# Chapter 1

# Time Series - Exercises

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**Exercise 14.1 For a scalar time series $Y_t$ define the sample autocovariance and autocorrelation**

$$\hat{\gamma}(k) = n^{-1} \sum_{t=k+1}^{n} \left(Y_t - \bar{Y}\right)\left(Y_{t-k} - \bar{Y}\right)$$

$$\hat{\rho}(k) = \frac{\hat{\gamma}(k)}{\hat{\gamma}(0)} = \frac{\sum_{t=k+1}^{n} \left(Y_t - \bar{Y}\right)\left(Y_{t-k} - \bar{Y}\right)}{\sum_{t=1}^{n} \left(Y_t - \bar{Y}\right)^2}.$$

**Assume the series is strictly stationary, ergodic, strictly stationary, and $E\left[Y_t^2\right] < \infty$. Show that $\hat{\gamma}(k) \xrightarrow{p} \gamma(k)$ and $\hat{\rho}(k) \xrightarrow{p} \gamma(k)$ as $n \to \infty$. (Use the Ergodic Theorem.)**

**Solution. 14.1**

(1) Since $Y_t$ is strictly stationarity, its transformation $\left(Y_t - \bar{Y}\right)\left(Y_{t-k} - \bar{Y}\right)$ is also strictly stationarity.

Use the Ergodic Theorem,

$$(n-k-1)^{-1} \sum_{t=k+1}^{n} \left(Y_t - \bar{Y}\right)\left(Y_{t-k} - \bar{Y}\right) \xrightarrow{p} E\left[\left(Y_t - \bar{Y}\right)\left(Y_{t-k} - \bar{Y}\right)\right] = \gamma(k)$$

as $n \to \infty$, and therefore

$$\hat{\gamma}(k) = n^{-1} \sum_{t=k+1}^{n} \left(Y_t - \bar{Y}\right)\left(Y_{t-k} - \bar{Y}\right) = \frac{n-k-1}{n} \times \frac{1}{n-k-1} \sum_{t=k+1}^{n} \left(Y_t - \bar{Y}\right)\left(Y_{t-k} - \bar{Y}\right) \xrightarrow{p} \gamma(k).$$

(2)

$$\hat{\rho}(k) = \frac{\hat{\gamma}(k)}{\hat{\gamma}(0)} \xrightarrow{p} \frac{\gamma(k)}{\gamma(0)} = \rho(k)$$

**Exercise 14.2 Show that if $(e_t, \mathscr{F}_t)$ is a MDS and $X_t$ is $\mathscr{F}_t$-measurable then $u_t = X_{t-1}e_t$ is a MDS.**

**Solution. 14.2**

Since $X_t$ is $\mathscr{F}_t$-measurable, we have $u_t$ adapted to $\mathscr{F}_t$ in that

$$E\left[u_t \mid \mathscr{F}_t\right] = E\left[X_{t-1}e_t \mid \mathscr{F}_t\right] = X_{t-1}e_t.$$

as $x_{t-1}$is $\mathscr{F}_t$ measurable. Moreover,

$$E\left[u_t \mid \mathscr{F}_{t-1}\right]=E\left[X_{t-1} e_t \mid \mathscr{F}_{t-1}\right]=X_{t-1} E\left[e_t \mid \mathscr{F}_{t-1}\right]=0$$

We thus have verified $u_t=X_{t-1} e_t$ being a MDS.

**Exercise 14.3 Let $\sigma_t^2=E\left[e_t^2 \mid \mathscr{F}_{t-1}\right]$. Show that $u_t=e_t^2-\sigma_t^2$ is a MDS.**

**Solution. 14.3**

(1) $\sigma_t^2=E\left[e_t^2 \mid \mathscr{F}_{t-1}\right]$ is adapted to $\mathscr{F}_t$.
(2) $E\left|\sigma_t^2\right|=E\left|E\left[e_t^2 \mid \mathscr{F}_{t-1}\right]\right|<\infty$.
(3)

$$\begin{aligned}
E\left[u_t \mid \mathscr{F}_{t-1}\right] &=E\left[e_t^2-\sigma_t^2 \mid \mathscr{F}_{t-1}\right] \\
&=E\left[e_t^2-E\left[e_t^2 \mid \mathscr{F}_{t-1}\right] \mid \mathscr{F}_{t-1}\right] \\
&=E\left[e_t^2 \mid \mathscr{F}_{t-1}\right]-E\left[e_t^2 \mid \mathscr{F}_{t-1}\right]=0
\end{aligned}$$

Therefore, $u_t=e_t^2-\sigma_t^2$ is a MDS.

**Exercise 14.8 Suppose $Y_t=Y_{t-1}+e_t$ with $e_t$ i.i.d. (0,1) and $Y_0=0$. Find $var\left[Y_t\right]$. Is $Y_t$ stationary?**

**Solution. 14.8**

(1)

$$\begin{aligned}
Y_t &= Y_{t-1}+e_t \\
&= Y_{t-2}+e_{t-1}+e_t \\
&= Y_{t-3}+e_{t-2}+e_{t-1}+e_t \\
&\vdots \\
&= Y_0+e_1 \cdots+e_{t-2}+e_{t-1}+e_t \\
&= \sum_{i=1}^t e_i
\end{aligned}$$

$$var\left[Y_t\right]=var\left[\sum_{i=1}^t e_i\right]=\sum_{i=1}^t var\left[e_i\right]=\sum_{i=1}^t 1=t$$

(2) $Y_t$ is not stationary because $var\left[Y_t\right]$ varies with $t$.