5

Transverse Wave Motion (1)

Introduction

We started this book with simple harmonic oscillators and ended the last chapter by deriving the wave equation. These are the tools which we now use in discussing waves. We have seen that the energy of a simple harmonic oscillator can be transferred by coupling to a neighbour and as we increase the number of oscillators we end up with a medium through which a wave propagates. In particular, the oscillators or particles in a medium do not move through the medium but only vibrate about their equilibrium positions so that what we observe as waves is the changing relative displacements of neighbouring oscillators.

We shall show by treating the string as a forced oscillator how it behaves as a medium with an impedance which stores wave energy, how power fed into one end of the string propagates and maintains waves along the string and how the wave energy is distributed along the string.

When the wave meets a boundary between two different impedances some energy is reflected and some is transmitted. We begin by extending our familiarity with partial differentiation using a range of different examples.

5.1 Partial Differentiation

From this chapter onwards we shall often need to use the notation of partial differentiation.

When we are dealing with a function of only one variable, $y = f(x)$ say, we write the differential coefficient

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

but if we consider a function of two or more variables, the value of this function will vary with a change in any or all of the variables. For instance, the value of the coordinate z on the surface of a sphere whose

Introduction to Vibrations and Waves, First Edition. H. J. Pain and Patricia Rankin.

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equation is $x^2 + y^2 + z^2 = a^2$, where a is the radius of the sphere, will depend on x and y so that z is a function of x and y written $z = z(x, y)$. The differential change of z which follows from a change of x and y may be written

$$dz = \left(\frac{\partial z}{\partial x}\right)_y dx + \left(\frac{\partial z}{\partial y}\right)_x dy$$

where $(\partial z/\partial x)_y$ means differentiating z with respect to x whilst y is kept constant, so that

$$\left(\frac{\partial z}{\partial x}\right)_y = \lim_{\delta x \to 0} \frac{z(x + \delta x, y) - z(x, y)}{\delta x}$$

The total change dz is found by adding the separate increments due to the change of each variable in turn whilst the others are kept constant. In Figure 5.1 we can see that keeping y constant isolates a plane which cuts the spherical surface in a curved line, and the incremental contribution to dz along this line is exactly as though z were a function of x only. Now by keeping x constant we turn the plane through 90° and repeat the process with y as a variable so that the total increment of dz is the sum of these two processes.

If only two independent variables are involved, the subscript showing which variable is kept constant is omitted without ambiguity.

In wave motion our functions will be those of variables of distance and time, and we shall write $\partial/\partial x$ and $\partial^2/\partial x^2$ for the first or second derivatives with respect to x, whilst the time t remains constant. Again, $\partial/\partial t$ and $\partial^2/\partial t^2$ will denote first and second derivatives with respect to time, implying that x is kept constant.

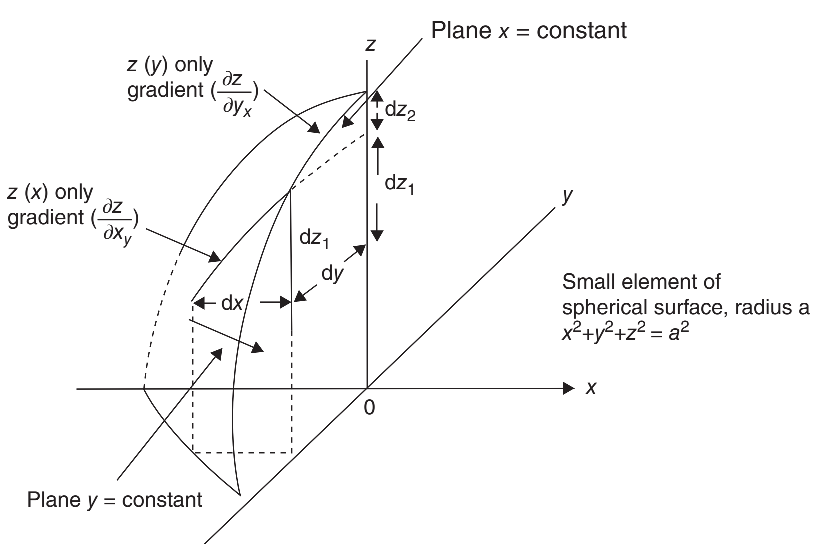

Figure 5.1 *Small element of a spherical surface showing* $dz = dz_1 + dz_2 = (\partial z/\partial x)_y dx + (\partial z/\partial y)_x dy$ *where each gradient is calculated with one variable remaining constant.*

Examples of Partial Differentiation

We now consider the partial differentiation of functions of z where z is itself a function of two variables, e.g. x and y or x and t, that is

$$f(z) \text{ where } z = z(x, y) \text{ or } z = z(x, t).$$

The rate of change of $f(z)$ with x if y remains constant is

$$\left(\frac{\partial \mathrm{f}(z)}{\partial x}\right)_y = \frac{\mathrm{df}(z)}{\mathrm{d}z}\left(\frac{\partial z}{\partial x}\right)_y,$$

that is the change of $f(z)$ with z times the change of z with x with y constant.

Similarly

$$\left(\frac{\partial \mathrm{f}(z)}{\partial y}\right)_x = \frac{\mathrm{df}(z)}{\mathrm{d}z}\left(\frac{\partial z}{\partial y}\right)_x$$

(a) $f(z) = z$ where $z = (3x - 2y)$

$$\left(\frac{\partial \mathrm{f}(z)}{\partial x}\right)_y = \frac{\mathrm{df}(z)}{\mathrm{d}z}\left(\frac{\partial z}{\partial x}\right)_y = 1 \cdot 3 = 3$$
$$\left(\frac{\partial \mathrm{f}(z)}{\partial y}\right)_x = \frac{\mathrm{df}(z)}{\mathrm{d}z}\left(\frac{\partial z}{\partial y}\right)_x = 1 \cdot -2 = -2$$

(b) $f(z) = z^2$ where $z = (3x - 2y)$

$$\left(\frac{\partial \mathrm{f}(z)}{\partial x}\right)_y = \frac{\mathrm{df}(z)}{\mathrm{d}z}\left(\frac{\partial z}{\partial x}\right)_y = 2z \cdot 3 = 6(3x - 2y)$$
$$\left(\frac{\partial \mathrm{f}(z)}{\partial y}\right)_x = \frac{\mathrm{df}(z)}{\mathrm{d}z}\left(\frac{\partial z}{\partial y}\right)_x = 2z \cdot -2 = -4(3x - 2y)$$

(c) $f(z) = e^z$ where $z = x + \mathrm{i}y$ so $\mathrm{e}^z = \mathrm{e}^{x+\mathrm{i}y}$

$$\left(\frac{\partial \mathrm{f}(z)}{\partial x}\right)_y = \frac{\mathrm{df}(z)}{\mathrm{d}z}\left(\frac{\partial z}{\partial x}\right)_y = \mathrm{e}^z \cdot 1 = \mathrm{e}^z = \mathrm{e}^{x+\mathrm{i}y}$$
$$\left(\frac{\partial \mathrm{f}(z)}{\partial y}\right)_x = \frac{\mathrm{df}(z)}{\mathrm{d}z}\left(\frac{\partial z}{\partial y}\right)_x = \mathrm{e}^z \cdot \mathrm{i} = \mathrm{i}\,\mathrm{e}^z = \mathrm{i}\,\mathrm{e}^{x+\mathrm{i}y}$$

The following function is very important in wave motion

(d) $f(z) = \mathrm{e}^z$ where $z = \mathrm{i}(\omega t - kx)$

$$\left(\frac{\partial \mathrm{f}(z)}{\partial t}\right)_x = \frac{\mathrm{df}(z)}{\mathrm{d}z}\left(\frac{\partial z}{\partial t}\right)_x = \mathrm{i}\,\omega \mathrm{e}^z = \mathrm{i}\,\omega \mathrm{e}^{(\omega t - kx)}$$

$$\left(\frac{\partial \mathrm{f}(z)}{\partial x}\right)_t = \frac{\mathrm{df}(z)}{\mathrm{d}z}\left(\frac{\partial z}{\partial x}\right)_t = -\mathrm{i}\,k\mathrm{e}^z = -\mathrm{i}\,k\mathrm{e}^{(\omega t - kx)}$$

5.2 Waves

One of the simplest ways to demonstrate wave motion is to take the loose end of a long rope which is fixed at the other end and to move the loose end quickly up and down. Crests and troughs of the waves move down the rope, and if the rope were infinitely long such waves would be called progressive waves – these are waves travelling in an unbounded medium free from possible reflection (Figure 5.2).

If the medium is limited in extent, for example, if the rope were reduced to a violin string, fixed at both ends, the progressive waves travelling on the string would be reflected at both ends; the vibration of the string would then be the combination of such waves moving to and fro along the string and standing waves would be formed.

Waves on strings are transverse waves where the displacements or oscillations in the medium are transverse to the direction of wave propagation. When the oscillations are parallel to the direction of wave propagation the waves are longitudinal. Sound waves are longitudinal waves; a gas can sustain only longitudinal waves because transverse waves require a shear force to maintain them. Both transverse and longitudinal waves can travel in a solid.

In this book we are going to discuss plane waves only. When we see wave motion as a series of crests and troughs we are in fact observing the vibrational motion of the individual oscillators in the medium, and in particular all of those oscillators in a plane of the medium which, at the instant of observation, have the same phase in their vibrations. When all the vibrations are restricted to one plane the wave is said to be *plane polarized*.

If we take a plane perpendicular to the direction of wave propagation and all oscillators lying within that plane have a common phase, we shall observe with time how that plane of common phase progresses through the medium. Over such a plane, all parameters describing the wave motion remain constant. The crests and troughs are planes of maximum amplitude of oscillation which are π rad out of phase; a crest is a plane of maximum positive amplitude, while a trough is a plane of maximum negative amplitude. In formulating such wave motion in mathematical terms we shall have to relate the phase difference between any two planes to their physical separation in space. We have, in principle, already done this in our discussion on oscillators.

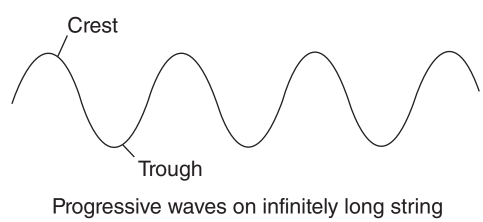

Figure 5.2 Progressive transverse waves moving along a string.

Spherical waves are waves in which the surfaces of common phase are spheres and the source of waves is a central point, e.g. an explosion; each spherical surface defines a set of oscillators over which the radiating disturbance has imposed a common phase in vibration. In practice, spherical waves become plane waves after travelling a very short distance. A small section of a spherical surface is a very close approximation to a plane.

5.3 Velocities in Wave Motion

At the outset we must be very clear about one point. The individual oscillators which make up the medium do not progress through the medium with the waves. Their motion is simple harmonic, limited to oscillations, transverse or longitudinal, about their equilibrium positions. It is their phase relationships we observe as waves, not their progressive motion through the medium.

There are three velocities in wave motion which are quite distinct although they are connected mathematically. They are

(1) The particle velocity, which is the simple harmonic velocity of the oscillator about its equilibrium position.
(2) The wave or phase velocity, the velocity with which planes of equal phase, crests or troughs, progress through the medium.
(3) The group velocity. A number of waves of different frequencies, wavelengths and velocities may be superposed to form a group. Waves rarely occur as single monochromatic components; a white light pulse consists of an infinitely fine spectrum of frequencies and the motion of such a pulse would be described by its group velocity. Such a group would, of course, 'disperse' with time because the wave velocity of each component would be different in all media except free space. Only in free space would it remain as white light. We shall discuss group velocity as a separate topic in Chapter 6. Its importance is that it is the velocity with which the energy in the wave group is transmitted. For a monochromatic wave the group velocity and the wave velocity are identical. Here we shall concentrate on particle and wave velocities.

5.4 The Wave Equation

This equation will dominate the rest of this text and we shall derive it, first of all, by considering the motion of transverse waves on a string.

We shall consider the vertical displacement y of a very short section of a uniform string. This section will perform vertical simple harmonic motions; it is our simple oscillator. The displacement y will, of course, vary with the time and also with x, the position along the string at which we choose to observe the oscillation.

The wave equation therefore will relate the displacement y of a single oscillator to distance x and time t. We shall consider oscillations only in the plane of the paper, so that our transverse waves on the string are *plane polarized*.

The mass of the uniform string per unit length or its linear density is ρ, and a constant tension T exists throughout the string although it is slightly extensible.

This requires us to consider such a short length and such small oscillations that we may linearize our equations. The effect of gravity is neglected.

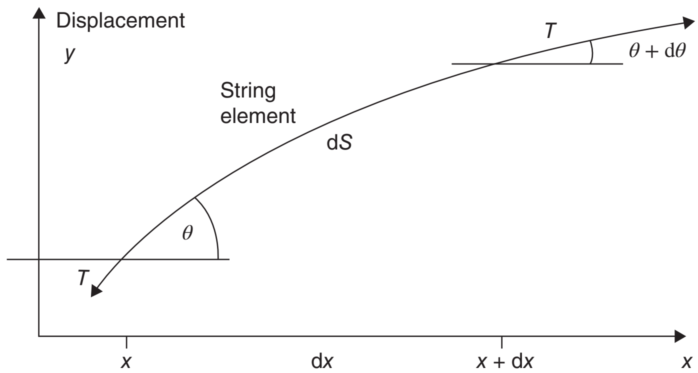

Figure 5.3 Displaced element of string of length $\mathrm{d}s \approx \mathrm{d}x$ with tension *T* acting at an angle θ at *x* and at $\theta + \mathrm{d}\theta$ at $x + \mathrm{d}x$.

Thus in Figure 5.3 the forces acting on the curved element of length $\mathrm{d}s$ are T at an angle θ to the axis at one end of the element, and T at an angle $\theta + \mathrm{d}\theta$ at the other end. The length of the curved element is

$$ds = \left[1 + \left(\frac{\partial y}{\partial x}\right)^2\right]^{1/2} \mathrm{d}x$$

but within the limitations imposed $\partial y/\partial x$ is so small that we ignore its square and take $\mathrm{d}s = \mathrm{d}x$. The mass of the element of string is therefore $\rho \mathrm{d}s = \rho \mathrm{d}x$. Its equation of motion is found from Newton's Law, force equals mass times acceleration.

The perpendicular force on the element dx is $T \sin(\theta + \mathrm{d}\theta) - T \sin\theta$ in the positive y direction, which equals the product of $\rho \mathrm{d}x$ (mass) and $\partial^2 y/\partial t^2$ (acceleration).

Since θ is very small $\sin\theta \approx \tan\theta = \partial y/\partial x$, so that the force is given by

$$T\left[\left(\frac{\partial y}{\partial x}\right)_{x+\mathrm{d}x} - \left(\frac{\partial y}{\partial x}\right)_x\right]$$

where the subscripts refer to the point at which the partial derivative is evaluated. The difference between the two terms in the bracket defines the differential coefficient of the partial derivative $\partial y/\partial x$ times the space interval $\mathrm{d}x$, so that the force is

$$T\frac{\partial^2 y}{\partial x^2}\mathrm{d}x$$

The equation of motion of the small element $\mathrm{d}x$ then becomes

$$T\frac{\partial^2 y}{\partial x^2}\mathrm{d}x = \rho \mathrm{d}x \frac{\partial^2 y}{\partial t^2}$$

or

$$\frac{\partial^2 y}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 y}{\partial t^2}$$

giving

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

where T/ρ has the dimensions of a velocity squared, so c in the preceding equation is a velocity. **This is the wave equation**.

It relates the acceleration of a simple harmonic oscillator in a medium to the second derivative of its displacement with respect to its position, x, in the medium. The position of the term c^2 in the equation is always shown by a rapid dimensional analysis.

So far we have not explicitly stated which velocity c represents. We shall see that it is the wave or phase velocity, the velocity with which planes of common phase are propagated. In the string the velocity arises as the ratio of the tension to the inertial density of the string. We shall see, whatever the waves, that the wave velocity can always be expressed as a function of the elasticity or potential energy storing mechanism in the medium and the inertia of the medium through which its kinetic or inductive energy is stored. For longitudinal waves in a solid the elasticity is measured by Young's modulus, in a gas by γP, where γ is the specific heat ratio and P is the gas pressure.

5.5 Solution of the Wave Equation

The solution of the wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

will, of course, be a function of the variables x and t. We are going to show that any function of the form $y = f_1(ct - x)$ is a solution. Moreover, any function $y = f_2(ct + x)$ will be a solution so that, generally, their superposition $y = f_1(ct - x) + f_2(ct + x)$ is the complete solution.

If f_1' represents the differentiation of the function with respect to the bracket $(ct - x)$, then using the chain rule which also applies to partial differentiation

$$\frac{\partial y}{\partial x} = -f_1'(ct - x)$$

and

$$\frac{\partial^2 y}{\partial x^2} = f_1''(ct - x)$$

also

$$\frac{\partial y}{\partial t} = cf_1'(ct - x)$$

and

$$\frac{\partial^2 y}{\partial t^2} = c^2 f_1''(ct - x)$$

so that

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

for $y = f_1(ct - x)$. When $y = f_2(ct + x)$ a similar result holds.

Worked Example

$$y = f_2(ct + x)$$
$$\frac{\partial y}{\partial x} = f_2'(ct + x) \quad \frac{\partial^2 y}{\partial x^2} = f_2''(ct + x)$$
$$\frac{\partial y}{\partial t} = cf_2'(ct + x) \quad \frac{\partial^2 y}{\partial t^2} = c^2 f_2''(ct + x)$$
$$\therefore \frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

If y is the simple harmonic displacement of an oscillator at position x and time t we would expect, from Chapter 1, to be able to express it in the form $y = a\sin(\omega t - \phi)$, and in fact all of the waves we discuss in this book will be described by sine or cosine functions.

The bracket $(ct - x)$ in the expression $y = f(ct - x)$ has the dimensions of a length and, for the function to be a sine or cosine, its argument must have the dimensions of radians so that $(ct - x)$ must be multiplied by a factor $2\pi/\lambda$, where λ is a length to be defined.

We can now write

$$y = a\sin(\omega t - \phi) = a\sin\frac{2\pi}{\lambda}(ct - x)$$

as a solution to the wave equation if $2\pi c/\lambda = \omega = 2\pi\nu$, where ν is the oscillation frequency and $\phi = 2\pi x/\lambda$.

This means that if a wave, moving to the right, passes over the oscillators in a medium and a photograph is taken at time $t = 0$, the locus of the oscillator displacements (Figure 5.4) will be given by the expression

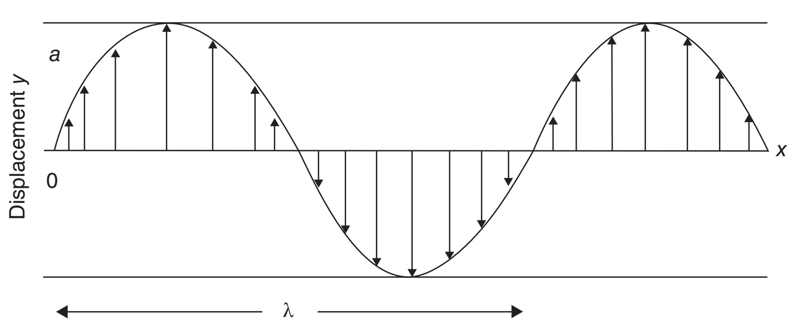

Figure 5.4 Locus of oscillator displacements in a continuous medium as a wave passes over them travelling in the positive x direction. The wavelength λ is defined as the distance between any two oscillators having a phase difference of 2π rad.

$y = a\sin(\omega t - \phi) = a\sin 2\pi(ct - x)/\lambda$. If we now observe the motion of the oscillator at the position $x = 0$ it will be given by $y = a\sin\omega t$.

Any oscillator to its right at some position x will be set in motion at some later time by the wave moving to the right; this motion will be given by

$$y = a\sin(\omega t - \phi) = a\sin\frac{2\pi}{\lambda}(ct - x)$$

having a phase lag of ϕ with respect to the oscillator at $x = 0$. This phase lag $\phi = 2\pi x/\lambda$, so that if $x = \lambda$ the phase lag is 2π rad that is, equivalent to exactly one complete vibration of an oscillator.

This defines λ as the wavelength, the separation in space between any two oscillators with a phase difference of 2π rad. The expression $2\pi c/\lambda = \omega = 2\pi\nu$ gives $c = \nu\lambda$, where c, the wave or phase velocity, is the product of the frequency and the wavelength. Thus, $\lambda/c = 1/\nu = \tau$, the period of oscillation, showing that the wave travels one wavelength in this time. An observer at any point would be passed by ν wavelengths per second, a distance per unit time equal to the velocity c of the wave.

If the wave is moving to the left the sign of ϕ is changed because the oscillation at x begins before that at $x = 0$. Thus, the bracket

$$(ct - x) \text{ denotes a wave moving to the right}$$

and

$$(ct + x) \text{ gives a wave moving in the direction of negative } x.$$

There are several equivalent expressions for $y = f(ct - x)$ which we list here as sine functions, although cosine functions are equally valid.

They are:

$$\begin{aligned}
y &= a\sin\frac{2\pi}{\lambda}(ct - x)\\
y &= a\sin 2\pi\left(\nu t - \frac{x}{\lambda}\right)\\
y &= a\sin\omega\left(t - \frac{x}{c}\right)\\
y &= a\sin(\omega t - kx)
\end{aligned}$$

where $k = 2\pi/\lambda = \omega/c$ is called the wave number; also $y = a\mathrm{e}^{\mathrm{i}(\omega t - kx)}$, the exponential representation of both sine and cosine.

Each of the expressions above is a solution to the wave equation giving the displacement of an oscillator and its phase with respect to some reference oscillator. The changes of the displacements of the oscillators and the propagation of their phases are what we observe as wave motion.

The wave or phase velocity is, of course, $\partial x/\partial t$, the rate at which the disturbance moves across the oscillators; the oscillator or particle velocity is the simple harmonic velocity $\partial y/\partial t$.

Choosing any one of the expressions above for a right-going wave, e.g.

$$y = a\sin(\omega t - kx)$$

we have

$$\frac{\partial y}{\partial t} = \omega a \cos(\omega t - kx)$$

and

$$\frac{\partial y}{\partial x} = -ka \cos(\omega t - kx)$$

so that

$$\frac{\partial y}{\partial t} = -\frac{\omega}{k}\frac{\partial y}{\partial x} = -c\frac{\partial y}{\partial x}\left(= -\frac{\partial x}{\partial t}\frac{\partial y}{\partial x}\right)$$

The particle velocity $\partial y/\partial t$ is therefore given as the product of the wave velocity

$$c = \frac{\partial x}{\partial t}$$

and the gradient of the wave profile preceded by a negative sign for a right-going wave

$$y = f(ct - x)$$

In Figure 5.5 the arrows show the direction of the particle velocity at various points of the right-going wave. It is evident that the particle velocity increases in the same direction as the transverse force in the wave and we shall see in the next section that this force is given by

$$-T\partial y/\partial x$$

where T is the tension in the string.

Worked Example

Show that, for a left-going wave

$$\frac{\partial y}{\partial t} = c\frac{\partial y}{\partial x}$$

Solution

A left-going wave is

$$y = a \sin(\omega t + kx)$$

with

$$\frac{\partial y}{\partial x} = +ka \cos(\omega t + kx)$$

and

$$\frac{\partial y}{\partial t} = +\omega a \cos(\omega t + kx)$$

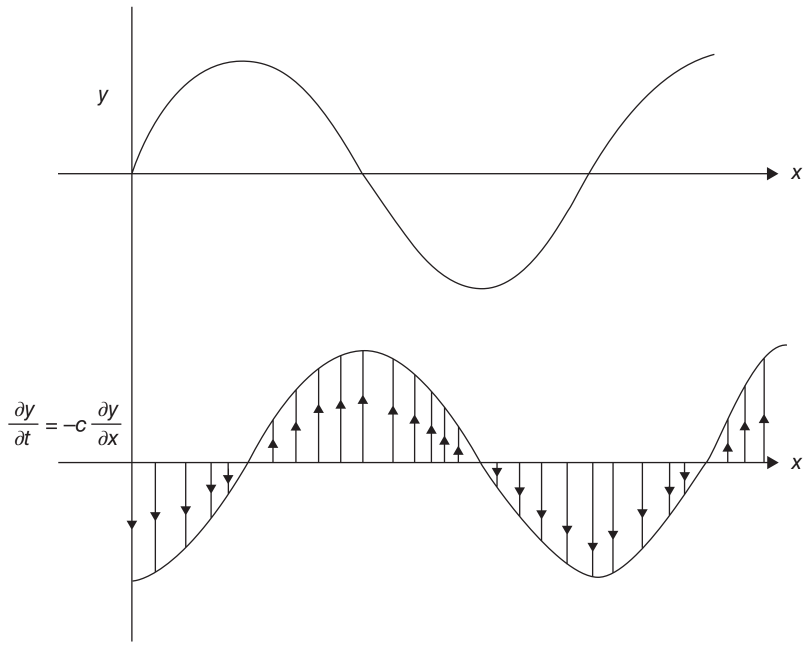

Figure 5.5 *The magnitude and direction of the particle velocity $\partial y/\partial t = -c(\partial y/\partial x)$ at any point x is shown by an arrow in the right-going sine wave above.*

so

$$\frac{\partial y}{\partial t} = \frac{\omega}{k}\frac{\partial y}{\partial x} = c\frac{\partial y}{\partial x}$$

c is a magnitude with no sign.

5.6 Characteristic Impedance of a String (the String as a Forced Oscillator)

Any medium through which waves propagate will present an impedance to those waves. If the medium is lossless, and possesses no resistive or dissipation mechanism, this impedance will be determined by the two energy storing parameters, inertia and elasticity, and it will be real. The presence of a loss mechanism will introduce a complex term into the impedance.

A string presents such an impedance to progressive waves and this is defined, because of the nature of the waves, as the transverse impedance

$$Z = \frac{\text{transverse force}}{\text{transverse velocity}} = \frac{F}{v}$$

The following analysis will emphasize the dual role of the string as a medium and as a forced oscillator.

In Figure 5.6 we consider progressive waves on the string which are generated at one end by an oscillating force, $F_0e^{i\omega t}$, which is restricted to the direction transverse to the string and operates only in the

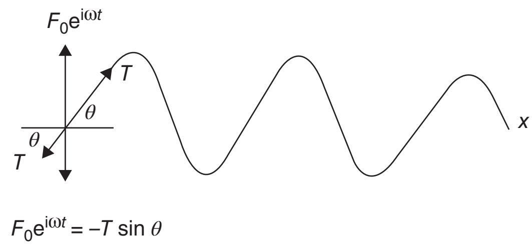

Figure 5.6 *The string as a forced oscillator with a vertical force $F_0e^{i\omega t}$ driving it at one end.*

plane of the paper. The tension in the string has a constant value, T, and at the end of the string the balance of forces shows that the applied force is equal and opposite to $T\sin\theta$ at all time, so that

$$F_0e^{i\omega t} = -T\sin\theta \approx -T\tan\theta = -T\left(\frac{\partial y}{\partial x}\right)$$

where θ is small.

The displacement of the progressive waves may be represented exponentially by

$$\boldsymbol{y} = \boldsymbol{A}e^{i(\omega t - kx)}$$

where the amplitude A may be complex because of its phase relation with F. At the end of the string, where $x = 0$,

$$F_0e^{i\omega t} = -T\left(\frac{\partial y}{\partial x}\right)_{x=0} = ikT\boldsymbol{A}e^{i(\omega t - k\cdot 0)}$$

giving

$$\boldsymbol{A} = \frac{F_0}{ikT} = \frac{F_0}{i\omega}\left(\frac{c}{T}\right)$$

and

$$\boldsymbol{y} = \frac{F_0}{i\omega}\left(\frac{c}{T}\right)e^{i(\omega t - kx)}$$

(since $c = \omega/k$).

The transverse velocity

$$\boldsymbol{v} = \dot{\boldsymbol{y}} = F_0\left(\frac{c}{T}\right)e^{i(\omega t - kx)}$$

where the velocity amplitude $v = F_0/Z$, gives a transverse impedance

$$Z = \frac{T}{c} = \rho c(\text{since } T = \rho c^2)$$

or characteristic impedance of the string.

Since the velocity c is determined by the inertia and the elasticity, the impedance is also governed by these properties.

(We can see that the amplitude of displacement $y = F_0/\omega Z$, with the phase relationship $-\mathrm{i}$ with respect to the force, is in complete accord with our discussion in Chapter 3.)

Rate of Wave Energy Transmission along the String

In moving the end of the string vertically up and down to sustain the wave motion along the string, the power, that is the *work rate* by the force is $F_0\,\mathrm{e}^{\mathrm{i}\omega t}v$ where v is the transverse simple harmonic velocity $\partial y/\partial t$, so

$$F_0\,\mathrm{e}^{\mathrm{i}\omega t}v = -T\frac{\partial y}{\partial x}\frac{\partial y}{\partial t}$$

From the worked example at the end of section 5.5, for a right-going wave we have

$$\frac{\partial y}{\partial t} = -c\frac{\partial y}{\partial x}$$

so the rate of working =

$$-T\frac{\partial y}{\partial x}\frac{\partial y}{\partial t} = -\rho c^2\frac{\partial y}{\partial x}\frac{\partial y}{\partial t} = \rho c\left(\frac{\partial y}{\partial t}\right)^2$$

where c is the phase velocity of the wave.

But $\rho\mathrm{d}x(\partial y/\partial t)^2_{\max}$ is the total harmonic energy of an elemental length $\mathrm{d}x$ of the oscillating string so

$$F_0\,\mathrm{e}^{\mathrm{i}\omega t}v = \rho c\left(\frac{\partial y}{\partial t}\right)^2_{\max}$$

equals the amount of wave energy travelling down the string per second which is stored and maintained in the string via its impedance as a medium.

Distribution of Wave Energy along a Vibrating String

A vibrating string possesses both kinetic and potential energy. The kinetic energy of an element of length $\mathrm{d}x$ and linear density ρ is given by

$$E_{\mathrm{kin}} = \frac{1}{2}\rho\mathrm{d}x\left(\frac{\partial y}{\partial t}\right)^2.$$

The potential energy is the work done by the tension T in extending an element $\mathrm{d}x$ to a new length $\mathrm{d}s$ when the string is vibrating. Thus

$$E_{\mathrm{pot}} = T(ds - dx) = T\left\{\left[1 + \left(\frac{\partial y}{\partial x}\right)^2\right]^{\frac{1}{2}} - 1\right\}\mathrm{d}x = \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2\mathrm{d}x$$

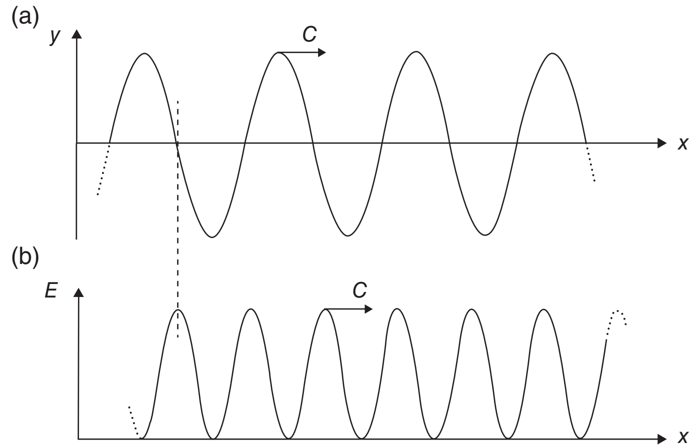

Figure 5.7 Distribution of total energy E in a wave (b) versus wavelengths (a). The wave velocity is c. The peaks of E coincide with the wave amplitude zeros and the zeros of E coincide with the crests and troughs of the waves.

Provided $\left(\frac{\partial y}{\partial x}\right)$ in the wave is of the first order of small quantities, the change in T is of the second order and T may be considered constant. But

$$\frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2 \mathrm{d}x = \frac{1}{2}\rho c^2\left(\frac{\partial y}{\partial x}\right)^2 \mathrm{d}x = \frac{1}{2}\rho \mathrm{d}x\left(\frac{\partial y}{\partial t}\right)^2$$

so the instantaneous values of the kinetic and potential energies in the wave are equal at all points.

In particular their maximum values occur at $x = 0$ where $\partial y/\partial t$ is a maximum and $\partial y/\partial x$ has a maximum and minimum value of ± 1.

Note that both $\partial y/\partial t$ and $\partial y/\partial x$ are zero at the crests and troughs of the waves.

Figure 5.7 shows the total wave energy distribution along wavelengths of the string.

Treating the string as a forced oscillator has allowed us to demonstrate (a) its function as a medium with an impedance capable of storing wave energy, (b) the rate at which the wave energy propagates in the medium and (c) the distribution of that energy within the medium.

5.7 Reflection and Transmission of Waves on a String at a Boundary

We have seen that a string presents a characteristic impedance ρc to waves travelling along it, and we ask how the waves will respond to a sudden change of impedance; that is, of the value ρc. We shall ask this question of all the waves we discuss, acoustic waves, voltage and current waves and electromagnetic waves, and we shall find a remarkably consistent pattern in their behaviour.

We suppose that a string consists of two sections smoothly joined at a point $x = 0$ with a constant tension T along the whole string. The two sections have different linear densities ρ_1 and ρ_2, and therefore different wave velocities $T/\rho_1 = c_1^2$ and $T/\rho_2 = c_2^2$. The specific impedances are $\rho_1 c_1$ and $\rho_2 c_2$, respectively.

An incident wave travelling along the string meets the discontinuity in impedance at the position $x = 0$ in Figure 5.8. At this position, $x = 0$, a part of the incident wave will be reflected and part of it will be transmitted into the region of impedance $\rho_2 c_2$.

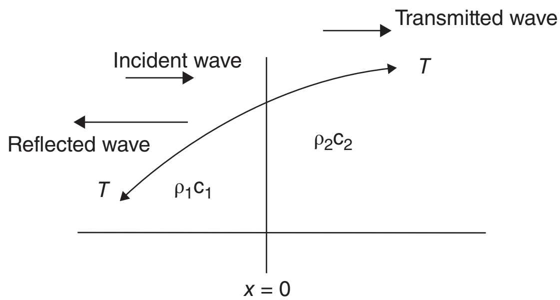

Figure 5.8 *Waves on a string of impedance $\rho_1 c_1$ reflected and transmitted at the boundary $x = 0$ where the string changes to impedance $\rho_2 c_2$.*

We shall denote the impedance $\rho_1 c_1$ by Z_1 and the impedance $\rho_2 c_2$ by Z_2. We write the displacement of the incident wave as $y_i = A_1\,\mathrm{e}^{\mathrm{i}(\omega t - kx)}$, a wave of real (not complex) amplitude A_1 travelling in the positive x direction with velocity c_1. The displacement of the reflected wave is $y_r = B_1\,\mathrm{e}^{\mathrm{i}(\omega t + k_1 x)}$, of amplitude B_1 and travelling in the negative x direction with velocity c_1.

The transmitted wave displacement is given by $y_t = A_2\,\mathrm{e}^{\mathrm{i}(\omega t - k_2 x)}$, of amplitude A_2 and travelling in the positive x direction with velocity c_2.

We wish to find the reflection and transmission amplitude coefficients; that is, the relative values of B_1 and A_2 with respect to A_1. We find these via two boundary conditions which must be satisfied at the impedance discontinuity at $x = 0$.

The boundary conditions which apply at $x = 0$ are:

(1) A geometrical condition that the displacement is the same immediately to the left and right of $x = 0$ for all time, so that there is no discontinuity of displacement.
(2) A dynamical condition that there is a continuity of the transverse force $T(\partial y/\partial x)$ at $x = 0$, and therefore a continuous slope. This must hold, otherwise a finite difference in the force acts on an infinitesimally small mass of the string giving an infinite acceleration; this is not permitted.

Condition (1) at $x = 0$ gives

$$y_i + y_r = y_t$$

or

$$A_1\,\mathrm{e}^{\mathrm{i}(\omega t - k_1 x)} + B_1\,\mathrm{e}^{\mathrm{i}(\omega t + k_1 x)} = A_2\,\mathrm{e}^{\mathrm{i}(\omega t - k_2 x)}$$

At $x = 0$ we may cancel the exponential terms giving

$$A_1 + B_1 = A_2 \tag{5.1}$$

Condition (2) gives

$$T\frac{\partial}{\partial x}(y_i + y_r) = T\frac{\partial}{\partial x}y_t$$

at $x = 0$ for all t, so that

$$-k_1TA_1 + k_1TB_1 = -k_2TA_2 \tag{5.1a}$$

or

$$-\omega\frac{T}{c_1}A_1 + \omega\frac{T}{c_1}B_1 = -\omega\frac{T}{c_2}A_2 \tag{5.1b}$$

after cancelling exponentials at $x = 0$. But $T/c_1 = \rho_1c_1 = Z_1$ and $T/c_2 = \rho_2c_2 = Z_2$, so that

$$Z_1(A_1 - B_1) = Z_2A_2 \tag{5.2}$$

Equations (5.1) and (5.2) give the

$$\textit{Reflection coefficient of amplitude},\ \frac{B_1}{A_1} = \frac{Z_1 - Z_2}{Z_1 + Z_2}$$

and the

$$\textit{Transmission coefficient of amplitude},\ \frac{A_2}{A_1} = \frac{2Z_1}{Z_1 + Z_2}$$

We see immediately that these coefficients are independent of ω and hold for waves of all frequencies; they are real and therefore free from phase changes other than that of π rad which will change the sign of a term. Moreover, these ratios depend entirely upon the ratios of the impedances. (See summary in Appendix 8). If $Z_2 = \infty$, this is equivalent to $x = 0$ being a fixed end to the string because no transmitted wave exists. This gives $B_1/A_1 = -1$, so that the incident wave is completely reflected (as we expect) with a phase change of π (phase reversal) – conditions we shall find to be necessary for standing waves to exist. A group of waves having many component frequencies will retain its shape upon reflection at $Z_2 = \infty$, but will suffer reversal (Figure 5.9). If $Z_2 = 0$, so that $x = 0$ is a free end of the string, then $B_1/A_1 = 1$ and $A_2/A_1 = 2$. This explains the 'flick' at the end of a whip or free ended string when a wave reaches it.

The use of a pulse is a convenient (but artificial) way of showing that the same phase change of all its component frequencies inverts the shape of the pulse. Without energy input this does not happen in practice and the pulse changes shape as it travels.

Replacement of Z by k and $\sqrt{\rho}$ in Transmission and Reflection Coefficients

Particular wave properties may replace the symbol Z where more convenient in a problem, e.g. equations 5.1 and 5.1a may be used to show that the reflection coefficient of amplitude

$$\frac{B_1}{A_1} = \frac{k_1 - k_2}{k_1 + k_2}$$

and the transmission coefficient of amplitude

$$\frac{A_2}{A_1} = \frac{2k_1}{k_1 + k_2}$$

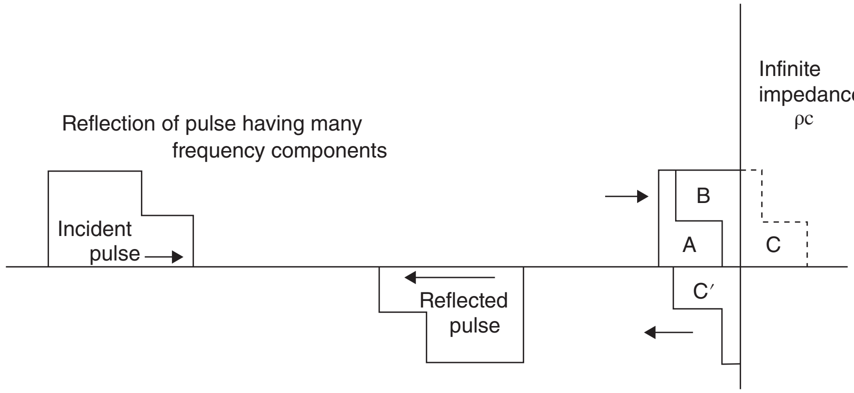

Figure 5.9 *A pulse of arbitrary shape is reflected at an infinite impedance with a phase change of π rad, so that the reflected pulse is the inverted and reversed shape of the initial waveform. The pulse at reflection is divided in the figure into three sections* A, B, *and* C. *At the moment of observation section* C *has already been reflected and suffered inversion and reversal to become* C′. *The actual shape of the pulse observed at this instant is A being* A + B − C′ *where* B = C′. *The displacement at the point of reflection must be zero.*

Moreover on a string

$$T = \rho c^2 = \rho \frac{\omega^2}{k^2}$$

so $k \propto \sqrt{\rho}$ gives

$$\frac{B_1}{A_1} = \frac{\sqrt{\rho_1} - \sqrt{\rho_2}}{\sqrt{\rho_1} + \sqrt{\rho_2}}$$

and

$$\frac{A_2}{A_1} = \frac{2\sqrt{\rho_1}}{\sqrt{\rho_1} + \sqrt{\rho_2}}$$

In electromagnetic waves we shall find

$$Z = \frac{1}{n} = \frac{v}{c} \quad \text{where} \quad n = \frac{c}{v}$$

is the refractive index of the material.

Worked Example

A transverse sinusodial wave of amplitude 3.0 cm and wavelength 25 cm travels along a light string of 1 gram · cm^{-1} mass, which is joined to a heavier string of 4.0 gram · cm^{-1} mass. The joined strings are held under constant tension. (a) What is the wavelength and amplitude of the wave as it travels along the heavier string and (b) what fraction of wave power is reflected at the boundary of the two strings?

Solution

$$\frac{\lambda_2}{\lambda_1} = \sqrt{\frac{\rho_1}{\rho_2}} \quad \therefore\ \lambda_2 = 12.5\,\text{cm}.$$

$$\frac{A_2}{A_1} = \frac{2\sqrt{\rho_1}}{\sqrt{\rho_1}+\sqrt{\rho_2}} \quad \therefore\ A_2 = 2\,\text{cm}.$$

$$\frac{B_1}{A_1} = \frac{\sqrt{\rho_1}-\sqrt{\rho_2}}{\sqrt{\rho_1}+\sqrt{\rho_2}} = -\frac{1}{3} \quad \therefore\ \left(\frac{B_1}{A_1}\right)^2 = \frac{1}{9}.$$

5.8 Reflection and Transmission of Energy

Our interest in waves, however, is chiefly concerned with their function of transferring energy throughout a medium, and we shall now consider what happens to the energy in a wave when it meets a boundary between two media of different impedance values.

If we consider each unit length, mass ρ, of the string as a simple harmonic oscillator of maximum amplitude A, we know that its total energy will be $E = \frac{1}{2}\rho\omega^2A^2$, where ω is the wave frequency.

The wave is travelling at a velocity c so that as each unit length of string takes up its oscillation with the passage of the wave the rate at which energy is being carried along the string is

$$(\text{energy} \times \text{velocity}) = \frac{1}{2}\rho\omega^2A^2c$$

Thus, the rate of energy arriving at the boundary $x = 0$ is the energy arriving with the incident wave; that is

$$\frac{1}{2}\rho_1c_1\omega^2A_1^2 = \frac{1}{2}Z_1\omega^2A_1^2$$

The rate at which energy leaves the boundary, via the reflected and transmitted waves, is

$$\frac{1}{2}\rho_1c_1\omega^2B_1^2 + \frac{1}{2}\rho_2c_2\omega^2A_2^2 = \frac{1}{2}Z_1\omega^2B_1^2 + \frac{1}{2}Z_2\omega^2A_2^2$$

which, from the ratio B_1/A_1 and A_2/A_1,

$$= \frac{1}{2}\omega^2A_1^2\frac{Z_1(Z_1-Z_2)^2 + 4Z_1^2Z_2}{(Z_1+Z_2)^2} = \frac{1}{2}Z_1\omega^2A_1^2$$

Thus, energy is conserved, and all energy arriving at the boundary in the incident wave leaves the boundary in the reflected and transmitted waves.

5.9 The Reflected and Transmitted Intensity Coefficients

These are given by

$$\frac{\text{Reflected Energy}}{\text{Incident Energy}} = \frac{Z_1 B_1^2}{Z_1 A_1^2} = \left(\frac{B_1}{A_1}\right)^2 = \left(\frac{Z_1 - Z_2}{Z_1 + Z_2}\right)^2$$

$$\frac{\text{Transmitted Energy}}{\text{Incident Energy}} = \frac{Z_2 A_2^2}{Z_1 A_1^2} = \frac{4Z_1 Z_2}{(Z_1 + Z_2)^2}$$

We see that if $Z_1 = Z_2$ no energy is reflected and the impedances are said to be matched.

5.10 Matching of Impedances

We have just seen that at the boundary between two unequal impedances wave energy transport will be lost due to reflection and only a fraction of the energy will be transmitted. This happens in all media, on strings, acoustically, in optics, in electrical cables and when light waves enter a dielectric. The solution is common to all media. It is the insertion of a layer of a medium with an impedance equal to the harmonic mean of the unmatched impedances having a thickness of $\lambda/4$ of a wavelength measured in the intermediate impedance. Two unmatched impedances Z_1 and Z_3 are matched when a medium Z_2 is inserted between them, where $Z_2^2 = Z_1 Z_3$ of thickness $\lambda/4$ measured in Z_2.

We shall prove this statement in section 8.9, Matching Impedances, for the very common example of electrical cables.

Worked Example

For an electromagnetic wave travelling in a dielectric the impedance equals $1/n$ where n is the refractive index

$$\frac{c}{v} = \frac{\nu\lambda_0}{\nu\lambda_{\text{dielectric}}}$$

where λ_0 is the wavelength in free space.

To avoid reflection a camera lens ($n = 1.9$) is coated with a $\lambda/4$ thickness of a dielectric with refractive index n_2. Calculate the value of n_2 and the thickness of the layer if the wavelength in air is 550 nm.

Solution

$$n_2^2 = n_1 n_3 \ (n_1 = 1, n_3 = 1.9) \quad \therefore n_2 = 1.38$$

$$\text{Thickness} = \frac{1}{4}\frac{\lambda_{\text{air}}}{n_2} = \frac{550}{4 \times 1.38} = 99\,\text{nm}$$

5.11 Standing Waves on a String of Fixed Length

We have already seen that a progressive wave is completely reflected at an infinite impedance with a π phase change in amplitude. A string of fixed length l with both ends rigidly clamped presents an infinite impedance at each end; we now investigate the behaviour of waves on such a string. Let us consider the simplest case of a monochromatic wave of one frequency ω with an amplitude a travelling in the positive

x direction and an amplitude b travelling in the negative x direction. The displacement on the string at any point would then be given by

$$y = a\,\mathrm{e}^{\mathrm{i}(\omega t - kx)} + b\,\mathrm{e}^{\mathrm{i}(\omega t + kx)}$$

with the boundary condition that $y = 0$ at $x = 0$ and $x = l$ at all times.

The condition $y = 0$ at $x = 0$ gives $0 = (a + b)\,\mathrm{e}^{\mathrm{i}\omega t}$ for all t, so that $a = -b$. This expresses physically the fact that a wave in either direction meeting the infinite impedance at either end is completely reflected with a π phase change in amplitude. This is a general result for all wave shapes and frequencies.

Thus

$$y = a\,\mathrm{e}^{\mathrm{i}\omega t}(\mathrm{e}^{-\mathrm{i}kx} - \mathrm{e}^{\mathrm{i}kx}) = (-2\mathrm{i})a\,\mathrm{e}^{\mathrm{i}\omega t}\sin kx \tag{5.3}$$

an expression for y which satisfies the standing wave time-independent form of the wave equation

$$\partial^2 y/\partial x^2 + k^2 y = 0$$

because $(1/c^2)(\partial^2 y/\partial t^2) = (-\omega^2/c^2)y = -k^2 y$. The condition that $y = 0$ at $x = l$ for all t requires

$$\sin kl = \sin\frac{\omega l}{c} = 0 \quad \text{or} \quad \frac{\omega l}{c} = n\pi$$

limiting the values of allowed frequencies to

$$\omega_n = \frac{n\pi c}{l}$$

or

$$\nu_n = \frac{nc}{2l} = \frac{c}{\lambda_n}$$

that is

$$l = \frac{n\lambda_n}{2}$$

giving

$$\sin\frac{\omega_n x}{c} = \sin\frac{n\pi x}{l}$$

These frequencies are the normal frequencies or modes of vibration we first met in Chapter 4. They are often called eigenfrequencies, particularly in wave mechanics.

Such allowed frequencies define the length of the string as an exact number of half wavelengths, and Figure 5.10 shows the string displacement for the first four harmonics ($n = 1, 2, 3, 4$). The value for $n = 1$ is called the fundamental.

As with the loaded string of Chapter 4, all normal modes may be present at the same time and the general displacement is the superposition of the displacements at each frequency. This is a more complicated problem which we discuss in Chapter 11 (Fourier Methods).

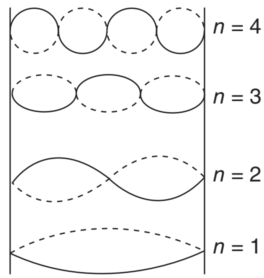

Figure 5.10 *The first four harmonics, n = 1, 2, 3, 4 of the standing waves allowed between the two fixed ends of a string.*

For the moment we see that for each single harmonic $n > 1$ there will be a number of positions along the string which are always at rest. These points occur where

$$\sin\frac{\omega_n x}{c} = \sin\frac{n\pi x}{l} = 0$$

or

$$\frac{n\pi x}{l} = r\pi \ (r = 0, 1, 2, 3, \dots n)$$

The values $r = 0$ and $r = n$ give $x = 0$ and $x = l$, the ends of the string, but between the ends there are $n - 1$ positions equally spaced along the string in the nth harmonic where the displacement is always zero. These positions are called nodes or nodal points, being the positions of zero motion in a system of standing waves. Standing waves arise when a single mode is excited and the incident and reflected waves are superposed. If the amplitudes of these progressive waves are equal and opposite (resulting from complete reflection), nodal points will exist. Often, however, the reflection is not quite complete and the waves in the opposite direction do not cancel each other to give complete nodal points. In this case we speak of a standing wave ratio which we shall discuss in the next section.

Whenever nodal points exist, however, we know that the waves travelling in opposite directions are exactly equal in all respects so that the energy carried in one direction is exactly equal to that carried in the other. This means that the total energy flux, that is, the energy carried across unit area per second in a standing wave system, is zero.

Returning to equation 5.3, we see that the complete expression for the displacement of the nth harmonic is given by

$$y_n = 2a(-\mathrm{i})(\cos\omega_n t + \mathrm{i}\sin\omega_n t)\sin\frac{\omega_n x}{c}$$

We can express this in the form

$$y_n = (A_n\cos\omega_n t + B_n\sin\omega_n t)\sin\frac{\omega_n x}{c} \tag{5.4}$$

where the amplitude of the nth mode is given by $(A_n^2 + B_n^2)^{1/2} = 2a$.

5.12 Standing Wave Ratio

When a wave is completely reflected the superposition of the incident and reflected amplitudes will give nodal points (zero amplitude) where the incident and reflected amplitudes cancel each other, and points of maximum displacement equal to twice the incident amplitude where they reinforce.

If a progressive wave system is partially reflected from a boundary let the amplitude reflection coefficient B_1/A_1 of the earlier section be written as r, where $r < 1$.

The maximum amplitude at reinforcement is then A_1+B_1; the minimum amplitude is given by A_1-B_1. In this case the ratio of maximum to minimum amplitudes in the standing wave system is called the

$$\textit{Standing Wave Ratio} = \frac{A_1 + B_1}{A_1 - B_1} = \frac{1+r}{1-r}$$

where $r = B_1/A_1$.

Measuring the values of the maximum and minimum amplitudes gives the value of the reflection coefficient for

$$r = B_1/A_1 = \frac{\mathrm{SWR} - 1}{\mathrm{SWR} + 1}$$

where SWR refers to the Standing Wave Ratio.

Worked Example

A travelling wave $y_1 = A\cos(\omega t - kx)$ combines with the reflected wave $y_2 = rA\cos(\omega t + kx)$ to produce a standing wave. Show that the standing wave can be represented by $y = 2rA\cos\omega t\cos kx + A(1-r)\cos(\omega t - kx)$. Show that SWR$= \frac{1+r}{1-r}$.

Solution

At reflection incident wave amplitude is reduced to $A(1-r)$ and reflected amplitude is rA. At reflection phase of reflected wave is $\cos(\omega t + kx) + \cos(\omega t - kx) = 2\cos\omega t\cos kx$ so reflected wave is $2rA\cos\omega t\cos kx$ and incident wave is $A(1-r)\cos(\omega t - kx)$. Max. amplitude $= 2rA + A(1-r)$ at antinode of the reflected wave. Max. amplitude $= A(1-r)$ at node of reflected wave.

$$\mathrm{SWR} = \frac{2rA + A(1-r)}{A(1-r)} = \frac{1+r}{1-r}$$

5.13 Energy in Each Normal Mode of a Vibrating String

The total displacement y in the string is the superposition of the displacements y_n of the individual harmonics and we can find the energy in each harmonic by replacing y_n for y in the results on the last page of section 5.11, Standing Waves on a String of Fixed Length. Thus, the kinetic energy in the nth harmonic is

$$E_n(\text{kinetic}) = \frac{1}{2}\int_0^l \rho\dot{y}_n^2 \mathrm{d}x$$

for a string of length l and the potential energy is

$$E_n(\text{potential}) = \frac{1}{2}T\int_0^l \left(\frac{\partial y_n}{\partial x}\right)^2 \mathrm{d}x$$

Since we have already shown for standing waves at the end of section 5.11, Standing Waves on a String of Fixed Length, that

$$y_n = (A_n \cos\omega_n t + B_n \sin\omega_n t)\sin\frac{\omega_n x}{c}$$

then

$$\dot{y}_n = (-A_n\omega_n \sin\omega_n t + B_n\omega_n \cos\omega_n t)\sin\frac{\omega_n x}{c}$$

and

$$\frac{\partial y_n}{\partial x} = \frac{\omega_n}{c}(A_n \cos\omega_n t + B_n \sin\omega_n t)\cos\frac{\omega_n x}{c}$$

Thus

$$E_n(\text{kinetic}) = \frac{1}{2}\rho\omega_n^2[-A_n \sin\omega_n t + B_n \cos\omega_n t]^2 \int_0^l \sin^2\frac{\omega_n x}{c}\mathrm{d}x$$

and

$$E_n(\text{potential}) = \frac{1}{2}T\frac{\omega_n^2}{c^2}[A_n \cos\omega_n t + B_n \sin\omega_n t]^2 \int_0^l \cos^2\frac{\omega_n x}{c}\mathrm{d}x$$

Remembering that $T = \rho c^2$ we have

$$\begin{aligned} E_n(\text{kinetic} + \text{potential}) &= \frac{1}{4}\rho l\omega_n^2(A_n^2 + B_n^2) \\ &= \frac{1}{4}m\omega_n^2(A_n^2 + B_n^2) \end{aligned}$$

where m is the mass of the string and $(A_n^2 + B_n^2)$ is the square of the maximum displacement (amplitude) of the mode. To find the exact value of the total energy E_n of the mode we would need to know the precise value of A_n and B_n and we shall evaluate these in Chapter 11 on Fourier Methods. The total energy of the vibrating string is the sum of all the E_n's of the normal modes.

Note that the distribution of energy along the normal mode of a vibration is the same as that of a travelling wave.

Problem 5.1. Show that the wave profile, that is,

$$y = f_1(ct - x)$$

remains unchanged with time when c is the wave velocity. To do this consider the expression for y at a time $t + \Delta t$ where $\Delta t = \Delta x/c$.

Repeat the problem for $y = f_2(ct + x)$.

Problem 5.2. A triangular shaped pulse of length l is reflected at the fixed end of the string on which it travels ($Z_2 = \infty$). Sketch the shape of the pulse (see Figure 5.9) after a length (a) $l/4$, (b) $l/2$, (c) $3l/4$ and (d) l of the pulse has been reflected.

Problem 5.3. An electrically driven oscillator at the end of string propagates a sinusoidal wave along the string which has a linear density $\rho = 30\ \mathrm{g \cdot m^{-1}}$ and is under a constant tension $T = 12$ N. What power is required to sustain a frequency of $\nu = 300\,\mathrm{Hz}$ and an amplitude of 1.5 cm? What power is required if (a) the frequency is doubled and (b) the amplitude is halved?

Problem 5.4. A cello string has a linear density of $\rho = 1.7\ \mathrm{g\,m^{-1}}$ and a length $L = 0.7$ m. A tension T in the string times it to 220 Hz. What is T?

Problem 5.5. A point mass M is concentrated at a point on a string of characteristic impedance ρc. A transverse wave of frequency ω moves in the positive x direction and is partially reflected and transmitted at the mass. The boundary conditions are that the string displacements just to the left and right of the mass are equal $(y_i + y_r = y_t)$ and that the difference between the transverse forces just to the left and right of the mass equal the mass times its acceleration. If A_1, B_1 and A_2 are respectively the incident, reflected and transmitted wave amplitudes the values

$$\frac{B_1}{A_1} = \frac{-\mathrm{i}q}{1 + \mathrm{i}q} \text{ and } \frac{A_2}{A_1} = \frac{1}{1 + \mathrm{i}q}$$

where $q = \omega M/2\rho c$ and $\mathrm{i}^2 = -1$. Writing $q = \tan\theta$, show that A_2 lags A_1 by θ and that B_1 lags A_1 by $(\pi/2 + \theta)$ for $0 < \theta < \pi/2$.

Show also that the reflected and transmitted energy coefficients are represented by $\sin^2\theta$ and $\cos^2\theta$, respectively.

Problem 5.6. A transverse harmonic force of peak value 0.3 N and frequency 5 Hz initiates waves of amplitude 0.1 m at one end of a very long string of linear density $0.01\ \mathrm{kg/m}$. Show that the rate of energy transfer along the string is $3\pi/20$ W and that the wave velocity is $30/\pi\ \mathrm{m\,s^{-1}}$.

Problem 5.7. The tension in a string produces a fundamental frequency of 440 Hz. (a) What are the frequencies of the 2nd and 3rd harmonics? (b) The average human ear can register 16,000 kHz. How many harmonics does this represent? (c) If the violin string is 32 cm long how far from its end should the string be pressed to shorten its length and produce a fundamental of 523 Hz?

Problem 5.8. The relation between the impedance Z and the refractive index n of a dielectric is given by $Z = 1/n$. Light travelling in free space enters a glass lens which has a refractive index of 1.5 for a free space wavelength of 5.5×10^{-7} m. Show that reflections at this wavelength are avoided by a coating of refractive index 1.22 and thickness 1.12×10^{-7} m.

Problem 5.9. Prove that the displacement y_n of the standing wave expression in equation (5.4) satisfies the time-independent form of the wave equation

$$\frac{\partial^2 y}{\partial x^2} + k^2 y = 0.$$

Problem 5.10. The total energy E_n of a normal mode may be found by an alternative method. Each section dx of the string is a simple harmonic oscillator with total energy equal to the maximum kinetic energy of oscillation

$$k.e._{\max} = \frac{1}{2}\rho \mathrm{d}x(\dot{y}_n^2)_{\max} = \frac{1}{2}\rho \mathrm{d}x\omega_n^2(y_n^2)_{\max}$$

Now the value of $(y_n^2)_{\max}$ at a point x on the string is given by

$$(y_n^2)_{\max} = (A_n^2 + B_n^2)\sin^2\frac{\omega_n x}{c}$$

Show that the sum of the energies of the oscillators along the string, that is, the integral

$$\frac{1}{2}\rho\omega_n^2\int_0^l (y_n^2)_{\max}\mathrm{d}x$$

gives the expected result.

Problem 5.11. The displacement of a wave on a string which is fixed at both ends is given by

$$y(x,\ t) = A\cos(\omega t - kx) + rA\cos(\omega t + kx)$$

where r is the coefficient of amplitude reflection. Show that this may be expressed as the superposition of standing waves

$$y(x,\ t) = A(1+r)\cos\omega t\ \cos kx + A(1-r)\sin\omega t \sin kx.$$

6

Transverse Wave Motion (2)

Introduction

Waves are rarely monochromatic, that is, limited to a single frequency, but are usually made of a mixture of frequencies. First of all we consider the superposition of two waves of equal amplitudes and phase velocities but with slightly different frequencies. Then the two waves are allowed different phase velocities and finally multiple waves over a narrow frequency range are superposed to form a pulse. This leads to the concepts of group velocity, beats, and dispersion. The Bandwidth Theorem is derived and its connection to Heisenberg's Uncertainty Principle is explored. The propagation of transverse waves in a periodic structure such as an ionic crystal explains how infrared radiation is absorbed. The Diffusion Equation, applied to the periodic structure of a transmission line, is used to account for energy loss in wave propagation.

6.1 Wave Groups, Group Velocity and Dispersion

Our discussion so far has been limited to monochromatic waves – waves of a single frequency and wavelength. It is much more common for waves to occur as a mixture of a number or group of component frequencies; white light, for instance, is composed of a continuous visible wavelength spectrum extending from about 3000 Å in the blue to 7000 Å in the red. Examining the behaviour of such a group leads to the third kind of velocity mentioned early in the last chapter, that is, the group velocity.

6.1.1 Superposition of Two Waves of Almost Equal Frequencies

We begin by considering a group which consists of two components of equal amplitude a but frequencies ω_1 and ω_2 which differ by a small amount.

Their separate displacements are given by

$$y_1 = a\cos(\omega_1 t - k_1 x)$$

Introduction to Vibrations and Waves, First Edition. H. J. Pain and Patricia Rankin.

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and

$$y_2 = a\cos(\omega_2 t - k_2 x)$$

Superposition of amplitude and phase gives

$$y = y_1 + y_2 = 2a\cos\left[\frac{(\omega_1-\omega_2)t}{2} - \frac{(k_1-k_2)x}{2}\right]\cos\left[\frac{(\omega_1+\omega_2)t}{2} - \frac{(k_1+k_2)x}{2}\right]$$

a wave system with a frequency $(\omega_1+\omega_2)/2$ which is very close to the frequency of either component but with a maximum amplitude of $2a$, modulated in space and time by a very slowly varying envelope of frequency $(\omega_1-\omega_2)/2$ and wave number $(k_1-k_2)/2$.

This system is shown in Figure 6.1 and shows a behaviour similar to that of the equivalent coupled oscillators in Chapter 4. The velocity of the new wave is $(\omega_1-\omega_2)/(k_1-k_2)$ which, if the phase velocities $\omega_1/k_1 = \omega_2/k_2 = c$, gives

$$\frac{\omega_1-\omega_2}{k_1-k_2} = c\frac{(k_1-k_2)}{k_1-k_2} = c$$

so that the component frequencies and their superposition, or group will travel with the same velocity, the profile of their combination in Figure 6.1 remaining constant.

If the waves are sound waves the intensity is a maximum whenever the amplitude is a maximum of $2a$; this occurs twice for every period of the modulating frequency; that is, at a frequency $\nu_1 - \nu_2$.

The beats of maximum intensity fluctuations thus have a frequency equal to the difference $\nu_1 - \nu_2$ of the components. In the example here where the components have equal amplitudes a, superposition will produce an amplitude which varies between $2a$ and 0; this is called complete or 100% modulation.

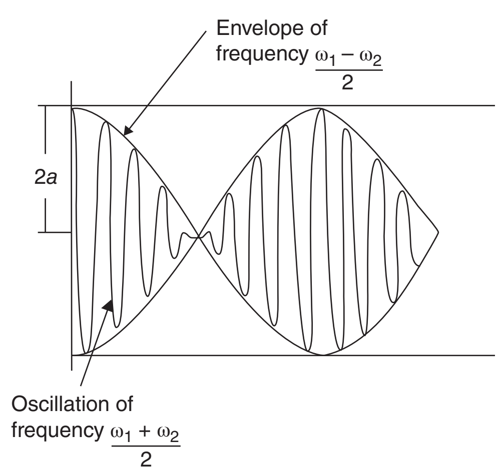

Figure 6.1 *The superposition of two waves of slightly different frequency ω_1 and ω_2 forms a group. The faster oscillation occurs at the average frequency of the two components $(\omega_1+\omega_2)/2$ and the slowly varying group envelope has a frequency $(\omega_1-\omega_2)/2$, half the frequency difference between the components.*

More generally an amplitude modulated wave may be represented by

$$y = A\cos(\omega t - kx)$$

where the modulated amplitude

$$A = a + b\cos\omega' t$$

This gives

$$y = a\cos(\omega t - kx) + \frac{b}{2}\{\cos[(\omega + \omega')t - kx] + \cos[(\omega - \omega')t - kx]\}$$

so that here amplitude modulation has introduced two new frequencies $\omega \pm \omega'$, known as combination tones or sidebands. Amplitude modulation of a carrier frequency is a common form of radio transmission, but its generation of sidebands has led to the crowding of radio frequencies and interference between stations.

6.1.2 Wave Groups, Group Velocity and Dispersion

Suppose now that the two frequency components of the last section have different phase velocities so that $\omega_1/k_1 \neq \omega_2/k_2$. The velocity of the maximum amplitude of the group, that is, the group velocity

$$\frac{\omega_1 - \omega_2}{k_1 - k_2} = \frac{\Delta\omega}{\Delta k}$$

is now different from each of these velocities; the superposition of the two waves will no longer remain constant and the group profile will change with time.

A medium in which the phase velocity is frequency dependent (ω/k not constant) is known as a dispersive medium and a dispersion relation expresses the variation of ω as a function of k. If a group contains a number of components of frequencies which are nearly equal the original expression for the group velocity is written

$$\frac{\Delta\omega}{\Delta k} = \frac{\mathrm{d}\omega}{\mathrm{d}k}$$

The group velocity is that of the maximum amplitude of the group so that it is the velocity with which the energy in the group is transmitted. Since $\omega = kv$, where v is the phase velocity, the group velocity

$$\begin{aligned} v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} &= \frac{\mathrm{d}}{\mathrm{d}k}(kv) = v + k\frac{\mathrm{d}v}{\mathrm{d}k} \\ &= v - \lambda\frac{\mathrm{d}v}{\mathrm{d}\lambda} \end{aligned}$$

where $k = 2\pi/\lambda$. Usually $\mathrm{d}v/\mathrm{d}\lambda$ is positive, so that $v_g < v$. This is called normal dispersion, but anomalous dispersion can arise when $\mathrm{d}v/\mathrm{d}\lambda$ is negative, so that $v_g > v$.

We shall see when we discuss electromagnetic waves that an electrical conductor is anomalously dispersive to these waves whilst a dielectric is normally dispersive except at the natural resonant frequencies of its atoms. In the chapter on forced oscillations we saw that the wave then acted as a driving force upon

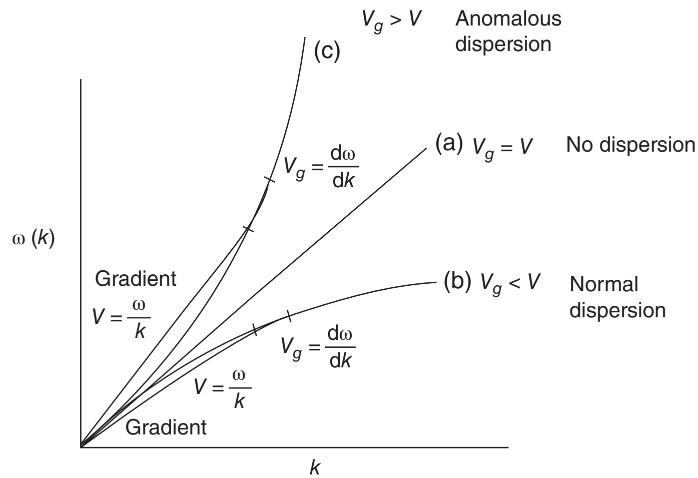

Figure 6.2 *Curves illustrating dispersion relations: (a) a straight line representing a non-dispersive medium, $v = v_g$; (b) a normal dispersion relation where the gradient $v = \omega/k > v_g = \mathrm{d}\omega/\mathrm{d}k$; (c) an anomalous dispersion relation where $v < v_g$.*

the atomic oscillators and that strong absorption of the wave energy was represented by the dissipation fraction of the oscillator impedance, whilst the anomalous dispersion curve followed the value of the reactive part of the impedance.

The three curves of Figure 6.2 represent

- A non-dispersive medium where ω/k is constant, so that $v_g = v$, for instance free space behaviour towards light waves.
- A normal dispersion relation $v_g < v$.
- An anomalous dispersion relation $v_g > v$.

Worked Example

The electric vector of an electromagnetic wave propagates in a dielectric with a velocity $v = (\mu\varepsilon)^{-1/2}$ where μ is the permeability and ε is the permittivity. In free space the velocity is that of light, $c = (\mu_0\varepsilon_0)^{-1/2}$. The refractive index $n = c/v = \sqrt{\mu\varepsilon/\mu_0\varepsilon_0} = \sqrt{\mu_r\varepsilon_r}$ where $\mu_r = \mu/\mu_0$ and $\varepsilon_r = \varepsilon/\varepsilon_0$. For many substances μ_r is constant and ~ 1, but ε_r is frequency dependent, so that v depends on λ.

The group velocity

$$\upsilon_g = v - \lambda \mathrm{d}v/\mathrm{d}\lambda = v\left(1 + \frac{\lambda}{2\varepsilon_r}\frac{\partial\varepsilon_r}{\partial\lambda}\right)$$

so that $v_g > v$ (anomalous dispersion) when $\partial\varepsilon_r/\partial\lambda$ is $+ve$. Figure 6.3 shows the behaviour of the refractive index $n = \sqrt{\varepsilon_r}$ versus ω, the frequency, and λ, the wavelength, in the region of anomalous dispersion associated with a resonant frequency. The dotted curve shows the energy absorption (compare this with Figure 3.11).

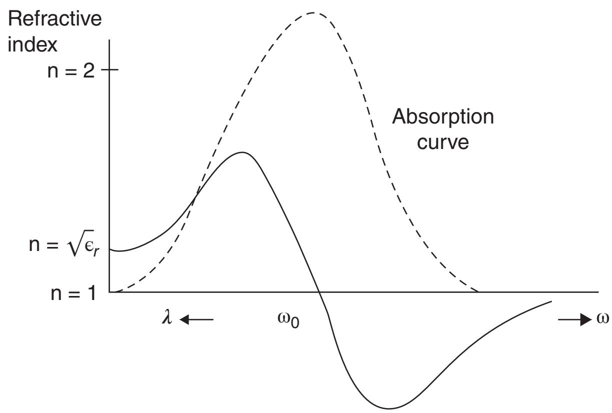

Figure 6.3 Anomalous dispersion showing the behaviour of the refractive index $n = \sqrt{\varepsilon_r}$ versus ω and λ, where ω_0 is a resonant frequency of the atoms of the medium. The absorption in such a region is shown by the dotted line (see Figure 3.11).

6.2 Wave Group of Many Components. The Bandwidth Theorem

We have so far considered wave groups having only two frequency components. We may easily extend this to the case of a group of many frequency components, each of amplitude a, lying within the narrow frequency range $\Delta\omega$.

The essential physics of this problem is shown in Appendix 3, where we find the sum of the series, with δ as the constant phase difference between n successive equal components to be

$$R = \sum_{0}^{n-1} a \cos(\omega t + n\delta)$$

Here we are concerned with the constant phase difference $(\delta\omega)t$ which results from a constant frequency difference $\delta\omega$ between successive components. The spectrum or range of frequencies of this group is shown in Figure 6.4a and we wish to follow its behaviour with time.

We seek the amplitude which results from the superposition of the frequency components and write it

$$\begin{aligned} R = {} & a\cos\omega_1 t + a\cos(\omega_1 + \delta\omega)t + a\cos(\omega_1 + 2\delta\omega)t + \cdots \\ & + a\cos[\omega_1 + (n-1)(\delta\omega)]t \end{aligned}$$

The result is given in Appendix 3 as

$$R = a\frac{\sin[n(\delta\omega)t/2]}{\sin[(\delta\omega)t/2]}\cos\overline{\omega}t$$

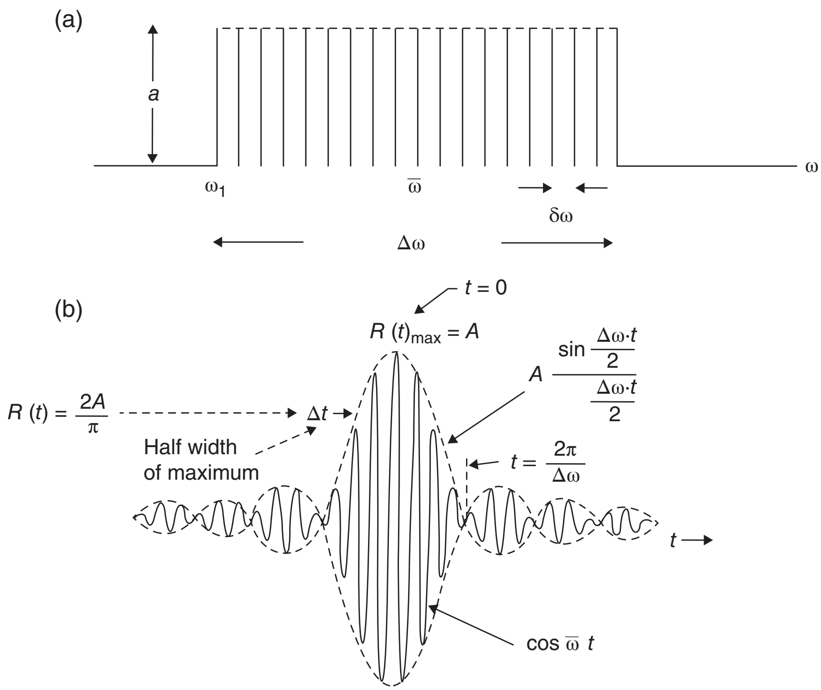

Figure 6.4 *A rectangular wave band of width $\Delta\omega$ having n frequency components of amplitude a with a common frequency difference $\delta\omega$. (b) Representation of the frequency band on a time axis is a cosine curve at the average frequency $\overline{\omega}$, amplitude modulated by a $\sin\alpha/\alpha$ curve where $\alpha = \Delta\omega \cdot t/2$. After a time $t = 2\pi/\Delta\omega$ the superposition of the components gives a zero amplitude.*

where the average frequency in the group or band is

$$\overline{\omega} = \omega_1 + \frac{1}{2}(n-1)(\delta\omega)$$

Now $n(\delta\omega) = \Delta\omega$, the bandwidth, so the behaviour of the resultant R with time may be written

$$R(t) = a\frac{\sin(\Delta\omega \cdot t/2)}{\sin(\Delta\omega \cdot t/n2)}\cos\overline{\omega}t = na\frac{\sin(\Delta\omega \cdot t/2)}{\Delta\omega \cdot t/2}\cos\overline{\omega}t$$

when n is large, and

$$\sin(\Delta\omega \cdot t/n2) \to \frac{\Delta\omega \cdot t}{n2}$$

or

$$R(t) = A\frac{\sin\alpha}{\alpha}\cos\overline{\omega}t$$

where $A = na$ and $\alpha = \Delta\omega \cdot t/2$ is half the phase difference between the first and last components at time t.

This expression gives us the time behaviour of the band and is displayed on a time axis in Figure 6.4b. We see that the amplitude $R(t)$ is given by the cosine curve of the average frequency $\overline{\omega}$ modified by the $A\sin\alpha/\alpha$ term.

At $t = 0$, $\sin\alpha/\alpha \to 1$ and all the components superpose with zero phase difference to give the maximum amplitude $R(t) = A = na$. After some time interval Δt when

$$\alpha = \frac{\Delta\omega\Delta t}{2} = \pi$$

the phases between the frequency components are such that the resulting amplitude $R(t)$ is zero.

The time Δt which is a measure of the width of the central pulse of Figure 6.4b is therefore given by

$$\frac{\Delta\omega\Delta t}{2} = \pi$$

or $\Delta\nu\,\Delta t = 1$ where $\Delta\omega = 2\pi\Delta\nu$.

The true width of the base of the central pulse is $2\Delta t$ but the interval Δt is taken as an arbitrary measure of time, centred about $t = 0$, during which the amplitude $R(t)$ remains significantly large ($>A/2$). With this arbitrary definition the exact expression

$$\Delta\nu\,\Delta t = 1$$

becomes the approximation

$$\Delta\nu\,\Delta t \approx 1 \quad \text{or} \quad (\Delta\omega\,\Delta t \approx 2\pi)$$

and this approximation is known as the Bandwidth Theorem.

It states that the components of a band of width $\Delta\omega$ in the frequency range will superpose to produce a significant amplitude $R(t)$ only for a time Δt before the band decays from random phase differences. The greater the range $\Delta\omega$ the shorter the period Δt.

Alternatively, the theorem states that a single pulse of time duration Δt is the result of the superposition of frequency components over the range $\Delta\omega$; the shorter the period Δt of the pulse the wider the range $\Delta\omega$ of the frequencies required to represent it.

When $\Delta\omega$ is zero we have a single frequency, the monochromatic wave which is therefore required (in theory) to have an infinitely long time span.

We have chosen to express our wave group in the two parameters of frequency and time (having a product of zero dimensions), but we may just as easily work in the other pair of parameters wave number k and distance x.

Replacing ω by k and t by x would define the length of the wave group as Δx in terms of the range of component wavelengths $\Delta(1/\lambda)$.

The Bandwidth Theorem then becomes

$$\Delta x\,\Delta k \approx 2\pi$$

or

$$\Delta x\Delta(1/\lambda) \approx 1 \ \text{ i.e. } \ \Delta x \approx \lambda^2/\Delta\lambda$$

Note again that a monochromatic wave with $\Delta k = 0$ requires $\Delta x \to \infty$; that is, an infinitely long wavetrain.

In the wave group we have just considered the problem has been simplified by assuming all frequency components to have the same amplitude a. When this is not the case, the different values $a(\omega)$ are treated by Fourier methods as we shall see in Chapter 11.

We shall meet the ideas of this section several times in the course of this text, noting particularly that in modern physics the Bandwidth Theorem becomes Heisenberg's Uncertainty Principle.

Worked Example

A pulse of white light has a frequency range $\Delta\nu$ between 769 and 384 times 10^{12} Hz, i.e. $\Delta\nu \approx 385 \times 10^{12}$ Hz. The Bandwidth Theorem gives $\Delta\nu\Delta t \approx 1 \therefore \Delta\nu = 1/\Delta t$ and the coherent length of the wavetrain of such a pulse is $c\Delta t = c/\Delta\nu = 779 \times 10^{-9}$ m, that is, one wavelength at the red end of the visible spectrum.

6.3 Heisenberg's Uncertainty Principle

Compton (in 1922–23) fired X-rays of a known frequency at thin foils of different materials and found that the scattered radiation was independent of the foil material and that his results were consistent only if momentum and energy were conserved in an elastic collision between two 'particles', an electron and an X-ray of energy $h\nu$, rest mass m_0 and (from Einstein's relativistic energy equation) a momentum $p = E/c = h\nu/c = h/\lambda$ where $c = \nu\lambda$ and h is Planck's constant.

In 1924 de Broglie proposed that if the dual wave particle nature of electromagnetic fields (X-rays) required a particle momentum of $p = h/\lambda$ it was possible that a wavelength λ of a 'matter' field could be associated with *any* particle $p = mv$ to give the relation $p = h/\lambda$. He showed that the velocity v in mv was the *group velocity* of a pulse (not a single frequency) so

$$p = \frac{h}{\lambda} = \frac{hk}{2\pi}$$

and

$$\Delta p = \frac{h}{2\pi}\Delta k$$

But the Bandwidth Theorem shows that a group in the wave number range Δk superposed in space over a distance Δx obeys the relation

$$\Delta x \Delta k \approx 2\pi$$

so

$$\Delta x \Delta p \approx h$$

This is Heisenberg's Uncertainty Principle.

This relation sets a fundamental limit on the ultimate precision with which we can know the position x of a particle and the x component of its momentum simultaneously (Figure 6.5). More advanced mathematics shows that a 'wave packet' of typical shape (Gaussian in Figure 6.5), representing an electron localized at time $t = 0$ to within a distance of $\Delta x = 10^{-10}$ m (atomic dimensions) with $\Delta p_x = h/\Delta x \approx 10^{-24}\,\mathrm{kg \cdot m \cdot s^{-1}}$ will spread to twice its length in time $t = 10^{-16}$ sec.