Chapter 1

Nonstationary Times Series - Exercises

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Exercise 16.3 Suppose $Y_t = X_t + u_t$ where $X_t = X_{t-1} + e_t$ with $(e_t, u_t) \sim I(0)$.

(a) Is Y_t $I(0)$ or $I(1)$?.

(b) Find the asymptotic functional distribution of $n^{-1/2}Y_{\lfloor nr \rfloor}$.

Solution. 16.3

(a)

$$\begin{aligned} Y_t &= X_t + u_t \\ &= X_{t-1} + e_t + u_t \\ &= X_{t-2} + e_{t-1} + e_t + u_t \\ &= ... \\ &= X_0 + \sum_{i=1}^{t} e_i + u_t \end{aligned}$$

$$\begin{aligned} \Delta Y_t &= Y_t - Y_{t-1} \\ &= X_t - X_{t-1} + u_t - u_{t-1} \\ &= e_t + \Delta u_t \end{aligned}$$

Since $(e_t, u_t) \sim I(0)$, ΔY_t is stationary with positive long-run variance. Thus, Y_t is $I(1)$.

(b)

$$\begin{aligned} Y_t &= \sum_{i=1}^{t} e_i + u_t + X_0 \\ &= S_t + u_t + X_0 \end{aligned}$$

Because ΔY_t is stationary and $E[\Delta Y_t] = 0$,

$$\begin{aligned} n^{-1/2}Y_{\lfloor nr \rfloor} &= S_n(r) + n^{-1/2}u_{\lfloor nr \rfloor} + n^{-1/2}X_0 \\ &= S_n(r) + o_p(1) \\ &\xrightarrow{d} B(r) \end{aligned}$$

with covariance matrix Ω which is the long-run variance of ΔY_t.

Exercise 16.4 Let $Y_t = e_t$ be i.i.d. and $X_t = \Delta Y_t$.

(a) Show that Y_t is stationary and $I(0)$.

(b) Show that X_t is stationary but not $I(0)$.

Solution. 16.4

(a) Since e_t is i.i.d. with finite variance, $Y_t = e_t$ is stationary with constant mean and variance

$$\gamma_Y(\ell) = Cov(Y_t, Y_{t-\ell}) = Cov(e_t, e_{t-\ell}) = \begin{cases} var(e_t) & \text{if } \ell = 0 \\ 0 & \text{if } \ell \neq 0 \end{cases}$$

and thus its long-run variance $\sum_{\ell=-\infty}^{\infty} \gamma_Y(\ell) > 0$ is positive. As a result, Y_t is $I(0)$.

(b) $X_t = \Delta Y_t$ is stationary because Y_t is stationary. Its variance is

$$\begin{aligned} \gamma_X(\ell) &= Cov(X_t, X_{t-\ell}) = Cov(\Delta Y_t, \Delta Y_{t-\ell}) \\ &= Cov(Y_t - Y_{t-1}, Y_{t-\ell} - Y_{t-\ell-1}) \\ &= 2\gamma_Y(\ell) - \gamma_Y(\ell+1) - \gamma_Y(\ell-1) \\ &= \begin{cases} -var(e_t) & \text{if } \ell = -1 \\ 2var(e_t) & \text{if } \ell = 0 \\ -var(e_t) & \text{if } \ell = 1 \\ 0 & \text{if } \ell \neq -1, 0, 1 \end{cases} \end{aligned}$$

Because its long-run variance $\sum_{\ell=-\infty}^{\infty} \gamma_X(\ell) = 0$, we conclude that X_t is not $I(0)$.

Exercise 16.6 Take the AR(1) model $Y_t = \alpha Y_{t-1} + e_t$ with i.i.d. e_t and the least squares estimator $\hat{\alpha}$. In Chaper 14 we learned that the asymptotic distribution when $|\alpha| < 1$ is $\sqrt{n}(\hat{\alpha} - \alpha) \xrightarrow{d} N(0, 1-\alpha^2)$. How do you reconcile this with Theorem 16.9, especially for α close to one?

Solution. 16.6

(1) When $|\alpha| < 1$,

$$var(Y_t) = var\left(\sum_{i=0}^{\infty} \alpha^i e_{t-i}\right) = \sum_{i=0}^{\infty} (\alpha^i)^2 var(e_{t-i}) = \sigma^2 \sum_{i=0}^{\infty} (\alpha^i)^2 = \frac{\sigma_e^2}{1-\alpha^2}$$

$$\sqrt{n}(\hat{\alpha} - \alpha) = \frac{\frac{1}{\sqrt{n}} \sum Y_{t-1} e_t}{\frac{1}{n} \sum Y_t^2}$$

The numerator $\frac{1}{\sqrt{n}} \sum Y_{t-1} e_t \xrightarrow{d} N(0, var(Y_{t-1} e_t)) = N(0, var(Y_{t-1}) \sigma_e^2)$. The denominator $\frac{1}{n} \sum Y_t^2 \xrightarrow{p} var(Y_t)$. Therefore,

$$\sqrt{n}(\hat{\alpha} - \alpha) = \frac{\frac{1}{\sqrt{n}} \sum Y_{t-1} e_t}{\frac{1}{n} \sum Y_t^2} \sim N\left(0, \frac{\sigma_e^2}{var(Y_t)}\right) = N(0, 1-\alpha^2).$$

(2) The last expression implies that when $\alpha \to 1$,

$$\sqrt{n}\left(\hat{\alpha}-\alpha\right)=\frac{\frac{1}{\sqrt{n}}\sum Y_{t-1}e_t}{\frac{1}{n}\sum Y_t^2}\sim N\left(0,\frac{\sigma_e^2}{var\left(Y_t\right)}\right)\sim N\left(0,0\right)=0$$

In other words, the scaling factor $\sqrt{n}$ is not big enough to blow $\left(\hat{\alpha}-\alpha\right)$ into an non-degenerate random variable. Actually, then $\alpha = 1$, the proper scaling factor should be n, as we will see in the next question.

(3) When $\alpha = 1$,

$$n\left(\hat{\alpha}-\alpha\right)=\frac{\frac{1}{n}\sum Y_{t-1}e_t}{\frac{1}{n^2}\sum Y_t^2}.$$

The numerator

$$\sum_{t=0}^{n-1}\frac{Y_t}{\sqrt{n}}\frac{Y_{t+1}-Y_t}{\sqrt{n}}=\int S_n\left(r\right)\mathrm{d}S_n\left(r\right)\xrightarrow{d}\int_0^1 B\mathrm{d}B=\sigma_e^2\int W\mathrm{d}W.$$

The denominator

$$\frac{1}{n}\sum_{t=0}^{n-1}\left(\frac{Y_t}{\sqrt{n}}\right)^2=\sum_{t=0}^{n-1}\frac{1}{n}S_n^2\left(r\right)\xrightarrow{d}\int_0^1 B^2=\sigma^2\int_0^1 W^2.$$

Therefore,

$$n\left(\hat{\alpha}-1\right)\xrightarrow{d}\frac{\int_0^1 W\mathrm{d}W}{\int_0^1 W^2}$$