(i) $f(x) \geqslant 0$ for all x ✓ we can also see it from the graph

(ii) $\int_{-\infty}^{+\infty} f(x)\,dx = 1$ ✓

$$\Rightarrow 0.006\int_{-\infty}^{\infty}(25-x^2)\,dx = 0 + 0.006\int_{-5}^{5}(25-x^2)\,dx = 0.006\left[25x-\frac{x^3}{3}\right]_{-5}^{5} = 0.006\left[125\left(1-\frac{1}{3}\right)-125\left(\frac{1}{3}-1\right)\right]$$

$$= \frac{\cancel{6}^{\,2}}{\cancel{1000}_{\,4}}\,(\cancel{250})\left(\frac{2}{\cancel{3}}\right) = \frac{4}{4} = 1 \checkmark$$

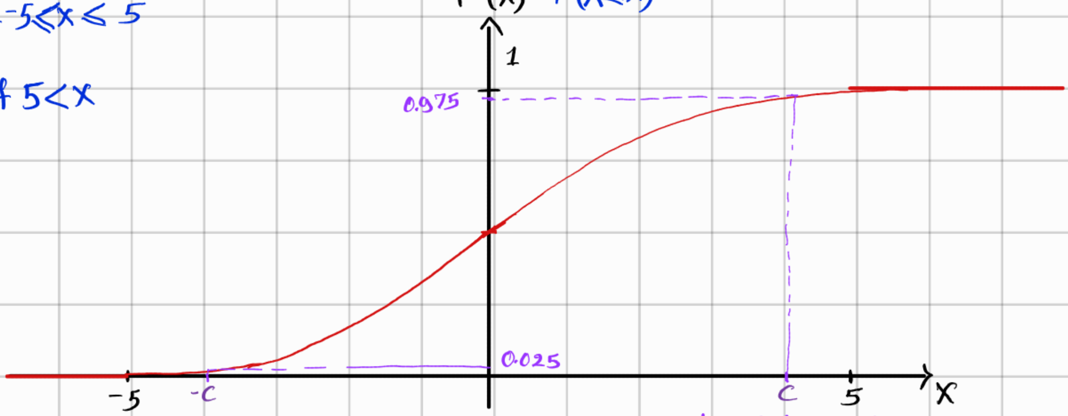
It's a Pdf function.

b) $$F(x) = \int_{-\infty}^{x} f(k)\,dk = 0.006\int_{-5}^{x}(25-k^2)\,dk = 0.006\left[25k-\frac{k^3}{3}\right]_{-5}^{x} = 0.006\left(25x-\frac{x^3}{3}\right) + \frac{\cancel{6}^{\,2}}{\cancel{1000}_{\,4}}\cdot\frac{250}{\cancel{3}}$$

$$= 0.006x\left(25-\frac{x^2}{3}\right) + 0.5$$

$$F(x) = \begin{cases} 0 & \text{if } x < -5 \\ 0.006x\left(25-\frac{x^2}{3}\right)+0.5 & \text{if } -5 \leqslant x \leqslant 5 \\ 1 & \text{if } 5 < x \end{cases}$$

Cdf is Always Continuous for a Continuous R.V.



Make sure that $1-0.95 = 0.05$ is devided equally for the upper and lower part of cdf graph.

C) From cdf graph and equation

$$0.006(25)c - 0.006\frac{c^3}{3} + 0.5 = 0.975$$

$$-0.002c^3 + 0.15c - 0.475 = 0$$

$$\Rightarrow 2c^3 - 150c + 475 = 0$$

Newton's method:

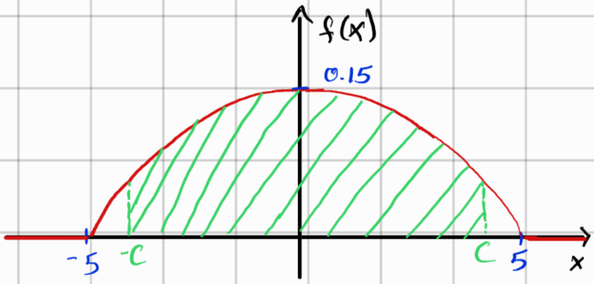
with initial guess $c_0 = 4$

$f(c=4) = 128 - 600 + 475 = 3$

$f'(c=4) = 6(16) - 150 = -54$

$$c_{n+1} = c_0 - \frac{f(c_0)}{f'(c_0)} = 4 + \frac{3}{54} \approx 4.055$$

Thus $P(-4.055 \leqslant X \leqslant 4.055) = 95\%$



$$\int_{-c}^{c} f(x)\,dx = 0.95$$

$$0.006\left[25x-\frac{x^3}{3}\right]_{-c}^{c} = 0.006\left[\left(25c-\frac{c^3}{3}\right)+\left(25c-\frac{c^3}{3}\right)\right]$$

$$\frac{\cancel{12}^{\,3}}{\cancel{1000}_{\,10\cancel{40}}}\cdot\cancel{25}c\ \ \frac{\cancel{12}^{\,4}}{1000}\ \frac{c^3}{\cancel{3}} = 0.95$$

$$\Rightarrow -0.004c^3 + 0.3c - 0.95 = 0$$

The expected value (= The mean) of a R.V. is

$$E[X] = \sum_x x \cdot P(X=x) \quad \text{, if } X \text{ is a Discrete R.V.}$$

$$E[X] = \int_{-\infty}^{\infty} x\, f(x)\, dx \quad \text{, if } X \text{ is continuous with pdf } f(x).$$

Example: Computing $E[X]$ for Discrete X.

If X has few outcomes use a spreadseet

(Rolling a fair 20-sided die.

x	$P(X=x)$	$x \cdot P(X=x)$
1	1/20	1/20
2	1/20	2/20
⋮	⋮	⋮
20	1/20	20/20

$$E[X] = \frac{1+2+3+\cdots+20}{20}$$

$$= \frac{1}{\cancel{20}} \cdot \frac{\cancel{20}(20+1)}{2} = 10.5$$

$$\sum_{K=1}^{n} K = 1+2+\cdots+n = \frac{n(n+1)}{2}$$

*Expected value doesn't have to be a value of the random variable.

$$\boxed{E[X+Y] = E[X] + E[Y]}$$

Linear Transformation

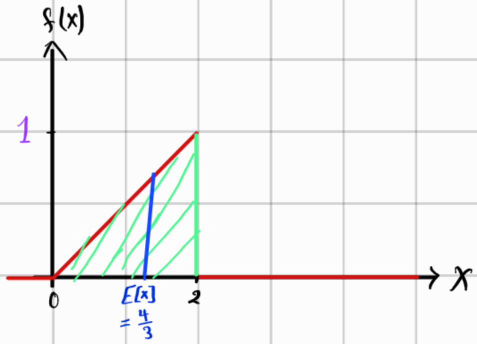
$$\boxed{E[a \cdot X + b] = a \cdot E[X] + b}$$

Valid for both discrete & continuous R.V.

Non-Linear Transformation

$$E[g(x)] = \begin{cases} \sum_x g(x) \cdot P(X=x) \\ \int_{-\infty}^{\infty} g(x) \cdot f(x)\, dx \end{cases}$$

Example: Continuous R.V. X with pdf f



in order to have $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$f(x) = \begin{cases} \frac{1}{2}x & \text{, if } 0 \leq x \leq 2 \\ 0 & \text{, otherwise} \end{cases}$$

$$E[X] = \int_{-\infty}^{\infty} x \cdot f(x)\, dx = \frac{1}{2}\int_0^2 x^2\, dx = \frac{1}{2}\left[\frac{x^3}{3}\right]_0^2 = \frac{1}{2} \cdot \frac{8}{3} = \frac{4}{3}$$

Independant Random Variable:

$P(A|B) = P(A)$ $\longrightarrow$ because A & B are independant events, so the given event B doesn't affect the possibility of event A.

Two events A & B are said to be Independent if:

$$\boxed{P(A \cap B) = P(A) \cdot P(B)}$$

since we always have:

$$P(A \cap B) = P(A|B) \cdot P(B) = P(B|A) \cdot P(A)$$

$P(A) \neq 0$

$P(B) \neq 0$

Independence Implies:

$$P(A|B) = P(A) \quad \text{and} \quad P(B|A) = P(B)$$

Random Variables X and Y are Independent if

$$P(X \leqslant x \text{ and } Y \leqslant y) = P(X \leqslant x) \cdot P(Y \leqslant y) \quad \text{for all } x, y$$

When Knowing about one event (R.V.) does not tell us anything about the other event (R.V.), then these events (Random variables) are Independent.

Variance:

To avoid the balancing we define:

The variance of a R.V. X is

expected value $E[X]$

$\sigma^2 = Var(X) = E[(X-E[X])^2] = E[(X-\mu)^2]$ ("Excepected square deviation")

It's square root is called the Standard Deviation σ (or σ_x).

$$E[(X-\mu)^2] = E[X^2-2\mu X+\mu^2] = E[X^2]+E[-2\mu X]+E[\mu^2]$$

$$\Rightarrow E[X^2]\underbrace{-2\mu E[X]}_{-2\mu^2}+\mu^2 = \boxed{E[X^2]-\mu^2}$$

with

$$E[X^2] = \begin{cases} \sum_x x^2 \cdot P(X=x) & \text{If } X \text{ is Discrete R.V.} \\ \int_{-\infty}^{+\infty} x^2 f(x)\,dx & \text{If } X \text{ is Continuous R.V. with Pdf } f(x) \end{cases}$$

$$\sigma^2 = Var(X) = E[X^2]-\mu^2 = E[X^2]-(E[X])^2$$

$$\sigma = \sqrt{Var(X)}$$ Standard Deviation

Formulas for variances:

$Var(X+Y) = Var(X)+Var(Y)+2CoV(X,Y)$

where the covariance of X and Y is

$CoV(X,Y) = E[(X-E[X])\cdot(Y-E[Y])] = E[X\cdot Y]-\mu_x\cdot\mu_y$

If X and Y are Independent

$Var(X+Y) = Var(X)+Var(Y)$

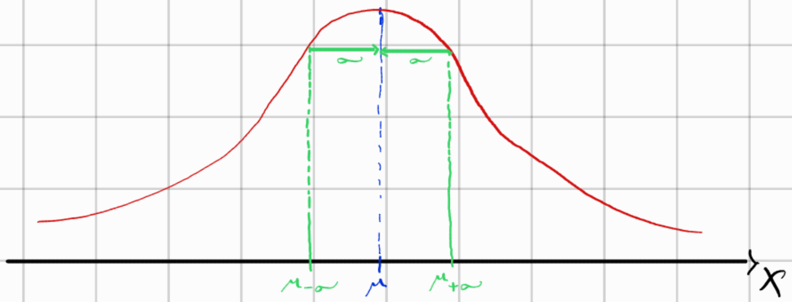
Formula for linear Transformation

$Var(aX+b) = a^2\cdot Var(X) = a^2(E[X^2]-(E[X])^2)$

$\sigma_{aX+b} = |a|\cdot\sigma_X$

A Random variable with a normal distribution is continuous with

Pdf $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ for $-\infty < x < +\infty$

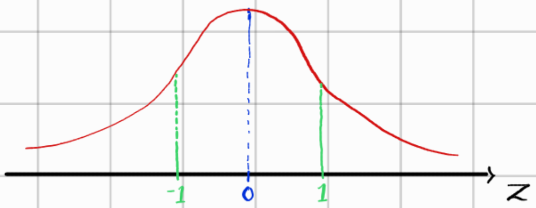


- Maximum at $\mu = E[x]$
- Symmetric about μ
- Points of inflection at $\mu \pm \sigma$
- everywhere positive
- Horizontal asymptote at $y = 0$

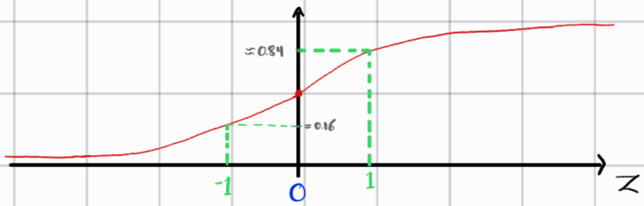
$$\boxed{Z = \frac{x - \mu_x}{\sigma_x}}$$

R.V. with standard Normal distribution is usually denoted by Z :-

Pdf $\phi(z) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{z^2}{2}}$

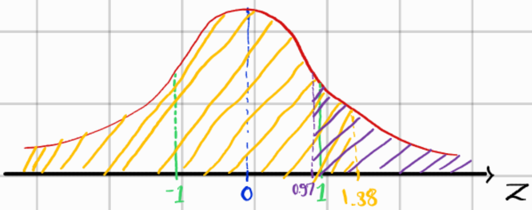


cdf $\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{t^2}{2}} \, dt$



Cumulative Standard Normal Distribution Function

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

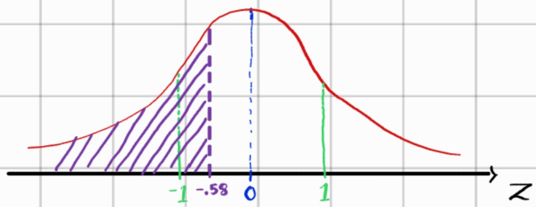


$P(Z \leq 1.38) = \Phi(1.38) = 0.9162$

$P(Z > 0.97) = 1 - P(Z \leq 0.97) = 1 - 0.834$
$= 0.166$

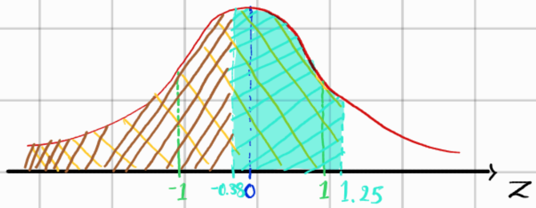
Cumulative Standard Normal Distributon Function

0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0.01	0.00	z
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	-3.9
0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	-3.8
0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	-3.7
0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0002	0.0002	-3.6
0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	-3.5
0.0002	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	-3.4
0.0003	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0005	0.0005	0.0005	-3.3
0.0005	0.0005	0.0005	0.0006	0.0006	0.0006	0.0006	0.0006	0.0007	0.0007	-3.2
0.0007	0.0007	0.0008	0.0008	0.0008	0.0008	0.0009	0.0009	0.0009	0.0010	-3.1
0.0010	0.0010	0.0011	0.0011	0.0011	0.0012	0.0012	0.0013	0.0013	0.0013	-3.0
0.0014	0.0014	0.0015	0.0015	0.0016	0.0016	0.0017	0.0018	0.0018	0.0019	-2.9
0.0019	0.0020	0.0021	0.0021	0.0022	0.0023	0.0023	0.0024	0.0025	0.0026	-2.8
0.0026	0.0027	0.0028	0.0029	0.0030	0.0031	0.0032	0.0033	0.0034	0.0035	-2.7
0.0036	0.0037	0.0038	0.0039	0.0040	0.0041	0.0043	0.0044	0.0045	0.0047	-2.6
0.0048	0.0049	0.0051	0.0052	0.0054	0.0055	0.0057	0.0059	0.0060	0.0062	-2.5
0.0064	0.0066	0.0068	0.0069	0.0071	0.0073	0.0075	0.0078	0.0080	0.0082	-2.4
0.0084	0.0087	0.0089	0.0091	0.0094	0.0096	0.0099	0.0102	0.0104	0.0107	-2.3
0.0110	0.0113	0.0116	0.0119	0.0122	0.0125	0.0129	0.0132	0.0136	0.0139	-2.2
0.0143	0.0146	0.0150	0.0154	0.0158	0.0162	0.0166	0.0170	0.0174	0.0179	-2.1
0.0183	0.0188	0.0192	0.0197	0.0202	0.0207	0.0212	0.0217	0.0222	0.0228	-2.0
0.0233	0.0239	0.0244	0.0250	0.0256	0.0262	0.0268	0.0274	0.0281	0.0287	-1.9
0.0294	0.0301	0.0307	0.0314	0.0322	0.0329	0.0336	0.0344	0.0351	0.0359	-1.8
0.0367	0.0375	0.0384	0.0392	0.0401	0.0409	0.0418	0.0427	0.0436	0.0446	-1.7
0.0455	0.0465	0.0475	0.0485	0.0495	0.0505	0.0516	0.0526	0.0537	0.0548	-1.6
0.0559	0.0571	0.0582	0.0594	0.0606	0.0618	0.0630	0.0643	0.0655	0.0668	-1.5
0.0681	0.0694	0.0708	0.0721	0.0735	0.0749	0.0764	0.0778	0.0793	0.0808	-1.4
0.0823	0.0838	0.0853	0.0869	0.0885	0.0901	0.0918	0.0934	0.0951	0.0968	-1.3
0.0985	0.1003	0.1020	0.1038	0.1056	0.1075	0.1093	0.1112	0.1131	0.1151	-1.2
0.1170	0.1190	0.1210	0.1230	0.1251	0.1271	0.1292	0.1314	0.1335	0.1357	-1.1
0.1379	0.1401	0.1423	0.1446	0.1469	0.1492	0.1515	0.1539	0.1562	0.1587	-1.0
0.1611	0.1635	0.1660	0.1685	0.1711	0.1736	0.1762	0.1788	0.1814	0.1841	-0.9
0.1867	0.1894	0.1922	0.1949	0.1977	0.2005	0.2033	0.2061	0.2090	0.2119	-0.8
0.2148	0.2177	0.2206	0.2236	0.2266	0.2296	0.2327	0.2358	0.2389	0.2420	-0.7
0.2451	0.2483	0.2514	0.2546	0.2578	0.2611	0.2643	0.2676	0.2709	0.2743	-0.6
0.2776	0.2810	0.2843	0.2877	0.2912	0.2946	0.2981	0.3015	0.3050	0.3085	-0.5
0.3121	0.3156	0.3192	0.3228	0.3264	0.3300	0.3336	0.3372	0.3409	0.3446	-0.4
0.3483	0.3520	0.3557	0.3594	0.3632	0.3669	0.3707	0.3745	0.3783	0.3821	-0.3
0.3859	0.3897	0.3936	0.3974	0.4013	0.4052	0.4090	0.4129	0.4168	0.4207	-0.2
0.4247	0.4286	0.4325	0.4364	0.4404	0.4443	0.4483	0.4522	0.4562	0.4602	-0.1
0.4641	0.4681	0.4721	0.4761	0.4801	0.4840	0.4880	0.4920	0.4960	0.5000	0.0



$P(Z\leqslant -0.58)=\Phi(-0.58)=0.281$

$P(Z>0.58)=1-P(Z\leqslant 0.58)=1-0.719=0.281$



$P(-0.38<Z<1.25)$

$=\boxed{P(Z\leqslant 1.25)}-\boxed{P(Z\leqslant -0.38)}$

$=\Phi(1.25)-\Phi(-0.38)$

$=0.8944-0.3520$

$=0.5424$

Example : X is normaly distribuled with $\mu=5$ & $\sigma=0.4$

Find $P(4.4\leqslant X\leqslant 5.2)$

$P(4.4\leqslant X\leqslant 5.2)=P(X\leqslant 5.2)-P(X\leqslant 4.4)$

$=\Phi\left(\frac{5.2-5}{0.4}\right)-\Phi\left(\frac{4.4-5}{0.4}\right)$

$=\Phi(0.5)-\Phi(-1.5)$

$=0.6915-0.0668$

$\frac{X-5}{0.4}$ $=0.6247$

1. A jar contains four marbles: three red, one white. Two marbles are drawn with replacement (ie. a marble is randomly selected, the colour noted, the marble replaced in the jar, then a second marble is drawn).

a. Write out, in curly bracket form, a sample space containing four outcomes

b. Write out a sample space with sixteen outcomes

c. What is the probability of each of the four outcomes in (a)?

d. What are the probabilities of the outcomes in (b)?

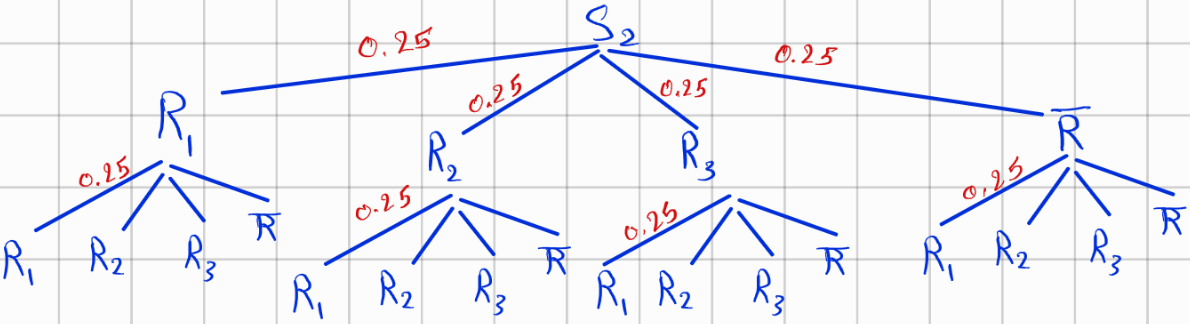
e. What is the probability the colours of the two marbles match?

f. What is the probability the same marble is drawn twice?

R = " The event of randomly pick a red Marble.

a) Sample space$_1$ = $S_1 = \{RR, R\bar{R}, \bar{R}R, \bar{R}\bar{R}\}$

b) $R_{1,2,3}$ = "The event of chosing the 1st, 2nd, or third red marbles randomly."



c) Using Laplace assumption :-

$P(R \cap R) = P(R) \cdot P(R|R) = \frac{3}{4} \cdot \frac{3}{4} = \frac{9}{16}$

$P(R \cap \bar{R}) = \frac{3}{4} \cdot \frac{1}{4} = \frac{3}{16}$

$P(\bar{R} \cap R) = \frac{1}{4} \cdot \frac{3}{4} = \frac{3}{16}$

$P(\bar{R} \cap \bar{R}) = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$

d) all of the 16 outcomes are equal to $\frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$

e) $P(R \cap R) + P(\bar{R} \cap \bar{R}) = \frac{9}{16} + \frac{1}{16} = \frac{10}{16}$

f) $P(R_1 \cap R_1) + P(R_2 \cap R_2) + P(R_3 \cap R_3) + P(\bar{R} \cap \bar{R}) = 4 \times \frac{1}{16} = \frac{1}{4}$

2. Imagine playing with a short deck of cards, as shown at right.
"H" is the event in which the card drawn is a heart.
"D" is the event in which the card drawn is a diamond.
"A" is the event in which the card drawn is an ace.

A♠	A♣	A♥	A♦
2♠	2♣	2♥	2♦
3♠	3♣	3♥	3♦
4♠	4♣	4♥	4♦

a. What are $P(H)$, $P(D)$, and $P(A)$?

b. Find $P(H \cup D)$

c. Find $P(H \cup A)$

d. Find $P(H \cap D)$

e. Find $P(H \cap A)$

f. Are H and D independent events?

g. Are H and A independent events?

Using Laplace assumption:

a) $P(H) = P(D) = P(A) = \frac{4}{16} = \frac{1}{4} = 25\%$

b) $P(H \cup D) = P(H) + P(D) = \frac{1}{2} = 50\%$

c) $P(H \cup A) = P(H) + P(A) - P(H \cap A) = \frac{1}{4} + \frac{1}{4} - \frac{1}{16}$

H = { A♥ 2♥ 3♥ 4♥ }

A = { A♠ A♣ A♥ A♦ }

D = { A♦ 2♦ 3♦ 4♦ }