

# Chapter 1

## M-Estimators

Except for some Bayesian estimators that require integration or simulation to get the values, almost all frequentist estimators in econometrics involve steps of minimizing some criterion functions. The m-estimator is a class of estimators that each individual observation contributes to the overall criterion function in an additive, separable manner.

We are familiar with OLS. The OLS estimator boasts an explicit solution in that it can be written as a function of the data only. This is due to the fact that the loss function of OLS is quadratic and the regressors are linearly combined by the slope coefficients. Most estimators do not enjoys such simplicity. For example, the logistic regression does not have a closed-form solution but must be solved numerically.

### 1.1 Formulation

Let the loss function be  $\rho_i(\theta) = \rho(z_i, \theta)$ , where  $z_i$  is the  $i$ th observation. It can be a scalar random variable or a multivariate one. The sample criterion is the average of  $\rho_i(\theta)$  over  $i = 1, 2, \dots, n$ :

$$S_n(\theta) = \frac{1}{n} \sum_{i=1}^n \rho_i(\theta)$$

The m-estimator minimizes the sample criterion function:

$$\hat{\theta}_n = \arg \min_{\theta \in \Theta} S_n(\theta).$$

The m-estimator includes many examples as special cases. For example, OLS, nonlinear least squares (NLS), maximum likelihood estimation (MLE), and quantile regressions are all m-estimators. GMM, another familiar econometric estimator, is not an m-estimator; it belongs to the broader class of minimum-distance estimators.

**Example 1.1.** The most familiar example of m-estimator is the OLS. Let  $y_i = x_i'\theta + e_i$ . The loss function is  $\rho_i(\theta) = \frac{1}{2}(y_i - x_i'\theta)^2$  where  $z_i = (y_i, x_i')'$ , and the sample criterion function is  $\frac{1}{2n} \sum_i (y_i - x_i'\theta)^2$ .

**Example 1.2.** Silva and Tenreyro (2006) propose a nonlinear conditional mean model  $E[y_i|x_i] = \exp(x_i'\theta)$ , where the non-negative  $\exp(\cdot)$  in the right-hand side is used to model the non-negative trade volumes between countries. NLS sets up the loss function  $\rho_i(\theta) = (y_i - \exp(x_i'\theta))^2$ .

**Example 1.3.** If  $y \sim \text{Poisson}(\lambda)$ , then  $\Pr(y = k) = \lambda^k \exp(-\lambda) / k!$  for  $k \in \{0, 1, 2, \dots\}$ . An alternative way to set up the criterion is to pretend that  $y_i$  is draw from a conditional Poisson model with mean  $\lambda_i = \exp(x_i' \theta)$  and thus

$$\rho_i(\theta) = -\log \Pr(y_i | x_i) = -y_i \cdot x_i' \theta + \exp(x_i' \theta)$$

where we drop  $y_i!$  from the above express as it is irrelevant to the parameter  $\theta$ . Here we use the “minus log-likelihood” to be consistent with the minimization for m-estimators, instead of maximizing the log-likelihood. This is the loss function of so called PPML (Pseudo Poisson Maximum Likelihood) in Silva and Tenreiro (2006).

For simplicity, in this lecture we work with iid data. Let

$$S(\theta) = E[S_n(\theta)] = E[\rho_i(\theta)]$$

be the population criterion function.

In econometrics, identification has different meanings in different contexts. Lewbel (2019) is an overview of the “identification zoo”. The definition here is rather mechanical and has nothing to do with economics. Let  $\Theta$  be the **parameter space**. We say  $\theta$  is **identified** if  $\theta_0 = \arg \min_{\theta \in \Theta} S(\theta)$  is unique. More formally:

**Definition 1.1** (Identification). For any  $\varepsilon > 0$  there exists a  $\delta = \delta(\varepsilon)$  such that

$$\inf_{\theta \in \Theta \setminus N_\delta(\theta_0)} S(\theta) > S(\theta_0) + \varepsilon$$

where  $N_\delta(\theta_0) := \{\theta \in \Theta : \|\theta - \theta_0\| < \delta\}$  is an open  $\delta$ -neighborhood around  $\theta_0$ .

Identification is a property of the underlying probabilistic model. It has nothing to do with the data or the randomness in sampling.

## 1.2 Consistency

To establish the consistency of the m-estimator, identification is a necessary condition. In the previous lecture we have covered the consistency of a single sequence of random variables. In estimation the model is index by  $\theta \in \Theta$ . Pointwise consistency  $S_n(\theta) \xrightarrow{P} S(\theta)$  is insufficient to guarantee the consistency of  $\hat{\theta}$  to the true parameter  $\theta_0$ . We need to strength it to the **uniform consistency** of the sample mean.

**Definition 1.2** (Uniform law of large numbers). For any  $\delta, \varepsilon > 0$ , there exists an  $N = N(\varepsilon, \delta)$  such that

$$\Pr \left\{ \sup_{\theta \in \Theta} |S_n(\theta) - S(\theta)| \geq \delta \right\} \leq \varepsilon$$

for all  $n > N$ . More concisely, we can write  $\sup_{\theta \in \Theta} |S_n(\theta) - S(\theta)| \xrightarrow{P} 0$  as  $n \rightarrow \infty$ .

Notice that uniform consistency is stronger than **pointwise consistency**:  $S_n(\theta) \xrightarrow{P} S(\theta)$  for every  $\theta \in \Theta$ .

*Remark 1.1.* Recall the definition of pointwise continuity and uniform continuity in undergraduate calculus. A function  $f(x)$  is **pointwisely continuous** in an open set  $A$ , if given any  $\varepsilon > 0$ , for every  $x \in A$  there exists a  $\delta = \delta(\varepsilon, x) > 0$  such that

$$|x' - x| \leq \delta \Rightarrow |f(x') - f(x)| < \varepsilon.$$

The function  $f(x)$  is **uniformly continuous** if  $\delta$  does not depend on  $x$ . In other words, there exists a finite bound  $L$  such that

$$\sup_{x, x' \in A, x \neq x'} \frac{|f(x') - f(x)|}{\|x - x'\|} \leq L$$

For example,  $\sin(x)$  is uniform continuous in  $\mathbb{R}$ ;  $1/x$  in  $(0, \infty)$  is pointwisely continuous but not uniformly continuous, and the same applies to  $x^2$  in  $(-\infty, \infty)$ .

**Theorem 1.1.** *If (i) ULLN:  $\sup_{\theta \in \Theta} |S_n(\theta) - S(\theta)| \xrightarrow{p} 0$ ; (ii)  $\theta_0$  is identified, then  $\hat{\theta} \xrightarrow{p} \theta_0$  as  $n \rightarrow \infty$ .*

*Proof.* We start from the condition of identification.

$$\begin{aligned} \Pr(|\hat{\theta} - \theta| > \delta) &\leq \Pr(S(\hat{\theta}) - S(\theta_0) > \varepsilon) \\ &= \Pr(S(\hat{\theta}) - S_n(\hat{\theta}) + S_n(\hat{\theta}) - S_n(\theta_0) + S_n(\theta_0) - S(\theta_0) > \varepsilon) \\ &\leq \Pr(S(\hat{\theta}) - S_n(\hat{\theta}) + S_n(\theta_0) - S(\theta_0) > \varepsilon) \\ &\leq \Pr(|S_n(\hat{\theta}) - S(\hat{\theta})| + |S(\theta_0) - S_n(\theta_0)| > \varepsilon) \\ &\leq \Pr\left(\sup_{\theta \in \Theta} |S_n(\theta) - S(\theta)| \geq \frac{\varepsilon}{2}\right) \rightarrow 0 \end{aligned}$$

where the second inequality follows from the definition of the m-estimator that  $S_n(\hat{\theta}) \leq S_n(\theta_0)$ .  $\square$

We have established consistent of the m-estimator  $\hat{\theta}$  to the true parameter  $\theta_0$ .

### 1.3 Asymptotic Normality

To further characterize the uncertainty of the estimator, we seek its asymptotic distribution.

We go with a heuristic argument. Define  $\psi_i(\theta) = \frac{\partial}{\partial \theta} \rho_i(\theta)$  and  $\bar{\psi}(\theta) = \frac{\partial}{\partial \theta} S_n(\theta)$ . A Taylor expansion of  $\bar{\psi}(\hat{\theta})$  around  $\theta_0$  gives

$$0 = \bar{\psi}(\hat{\theta}) = \bar{\psi}(\theta_0) + \frac{\partial^2}{\partial \theta \partial \theta'} S_n(\dot{\theta}) (\hat{\theta} - \theta_0)$$

where  $\dot{\theta}$  lies in between  $\hat{\theta}$  and  $\theta_0$ . If  $\frac{\partial^2}{\partial \theta \partial \theta'} S_n(\dot{\theta})$  is invertible, we can rearrange the above inequality,

$$\sqrt{n}(\hat{\theta} - \theta_0) = - \left[ \frac{\partial^2}{\partial \theta \partial \theta'} S_n(\dot{\theta}) \right]^{-1} \sqrt{n} \bar{\psi}(\theta_0).$$

Since  $\hat{\theta} \xrightarrow{p} \theta_0$ , we also have  $\dot{\theta} \xrightarrow{p} \theta_0$ . By the continuous mapping theorem:

$$\frac{\partial^2}{\partial \theta \partial \theta'} S(\dot{\theta}) \xrightarrow{p} \frac{\partial^2}{\partial \theta \partial \theta'} S(\theta_0) = Q$$

if  $\frac{\partial^2}{\partial \theta \partial \theta'} S(\cdot)$  is continuous. (Ultimately we want to show  $\frac{\partial^2}{\partial \theta \partial \theta'} S_n(\dot{\theta}) \xrightarrow{p} Q$  but our heuristic argument here has a gap, because  $\dot{\theta}$  is moving as  $n \rightarrow \infty$ . A rigorous proof needs to invoke the empirical process theory, which we do not pursue here.) In the population,  $E[\bar{\psi}(\theta_0)] = E[\psi(\theta_0)] = 0$ , and

$$\sqrt{n} \bar{\psi}(\theta_0) \xrightarrow{d} N(0, \Omega)$$

where  $\Omega = E[\psi_i(\theta_0)\psi_i'(\theta_0)]$ . As a result,

$$\sqrt{n}(\hat{\theta} - \theta_0) \xrightarrow{d} N(0, Q^{-1}\Omega Q^{-1})$$

where the asymptotic variance follows a sandwich form.

**Example 1.4.** We use the OLS estimator or demonstration. Its  $\psi_i(\theta) = -x_i(y_i - x_i'\theta)$  and  $\frac{\partial^2}{\partial\theta\partial\theta'}\rho_i(\theta) = x_i x_i'$ . Evaluated at  $\theta = \theta_0$ , we have

$$\psi_i(\theta_0) = -x_i(y_i - x_i'\theta_0) = -x_i e_i,$$

and obviously the assumption that  $x_i$  and  $e_i$  are orthogonal implies  $E[\psi_i(\theta_0)] = 0$ . When  $E[x_i x_i' e_i^2] < \infty$  is finite, the Lindeberg-Levy CLT gives

$$\sqrt{n}\bar{\psi}(\theta_0) \xrightarrow{d} N(0, E[x_i x_i' e_i^2]).$$

Moreover  $\Omega = E[x_i x_i']$ . Therefore we conclude

$$\sqrt{n}(\hat{\theta} - \theta_0) \xrightarrow{d} N\left(0, (E[x_i x_i'])^{-1} E[x_i x_i' e_i^2] (E[x_i x_i'])^{-1}\right).$$

This is the asymptotic distribution of the OLS estimator under conditional heteroskedasticity.

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# Bibliography

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