Chapter 1

CLT for inid Sequences

1.1 Notations and Definitions

A random variable z is rth *integrable* if $E[|z|^r] < \infty$. Equivalently,

$$\lim_{M\to\infty} E[|z|^r \, \mathbb{I}\{|z|^r > M\}] = 0,$$

where $\mathbb{I}\{\cdot\}$ is the indicator function. Without referring explicitly to the rth moment, we say z is *integrable* if $r = 1$, and *square integrable* if $r = 2$.

A *triangular array* $\{(x_{1n}, x_{2n}, x_{3n}, ..., x_{r_n n})\}_{n\in\mathbb{N}}$ stacks like a triangular:

$$\begin{pmatrix} x_{11} & x_{22} & \cdots & x_{r_1 1} & & & & \\ x_{12} & x_{22} & \cdots & \cdots & x_{r_2 2} & & & \\ x_{13} & x_{23} & \cdots & \cdots & \cdots & x_{r_3 3} & & \\ \vdots & & & & & & \ddots & \\ x_{1n} & x_{2n} & \cdots & \cdots & \cdots & \cdots & \cdots & x_{r_n n} \end{pmatrix}$$

Here r_n is an increasing number in n, and $r_n \to \infty$ as $n \to \infty$. Suppose for each n, the elements in $(x_{in})_{i=1}^{r_n}$ are independently non-identically distributed (inid). (Please keep a liberal mind and consider "identically distributed" as a special case of "non-identically distributed".)

Without loss of generality, assume $E[x_{in}] = 0$ for all i and n and denote $\sigma_{in}^2 = E[x_{in}^2]$. Define the *partial sum* (up to n) as $S_n = \sum_{i=1}^{r_n} x_{in}$ and and (the nth) *aggregate variance* as $\tilde{\sigma}_n^2 = \sum_{i=1}^{r_n} \sigma_{in}^2$.

1.2 Lindeberg Condition

Lindeberg-Lévy Central Limit Theorem is for independently and identically distributed (iid) sequences. In this lecture we consider independent but heterogeneous sequences.

Definition 1.1. Lindeberg Condition:

$$\lim_{n\to\infty} \frac{1}{\tilde{\sigma}_n^2} \sum_{i=1}^{r_n} E[x_{in}^2 \mathbb{I}\{x_{in}^2 \geq \varepsilon \tilde{\sigma}_n^2\}] = 0$$

for all $\varepsilon > 0$.

Theorem 1.1 (Lindeberg-Feller CLT)**.** *If the triangular array $\{(x_{in})_{i=1}^{r_n}\}_{n\in\mathbb{N}}$ satisfies the Lindeberg condition, then*

$$\frac{S_n}{\tilde{\sigma}_n} \xrightarrow{d} N(0,1)$$

Lindeber-Feller CLT allows heterogeneity across $i = 1, ..., r_n$. It includes *Lindeberg-Lévy CLT* as a special case. To see this fact, under iid let us use z to represent the homogeneous distribution. Denote $\text{var}(z) = \sigma_z^2 \in (0, \infty)$, and equivalently $\lim_{M\to\infty} E\left[z^2 \mathbb{I}\left\{z^2 \geq M\right\}\right] = 0$ (square integrability). Set $r_n = n$, and thus $\tilde{\sigma}_n^2 = n\sigma_z^2$. As a result,

$$\begin{aligned}\frac{1}{\tilde{\sigma}_n^2} \sum_{i=1}^{n} E\left[x_{in}^2 \mathbb{I}\left\{x_{in}^2 \geq \varepsilon \tilde{\sigma}_n^2\right\}\right] &= \frac{1}{n\sigma_1^2} \times nE\left[z^2 \mathbb{I}\left\{z^2 \geq n\sigma_z^2 \varepsilon\right\}\right] \\ &= const \times E\left[z^2 \mathbb{I}\left\{z^2 \geq n\sigma_z^2 \varepsilon\right\}\right] \to \infty\end{aligned}$$

since $n\sigma_1^2 \varepsilon \to \infty$ as $n \to \infty$.

With iid and $r_n = n$, we can drop the subscript n and write $z_i = x_{in}$. The ratio

$$\frac{S_n}{\tilde{\sigma}_n} = \frac{\sum_{i=1}^n z_i}{\sqrt{n\sigma_z^2}} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{z_i}{\sigma_z}$$

retains its familiar for in CLT.

1.3 Lyapunov Condition

Lindeberg condition is difficult to interpret. Lyapunov condition is a more interpretable sufficient condition.

Definition 1.2. Lyapunov Condition: There exists some $\delta > 0$ such that

$$\lim_{n\to\infty} \frac{1}{\tilde{\sigma}_n^{2+\delta}} \sum_{i=1}^{r_n} E[|x_{in}|^{2+\delta}] = 0$$

Lyapunov condition implies Lindeberg condition. To see this fact, we use the quantify in the Lindeberg condition as a starting point:

$$\begin{aligned}\frac{1}{\tilde{\sigma}_n^2} \sum_{i=1}^{r_n} E\left[x_{in}^2 \mathbb{I}\left\{x_{in}^2 \geq \varepsilon \tilde{\sigma}_n^2\right\}\right] &= \frac{1}{\tilde{\sigma}_n^2} \sum_{i=1}^{r_n} E\left[|x_{in}|^2 \mathbb{I}\left\{|x_{in}|^\delta \geq \varepsilon^{\frac{\delta}{2}} \tilde{\sigma}_n^\delta\right\}\right] \\ &= \frac{1}{\tilde{\sigma}_n^2} \sum_{i=1}^{r_n} E\left[\frac{|x_{in}|^{2+\delta}}{|x_{in}|^\delta} \mathbb{I}\left\{|x_{in}|^\delta \geq \varepsilon^{\frac{\delta}{2}} \tilde{\sigma}_n^\delta\right\}\right] \\ &\leq \frac{1}{\tilde{\sigma}_n^2} \times \frac{1}{\varepsilon^{\delta/2} \tilde{\sigma}_n^\delta} \sum_{i=1}^{r_n} E\left[|x_{in}|^{2+\delta} \mathbb{I}\left\{|x_{in}|^\delta \geq \varepsilon^{\frac{\delta}{2}} \tilde{\sigma}_n^\delta\right\}\right] \\ &\leq \frac{1}{\varepsilon^{\delta/2}} \times \frac{1}{\tilde{\sigma}_n^{2+\delta}} \sum_{i=1}^{r_n} E\left[|x_{in}|^{2+\delta}\right] \to 0\end{aligned}$$

as $n \to \infty$, where the limit follows from Lyapunov condition.

1.3.1 Sufficient Condition for Lyapunov Condition

- Condition1: $\sup_{i \leq r_n} E[|x_{in}|^{2+\delta}] \leq B < \infty$ for all sufficiently large n.

Let $\bar{\sigma}_n^2 = \tilde{\sigma}_n^2 / r_n$ be the *average variance*.

- Condition2: $\liminf_{n\to\infty} \bar{\sigma}_n^2 > b > 0$.

Under Condition 1 and Condition 2 we have

$$\begin{aligned}\frac{1}{\tilde{\sigma}_n^{2+\delta}} \sum_{i=1}^{r_n} E[|x_{in}|^{2+\delta}] &\leq \frac{1}{(\sqrt{r_n b})^{2+\delta}} \times r_n \max_{i\leq r_n} E[|x_{in}|^{2+\delta}] \\ &\leq \frac{r_n B}{(\sqrt{r_n b})^{2+\delta}} = const \times r_n^{-\delta/2} \to 0\end{aligned}$$

since $r_n \to \infty$ as $n \to \infty$.

If we further assume $\bar{\sigma}_n^2 \to \sigma_*^2$ as $n \to \infty$, then under Condition 1 we have $\frac{S_n}{\sqrt{n}} \xrightarrow{d} N\left(0, \sigma_*^2\right)$.

1.3.2 Uniform CLT

If $E\left[|z|^{2+\delta}\right] \leq B < \infty$ and $\mathrm{var}\,(z) \geq b > 0$ for all $f \in \mathscr{F}$, then

$$\sup_{f\in\mathscr{F}} \left| P_f\left(\frac{\sqrt{n}\left(\bar{z}_n - E\left(z\right)\right)}{\sqrt{\mathrm{var}\left(z\right)}} \leq a\right) - \Phi\left(a\right)\right| \to 0.$$

This is a uniform CLT over a class of distributions in $\mathscr{F}$, instead of a single distribution f. Here P_f means that the probability is computed under a specific distribution f.

In a direct proof, the approximation error is controlled by B and b. The textbook uses a counter-positive argument: If the statement is false, then there is a sequence $f_1, f_2, ... \in \mathscr{F}$ that violates the convergence. That contradicts with Lyapunov CLT.

1.4 Uniform Integrability

Definition 1.3. The sequence of random variables z_n is *uniformly integrable* if

$$\lim_{M\to\infty} \sup_{n\geq 1} E\left[|z_n| \mathbb{I}\left\{|z_n| > M\right\}\right] = 0.$$

The textbook uses $\limsup_{n\to\infty}$ instead of $\sup_{n\geq 1}$ in the definition. These two notations are equivalent in our context here, as $E\left[|z_n| \mathbb{I}\left\{|z_n| > M\right\}\right] \searrow 0$ for every n as $M \to \infty$. "$\sup_{n\geq 1}$" appears more often in probability theory textbooks, and literally adheres to the notation of "uniformity".

Uniform integrability requires that the triangular array z_n is tight under the rth norm.

Example 1.1. Consider a counterexample

$$z_n = \begin{cases} -\sqrt{n} & with\ \ probability\ 1/n \\ 0 & with\ \ probability\ 1 - 2/n \\ \sqrt{n} & with\ \ probability\ 1/n. \end{cases}$$

Notice that

$$E\left[z_n^2 \mathbb{I}\left\{z_n^2 > M\right\}\right] = 2 \times \left(n\mathbb{I}\left\{n > M\right\}\right) \times \frac{1}{n} = 2 \cdot \mathbb{I}\left\{n > M\right\}.$$

For each fixed n, this z_n is square integrable in that $2 \cdot \mathbb{I}\left\{n > M\right\} = 0$ for all $M \geq n$. However, as $\sup_{n\geq 1} E\left[z_n^2 \mathbb{I}\left\{z_n^2 > M\right\}\right] = 2 \sup_{n\geq 1} \mathbb{I}\left\{n > M\right\} = 2$ for any finite M, and thus

$$\lim_{M\to\infty} \sup_{n\geq 1} E\left[z_n^2 \mathbb{I}\left\{z_n^2 > M\right\}\right] = 2 \nrightarrow 0.$$

As a result, this z_n is NOT uniformly square integrable.

Definition 1.4. A triangular array of random variables is *uniformly integrable* if

$$\lim_{M\to\infty}\sup_{n\geq 1}\max_{i\leq r_n} E\left[|x_{in}|\,\mathbb{I}\left\{|x_{in}|>M\right\}\right]=0.$$

Compared with Definition 1.3, we replace $E\left[|z_n|\,\mathbb{I}\left\{|z_n|>M\right\}\right]$ with $\max_{i\leq r_n} E\left[|x_{in}|\,\mathbb{I}\left\{|x_{in}|>M\right\}\right]$ in Definition 1.4 to control the worst case among the heterogeneous $(x_{in})_{i=1}^{r_n}$.

Proposition 1.1. *If Condition2 and the triangular array* $\{(x_{in})_{i=1}^{r_n}\}_{n\in\mathbb{N}}$ *is uniform square integrable, then Lindeberg condition holds.*

Proof. For any $\varepsilon>0$, we have

$$\begin{aligned}
\frac{1}{\tilde{\sigma}_n^2}\sum_{i=1}^{r_n} E\left[x_{in}^2\mathbb{I}\left\{x_{in}^2\geq\varepsilon\tilde{\sigma}_n^2\right\}\right] &\leq \frac{1}{r_n b}\sum_{i=1}^{r_n} E\left[x_{in}^2\mathbb{I}\left\{x_{in}^2\geq\varepsilon r_n b\right\}\right]\\
&\leq \frac{1}{r_n b}\times r_n\max_{i\leq r_n} E\left[x_{in}^2\mathbb{I}\left\{x_{in}^2\geq r_n\varepsilon b\right\}\right]\\
&= \mathit{const}\times\max_{i\leq r_n} E\left[x_{in}^2\mathbb{I}\left\{x_{in}^2\geq r_n\varepsilon b\right\}\right]\to 0
\end{aligned}$$

by the definition of uniform integrability since $r_n\varepsilon b\to\infty$ as $n\to\infty$. □

1.4.1 Uniform Stochastic Bound

Theorem 1.2. *If*

$$\lim_{M\to\infty}\sup_{n\geq 1}\max_{i\leq r_n} E\left[|x_{in}|^r\,\mathbb{I}\left\{|x_{in}|^r>M\right\}\right]=0$$

holds, then

$$r_n^{-1/r}\max_{i\leq r_n}|x_{in}|\xrightarrow{p}0.$$

Proof. We start with the definition of convergence in probability:

$$\begin{aligned}
P\left(r_n^{-1/r}\max_{i\leq r_n}|x_{in}|>\varepsilon\right) &= P\left(\max_{i\leq n}|x_{in}|^r>r_n\varepsilon^r\right)\\
&\leq \sum_{i\leq r_n} P\left(|x_{in}|^r>r_n\varepsilon^r\right)\\
&= \sum_{i\leq r_n} E\left[\mathbb{I}\left\{|x_{in}|^r>r_n\varepsilon^r\right\}\right]\\
&\leq r_n\max_{i\leq r_n} E\left[\mathbb{I}\left\{|x_{in}|^r>r_n\varepsilon^r\right\}\right]\\
&\leq r_n\times\frac{1}{r_n\varepsilon^r}\max_{i\leq r_n} E\left[|x_{in}|^r\,\mathbb{I}\left\{|x_{in}|^r>r_n\varepsilon^r\right\}\right]\\
&= \mathit{const}\times\max_{i\leq r_n} E\left[|x_{in}|^r\,\mathbb{I}\left\{|x_{in}|^r\geq r_n\varepsilon^r\right\}\right]\\
&\to 0
\end{aligned}$$

under the uniform rth integrability, since $r_n\varepsilon^r\to\infty$ as $n\to\infty$. □

As a special case, if we set $r_n=n$, then $\max_{i\leq n}|x_{in}|=o_p\left(n^{1/r}\right)$ if x_{in} is rth uniformly integrable.

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