State-Space model :-

$$\boxed{\text{minimum number of state variables} = \text{order of differential equation}}$$

Newton´s Law yields

$$M\,\ddot{y}(t) + b\,\dot{y}(t) + k\,y(t) = u(t)$$

$$x_1(t) = y(t) \xrightarrow{d/dt} \dot{x}_1(t) = \dot{y}(t) \Rightarrow \dot{x}_1(t) = x_2(t)$$

$$x_2(t) = \dot{y}(t) \xrightarrow{d/dt} \dot{x}_2(t) = \ddot{y}(t) \Rightarrow \dot{x}_2(t) = -\frac{b}{M}x_2(t) - \frac{c}{M}x_1(t) + \frac{1}{M}u(t)$$

$$M\ddot{y}(t) + b\dot{y}(t) + c\,y(t) = u(t)$$

$$M\dot{x}_2(t) + b\,x_2(t) + c\,x_1(t) = u(t) \Rightarrow \underline{\dot{x}_2(t) = \frac{1}{M}u(t) - \frac{b}{M}x_2(t) - \frac{c}{M}x_1(t)}$$

output equation $y = x_1$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix}\cdot u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{c}{M} & -\frac{b}{M} \end{bmatrix}\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\substack{\text{state} \\ \text{vector} \\ \underline{x}}} + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix}\cdot \underbrace{u(t)}_{\substack{\text{Input/control} \\ \text{vector}}}$$

$$\begin{aligned}
\dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + b_{11}u_1 + \cdots + b_{1m}u_m \\
\dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + b_{21}u_1 + \cdots + b_{2m}u_m \\
&\;\;\vdots \\
\dot{x}_n &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + b_{n1}u_1 + \cdots + b_{nm}u_m
\end{aligned}$$

state vector $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

Input/control vector $\underline{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$

State differential equation

$$\boxed{\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}}$$

Output equation

$$\boxed{\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u}}$$

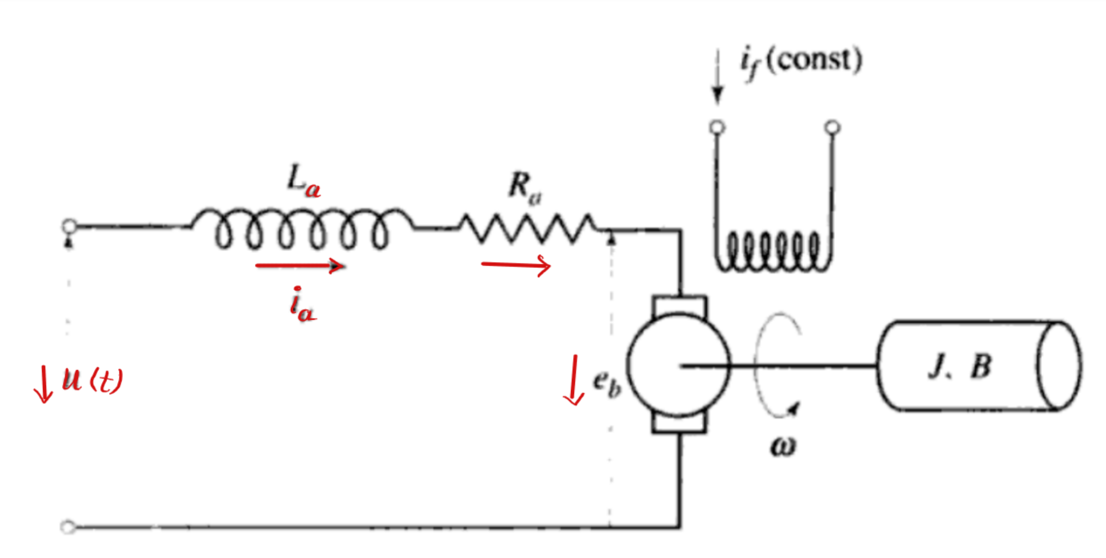
- $\boldsymbol{x}$:= State vector
- $\boldsymbol{A}$:= System matrix
- $\boldsymbol{u}$:= Control vector
- $\boldsymbol{B}$:= Input matrix
- $\boldsymbol{y}$:= Output vector
- $\boldsymbol{C}$:= Output matrix
- $\boldsymbol{D}$:= Feedforward matrix

$$\dot{\underline{x}} = \underline{\underline{A}}\;\underline{x} + \underline{\underline{B}}\;\underline{u}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = n\Bigg\{\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}}_{n}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + n\Bigg\{\underbrace{\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{bmatrix}}_{m}\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$$

$$y = \underline{\underline{C}}\;\underline{x} + \underline{\underline{D}}\;\underline{u}$$

$$y = 1\Big\{\underbrace{\begin{bmatrix} \quad & \quad & \quad \end{bmatrix}}_{n}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + 1\Big\{\underbrace{\begin{bmatrix} \quad & \quad & \quad \end{bmatrix}}_{m}\begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}$$



$$u(t) = R_a \cdot i_a(t) + L_a \frac{di_a}{dt} + e_b(t) \qquad (1)$$

e_b: back emf voltage

$$T_M = J \cdot \frac{d\omega}{dt} + B\,\omega(t) \qquad (2)$$

T_M: Torque from motor; J: moment of ineria; $\frac{d\omega}{dt}$: angular acceleration; B: viscos friction

$$T_M = K_T \cdot i_a(t) \qquad (3)$$

K_T: motor torque Const.

$$e_b(t) = K_b \cdot \omega(t) \qquad (4)$$

K_b: back emf const.

$x_1 = \omega \quad , \quad x_2 = i_a$

from (2) and (3)

$$\boxed{\frac{d\omega}{dt} = -\frac{B}{J}\omega(t) + \frac{K_T}{J} i_a(t)}$$

from (1) and (4)

$$\boxed{\frac{di_a}{dt} = -\frac{K_b}{L_a}\omega(t) - \frac{R_a}{L_a} i_a(t) + u(t)}$$

$$\dot{x}_1 = -\frac{B}{J}x_1 + \frac{K_T}{J}x_2 \qquad \dot{x}_2 = -\frac{K_b}{L_a}x_1 - \frac{R_a}{L_a}x_2 + u(t)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{B}{J} & \frac{K_T}{J} \\ -\frac{K_b}{L_a} & -\frac{R_a}{L_a} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot u$$

$$\omega = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} \cdot u$$

output equation

Phase variable canonical form:

$$\frac{Y(s)}{U(s)} = G(s) = \frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0} \cdot \frac{Z(s)}{Z(s)}$$

$$Y(s) = [b_3 S^3 + b_2 S^2 + b_1 S + b_0] \cdot Z(s)$$

$$U(s) = [S^4 + a_3 S^3 + a_2 S^2 + a_1 S + a_0] \cdot Z(s)$$

$$\mathcal{L}^{-1} \longrightarrow Y(t) = b_3 \dddot{Z}(t) + b_2 \ddot{Z}(t) + b_1 \dot{Z}(t) + b_0 Z(t)$$

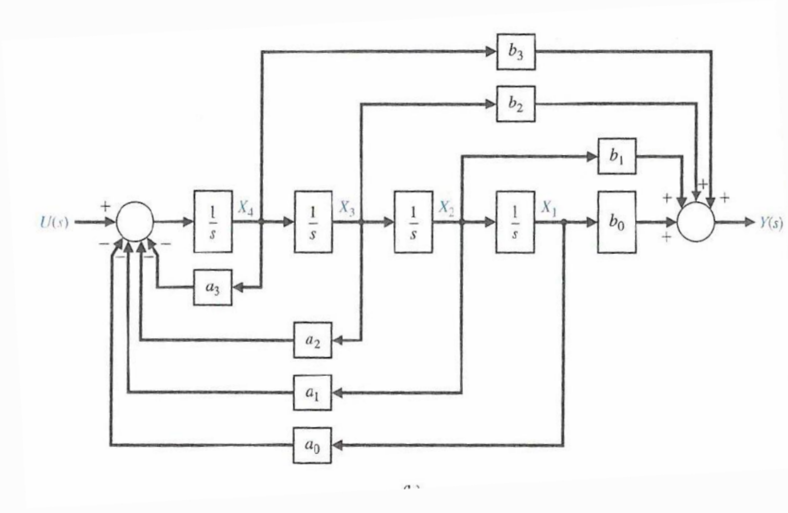
$$U(t) = \ddddot{Z}(t) + a_3 \dddot{Z}(t) + a_2 \ddot{Z}(t) + a_1 \dot{Z}(t) + a_0 Z(t)$$

$$\left.\begin{aligned} X_1 &= Z \\ X_2 &= \dot{Z} \\ X_3 &= \ddot{Z} \\ X_4 &= \dddot{Z} \end{aligned}\right\} \boxed{\begin{aligned} \dot{X}_1 &= X_2 \\ \dot{X}_2 &= X_3 \\ \dot{X}_3 &= X_4 \\ \dot{X}_4 &= -a_0 X_1 - a_1 X_2 - a_2 X_3 - a_3 X_4 + U(t) \end{aligned}}$$

output equation: $\boxed{Y = b_0 X_1 + b_1 X_2 + b_2 X_3 + b_3 X_4}$

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \\ \dot{X}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \cdot U$$

$$Y(t) = \begin{bmatrix} b_0 & b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} + [0] \cdot U$$



Converting from State Space to a TF

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}u$$
$$y = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}u$$

where y is the single output and u is the single input

$$G(s) = \frac{Y(s)}{U(s)}$$

Let´s apply the Laplace transform, we get

$$s\boldsymbol{X}(s) = \boldsymbol{A}\boldsymbol{X}(s) + \boldsymbol{B}U(s) \quad (1)$$
$$Y(s) = \boldsymbol{C}\boldsymbol{X}(s) + \boldsymbol{D}U(s) \quad (2)$$

$$\underline{X}(s)\left[s\underline{\underline{I}} - \underline{\underline{A}}\right] = \underline{B}\,U(s) \qquad \text{with } \left[s\underline{\underline{I}} - \underline{\underline{A}}\right]^{-1} = \underline{\underline{\Phi}}(s)$$

$$\underline{X}(s) = \left[s\underline{\underline{I}} - \underline{\underline{A}}\right]^{-1}\underline{B}\,U(s)$$

$$\left[s\underline{\underline{I}} - \underline{\underline{A}}\right]^{-1} = \frac{\mathrm{adj}(s\underline{\underline{I}} - \underline{\underline{A}})}{\det(s\underline{\underline{I}} - \underline{\underline{A}})}$$

$$\underline{X}(s) = \underline{\underline{\Phi}}(s)\,\underline{B}\,U(s)$$

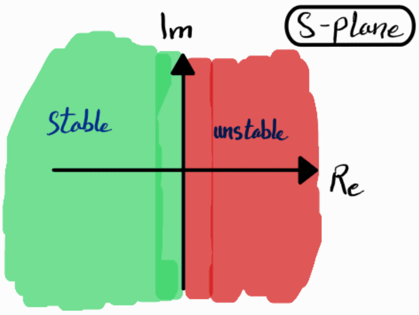
Into (2)

$$Y(s) = \underline{C}\,\underline{\underline{\Phi}}(s)\,\underline{B}\,U(s) + \underline{D}\,U(s)$$

$$\boxed{G(s) = \frac{Y(s)}{U(s)} = \underline{C}\,\underline{\underline{\Phi}}(s)\,\underline{B} + \underline{D}}$$

Stability: depends on the system's poles

denominator $= 0 \implies \boxed{\det(s\underline{\underline{I}} - \underline{\underline{A}}) = 0}$



$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

add the red product,
substract the green product.

Adjoint of 2x2 Matrix

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ — Interchange — Change signs

$$\text{adj } A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example

Given the system defined by the equation

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \boldsymbol{x}$$

find the TF $G(s) = \frac{Y(s)}{U(s)}$, where $U(s)$ is the input and $Y(s)$ the output.

$$[s\underline{\underline{I}} - A] = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 2 & s+3 \end{bmatrix}$$

$$\underline{\underline{M}}^{-1} = \frac{\operatorname{adj} \underline{\underline{M}}}{\det \underline{\underline{M}}} \quad , \quad \underline{\underline{M}}^{-1} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^{-1} = \frac{1}{\det(M)} \begin{bmatrix} ei-fh & ch-bi & bf-ce \\ fg-di & ai-cg & cd-af \\ dh-eg & bg-ah & ae-bd \end{bmatrix}$$

$$[s\underline{\underline{I}} - A]^{-1} = \frac{\begin{bmatrix} s(s+3)+2 & s+3 & 1 \\ -1 & s(s+3) & s \\ -s & -(1+2s) & s^2 \end{bmatrix}}{\underbrace{s(s^2+3s+2)+1(+1)}_{s^3+3s^2+2s+1 = \nabla}} = \underline{\underline{\Phi}}(s)$$

$$\underline{C} \cdot \underline{\underline{\Phi}} = [1 \quad 0 \quad 0] \cdot \underline{\underline{\Phi}}(s) = \frac{1}{\nabla} \cdot [s(s+3)+2 \quad (s+3) \quad 1]$$

$$G(s) = \underline{C} \cdot \underline{\underline{\Phi}} \cdot \underline{B} + \cancel{D}^{\,0} = \frac{1}{\nabla} \cdot [s(s+3)+2 \quad (s+3) \quad 1] \cdot \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} = \frac{10}{\nabla}(s^2+3s+2)$$

$$\boxed{G(s) = \frac{10(s^2+3s+2)}{s^3+3s^2+2s+1}}$$ ★

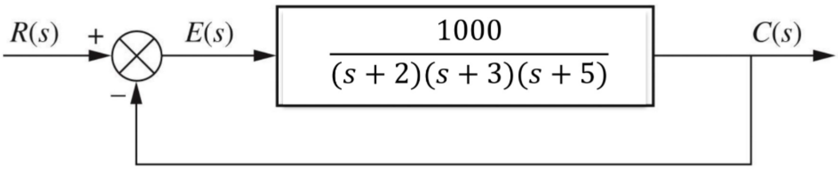
stability: $\det(s\underline{\underline{I}} - \underline{A}) = 0 \Rightarrow s^3+3s^2+2s+1 = 0$

s^3	1	2
s^2	3	1
s^1	$b_1 = \frac{-\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}}{3} = \frac{5}{3}$	$b_2 = \frac{-\begin{vmatrix} 1 & 0 \\ 3 & 0 \end{vmatrix}}{3} = 0$
s^0	$c_1 = \frac{-\begin{vmatrix} 3 & 1 \\ \frac{5}{3} & 0 \end{vmatrix}}{\frac{5}{3}} = 1$	$c_2 = \frac{-\begin{vmatrix} 3 & 0 \\ \frac{5}{3} & 0 \end{vmatrix}}{\frac{5}{3}} = 0$

No sign change in the 1st column
→ The system is stable

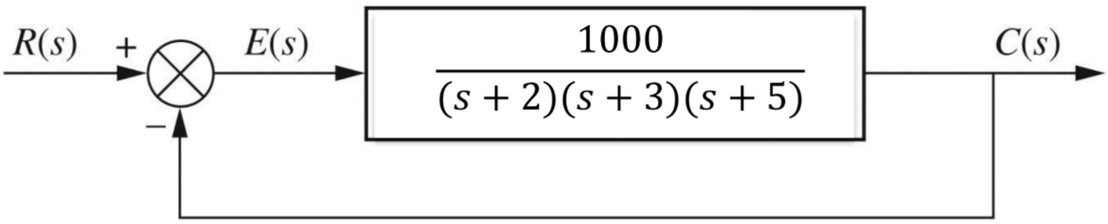
- System is stable if there are no sign changes in the first column of the Routh table

Example



s^3	a_3	a_1
s^2	a_2	a_0
s^1	$b_1 = \dfrac{-\begin{vmatrix} a_3 & a_1 \\ a_2 & a_0 \end{vmatrix}}{a_2}$	$b_2 = \dfrac{-\begin{vmatrix} a_3 & 0 \\ a_2 & 0 \end{vmatrix}}{a_2} = 0$
s^0	$c_1 = \dfrac{-\begin{vmatrix} a_2 & a_0 \\ b_1 & 0 \end{vmatrix}}{b_1}$	$c_2 = \dfrac{-\begin{vmatrix} a_2 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$

$$G(s) = \frac{\dfrac{1000}{(s+2)(s+3)(s+5)}}{1 + \dfrac{1000}{(s+2)(s+3)(s+5)}} = \frac{1000}{s^3+10s^2+31s+1030}$$



s^3	$a_3 = 1$	$a_1 = 31$
s^2	$a_2 = 10$	$a_0 = 1030$
s^1	$b_1 = \dfrac{-\begin{vmatrix} 1 & 31 \\ 10 & 1030 \end{vmatrix}}{10} = -72$	$b_2 = 0$
s^0	$c_1 = \dfrac{-\begin{vmatrix} 10 & 1030 \\ -72 & 0 \end{vmatrix}}{-72} = 1030$	$c_2 = 0$

Sign Change

Sign Change

$$G(s) = \frac{1000}{s^3+10s^2+31s+1030}$$

$$G(s) = \frac{1000}{a_3s^3+a_2s^2+a_1s^1+a_0}$$

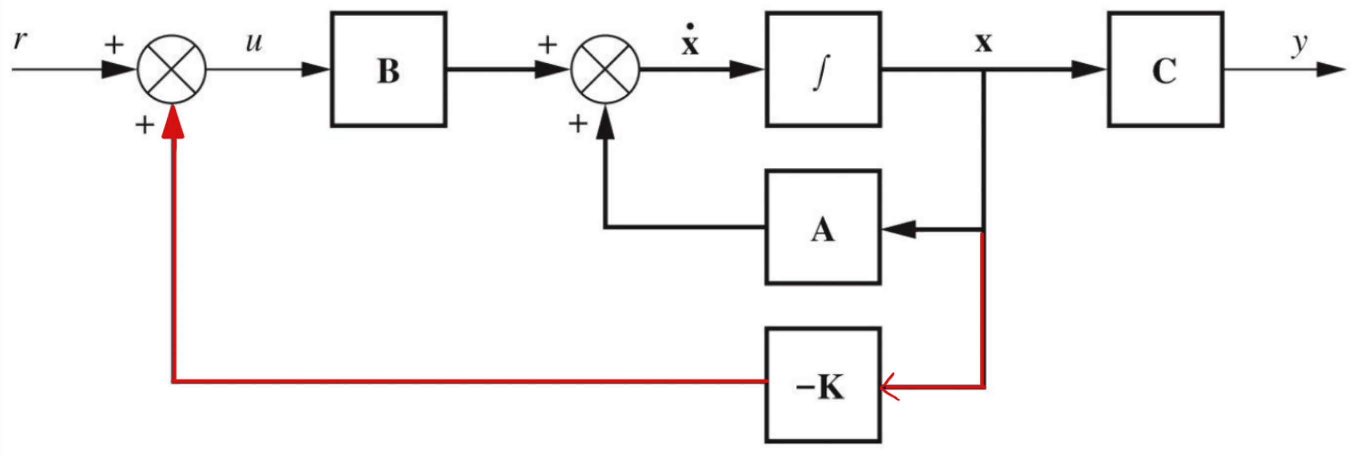
$\Rightarrow$ 2 sign change

$\Rightarrow$ 2 unstable poles.

Quadratic formula:

$$P_{1,2} = \frac{-b}{2} \pm \sqrt{\frac{b^2}{4} - c}$$

State equation for the closed-loop system:



$$\dot{x} = \underline{B}\,\underline{u} + \underline{A}\,\underline{x} = \underline{B}(r + (-\underline{K}\,\underline{x})) + \underline{A}\,\underline{x} = (\underline{A} - \underline{B}\,\underline{K})\,\underline{x} + \underline{B}\,r$$

$$y = \underline{C}\,\underline{X}$$

Comparizon:

Open-loop: $\dot{x} = Ax + Bu \longrightarrow$ system matrix is A

$\rightarrow$ Eigenvalues of system matrix = poles

Closed-loop: $\dot{x} = (A - BK)x + Br \longrightarrow$ system matrix is $(A - BK)$

Example: An unstable system, characterized through the state space model:

$\dot{x} = \underbrace{\begin{bmatrix} 4 & 8 \\ 1 & -5 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B} u$ is controlled by a feedback vector $K = [1 \quad k_2]$

Calculate k_2 so that the controlled system is stable

$$BK = \begin{bmatrix} 1 & K_2 \\ 0 & 0 \end{bmatrix}, \quad [A - BK] = \begin{bmatrix} 3 & 8 - K_2 \\ 1 & -5 \end{bmatrix}$$

$$\det\left(s\underline{I} - [A - BK]\right) = \begin{bmatrix} s - 3 & K_2 - 8 \\ -1 & s + 5 \end{bmatrix} = (s-3)(s+5) + (K_2 - 8)$$
$$= s^2 + 2s + (K_2 - 23)$$

Routh table

s^2	1	$K_2 - 23$
s^1	2	0
s^0	$\dfrac{-\begin{vmatrix} 1 & K_2 - 23 \\ 2 & 0 \end{vmatrix}}{2} = K_2 - 23$	

$$K_2 - 23 > 0$$

$$K_2 > 23$$

For a stable system with this feedback vector

Pole placement for plants in phase-variable form

(where the system is in Canonical form and the Poles are given, we need to determine the feedback vector K)

Step 1) Represent the plant in phase-variable (canonical form).

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}; \boldsymbol{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$G(s) = \frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}$$

$$\boldsymbol{C} = [c_1 \quad c_2 \quad \cdots \quad c_n]$$

The characteristic equation is thus

$$s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = 0$$

Step 2) $K = [K_1 \quad K_2 \quad \cdots\cdots \quad K_n]$

Step 3) characteristic equ. for closed-loop system

$$\underline{A} - \underline{B}\,\underline{K} = \begin{bmatrix} 0 & 1 & 0 & --- & 0 \\ 0 & 0 & 1 & --- & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -(a_0+K_1) & -(a_1+K_2) & -(a_2+K_3) & ---- & -(a_{n-1}+K_n) \end{bmatrix}$$

$$\det(s\boldsymbol{I} - (\boldsymbol{A} - \boldsymbol{BK})) = s^n + (a_{n-1} + k_n)s^{n-1} + (a_{n-2} + k_{n-1})s^{n-2} + \cdots + (a_1 + k_2)s + (a_0 + k_1) = 0$$

Notice the relationship to the open-loop system → open-loop characteristic equation by adding the appropriate k_i to each coefficient.

Step 4) Use the poles to get the polynomial equ. of the deminator.

Now assume that the desired characteristic equation for proper pole placement is;

$$(S-P_1)(S-P_2)(S-P_n) = s^n + d_{n-1}s^{n-1} + d_{n-2}s^{n-2} + \cdots + d_2 s^2 + d_1 s + d_0$$

Step 5) Equating Coefficients (K_i)

$$d_i = a_i + K_{i+1} \; ; \quad i = 0, 1, 2, \ldots, n-1$$

from which $\Rightarrow$ $\boxed{K_{i+1} = d_i - a_i}$

Controllability:

For SISO-Systems, the Controllability matrix is

$$C_M = [B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B] \quad , \text{rank}(C_M) = n$$

$\longrightarrow n \times n$ matrix $\longrightarrow$ The system is controllable if $\det(C_M) \neq 0$

Example: let's consider the system:

$$\dot{\underline{x}} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}}_{A} \underline{x} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{B} u$$

Is the system controllable?

$$C_M = [B \quad AB \quad A^2B]$$

$$AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -a_2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \\ a_0a_2 & a_1a_2 & a_2^2 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \\ a_0a_2 & a_1a_2 & a_2^2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -a_2 \\ a_2^2 \end{bmatrix}$$

$$C_M = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -a_2 \\ 1 & -a_2 & a_2^2 \end{bmatrix}$$

$$\det(C_M) = 1 \cdot \begin{vmatrix} 0 & 1 \\ 1 & -a_2 \end{vmatrix} = -1 \neq 0$$

$\therefore$ The system is controllable

Consider the system

$$G(s) = \frac{1}{s^2}$$

and check if the system is controllable. If so, determine the feedback gain to place the closed-loop poles at $p_{1,2} = -1 \pm j$.

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ \underbrace{0}_{-a_0} & \underbrace{0}_{-a_1} \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot u$$

$$y = [\underbrace{1}_{b_0} \quad \underbrace{0}_{b_1}] \underline{x} + [0] \cdot u$$

$$\underline{C_M} = [B \quad AB] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad , \det(C_M) = -1 \neq$$

$\therefore$ The system is Controllable.

step 2) $\underline{K} = [K_1 \quad K_2]$

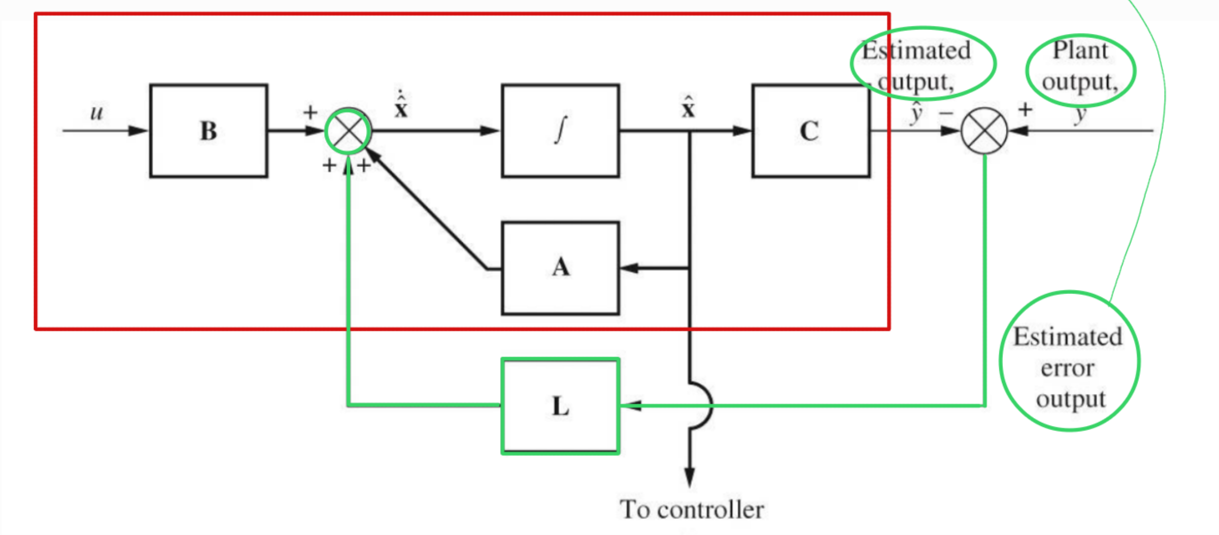
step 3) $\underline{\underline{A}} - \underline{B}\,\underline{K} = \begin{bmatrix} 0 & 1 \\ -K_1 & -K_2 \end{bmatrix}$

step 4) $(s+1-j)(s+1+j) = s^2+2s-\cancel{sj}+\cancel{sj}-\cancel{j}+\cancel{j}+1-(-1) = s^2\underbrace{+2}_{d_1}s+\underbrace{2}_{d_0}$

step 5) $d_1 = -K_2 \Longrightarrow \boxed{K_2 = -2}$, $d_0 = -K_1 \Longrightarrow \boxed{K_1 = -2}$

$\therefore \underline{K} = [-2 \quad -2]$ is the feedback gain to the closed-loop poles at $p_{1,2} = -1 \pm j$

In red is corrected with what is in green.



Digital Computer systems:

Application:

* Industry has grown over the past three decades.
* Transistor density
* Powerful
* Mobile capability
* Improved measurement sensitivty.
* Flexibility
* Reconfigure control algorithm in software

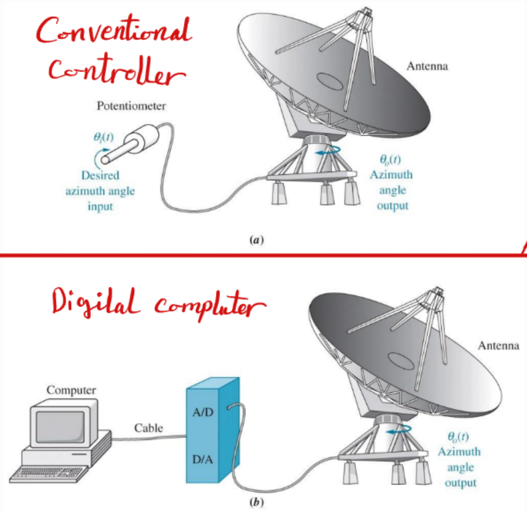
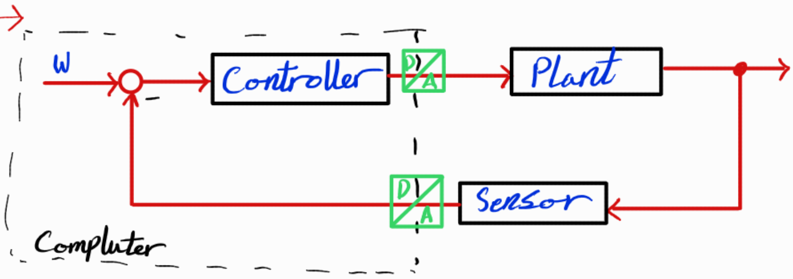


Figure 13.1



at input of digital comp.

at output

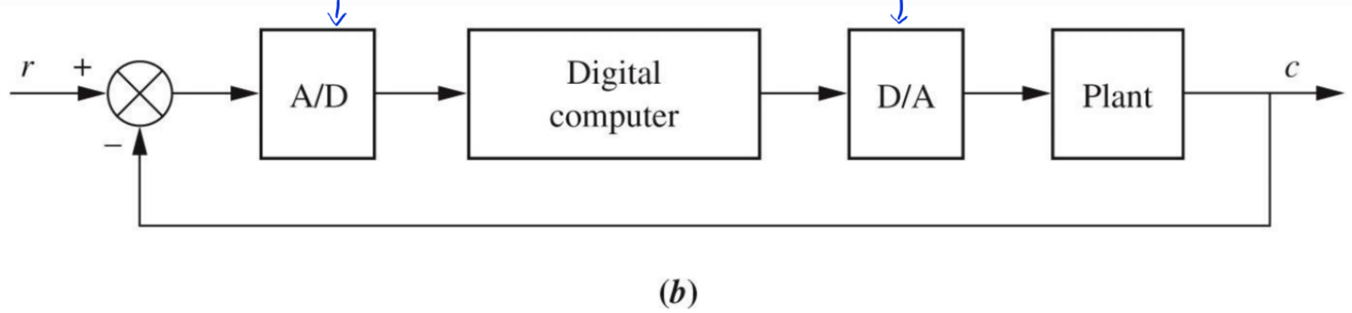


Figure 13.2

- Analog signal (a)
- Analog signal sampled at periodic intervals and held over the sampling interval (b)
- Device so called zero-order sample-and-hold (z.o.h.)
- After sampling and holding, the A/D-converter converts to a digital number (c)
- Quantization error
- Stability and transient response are now dependent on sampling rate

→ taking conversion into account during modeling

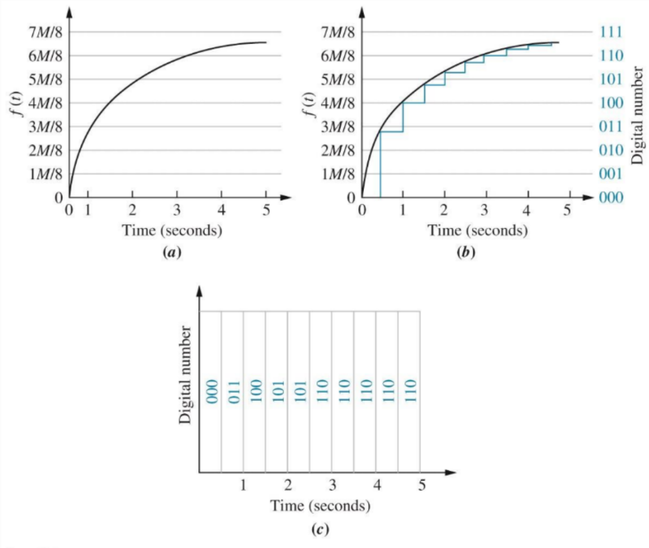
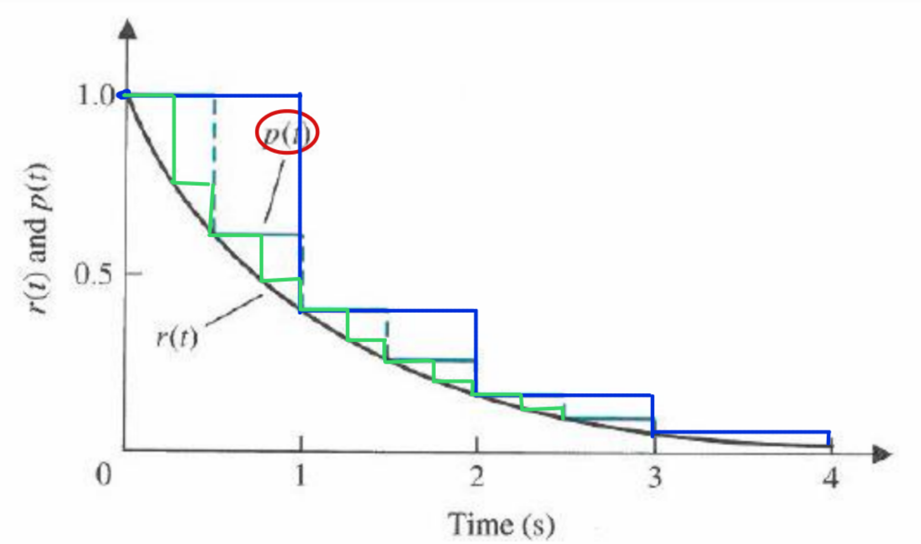


Figure 13.4



(a) $T = 0.5$ s

$T = 1s$

$T = 0.25$

Sampling time (T) is important for many factors, which stability is one of them.

$T \uparrow \Rightarrow$ stability $\downarrow$

$$Z.O.h(s) = \frac{1 - e^{-sT}}{s} \Rightarrow Z.O.h.(z) = (1 - z^{-1}) \cdot \mathcal{Z}\left\{\frac{1}{s}\right\}$$

$$\text{or} = \frac{z-1}{z}$$

➔ Difference equation

depends on $\alpha_n \cdots \alpha_0 + T$

$$y(k) + a_1 y(k-1) + a_2 y(k-2) + \cdots + a_m y(k-m)$$
$$= b_0 u(k) + \cdots + b_m u(k-m)$$

$$G(z) = \frac{y(z)}{u(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_m z^{-m}}{1 + a_1 z^{-1} + \cdots + a_m z^{-m}} = \frac{B(z)}{A(z)}$$

Matching coeffients with difference equation

TABLE 13.1 Partial table of z- and s-transforms

	$f(t)$	$F(s)$	$F(z)$	$f(kT)$
1.	$u(t)$	$\frac{1}{s}$	$\frac{z}{z-1}$	$u(kT)$
2.	t	$\frac{1}{s^2}$	$\frac{Tz}{(z-1)^2}$	kT
3.	t^n	$\frac{n!}{s^{n+1}}$	$\lim_{a\to 0}(-1)^n \frac{d^n}{da^n}\left[\frac{z}{z-e^{-aT}}\right]$	$(kT)^n$
4.	e^{-at}	$\frac{1}{s+a}$	$\frac{z}{z-e^{-aT}}$	e^{-akT}
5.	$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$	$(-1)^n \frac{d^n}{da^n}\left[\frac{z}{z-e^{-aT}}\right]$	$(kT)^n e^{-akT}$
6.	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$	$\frac{z \sin \omega T}{z^2 - 2z\cos\omega T + 1}$	$\sin \omega kT$
7.	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$	$\frac{z(z-\cos\omega T)}{z^2 - 2z\cos\omega T + 1}$	$\cos \omega kT$
8.	$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$	$\frac{ze^{-aT}\sin\omega T}{z^2 - 2ze^{-aT}\cos\omega T + e^{-2aT}}$	$e^{-akT}\sin \omega kT$
9.	$e^{-at}\cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$	$\frac{z^2 - ze^{-aT}\cos\omega T}{z^2 - 2ze^{-aT}\cos\omega T + e^{-2aT}}$	$e^{-akT}\cos \omega kT$

Table 13.1

TABLE 13.2 z-transform theorems

	Theorem	Name
1.	$z\{af(t)\} = aF(z)$	Linearity theorem
2.	$z\{f_1(t) + f_2(t)\} = F_1(z) + F_2(z)$	Linearity theorem
3.	$z\{e^{-aT}f(t)\} = F(e^{aT}z)$	Complex differentiation
4.	$z\{f(t-nT)\} = z^{-n}F(z)$	Real translation
5.	$z\{tf(t)\} = -Tz\frac{dF(z)}{dz}$	Complex differentiation
6.	$f(0) = \lim_{z\to\infty} F(z)$	Initial value theorem
7.	$f(\infty) = \lim_{z\to 1}(1-z^{-1})F(z)$	Final value theorem

Table 13.2

With $G_1(s) = \frac{1}{s+1}$ & $G_2(s) = \frac{1}{s+2}$

$$G_1(s)\cdot G_2(s) = \frac{1}{(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{s+2}$$

Partial fraction because we don't have a formula for this form

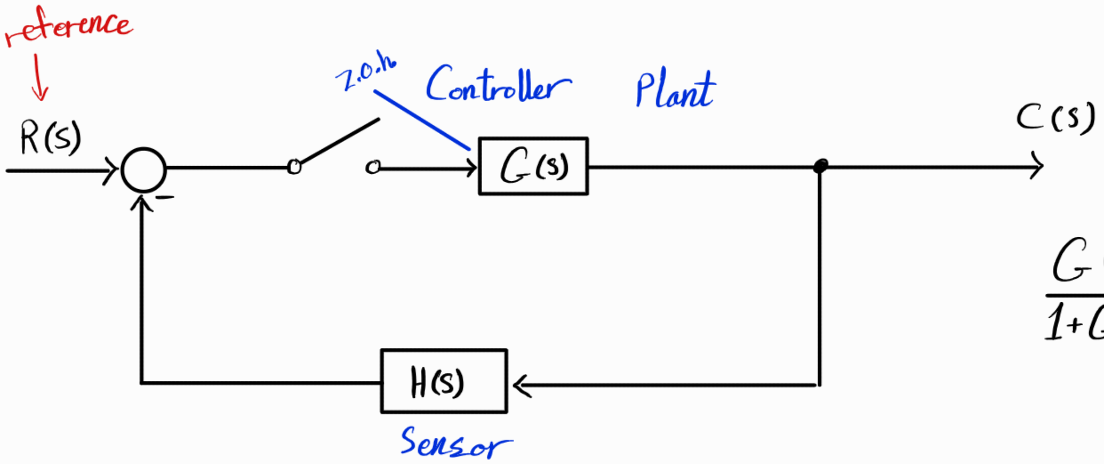
$$G_1(s)\cdot G_2(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$$

$$\mathcal{Z}\left\{G_1(s)\cdot G_2(s)\right\} = \mathcal{Z}\left\{\frac{1}{s+1} - \frac{1}{s+2}\right\} = \frac{z}{z-e^{-T}} - \frac{z}{z-e^{-2T}}$$

$$= \frac{z(z-e^{-2T}) - z(z-e^{-T})}{(z-e^{-T})(z-e^{-2T})} = \frac{z((\cancel{z}-e^{-2T}) - (\cancel{z}-e^{-T}))}{(z-e^{-T})(z-e^{-2T})} = \overset{G_1G_2(z)}{\boxed{\frac{z(e^{-T}-e^{-2T})}{(z-e^{-T})(z-e^{-2T})}}}$$

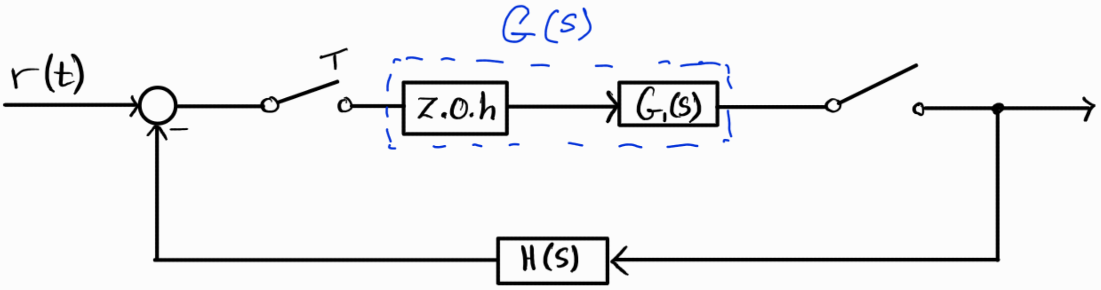
$$\mathcal{Z}\left\{G_1(s)\right\}\cdot\mathcal{Z}\left\{G_2(s)\right\} = \mathcal{Z}\left\{\frac{1}{s+1}\right\}\cdot\mathcal{Z}\left\{\frac{1}{s+2}\right\} = \frac{z}{z-e^{-T}}\cdot\frac{z}{z-e^{-2T}} = \overset{G_1(z)\cdot G_2(z)}{\frac{z^2}{(z-e^{-T})(z-e^{-2T})}}$$

$$\therefore \boxed{G_1G_2(z) \neq G_1(z)\cdot G_2(z)}$$



$$\frac{G(z)}{1+GH(z)}$$

Exam question:



with $G_1(s) = \frac{5}{s}$

$H(s) = \frac{10}{s+10}$

$G_{cL}(z) = \frac{C(z)}{R(z)}$

$$Z.O.h = \frac{1-e^{-sT}}{s} \Rightarrow Z.O.h.(z) = (1 - z^{-1})$$

$$G_{cL}(z) = \frac{G(z)}{\underbrace{1+G(z) \cdot H(z)}_{\text{because of the sampler in between}}}$$

$$G(z) = \mathcal{Z}\left\{\frac{1-e^{-sT}}{s} \cdot \frac{5}{s}\right\} = (1-z^{-1})\,\mathcal{Z}\left\{\frac{5}{s^2}\right\} = \frac{\cancel{z-1}}{\cancel{z}} \cdot \frac{5T\cancel{z}}{(z-1)^{\cancel{2}}}$$

$$\boxed{G(z) = \frac{5T}{(z-1)}}$$

$$H(s) = \frac{10}{s+10} \Longrightarrow \boxed{H(z) = \frac{10z}{z-e^{-10T}}}$$

$$G_{cl}(z) = \frac{G(z)}{1+G(z) \cdot H(z)} = \frac{5T/(z-1)}{1+\frac{50Tz}{(z-1)(z-e^{-10T})}} = \frac{5T\,(\cancel{z-1})(z-e^{-10T})}{(50Tz+(z-1)(z-e^{-10T}))(\cancel{z-1})}$$

$$= \boxed{\frac{5T(z-e^{-10T})}{z^2+z(50T-1-e^{-10T})+e^{-10T}}}$$

Student No.: ____________________

Points:

Question 8: 17

A closed-loop sampled system with the reference input $r(t)$ and the actual output $c(t)$, as shown in block diagram fig. 8.1, consists of a system with the transfer function $G_1(s)$ in the feedforward path, a zero-order-hold in the feedforward path, two samplers which are synchronized with the sampling time T, and a system with the transfer function $H(s)$ in the feedback path.

The following transfer functions are given:

$G_1(s) = \frac{5}{s}$ and $H(s) = \frac{10}{s+10}$

$\tilde{G}(z)\cdot H(z) \neq \tilde{G}H(z)$

$\tilde{G}(s) = \frac{1-e^{-sT}}{s}\cdot\frac{5}{s}$

If this is not there $G_{CL}(z) = \frac{\tilde{G}(z)}{1+\tilde{G}H(z)}$

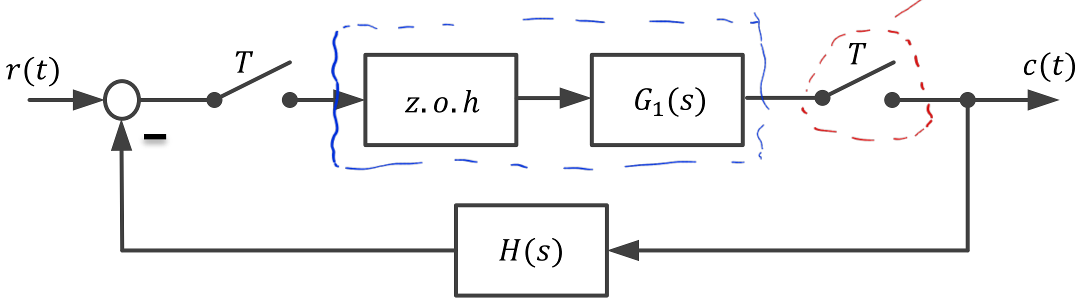


Figure 8.1: Block diagram of closed-loop system

a) Find the sampled-data transfer function $G(z) = \frac{C(z)}{R(z)}$ of the closed-loop system as a function of the sampling time T. *Hint: Check the sampler arrangement!*

b) Is the sampled closed-loop system stable for $T = 10\ [ms]$? Please explain.

$$\tilde{G}(z) = \mathcal{Z}\{1-e^{-sT}\}\cdot\mathcal{Z}\left\{\frac{5}{s^2}\right\} = \frac{\cancel{z}-1}{\cancel{z}}\cdot\frac{5T\cancel{z}}{(z-1)^{\cancel{2}}}$$

$$\boxed{\tilde{G}(z) = \frac{5T}{z-1}}$$

$$H(z) = \mathcal{Z}\left\{\frac{10}{s+10}\right\} = \frac{10z}{z-e^{-10T}}$$

$$G_{CL}(z) = \frac{C(z)}{R(z)} = \frac{\tilde{G}(z)}{1+H(z)\cdot\tilde{G}(z)} = \frac{\frac{5T}{z-1}}{1+\frac{50Tz}{(z-1)(z-e^{-10T})}} = \frac{\frac{5T}{\cancel{z-1}}}{\frac{z^2-z-e^{-10T}z+50Tz+e^{-10T}}{\cancel{(z-1)}(z-e^{-10T})}}$$

$$\boxed{G_{CL}(z) = \frac{5T(z-e^{-10T})}{z^2+z(50T-e^{-10T}-1)+e^{-10T}}}$$

3.2 Exercise

$\tilde{G}(s) \longrightarrow \tilde{G}(z$

A sampled-data system is given in Figure 3.1.

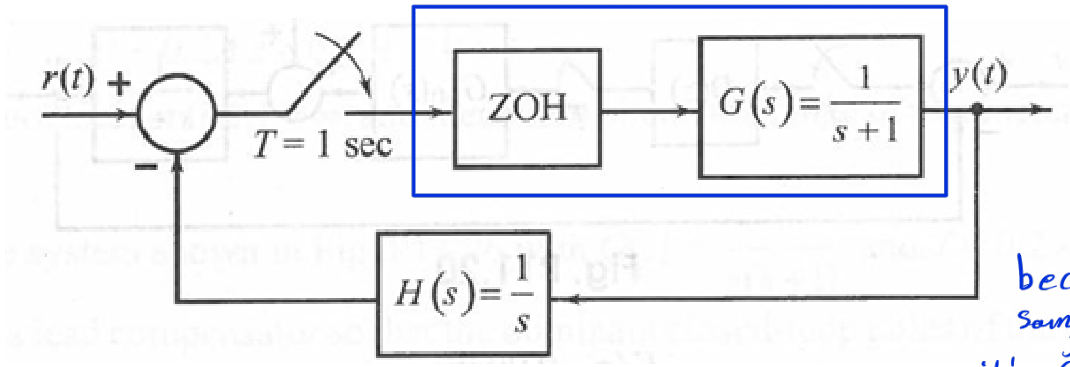


Figure 3.1. sampled-data system

because there is no sampler in between it's $\tilde{G}H(z)$

if there is a sampler $G_{cl}(z) = \frac{\tilde{G}(z)}{1+\tilde{G}(z)\cdot H(z}$

3.2.1 Task

Find out if the closed-loop system $G_{Cl}(z) = \frac{Y(z)}{R(z)}$ is stable or unstable and explain your result.

3.2.2 Solution

Please refer to ??.

$$G_{cl}(z) = \frac{\tilde{G}(z)}{1+\tilde{G}H(z)}$$

$$\tilde{G}(s) = \frac{1-e^{sT}}{s} \cdot \frac{1}{s+1}$$

$$\tilde{G}(z) = Z\left\{\frac{1-e^{sT}}{s} \cdot \frac{1}{s+1}\right\} = (1-z^{-1})\, Z\left\{\frac{1}{s} - \frac{1}{s+1}\right\} = \frac{1-e^{-T}}{z-e^{-T}}$$

$$\tilde{G}(z) = \frac{0.63}{z-0.37}$$

$$\tilde{G}H(z) = Z\left\{\frac{1-s^{sT}}{s} \cdot \frac{1}{s+1} \cdot \frac{1}{s}\right\} = (1-z^{-1})\, Z\left\{\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}\right\}$$

$$= \frac{z-1}{z}\left(\frac{Tz}{(z-1)^2} - \frac{z}{z-1} + \frac{z}{z-e^{-T}}\right) = \left(\frac{T}{z-1} - 1 + \frac{z-1}{z-e^{-T}}\right)$$

$$\tilde{G}H(z) = \frac{0.37z+0.26}{z^2-1.37z+0.37}$$

$$G_{cl}(z) = \frac{0.63(z-1)}{z^2-z+0.63}$$

Stable

3.3 Exercise

A sampled-data system is given in Figure 3.2.

$G(s)$

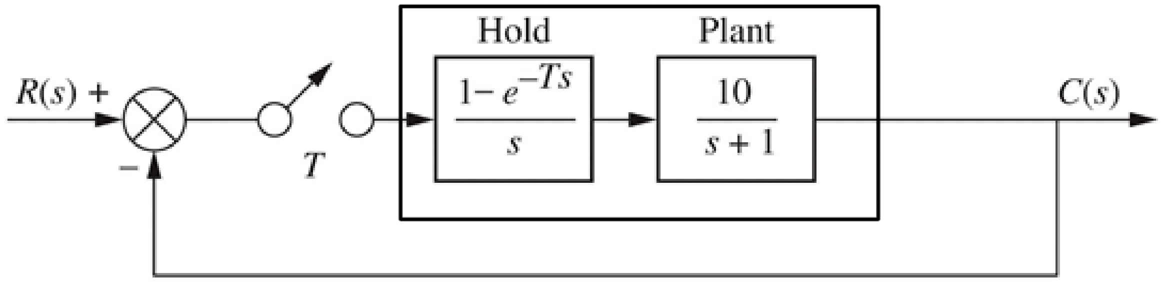


Figure 3.2. Diagram system

3.3.1 Task

Determine the range of sampling interval, T , which makes the system shown in Figure 3.2 stabel, and furthermore give the range that will lead to an unstable system.

3.3.2 Solution

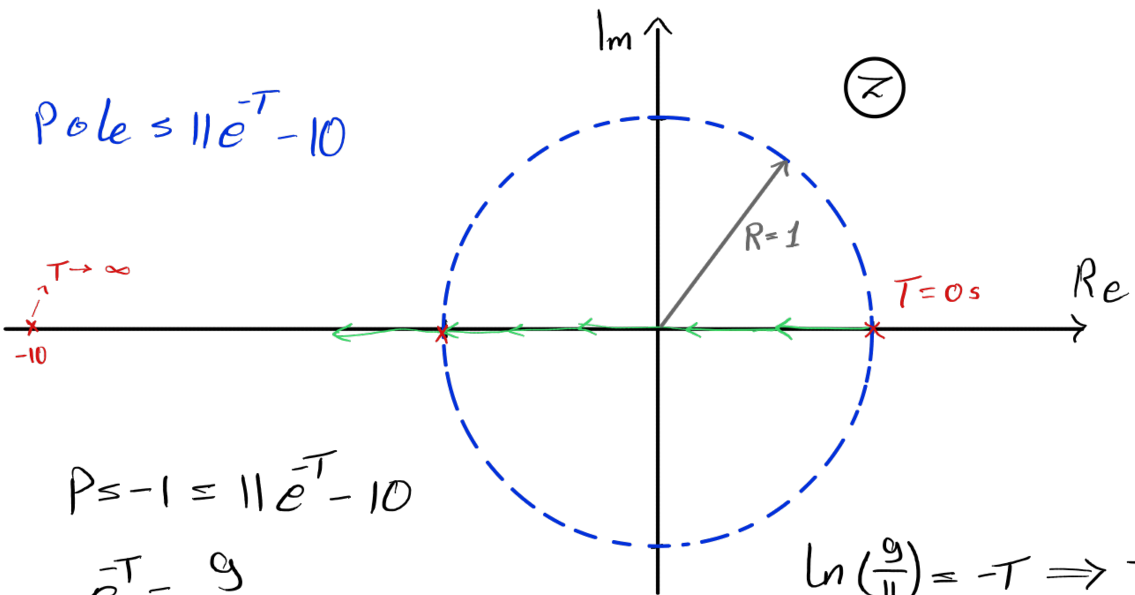
Please refer to ??.

$$G(z) = \mathcal{Z}\left\{\frac{1-e^{-Ts}}{s} \cdot \frac{10}{s+1}\right\} = \left(\frac{z-1}{z}\right)\mathcal{Z}\left\{\frac{1}{10s} - \frac{1}{10(s+1)}\right\}$$

$$G(z) = 10 \cdot \frac{1-e^{-T}}{z-e^{-T}}$$

$$G_{cl}(z) = \frac{10(1-e^{-T})}{z - e^{-T} + 10 - 10e^{-T}} = \frac{num}{z - 11e^{-T} + 10}$$

$$\text{Pole} = 11e^{-T} - 10$$



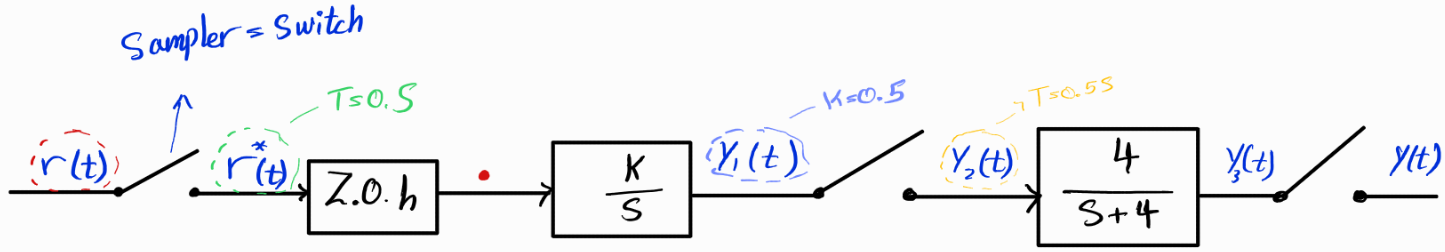
$$P = -1 = 11e^{-T} - 10$$

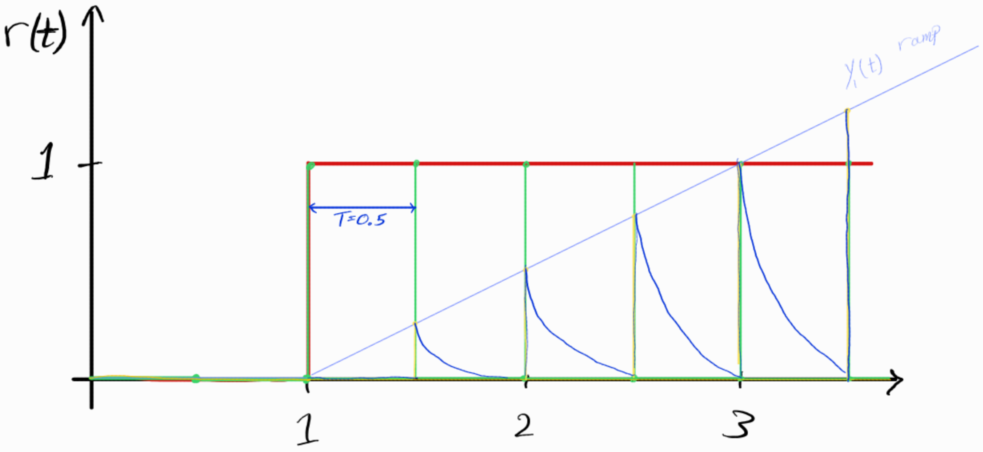
$$e^{-T} = \frac{9}{11}$$

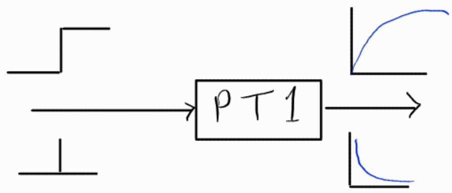
$$\ln\left(\frac{9}{11}\right) = -T \Rightarrow T = -\ln\left(\frac{9}{11}\right) = 0.2\ S$$

$$\therefore \boxed{0 < T < 0.2}$$

EXAM QUESTION:







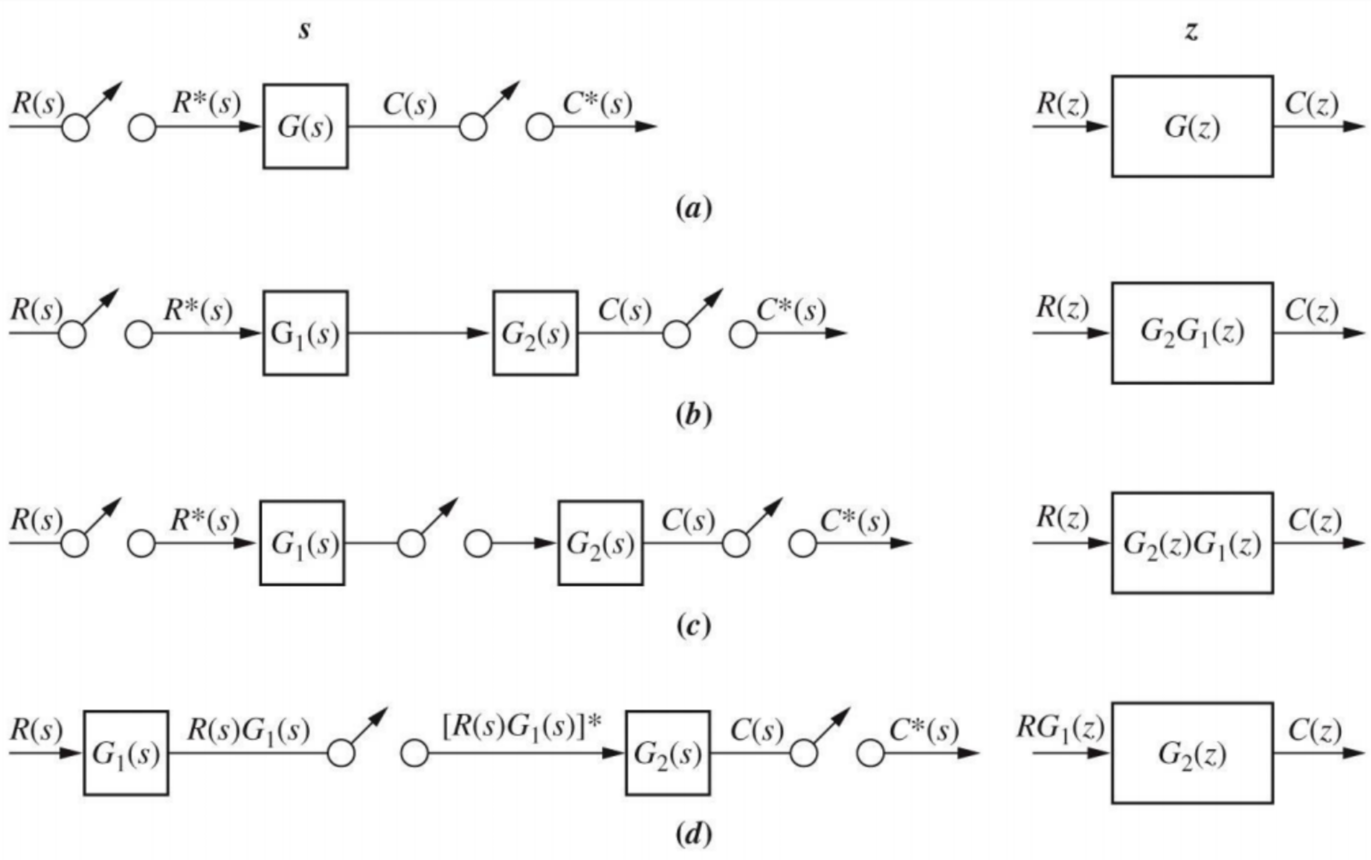


Figure 13.9

طب لما في sampler ما بينهم يبقى هيضربوا في z-domain

لو مافيش يبقى يضربوا في s-domin و بعدين يتحولوا الـ Z