Chapter 1

Asymptotic Theory I - Exercises

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Exercise 7.4 Consider a random variable Z_n with the probability distribution

$$Z_n = \begin{cases} -n & with \;\; probability \; 1/n \\ 0 & with \;\; probability \; 1-2/n \\ n & with \;\; probability \; 1/n \end{cases}$$

(a) Does $Z_n \xrightarrow{p} 0$ as $n \to \infty$?

(b) Calculate $E[Z_n]$.

(c) Calculate $Var[Z_n]$.

(d) Now suppose the distribution is

$$Z_n = \begin{cases} 0 & with \;\; probability \; 1-1/n \\ n & with \;\; probability \; 1/n \end{cases}$$

Calculate $E[Z_n]$.

(e) Conclude that $Z_n \xrightarrow{p} 0$ as $n \to \infty$ and $E[Z_n] \to 0$ are unrelated.

Solution. 7.4

(a) Take any $\delta > 0$, as $n \to \infty$

$$P\left(\omega : \left|Z_n\left(\omega\right) - 0\right| > \delta\right) = P\left(\omega : Z_n\left(\omega\right) = -n\right) + P\left(\omega : Z_n\left(\omega\right) = n\right) = 2/n \to 0$$

so that $Z_n \xrightarrow{p} 0$ as $n \to \infty$.

(b) $E[Z_n] = (-n) \times 1/n + 0 \times (1-2/n) + n \times 1/n = 0$.

(c) $Var[Z_n] = E\left(Z_n - E[Z_n]\right)^2 = (-n)^2 \times 1/n + 0^2 \times (1-2/n) + n^2 \times 1/n = 2n$.

(d) $E[Z_n] = 0 \times (1-1/n) + n \times 1/n = 1$.

(e) Take any $\delta > 0$, as $n \to \infty$

$$P\left(\omega : |Z_n(\omega) - 0| > \delta\right) = P\left(\omega : Z_n(\omega) = n\right) = 1/n \to 0$$

so that $Z_n \xrightarrow{p} 0$ as $n \to \infty$ and $E[Z_n] \to 0$ are unrelated.

Exercise 7.7 A weighted sample mean takes the form $\overline{X_n^*} = \frac{1}{n}\sum_{i=1}^n w_i X_i$ for some non-negative constants w_i satisfying $\frac{1}{n}\sum_{i=1}^n w_i = 1$. Assume X_i is i.i.d.

(a) Show that $\overline{X_n^*}$ is unbiased for $\mu = E[X]$.

(b) Calculate $Var[\overline{X_n^*}]$.

(c) Show that a sufficient condition for $\overline{X_n^*} \xrightarrow{p} \mu$ is that $n^{-2}\sum_{i=1}^n w_i^2 \to 0$.

(d) Show that a sufficient condition for the condition in part 3 is $\max_{i\le n} w_i$ →0 as $n \to \infty$.

Solution. 7.7

(a) $E[\overline{X_n^*}] = E[\frac{1}{n}\sum_{i=1}^n w_i X_i] = \frac{1}{n}\sum_{i=1}^n w_i E[X_i] = \frac{1}{n}\sum_{i=1}^n w_i \mu = \mu$.

(b)

$$\begin{aligned}
Var[\overline{X_n^*}] &= E\left(\overline{X_n^*} - E[\overline{X_n^*}]\right)^2 \\
&= E\left(\tfrac{1}{n}\textstyle\sum_{i=1}^n w_i X_i - \mu\right)^2 \\
&= E\left(\tfrac{1}{n}\textstyle\sum_{i=1}^n w_i \left(X_i - \mu\right)\right)^2 \\
&= Var\left(\tfrac{1}{n}\textstyle\sum_{i=1}^n w_i \left(X_i - \mu\right)\right) + E^2\left(\tfrac{1}{n}\textstyle\sum_{i=1}^n w_i \left(X_i - \mu\right)\right) \\
&= n^{-2}\textstyle\sum_{i=1}^n w_i^2 Var[X_i] \\
&= \sigma^2 n^{-2}\textstyle\sum_{i=1}^n w_i^2
\end{aligned}$$

(c) If $n^{-2}\sum_{i=1}^n w_i^2 \to 0$, we have $\sigma^2 n^{-2}\sum_{i=1}^n w_i^2 \to 0$.

So $E\left(\overline{X_n^*} - \mu\right)^2 = Var[\overline{X_n^*}] \to 0$

$\Rightarrow \overline{X_n^*} \xrightarrow{m.s.} \mu$

and square-mean convergence implies $\overline{X_n^*} \xrightarrow{p} \mu$.

(d) If $\max_{i\le n} w_i$ →0 as $n \to \infty$,

$$\frac{1}{n}\sum_{i=1}^{n} w_i^2 \leq \frac{1}{n} n \left(\max_i w_i\right)^2 = \left(\max_i w_i\right)^2 \to 0$$

so that $n^{-2}\sum_{i=1}^{n} w_i^2 \to 0$.

Exercise 7.11 Take a random sample $\{X_1, ..., X_n\}$ where $X > 0$ and $E\,|logX| < \infty$. Consider the sample geometric mean

$$\hat{\mu} = \left(\prod_{i=1}^{n} X_i\right)^{1/n}$$

and population geometric mean

$$\mu = \exp\left(E[\log X]\right)$$

Show that $\hat{\mu} \xrightarrow{p} \mu$ as $n \to \infty$.

Solution. 7.11

As $n \to \infty$ by WLLN,

$$\begin{aligned} E\,|\log X| < \infty \quad & \Rightarrow \tfrac{1}{n}\textstyle\sum_{i=1}^{n} \log X_i \xrightarrow{p} E[\log X] \\ & \Rightarrow \tfrac{1}{n}\log\left(\textstyle\prod_{i=1}^{n} X_i\right) \xrightarrow{p} E[\log X] \\ & \Rightarrow \log\left(\textstyle\prod_{i=1}^{n} X_i\right)^{1/n} \xrightarrow{p} E[\log X] \end{aligned}$$

Take $h(u) = \exp(u)$, by CMT,

$$\begin{aligned} \log\left(\textstyle\prod_{i=1}^{n} X_i\right)^{1/n} \xrightarrow{p} E[\log X] \quad & \Rightarrow h\left(\log\left(\textstyle\prod_{i=1}^{n} X_i\right)^{1/n}\right) \xrightarrow{p} h\left(E[\log X]\right) \\ & \Rightarrow \exp\left(\log\left(\textstyle\prod_{i=1}^{n} X_i\right)^{1/n}\right) \xrightarrow{p} \exp\left(E[\log X]\right) \\ & \Rightarrow \left(\textstyle\prod_{i=1}^{n} X_i\right)^{1/n} \xrightarrow{p} \exp\left(E[\log X]\right) \\ & \Rightarrow \hat{\mu} \xrightarrow{p} \mu \end{aligned}$$

Exercise 8.1 Let X be distributed Bernoulli $P[X=1]=p$ and $P[X=0]=1-p$.

(a) Show that $p=E[X]$.

(b) Write down the moment estimator $\hat{p}$ of p.

(c) Find $Var[\hat{p}]$.

(d) Find the asymptotic distribution of $\sqrt{n}(\hat{p}-p)$ as $n\to\infty$.

Solution. 8.1

(a) $E[X]=P[X=1]\times 1+P[X=0]\times 0=p.$

(b) $\because E[\overline{X}]=E[\frac{1}{n}\sum_{i=1}^{n}X_i]=\frac{1}{n}\sum_{i=1}^{n}E[X_i]=\frac{1}{n}\times np=p$

$\therefore \hat{p}=\overline{X}=\frac{1}{n}\sum_{i=1}^{n}X_i$

(c) $Var[\hat{p}]=Var[\frac{1}{n}\sum_{i=1}^{n}X_i]=\frac{1}{n^2}Var[\sum_{i=1}^{n}X_i]=\frac{1}{n^2}\sum_{i=1}^{n}Var[X_i]=\frac{1}{n^2}np\,(1-p)=\frac{1}{n}p\,(1-p).$

(d) By CLT, $\sqrt{n}(\hat{p}-p)\xrightarrow{d}N\left(0,p\,(1-p)\right)$ as $n\to\infty$.

Exercise 8.7 Assume $\sqrt{n}\left(\widehat{\theta}-\theta\right)\xrightarrow{d}N\left(0,v^2\right)$. Use the Delta Method to find the asymptotic distribution of the following statistics

(a) $\hat{\theta}^2$.

(b) $\hat{\theta}^3$.

(c) $\hat{\theta}^k$.

(d) $\hat{\theta}^2+\hat{\theta}^3$.

(e) $\frac{1}{1+\hat{\theta}^2}$.

(f) $\frac{1}{1+\exp(\hat{\theta})}$.

Solution. 8.7

(a) Take $h(u)=u^2$, so that by the Delta Method

$$\sqrt{n}\left(\widehat{\theta}^2-\theta^2\right)\xrightarrow{d}N\left(0,(2\theta)^2v^2\right)$$

(b) Take $h(u)=u^3$, so that by the Delta Method

$$\sqrt{n}\left(\widehat{\theta}^3-\theta^3\right)\xrightarrow{d}N\left(0,\left(3\theta^2\right)^2v^2\right)$$

(c) Take $h(u) = u^k$, so that by the Delta Method

$$\sqrt{n}\left(\widehat{\theta}^k - \theta^k\right) \xrightarrow{d} N\left(0, \left(k\theta^{k-1}\right)^2 v^2\right)$$

(d) Take $h(u) = u^2 + u^3$, so that by the Delta Method

$$\sqrt{n}\left(\left(\hat{\theta}^2 + \hat{\theta}^3\right) - \left(\theta^2 + \theta^3\right)\right) \xrightarrow{d} N\left(0, \left(2\theta + 3\theta^2\right)^2 v^2\right)$$

(e) Take $h(u) = \frac{1}{1+u^2}$, so that by the Delta Method

$$\sqrt{n}\left(\frac{1}{1+\hat{\theta}^2} - \frac{1}{1+\theta^2}\right) \xrightarrow{d} N\left(0, \frac{4\theta^2}{(1+\theta^2)^4} v^2\right)$$

(f) Take $h(u) = \frac{1}{1+\exp(u)}$, so that by the Delta Method

$$\sqrt{n}\left(\frac{1}{1+\exp(\hat{\theta})} - \frac{1}{1+\exp(\theta)}\right) \xrightarrow{d} N\left(0, \frac{\exp(2\theta)}{(1+\exp(\theta))^4} v^2\right)$$