Chapter 1

ARMA and Nonstationary Times Series - Exercises

Shu Shen

1.1 ARMA

Exercise 14.15 A Gaussian AR model is an autoregression with i.i.d. $N\left(0, \sigma^2\right)$ errors. Consider the Gaussian AR(1) model

$$Y_t = \alpha_0 + \alpha_1 Y_{t-1} + e_t$$

$$e_t \sim N\left(0, \sigma^2\right)$$

with $|\alpha_1| < 1$. Show that the marginal distribution of Y_t is also normal:

$$Y_t \sim N\left(\frac{\alpha_0}{1-\alpha_1}, \frac{\sigma^2}{1-\alpha_1^2}\right).$$

Hint: Use the MA representation of Y_t.

Solution. 14.15

$$\begin{aligned}
Y_t &= \alpha_0 + \alpha_1 Y_{t-1} + e_t \\
&= \alpha_0 + \alpha_1\left(\alpha_0 + \alpha_1 Y_{t-2} + e_{t-1}\right) + e_t \\
&= \alpha_0\left(1+\alpha_1\right) + \alpha_1^2 Y_{t-2} + \alpha_1 e_{t-1} + e_t \\
&= \alpha_0\left(1+\alpha_1+\alpha_1^2\right) + \alpha_1^3 Y_{t-3} + \alpha_1^2 e_{t-2} + \alpha_1 e_{t-1} + e_t \\
&= \cdots \\
&= \sum_{i=0}^{\infty} \alpha_1^i e_{t-i} + \frac{\alpha_0}{1-\alpha_1}
\end{aligned}$$

The mean of Y_t is:

$$
\begin{aligned}
E\left[Y_t\right] &= E\left[\sum_{i=0}^{\infty} \alpha_1^i e_{t-i} + \frac{\alpha_0}{1-\alpha_1}\right] \\
&= \frac{\alpha_0}{1-\alpha_1} + \sum_{i=0}^{\infty} \alpha_1^i E\left[e_{t-i}\right] \\
&= \frac{\alpha_0}{1-\alpha_1}
\end{aligned}
$$

The variance of Y_t is:

$$
\begin{aligned}
var\left(Y_t\right) &= var\left(\sum_{i=0}^{\infty} \alpha_1^i e_{t-i} + \frac{\alpha_0}{1-\alpha_1}\right) \\
&= var\left(\sum_{i=0}^{\infty} \alpha_1^i e_{t-i}\right) \\
&= \sum_{i=0}^{\infty} (\alpha_1^i)^2 var\left(e_{t-i}\right) \\
&= \sigma^2 \sum_{i=0}^{\infty} (\alpha_1^i)^2 \\
&= \frac{\sigma^2}{1-\alpha_1^2}
\end{aligned}
$$

Thus we conclude

$$
Y_t \sim N\left(\frac{\alpha_0}{1-\alpha_1}, \frac{\sigma^2}{1-\alpha_1^2}\right)
$$

1.2 Nonstationary Times Series

Exercise 16.1 Take $S_t = S_{t-1} + e_t$ with $S_0 = 0$ and e_t i.i.d. $N\left(0, \sigma^2\right)$.

(a) Calculate $E\left[S_t\right]$ and $var\left[S_t\right]$.

(b) Set $Y_t = \left(S_t - E\left[S_t\right]\right) / \sqrt{var\left[S_t\right]}$. By construction $E\left[Y_t\right] = 0$ and $var\left[Y_t\right] = 1$. Is Y_t stationary?

(c) Find the asymptotic distribution of $Y_{\lfloor nr \rfloor}$ for $r \in [\delta, 1]$.

Solution. 16.1

(a)

$$
\begin{aligned}
E\left[S_t\right] &= E\left[S_{t-1} + e_t\right] \\
&= E\left[e_1 + e_2 + \cdots + e_t\right] \\
&= E\left[\sum_{i=1}^{t} e_i\right] \\
&= \sum_{i=1}^{t} E\left[e_i\right] = 0
\end{aligned}
$$

$$
\begin{aligned}
var\left(S_t\right) &= var\left(\sum_{i=1}^{t} e_i\right) \\
&= \sum_{i=1}^{t} var\left(e_i\right) \\
&= t\sigma^2
\end{aligned}
$$

(b) Notice that

$$
Y_t = \frac{S_t - E\left[S_t\right]}{\sqrt{var\left[S_t\right]}} = \frac{S_t}{\sqrt{t}\sigma} = \frac{\sum_{i=1}^{t} e_i}{\sqrt{t}\sigma} = \frac{1}{\sigma}\frac{1}{\sqrt{t}}\sum_{i=1}^{t} e_i
$$

If Y_t is stationary, its covariance must not depend on t. However,

$$
\begin{aligned}
Cov\left(Y_t, Y_{t+k}\right) &= E\left[\left(Y_t - E\left[Y_t\right]\right)\left(Y_{t+k} - E\left[Y_{t+k}\right]\right)\right] \\
&= E\left[Y_t Y_{t+k}\right] \\
&= E\left[\frac{1}{\sigma}\frac{1}{\sqrt{t}}\sum_{i=1}^{t} e_i \frac{1}{\sigma}\frac{1}{\sqrt{t+k}}\sum_{i=1}^{t+k} e_i\right] \\
&= \frac{1}{\sigma^2\sqrt{t\left(t+k\right)}} E\left[\left(e_1 + \cdots + e_t\right)\left(e_1 + \cdots + e_t + \cdots e_{t+k}\right)\right] \\
&= \frac{1}{\sigma^2\sqrt{t\left(t+k\right)}} \sum_{i=1}^{t} E\left[e_i^2\right] \\
&= \sqrt{\frac{t}{t+k}}.
\end{aligned}
$$

Therefore, Y_t is not stationary.

(c)

$$
Y_{\lfloor nr \rfloor} = \frac{S_{\lfloor nr \rfloor} - E\left[S_{\lfloor nr \rfloor}\right]}{\sqrt{var\left[S_{\lfloor nr \rfloor}\right]}} = \frac{S_{\lfloor nr \rfloor}}{\sqrt{\lfloor nr \rfloor}\sigma} = \frac{\sum_{i=1}^{\lfloor nr \rfloor} e_i}{\sqrt{\lfloor nr \rfloor}\sigma} \sim N\left(0, 1\right)
$$