}$	$-\frac{1}{3}=-0.33333$
2	$\frac{1}{18}$	-0.27777
3	$\frac{-1}{162}$	$-0.2839506\ldots$
4	$\frac{1}{1944}$	-0.283436
5	$\frac{-1}{29160}$	$-0.28347\ldots$

$$b_5 = 0.00003429\ldots$$

Section $\boxed{11.6}$ from the book:

1 Definition A series $\Sigma\, a_n$ is called **absolutely convergent** if the series of absolute values $\Sigma\, |a_n|$ is convergent.

2 Definition A series $\Sigma\, a_n$ is called **conditionally convergent** if it is convergent but not absolutely convergent.

3 Theorem If a series $\Sigma\, a_n$ is absolutely convergent, then it is convergent.

The Ratio Test

(i) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\Sigma\, a_n$.

2) $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^2}$ let $a_n = \frac{(-2)^n}{n^2}$, using ration test $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-2)^{n+1}}{(n+1)^2}\cdot\frac{n^2}{(-2)^n}\right|$

$\Rightarrow \lim_{n\to\infty}\left|\frac{(-2)\cdot\cancel{n^2}}{\cancel{n^2}(1+\frac{1}{n})^2}\right| = \frac{2}{(1+0)^2} = 2 > 1$ $\therefore \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}\frac{(-2)^n}{n^2}$ diverges by Ratio test.

3) $\sum_{n=1}^{\infty}\frac{n}{5^n}$ $\sum_{n=1}^{\infty}\left|\frac{n}{5^n}\right| = \sum_{n=1}^{\infty}\frac{n}{5^n}$, $\frac{n}{5^n} < \frac{4^n}{5^n}$ for all $n \geq 1$.

$\therefore$ The series $\sum_{n=1}^{\infty}\frac{n}{5^n}$ is absolutely convergent by comparizon Test with $\sum_{n=1}^{\infty}\left|\frac{4^n}{5^n}\right| = \sum_{n=1}^{\infty}\left(\frac{4}{5}\right)^n$ which is a convergent geometric series with $|q| = \frac{4}{5} < 1$

Another approach (Ratio test): $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{n+1}{5^{n+1}}\cdot\frac{5^n}{n}\right| = \lim_{n\to\infty}\left|\frac{\cancel{n}(1+\frac{1}{n})}{5\cdot\cancel{n}}\right| = \left|\frac{1+0}{5}\right| = \frac{1}{5} < 1$

$\therefore$ By Ratio test, the series $\sum_{n=1}^{\infty}\frac{n}{5^n}$ is absolutely convergent.

4) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{n^2+4} = \sum_{n=1}^{\infty} a_n$, $|a_n| > 0$ for all $n \geq 1$, $|a_n|$ is decreasing for all $n \geq 1$, and $\lim_{n\to\infty}|a_n| = 0$, so $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{n^2+4}$ is convergent by Alternating series Test.

To determine absolute convergence:

$\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\frac{n}{n^2+4}$ using limit comparizon test with $|a_n| = \frac{n}{n^2+4}$ & $|b_n| = \frac{1}{n}$

$\lim_{n\to\infty}\frac{|a_n|}{|b_n|} = \lim_{n\to\infty}\frac{n}{n^2+4}\cdot\frac{n}{1} = \lim_{n\to\infty}\frac{\cancel{n^2}\cdot 1}{\cancel{n^2}(1+\frac{4}{n^2})} = \frac{1}{1+0} = 1 > 0$ since $\sum_{n=1}^{\infty}|b_n|$ is a divergent harmonic series, so is $\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\frac{n}{n^2+4}$

Thus $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{n^2+4}$ is conditionally convergent.

5) $\sum_{n=0}^{\infty} \frac{(-1)^n}{5n+1} = \sum_{n=0}^{\infty} a_n$, $a_n = \frac{1}{5n+1} > 0$ for all $n \geq 0$, b_n is decreasing for $n \geq 0$ and $\lim_{n\to\infty} a_n = 0$, so $\sum_{n=0}^{\infty} \frac{(-1)^n}{5n+1}$ converges by alternating series test.

To determine absolute convergence: let $b_n = \frac{1}{n}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{5n+1} = \lim_{n\to\infty} \frac{\cancel{n} \cdot 1}{\cancel{n}(5+\frac{1}{n})} = \frac{1}{5} > 0$, so $\sum_{n=0} \frac{1}{5n+1}$ diverges by limit Comparizon Test with a divergent harmonic series. Thus $\sum_{n=0}^{\infty} \frac{(-1)^n}{5n+1}$ is conditionally convergent.

6) $\sum_{n=0}^{\infty} \frac{(-3)^n}{(2n+1)!} = \sum_{n=0}^{\infty} a_n$, $|a_n| > 0$ for $n \geq 0$, $|a_n|$ is decreasing for $n > 0$, & $\lim_{n\to\infty} |a_n| = 0$

So $\sum_{n=0}^{\infty} \frac{(-3)^n}{(2n+1)!}$ is convergent by Alternating Series test.

To determine absolute convergence: $(2n+1)! \geq (4)^n$ for $n \geq 0$

$\therefore |a_n| = \frac{(3)^n}{(2n+1)!} \leqslant \frac{(3)^n}{(4)^n}$ for $n \geq 0$ Thus $\sum_{n=0}^{\infty} \frac{3^n}{(2n+1)!}$ is Convergent by comparizon Test with a convergent geometric series with $|q| = \frac{3}{4} < 1$

Thus $\sum_{n=0}^{\infty} \frac{(-3)^n}{(2n+1)!}$ is absolutely convergent. (We could have only done the second part)

7) $\sum_{K=1}^{\infty} K\left(\frac{2}{3}\right)^K = \sum_{K=1}^{\infty} a_K$ using Ratio Test $\lim_{K\to\infty} \left|\frac{a_{K+1}}{a_K}\right| = \lim_{K\to\infty} \left|\frac{(K+1)\left(\frac{2}{3}\right)^{K+1}}{K\left(\frac{2}{3}\right)^K}\right| = \lim_{K\to\infty} \left|\frac{\cancel{K}(1+\frac{1}{K})\cancel{(\frac{2}{3})^K}(\frac{2}{3})}{\cancel{K} \cdot \cancel{(\frac{2}{3})^K} \cdot 1}\right|$

$= \frac{(1+0)}{1} \cdot \frac{2}{3} = \frac{2}{3} < 1$ Thus, By Ratio Test $\sum_{K=1}^{\infty} K\left(\frac{2}{3}\right)^K$ is absolutely Convergent.

8) $\sum_{n=1}^{\infty} \frac{n!}{100^n} = \sum_{n=1}^{\infty} a_n$ using Ratio test $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!}{100^{n+1}} \cdot \frac{100^n}{n!}\right| = \lim_{n\to\infty} \left|\frac{\cancel{n!}(n+1) \cdot \cancel{100^n}}{\cancel{100^n}(100) \cdot \cancel{n!}}\right| = \infty$

Thus $\sum_{n=1}^{\infty} \frac{n!}{100^n}$ diverges by Ratio test.

9) $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{(1.1)^n}{n^4}$ $f(x) = \frac{(1.1)^x}{x^4} \Rightarrow f'(x) = \frac{(1.1)^x \ln(1.1) x^4 - 4x^3 (1.1)^x}{x^8} = \frac{\cancel{x^3}(1.1)^x(\ln(1.1) \cdot x - 4)}{x^{\cancel{8}5}}$

$(1.1)^x\left(x - \frac{4}{\ln(1.1)}\right) < 0$

$(1.1)^x \neq 0 \longrightarrow x < \frac{4}{\ln(1.1)}$

$= \sum_{n=1}^{\infty} (-1)^n a_n$ $a_n > 0$ for $n \geq 1$, a_n is increasing for $n > \frac{4}{\ln(1.1)}$

$\therefore$ By Alternating Test $\sum_{n=1}^{\infty} (-1)^n \frac{(1.1)^n}{n^4}$ diverges.

Another approach: Ratio Test : $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(1.1)^{n+1}}{(n+1)^4} \cdot \frac{n^4}{(1.1)^n} = \lim_{n\to\infty} \frac{1.1\cancel{(1.1)^n}}{\frac{(n+1)^4}{n^4}} \cdot \frac{1}{\cancel{(1.1)^n}} = \lim_{n\to\infty} \frac{1.1}{(1+\frac{1}{n})^4} = \frac{1.1}{(1+0)^4} = 1.1 > 1$

$\therefore$ The series diverges by Ratio test.

10) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{\sqrt{n^3+2}} = \sum_{n=1}^{\infty} (-1)^n \cdot a_n$, $a_n > 0$ for $n \geq 1$, a_n is decreasing for $n \geq 1$, and $\lim_{n\to\infty} a_n = 0$

So $\sum_{n=1}^{\infty} (-1)^n \frac{n}{\sqrt{n^3+2}}$ is convergent by Alternating series Test.

$\sum_{n=1}^{\infty} \left| (-1)^n \cdot \frac{n}{\sqrt{n^3+2}} \right| = \sum_{n=1}^{\infty} |b_n|$ using limit comparizon Test with $|b_n| = \frac{n}{\sqrt{n^3+2}}$ & $d_n = \frac{1}{n^{\frac{1}{2}}}$

$\lim_{n\to\infty} \frac{|b_n|}{d_n} = \lim_{n\to\infty} \frac{n}{\sqrt{n^3(1+\frac{2}{n^3})}} \cdot \frac{n^{\frac{1}{2}}}{1} = \lim_{n\to\infty} \frac{\cancel{n^{\frac{3}{2}}} \cdot 1}{\cancel{n^{\frac{3}{2}}} \sqrt{1+\frac{2}{n^3}}} = \frac{1}{\sqrt{1+0}} = 1 > 0$ so $\sum_{n=1}^{\infty} |b_n|$ diverges by limit comparizon

Test with a divergent p-series ($P = \frac{1}{2} \leq 1$). Thus the series $\sum_{n=1}^{\infty} (-1)^n \frac{n}{\sqrt{n^3+2}}$ is Conditionally Convergent.

11) $\sum_{n=1}^{\infty} \frac{(-1)^n e^{\frac{1}{n}}}{n^3} = \sum_{n=1}^{\infty} (-1)^n \cdot a_n$ $a_n > 0$ for $n \geq 1$, a_n is decreasing for $n > 1$, and $\lim_{n\to\infty} a_n = 0$

So $\sum_{n=1}^{\infty} (-1)^n \frac{e^{\frac{1}{n}}}{n^3}$ is Convergent by Alternating Test.

We only need this part if it's conditionally convergent!

$\sum_{n=1}^{\infty} \left| (-1)^n \frac{e^{\frac{1}{n}}}{n^3} \right| = \sum_{n=1}^{\infty} a_n$ $0 \leq \frac{e^{\frac{1}{n}}}{n^3} \leq \frac{e}{n^3}$ for $n \geq 1$, since $\sum_{n=1}^{\infty} \frac{e}{n^3}$ is a convergent p-series ($P = 3 > 1$),

so $\sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}}{n^3}$ also converges by comparizon Test. This part is enough.

Thus $\sum_{n=1}^{\infty} \frac{(-1)^n e^{\frac{1}{n}}}{n^3}$ is absolutely Convergent

12) $\sum_{n=1}^{\infty} \frac{\sin(4n)}{4^n}$ $|\sin(4n)| \leq 1$ $\therefore 0 \leq \left| \frac{\sin(4n)}{4^n} \right| \leq \frac{1}{4^n}$ for all $n \geq 1$ since $\sum_{n=1}^{\infty} \frac{1}{4^n}$ is a convergent geometric series ($|q| = \frac{1}{4} < 1$)

$\therefore$ Thus $\sum_{n=1}^{\infty} \frac{\sin(4n)}{4^n}$ is absolutely convergent.

13) $\sum_{n=1}^{\infty} \frac{10^n}{(n+1) \cdot 4^{2n+1}} = \sum_{n=1}^{\infty} a_n$ using Ratio test $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{10^{n+1}}{(n+2)\, 4^{2n+3}} \cdot \frac{(n+1) \cdot 4^{2n+1}}{10^n} \right| = \lim_{n\to\infty} \frac{10\,(1+\frac{1}{n})}{4^2 (1+\frac{2}{n})} = \frac{10\,(1+0)}{16\,(1+0)} = \frac{5}{8} < 1$

So the series $\sum_{n=1}^{\infty} a_n$ is Absolutely convergent by Ratio Test.

14) $\sum_{n=1}^{\infty} \frac{n^{10}}{(-10)^{n+1}} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^{10}}{(10)^{n+1}} = \sum_{n=1}^{\infty} a_n$ using Ratio Test $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{(n+1)^{10}}{(10)^{n+2}} \cdot \frac{(10)^{n+1}}{n^{10}}$

$\Longrightarrow = \lim_{n\to\infty} \frac{\frac{(n+1)^{10}}{n^{10}} \cdot \cancel{(10)^{n+1}}}{\cancel{(10)^{n+1}} \cdot 10} = \lim_{n\to\infty} \frac{(1+\frac{1}{n})^{10}}{10} = \frac{1+0}{10} = 0.1 < 1$

Thus, the series $\sum_{n=1}^{\infty} \frac{n^{10}}{(-10)^{n+1}}$ is absolutely convergent.

15) $\sum_{n=1}^{\infty} \frac{(-1)^n \arctan(n)}{n^2} = \sum_{n=1}^{\infty} a_n$ $\arctan(n) \leq \frac{\pi}{2}$, $0 \leq |a_n| = \frac{\arctan(n)}{n^2} \leq \frac{2}{n^2}$ for $n \geq 1$

Since $\sum_{n=1}^{\infty} 2 \cdot \frac{1}{n^2}$ is a convergent p-series ($P = 2 > 1$), thus $\sum_{n=1}^{\infty} a_n$ is absolutely convergent by Comparizon Test.

16) $\sum_{n=1}^{\infty} \frac{3-\cos(n)}{n^{2/3}-2} = \sum_{n=1}^{\infty} a_n$ $\quad 2 \leqslant 3-\cos(n) \leqslant 4 \Rightarrow |a_n| = \left|\frac{3-\cos(n)}{n^{2/3}-2}\right| \geqslant \frac{1}{n^{2/3}}$

Since $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ diverges because it's a p-series with $p=\frac{2}{3}<1$,

Thus, the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{3-\cos(n)}{n^{2/3}-2}$ diverges as well by comparizon Test.

17) $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)} = \sum_{n=1}^{\infty} a_n$ $\quad$ Since $|a_n|$ is decreasing, & $\lim_{n\to\infty} |a_n| = 0$ so $\sum_{n=2}^{\infty} a_n$ is convergent by Alternating Series Test.

$|a_n| = \frac{1}{\ln(n)} > \frac{1}{n}$ for $n \geqslant 2$, since $\sum_{n=2}^{\infty} \frac{1}{n}$ is a divergent harmonic series, so $\sum_{n=2}^{\infty} |a_n|$ diverges as well.

Thus $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$ is Conditionally convergent.

18) $\sum_{n=1}^{\infty} \frac{n!}{n^n} = \sum_{n=1}^{\infty} a_n$ $\quad$ using Ratio Test $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \lim_{n\to\infty} \frac{\cancel{n!}\cancel{(n+1)} \cdot n^n}{(n+1)^n \cdot \cancel{(n+1)} \cdot \cancel{n!}} = \lim_{n\to\infty} \frac{n^n}{(n+1)^n}$

$\Rightarrow \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = e^{-1} = \frac{1}{e} < 1$ $\quad \left(\lim_{x\to\infty} \left(\frac{x}{x+k}\right)^x = e^{-k}\right)$

By Ratio test the series $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ is Absolutely convergent, & therefore convergent.

19) $\sum_{n=1}^{\infty} \frac{\cos(n\pi/3)}{n!} = \sum_{n=1}^{\infty} a_n$ $\quad \frac{|\cos(n\pi/3)|}{n!} \leqslant \frac{1}{n!}$ for all $n \geq 1$

if we used Ratio Test with $b_n = \frac{1}{n!} \Rightarrow \lim_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1.$

Since $\sum_{n=1}^{\infty} \frac{1}{n!}$ Converges by Ratio Test, The series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{\cos(n\pi/3)}{n!}$ is absolutely convergent by comparizon Test.

20) $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^n} = \sum_{n=1}^{\infty} a_n$ $\quad$ Ratio Test, $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-2)^{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{(-2)^n}\right| = \lim_{n\to\infty} \frac{2n^n}{(n+1)^n (n+1)} = 2\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n \cdot \frac{1}{n+1} = e^{-1} \cdot 0 = 0 < 1$

By Ratio test, the series $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^n}$ is absolutely convergent.

21) $\sum_{n=1}^{\infty} \left(\frac{n^2+1}{2n^2+1}\right)^n = \sum_{n=1}^{\infty} \left|\left(\frac{n^2+1}{2n^2+1}\right)^n\right|$ $\quad$ using limit comparizon Test with $a_n = \left(\frac{n^2+1}{2n^2+1}\right)^n$ & $b_n = \left(\frac{1}{2}\right)^n$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{n^2+1}{2n^2+1}\right)^n \cdot \frac{2^n}{1} = \lim_{n\to\infty} \left(\frac{\cancel{n^2}(1+\frac{1}{n^2})}{\cancel{n^2}(2+\frac{1}{n^2})}\right)^n \cdot \frac{2^n}{1} = \lim_{n\to\infty} \left(\frac{1+0}{2+0}\right)^n \cdot \frac{2^n}{1} = \lim_{n\to\infty} \frac{1}{\cancel{2^n}} \cdot \frac{\cancel{2^n}}{1} = 1$

Since $\sum_{n=1}^{\infty} b_n$ is a convergent geometric series with $(|r| = \frac{1}{2} < 1)$, thus the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} |a_n|$

Converges by Limit comparizon Test.

Power Series:

Polynomials: Finite Linear combination of powers of x

$$C_0 + C_1 x + C_2 x^2 + C_3 x^3 \cdots + C_n x^n$$

Coefficients

Extend to infinite sums:

$$C_0 + C_1 x + C_2 x^2 + \cdots = \sum_{k=0}^{\infty} C_k x^k$$

Definition:

$$C_0 + C_1(x-a) + C_2(x-a)^2 + C_3(x-a)^2 + \cdots = \sum_{k=0}^{\infty} C_k (x-a)^k$$

Is a power series with center a,

and coefficients $C_0, C_1, C_2, \cdots$

Example: The Geometric series

$$1 + x + x^2 + x^3 + \cdots = \sum_{k=0}^{\infty} x^k$$

Convergence for $|x| < 1$, so it defines a function

with the domain $(-1, 1)$

and the values:

$$f(x) = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

Example: $\sum_{k=1}^{\infty} \frac{(x-3)^k}{k} = (x-3) + \frac{(x-3)^2}{2} + \frac{(x-3)^3}{3} + \cdots$

For which x does this series converges?

$x=3$: $\sum_{k=1}^{\infty} \frac{(3-3)^k}{k} = 0 + 0 + 0 + \cdots = 0$

Every power series converges at its center.

$$\sum_{n=0}^{\infty} C_n (x-a)^n \xrightarrow[x=a]{} \sum_{n=0}^{\infty} C_n (a-a)^n = C_0$$

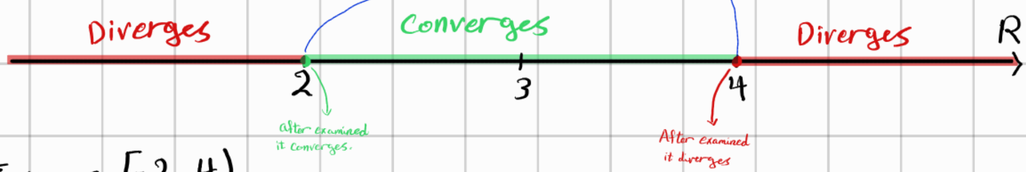
$x=4$: $\sum_{k=1}^{\infty} \frac{(4-3)^k}{k} = \sum_{k=1}^{\infty} \frac{1}{k}$ Diverges (Harmonic series)

$x=2$: $\sum_{k=1}^{\infty} \frac{(2-3)^k}{k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ (Negative alternating harmonic series) Converges.

End points must be examined separately.

Ratio Test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x-3)^{n+1}}{n+1} \cdot \frac{n}{(x-3)^n}\right| = \lim_{n\to\infty} |x-3| \frac{n}{n+1} = |x-3| \cdot 1 = |x-3|$

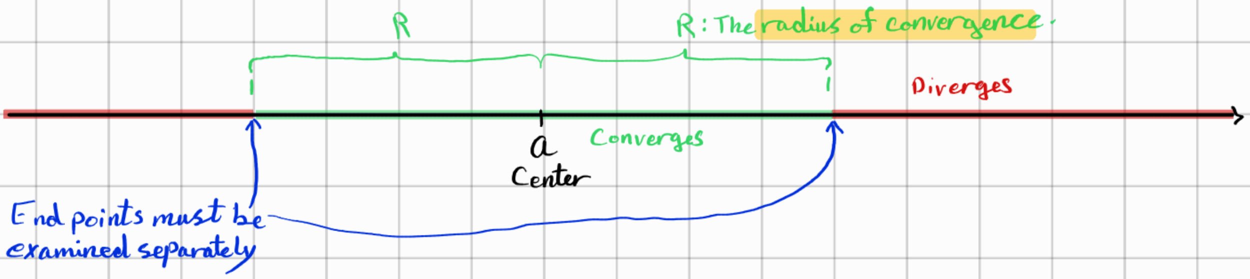
$$|x-3| \begin{cases} < 1 & \text{Converges} \\ > 1 & \text{Diverges} \end{cases}$$

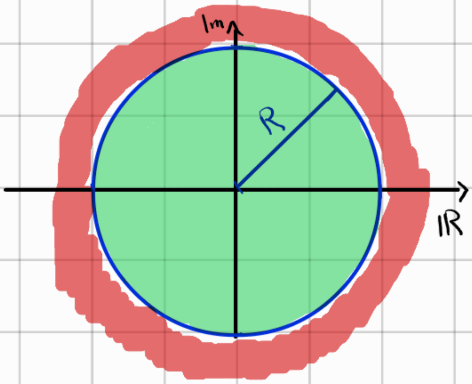


So $\sum_{k=1}^{\infty} \frac{(x-3)^k}{k}$ defines a function on $[2, 4)$

Interval of convergence

This Example is typical of all power series:

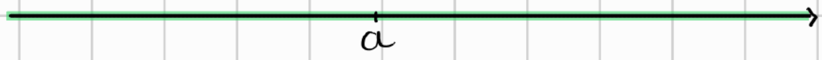




Two extreme cases for the radius of Convergence are possible.

$R = 0$ Power series only converges at center.

$R = \infty$ Power series converges everywhere.



A Power Series with center a and radius of convergence R define a function defined on $(a-R, a+R)$ (sometimes one or both end points can be included.

Power Series form a whole class of functions with very nice properties:

- Easy domain, interval (Symmetric about center, except maybe at end points).
- Continuous on the whole domain.
- Infinitely many times differentiable $\longrightarrow \frac{d}{dx}\left[C_0 + C_1(x-a) + C_2(x-a)^2 + \cdots\right] = C_1 + 2C_2(x-a) + 3C_3(x-a)^2 + \cdots$

  as well

  With same radius of convergence.
- Integration is easy too. $\longrightarrow \int\left[C_0 + C_1(x-a) + C_2(x-a)^2 + \cdots\right]dx = C + C_0(x-a) + C_1\frac{(x-a)^2}{2} + C_2\frac{(x-a)^3}{3} + \cdots$

  With the same radius of Convergence.
- The function values can be computed within any desired tolerence using partial sums. These Partial Sums are polynomials and require only addition, Substraction, and Multiplication.
- Many important functions can be written as Power Series. ☆
  (e.g. Sine, Cosine, Tangent, Arctangent, some Root, Exponential, Logarithmic, many engineering functions, for which there is no Key on the calculator) [e.g. Bessel Func]

Series 2 Function:

Geometric Series:

$$1 + x + x^2 + x^3 + \cdots = \sum_{K=0}^{\infty} x^K = \frac{1}{1-x} \quad \text{for } |x|<1$$

Differentiating Both sides:

$$0 + 1 + 2x + 3x^2 + 4x^3 + \cdots = \sum_{\substack{K=0\\K=1}}^{\infty} K x^{K-1} = \frac{0(1-x)-(1)(-1)}{(1-x)^2} = \frac{1}{(1-x)^2}$$

$$\therefore 1 + 2x + 3x^2 + \cdots = \sum_{K=1}^{\infty} K x^{K-1} = \frac{1}{(1-x)^2} \quad \text{for } |x|<1$$

Alternatively, let's Integerate the geometric series formula

$$C + x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \tfrac{1}{4}x^4 + \cdots = \sum_{K=0}^{\infty} \frac{1}{K+1} x^{K+1} = -\ln|1-x| \quad \text{for } |x|<1$$

Sufficient on one side of the equation.

$$= \sum_{K=1}^{\infty} \frac{x^K}{K} = \ln\left|\frac{1}{1-x}\right| \quad \text{for } |x|<1$$

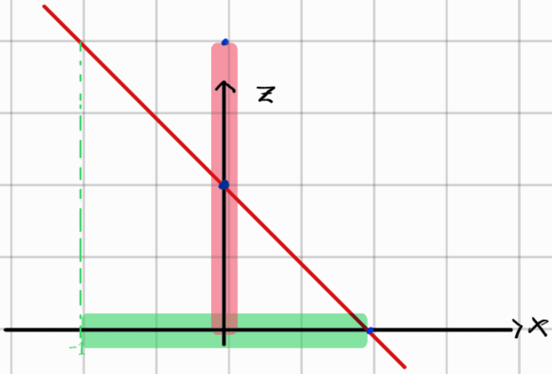
Compute C: plug in the center $x=a=0$

$$C + 0 + 0 + 0 + \cdots = -\ln|1| = 0 \Longrightarrow \boxed{C=0}$$

$$\therefore \ln|1-x| = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots \quad \text{for } |x|<1$$

Substitution $z = 1 - x$
$\rightarrow x = 1 - z$

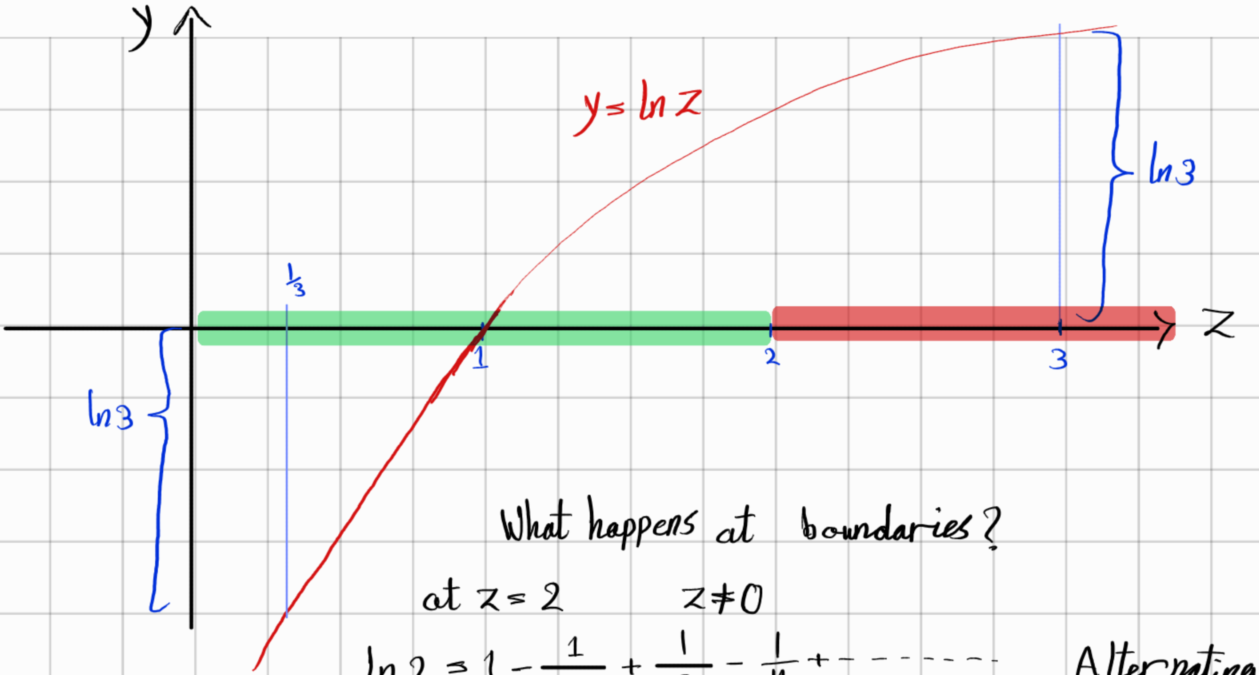
$-1 < x < 1 \Rightarrow 0 < z < 2$



$$\therefore \ln z = -(1-z) - \frac{(1-z)^2}{2} - \frac{(1-z)^3}{3} - \frac{(1-z)^4}{4} - \cdots \quad \text{for } 0<z<2$$

$$\boxed{\ln z = (z-1) - \frac{(z-1)^2}{2} + \frac{(z-1)^3}{2} - \frac{(z-1)^4}{4} + \cdots \quad \text{for } 0<z<2}$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(z-1)^n}{n}$$



What happens at boundaries?

at $z = 2$ $\qquad$ $z \neq 0$

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + - - - - - -$$

Alternating harmonic series

Converges

How to compute $\ln 3$?

$$\ln 3 = \ln\left(\tfrac{1}{3}\right)^{-1} = -\ln\left(\tfrac{1}{3}\right)$$

$\ln(\frac{1}{3})$ inside our radius of convergence.

To find $(\ln 3)$ with a tolerance (approximation of $\ln 3$), we use partial sums.

$$\ln 3 = \frac{2}{3} + \frac{1}{2}\left(\frac{2}{3}\right)^2 + \frac{1}{3}\left(\frac{2}{3}\right)^3 + \frac{1}{4}\left(\frac{2}{3}\right)^4 + - - - -$$

n	Sumands	Partial sums
1	$\frac{2}{3} \approx 0.666$	0.6666---
2	$\frac{2}{9} \approx 0.2222$	0.8888---
3	$\frac{8}{81} \approx 0.09876$--	0.9876---
4	$\frac{4}{81} \approx 0.0493827$	1.037037---
5	⋮	⋮
	$\to 0$	$\to \ln 3$

When all summands are positive, the partial sums are Lower Bounds for the sum of the series.

$$1.037 < \ln 3 < ???$$

$1.037 + 0.08 = 1.117$
≈ 1.12

$$\ln 3 = \underbrace{\frac{2}{3} + \frac{1}{2}\left(\frac{2}{3}\right)^2 + \frac{1}{3}\left(\frac{2}{3}\right)^3 + \frac{1}{4}\left(\frac{2}{3}\right)^4}_{\substack{\text{Fourth partial sum} \\ \approx 1.037}} + \underbrace{\frac{1}{5}\left(\frac{2}{3}\right)^5 + \frac{1}{6}\left(\frac{2}{3}\right)^6 + \frac{1}{7}\left(\frac{2}{3}\right)^7 + - - - -}_{\text{Reminder}}$$

$$\text{Reminder} < \frac{1}{5}\left(\frac{2}{3}\right)^5 + \frac{1}{5}\left(\frac{2}{3}\right)^6 + \frac{1}{5}\left(\frac{2}{3}\right)^7 + - - - -$$

$$\text{//} \quad < \frac{1}{5}\left(\frac{2}{3}\right)^5 \left[1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + - - -\right]$$

→ geometric series

$$\text{//} \quad < \frac{1}{5}\left(\frac{2}{3}\right)^5 \cdot \frac{1}{1 - \frac{2}{3}} = 0.079 - - - \approx \underline{0.08}$$

↳ we round up to make sure it's an upper bounds

Example: Use of Series in Integration:

$$\int_0^{1/2} \frac{1}{1+x^7}\,dx = \int_0^{1/2} \frac{1}{1-(-x^7)}\,dx = \int_0^{1/2} \left(1+(-x^7)+(-x^7)^2+(-x^7)^3+\cdots\right)dx \qquad \text{for } -1<x<1$$

$$= \left[x-\frac{x^8}{8}+\frac{x^{15}}{15}-\frac{x^{22}}{22}+\cdots\right]_0^{\frac{1}{2}} = \underline{\left(\frac{1}{2}-\frac{1}{8}\left(\frac{1}{2}\right)^8+\frac{1}{15}\left(\frac{1}{2}\right)^{15}-\frac{1}{22}\left(\frac{1}{2}\right)^{22}+\cdots\right)}-0$$

approximate it using partial sums.

Summands	partial sums
$\frac{1}{2}=0.5$	0.5
$-\frac{1}{8(2)^8}=-\frac{1}{1024} \approx 0.0005$	≈ 0.4995

$$0.4995 < \int_0^{1/2} \frac{1}{1+x^7}\,dx < 0.5$$

960
+64

next value is very small $\Rightarrow$ $\displaystyle\int_0^{1/2} \frac{1}{1+x^7}\,dx \approx 0.4995$

Example: Arctangent

$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$$

Geometric series $1+z+z^2+\cdots = \frac{1}{1-z}$ for $|z|<1$

Substitution: $z=-x^2$ for $-1<x<1$

$$\frac{1}{1-(-x^2)} = 1-x^2+(-x^2)^2+(-x^2)^3+(-x^2)^4+\cdots$$

$$= 1-x^2+x^4-x^6+x^8-+\cdots = \sum_{n=0}^{\infty}(-1)^n x^{2n} \qquad \text{for } |x|<1$$

Integerat both sides:

$$\arctan x = C + x - \frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots \qquad \text{for } -1<x<1$$

Find C by pluging in $x=0$:

$$0 = C+0+0+\cdots \Longrightarrow \boxed{C=0}$$

Hence

$$\boxed{\arctan x = x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots \qquad \text{for } |x|<1}$$

Important Aspect:

Power Series Allow us to approximate Complicated function by simpter Partial sums (= Polynomials)

Taylor Series:

How can we find power Series systematically?

Start with some function f. It must be infinitely many times differentiable to allow a power series representation

$$f(x) = C_0 + C_1(x-a) + C_2(x-a)^2 + C_3(x-a)^3 + \cdots$$

Can we determine the coefficients systematically?

Plug in the center a:

$$f(a) = C_0 + 0 + 0 + 0 + \cdots \Rightarrow \boxed{C_0 = f(a)}$$

$$f(x) = C_0 + C_1(x-a) + C_2(x-a)^2 + C_3(x-a)^3 + \cdots$$

Differentiate:

$$f'(x) = C_1 + 2C_2(x-a) + 3C_3(x-a)^2 + \cdots$$

Plug in a: $\boxed{f'(a) = C_1}$

Differentiate once more

$$f''(x) = 2C_2 + 3\cdot 2C_3(x-a) + 4\cdot 3C_4(x-a)^2 + \cdots$$

Plug in a; $f''(a) = 2C_2$

Differentiate once more

$$f'''(x) = 3\cdot 2C_3 + 4\cdot 3\cdot 2C_4(x-a) + 5\cdot 4\cdot 3C_5(x-a)^2 + \cdots$$

Plug in a; $\boxed{f'''(a) = 3\cdot 2C_3}$

Generally: $\boxed{C_n = \dfrac{f^{(n)}(a)}{n!}}$ Note $0! = 1$

Good will hunting! Movie.

Definition

The Tylor series of f about a is

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

A Tylor series about 0,

$$f(0) + f'(0)\cdot x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

is called Maclaurin Series

If a function f has a power series representation with center a, then it's given by the Tylor series.

Problem 1: One of the derivatives may not exist.

Prolem 2: Some Tylor series have radius of convergence 0.

Problem 3: The Tylor series may exist and may have positive raduis of convergence, but need not converge to the original function.

For example:

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$$

f(x) is infinitely many times differentiable at 0. The Taylor series has coefficients $\dfrac{f^{(n)}(a)}{n!} = 0$.

It is the constant zero function (with infinite radius of convergence) and differs from f at every $x \neq 0$.

Luckily, In engineering such examples never occur.
It is safe to use:

Engineering assumption:

If the tylor series exists and has positive radius of Convergence R, then

$$f(x) = f(a) + f'(a)\cdot(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \quad \text{for all } a-R<x<a+R$$

I.E. we assume the tylor series converges to the original function.

Example: $f(x) = e^x$, find Maclaurin Series:

$f(0) = e^0 = 1$ Since $\frac{f^{(n)}(a)}{n!} = e^a$

$f'(x) = e^x$

$f''(x) = e^x$ $\therefore$ all coefficients $\frac{f^{(n)}(0)}{n!} = \frac{e^0}{n!} = \frac{1}{n!}$

$\therefore$ Tylor series about 0 (Maclaurin series)

$$f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots \quad \text{for all } a-R<x<a+R$$

$$R<x<R$$

$$\Rightarrow 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{k=0}^{\infty}\frac{x^k}{k!}$$

Radius of convergence: Ratio Test:

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{x^{k+1}}{(k+1)!}\cdot\frac{k!}{x^k}\right| = \lim_{k\to\infty}\left|\frac{x\cdot\cancel{k!}}{\cancel{k!}(k+1)}\right| = \lim_{k\to\infty}\frac{|x|}{k+1} = 0<1 \quad \text{Regardless of } x$$

This series converges for all x, I.E. it's Radius of convergence is $R=\infty$.

By Engineering assumption we obtain:

Know by heart

$$\boxed{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty}\frac{x^k}{k!} \quad \text{for } -\infty<x<\infty}$$

In particular:

$$e = e^1 = \underbrace{\underbrace{1 + 1 + \frac{1}{2!}}_{2.5} + \frac{1}{3!}}_{2.666} + \frac{1}{4!} + \cdots \approx 2.71815$$

Euler's number.

Neater

$$e^{-1} = \frac{1}{e} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \cdots$$

Summands	Partiale sums
1	1
-1	0
$\frac{1}{2}$	$\frac{1}{2}$
$-\frac{1}{6}$	$\frac{1}{3}$

$\frac{1}{3} < \frac{1}{e} < \frac{1}{2} \Rightarrow 2 < e < 3$

continue till desired accuricy

Example:

$$\int_0^1 e^{-x^2} dx \qquad e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$= \int_0^1 \left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - + \cdots\right) dx$$

$$= \left[x - \frac{x^3}{3} + \frac{1}{5}\frac{x^5}{2!} - \frac{1}{7}\frac{x^7}{3!} + \cdots\right]_0^1$$

$$= \left[\left(x - \frac{1}{3\cdot 1} + \frac{1}{5\cdot 2} - \frac{1}{7\cdot 6} + \frac{1}{9\cdot 24} - + \cdots\right) - 0\right] = 1 - \frac{1}{3\cdot 1} + \frac{1}{5\cdot 2} - \frac{1}{7\cdot 6} + \frac{1}{9\cdot 24} - + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)n!}$$

Summands	Partial sums
1	1
$-\frac{1}{3}$	$\frac{2}{3}$
$+\frac{1}{10}$	$0.7666 \cdots$
$-\frac{1}{42}$	

$0.6666 < \int_0^1 e^{-x^2} dx < 0.7666$

Formulas:

Require Radian measurement

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad \text{for all } x$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad \text{for all } x$$

(Do not use them for large x; No problem, we only need them till $\frac{\pi}{4}$.)

Binomial series

$$(1+x)^K = 1 + \frac{K}{1}x + \frac{K(K-1)}{2\cdot 1}x^2 + \frac{K\cdot(K-1)(K-2)}{3\cdot 2\cdot 1}x^3 + \cdots \quad \text{for } -1<x<1$$

(for K = positive integer this is a finite sum)

$$\sqrt{1+x} = (1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{\frac{1}{2}\cdot(-\frac{1}{2})}{2\cdot 1}\cdot x^2 + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{3\cdot 2\cdot 1}x^3 + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{K=0}^{\infty} \frac{x^K}{K!} \quad \text{for } -\infty < x < \infty$$

$$\ln z = (z-1) - \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3} - \frac{(z-1)^4}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{(z-1)^n}{n} \quad \text{for } 0<z<2$$

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

Know by heart

Reminder: We use these series to prove Euler's formula:

$$e^{ix} = \cos x + i\sin x$$

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \cdots$$

Then seperate into real and imaginary part.

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

From section $\boxed{11.8}$ in the book

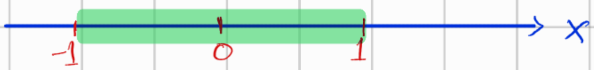
3) $\displaystyle\sum_{n=1}^{\infty} (-1)^n n x^n$

Ratio test $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)x^{n+1}}{n x^n}\right| = \lim_{n\to\infty}\left|\frac{\cancel{n}(1+\frac{1}{n})\cancel{x^n}\cdot x}{\cancel{n}\cancel{x^n}\cdot 1}\right| = \frac{1+0}{1}|x| = |x| < 1$

$-1 < x < 1$

The radius of convergence $R = 1$

Check boundaries



at $x=-1$ $\displaystyle\sum_{n=1}^{\infty} (-1)^n n(-1)^n = \sum_{n=1}^{\infty} n$

diverges by divergence Test, since $\displaystyle\lim_{n\to\infty} n = \infty$.

at $x = 1$ $\displaystyle\sum_{n=1}^{\infty} (-1)^n n (1)^n = \sum_{n=1}^{\infty} (-1)^n n$ // // // //

Then the interval of convergence for x is $(-1, 1)$

4) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\sqrt[3]{n}}$

Ratio Test $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)^{\frac{3}{2}}}\cdot\frac{n^{\frac{3}{2}}}{x^n}\right| = \lim_{n\to\infty}\left|\frac{x}{(1+\frac{1}{n})^{\frac{3}{2}}}\right| = \frac{|x|}{1+0} = |x| < 1$

$-1 < x < 1$

Radius of convergence $R = 1$

Check boundaries:

at $x=-1$ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ A convergent p series ($P = \frac{3}{2} > 1$)

at $x = 1$ $\displaystyle\sum \frac{(-1)^n}{n^{\frac{3}{2}}}$ $\displaystyle\left|\frac{(-1)^n}{n^{\frac{3}{2}}}\right| = \frac{1}{n^{\frac{3}{2}}}$ a convergent p-series ($P = \frac{3}{2} > 1$)

$\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\sqrt[3]{n}}$ Converges at the interval $x \in [-1, 1]$, with $R=1$

5) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{2n-1}$ Ratio Test $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{2n+1}\cdot\frac{2n-1}{x^n}\right| = \lim_{n\to\infty}\left|\frac{x(2-\frac{1}{n})}{(2+\frac{1}{n})}\right| = \frac{2-0}{2+0}|x| = |x| < 1$

By Ratio test, the series $\displaystyle\sum_{n=1}^{\infty}\frac{x^n}{2n-1}$ converges when $|x|<1$, So the Radius of convergence $R=1$

check end points

$\displaystyle\frac{1}{2n-1} > \frac{1}{2n}$ for all $n \geq 1$

at $x = 1$ $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2n-1}$

Since $\displaystyle\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}$ is a divergent harmonic series

$\sum_{n=1}^{\infty} \frac{1}{2n-1}$ diverges by divergence Test

at $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} = \sum_{n=1}^{\infty} b_n$ $\quad \lim_{n\to\infty} |b_n| = 0$, and $|b_n|$ is decreasing for $n > 1$

$\therefore \sum_{n=1}^{\infty} b_n$ converges by Alternating series Test.

Thus, The interval of convergence is $I = [-1, 1)$

6) $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n^2}$ Ratio test $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)^2} \cdot \frac{n^2}{x^n} \right| = \lim_{n\to\infty} \frac{|x|}{(1+\frac{1}{n})^2} = \frac{|x|}{(1+0)^2} = |x| < 1$

By Ratio test, the series $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n^2}$ converges when $|x| < 1$, so radius of convergence is $R = 1$

at end points:

for both $x = -1$ & $x = 1$ the absolute value of the series is a p-series with $(P = 2)$

Therefore, the interval of convergence $I = [-1, 1]$

7) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ Ratio test $\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n\to\infty} \frac{|x|}{|n+1|} = 0$

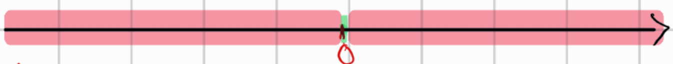
By Ratio test, the series $\sum_{n=0} \frac{x^n}{n!}$ converges regardless of the value of x

So, The radius of convergence $R = \infty$

Thus, The interval of convergence for x, $I = (-\infty, \infty)$

8) $\sum_{n=1}^{\infty} n^n x^n$ Ratio test $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{n^{n+1} x^{n+1}}{n^n x^n} \right| = \lim_{n\to\infty} |n x| = \infty \cdot |x|$

By Ratio test, the series $\sum_{n=1}^{\infty} n^n x^n$ diverges everywhere except for the center, So $R = 0$



Since the center $a = 0$; The interval of convergence $I = [a] \Rightarrow I = [0]$

9) $\sum_{n=1}^{\infty} (-1)^n \frac{n^2 x^n}{2^n}$ Ratio Test $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^2 x^{n+1}}{2^{n+1}} \cdot \frac{2^n}{n^2 \cdot x^n} \right| = \lim_{n\to\infty} \left| \frac{x(1+\frac{1}{n})^2}{2} \right| = \frac{|x|(1+0)^2}{2} = \frac{|x|}{2} < 1$

$|x| < 2$

By Ratio Test, the series $\sum_{n=1}^{\infty} (-1)^n \frac{n^2 x^n}{2^n}$ converges when $|x| < 2$, So $R = 2$

Checking the Boundaries

when $x = 2$ $\quad \sum_{n=1}^{\infty} (-1)^n \frac{n^2 \cancel{2^n}}{\cancel{2^n}}$

when $x = -2$ $\quad \sum_{n=1}^{\infty} \frac{n^2 \cancel{2^n}}{\cancel{2^n}}$

$\Big\}$ Both diverges by divergence test $\lim_{n\to\infty} a_n = \infty$

Thus, the interval of convergence $I = (-2, 2)$

10) $\sum_{n=1}^{\infty} \frac{10^n x^n}{n^3}$ Ratio test $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{10^{n+1}x^{n+1}}{(n+1)^3}\cdot\frac{n^3}{10^n x^n}\right| = \lim_{n\to\infty}\frac{|10x|}{(1+\frac{1}{n})^3} = |10x| < 1$

$|x| < \frac{1}{10}$

By Ratio Test, the series $\sum_{n=1}^{\infty}\frac{10^n x^n}{n^3}$ converges when $|x|<\frac{1}{10}$, so the radius of convergence $R=\frac{1}{10}$

at the end points

at $x=\frac{1}{10}$ $\sum_{n=1}^{\infty}\frac{\cancel{10^n}\cdot 1}{\cancel{10^n}n^3}$

at $x=-\frac{1}{10}$ $\sum_{n=1}^{\infty}(-1)^n\frac{\cancel{10^n}}{\cancel{10^n}n^3}$

Both absolute value of the series are convergent p-series (with $p=3>1$)

Thus, the interval of convergence $I=[-\frac{1}{10},\frac{1}{10}]$

11) $\sum_{n=1}^{\infty}\frac{(-3)^n}{n\sqrt{n}}x^n = \sum_{n=1}^{\infty}(-1)^n\frac{3^n\cdot x^n}{n^{\frac{3}{2}}}$ Ratio test $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{3^{n+1}|x^{n+1}|}{(n+1)^{3/2}}\cdot\frac{n^{3/2}}{3^n|x^n|} = \lim_{n\to\infty}\frac{3|x|}{(1+\frac{1}{n})^{3/2}} = 3|x| < 1$

$|x| < \frac{1}{3}$

By Ratio test the series $\sum_{n=1}^{\infty}\frac{(-3)^n x^n}{n^{3/2}}$ converges when $|x|<\frac{1}{3}$, so the radius of convergence $R=\frac{1}{3}$

Check end points

When $x=\frac{1}{3}$ $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{3/2}}$

When $x=-\frac{1}{3}$ $\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$

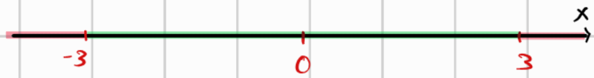
Both absolute values of the series are convergent p-series ($P=\frac{3}{2}>1$)

Thus the interval of convergence $I=[-\frac{1}{3},\frac{1}{3}]$

12) $\sum_{n=1}^{\infty}\frac{x^n}{n3^n}$ Ratio test $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)3^{n+1}}\cdot\frac{n3^n}{x^n}\right| = \lim_{n\to\infty}\frac{|x|}{3(1+\frac{1}{n})} = \frac{|x|}{3(1+0)} = \frac{|x|}{3} < 1 \Rightarrow |x|<3$

By Ratio test, the series $\sum_{n=1}^{\infty}\frac{x^n}{n3^n}$ converges when $|x|<3$, so radius of convergence $R=3$

check end points



when $x=3$ $\sum_{n=1}^{\infty}\frac{1}{n}$ A divergent harmonic series

when $x=-3$ $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ Alternating harmonic series $\to$ converges

Thus, the interval of convergence is $I=[-3,3)$

14) $\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}a_n$ Ratio test $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|x^{2n+3}|}{(2n+3)!}\cdot\frac{(2n+1)!}{|x^{2n+1}|} = \lim_{n\to\infty}\frac{|x^2|}{(2n+3)(2n+2)} = 0 < 1$

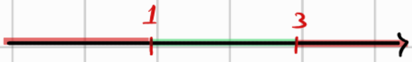
By Ratio Test, the series $\sum_{n=0}^{\infty}a_n$ converges regardless of x, so the radius of convergence $R=\infty$

Thus, the Interval of convergence $I=(-\infty,\infty)$.

15) $\sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2+1} = \sum_{n=0}^{\infty} a_n$ Ratio test $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x-2)^{\cancel{n+1}}}{(n+1)^2+1} \cdot \frac{n^2+1}{\cancel{(x-2)^n}}\right| = \lim_{n\to\infty} \frac{|x-2|(1+\cancel{\frac{1}{n^2}}^{0})}{(1+\cancel{\frac{1}{n}}_{0})^2+\cancel{\frac{1}{n^2}}_{0}} = |x-2| < 1$

By Ratio test, the series $\sum_{n=0}^{\infty} a_n$ converges when $|x-2|<1 \Rightarrow |x|<3$, so $R=1$

checking end points $1 < x < 3$



when $x=3$ $\sum_{n=0}^{\infty} \frac{1}{n^2+1}$ $\frac{1}{n^2+1} < \frac{1}{n^2}$ for all $n \geq 1$ since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series ($p=2>1$)

$\therefore \sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges by comparizon Test

when $x=1$ $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1}$ $\left|\frac{(-1)^n}{n^2+1}\right| < \frac{1}{n^2}$ for all $n \geq 1$ since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series ($p=2>1$)

Thus, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$ Converges by comparizon Test.

Therefor, the interval of convergence $I=[1, 3]$

16) $\sum_{n=0}^{\infty} (-1)^n \frac{(x-3)^n}{2n-1}$ Ratio Test $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|(x-3)^{n+1}|}{2n+1} \cdot \frac{2n-1}{|(x-3)^n|} = \lim_{n\to\infty} \frac{|x-3|(2-\cancel{\frac{1}{n}}^{0})}{(2+\cancel{\frac{1}{n}}_{0})} = |x-3| < 1$

$|x-a| < R$

By Ratio test, the series $\sum_{n=0}^{\infty} a_n$ Converges when $|x-3|<1$, so $R=1$

check end points $2<x<4$

when $x=4$ $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n-1} = \sum_{n=0}^{\infty} b_n$ since $\lim_{n\to\infty} |b_n| = 0$, & $|b_n|$ is decreasing for $n \geq 1$,

thus, the series $\sum_{n=1}^{\infty} b_n$ Converges by Alternating series test.

When $x=2$ $\sum_{n=0}^{\infty} \frac{1}{2n-1} = \sum_{n=0}^{\infty} C_n$ $\frac{1}{2n-1} > \frac{1}{2n}$ for $n>1$, since $\frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent harmonic series time a constant.

Thus, the series $\sum_{n=1}^{\infty} C_n$ diverges by Comparizon test.

Thus, The interval of convergence is $I=(2, 4]$

17) $\sum_{n=1}^{\infty} \frac{3^n (x+4)^n}{n^{1/2}} = \sum_{n=1}^{\infty} a_n$ Ratio test $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{3^{n+1}(x+4)^{n+1}}{(n+1)^{1/2}} \cdot \frac{n^{1/2}}{3^n (x+4)^n}\right| = \lim_{n\to\infty} \frac{3|x+4|}{(1+\frac{1}{n})^{1/2}} = 3|x+4| < 1 \Rightarrow |x+4| < \frac{1}{3}$

$-R < x+a < R \Rightarrow -\frac{13}{3} < x < -\frac{11}{3}$

By Ratio Test, The Series $\sum_{n=1}^{\infty} a_n$ Converges when $|x+4|<\frac{1}{3}$, So $R=\frac{1}{3}$

Checking end Points $-\frac{13}{3} < x < -\frac{11}{3}$

When $x=\frac{-11}{3}$ $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ Divergent p-series ($p=\frac{1}{2} \leq 1$)

When $x=-\frac{13}{3}$ $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/2}} = \sum_{n=1}^{\infty} b_n$ Since $\lim_{n\to\infty} b_n = 0$, & $|b_n|$ is decreasing for $n>1$;

Thus $\sum_{n=1}^{\infty} b_n$ converges by Alternating Series Test.

Therefor, the interval of convergence is $I=\left(-\frac{13}{3}, -\frac{11}{3}\right]$

18) $\sum_{n=1}^{\infty} \frac{n}{4^n}(x+1)^n = \sum_{n=1}^{\infty} a_n$ Ratio test $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)(x+1)^{n+1}}{4^{n+1}}\cdot\frac{4^n}{n(x+1)^n}\right| = \lim_{n\to\infty}\frac{|x+1|(1+\frac{1}{n})}{4} = \frac{1}{4}|x+1|<1 \Rightarrow |x+1|<4$

By Ratio test, the series $\sum_{n=1}^{\infty} a_n$ converges when $|x+1|<4$, so $R=4$

check end points $-5<x<3$

when $x=3$ $\sum_{n=1}^{\infty} n$ diverges by divergence Test since $\lim_{n\to\infty} n = \infty$

when $x=-5$ $\sum_{n=1}^{\infty}(-1)^n n$ since $\lim_{n\to\infty} n=\infty$, the series $\sum_{n=1}^{\infty}(-1)^n * n$ diverges by alternating series test.

Thus, The interval of convergence is $I=(-5,3)$

19) $\sum_{n=1}^{\infty}\frac{(x-2)^n}{n^n} = \sum_{n=1}^{\infty} a_n$ Ratio test $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{(x-2)^n}\right| = \lim_{n\to\infty}\frac{|x-2|}{(n+1)(1+\frac{1}{n})^n} = 0<1$

By Ratio test, the series $\sum_{n=1}^{\infty} a_n$ converges always regardless of x value, $R=\infty$

Thus, the interval of convergence $I=(-\infty,\infty)$

20) $\sum_{n=1}^{\infty}\frac{(2x-1)^n}{5^n\sqrt{n}} = \sum_{n=1}^{\infty} a_n$ Ratio test $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(2x-1)^{n+1}}{5^{n+1}(n+1)^{\frac{1}{2}}}\cdot\frac{5^n n^{\frac{1}{2}}}{(2x-1)^n}\right| = \lim_{n\to\infty}\frac{|2x-1|}{5(1+\frac{1}{n})^{\frac{1}{2}}} = |2x-1|<1 \Rightarrow |x-\frac{1}{2}|<\frac{1}{2}$

By Ratio test, the series $\sum_{n=1}^{\infty} a_n$ converges when $|x-\frac{1}{2}|<\frac{1}{2}$, so $R=\frac{1}{2}$

Check end points $0<x<1$

when $x=0$ $\sum_{n=1}^{\infty}\frac{(-1)^n}{5^n\sqrt{n}} = \sum_{n=1}^{\infty} C_n$, Since $\lim_{n\to\infty} C_n = 0$, & $|C_n|$ decreases for all $n\geq 1$, so $\sum_{n=1}^{\infty} C_n$ converges by alternating series test.

when $x=1$ $\sum_{n=1}^{\infty}\frac{1}{5^n\sqrt{n}} = \sum_{n=1}^{\infty} b_n$ $\lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty}\left|\frac{5^n n^{\frac{1}{2}}}{5^{n+1}(n+1)^{\frac{1}{2}}}\right| = \lim_{n\to\infty}\frac{1}{5(1+\frac{1}{n})^{\frac{1}{2}}} = \frac{1}{5(1+0)} = \frac{1}{5}<1$

Thus $\sum_{n=1}^{\infty} b_n$ converges by ratio test.

Thus, the interval of convergence is $I=[0,1]$

From section $\boxed{11.9}$ in the book

25) $\int\frac{t}{1-t^8}\,dt$ $\qquad t\cdot\frac{1}{1-t^8} = t\sum_{n=0}^{\infty}(t^8)^n$ for $|t^8|<1$

$$= \sum_{n=0}^{\infty} t^{8n+1}$$

$$\int t\cdot\frac{1}{1-t^8}\,dt = \int\sum_{n=0}^{\infty} t^{8n+1}\,dt$$

$$= C+\sum_{n=0}^{\infty}\frac{t^{8n+2}}{8n+2} \quad \text{for all } |t|<1$$

Thus, radius of convergence $R=1$

26) $\int\frac{t}{1+t^3}\,dt$ $\qquad t\cdot\frac{1}{1-(-t^3)} = t\sum_{n=0}^{\infty}(-t^3)^n = \sum_{n=0}^{\infty}(-1)^n t^{3n+1}$ for $|-t^3|<1 \Longrightarrow |t|<1$

$\int t\cdot\frac{1}{1-(-t^3)}\,dt = \int\sum_{n=0}^{\infty}(-1)^n t^{3n+1}\,dt = \sum_{n=0}^{\infty}\left[(-1)^n\cdot\int t^{3n+1}\,dt\right] = C+\sum_{n=0}^{\infty}(-1)^n\frac{t^{3n+2}}{3n+2}$ for $|t|<1=R$

Thus, $\int\frac{t}{1+t^3}\,dt = C+\sum_{n=0}^{\infty}(-1)^n\frac{t^{3n+2}}{3n+2}$ with a radius of convergence $R=1$

27) $\int x^2 \ln(1+x)\,dx$

$$\ln(z) = (z-1) - \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3} - + \cdots$$

$$\therefore \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

$$x^2 \ln(1+x) = x^3 - \frac{x^4}{2} + \frac{x^5}{3} - + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{n+2}}{n}$$

$$\int x^2 \ln(1+x)\,dx = \int \left(\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{n+2}}{n}\right) dx = \sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}}{n} \cdot \int x^{n+2}\,dx\right] = C + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{n+3}}{n(n+3)}$$

Ratio test $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \frac{|x^{n+4}|}{(n+1)(n+4)} \cdot \frac{n(n+3)}{|x^{n+3}|} = \lim\limits_{n\to\infty} \frac{|x|\,(1+\frac{3}{n})}{(1+\frac{1}{n})(1+\frac{4}{n})} = |x| < 1$

By Ratio test, the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{n+3}}{n(n+3)}$ converges only when $|x|<1$, so the radius of convergence $R=1$

& the interval of convergence $I=[-1,1]$ where the end points are included because in both cases

The absolute value of the series will be convergent by comparizon test with $\frac{1}{n^2}$ which is also convergent p-series ($P=2>1$)

28) $\int \frac{\tan^{-1}x}{x}\,dx$

$$\frac{d \tan^{-1}x}{dx} = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n \quad \text{for } |x^2|<1$$

$$\therefore \tan^{-1}x = \int \sum_{n=0}^{\infty} (-1)^n x^{2n}\,dx = \sum_{n=0}^{\infty}\left[(-1)^n \cdot \int x^{2n}\,dx\right] = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$$

Plug in $x=0$

$\tan^{-1}(0) = C + 0 + 0 + 0 + \cdots \Rightarrow C = 0$

$$\frac{\tan^{-1}x}{x} = \frac{\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n}$$

$$\int \frac{\tan^{-1}x}{x}\,dx = \int \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n}}{2n+1}\,dx = \sum_{n=0}^{\infty}\left[\frac{(-1)^n}{2n+1} \cdot \int x^{2n}\,dx\right] = C + \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)^2}$$

Ratio test with $a_n = (-1)^n \frac{x^{2n+1}}{(2n+1)^2}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|x^{2n+3}|}{(2n+3)^2} \cdot \frac{(2n+1)^2}{|x^{2n+1}|} = \lim_{n\to\infty} \frac{|x^2| \cdot (2+\frac{1}{n})^2}{(2+\frac{3}{n})^2} = \frac{x^2 (2+0)^2}{(2+0)^2} = x^2 < 1 \Rightarrow -1 < x < 1$$

Thus $\int \frac{\tan^{-1}x}{x}\,dx = C + \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)^2}$, and the radius of convergence is 1 ($R=1$)

29) $\int_0^{0.2} \frac{1}{1+x^5}\,dx$ (use power series to approximate the series)

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots$$

$$\frac{1}{1-(-x^5)} = 1 - x^5 + x^{10} - x^{15} + x^{20} - + \cdots$$

$$\int_0^{0.2} \frac{1}{1-(-x^5)}\,dx = \int_0^{0.2} (1 - x^5 + x^{10} - x^{15} + \cdots)\,dx$$

$$= \left[x - \frac{x^6}{6} + \frac{x^{11}}{11} - \frac{x^{16}}{16} + \cdots\right]_0^{0.2} = \left[(0.2 - \frac{(0.2)^6}{6} + \frac{(0.2)^{11}}{11} - \frac{(0.2)^{16}}{16} + \cdots) - 0\right]$$

Summands	Partial sums
$0.2 = \frac{2}{10}$	0.2
$\frac{-\frac{64}{10^6}}{6} = -0.0000106\overline{6}$	0.1999893
$\frac{\frac{2048}{10^{11}}}{11} \simeq 0.2 \times 10^{-8}$	$\simeq 0.1999893352$

$$0.1999893 < \int_0^{0.2} \frac{1}{1+x^5}\,dx < 0.1999893352$$

$$\int_0^{0.2} \frac{1}{1+x^5}\,dx \simeq 0.1999893$$

30) $\int_0^{0.4} \ln(1+x^4)\,dx$

$$\ln z = (z-1) - \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3} - \frac{(z-1)^4}{4} + \cdots$$

$$\ln(1+x^4) = x^4 - \frac{(x^4)^2}{2} + \frac{(x^4)^3}{3} - \frac{(x^4)^4}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{4n}}{n}$$

$$\int_0^{0.4} \ln(1+x^4)\,dx = \int_0^{0.4} \sum_{n=1}^{\infty} (-1)^n \frac{x^{4n}}{n}\,dx = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \int_0^{0.4} x^{4n}\,dx = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left[\frac{x^{4n+1}}{4n+1}\right]_0^{0.4} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}\left[\frac{(0.4)^{4n+1}}{4n+1} - 0\right]$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n (0.4)^{4n+1}}{n(4n+1)} = \sum_{n=1}^{\infty} (-1)^n b_n$$

Since $\lim_{n\to\infty} b_n = 0$, & b_n is decreasing for $n > 1$

The series $\sum_{n=1}^{\infty} (-1)^n b_n$ Converges by Alternating series Test.

31) $\int_0^{0.1} x \arctan(3x)\,dx$

$$\tan(z) = \sum_{n=1}^{\infty} (-1)^n \frac{z^{2n+1}}{2n+1} \Rightarrow \tan(3x) = \sum_{n=1}^{\infty} (-1)^n \frac{(3x)^{2n+1}}{2n+1}$$

$$x\tan(3x) = 3\sum_{n=1}^{\infty} (-1)^n \frac{3^{2n} x^{2(n+1)}}{2n+1}$$

$$\Rightarrow 3\int_0^{0.1} \sum_{n=1}^{\infty} (-1)^n \frac{3^{2n} x^{2n+2}}{2n+1}\,dx = 3\sum_{n=1}^{\infty} \frac{(-1)^n 3^{2n}}{2n+1} \int_0^{1} x^{2n+2}\,dx = 3\sum_{n=1}^{\infty} \frac{(-1)^n (3)^{2n}}{2n+1} \left[\frac{x^{2n+3}}{2n+3}\right]_0^{0.1} = \sum_{n=1}^{\infty} \frac{(-1)^n (3)^{2n+1} (0.1)^{2n+3}}{(2n+1)(2n+3)}$$

This series diverges by divergence Test

$$0.01\sum_{n=1}^{\infty} \frac{(-1)^n (0.3)^{2n+1}}{(2n+1)(2n+3)}$$

32) $\int_0^{0.3} \frac{x^2}{1+x^4}\,dx$

$$\frac{1}{1-(-x^4)} = \sum_{n=0}^{\infty} (-1)^n (x^4)^n \Rightarrow \frac{x^2}{1-(-x^4)} = \sum_{n=0}^{\infty} (-1)^n x^{4n+2}$$

$$\Rightarrow \int_0^{0.3} \sum_{n=0}^{\infty} (-1)^n x^{4n+2}\,dx = \sum_{n=0}^{\infty} (-1)^n \left[\frac{x^{4n+3}}{4n+3}\right]_0^{0.3} = \sum_{n=0}^{\infty} (-1)^n \frac{(0.3)^{4n+3}}{4n+3} = \sum_{n=0}^{\infty} (-1)^n b_n$$

Since $\lim_{n\to\infty} b_n = 0$, & b_n is decreasing for $n > 1$, then the series $\sum_{n=0}^{\infty} (-1)^n b_n$ Converges by Alternating series Test.

$\boxed{\text{* Important: } \lim_{n\to\infty} \frac{x^n}{n!} = 0 \text{ for every real number } x}$

$$\lim_{n \to 0} \frac{1 - \cos x}{1 + x - e^{x}}$$

$$\lim_{n \to \infty} \frac{1 - \cos x}{1 + x - e^{x}}$$

$$\int x^{2} \ln(1 + x)\, dx$$