Chapter 1

Empirical Process Theory - Exercises

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Exercise 18.1 Let $g(x,\theta) = \mathbb{I}(x \leq \theta)$ **for** $\theta \in [0,1]$ **and assume** $X \sim F = \mathcal{U}[0,1]$**. Let** $N_1(\varepsilon, F)$ **be** L_1 **packing numbers.**

(a) Show that $N_1(\varepsilon, F)$ **equal the packing numbers constructed with respect to the Euclidean metric** $d(\theta_1, \theta_2) = |\theta_1 - \theta_2|$**.**

(b) Verify that $N_1(\varepsilon, F) \leq \lceil \frac{1}{\varepsilon} \rceil$**.**

Solution. 18.1

(a) When $\theta_1 \geq \theta_2$,

$$\begin{aligned}
\|g(x,\theta_1) - g(x,\theta_2)\|_{F,1} &= E_F[\|g(x,\theta_1) - g(x,\theta_2)\|] \\
&= \int \|g(x,\theta_1) - g(x,\theta_2)\| \, \mathrm{d}F \\
&= \int (\mathbb{I}(x \leq \theta_1) - \mathbb{I}(x \leq \theta_2)) \, \mathrm{d}F \\
&= \int \mathbb{I}(x \leq \theta_1) \, \mathrm{d}F - \int \mathbb{I}(x \leq \theta_2) \, \mathrm{d}F \\
&= F(\theta_1) - F(\theta_2) \\
&= \theta_1 - \theta_2 = d(\theta_1, \theta_2)
\end{aligned}$$

When $\theta_1 < \theta_2$, a similar calculation is given $d(\theta_1, \theta_2)$.

(b) Set $N = \frac{1}{\varepsilon}$ as an integer, and let $\theta_j = \frac{j}{N}$ be the $\frac{j}{N}$-th quantile.

θ_j can be packed into $[0,1]$ with each pair satisfying $d(\theta_{j+1}, \theta_j) = |\theta_{j+1} - \theta_j| = \varepsilon$

Thus $N_1(\varepsilon, F) \leq \lceil \frac{1}{\varepsilon} \rceil$.

Exercise 18.3 Define $\nu_n(\theta) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i \mathbb{I}(X_i \le \theta)$ **for** $\theta \in [0,1]$ **where** $E[X] = 0$ **and** $E[X^2] = 1$**.**

(a) Show that $\nu_n(\theta)$ **is stochastically equicontinuous.**

(b) Find the stochastic process $\nu(\theta)$ **which has asymptotic finite dimensional distributions of** $\nu_n(\theta)$**.**

(c) Show that $\nu_n \xrightarrow{d} \nu$**.**

Solution. 18.3

(a) $\nu_n(\theta) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (X_i \mathbb{I}(X_i \le \theta) - E[X_i \mathbb{I}(X_i \le \theta)])$. Suppose $X_i \sim F$, where F can be either continuous or discrete. We can compute the bracket integral to check the complexity of the function $x\mathbb{I}(x \le \theta)$ for $\theta \in [0,1]$. Set $N = \varepsilon^{-1}$ as an integer, and let $\theta_j = F^{-1}\left(\frac{j}{N}\right)$ be the $\frac{j}{N}$-th quantile. We construct a bracket with the lower bound function $l_j(x) = x\mathbb{I}(x \le \theta_j)$ and the upper bound function $u_j(x) = x\mathbb{I}(x \le \theta_{j+1})$. Obviously, for any $\theta \in [0,1]$, there exists a pair $[l_j, u_j]$ such that $l_j(x) \le x\mathbb{I}(x \le \theta) \le u_j(x)$. The size of such a bracket is

$$\begin{aligned}\left\|u_j(x) - l_j(x)\right\|_{F,2}^2 &= E\left[\left[x\mathbb{I}\left(\theta_j \le x \le \theta_{j+1}\right)\right]^2\right] \\ &\le E\left[\mathbb{I}\left(\theta_j \le x \le \theta_{j+1}\right)\right] \\ &= F\left(\theta_{j+1}\right) - F\left(\theta_j\right) \le \varepsilon\end{aligned}$$

where the inequality holds as $x \in [\theta_j, \theta_{j+1}] \subseteq [0,1]$. Thus $N_{[]}(\varepsilon, L_2(F)) \le N = \varepsilon^{-1} = O(\varepsilon^{-1})$, we have $J_{[\,]}(1, L_2(F)) < \infty$. We invoke the textbook's Theorem 18.4 to establish stochastic equicontinuity.

(b) Let $u_i(\theta) = X_i \mathbb{I}(X_i \le \theta) - E[X_i \mathbb{I}(X_i \le \theta)]$. Denote $m(\theta) := E[X^2 \mathbb{I}(X \le \theta)]$ and $\mu(\theta) = E[X\mathbb{I}(X \le \theta)]$.

$$\sigma^2(\theta) := var[u_i(\theta)] = E\left[X^2 \mathbb{I}(X \le \theta)\right] - (E[X\mathbb{I}(X \le \theta)])^2 = m^2(\theta) - \mu^2(\theta)$$

and

$$\begin{aligned}
\sigma(\theta_1,\theta_2) &:= Cov\left(u_i(\theta_1), u_i(\theta_2)\right) \\
&= Cov\left(X\mathbb{I}(X \le \theta_1), X\mathbb{I}(X \le \theta_2)\right) \\
&= E\left[X^2\mathbb{I}(X \le \theta_1)\mathbb{I}(X \le \theta_2)\right] - E\left[X\mathbb{I}(X \le \theta_1)\right]E\left[X\mathbb{I}(X \le \theta_2)\right] \\
&= E\left[X^2\mathbb{I}(X \le \theta_1 \wedge \theta_2)\right] - E\left[X\mathbb{I}(X \le \theta_1)\right]E\left[X\mathbb{I}(X \le \theta_2)\right] \\
&= m^2(\theta_1 \wedge \theta_2) - \mu(\theta_1)\mu(\theta_2).
\end{aligned}$$

Since $u_i(\theta)$ is i.i.d, we have the CLT

$$\begin{pmatrix} \nu_n(\theta_1) \\ \nu_n(\theta_2) \end{pmatrix} = \sqrt{n}\sum_{i=1}^{n}\begin{pmatrix} X\mathbb{I}(X \le \theta_1) \\ X\mathbb{I}(X \le \theta_2) \end{pmatrix} \xrightarrow{d} N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma^2(\theta_1) & \sigma(\theta_1,\theta_2) \\ \sigma(\theta_1,\theta_2) & \sigma^2(\theta_2) \end{pmatrix}\right)$$

The joint distribution is satisfied for any finite $\theta_1, \theta_2, ..., \theta_m$, so

$$\left(\nu_n(\theta_1), \nu_n(\theta_2), ..., \nu_n(\theta_m)\right) \xrightarrow{d} \left(\nu(\theta_1), \nu(\theta_2), ..., \nu(\theta_m)\right)$$

for every finite set $\theta_1, \theta_2, ..., \theta_m \in \theta \in [0,1]$

(c) Given (a) and (b), we invoke the textbook's Theorem 18.3 to establish $\nu_n \xrightarrow{d} \nu$.