2022

British Algorithmic Olympiad

MARK SCHEME

GUIDE FOR TEACHERS

- If a question requires a coded solution, you should test the candidate's code using the given test cases. Marks are allocated depending on the program's output. Correct outputs are formatted in **`Bold`** and the marks for each line of output is stated.
- Some questions will award partial marks for solutions that are close to correct. Some questions will award working marks.
- If a test case seems to have more input lines than necessary, this is because the input is large and requires multiple lines to be displayed in this document.
- You are advised to read through the entire exam paper before marking any exams.

INSTRUCTIONS

- A correct line of output from a candidate's program should be disallowed if the program takes longer than **1 second**.
- Please mark all the candidate's working, and ignore crossed out working.
- Mark positively; for written answers, you should give marks depending on whether the candidate's reasoning is similar to the mark scheme. You should award marks to alternative solutions which provide a correct answer.
- If a candidate's answer to a written question differs from the mark scheme but seems to be correct, award marks depending on how complete the answer is. Do not award marks if the approach is incorrect.

Question 1. Binary Rotors

A

	Input	Output	

1	`10`
	`00`
	`1`
		`11`	**(1)**

2	`1110`
	`1000`
	`5`
		`1011`	**(1)**

3	`10010101101`
	`00000000000`
	`-85`
		`11111110100`	**(2)**

4	`101000000`
	`111111111`
	`-2000001`
		`100100100`	**(2)**

5	`1111111111110000000111111`
	`1001001001001001001001001`
	`2147483`
		`1001001001001101011001001`	**(2)**

6	`010000100100000101001111`
	`111111111111111111111110`
	`-1844674407370`
		`110111000000110111000001`	**(3)**

7	`101010101111111110100`
	`000011100011001010001`
	`58446744073709`
		`100001111110111101000`	**(3)**

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B I, II and IV **(1)**

C 35,184,367,894,528 **(4)** allow any 14 digit number for **(1)**

The mark scheme for question **1** is continued on the next page.

D	Recognises that $\Gamma(S)$ cycles between $\{3, 3, 2, 0, 1, 3\}$	**(1)**
	Correct expansion of infinite series: $$\sum_{n=1}^{\infty}\left(\frac{\Gamma(S_n)}{2L}\right)^n = \left(\frac{3}{12}\right)^1 + \left(\frac{3}{12}\right)^2 + \left(\frac{2}{12}\right)^3 + (0)^4 + \left(\frac{1}{12}\right)^5 + \left(\frac{3}{12}\right)^6 + \cdots$$	**(1)**
	$$\sum_{n=1}^{\infty}\left(\frac{\Gamma(S_n)}{2L}\right)^n = \underbrace{\sum_{k=0}^{\infty}\left(\frac{3}{12}\right)\left(\frac{3}{12}\right)^{6k}}_{Z_1} + \underbrace{\sum_{k=0}^{\infty}\left(\frac{3}{12}\right)^2\left(\frac{3}{12}\right)^{6k}}_{Z_2} + \underbrace{\sum_{k=0}^{\infty}\left(\frac{2}{12}\right)^3\left(\frac{2}{12}\right)^{6k}}_{Z_3} + 0 + \underbrace{\sum_{k=0}^{\infty}\left(\frac{1}{12}\right)^5\left(\frac{1}{12}\right)^{6k}}_{Z_4} + \underbrace{\sum_{k=0}^{\infty}\left(\frac{3}{12}\right)^6\left(\frac{3}{12}\right)^{6k}}_{Z_5}$$	**(2)**
	Uses $S_\infty = \dfrac{a}{1-r}$ to calculate each sum individually. At least 1 of the following must be correct. $$Z_1 = \frac{\frac{3}{12}}{1-\left(\frac{3}{12}\right)^6} = \frac{1024}{4095} \qquad Z_2 = \frac{\left(\frac{3}{12}\right)^2}{1-\left(\frac{3}{12}\right)^6} = \frac{256}{4095} \qquad Z_3 = \frac{\left(\frac{2}{12}\right)^3}{1-\left(\frac{2}{12}\right)^6} = \frac{216}{46655}$$ $$Z_4 = \frac{\left(\frac{1}{12}\right)^5}{1-\left(\frac{1}{12}\right)^6} = \frac{12}{2985983} \quad Z_5 = \frac{\left(\frac{3}{12}\right)^6}{1-\left(\frac{3}{12}\right)^6} = \frac{1}{4095}$$	**(1)**
	Calculates $mL(Z_1+Z_2+Z_3+Z_4+Z_5) = m\left(\dfrac{7581408374}{3980315339}\right) \implies m = 3980315339$	**(1)**

3,980,315,339 **(6)**

Question 2. Sonar

A

	Input	Output	
1	2 1 1	**8**	**(1)**
2	1 0 0	**3**	**(1)**
3	5 2 3	**24**	**(1)**
4	8 8 7	**50**	**(1)**
5	22 11 11	**336**	**(2)**
6	150 150 0	**13717**	**(2)**
7	250 119 1	**38407**	**(3)**
8	672 345 345	**275094**	**(3)**
9	1002 501 501	**611584**	**(4)**

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B "No" **and** any **two** from:

The idea that x and y are not coprime (e.g. 3 divides x **and** y)	**(1)**
$\gcd(x, y) \neq 1$ **or** $\mathrm{hcf}(x, y) \neq 1$	**(1)**
The gradient of the line from $(0, 0)$ to $(x, y) = \frac{123456789}{987654321}$, which is not in its lowest terms	**(1)**

C

Attempt to find the combinations of 2 points with a charge on each side of the grid	**(1)**
The left and bottom sides both have $\binom{1000}{2}$ $(= 499500)$ combinations of 2 such points.	**(1)**
Works out by writing a program, or by other means, that the top and right side each have k such points, where k is the number of coprime pairs on each respective border. $(k = 499)$	**(1)**
The number of combinations of 2 of those k points is $\binom{499}{2}$ $(= 124251)$.	**(1)**
The total number is $2(499500 + 124251) + 1 = 1247503$	**(1)**

1247503 **(5)** allow 1247502 for **(4)**
reward a recursive function approach for appropriate marks

Question 3. Pairs and Triples

A

A typical solution is proof by exhaustion.	
Both cases where the result is a multiple of 4: $x = 2a,\ y = 2b \quad (a, b \in \mathbb{Z})$ $x^2 - y^2 = 4a^2 - 4b^2 = 4(a^2 - b^2)$ $x = 2a + 1,\ y = 2b + 1 \quad (a, b \in \mathbb{Z})$ $x^2 - y^2 = (2a + 1)^2 - (2b + 1)^2 = 4a^2 + 4a + 1 - 4b^2 - 4b - 1 = 4(a^2 - b^2 + a - b)$	**(1)**
A case where the result is odd, e.g. $x = 2a,\ y = 2b + 1$ $x^2 - y^2 = 4a^2 - 4b^2 - 2b + 1 = 2(2a^2 - 2b^2 - 2b) + 1$	**(1)**

(2) allow any suitable mathematical proof such as the use of modular arithmetic for **(2)**

B

$a^2 = c^2 - b^2$	
$a^2 = (c + b)(c - b)$	**(1)**
We know that $\dfrac{c+b}{a} = \dfrac{m}{n}$ so $\dfrac{a}{c-b} = \dfrac{m}{n}$ **or** $\dfrac{c-b}{a} = \dfrac{n}{m}$	**(1)**
$\dfrac{b}{a} = \dfrac{1}{2}\left(\dfrac{m}{n} - \dfrac{n}{m}\right) = \dfrac{m^2 - n^2}{2mn}$	**(1)**
$\dfrac{c}{a} = \dfrac{1}{2}\left(\dfrac{m}{n} + \dfrac{n}{m}\right) = \dfrac{m^2 + n^2}{2mn}$	**(1)**
$m^2 - n^2$ and $2mn$ have a greatest common divisor / highest common factor of 1 or 2	**(1)**
If $m^2 - n^2$ has a factor of 2 then it has a factor of 4 meaning b would be even, but b is odd. $\Rightarrow\Leftarrow$ (contradiction) Therefore $m^2 - n^2$ is odd.	**(1)**
$a = 2mn$, $b = m^2 - n^2$ **and** $c = m^2 + n^2$	**(1)**
$(2mn)^2 + (m^2 - n^2)^2 = 4m^2n^2 + m^4 - 2m^2n^2 + n^4 = (m^2 + n^2)^2$	**(1)**

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C

	Input	Output	
1	`42`	**`41`**	**(1)**
2	`100`	**`86`**	**(1)**
3	`4194304`	**`3296229`**	**(3)**
4	`134217725`	**`105425906`**	**(3)**
5	`2147483646`	**`1686675768`**	**(3)**

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D	$\frac{\pi}{4}$ **(1)**	
E	$c_1(n) = 2\left(c(n) - t(\sqrt{n})\right)$ **(3)**	allow equal expression for **(3)** allow $c_1(n) = c(n) - t(\sqrt{n})$ for **(1)**

Question 4. Wolves

A For this question, the number of marks given should be **halved** if only one number is correct.

	Input	Output	
1	`abced`	**`23 4`**	**(2)**
2	`gjhadeibcfk`	**`87 5`**	**(2)**
3	`hndemgijcbpoafkl`	**`280 7`**	**(2)**
4	`stmcahirdlnkojebpgfq`	**`779 8`**	**(4)**
5	`ivhsowebftdqarjxnlpugkzcym`	**`1706 7`**	**(4)**
6	`abcdefghijklmnopqrstuvwxzy`	**`50331647 25`**	**(4)**

(18)

B 3.77 **(2)**

C This question requires the candidate to interpret the code in their own way.
Award **(5)** if the candidate's reasoning reaches a valid answer. You should validate their reasoning.

An **example** of a solution is below.

The candidate may choose to take each 'bark' as a binary number, e.g. the lowercase letters as 0s and the uppercase letters as 1s. The candidate may also choose to interpret the subscript digits as additions. E.g. 'bARK$_1$' means the base 10 value of $0111 + 1 = 8$ (= `h`, say, as `a` = 1, `b` = 2, etc.). The slashes could mean multiple options of endpoints, and this is where the candidate could reason an original list of endpoints. The answer in this case would be `aehgdcfb`.

Total Marks: **100**