Let´s control our plant with PT1 characteristic by using a PI-controller:

$$G_P(s) = \frac{K}{T_1 s + 1}\ ;\ G_C(s) = \frac{K_I}{s} + K_P$$

Reference TF:

PDT$_2$ - System

PT$_2$ DT$_2$

$$G_R(S) = \frac{\frac{K_I\ K}{S(T_1 S+1)} + \frac{K_P\ K}{T_1 S+1}}{1+\frac{K_I\ K}{S(T_1 S+1)} + \frac{K_P\ K}{T_1 S+1}} = \frac{K_I K + K_P K S}{T_1 S^2 + (K_P K+1) S + K_I K} = \frac{\frac{K}{T_1}(K_P S + K_I)}{S^2 + \underbrace{\frac{(K_P K+1)}{T_1}}_{2D\omega_0} S + \underbrace{\frac{K_I\ K}{T_1}}_{\omega_0^2}}$$

$$\omega_0 = \sqrt{\frac{K_I\ K}{T_1}}$$

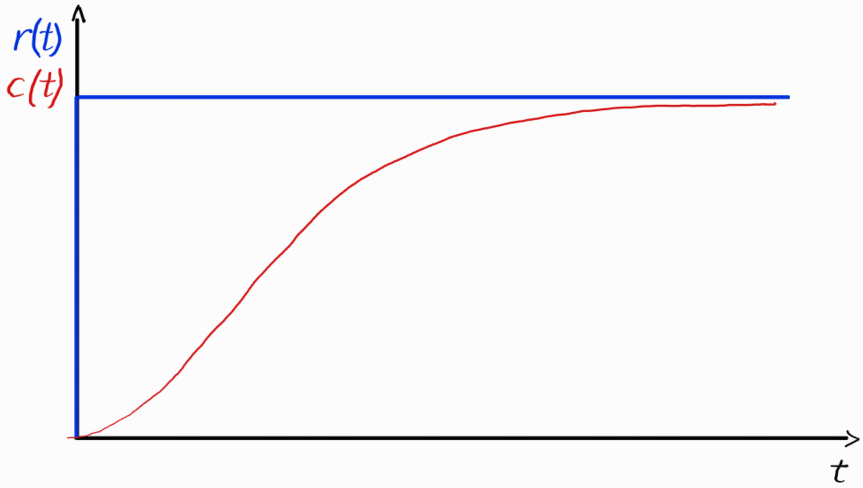
$$\omega_0 \uparrow \Rightarrow K_I \uparrow$$

$$2D\sqrt{\frac{K_I K}{T_1}} = \frac{K_P K + 1}{T_1}$$

$$\Rightarrow D = \frac{K_P K + 1}{2\sqrt{T_1 K_I \cdot K}}$$

$$K_I \uparrow \Rightarrow D \downarrow$$

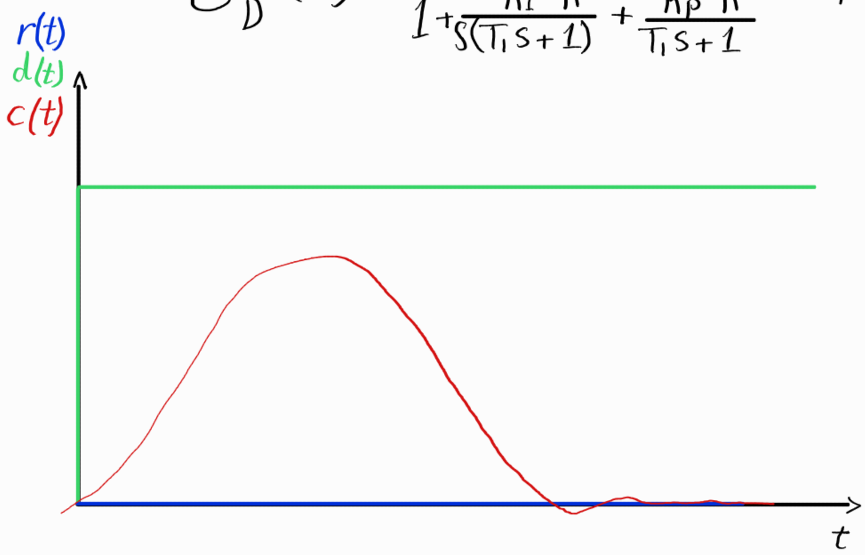
$$D \uparrow \Rightarrow K_P \uparrow$$



Disturbance TF:

DT2 - system

$$G_D(S) = \frac{\frac{K}{(T_1 S+1)}}{1+\frac{K_I\ K}{S(T_1 S+1)} + \frac{K_P\ K}{T_1 S+1}} = \frac{K \cdot S}{T_1 S^2 + (K_P K+1) S + K_I K}$$



Important note: Ideal sensor $G_{sensor}(s) = 1$

Frequency Domain Techniques:

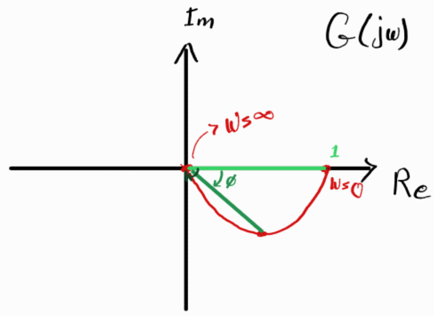
1- Polar plot / Nyquist-plot

a) Find $G(j\omega) = G(s=j\omega)$

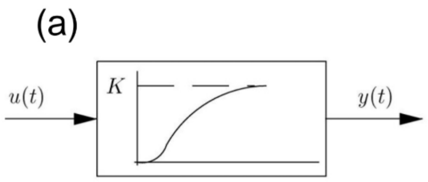
b) bring the amplitude $|G(j\omega)| = \sqrt{Re^2 + Im^2}$, & the phase shift $\phi = \tan^{-1}\left(\frac{Im}{Re}\right)$

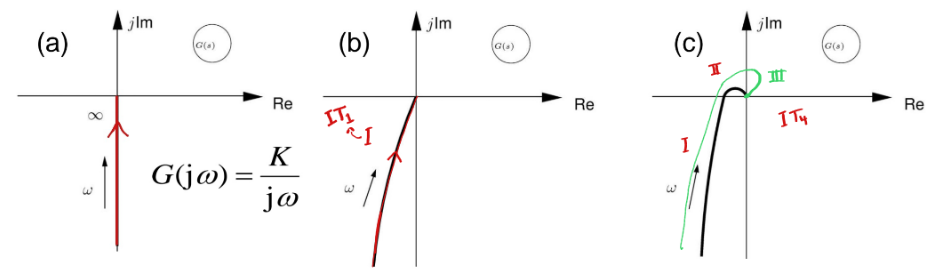
C) write the data table with three points at least

ω	0		∞
$\lvert G(j\omega)\rvert$			
ϕ			



PT*n*-system





Nyquist plot of an I-system

(a) I-system

(b) I-system with delay 1st order (IT1)

(c) I-system with delay 2nd order (IT2)

I-system (ideal integral)

$$G(s) = \frac{K_I}{s}$$

$$G(j\omega) = \frac{K_I}{j\omega} * \frac{j\omega}{j\omega}$$

$$G(j\omega) = \frac{K_I \cdot j\omega}{-\omega^2} = \frac{-K_I}{\omega}\, j$$

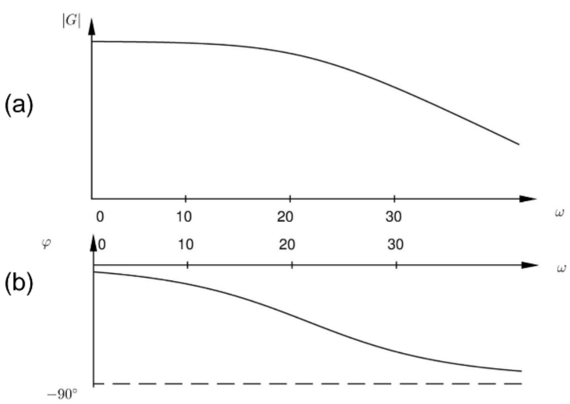
$$|G(j\omega)| = \frac{K_I}{\omega}$$

$$\phi = -90$$

2- Magnitude & Phase plot

(a) Magnitude plot $|G(j\omega)|$ dependent on frequency ω

(b) Phase plot $\phi(\omega)$ dependent on frequency ω

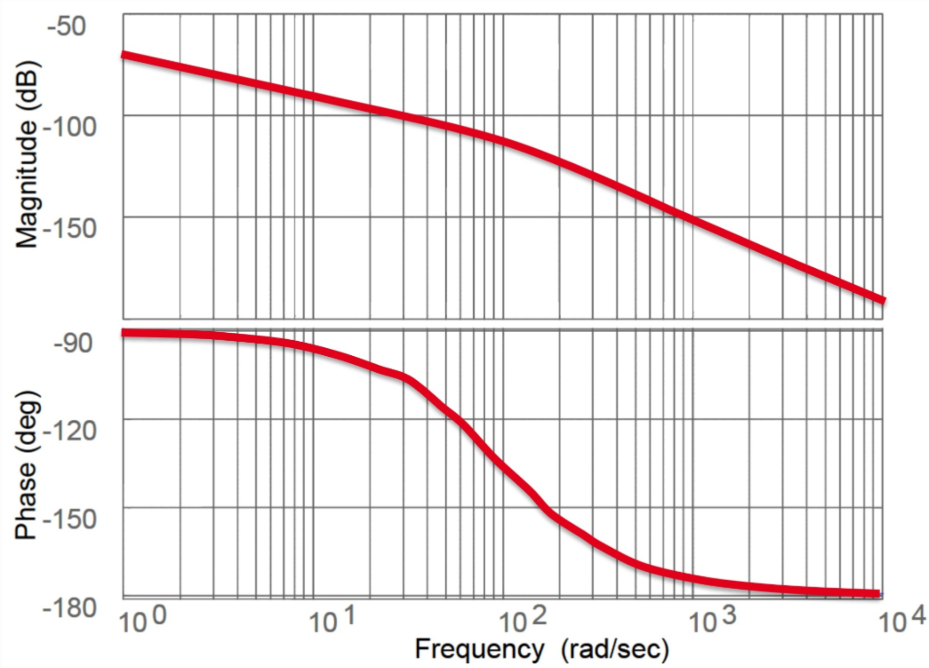


3- Bode-Diagram

$* \; A(\omega)_{dB} = 20 \log_{10}(A(\omega)) \implies A(\omega) = 10^{\frac{A(\omega)_{dB}}{20}}$

Logarithmic magnitude axis

Linear Phase-Axis



Logarithmic Frequency Axis

Advantages of Working with Frequency Response in Terms of Bode Plots

1. Dynamic compensator design can be based entirely on Bode plots.
2. Bode plots can be determined experimentally.
3. Bode-Diagrams of systems in series (or tandem) simply add.
4. The use of a log scale permits a much wider range of frequencies to be displayed on a single plot than is possible with linear scales.

Bode-Diagrams of systems in series (or tandem) simply add:

$$\ln G(j\omega) = \ln|G_1(j\omega)| + \ln|G_2(j\omega)| + j\left(\phi_{G_1}(\omega) + \phi_{G_2}(\omega)\right)$$

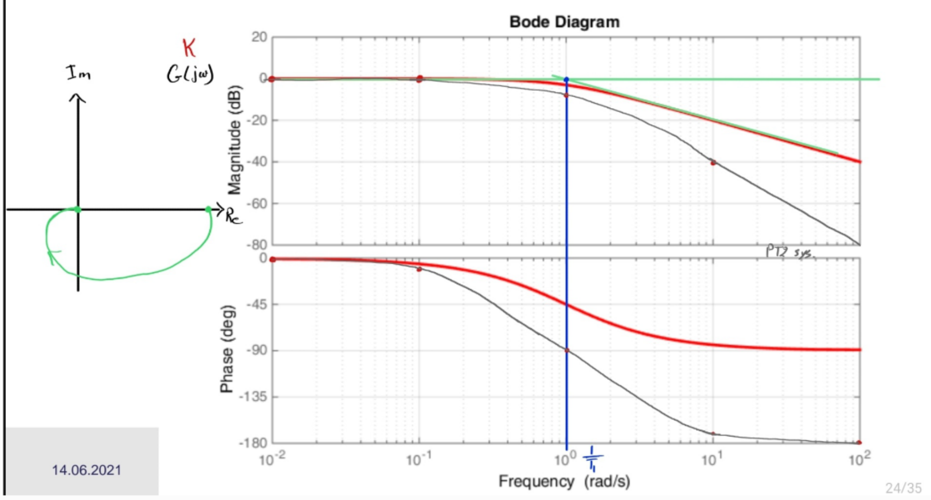
Proof …

$$G(j\omega) = G_1(j\omega)\cdot G_2(j\omega) = |G_1(j\omega)|e^{j\phi_1(\omega)}\cdot|G_2(j\omega)|\cdot e^{j\phi_2(\omega)} = |G_1(j\omega)|\cdot|G_2(j\omega)|e^{j(\phi_1(\omega)+\phi_2(\omega))}$$

$$\ln(G(j\omega)) = \ln\left(|G_1(j\omega)|\cdot|G_2(j\omega)|e^{j(\phi_1(\omega)+\phi_2(\omega))}\right) = \ln|G_1(j\omega)| + \ln|G_2(j\omega)| + j(\phi_1(\omega)+\phi_2(\omega))$$

6.4 The Bode-Diagram

Derive the Bode-diagram of a PT2-system, consisting of two PT1-systems in series.



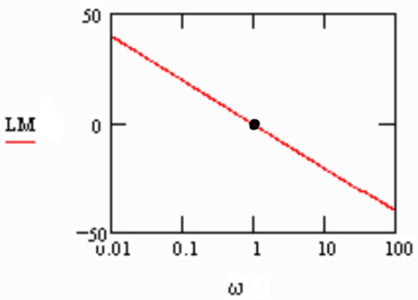
Drawing Bode diagram :

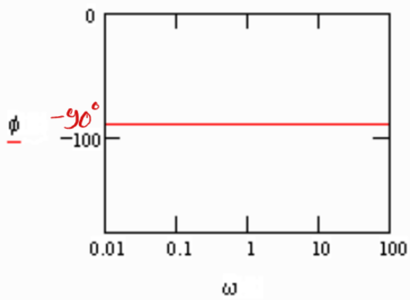
* Constant function $G(s) = K : |G(j\omega)| = K \Rightarrow G_{dB} = 20\log^{K}$, $\phi = 0^\circ$ for $K > 0$ & $\phi = -180^\circ$ for $K < 0$

* $G(s) = \frac{1}{S}$: $|G(j\omega)| = \frac{1}{\omega} \Rightarrow G_{dB} = 20\log(\frac{1}{\omega})$, $\phi = -90^\circ$

$W=1 \Rightarrow gain = 1 \Rightarrow G_{dB} = 0dB$ / at $w = 10$ rad/s $\Rightarrow gain = 0.1 \Rightarrow G_{dB} = -20dB$ / at $w = 100$ r/s $\Rightarrow G_{dB} = -40$ dB

$G(S) = \frac{1}{S} \longrightarrow$



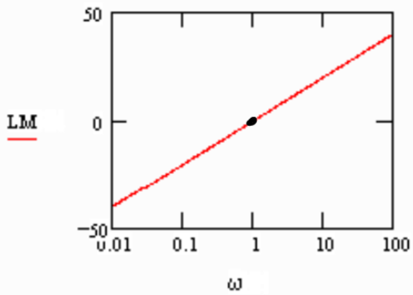


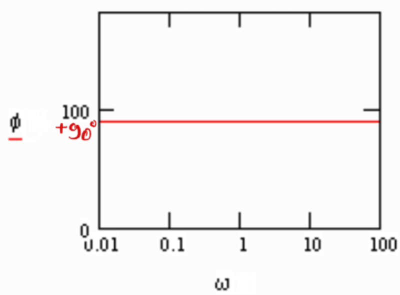
* $G(S) = \frac{5}{S^2}$; $\Rightarrow G(s) = \underbrace{5}_{1} \cdot \underbrace{\frac{1}{S}}_{2} \cdot \underbrace{\frac{1}{S}}_{3} \Rightarrow G_{dB} = G_{dB_1} + G_{dB_2} + G_{dB_3}$ / $\phi = \phi_1 + \phi_2 + \phi_3$

* $G(s) = S = \frac{1}{1/s}$: $\Rightarrow$ $\underbrace{\text{respose of } 1}_{G_{dB}=0 \ \& \ \phi = 0^\circ}$ $-$ respose of $\frac{1}{S}$ $= -$response of $\frac{1}{S}$

make it the reflection of respond $\frac{1}{s}$ around 0dB & 0°

$G(S) = S \longrightarrow$





$*G(s) = \frac{1}{1 \pm \frac{s}{w_0}}$: $\Rightarrow G_{dB} = -20 \log\left(\sqrt{1+\frac{w^2}{w_0^2}}\right)$ & $\phi = \tan^{-1}\left(\mp \frac{w}{w_0}\right)$

- Case 1: $W << W_0$

Gain $= -20 \log(1) = 0\,dB$

Phase$(\phi) = \tan^{-1}(0) = 0^\circ$

- Case 2: $W = W_0$

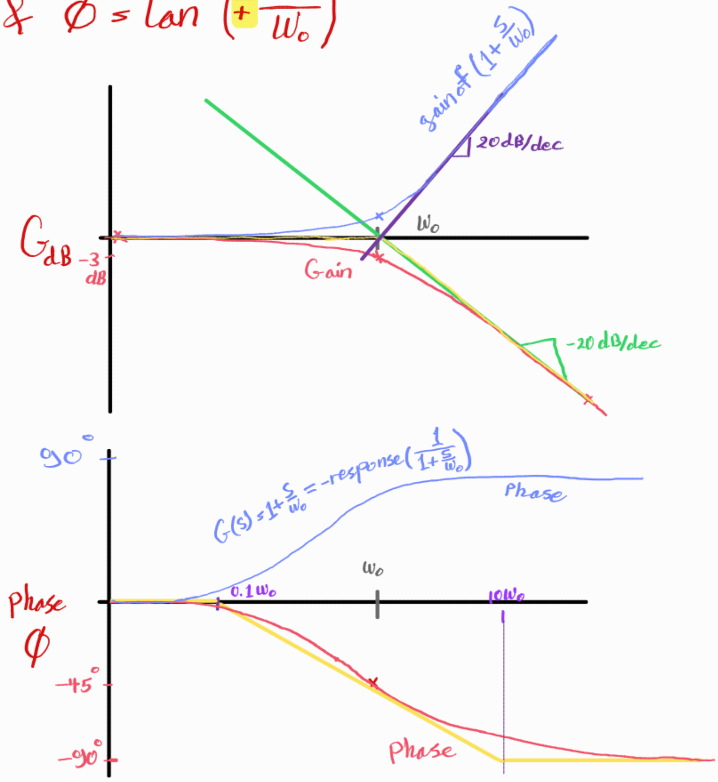
Gain $= -20 \log(\sqrt{2}) = -3\,dB$

Phase $= \tan^{-1}(-1) = -45^\circ$

- Case 3: $W >> W_0$

Gain $= -20 \log\left(\frac{W}{W_0}\right) =$ -20 dB/dec slope through 0 at w_0

Phase $= \tan^{-1}(-\infty) = -90^\circ$



$*G(s) = 1 \pm \frac{S}{W_0} = \frac{1}{\frac{1}{1 \pm \frac{S}{W_0}}} = -\text{response}\left(\frac{1}{1+\frac{s}{w_0}}\right)$

↳ reflection of this around 0 dB & 0° phase.

$\Rightarrow G_{dB} = 20 \log\left(\sqrt{1+\frac{w^2}{w_0^2}}\right)$, $\phi = \tan^{-1}\left(\pm \frac{w}{w_0}\right)$

*Complex poles or Zeros : $TF = \frac{W_0^2}{S^2 + 2DW_0S + W_0^2} = \frac{1}{\left(\frac{S}{W_0}\right)^2 + 2D\left(\frac{S}{W_0}\right) + 1}$

$$\boxed{\text{Real} = \frac{1-\left(\frac{\omega}{\omega_0}\right)^2}{\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \left(2\zeta\left(\frac{\omega}{\omega_0}\right)\right)^2}}$$

$$\boxed{\text{Imag} = \frac{-\left(2\zeta\left(\frac{\omega}{\omega_0}\right)\right)}{\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \left(2\zeta\left(\frac{\omega}{\omega_0}\right)\right)^2} \cdot j}$$

Whoa!

Case 1: $\omega << \omega_0$ Real = 1 Imag = 0

$|H(j\omega)| = 20 \log_{10}\left(\sqrt{1^2+0^2}\right) = \boxed{0\,dB}$

$\arg(H(j\omega)) = \text{atan}\left(\frac{0}{1}\right) = \boxed{0 \text{ degrees}}$

Case 2: $\omega >> \omega_0$ Real $= \left(\frac{\omega}{\omega_0}\right)^{-2}$ Imag $= -2\zeta\left(\frac{\omega}{\omega_0}\right)^{-3}$

damping ratio

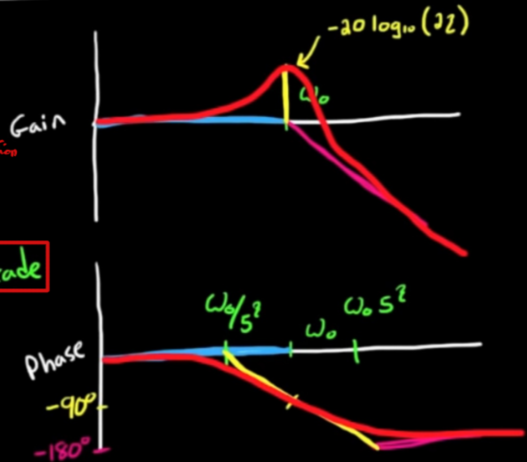
$|H(j\omega)| = 20 \log_{10}\left(\left(\frac{\omega}{\omega_0}\right)^{-2}\right) = -40 \log_{10}\left(\frac{\omega}{\omega_0}\right)$ $\boxed{-40\ dB/decade}$

$\arg(H(j\omega)) = \text{atan}\left(2\zeta\left(\frac{\omega}{\omega_0}\right)^{-1}\right) = \boxed{-180 \text{ degrees}}$

Case 3: $\omega = \omega_0$ Real = 0 Imag $= -\frac{1}{2\zeta}$

$|H(j\omega)| = 20 \log_{10}\left(\frac{1}{2\zeta}\right) = \boxed{-20 \log_{10}(2\zeta)}$

$\arg(H(j\omega)) = \text{atan}(\infty) = \boxed{-90 \text{ degrees}}$



Stability:

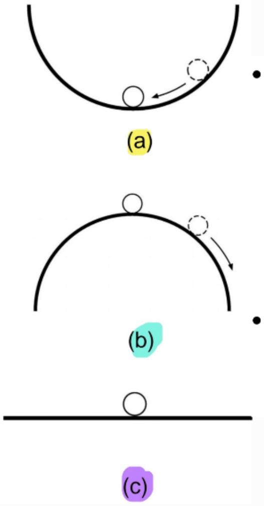
Introduction

Definition of **stability**

(a) A LTI system is stable if the natural response approaches zero as time approaches infinity

(b) A LTI system is unstable if the natural response grows w/o bound as time approaches infinity

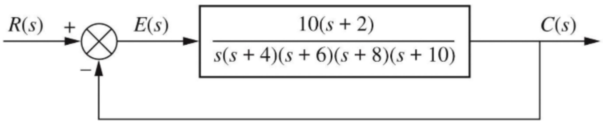
(c) A LTI system is marginally stable if the natural response neither decays nor grows but remains constant or oscillates as time approaches infinity



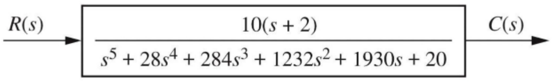
* A ramp input is not a bounded I/O

- Alternate definition of **stability** (one that regards the total response and implies the first definition based upon the natural response): → limited
  A system is stable if **every** bounded input yields a bounded output.
  We call this statement the bounded-input, bounded-output (BIBO) definition of stability.
- Alternate definition of **instability**
  A system is unstable if **any** bounded input yields an unbounded output

➔ Knowing the poles of the forward TF, does not imply the poles of the equivalent closed-loop TF w/o factoring or otherwise solving the roots.



(a)



(b)

Therefore we use different methods to check the stability of the system (controller):

1- Routh Hurwitz methode:

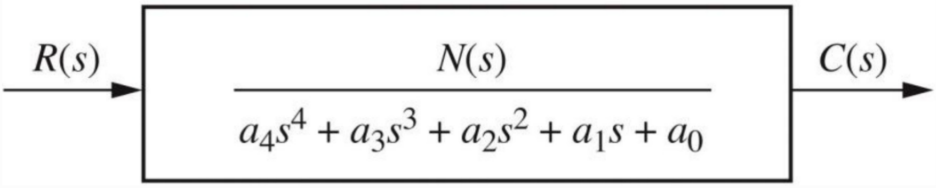
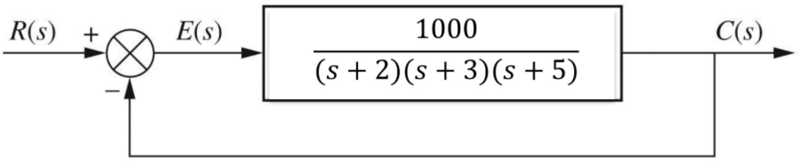


TABLE 6.2 Completed Routh table

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$\dfrac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$	$\dfrac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$\dfrac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$
s^1	$\dfrac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$\dfrac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$\dfrac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$
s^0	$\dfrac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$\dfrac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$\dfrac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

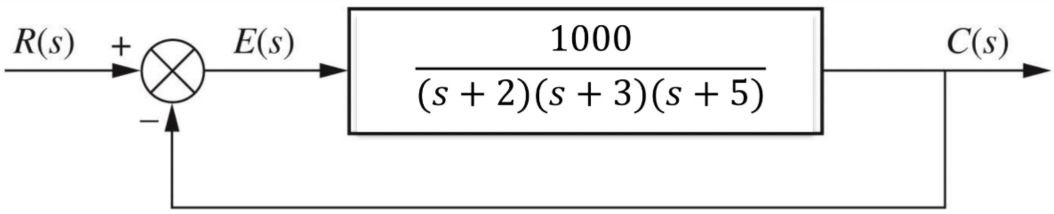
- System is stable if there are no sign changes in the first column of the Routh table

Example



s^3	a_3	a_1
s^2	a_2	a_0
s^1	$b_1 = \dfrac{-\begin{vmatrix} a_3 & a_1 \\ a_2 & a_0 \end{vmatrix}}{a_2}$	$b_2 = \dfrac{-\begin{vmatrix} a_3 & 0 \\ a_2 & 0 \end{vmatrix}}{a_2} = 0$
s^0	$c_1 = \dfrac{-\begin{vmatrix} a_2 & a_0 \\ b_1 & 0 \end{vmatrix}}{b_1}$	$c_2 = \dfrac{-\begin{vmatrix} a_2 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$

$$G(s) = \frac{\dfrac{1000}{(s+2)(s+3)(s+5)}}{1 + \dfrac{1000}{(s+2)(s+3)(s+5)}} = \frac{1000}{s^3 + 10s^2 + 31s + 1030}$$



s^3	$a_3 = 1$	$a_1 = 31$
s^2	$a_2 = 10$	$a_0 = 1030$
s^1	$b_1 = \dfrac{-\begin{vmatrix} 1 & 31 \\ 10 & 1030 \end{vmatrix}}{10} = -72$	$b_2 = 0$
s^0	$c_1 = \dfrac{-\begin{vmatrix} 10 & 1030 \\ -72 & 0 \end{vmatrix}}{-72} = 1030$	$c_2 = 0$

Sign Change

$$G(s) = \frac{1000}{s^3 + 10s^2 + 31s + 1030}$$

$$G(s) = \frac{1000}{a_3 s^3 + a_2 s^2 + a_1 s^1 + a_0}$$

$\Rightarrow$ 2 sign change

$\Rightarrow$ 2 unstable poles.

Another example:

Stability design via Routh-Hurwitz

Find the range of the gain K, that will cause the system to be stable. Assume K > 0.

1. TF of closed-loop system

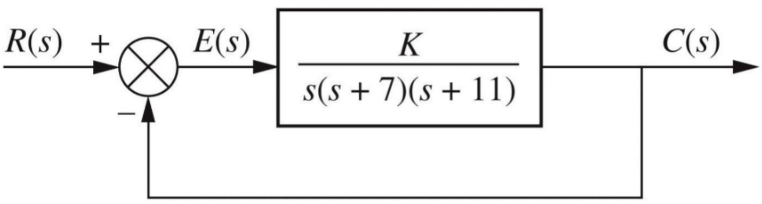


Figure 6.10

$$G(s) = \frac{\dfrac{K}{s(s+7)(s+11)}}{1 + \dfrac{K}{s(s+7)(s+11)}}$$

$$G(s) = \frac{K}{s(s+7)(s+11) + K}$$

$$G(s) = \frac{K}{s^3 + 18s^2 + 77s + K}$$

s^3	$a_3 = 1$	$a_1 = 77$
s^2	$a_2 = 18$	$a_0 = K$
s^1	$b_1 = \dfrac{-\begin{vmatrix} 1 & 77 \\ 18 & K \end{vmatrix}}{18} = 77 - \frac{K}{18}$	$b_2 = 0$
s^0	$C_1 = \dfrac{-\begin{vmatrix} 18 & K \\ 77-\frac{K}{18} & 0 \end{vmatrix}}{77-\frac{K}{18}} = \dfrac{K\cancel{(77-\frac{K}{18})}}{\cancel{77-\frac{K}{18}}} = K$	$C_2 = 0$

$$G(s) = \frac{K}{s^3 + 18s^2 + 77s + K}$$

$b_1 > 0$ & $C_1 > 0$

$77 - \frac{K}{18} > 0$

$\Rightarrow K < 1386$ & $K > 0$

$\therefore \boxed{0 < K < 1386}$

for stable system

2- Stability via Nyquist plot:

Stability of a closed-loop controller guaranteed in case of:

If critical point $(-1, j0)$ of open-loop Nyquist plot is located to the left of the Nyquist plot for frequency changes from 0 to $+\infty$

Critical point $\longrightarrow (-1, j0)$ or at 1 & phaseshift $-180°$

Example

For the unity feedback system of the figure, where $G(s) = \frac{K}{s(s+3)(s+5)}$, find the range of gain K, for stability, instability and the value of gain for marginal stability. For marginal stability also find the frequency of oscillation. Use the Nyquist criterion.

$$G(s) = \frac{K}{s(s+3)(s+5)}$$

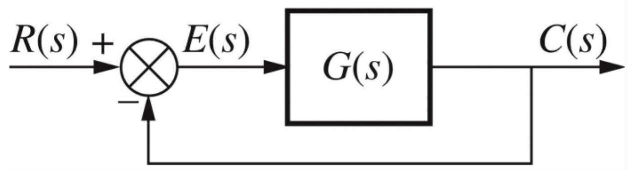


Figure 10.10

$$G_{OL}(j\omega) = \frac{K}{(j\omega)(j\omega+3)(j\omega+5)} = \frac{K}{j\omega(15-\omega^2) - 8\omega^2} * \frac{j\omega(15-\omega^2)+8\omega^2}{j\omega(15-\omega^2)+8\omega^2} = \frac{jK\omega(15-\omega^2) + 8K\omega^2}{-\omega^2(15-\omega^2)^2 - 64\omega^4}$$

$$Re = \frac{-8K\omega^2}{\omega^2(15-\omega^2)^2 + 64\omega^4}$$

$$Im = \frac{-K\omega(15-\omega^2)}{\omega^2(15-\omega^2)^2 + 64\omega^4} j$$

$Im = 0 \Rightarrow 15 - \omega^2 = 0 \Rightarrow \omega = \sqrt{15}$

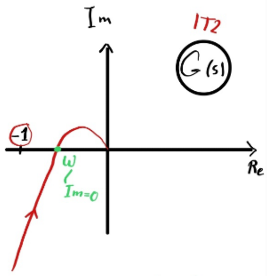
Plug $\omega = \sqrt{15}$ in Re part to find the range of K where the system will be stable!

$$Re = \frac{-\cancel{8}K\cancel{(15)}}{(15)(0) + \cancel{64}(15)^{\cancel{2}}} = \frac{-K}{8(15)} > -1$$

(with 64 reduced to 8)

$\boxed{0 < K < 120}$

For a stable closed-loop Controller.



we want make sure -1 is to left of the openloop plot

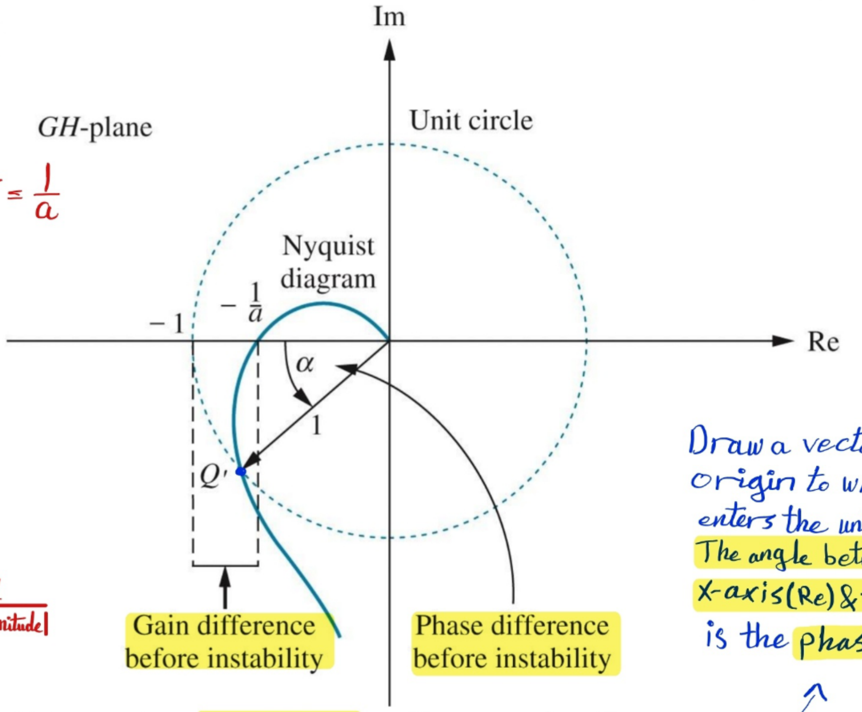
based upon that, we can even define how stable the system is, using:

a) Gain margin (G_M in dB)

b) Phase margin (ϕ_M)

Systems with greater gain and phase margins can withstand greater changes in system parameters before becoming unstable

Gain and Phase Margin via Nyquist Diagram



if Magnitude $= 0.5 = \frac{1}{a}$

@ −180° phase

$\therefore G_M = 20\log(2)$

$= 6\,dB$

$G_M = 20\log(a)$

$a = \frac{1}{|\text{magnitude}|}$

Gain margin = $G_M = 20\log a$ Phase margin = $\Phi_M = \alpha$

Draw a vector from the origin to where the plot enters the unit circle. The angle between the negative x-axis (Re) & this vetor is the phase margin

Example:

Find the gain margin for the system

$$G(s) = \frac{K}{(s^2+2s+2)(s+2)}$$

if K = 6.

$Re|_{K=1} = \frac{-1}{20}$

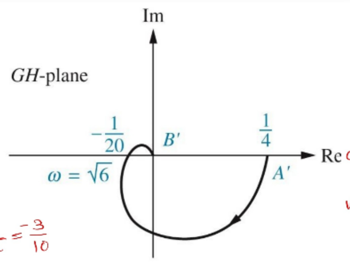
$Re|_{K=6} = 6 \cdot \frac{-1}{20} = \frac{-3}{10}$

at $K=6 \Rightarrow -\frac{1}{a} = \frac{-3}{10}$

$\Rightarrow a = \frac{10}{3}$

$G_M = 20\log(a)$

$= 20\log(\frac{10}{3}) \simeq 10.45\,dB$



Nyquist diagram for K = 1

$$G(j\omega) = \frac{K[4(1-\omega^2) - j\omega(6-\omega^2)]}{16(1-\omega^2)^2 + \omega^2(6-\omega^2)^2}$$

Another way:

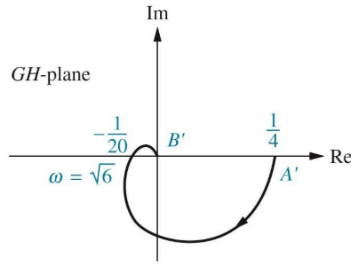
at $K=1$ & $w=\sqrt{6}$ $|G(j\omega)| = \frac{1}{20}$

while at $K=6$ & same freq. $w=\sqrt{6}$ $|G(j\omega)| = \sqrt{(\frac{-3}{10})^2} = \frac{3}{10}$

Since $\frac{1}{a}$ = magnitude $= \frac{3}{10}$

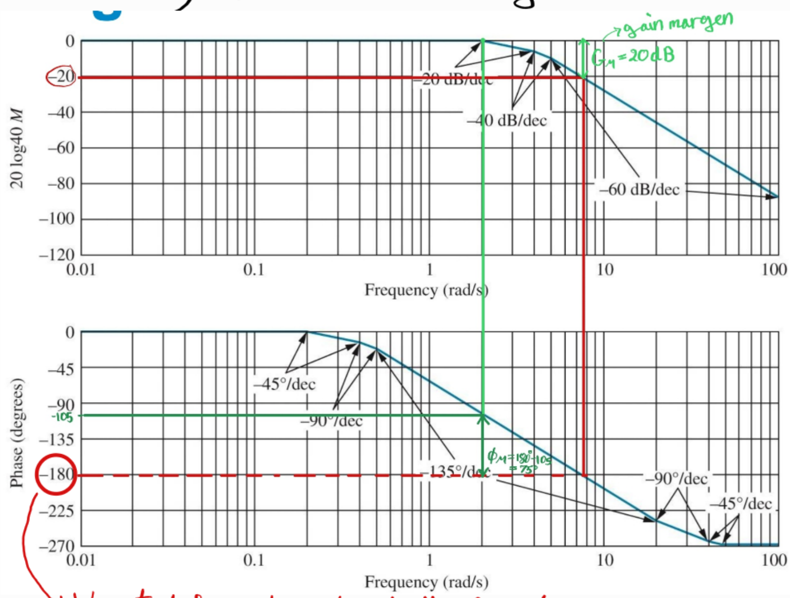
$\therefore a = \frac{10}{3}$

$G_M = 20\log(\frac{10}{3}) \simeq 10.5\,dB$



Nyquist diagram for K = 1

3- Stability via Bode diagram :-



$|G_{dB}| \simeq -20\,dB$

(below 0 dB line, which means in the open-loop Nyquist plot, it's between the 0 point & critical point (-1) on the Re-axis.)

⟹ Which means The critical Point (-1) is to the left of the plot ⟹ Stable

We start from here to get all information.

Evaluating Gain and Phase Margins

- Gain margin found by using the phase plot, where the phase angle is -180°
- Phase margin is found by using the magnitude plot, where the gain is 0 dB

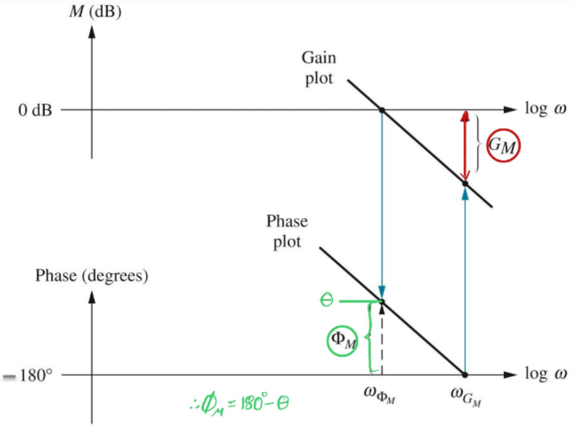


Figure 10.37

Open-loop frequency response curves can be used to determine

- Stability of a system Stable or not (to the left or to the right of the critical point)
- Range of loop gain that will ensure stability range of K where the sys. is stable
- Gain and phase margin How stable our system is?

Approximate values for gain and phase margins:

- G_M = 12 dB – 20 dB for reference performance
- G_M = 3.5 dB – 9.5 dB for disturbance response
- Φ_M = 40° - 60° for reference performance
- Φ_M = 20° - 50° for disturbance performance