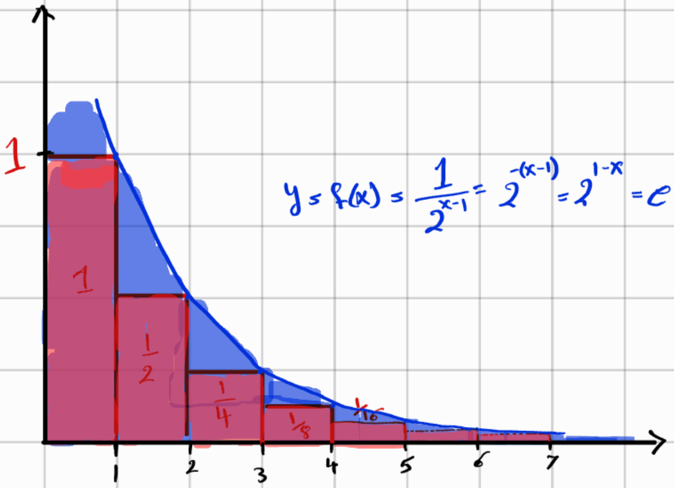
Series:

Example: $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots\cdots=2$



$y=f(x)=\frac{1}{2^{x-1}}=2^{-(x-1)}=2^{1-x}=e^{\ln(2^{1-x})}=e^{(1-x)\cdot\ln 2}$

If the blue area (the improper integral) is finite,
then the red area is finite as well.

Area under blue curve:

$$\int_0^\infty e^{(1-x)\ln 2}\,dx=\lim_{b\to\infty}\int_0^b e^{(1-x)\ln 2}\,dx=\lim_{b\to\infty}\left[\frac{-1}{\ln 2}e^{(1-x)\ln 2}\right]_0^b$$

$$\lim_{b\to\infty}\left[\frac{-1}{\ln 2}e^{(1-b)\ln 2}+\frac{1}{\ln 2}e^{\ln 2}\right]=\frac{-1}{\ln 2}\cdot\underbrace{\cancel{\lim_{b\to\infty}e^{(1-b)\ln 2}}}_{0}+\frac{2}{\ln 2}$$

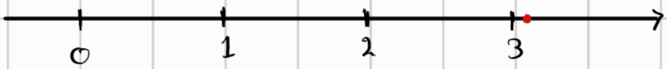
$$\Longrightarrow\int_0^\infty e^{(1-x)\ln 2}\,dx=\frac{2}{\ln(2)}\approx 2.885$$

$\therefore$ The red area is finit as well & less than $\frac{2}{\ln(2)}$

Another Example

$$\frac{(3)^{0.5+0.5(-1)^{x-1}}+1(0.5+0.5(-1)^{x-1})}{10^{x-1}}$$

$\pi=3.14159\ldots\ldots=3+\frac{1}{10}+\frac{4}{100}+\frac{1}{1000}+\frac{5}{10000}+\cdots\cdots$



IT is possible to add infinitely many numbers & still obtain a finite sum.

Definition:

$a_1+a_2+a_3+\cdots\cdots=\sum_{K=1}^{\infty}a_K$ is called an ifinite series.

$$\underbrace{\sum_{K=1}^{\infty}a_K}_{\text{Infinite Series}}=\underbrace{\underbrace{\lim_{n\to\infty}}_{\text{Limit Process}}\underbrace{\sum_{K=1}^{n}a_K}_{\text{Finite sum}}}$$

(we check a finite sums, & see what happens to those sums when we let the numbers of summands go to infinity

$\Rightarrow \lim_{n\to\infty}\underbrace{(a_1+a_2+a_3\cdots a_n)}_{\text{Partial Sum}}$

If this limit exists (as a number) then the infinite series Converges. Otherwise the series Diverges.

In case the series converges, this limit is called the sum of the series.

It is also written as $\sum_{k=1}^{\infty} a_k$.

The sum of a Series is the limit of it's partial sums.

Back to the introductory example:

$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots$

Partial sums : $S_n = \frac{2^{n+1}-1}{2^n}$

Sequence of summands $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \cdots$

list

	Summands	Partial sums	
0	1	1	$= 1$
1	$\frac{1}{2}=0.5$	$1+0.5=1.5$	$=\frac{3}{2}$
2	$\frac{1}{4}=0.25$	$1.5+0.25=1.75$	$=\frac{7}{4}$
3	$\frac{1}{8}=0.125$	$1.75+0.125=1.875$	$=\frac{15}{8}$
4	$\frac{1}{16}=0.0625$	$1.875+0.0625=1.9375$	$=\frac{31}{16}$
$\vdots$	$\vdots$	$\vdots$	
		2	

Very useful Table

$S_0 = \frac{2-1}{1} = 1$

$S_1 = \frac{4-1}{2} = \frac{3}{2}$

$S_2 = \frac{8-1}{4} = \frac{7}{4}$

$\vdots$

limit of partial sums:

$$\lim_{n\to\infty} \frac{2^{n+1}-1}{2^n} = \lim_{n\to\infty} \frac{\cancel{2^n}\cdot 2^1}{\cancel{2^n}} - \lim_{n\to\infty} \frac{1}{2^n} = 2 - 0 = 2$$

Divergence Test for series:

$1, 1, 1, 1, 1, \cdots$ sequence

$1+1+1+1+1+\cdots$ Series

$\lim_{n\to\infty} a_n = 0 \Longrightarrow$ series may converge

$\hookrightarrow$ sequence (summands)

Telescoping series:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \cdots$$

	Summands	Partial sums
0	$\frac{1}{2} = 0.5$	$\frac{1}{2} = 0.5$
1	$\frac{1}{6}$	$\frac{2}{3}$
2	$\frac{1}{12}$	$\frac{3}{4}$
3	$\frac{1}{20}$	$\frac{4}{5}$
4	$\frac{1}{30}$	$\frac{5}{6}$
$\vdots$	$\vdots$ 0	$\vdots$

It may converge

Educated geuss: partial sums $S_n = \frac{n}{n+1}$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{\cancel{n}\cdot 1}{\cancel{n}(1+\frac{1}{n})} = \frac{1}{1+0} = \boxed{1}$$

Limit of the partial sums is the limit of the series.
Thus, if the geuss is correct, the series converges.

Let's confirm our geuss about the partial sums:

$$\boxed{\sum_{n=1}^{K} \frac{1}{n(n+1)} = \frac{K}{K+1}}$$

Integration is similar to summation, so we try partial fration decomposition

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} \Longrightarrow 1 = A(n+1) + B\cdot n$$

by comparing coefficients: $A = 1,\ B = -1$

$$\therefore \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

Telescoping

$$\therefore \sum_{n=1}^{K} \frac{1}{n(n+1)} = \sum_{n=1}^{K}\left(\frac{1}{n} - \frac{1}{(n+1)}\right) = \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{5}}\right) \cdots$$

$$+ \left(\cancel{\frac{1}{K}} - \frac{1}{K+1}\right)$$

$$= 1 - \frac{1}{K+1} = \frac{K+1}{K+1} - \frac{1}{K+1} = \boxed{\frac{K}{K+1}}$$

Since the educated geuss of the partial sums is correct, & its limit is 1, then the series converges.

Geometric series:

Introductory Example: $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$

$\cdot\frac{1}{2}$ $\quad \cdot\frac{1}{2}$ $\quad \cdot\frac{1}{2}$

here it's $\frac{1}{2}$

The ratio of neighboring summands is constant: $a_{n+1} = q \cdot a_n$

Definition:

A Geometric series has the form:

$$a + a\cdot q + a\cdot q^2 + a\cdot q^3 + \cdots = \sum_{n=0}^{\infty} a\cdot q^n$$

Simpler: $a\underbrace{(1+q+q^2+q^3+\cdots)}_{\text{Always start with 1 by factoring } a}$

$1+q+q^2+q^3+\cdots\cdots$

n	Summands	partial sums
	1	1
	q	$1+q$
	q^2	$1+q+q^2$
	q^3	$1+q+q^2+q^3$
	q^4	$1+q+q^2+q^3+q^4$

$\vdots$

$\longrightarrow 0$

To have a chance to converge

We need $|q|<1$

$\therefore\ -1<q<1$

The geometric series Diverges For $q<-1$ or $q>+1$

Because in these cases:

$\lim_{n\to\infty} q^n \neq 0$

Do we get convergence for $|q|<1$?

Examine the partial sums:

$S_n = 1+q+q^2+q^3+\cdots+q^{n-1}+q^n$

$q\cdot S_n = q+q^2+q^3+q^4+\cdots\cdots+q^{n-1}+q^n+q^{n+1}$

$S-q\cdot S_n = 1-q^{n+1} \Longrightarrow S_n(1-q)=1-q^{n+1}$

$\Longrightarrow \boxed{S_n = \frac{1-q^{n+1}}{1-q}}\ , q\neq 1$

Know by Heart

Finite Geometric sum: $1+q+q^2+q^3+\cdots+q^n = \begin{cases} \frac{1-q^{n+1}}{1-q} & , q\neq 1 \\ n+1 & , q=1 \end{cases}$

With this formula for the partial sums, we get:

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1-q^{n+1}}{1-q} = \frac{1-\lim_{n\to\infty} q^{n+1}}{1-q} = \frac{1}{1-q}$

$\to 0$ When $|q|<1$

By Heart

For Geometric series, IF $|q|<1 \longrightarrow$ then it converges

$1+q+q^2+q^3+q^4+q^5+\cdots\cdots = \frac{1}{1-q}\ , \ |q|<1.$

& Diverges For $|q|\geq 1$

Examples:

(i) First example we took: $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots$ is A Geometric series

With $q=\frac{1}{2}$, so it converges to $\frac{1}{1-q}=\frac{1}{1-\frac{1}{2}}=2$

(ii) $0.343434\ldots = \frac{34}{100}+\frac{34}{10000}+\frac{34}{1000000}+\cdots$

$= \frac{34}{100}\left(1+\frac{1}{100}+\frac{1}{10000}+\frac{1}{1000000}+\cdots\right)$

since $q=\frac{1}{100}<1$, $\Rightarrow \frac{34}{100}\cdot\frac{1}{1-q}=\frac{34}{100}\cdot\frac{1}{1-\frac{1}{100}}=\frac{34}{100-1}=\frac{34}{99}$

(iii) $5-\frac{10}{3}+\frac{20}{9}-\frac{40}{27}+\frac{80}{81}-\frac{160}{243}+\cdots$

$5\left(1-\frac{2}{3}+\frac{4}{9}-\frac{8}{27}+\frac{16}{81}-\frac{32}{243}+\cdots\right)=\sum_{n=0}^{\infty} a\cdot q^n = a\cdot\frac{1}{1-q}=\sum_{n=0}^{\infty}5\left(-\frac{2}{3}\right)^n=\frac{5}{1-\left(-\frac{2}{3}\right)}=3$

(iv) $\sum_{n=1}^{\infty} 2^{2n}\cdot 3^{(1-n)} = \sum_{n=1}^{\infty} 4^n\cdot 3^{-n}\cdot 3 = \sum_{n=1}^{\infty} 3\cdot\left(\frac{4}{3}\right)^n = \sum_{n=0}^{\infty} 3\cdot\left(\frac{4}{3}\right)^{n+1}$

$\Longrightarrow \sum_{n=0}^{\infty} \cancel{3}\cdot\frac{4}{\cancel{3}}\left(\frac{4}{3}\right)^n$, Geometric series with $a=4$ & $q=\frac{4}{3}>1$

$\Longrightarrow$ The series Diverges.

Alternating series:

$\underbrace{1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots}_{\text{Alternating Harmonic series}}$

Pattern $+-+-+-+-$ or $-+-+-+-+$ "Alternating"

Alternating series Test:

If the Alternating series

$b_1-b_2+b_3-b_4+b_5-+\cdots$

Satisfies:

(i) $\lim_{n\to\infty} b_n = 0$ (Summands must approach 0)

(ii) $b_1 \geq b_2 \geq b_3 \geq b_4 \geq b_5 \geq \cdots \geq 0$

then it converges

Moreover, the partial sums provide upper & lower bounds for the sum.

Eample:

$$\boxed{\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\cdots \qquad \text{For all } x \in R}$$

(But only efficient when x close to 0) Requires Radians!

Let's do $x = 1$ ($\approx 60°$)

$$\cos 1 = 1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \cdots\cdots = 1 - \frac{1}{2} + \frac{1}{24} - \frac{1}{720}$$

Summands	Partial sums
1	1
$-\frac{1}{2}$	$\frac{1}{2}$
$+\frac{1}{24}$	$0.541\overline{6}$
$-\frac{1}{720}$	$0.5402\overline{7}$

$\Rightarrow \frac{1}{2} < \cos 1 < 1$

$0.5 < \cos 1 < 0.541\overline{6}$

$0.541\overline{6} < \cos 1 < 0.5402\overline{7}$

Playing around with series:

$$\frac{1}{19} \approx \frac{1}{20} = \frac{5}{100} = 0.05$$

$\rightarrow$ geometric series

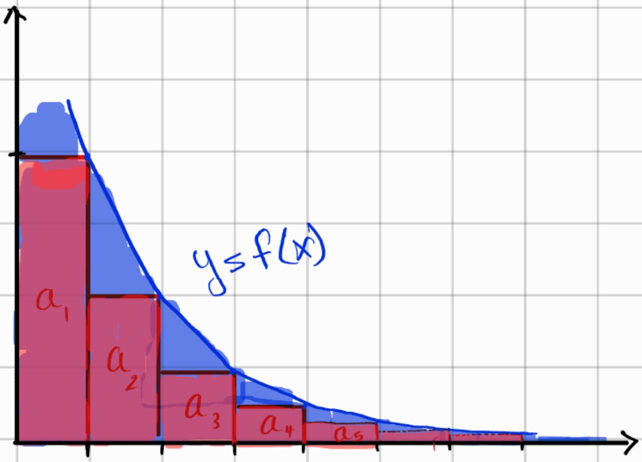
$$\frac{1}{19} = \frac{1}{20-1} = \frac{1}{20\left(1-\frac{1}{20}\right)} = \frac{1}{20}\left(1 + \left(\frac{1}{20}\right) + \left(\frac{1}{20}\right)^2 + \left(\frac{1}{20}\right)^3 + \cdots\right)$$

$$\Rightarrow 0.05\left(1 + 0.05 + (0.05)^2 + (0.05)^3 + \cdots\right) = 0.05 + 0.0025 + 0.000125 + \cdots$$

$$\approx 0.0526\ldots\ldots$$

Often we use about three summands to obtain a good aproximation.

Integral Test compares series with improper integrals.



Improper integral

$$\int_a^\infty f(x)\,dx$$

non-negative Function, $f(x) \geq 0$

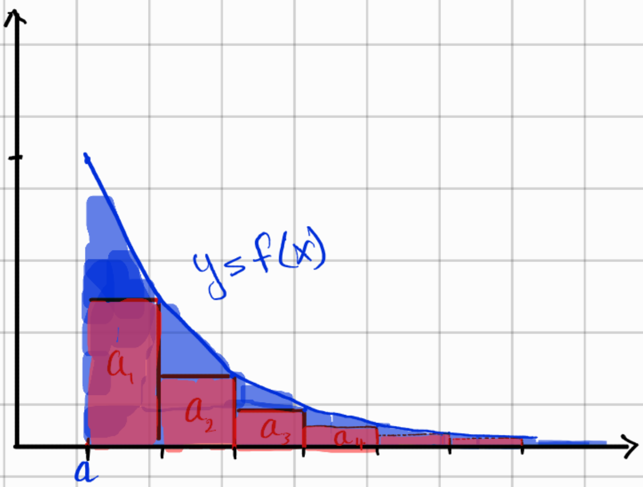
Converges if Area is finite

$$a_1 + a_2 + a_3 + a_4 + \cdots = \sum_{n=1}^{\infty} a_n$$

needs non-negative summands, $a_n \geq 0$

Converges if Area is finite

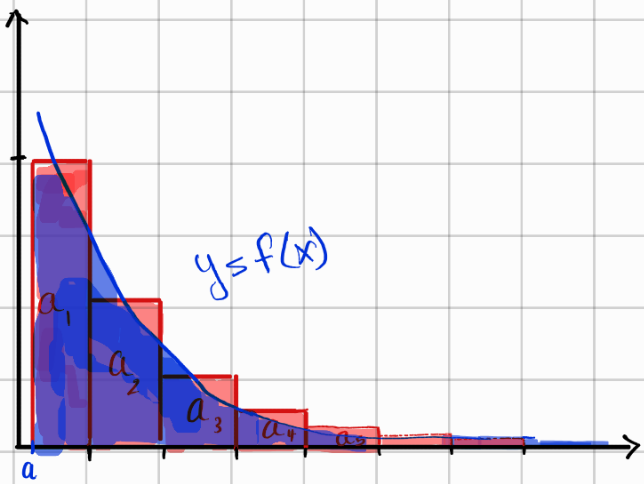
Two possible comparisons



Rectangles are inside the function area from some a on.

If the blue area is finite, then the Red area is finite too.

$$\sum a_n \leq \int_a^\infty f(x)\,dx < \infty$$



Function Area is inside the rectangles From a on.

If the blue area is infinite, then the red area is infinite too.

$$\sum a_n \geq \int_a^\infty f(x)\,dx = \infty$$

Let's consider P-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$$

Consider the improper integral:

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^p}\,dx \quad \text{if } p\neq 1 \rightarrow \lim_{b\to\infty} \left[\frac{1}{1-p}\cdot x^{1-p}\right]_1^b$$

$$\lim_{b\to\infty}\left[\frac{1}{1-p}\left(\underbrace{b^{1-p}}_{0 \text{ if } p>0} - \cancel{1^{1-p}}^{\,1}\right)\right] = \frac{-1}{1-p} = \frac{1}{p-1} \qquad \text{Converges if } p>1$$

$$\boxed{\text{Hence } \sum_{n=1}^{\infty}\frac{1}{n^p} \text{ converges if } p>1}$$

Case: $p=1$

$$\int_1^{\infty}\frac{1}{x}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty}\left[\ln|x|\right]_1^b = \lim_{b\to\infty} \ln|b| = +\infty \quad \text{Diverges}$$

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots \geq \int_1^{\infty}\frac{1}{x}\,dx = \infty$$

We see again why harmonic series diverges!

Case: $0<p<1$ (roots)

$$\boxed{\text{P-series } \sum_{n=1}^{\infty}\frac{1}{n^p} \text{ Diverges for } p<1}$$ $\Rightarrow$ same reason of the last case using integral test.

For $p\leq 0$ Test for Divergence applies.

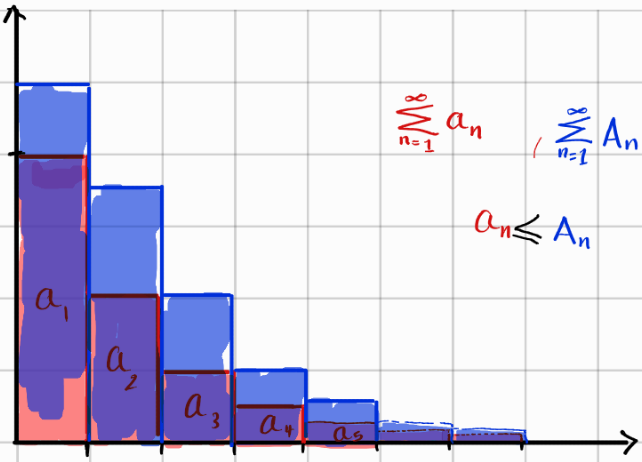
Comparison Test:

Suppose that $\sum a_n$ & $\sum b_n$ are series with positive terms.

(i) If $\sum b_n$ is convergent & $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.

sufficient: for all $n \geq N$

(ii) If $\sum b_n$ is Divergent & $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent.



$\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} A_n$

$a_n \leqslant A_n$

(i) If the blue area is finite, so is the red one. If $\sum_{n=1}^{\infty} A_n$ Converges, so does $\sum_{n=1}^{\infty} a_n$

(ii) If the red area is infinite, so is the blue one.

If $\sum_{n=1}^{\infty} a_n$ diverges, so does $\sum_{n=1}^{\infty} A_n$.

* If the Blue area is infinite, that doesn't tell us anything about the red area.

* The same if the red are is finite.

We often use A p-series or A Geometric series for Comparison.

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \begin{array}{l} \text{Converges if } p>1 \\ \text{Diverges if } p \leqslant 1 \end{array}}$$

$$\boxed{1+q+q^2+q^3+\cdots \quad \begin{array}{l} \text{Converges to } \frac{1}{1-q} \text{ if } |q|<1 \\ \text{Diverges if } |q| \geqslant 1 \end{array}}$$

Example: $\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$ (We see some p-series in this fourmula)

$\sum_{n=1}^{\infty} \frac{5}{2n^2} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$ which converges.

& we have $\frac{5}{2n^2+4n+3} < \frac{5}{2n^2}$

By comparison, $\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$ converges as well.

* $\sum_{K=1}^{\infty} \frac{\ln K}{K}$

Summands	Partial sums
$\frac{\ln 1}{1} = 0$	0
$\frac{\ln 2}{2} = \frac{0.69\ldots}{2}$	0.34...
$\frac{\ln 3}{3} = \frac{1.098\ldots}{3}$	0.712....
$\frac{\ln 4}{4} = \frac{1.38\ldots}{4}$	....

Harmonic series

$\sum_{n=1}^{\infty} \frac{1}{K}$ Diverges

(also p-series with $p \leqslant 1$)

$\frac{1}{K} < \frac{\ln K}{K}$ For all $K \geqslant 3$

By comparison, $\sum_{K=1}^{\infty} \frac{\ln K}{K}$ diverges as well.

$$* \sum_{n=1}^{\infty} \frac{1}{2^n-1} = \frac{1}{1} + \frac{1}{3} + \frac{1}{7} + \frac{1}{15} + \frac{1}{31} + \cdots\cdots$$

Geometric series $\sum_{n=1}^{\infty} 2\left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{2}{2^n}$ since $|q|<1$ it converges

$$\frac{1}{2^n-1} \leq \frac{2}{2^n} \quad \text{for all } n$$

By comparison, $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ converges as well

The limit comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

If $\lim_{n\to\infty} \frac{a_n}{b_n} = C$, where C is a finite number & $C>0$ [i.e. $C \neq 0$]

Then either both series converges or both diverges.

again with $\sum_{n=1}^{\infty} \frac{1}{2^n-1} = \sum_{n=1}^{\infty} a_n$

& $\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} b_n$ which converges.

Try $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{2^n-1} \cdot \frac{2^n}{1} = \lim_{n\to\infty} \frac{2^n}{2^n-1} = \lim_{n\to\infty} \frac{\cancel{2^n}\cdot 1}{\cancel{2^n}(1-\underbrace{\tfrac{1}{2^n}}_{0})} = 1 > 0$

By limit comparizon, $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ converges.

$$* \sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}} = \sum_{n=1}^{\infty} a_n$$

Consider the dominating terms in numerator and denominator

$\sum_{n=1}^{\infty} \frac{2n^2}{\sqrt{n^5}} = \sum_{n=1}^{\infty} 2n^{2} \cdot n^{-\frac{5}{2}} = \sum_{n=1}^{\infty} \frac{2}{n^{\frac{1}{2}}} = \sum_{n=1}^{\infty} b_n$ P-series diverges because $P \leq 1$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^2+3n}{\sqrt{5+n^5}} \cdot \frac{\sqrt{n^5}}{2n^2} = \lim_{n\to\infty} \frac{(1+\overbrace{\tfrac{3}{2n}}^{0})\cdot \cancel{n^{\frac{5}{2}}}}{\cancel{n^{\frac{5}{2}}}\sqrt{\underbrace{\tfrac{5}{n^5}}_{0}+1}} = \frac{1}{\sqrt{1}} = 1$$

By limit comparizon test, $\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}$ Diverges.

From the book, section 11.2

17) $3-4+\frac{16}{3}-\frac{64}{9}+\cdots = \sum_{n=0}^{\infty} 3\cdot\left(-\frac{4}{3}\right)^n$

$\cdot\frac{-4}{3}\quad \cdot\frac{-4}{3}\quad \cdot\frac{-4}{3}$

Diverges because $\left|-\frac{4}{3}\right| \geq 1$

18) $\frac{1}{8}-\frac{1}{4}+\frac{1}{2}-1+\cdots = \sum_{n=0}^{\infty} \frac{1}{8}(-2)^n$

$(-2)\quad(-2)\quad(-2)$

Diverges because $|-2| \geq 1$

19) $10-2+\frac{4}{10}-\frac{8}{100}+\cdots = \sum_{n=0}^{\infty} 10\left(-\frac{2}{10}\right)^n = 10\cdot\frac{1}{1-\left(\frac{-2}{10}\right)} = \frac{100}{12} = \frac{25}{3}$

$\frac{-2}{10}\quad\frac{-2}{10}\quad\frac{-2}{10}$

Converges because $\left|\frac{-2}{10}\right| < 1$

20) $1+\frac{4}{10}+\frac{16}{100}+\frac{64}{1000}+\cdots = \sum_{n=0}^{\infty}\left(\frac{4}{10}\right)^n = \frac{1}{1-\frac{2}{5}} = \frac{5}{3}$

$\frac{4}{10}\quad\frac{4}{10}\quad\frac{4}{10}$

Converges because $\left|\frac{4}{10}\right| < 1$

21) $\sum_{n=1}^{\infty} 6(0.9)^{n-1} = 6\cdot\frac{1}{1-\frac{9}{10}} = 60$ Converges because $\frac{9}{10} < 1$

22) $\sum_{n=1}^{\infty}\frac{10^n}{(-9)^{n-1}} = \sum_{n=1}^{\infty}\frac{10^n}{(-9)^n}\cdot\frac{1}{(-9)^{-1}} = \sum_{n=1}^{\infty} -9\left(-\frac{10}{9}\right)^n = \sum_{n=1}^{\infty}(-9)\left(-\frac{10}{9}\right)\left(-\frac{10}{9}\right)^{n-1}$

Diverges because $\left|-\frac{10}{9}\right| \geq 1$

23) $\sum_{n=1}^{\infty}\frac{(-3)^{n-1}}{4^n} = \sum_{n=1}^{\infty}\cancel{(-3)^{-1}}\left(\frac{\cancel{-3}}{4}\right)\left(\frac{-3}{4}\right)^{n-1} = \sum_{n=1}^{\infty}\frac{1}{4}\left(\frac{-3}{4}\right)^{n-1} = \frac{1}{4}\cdot\frac{1}{1+\frac{3}{4}} = \frac{1}{7}$ Converges because $\left|\frac{-3}{4}\right| < 1$

24) $\sum_{n=0}^{\infty}\frac{1}{(\sqrt{2})^n} = \sum_{n=0}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^n = \frac{1}{1-\sqrt{2}} = \frac{1+\sqrt{2}}{1-2} = -(1+\sqrt{2})$ Converges because $\left|\frac{1}{\sqrt{2}}\right| < 1$

25) $\sum_{n=0}^{\infty}\frac{\pi^n}{3^{n+1}} = \sum_{n=0}^{\infty}\frac{1}{3}\left(\frac{\pi}{3}\right)^n$ Diverges because $\left|\frac{\pi}{3}\right| \geq 1$

26) $\sum_{n=1}^{\infty}\frac{e^n}{3^{n-1}} = \sum_{n=1}^{\infty}\frac{1}{\cancel{(3)^{n}}}\cdot\frac{e}{\cancel{3}}\left(\frac{e}{3}\right)^{n-1} = \frac{e}{1-\frac{e}{3}} = \frac{3e}{3-e}$ Converges because $\left|\frac{e}{3}\right| < 1$

27) $\frac{1}{3}+\frac{1}{6}+\frac{1}{9}+\frac{1}{12}+\frac{1}{15}+\cdots = \sum_{n=1}^{\infty}\frac{1}{3}\left(\frac{1}{n}\right)$ Harmonic serices diverges

28) $\frac{1}{3}+\frac{2}{9}+\frac{1}{27}+\frac{2}{81}+\frac{1}{243}+\frac{2}{729}+\cdots = \sum_{n=1}^{\infty}\left(\frac{1}{3^{2n-1}}+\frac{2}{3^{2n}}\right)$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{9^n\cdot(3)^{-1}}+\sum_{n=1}^{\infty}\frac{2}{9^n} = 3\sum_{n=1}^{\infty}\left(\frac{1}{9}\right)^n+2\sum_{n=1}^{\infty}\left(\frac{1}{9}\right)^n$$

$$\Rightarrow \frac{1}{3}\sum_{n=1}^{\infty}\left(\frac{1}{9}\right)^{n-1}+\frac{2}{9}\sum_{n=1}^{\infty}\left(\frac{1}{9}\right)^{n-1}$$

both series are Geometric & converges because $\left|\frac{1}{9}\right|<1$

$$\Rightarrow \frac{1}{3}\cdot\frac{1}{1-\frac{1}{9}}+\frac{2}{9}\cdot\frac{1}{1-\frac{1}{9}} = \frac{1}{\cancel{3}}\cdot\frac{\cancel{9}^{3}}{8}+\frac{2}{\cancel{9}}\cdot\frac{\cancel{9}}{8} = \frac{5}{8}$$

29) $\sum_{n=1}^{\infty}\frac{n-1}{3n-1}$

Divergence test: $\lim_{n\to\infty}\frac{n-1}{3n-1} = \lim_{n\to\infty}\frac{\cancel{n}(1-\cancel{\frac{1}{n}}^{0})}{\cancel{n}(3-\cancel{\frac{1}{n}}_{0})} = \frac{1}{3}\neq 0$

This series diverges

30) $\sum_{K=1}^{\infty}\frac{K(K+2)}{(K+3)^2}$ (When we have fraction of two polynomials with the same order, it Diverges)

Diverges by the Test for Divergence, since $\lim_{K\to\infty}a_K = \lim_{K\to\infty}\frac{K^2+2}{K^2+6K+9} = 1\neq 0$

31) $\sum_{n=1}^{\infty}\frac{1+2^n}{3^n} = \sum_{n=1}^{\infty}\left(\frac{1}{3^n}+\frac{2^n}{3^n}\right) = \frac{1}{3}\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^{n-1}+\frac{2}{3}\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^{n-1}$

Both Geometric series converges, because $\left|\frac{1}{3}\right|$ and $\left|\frac{2}{3}\right|$ are less than 1

$$\Rightarrow \frac{1}{3}\cdot\frac{1}{1-\frac{1}{3}}+\frac{2}{3}\cdot\frac{1}{1-\frac{2}{3}} = \frac{1}{2}+2 = \frac{5}{2}$$

32) $\sum_{n=1}^{\infty}\frac{1+3^n}{2^n} = \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n+\sum_{n=1}^{\infty}\left(\frac{3}{2}\right)^n$

Diverges because one of the series is Divergent $\left|\frac{3}{2}\right|\geqslant 1$

33) $\sum_{n=1}^{\infty}\sqrt[n]{2} = \sum_{n=1}^{\infty}(2)^{\frac{1}{n}}$

Diverges by Test for Divergence,

since $\lim_{n\to\infty}a_n = \lim_{n\to\infty}(2)^{\frac{1}{n}} = 2^0 = 1\neq 0$

34) $\sum_{n=1}^{\infty}\left[(0.8)^{n-1}-(0.3)^{n}\right] = \sum_{n=1}^{\infty}\left(\frac{4}{5}\right)^{n-1} - \frac{3}{10}\sum_{n=1}^{\infty}\left(\frac{3}{10}\right)^{n-1}$ Sum of two Convergent Geometric series

$\Rightarrow \frac{1}{1-\frac{4}{5}} - \frac{3}{10}\cdot\frac{1}{1-\frac{3}{10}} = 5 - \frac{3}{\cancel{10}}\cdot\frac{\cancel{10}}{7} = \frac{32}{7}$

35) $\sum_{n=1}^{\infty}\ln\left(\frac{n^2+1}{2n^2+1}\right)$ Diverges by the Test of divergence,

since $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\ln\left(\frac{\cancel{n^2}(1+\frac{1}{n^2})}{\cancel{n^2}(2+\frac{1}{n^2})}\right) = \ln\left(\frac{1}{2}\right) \neq 0$

36) $\sum_{n=1}^{\infty}\frac{1}{1+\left(\frac{2}{3}\right)^n}$ Diverges by the Test for Divergence,

Since $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{1+\left(\frac{2}{3}\right)^n} = \frac{1}{1+0} = 1 \neq 0$

37) $\sum_{K=0}^{\infty}\left(\frac{\pi}{3}\right)^K$ A Divergent Geometric series, since $\left|\frac{\pi}{3}\right| \geq 1$

38) $\sum_{K=1}^{\infty}(\cos 1)^K = \cos(1)\cdot\sum_{K=1}^{\infty}(\cos 1)^{K-1}$ A Convergent Geometric series, since $|\cos(1)| < 1$

$\Rightarrow = \frac{\cos 1}{1-\frac{1}{\cos 1}} = \frac{(\cos 1)^2}{\cos(1)-1}$

39) $\sum_{n=1}^{\infty}\arctan(n)$ Diverges by the Test for Divergence, since $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\arctan(n) = \frac{\pi}{2} \neq 0$

40) $\sum_{n=1}^{\infty}\left(\frac{3}{5^n}+\frac{2}{n}\right) = \frac{3}{5}\sum_{n=1}^{\infty}\left(\frac{1}{5}\right)^{n-1} + \underline{2\sum_{n=1}^{\infty}\frac{1}{n}}$

Diverges because one of the sumed series is a Divergent Harmonic series.

41) $\sum_{n=1}^{\infty}\left(\frac{1}{e^n}+\frac{1}{n(n+1)}\right) = \frac{1}{e}\cdot\sum_{n=1}^{\infty}\left(\frac{1}{e}\right)^{n-1} + \sum_{n=1}^{\infty}\frac{1}{n(n+1)}$ check example 7, or telescoping series $\longrightarrow \frac{n}{n+1}$

Sum of 2 Convergent series, Geomelric series with $a = \frac{1}{e}$ & $q = \left|\frac{1}{e}\right| < 1$, and Telescoping Series.

$\Rightarrow \frac{1}{e}\cdot\frac{1}{1-\frac{1}{e}} + \lim_{n\to\infty}\frac{n}{n+1} = \frac{\cancel{e}\cdot 1}{\cancel{e}(e-1)} + \lim_{n\to\infty}\frac{\cancel{n}\,1}{\cancel{n}(1+\frac{1}{n})} = \frac{e}{(e-1)} + 1$

42) $\sum_{n=1}^{\infty}\frac{e^n}{n^2} = \frac{2.7183\ldots}{1} + \frac{7.39}{4} + \frac{20.08\ldots}{8} + \frac{54.6}{16} + \cdots$

It is very clear that e^n is growing much faster than n^2

$\therefore$ The series diverges by Divergence Test,

since $\lim_{n\to\infty}\frac{e^n}{n^2} = \infty \neq 0$

43) $\displaystyle\sum_{n=2}^{\infty} \frac{2}{n^2-1}$

$$\frac{2}{n^2-1} = \frac{2}{(n-1)(n+1)} = \frac{A}{(n-1)} + \frac{B}{(n+1)}$$

$$2 = A(n+1) + B(n-1)$$

$$\boxed{A=1}\ ,\ \boxed{B=-1}$$

$$\Rightarrow \sum_{n=2}^{K} \frac{2}{n^2-1} = \sum_{n=2}^{K}\left(\frac{1}{(n-1)} - \frac{1}{(n+1)}\right) = \left(\frac{1}{1} - \cancel{\frac{1}{3}}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right)$$

$$+\left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{6}}\right) + \left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{7}}\right) + \cancel{\ } - - - \cancel{\ } + \left(\frac{1}{K-1} - \frac{1}{K+1}\right)$$

$$= 1 + \frac{1}{2} + \frac{1}{K-1} - \frac{1}{K+1} = \frac{3}{2} + \frac{2}{(K^2-1)}$$

$$S_K = \frac{3}{2} + \frac{2}{K^2-1}$$

$\displaystyle\lim_{K\to\infty} S_K = \frac{3}{2}$ $\quad\therefore$ Converges as $\displaystyle\sum_{n=2}^{\infty} \frac{2}{n^2-1} = \frac{3}{2}$

44) $\displaystyle\sum_{n=1}^{\infty} \ln\frac{n}{n+1} = \sum_{n=1}^{\infty}(\ln(n) - \ln(n+1))$

Partial sum $\displaystyle S_K = \sum_{n=1}^{K}(\ln(n) - \ln(n+1)) = (\ln(1) - \cancel{\ln(2)}) + (\cancel{\ln(2)} - \cancel{\ln(3)}) + (\cancel{\ln(3)} - \cancel{\ln(4)}) + \cancel{\ } - - - \cancel{\ } + (\cancel{\ln(K)} - \ln(K+1))$

$$S_K = \ln(1) - \ln(K+1)$$

$$\lim_{K\to\infty} S_K = \lim_{K\to\infty} \ln(1) - \lim_{K\to\infty} \ln(K+1) = 1 - \infty = -\infty$$

The series Diverges

45) $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n(n+3)} \Rightarrow \frac{3}{n(n+3)} = \frac{A}{n} + \frac{B}{n+3} \Rightarrow 3 = A(n+3) + B\cdot n$

By comparing Coeffiecents $\boxed{A=1}$, $\boxed{B=-1}$

$\displaystyle\sum_{n=1}^{\infty} \frac{3}{n(n+3)} = \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+3}\right) \Rightarrow$ Partial sums $\displaystyle S_K = \sum_{n=1}^{K}\left(\frac{1}{n} - \frac{1}{n+3}\right) = \left(\frac{1}{1} - \cancel{\frac{1}{4}}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{5}}\right) + \left(\frac{1}{3} - \cancel{\frac{1}{6}}\right) + \left(\cancel{\frac{1}{4}} - \frac{1}{7}\right)$

$$+\left(\cancel{\frac{1}{5}} - \frac{1}{8}\right) + \cancel{\ } - - - \cancel{\ } + \left(\cancel{\frac{1}{K}} - \frac{1}{K+3}\right)$$

$$S_K = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{K+3} = \frac{11}{6} + \frac{1}{K+3}$$

Thus, $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n(n+3)} = \lim_{K\to\infty} S_K = \lim_{K\to\infty}\left(\frac{11}{6} + \frac{1}{K+3}\right) = \frac{11}{6}$ converges

46) $\sum_{n=1}^{\infty}\left(\cos\left(\frac{1}{n^2}\right)-\cos\left(\frac{1}{(n+1)^2}\right)\right) = \left(\cos(1)-\cos\left(\frac{1}{4}\right)\right)+\left(\cos\left(\frac{1}{4}\right)-\cos\left(\frac{1}{9}\right)\right)+\cdots$

$$S_K=\sum_{n=1}^{K}\left(\cos\left(\frac{1}{n^2}\right)-\cos\left(\frac{1}{(n+1)^2}\right)\right) = \left(\cos(1)-\cancel{\cos\left(\tfrac{1}{4}\right)}\right)+\left(\cancel{\cos\left(\tfrac{1}{4}\right)}-\cancel{\cos\left(\tfrac{1}{9}\right)}\right)+\left(\cancel{\cos\left(\tfrac{1}{9}\right)}-\cancel{\cos\left(\tfrac{1}{16}\right)}\right)+\cdots$$
$$+\left(\cos\left(\frac{1}{K^2}\right)-\cos\left(\frac{1}{(K-1)^2}\right)\right) = \cos 1-\cos\left(\frac{1}{(K+1)^2}\right)$$

$$\lim_{K\to\infty} S_K=\lim_{K\to\infty}\left(\cos 1-\cos\left(\frac{1}{(K+1)^2}\right)\right) = \cos 1-\cos\left(\frac{1}{\lim_{K\to\infty}(K+1)^2}\right) = \cos 1-\cos 0=\cos 1-1=-0.4597$$

$\therefore$ The Telescoping series $\sum_{n=1}^{\infty}\left(\cos\left(\frac{1}{n^2}\right)-\cos\left(\frac{1}{(n+1)^2}\right)\right)$ converges to -0.4597

47) $\sum_{n=1}^{\infty}\left(e^{\frac{1}{n}}-e^{\frac{1}{(n+1)}}\right)$

$$S_K=\sum_{n=1}^{K}\left(e^{\frac{1}{n}}-e^{\frac{1}{n+1}}\right)=\left(e^{\frac{1}{1}}-\cancel{e^{\frac{1}{2}}}\right)+\left(\cancel{e^{\frac{1}{2}}}-\cancel{e^{\frac{1}{3}}}\right)+\left(\cancel{e^{\frac{1}{3}}}-\cancel{e^{\frac{1}{4}}}\right)+\cdots+\left(\cancel{e^{\frac{1}{K}}}-e^{\frac{1}{K+1}}\right)=e-e^{\frac{1}{K+1}}$$

$$\sum_{n=1}^{\infty}\left(e^{\frac{1}{n}}-e^{\frac{1}{n+1}}\right)=\lim_{n\to\infty}S_n=\lim_{n\to\infty}\left(e-e^{\frac{1}{n+1}}\right)=e-e^0=e-1\approx 1.7183$$

This telescoping series converges to 1.7183

48) $\sum_{n=2}^{\infty}\frac{1}{n^3-n}$, $\frac{1}{n^3-n}=\frac{1}{n(n^2-1)}=\frac{1}{n(n-1)(n+1)}=\frac{A}{n}+\frac{B}{(n+1)}+\frac{C}{(n-1)}$

$$\Longrightarrow\ 1=A(n+1)(n-1)+B\cdot n(n-1)+C\cdot n(n+1)$$

By comparing coefficients $\boxed{A=-1}$, $\boxed{B=\frac{1}{2}}$, $\boxed{C=\frac{1}{2}}$

$$\Rightarrow \frac{1}{n^3-n}=\frac{-1}{n}+\frac{1}{2(n+1)}+\frac{1}{2(n-1)}\ ,\quad -\frac{1}{n}=\frac{-1}{2n}-\frac{1}{2n}$$

$$\sum_{n=2}^{K}\left(\frac{1}{n^3-n}\right)=\sum_{n=2}^{K}\left(\frac{-1}{2n}+\frac{1}{2(n+1)}-\frac{1}{2n}+\frac{1}{2(n-1)}\right)=\left(-\frac{1}{4}+\cancel{\frac{1}{6}}-\cancel{\frac{1}{4}}+\frac{1}{2}\right)+\left(-\cancel{\frac{1}{6}}+\cancel{\frac{1}{8}}-\cancel{\frac{1}{6}}+\cancel{\frac{1}{4}}\right)$$
$$+\left(-\cancel{\frac{1}{8}}+\cancel{\frac{1}{10}}-\cancel{\frac{1}{8}}+\cancel{\frac{1}{6}}\right)+\cdots\left(-\frac{1}{2K}+\frac{1}{2(K+1)}-\cancel{\frac{1}{2K}}+\cancel{\frac{1}{2(K-1)}}\right)=-\frac{1}{4}+\frac{1}{2}-\frac{1}{2K}+\frac{1}{2(K+1)}$$

$$S_K=\frac{1}{4}+\frac{\cancel{K}-\cancel{K}-1}{2K(K+1)}=\frac{1}{4}-\frac{1}{2K(K+1)}$$

The series converges to $\lim_{K\to\infty}S_K=\lim_{K\to\infty}\left(\frac{1}{4}-\frac{1}{2K(K+1)}\right)=\frac{1}{4}$

$$\therefore \sum_{n=2}^{\infty}\left(\frac{1}{n^3-n}\right)=\frac{1}{4}$$

$\lim_{n\to\infty} S_n = S$

$\boxed{2}$ **Remainder Estimate for the Integral Test** Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geq n$ and Σa_n is convergent. If $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx$$

since $R_n = S - S_n$

$$\int_{n+1}^{\infty} f(x)\,dx \leq S - S_n \leq \int_{n}^{\infty} f(x)\,dx$$

$$\therefore \boxed{S_n + \int_{n+1}^{\infty} f(x)\,dx \leq S \leq S_n + \int_{n}^{\infty} f(x)\,dx}$$

Section $\boxed{11.3}$ from the book:

3) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}} = \sum_{n=1}^{\infty} (n)^{-\frac{1}{5}}$ $f(x) = x^{-\frac{1}{5}}$ is a continuous, positive, & decreasing function on $[1, \infty)$, then integral test applies.

$$\int_1^{\infty} f(x) = \lim_{t\to\infty} \int_1^t x^{-\frac{1}{5}}\,dx = \lim_{t\to\infty} \left[\frac{5}{4} x^{\frac{4}{5}}\right]_1^t = \lim_{t\to\infty} \left[\frac{5}{4} t^{\frac{4}{5}} - \frac{5}{4}\right] = \infty$$

since $\int_1^{\infty} f(x)$ diverges, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}}$ diverges as well

also because it's a p-series with $P = -\frac{1}{5} \leq 1$.

5) $\sum_{n=1}^{\infty} \frac{1}{(2n+1)^3}$, The function $f(x) = \frac{1}{(2x+1)^3}$ is continuous, positive, & decreasing on $[1,\infty)$, then Integral test applies.

$$\int_1^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_1^t (2n+1)^{-3}\,dx = \lim_{t\to\infty} \left[\frac{1}{2} \cdot \frac{-1}{2} (2n+1)^{-2}\right]_1^t = \lim_{t\to\infty} \left[\frac{-1}{4(2t+1)^2} + \frac{1}{4(2+1)^2}\right] = \frac{1}{4(9)} = \frac{1}{36}$$

Since the improper integral is convergent, the series $\sum_{n=1}^{\infty} \left(\frac{1}{(2n+1)^3}\right)$ is also convergent by integral test.

7) $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$, the function $f(x) = \frac{x}{x^2+1}$ is continuous, positive, & decreasing on $[1,\infty)$ then integral test applies.

$$\int_1^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_1^t \frac{x}{x^2+1}\,dx = \lim_{t\to\infty} \left[\frac{1}{2} \ln|x^2+1|\right]_1^t = \lim_{t\to\infty} \left[\frac{1}{2} \ln|t^2+1| - \frac{1}{2} \ln|2|\right] = \infty$$

Since the improper integral is divergent, the series $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ diverges also by the integral test.

8) $\sum_{n=1}^{\infty} n^2 \cdot e^{-n^3}$, The function $f(x) = x^2 e^{-x^3}$ is Continuous, Positive, decreasing, then integral test applies

$$\int_1^{\infty} f(x)dx = \lim_{t\to\infty} \int_1^t x^2 e^{-x^3} dx = \lim_{t\to\infty} \left[-\frac{1}{3} e^{-x^3}\right]_1^t = \lim_{t\to\infty} \left[-\frac{1}{3} e^{-t^3} + \frac{1}{3e}\right] = 0 + \frac{1}{3e} \approx 0.1226\ldots$$

Since the improper integral is convergent, the series $\sum_{n=1}^{\infty} n^2 \cdot e^{-n^3}$ is also convergent by integral test.

9) $\sum_{n=1}^{\infty} \frac{1}{n^{\sqrt{2}}}$ p-series with $p \leq 1 \Longrightarrow$ The series diverges

10) same answer

11) $1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^3}$ Convergent p-series, because $P > 1$

12) $1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \frac{1}{5\sqrt{5}} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} n^{-1} \cdot n^{-\frac{1}{2}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$

(2^3, 3^3, 4^3, 5^3)

Convergent p-series with $p = \frac{3}{2} > 1$

13) $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \cdots = \sum_{n=1}^{\infty} \frac{1}{2n-1}$

The function $f(x) = \frac{1}{2x-1}$ is continuous, positive, & decreasing on for $x \in [1, \infty)$, then integral test applies

$$\int_1^{\infty} f(x)dx = \lim_{t\to\infty} \int_1^t \frac{1}{2x-1} dx = \lim_{t\to\infty} \left[\frac{1}{2} \ln|2x-1|\right]_1^t = \lim_{t\to\infty} \left[\frac{1}{2} \ln|2t-1| - \frac{1}{2} \ln|2-1|\right] = \infty - 0 = \infty$$

Since the improper integral diverges, the series $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ diverges as well by integral test

14) $\frac{1}{5} + \frac{1}{8} + \frac{1}{11} + \frac{1}{14} + \frac{1}{17} + \cdots = \sum_{n=2}^{\infty} \frac{1}{3n-1}$ Same as the previous, diverges.

15) $\sum_{n=1}^{\infty} \frac{\sqrt{n} + 4}{n^2} = \sum_{n=1}^{\infty} \left(\frac{n^{\frac{1}{2}}}{n^2} + \frac{4}{n^2}\right) = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}} + 4\sum_{n=1}^{\infty} \frac{1}{n^2}$

Sum of two convergents p-series, first with $p = \frac{3}{2} > 1$, & second with $p = 2 > 1$

$\therefore \sum_{n=1}^{\infty} \frac{\sqrt{n} + 4}{n^2}$ is convergent

16) $\sum_{n=1}^{\infty} \frac{n^2}{n^3+1} = \sum_{n=1}^{\infty} a_n$, very close to $\sum_{n=1}^{\infty} \frac{n^2}{n^3} = \sum_{n=1}^{\infty} b_n$ which is a divergent p-series with $p = 1$

limit comparisson: $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\cancel{n^2}}{n^3+1} \cdot \frac{n^3}{\cancel{n^2}} = \lim_{n\to\infty} \frac{\cancel{n^3} \cdot 1}{\cancel{n^3}(1+\underbrace{\frac{1}{n^3}}_{0})} = 1$

by limit comparison test, $\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$ diverges

19) $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^3}$, $f(x)=\frac{\ln(x)}{x^3}$ is continuous, positive, & decrease function on $x\in[2,\infty)$; thus integral test applies.

$$\int_2^{\infty} f(x)\,dx = \lim_{t\to\infty}\int_2^t \frac{\ln(x)}{x^3}\,dx = \lim_{t\to\infty}\left[-\frac{1}{2}x^{-2}\cdot \ln(x) - \frac{1}{4}x^{-2}\right]_2^t$$

$u=\ln(x)$ $\quad dv = x^{-3}\,dx$

$du=\frac{1}{x}$ $\quad v=\frac{-1}{2}x^{-2}$

$$\Rightarrow \lim_{t\to\infty}\left[-\frac{\ln(t)}{2t^2} - \frac{1}{4t^2} + \frac{1}{4}\right] = \lim_{t\to\infty}\left[\overset{0}{\cancel{\frac{-1}{4t^2}}}\left(\ln(t)+1\right) + \frac{1}{4}\right] = \frac{1}{4} \cdot \frac{1}{16}$$

Since the improper integral converges, then the series $\sum_{n=1}^{\infty}\frac{\ln(n)}{n^3}$ is convergent.

20) $\sum_{n=1}^{\infty} \frac{1}{n^2+6n+13}$, very similar to the convergent p-series with $p=2>1$

$$\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\frac{1}{n^2} > \frac{1}{n^2+6n+13}$$

By compasizon Test, $\sum_{n=1}^{\infty}\frac{1}{n^2+6n+13}$ is convergent as well.

21) $\sum_{n=2}^{\infty}\frac{1}{n\ln(n)}$, $f(x)=\frac{1}{x\ln(x)}$ is a continuous, positive, decreasing function on $x\in[2,\infty)$, then integral test applies.

$$\int_2^{\infty} f(x)\,dx = \lim_{t\to\infty}\int_2^t \frac{1}{x\ln(x)}\,dx = \lim_{t\to\infty}\left[\ln(\ln(x))\right]_2^t = \lim_{t\to\infty}\left[\ln(\ln(t)) - \ln(\ln(2))\right] = \infty$$

since the improper integral is divergent, then the series $\sum_{n=2}^{\infty}\frac{1}{n\ln(n)}$ is Divergent as well, by integral test.

22) $\sum_{n=2}^{\infty}\frac{1}{n(\ln(n))^2}$

Same as 21, but convergent because $\int_2^{\infty} f(x) = \lim_{t\to\infty}\left[\frac{1}{\ln(t)} - \frac{1}{\ln(2)}\right] = -\frac{1}{\ln(2)}$

23) $\sum_{n=1}^{\infty}\frac{e^{1/n}}{n^2}$, $f(x)=\frac{e^{1/x}}{x^2}$ is a continuous, positive, decreasing function on $[1,\infty)$, then integral test applies.

$$\int_1^{\infty} f(x)\,dx = \lim_{t\to\infty}\int_1^t e^{\frac{1}{x}}\cdot x^{-2}\,dx = \lim_{t\to\infty}\left[-e^{\frac{1}{x}}\right]_1^t = \lim_{t\to\infty}\left[-e^{\overset{0}{\frac{1}{t}}} + e^{1}\right] = e^1 - e^0 = e-1$$

since the improper integral converges, then the series $\sum_{n=1}^{\infty}\frac{e^{1/n}}{n^2}$ Converges as well by integral test.

24) $\sum_{n=3}^{\infty}\frac{n^2}{e^n}$, $f(x)=\frac{x^2}{e^x}$ is a continuous, positive, & decreasing function on $[3,\infty)$, thus integral test applies.

$$\int_3^{\infty} f(x)\,dx = \lim_{t\to\infty}\int_3^t x^2 e^{-x}\,dx = \lim_{t\to\infty}\left[-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\right]_3^t = \lim_{t\to\infty}\left[-e^{-x}(x^2+2x+2)\right]_3^t$$

u		dv
x^2	+	e^{-x}
$2x$	−	$-e^{-x}$
2	+	e^{-x}
0		$-e^{-x}$

0.8463801623

$$\Rightarrow \lim_{t\to\infty}\left[\overset{0}{\cancel{-e^{-t}}}(t^2+2t+2) + e^{-3}((3)^2+2(3)+2)\right] =$$

Since improper integral converges, the series $\sum_{n=3}^{\infty}\frac{n^2}{e^n}$ converges as well by integral test.

25) $\sum_{n=1}^{\infty} \frac{1}{n^2+n^3} = \sum_{n=1}^{\infty} \frac{1}{n^3(1+\frac{1}{n})}$ very close to the convergent p-series with $p=3>1$ $\sum_{n=}^{\infty} \frac{1}{n^3}$

$\because \frac{1}{n^3} \geqslant \frac{1}{n^3(1+\frac{1}{n})}$ for all n

By comparizon test, the series $\sum_{n=1}^{\infty} \frac{1}{n^2+n^3}$ converges.

26) $\sum_{n=1}^{\infty} \frac{n}{n^4+1} = \sum_{n=1}^{\infty} a_n$, using limit comparizon test with the convergent p-series $\sum_{n=1}^{\infty} \frac{1}{n^3} = \sum_{n=1}^{\infty} b_n$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{n^4+1} \cdot \frac{n^3}{1} = \lim_{n\to\infty} \frac{\cancel{n^4} \cdot 1}{\cancel{n^4}(1+\cancel{\frac{1}{n^4}})} = 1$ Both series converges

By limit comparizon Test, the series $\sum_{n=1}^{\infty} \frac{n}{n^4+1}$ converges

Section $\boxed{11.4}$ from the book:

The Comparison Test Suppose that Σa_n and Σb_n are series with positive terms.

(i) If Σb_n is convergent and $a_n \leqslant b_n$ for all n, then Σa_n is also convergent.

(ii) If Σb_n is divergent and $a_n \geqslant b_n$ for all n, then Σa_n is also divergent.

The Limit Comparison Test Suppose that Σa_n and Σb_n are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

3) $\sum_{n=1}^{\infty} \frac{n}{2n^3+1}$

$\frac{n}{2n^3+1} < \frac{n}{2n^3} = \frac{1}{2n^2} < \frac{1}{n^2}$ for all $n \geq 1$,

thus $\sum_{n=1}^{\infty} \frac{n}{2n^3+1}$ is convergent by comparizon with $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which converges because it's a p-series with $p=2>1$

4) $\sum_{n=2}^{\infty} \frac{n^3}{n^4-1}$, $\frac{n^3}{n^4-1} > \frac{n^3}{n^4} = \frac{1}{n}$ for all $n \geqslant 2$, so $\sum_{n=2}^{\infty} \frac{n^3}{n^4-1}$ is divergent by comparizon with $\sum_{n=2}^{\infty} \frac{1}{n}$, which also diverges because it's a harmonic series.

5) $\sum_{n=1}^{\infty} \frac{n+1}{n\cdot\sqrt{n}}$, $\frac{n+1}{n^{\frac{3}{2}}} > \frac{n}{n^{\frac{3}{2}}} = \frac{1}{n^{\frac{1}{2}}}$ for all $n \geq 1$, so $\sum_{n=1}^{\infty} \frac{n+1}{n\sqrt{n}}$ is divergent by comparizon with $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$, which diverges because it's a p-series with $P = \frac{1}{2} \leq 1$

6) $\sum_{n=1}^{\infty} \frac{n-1}{n^2\sqrt{n}}$, $\frac{n-1}{n^2\sqrt{n}} < \frac{\cancel{n}}{n^{\cancel{2}}\sqrt{n}} = \frac{1}{n^{\frac{3}{2}}}$ for all $n \geq 1$, so $\sum_{n=1}^{\infty} \frac{n-1}{n^2\sqrt{n}}$ converges by comparizon with $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$, which converges because it's a p-series with $P = \frac{3}{2} > 1$

7) $\sum_{n=1}^{\infty} \frac{9^n}{3+10^n}$, $\frac{9^n}{3+10^n} < \frac{9^n}{10^n} = \left(\frac{9}{10}\right)^n$ for all $n \geq 1$, so $\sum_{n=1}^{\infty} \frac{9^n}{3+10^n}$ converges by comparizon with $\sum_{n=1}^{\infty} \left(\frac{9}{10}\right)^n$ which also converges because it's a geometric series with $|q| = \frac{9}{10} < 1$

8) $\sum_{n=1}^{\infty} \frac{6^n}{5^n-1}$, $\frac{6^n}{5^n-1} > \frac{6^n}{5^n} = \left(\frac{6}{5}\right)^n$ for all $n \geq 1$, thus $\sum_{n=1}^{\infty} \frac{6^n}{5^n-1}$ diverges by comparizon with $\sum_{n=1}^{\infty} \left(\frac{6}{5}\right)^n$ which also diverges because it's a geometric series with $|q| = \frac{6}{5} \geq 1$

9) $\sum_{K=1}^{\infty} \frac{\ln K}{K}$, $\frac{\ln K}{K} > \frac{1}{K}$ for all $K \geq 3$, so $\sum_{K=1}^{\infty} \frac{\ln(K)}{K}$ diverges by comparizon with $\sum_{K=3}^{\infty} \frac{1}{K}$ which diverges because it a harmonic series.

10) $\sum_{n=1}^{\infty} \frac{K\sin^2 K}{1+K^3}$, $\sin^2 K \leq 1 \Rightarrow \frac{K\cdot\sin^2 K}{1+K^3} \leq \frac{K}{K^3} = \frac{1}{K^2}$ for all $K \geq 1$, so $\sum_{K=1}^{\infty} \frac{K\cdot\sin^2 K}{1+K^3}$ diverges by Comparizon with $\sum_{K=1}^{\infty} \frac{1}{K^2}$ which converges because it's a p-series with $P = 2 > 1$

11) $\sum_{K=1}^{\infty} \frac{\sqrt[3]{K}}{\sqrt{K^3+4K+3}}$, $\frac{\sqrt[3]{K}}{\sqrt{K^3+4K+3}} < \frac{\sqrt[3]{K}}{\sqrt{K^3}} = \frac{1}{K^{\frac{7}{6}}}$ for all $K \geq 1$, so $\sum_{K=1}^{\infty} \frac{\sqrt[3]{K}}{\sqrt{K^3+4K+3}}$ converges by comparizon with $\sum_{K=1}^{\infty} \frac{1}{K^{7/6}}$ which converges because it's p-series with $p = \frac{7}{6} > 1$

13) $\sum_{n=1}^{\infty} \frac{\arctan(n)}{n^{1.2}}$, $\frac{\arctan(n)}{n^{1.2}} \leq \frac{\pi/2}{n^{1.2}}$ for all $n \geq 1$, so $\sum_{n=1}^{\infty} \frac{\arctan(n)}{n^{1.2}}$ converges by comparizon with $\sum_{n=1}^{\infty} \frac{1}{n^{1.2}}$ which converges because it's a p-series with $p = 1.2 > 1$

14) $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{n-1}$ diverges like number 4, $P = \frac{1}{2} < 1$

15) $\sum_{n=1}^{\infty} \frac{4^{n+1}}{3^n-2}$, $\frac{4^{n+1}}{3^n-2} \geq \frac{4^{n+1}}{3^n} = \frac{16}{3}\left(\frac{4}{3}\right)^{n-1}$ for all $n \geq 1$, so $\sum_{n=1}^{\infty} \frac{4^{n+1}}{3^n-2}$ diverges by comparizon test with $\sum_{n=1}^{\infty} \left(\frac{4}{3}\right)^{n-1}$ which diverges because it's geometric series with $|q| = \frac{4}{3} \geq 1$

16) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{3n^4+1}}$, $\frac{1}{\sqrt[3]{3n^4+1}} < \frac{1}{\sqrt[3]{3} \cdot n^{4/3}}$ for all $n \geq 1$, so $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{3n^4+1}}$ Converges by comparizon Test with $\sum_{n=1}^{\infty} \frac{1}{n^{4/3}}$ which also converges because it's a p-series where $p = \frac{4}{3} > 1$

17) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$ using limit comparizon test where $a_n = \frac{1}{\sqrt{n^2+1}}$ & $b_n = \frac{1}{n}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\sqrt{n^2+1}} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{\cancel{n} \cdot 1}{\cancel{n}\sqrt{1+\underbrace{\frac{1}{n^2}}_{0}}} = 1 > 0$, since $\sum_{n=1}^{\infty} b_n$ diverges as a harmonic series

So does $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$!

18) $\sum_{n=1}^{\infty} \frac{1}{2n+3}$ diverges as 17

19) $\sum_{n=1}^{\infty} \frac{1+4^n}{1+3^n}$ using limit comparizon Test with $a_n = \frac{1+4^n}{1+3^n}$ & $b_n = \frac{4^n}{3^n}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1+4^n}{1+3^n} \cdot \frac{3^n}{4^n} = \lim_{n\to\infty} \frac{\cancel{4^n}(\underbrace{\frac{1}{4^n}}_{0}+1)}{\cancel{3^n}(\underbrace{\frac{1}{3^n}}_{0}+1)} \cdot \frac{\cancel{3^n}}{\cancel{4^n}} = 1 > 0$, since $\sum_{n=1}^{\infty} b_n$ diverges because it's a geometric series with $|q| = \frac{4}{3} \geq 1$, so does $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1+4^n}{1+3^n}$

(We could use also divergence Test easily.)

20) $\sum_{n=1}^{\infty} \frac{n+4^n}{n+6^n}$ using limit comparizon test with $a_n = \frac{n+4^n}{n+6^n}$ & $b_n = \frac{4^n}{6^n}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\cancel{4^n}(\frac{n}{4^n}+1)}{\cancel{6^n}(\frac{n}{6^n}+1)} \cdot \frac{\cancel{6^n}}{\cancel{4^n}} = \frac{0+1}{0+1} = 1 > 0$, since $\sum_{n=1}^{\infty} b_n$ converges because it's a geometric series with $|q| = \frac{4}{6} < 1$, so does $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n+4^n}{n+6^n}$

21) $\sum_{n=1}^{\infty}\frac{\sqrt{n+2}}{2n^2+n+1}$ using limit comparizon Test with $a_n=\frac{\sqrt{n+2}}{2n^2+n+1}$ & $b_n=\frac{1}{n^{3/2}}$

$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{\sqrt{n+2}}{2n^2+n+1}\cdot\frac{n^{\frac{3}{2}}}{1}=\lim_{n\to\infty}\frac{\cancel{n^{\frac12}}\sqrt{1+\frac{2}{n}}}{\cancel{n^2}(2+\frac{1}{n}+\frac{1}{n^2})}\cdot\frac{\cancel{n^{\frac32}}}{1}=\frac{\sqrt{1+0}}{2+0+0}=\frac{1}{2}>0$ since $\sum_{n=1}^{\infty}b_n$ is a convergent p-series with $p=\frac{3}{2}>1$, $\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}\frac{\sqrt{n+2}}{2n^2+n+1}$ is also convergent.

☆

22) $\sum_{n=1}^{\infty}\frac{n+2}{(n+1)^3}$ using limit comparizon test where $a_n=\frac{n+2}{(n+1)^3}=\frac{n+2}{n^3+3n^2+3n+1}$ & $b_n=\frac{1}{n^2}$

$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{\cancel{n}(1+\frac{2}{n})}{\cancel{n^3}(1+\frac{3}{n}+\frac{3}{n^2}+\frac{1}{n^3})}\cdot\frac{1\cdot\cancel{n^2}}{1}=\frac{1+0}{1+0+0+0}=1>0$ since $\sum_{n=1}^{\infty}b_n$ Converges as a p-series with $p=2>1$

So deos $\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}\frac{n+2}{(n+1)^2}$

23) $\sum_{n=1}^{\infty}\frac{5+2n}{(1+n^2)^2}$ using limit comparizon test with $a_n=\frac{5+2n}{(1+n^2)^2}$ & $b_n=\frac{1}{n^3}$

$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{n(\frac{5}{n}+2)}{\cancel{n^4}(\frac{1}{n^2}+1)^2}\cdot\frac{\cancel{n^3}}{1}=\frac{0+2}{(0+1)^2}=2>0$ since $\sum_{n=1}^{\infty}b_n$ converges as a p-series with $p=3>1$

so does $\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}\frac{5+2n}{(1+n^2)^2}$

24) $\sum_{n=1}^{\infty}\frac{n^2-5n}{n^3+n+1}$ Diverges by limit comparizon test with $b_n=\frac{1}{n}$

25) $\sum_{n=1}^{\infty}\frac{\sqrt{n^4+1}}{n^3+n^2}$ using limit comparizon Test with $a_n=\frac{\sqrt{n^4+1}}{n^3+n^2}$ & $b_n=\frac{1}{n}$

$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{\sqrt{n^4+1}}{n^3+n^2}\cdot\frac{n}{1}=\lim_{n\to\infty}\frac{\cancel{n^2}\sqrt{1+\frac{1}{n^4}}\cdot\cancel{n}}{\cancel{n^3}(1+\frac{1}{n})}=\frac{\sqrt{1+0}}{1+0}=1>0$ since $\sum_{n=1}^{\infty}b_n$ diverges as a harmonic series

so does $\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}\frac{\sqrt{n^4+1}}{n^3+n^2}$

26) $\sum_{n=2}^{\infty}\frac{1}{n\sqrt{n^2-1}}$ Converges by limit comparizon Test with $b_n=\frac{1}{n^2}$

27) $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^2e^{-n}$ using limit comparizon Test with $a_n=(1+\frac{1}{n})^2e^{-n}$ & $b_n=\frac{1}{e^n}$

$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{(1+\frac{1}{n})^2}{\cancel{e^n}}\cdot\frac{\cancel{e^n}}{1}=(1+0)^2=1>0$ since $\sum_{n=1}^{\infty}b_n$ converges because it's a geometric series with $|r|=\frac{1}{e}<1$

So does $\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}(1+\frac{1}{n})^2e^{-n}$

28) $\sum_{n=1}^{\infty}\frac{e^{1/n}}{n}$ Diverges by limit comparizon Test with $b_n=\frac{1}{n}$

29) $\sum_{n=1}^{\infty} \frac{1}{n!}$ Clearly $n! = n(n-1)(n-2)\cdots(3)(2) \geq 2\cdot 2\cdot 2\cdots 2\cdot 2 = 2^{n-1}$

so $\frac{1}{n!} \leq \frac{1}{2^{n-1}}$ for all $n \geq 1$

Since $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ is a convergent geometric series ($|q| = \frac{1}{2} < 1$),

thus $\sum_{n=1}^{\infty} \frac{1}{n!}$ is also convergent by comparizon test.

30) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$, $\frac{n!}{n^n} \leq \frac{19^{n-1}}{20^n}$ for $n \geq 20$, so $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges by comparizon Test

31) $\sum_{n=1}^{\infty} \sin(\frac{1}{n})$ using limit comparizon Test with $a_n = \sin(\frac{1}{n})$ & $b_n = \frac{1}{n}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin(\frac{1}{n})}{1/n} = \lim_{x\to 0} \frac{\sin x}{x} = 1 > 0$ Since $\sum_{n=1}^{\infty} b_n$ is a divergent harmonic series

$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \sin(\frac{1}{n})$ is also divergent.

32) $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}} = \sum_{n=1}^{\infty} \frac{1}{n \cdot n^{1/n}}$ using limit comparizon Test with $a_n = \frac{1}{n^{1+1/n}}$ & $b_n = \frac{1}{n}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\cancel{n} \cdot n^{1/n}} \cdot \frac{\cancel{n}}{1} = \frac{1}{n^0} = 1 > 0$ Since $\sum_{n=1}^{\infty} b_n$ is a divergent harmonic series

$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$ is also divergent.

Section $\boxed{11.5}$ from the book:

Alternating Series Test If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \qquad b_n > 0$$

satisfies

(i) $b_{n+1} \leq b_n$ for all n

(ii) $\lim_{n\to\infty} b_n = 0$

then the series is convergent.

8) $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{n}{\sqrt{n^3+2}}$, $f(x) = \frac{x}{\sqrt{x^3+2}}$

$$f'(x) = \frac{1 \cdot \sqrt{x^3+2} - \frac{3}{2}x^3 \frac{1}{\sqrt{x^3+2}}}{(\sqrt{x^3+2})^2} * \frac{\sqrt{x^3+2}}{\sqrt{x^3+2}} = \frac{x^3+2-\frac{3}{2}x^3}{(x^3+2)^{\frac{3}{2}}}$$

$$f'(x) = \frac{2-\frac{1}{2}x^3}{(x^3+2)^{\frac{3}{2}}}$$

$2-\frac{1}{2}x^3 < 0 \longrightarrow$ when f(x) is decreasing

$\sum_{n=1}^{\infty} (-1)^n \cdot b_n$

$x^3 > 4 \Rightarrow x > \sqrt[3]{4} \approx 1.6$

f(x) decreasing on the interval $(\sqrt[3]{4}, \infty)$

(i) $b_{n+1} < b_n$ for all $n > \sqrt[3]{4}$

(ii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\cancel{n} \cdot 1}{n^{\cancel{\frac{3}{2}}} \sqrt{1+\frac{2}{n^3}}} = \lim_{n\to\infty} \frac{1}{\sqrt{n} \cdot \sqrt{1+\frac{2}{n^3}}} = 0$

Since both conditions are satisfied, the series $\sum_{n=1}^{\infty} (-1)^n \frac{n}{\sqrt{n^3+2}}$ is convergent by Alternating series Test.

11) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+4} = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot b_n$

$f(x) = \frac{x^2}{x^3+4}$, $f'(x) = \frac{2x(x^3+4) - 3x^2 \cdot x^2}{(x^3+4)^2} = \frac{8x - x^4}{(x^3+4)^2} = \frac{x(8-x^3)}{(x^3+4)^2}$

f(x) is decreasing when $8-x^3 < 0$

$\therefore x > \sqrt[3]{8} = 2$

f(x) is decreasing in the interval $(2, \infty)$

(i) $\therefore$ $b_{n+1} < b_n$ for all $n > 2$

(ii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n^2}{n^3+4} = 0$

since (i) & (ii) are satisfied, the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+4}$ is convergent by Alternating Series Test.