Chapter 1

CLT for inid Sequences

1.1 Notations and Definitions

Without loss of generality, assume $E[x_{in}] = 0$ for all i and n and denote $\sigma_{in}^2 = E[x_{in}^2]$. Define the *partial sum* (up to n) as $S_n = \sum_{i=1}^{r_n} x_{in}$ and and (the nth) *aggregate variance* as $\tilde{\sigma}_n^2 = \sum_{i=1}^{r_n} \sigma_{in}^2$.

A *triangular array* $\{(x_{1n}, x_{2n}, x_{3n}, ..., x_{r_n n})\}_{n\in\mathbb{N}}$ stacks like a triangle:

$$\begin{pmatrix} x_{11} & x_{22} & \cdots & x_{r_1 1} & & & & \\ x_{12} & x_{22} & \cdots & \cdots & x_{r_2 2} & & & \\ x_{13} & x_{23} & \cdots & \cdots & \cdots & x_{r_3 3} & & \\ \vdots & & & & & & \ddots & \\ x_{1n} & x_{2n} & \cdots & \cdots & \cdots & \cdots & \cdots & x_{r_n n} \end{pmatrix}.$$

Here r_n is an increasing number in n, and $r_n \to \infty$ as $n \to \infty$. (Think about the special case $r_n = n$, which makes an exact triangle.) Suppose for each n, the elements in $(x_{in})_{i=1}^{r_n}$ are independently non-identically distributed (inid). (Please keep a liberal mind and take "identically distributed" as a special case of "non-identically distributed".)

1.2 Lindeberg Condition

Lindeberg-Lévy Central Limit Theorem is for independently and identically distributed (iid) sequences. In this lecture we consider independent, heterogeneous sequences.

Definition 1.1. Lindeberg Condition:

$$\lim_{n\to\infty} \frac{1}{\tilde{\sigma}_n^2} \sum_{i=1}^{r_n} E[x_{in}^2 \mathbb{I}\left\{x_{in}^2 \geq \varepsilon \tilde{\sigma}_n^2\right\}] = 0$$

for all $\varepsilon > 0$.

Theorem 1.1 (Lindeberg-Feller CLT)**.** *If the triangular array $\{(x_{in})_{i=1}^{r_n}\}_{n\in\mathbb{N}}$ satisfies the Lindeberg condition, then*

$$\frac{S_n}{\tilde{\sigma}_n} \xrightarrow{d} N(0, 1)$$

Lindeber-Feller CLT allows heterogeneity across $i = 1, ..., r_n$. It includes *Lindeberg-Lévy CLT* as a special case. To see this fact, under iid let us use z to represent the homogeneous distribution. Denote $\text{var}(z) = \sigma_z^2 \in (0, \infty)$, and equivalently $\lim_{M\to\infty} E\left[z^2 \mathbb{I}\left\{z^2 \geq M\right\}\right] = 0$ (square integrability). Set $r_n = n$, and thus $\tilde{\sigma}_n^2 = n\sigma_z^2$. As a result,

$$\begin{aligned}\frac{1}{\tilde{\sigma}_n^2}\sum_{i=1}^{n} E\left[x_{in}^2 \mathbb{I}\left\{x_{in}^2 \geq \varepsilon\tilde{\sigma}_n^2\right\}\right] &= \frac{1}{n\sigma_z^2} \times nE\left[z^2 \mathbb{I}\left\{z^2 \geq n\sigma_z^2\varepsilon\right\}\right] \\ &= const \times E\left[z^2 \mathbb{I}\left\{z^2 \geq n\sigma_z^2\varepsilon\right\}\right] \to 0\end{aligned}$$

since $n\sigma_1^2\varepsilon \to \infty$ as $n \to \infty$.

With iid and $r_n = n$, we can drop the subscript n and write $z_i = x_{in}$. The ratio

$$\frac{S_n}{\tilde{\sigma}_n} = \frac{\sum_{i=1}^n z_i}{\sqrt{n\sigma_z^2}} = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{z_i}{\sigma_z}$$

retains its familiar form in CLT.

1.3 Lyapunov Condition

Lindeberg condition is mathematical artifact that is difficult to interpret. Lyapunov condition is a more interpretable sufficient condition.

Definition 1.2. Lyapunov Condition: There exists some $\delta > 0$ such that

$$\lim_{n\to\infty}\frac{1}{\tilde{\sigma}_n^{2+\delta}}\sum_{i=1}^{r_n} E[|x_{in}|^{2+\delta}] = 0$$

Lyapunov condition implies Lindeberg condition. To see this fact, we use the quantity in the Lindeberg condition as a starting point:

$$\begin{aligned}\frac{1}{\tilde{\sigma}_n^2}\sum_{i=1}^{r_n} E\left[x_{in}^2 \mathbb{I}\left\{x_{in}^2 \geq \varepsilon\tilde{\sigma}_n^2\right\}\right] &= \frac{1}{\tilde{\sigma}_n^2}\sum_{i=1}^{r_n} E\left[|x_{in}|^2 \mathbb{I}\left\{|x_{in}|^\delta \geq \varepsilon^{\frac{\delta}{2}}\tilde{\sigma}_n^\delta\right\}\right] \\ &= \frac{1}{\tilde{\sigma}_n^2}\sum_{i=1}^{r_n} E\left[\frac{|x_{in}|^{2+\delta}}{|x_{in}|^\delta}\mathbb{I}\left\{|x_{in}|^\delta \geq \varepsilon^{\frac{\delta}{2}}\tilde{\sigma}_n^\delta\right\}\right] \\ &\leq \frac{1}{\tilde{\sigma}_n^2} \times \frac{1}{\varepsilon^{\delta/2}\tilde{\sigma}_n^\delta}\sum_{i=1}^{r_n} E\left[|x_{in}|^{2+\delta}\mathbb{I}\left\{|x_{in}|^\delta \geq \varepsilon^{\frac{\delta}{2}}\tilde{\sigma}_n^\delta\right\}\right] \\ &\leq \frac{1}{\varepsilon^{\delta/2}} \times \frac{1}{\tilde{\sigma}_n^{2+\delta}}\sum_{i=1}^{r_n} E\left[|x_{in}|^{2+\delta}\right] \to 0\end{aligned}$$

as $n \to \infty$, where the limit follows from Lyapunov condition.

1.3.1 Sufficient Condition for Lyapunov Condition

- Condition1: $\sup_{i\leq r_n} E[|x_{in}|^{2+\delta}] \leq B < \infty$ for all sufficiently large n.
- Condition2: $\liminf_{n\to\infty}\bar{\sigma}_n^2 > b > 0$, where $\bar{\sigma}_n^2 = \tilde{\sigma}_n^2/r_n$ is the *average variance*.

Under Condition 1 and Condition 2 we have

$$\begin{aligned}\frac{1}{\tilde{\sigma}_n^{2+\delta}}\sum_{i=1}^{r_n} E[|x_{in}|^{2+\delta}] &\leq \frac{1}{(\sqrt{r_n b})^{2+\delta}} \times r_n \max_{i\leq r_n} E[|x_{in}|^{2+\delta}] \\ &\leq \frac{r_n B}{(\sqrt{r_n b})^{2+\delta}} = const \times r_n^{-\delta/2} \to 0\end{aligned}$$

since $r_n \to \infty$ as $n \to \infty$.

If we further assume $\bar{\sigma}_n^2 \to \sigma_*^2$ as $n \to \infty$, then under Condition 1 we have $\frac{S_n}{\sqrt{n}} \xrightarrow{d} N\left(0, \sigma_*^2\right)$.

1.3.2 Uniform CLT

If $E\left[|z|^{2+\delta}\right] \leq B < \infty$ and $\mathrm{var}\,(z) \geq b > 0$ for all $f \in \mathscr{F}$, then

$$\sup_{f\in\mathscr{F}}\left|P_f\left(\frac{\sqrt{n}\,(\bar{z}_n - E\,(z))}{\sqrt{\mathrm{var}\,(z)}} \leq a\right) - \Phi\,(a)\right| \to 0.$$

This is a uniform CLT over a class of distributions in $\mathscr{F}$, instead of a single distribution f. Here P_f means that the probability is computed under a specific distribution f.

The textbook uses a counter-positive argument. Let $f_1, f_2, ... \in \mathscr{F}$ be a sequence of distributions, and each f_n forms a row in the triangular array. If the statement is false, then there exists a sequence $f_1, f_2, ..$ that contradicts with the Lyapunov CLT.

1.4 Uniform Integrability

Definition 1.3. A random variable z is rth *integrable* if $E[|z|^r] = \int_{-\infty}^{\infty} |z|^r \, dF(z) < \infty$. Equivalently,

$$\lim_{M\to\infty} E[|z|^r \,\mathbb{I}\{|z|^r > M\}] = 0,$$

where $\mathbb{I}\{\cdot\}$ is the indicator function.

Without referring explicitly to the rth moment, we say z is *integrable* if $r = 1$, and *square integrable* if $r = 2$.

Definition 1.4. The sequence of random variables z_n is *uniformly integrable* if

$$\lim_{M\to\infty} \sup_{n\geq 1} E\left[|z_n|\,\mathbb{I}\{|z_n| > M\}\right] = 0.$$

I prefer the above definition (with $\sup_{n\geq 1}$) to that of the textbook (with $\limsup_{n\to\infty}$). These two notations are equivalent in our context here, as $E\left[|z_n|\,\mathbb{I}\{|z_n| > M\}\right] \searrow 0$ for every n as $M \to \infty$. "$\sup_{n\geq 1}$" appears more often in probability theory textbooks, and literally adheres to the notation of "uniformity".

Example 1.1. Consider a counterexample

$$z_n = \begin{cases} -\sqrt{n} & \text{with probability } 1/n \\ 0 & \text{with probability } 1 - 2/n \\ \sqrt{n} & \text{with probability } 1/n. \end{cases}$$

Notice that

$$E\left[z_n^2\mathbb{I}\left\{z_n^2 > M\right\}\right] = 2 \times (n\mathbb{I}\{n > M\}) \times \frac{1}{n} = 2 \cdot \mathbb{I}\{n > M\}.$$

For each fixed n, this z_n is square integrable in that $2 \cdot \mathbb{I}\{n > M\} = 0$ for all $M \geq n$. However, as $\sup_{n\geq 1} E\left[z_n^2\mathbb{I}\left\{z_n^2 > M\right\}\right] = 2\sup_{n\geq 1}\mathbb{I}\{n > M\} = 2$ for any finite M, and thus

$$\lim_{M\to\infty}\sup_{n\geq 1} E\left[z_n^2\mathbb{I}\left\{z_n^2 > M\right\}\right] = 2 \nrightarrow 0.$$

As a result, this sequence z_n is NOT uniformly square integrable.

Definition 1.5. A triangular array of random variables is *uniformly integrable* if

$$\lim_{M\to\infty}\sup_{n\geq 1}\max_{i\leq r_n} E\left[|x_{in}|\,\mathbb{I}\{|x_{in}| > M\}\right] = 0.$$

Compared with Definition 1.4, we replace $E\left[|z_n|\,\mathbb{I}\{|z_n| > M\}\right]$ with $\max_{i\leq r_n} E\left[|x_{in}|\,\mathbb{I}\{|x_{in}| > M\}\right]$ in Definition 1.5 to control the worst case among the heterogeneous $(x_{in})_{i=1}^{r_n}$.

Proposition 1.1. *If Condition2 holds and the triangular array $\{(x_{in})_{i=1}^{r_n}\}_{n\in\mathbb{N}}$ is uniformly square integrable, then the Lindeberg condition holds.*

Proof. For any $\varepsilon > 0$, we have

$$\begin{aligned}
\frac{1}{\tilde{\sigma}_n^2}\sum_{i=1}^{r_n} E\left[x_{in}^2\mathbb{I}\left\{x_{in}^2 \geq \varepsilon\tilde{\sigma}_n^2\right\}\right] &\leq \frac{1}{r_n b}\sum_{i=1}^{r_n} E\left[x_{in}^2\mathbb{I}\left\{x_{in}^2 \geq \varepsilon r_n b\right\}\right] \\
&\leq \frac{1}{r_n b} \times r_n \max_{i\leq r_n} E\left[x_{in}^2\mathbb{I}\left\{x_{in}^2 \geq r_n\varepsilon b\right\}\right] \\
&= \mathit{const} \times \max_{i\leq r_n} E\left[x_{in}^2\mathbb{I}\left\{x_{in}^2 \geq r_n\varepsilon b\right\}\right] \to 0
\end{aligned}$$

by the definition of uniform integrability since $r_n\varepsilon b \to \infty$ as $n \to \infty$. $\square$

1.4.1 Uniform Stochastic Bound

Theorem 1.2. *If*

$$\lim_{M\to\infty}\sup_{n\geq 1}\max_{i\leq r_n} E\left[|x_{in}|^r\,\mathbb{I}\{|x_{in}|^r > M\}\right] = 0$$

holds, then

$$r_n^{-1/r}\max_{i\leq r_n}|x_{in}| \xrightarrow{p} 0.$$

Proof. We start with the definition of convergence in probability:

$$\begin{aligned}
P\left(r_n^{-1/r}\max_{i\leq r_n}|x_{in}| > \varepsilon\right) &= P\left(\max_{i\leq n}|x_{in}|^r > r_n\varepsilon^r\right) \leq \sum_{i\leq r_n} P\left(|x_{in}|^r > r_n\varepsilon^r\right) \\
&\leq r_n \max_{i\leq r_n} P\left(|x_{in}|^r > r_n\varepsilon^r\right) = r_n \max_{i\leq r_n} E\left[\mathbb{I}\{|x_{in}|^r > r_n\varepsilon^r\}\right] \\
&\leq r_n \times \frac{1}{r_n\varepsilon^r}\max_{i\leq r_n} E\left[|x_{in}|^r\,\mathbb{I}\{|x_{in}|^r > r_n\varepsilon^r\}\right] \\
&= \mathit{const} \times \max_{i\leq r_n} E\left[|x_{in}|^r\,\mathbb{I}\{|x_{in}|^r \geq r_n\varepsilon^r\}\right] \to 0
\end{aligned}$$

under the uniform rth integrability, since $r_n\varepsilon^r \to \infty$ as $n \to \infty$. $\square$

As a special case, if we set $r_n = n$, then $\max_{i\leq n}|x_{in}| = o_p\left(n^{1/r}\right)$ if x_{in} is rth uniformly integrable.

1.5 Summary

Inid sequence is the first step out of the restrictive iid world. Uniform integrability is an important assumption in various advanced asymptotic techniques, for example bootstrap and multi-index panel data.

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