Multivariable Calculus

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Contents

Chapter 1

Vectors and the Geometry of Space

1.1 Three-Dimensional Coordinate Systems

We would use an ordered tuple of three numbers (x, y, z) to represent a point in three-dimensional space. The three numbers correspond to the distances along the x-axis, y-axis, and z-axis respectively.

Moreover, we can use a vector to represent a point in space. A vector $\mathbf{v}$ can be expressed as:

$$\mathbf{v} = \langle x, y, z \rangle = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are the unit vectors along the x-, y-, and z-axes respectively.

Remark. Unit vectors are vectors with a magnitude of 1. They are often used to indicate direction.

The distance, or norm, of the vector $\mathbf{v}$ from the origin can be calculated using the formula:

$$\|\mathbf{v}\|_2 = \|\mathbf{v}\| = \sqrt{x^2 + y^2 + z^2}$$

This is also known as the Euclidean norm.

As we are used to consider two-dimensional planes, we always consider the following equations as circles in two-dimensional space:

$$x^2 + y^2 = r^2$$

However, in three-dimensional space, this equation represents a cylinder extending infinitely along the z-axis. As implicitly, the equation does not restrict the value of z. Then the set of points satisfying the equation forms a cylinder.

In two-dimensional case, the set of points satisfying the equation $x^2 + y^2 = r^2$ represents a circle of radius r centered at the origin:

$$S^1 = \{(x, y) \mid x^2 + y^2 = r^2\}$$

In three-dimensional case, the set of points satisfying the equation $x^2 + y^2 = r^2$ represents a cylinder of radius r centered along the z-axis:

$$C = \{(x, y, z) \mid x^2 + y^2 = r^2, z \in \mathbb{R}\}$$

So if we want to represent a two-dimensional circle in three-dimensional space, we need to add an additional constraint on z. For example, the set of points satisfying the equations $x^2 + y^2 = r^2$ and $z = 0$ represents a circle of radius r in the xy-plane:

$$S^1 = \{(x, y, z) \mid x^2 + y^2 = r^2, z = 0\}$$

For vector operations, we have:

- Vector Addition: $\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$
- Scalar Multiplication: $c\mathbf{a} = \langle ca_1, ca_2, ca_3 \rangle$

Also, we have the dot product and cross product defined as:

- Dot Product: $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$
- Cross Product: $\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$

Moreover, the dot product can also be expressed in terms of the magnitudes of the vectors and the angle θ between them:

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\| \cos\theta$$

and the magnitude of the cross product can be expressed as:

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\| \sin\theta$$

It represents the area of the parallelogram formed by the two vectors.

If we want to project vector $\mathbf{b}$ onto vector $\mathbf{a}$, we can use the formula:

$$\mathrm{proj}_{\mathbf{a}}\mathbf{b} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|}\right) \frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|^2}\mathbf{a}$$

The scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is given by:

$$\mathrm{comp}_{\mathbf{a}}\mathbf{b} = \|\mathbf{b}\| \cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|}$$

For the cross product, we can use the following determinant form:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Remark. The cross product of two vectors results in a vector that is orthogonal (perpendicular) to both original vectors. The direction of the resulting vector is determined by the right-hand rule.

1.2 Lines and Planes

1.2.1 Lines

To represent a line in three-dimensional space, we can use a point and a direction vector. If we have a point $P_0(x_0, y_0, z_0)$ on the line and a direction vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, then any point $P(x, y, z)$ on the line, the vector $\overrightarrow{P_0P}$ is parallel to $\mathbf{v}$, i.e., $\overrightarrow{P_0P} = t\mathbf{v}$ for some scalar t. Then we have the parametric equations of the line as:

$$\langle x, y, z \rangle - \langle x_0, y_0, z_0 \rangle = t\langle v_1, v_2, v_3 \rangle$$

or equivalently,

$$\begin{cases} x = x_0 + tv_1 \\ y = y_0 + tv_2 \\ z = z_0 + tv_3 \end{cases}$$

which are called the *parametric equations* of the line. The t is called the *parameter* of the line.

To visualize the parametric equation of a line in 3D, consider Figure 1.1 below.

From Figure 1.1, we can also write the parametric equations as:

$$\mathbf{r}(t) = \overrightarrow{OP_0} + t\mathbf{v}$$

which is called the *vector form* of the line.

If $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ where none of v_1, v_2, v_3 is zero, we can also express the line in *symmetric form* as:

$$\frac{x - x_0}{v_1} = \frac{y - y_0}{v_2} = \frac{z - z_0}{v_3}$$

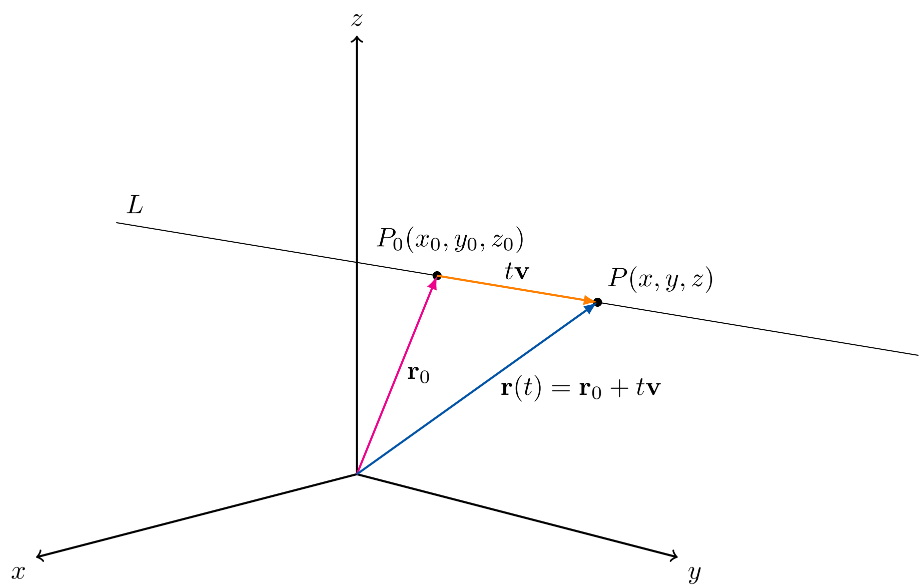


Figure 1.1: Parametric Equation of a Line in 3D

Example 1.1. *Find the parametric equations of the line that passes through the points* $A(1, 2, 3)$ *and* $B(4, 5, 6)$*. Express the line in vector form, parametric form and symmetric forms.*

Solution. In order to find the equation of the line, we need

- A point on the line: $A(1, 2, 3)$;
- A direction vector: $\mathbf{v} = \overrightarrow{AB} = \langle 4-1, 5-2, 6-3\rangle = \langle 3, 3, 3\rangle$.

Therefore, the vector form of the line is:

$$\mathbf{r}(t) = \langle 1, 2, 3\rangle + t\langle 3, 3, 3\rangle$$

The parametric form of the line is:

$$\begin{cases} x = 1 + 3t \\ y = 2 + 3t \\ z = 3 + 3t \end{cases}$$

The symmetric form of the line is:

$$\frac{x-1}{3} = \frac{y-2}{3} = \frac{z-3}{3}$$

1.2.2 Planes

A plane in three-dimensional space can be defined using a point and a normal vector. If we have a point $P_0(x_0, y_0, z_0)$ on the plane and a normal vector $\mathbf{n} = \langle A, B, C\rangle$, then any point $P(x, y, z)$ on the plane satisfies the condition that the vector $\overrightarrow{P_0P}$ is orthogonal to the normal vector $\mathbf{n}$, i.e., $\mathbf{n} \cdot \overrightarrow{P_0P} = 0$. This leads to the equation of the plane:

$$\langle A, B, C\rangle \cdot (\langle x, y, z\rangle - \langle x_0, y_0, z_0\rangle) = 0$$

or equivalently,

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

which is called the *scalar equation* of the plane.

Expanding this, we get:

$$Ax + By + Cz = Ax_0 + By_0 + Cz_0$$

or equivalently,

$$Ax + By + Cz + D = 0$$

where $D = -(Ax_0 + By_0 + Cz_0)$ is a constant. It is called a *linear equation* in x, y and z.

To visualize the equation of a plane in 3D, consider Figure 1.2 below.

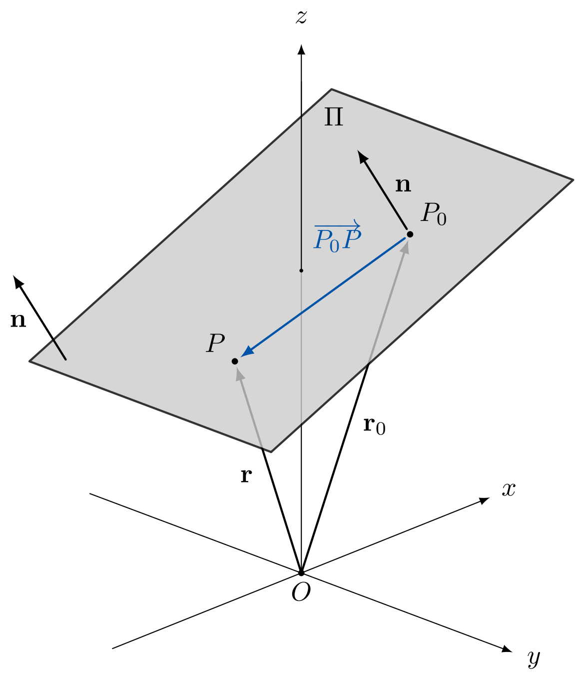


Figure 1.2: Equation of a Plane in 3D

In order to find $\mathbf{n}$, we can use the cross product.

Example 1.2. *Find the equation of the plane that passes through the points:*

$$A(1, 2, 3), \quad B(4, 5, 6), \quad C(7, 8, 0).$$

Solution. In order to find the equation of the plane, we need

- A point on the plane: $A(1, 2, 3)$;
- A normal vector: $\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC}$.

First, we calculate the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$:

$$\begin{aligned}\overrightarrow{AB} &= \langle 4-1, 5-2, 6-3\rangle = \langle 3, 3, 3\rangle,\\ \overrightarrow{AC} &= \langle 7-1, 8-2, 0-3\rangle = \langle 6, 6, -3\rangle.\end{aligned}$$

Taking the cross product, we have:

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 3 & 3 \\ 6 & 6 & -3 \end{vmatrix} = \langle 0, 0, -9\rangle.$$

For simplicity, we can take the normal vector as $\mathbf{n} = \langle 0, 0, 1\rangle$. Therefore, the equation of the plane is:

$$\begin{aligned}0(x-1) + 0(y-2) + 1(z-3) &= 0\\ z - 3 &= 0\\ z &= 3.\end{aligned}$$

If we have a point $P_1(x_1, y_1, z_1)$ not on the plane, we can calculate the distance from the point to the plane using the formula:

$$\text{Distance} = \frac{\|\mathbf{n}\cdot\mathbf{b}\|}{\|\mathbf{n}\|} = \frac{A(x_1-x_0)+B(y_1-y_0)+C(z_1-z_0)}{\sqrt{A^2+B^2+C^2}} = \frac{|Ax_1+By_1+Cz_1+D|}{\sqrt{A^2+B^2+C^2}}$$

where $\mathbf{b} = \overrightarrow{P_0P_1} = \langle x_1-x_0, y_1-y_0, z_1-z_0\rangle$.

1.3 Cylinders and Quadric Surfaces

1.3.1 Cylinders

A cylinder is a surface that consists of all lines that are parallel to a given line and pass through a given curve. The given line is called the *generatrix* of the cylinder, and the given curve is called the *directrix* of the cylinder.

Example 1.3. *Sketch the graph of the surface defined by the equation:*

$$z = x^2$$

Solution. This equation represents a parabolic cylinder. For any fixed value of y, the cross-section in the xz-plane is a parabola defined by $z = x^2$. The surface extends infinitely along the y-axis, forming a cylinder-like shape. Consider the Figure 1.3 below, which illustrates the parabolic cylinder defined by the equation $z = x^2$. If we take cross-sections at different values of y, we obtain parabolas that open upwards in the xz-plane.

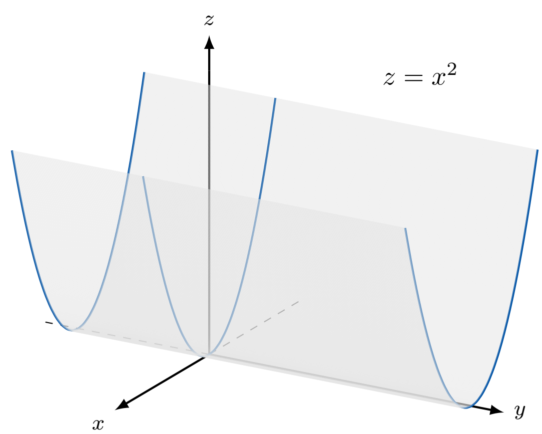


Figure 1.3: Parabolic Cylinder of $z = x^2$

Example 1.4. *Sketch the graph of the surface defined by the equation:*

$$x^2 + y^2 = 1$$

Solution. This equation represents a circular cylinder. For any fixed value of z, the cross-section in the xy-plane is a circle defined by $x^2 + y^2 = 1$. The surface extends infinitely along the z-axis, forming a cylinder-like shape. Consider the Figure 1.4 below, which illustrates the circular cylinder defined by the equation $x^2 + y^2 = 1$. If we take cross-sections at different values of z, we obtain circles in the xy-plane.

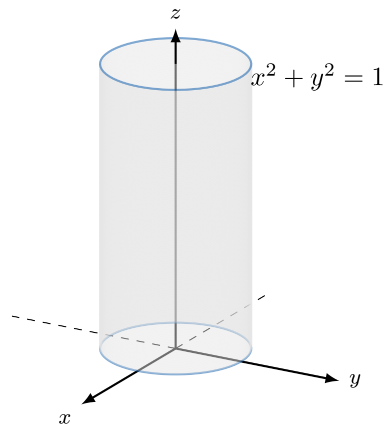


Figure 1.4: Circular Cylinder of $x^2 + y^2 = 1$

1.3.2 Quadric Surfaces

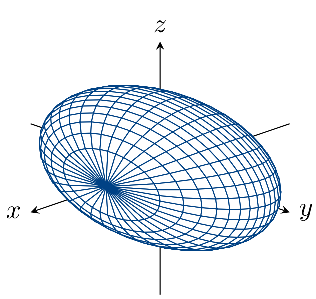
A quadric surface is a surface in three-dimensional space defined by a second-degree polynomial equation in three variables x, y, and z. The general form of a quadric surface equation is:

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0.$$

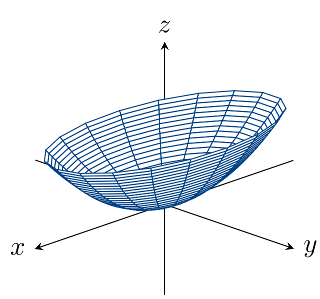
By simple translation or rotations, it can be brought into one of the following forms:

$$Ax^2 + By^2 + Cz^2 + J = 0, \quad Ax^2 + By^2 + Iz = 0$$

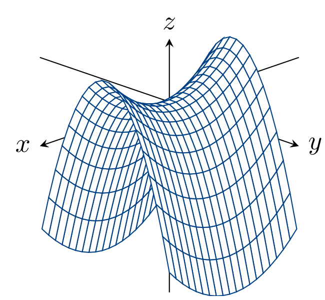
There are 6 kinds of quadric surfaces, as shown below:



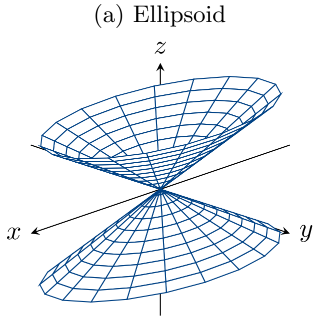
(a) Ellipsoid



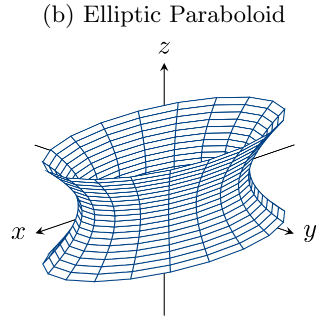
(b) Elliptic Paraboloid



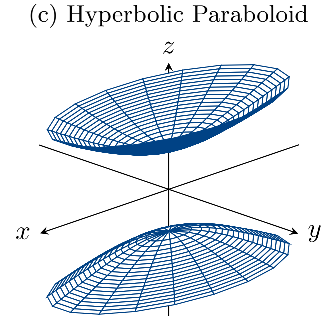
(c) Hyperbolic Paraboloid



(d) Cone



(e) Hyperboloid of One Sheet



(f) Hyperboloid of Two Sheets

1.4 Vector Functions

A vector function is a function that takes one or more variables and returns a vector. In three-dimensional space, a vector function can be represented as:

$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$$

The limit of the vector function $\mathbf{r}(t)$ as t approaches t_0 is defined as:

$$\lim_{t \to t_0} \mathbf{r}(t) = \left\langle \lim_{t \to t_0} x(t), \lim_{t \to t_0} y(t), \lim_{t \to t_0} z(t) \right\rangle$$

The derivatives of the vector function $\mathbf{r}(t)$ is defined as:

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} = \langle x'(t), y'(t), z'(t) \rangle$$

There are some properties for derivatives of vector functions:

- $\frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$
- $\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$
- $\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$
- $\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$
- $\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$
- $\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$

The definite integral of vector functions $\mathbf{r}(t)$ from a to b is defined as:

$$\int_a^b \mathbf{r}(t)dt = \left\langle \int_a^b x(t)dt, \int_a^b y(t)dt, \int_a^b z(t)dt \right\rangle$$

Arc length:

$$L = \int_a^b \|\mathbf{r}'(t)\|dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}dt$$

Arc length parametrisation:
Given a curve $\mathbf{r}(t)$, compute the integral:

$$s = s(t) = \int_a^t \|\mathbf{r}'(\tau)\|d\tau$$

Then express t as a function of s, i.e., $t = t(s)$. Finally replace all t in $\mathbf{r}(t)$ as $\mathbf{r}(t(s))$, a function in terms of s.

Example 1.5. *Find the arc-length parametrisation of the curve:*

$$\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle, \qquad t \in [0, 2\pi].$$

Solution. We have:

$$\|\mathbf{r}'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2} = \sqrt{2}.$$

So,

$$s = \int_0^t \sqrt{2} d\tau = \sqrt{2}t.$$

Express t in terms of s, we get $t = \frac{s}{\sqrt{2}}$. Replace all t's in $\mathbf{r}(t)$, we have the arc-length parametrisation:

$$\tilde{\mathbf{r}}(s) = \left\langle \cos\left(\frac{s}{\sqrt{2}}\right), \sin\left(\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}} \right\rangle, \qquad s \in [0, 2\pi\sqrt{2}].$$

Chapter 2

Partial Derivatives

2.1 Functions of Several Variables

For a function of two variables $z = f(x, y)$, the domain is a subset of the xy-plane, and the range is a subset of the z-axis. The graph of the function is a surface in three-dimensional space defined by the set of points (x, y, z) such that $z = f(x, y)$.

We can consider the "natural domain" of the function, which is the largest possible domain on $\mathbb{R}^n$ for which the function is defined for n variable functions. For example, the natural domain of the function $f(x, y) = \sqrt{9 - x^2 - y^2}$ is the disk defined by $x^2 + y^2 \leq 9$. It is to find the largest possible domain on $\mathbb{R}^2$ such that the expression under the square root is non-negative. Then the natural domain is:

$$D = \{(x, y) \mid x^2 + y^2 \leq 9\}$$

2.2 Level Sets

Instead of visualising the graph of a function of two variables in three-dimensional space, we can also visualise the function using level curves (or contour curves). A level set of a function $f : \mathbb{R}^n \to \mathbb{R}$ is a subset of the domain where the function takes on a constant value. For a function of two variables $z = f(x, y)$, the level curves are defined by the equation:

$$f(x, y) = k$$

Given $f(x, y) = x^2 + y^2$, an example of level curves is $x^2 + y^2 = 1$, which is the unit circle on $\mathbb{R}^2$ centered at the origin. The level set diagram of the two variables function consists of some representative level sets of function on $\mathbb{R}^2$. The level set diagram of the function $f(x, y) = x^2 + y^2$ is shown in Figure 2.1.

2.3 Limit and Continuity

Definition 2.1 (Limits)**.** *The limit of a function of two variables $f(x, y)$ as (x, y) approaches (x_0, y_0) is L and we write*

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L.$$

if for every $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $0 < \|\vec{x} - \vec{x_0}\| < \delta$, it follows that $|f(\vec{x}) - L| < \epsilon$.

Example 2.1. *Show that the limit below does not exists:*

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}.$$

Solution. Let $f(x, y) = \frac{x^2-y^2}{x^2+y^2}$. We will approach the point $(0, 0)$ along two different paths: x-axis and y-axis.

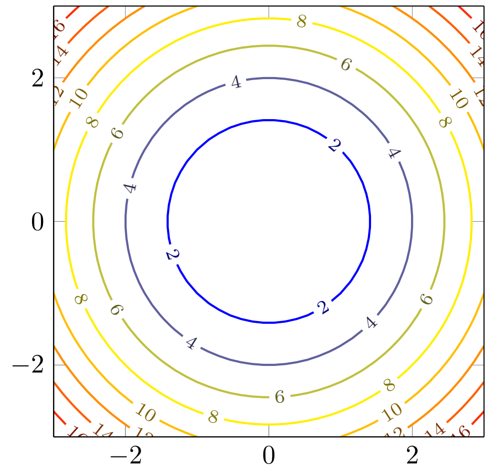


Figure 2.1: Level Sets of $f(x,y) = x^2 + y^2$

- Along the x-axis ($y = 0$):

$$f(x,0) = \frac{x^2 - 0^2}{x^2 + 0^2} = \frac{x^2}{x^2} = 1.$$

Thus,

$$\lim_{x \to 0} f(x,0) = 1.$$

- Along the y-axis ($x = 0$):

$$f(0,y) = \frac{0^2 - y^2}{0^2 + y^2} = \frac{-y^2}{y^2} = -1.$$

Thus,

$$\lim_{y \to 0} f(0,y) = -1.$$

Since the limits along the two different paths are not equal (1 and -1), the limit $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Example 2.2. *Does the limit below exist? If it exists, find the limit.*

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2}.$$

Solution. Although approaching along the x-axis and y-axis both give the limit 0, we need to check other paths to confirm the existence of the limit.

Let's approach the point $(0,0)$ along the line $y = mx$, where m is a constant. Substituting $y = mx$ into the function, we have:

$$f(x,mx) = \frac{x(mx)}{x^2 + (mx)^2} = \frac{mx^2}{x^2 + m^2x^2} = \frac{mx^2}{x^2(1 + m^2)} = \frac{m}{1 + m^2}.$$

As $x \to 0$, the expression $\frac{m}{1+m^2}$ remains constant and depends on the value of m. Since the limit depends on the slope m of the line we choose to approach $(0,0)$, the limit does not exist.

Example 2.3. *Find the limit below, if it exists:*

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2 + y^2}.$$

Solution. Let $\epsilon > 0$. We need to find a $\delta > 0$ such that whenever $0 < \sqrt{x^2+y^2} < \delta$, it follows that

$$\left|\frac{3x^2y}{x^2+y^2} - 0\right| < \epsilon \Leftrightarrow \frac{3x^2|y|}{x^2+y^2} < \epsilon.$$

Note that $x^2 \leq x^2 + y^2$, so we have

$$\frac{3x^2|y|}{x^2+y^2} \leq 3|y| = 3\sqrt{y^2} \leq 3\sqrt{x^2+y^2}.$$

Thus, we choose $\delta = \frac{\epsilon}{3}$. Then, whenever $0 < \sqrt{x^2+y^2} < \delta$, we have

$$\frac{3x^2|y|}{x^2+y^2} \leq 3\sqrt{x^2+y^2} < 3 \cdot \frac{\epsilon}{3} = \epsilon.$$

Therefore, the limit is:

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0.$$

If we drop the condition that $0 < \|\vec{x} - \vec{x_0}\|$, we get the definition of continuity.

Definition 2.2 (Continuity)**.** *A function $f(x, y)$ is continuous at the point (x_0, y_0) if for every $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $\|\vec{x} - \vec{x_0}\| < \delta$, it follows that $|f(\vec{x}) - f(\vec{x_0})| < \epsilon$.*

Note that any polynomial function of several variables is continuous everywhere in its domain.

2.4 Partial Derivatives

Definition 2.3 (Partial Derivatives)**.** *The partial derivative of a function $f(x, y)$ with respect to x at the point (x_0, y_0) is defined as:*

$$f_x(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$$

Similarly, the partial derivative of $f(x, y)$ with respect to y at the point (x_0, y_0) is defined as:

$$f_y(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0, y_0+h) - f(x_0, y_0)}{h}$$

If we let (x_0, y_0) be any point in the domain of $f(x, y)$, then the partial derivatives $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ represent the rates of change of the function $f(x, y)$ in the x and y directions, respectively, at that point. We have the following notations for partial derivatives:

$$f_x = \frac{\partial f}{\partial x} = \partial_x f = D_x f, \quad f_y = \frac{\partial f}{\partial y} = \partial_y f = D_y f.$$

For higher order partial derivatives, we can interchange the order of differentiation if the function is sufficiently smooth (i.e., the mixed partial derivatives are continuous). This is known as Clairaut's theorem or Schwarz's theorem:

$$f_{xy} = f_{yx}$$

2.5 Tangent Planes and Linear Approximations

Given a function of two variables $z = f(x, y)$, the tangent plane to the surface at the point (x_0, y_0, z_0) is given by:

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

If $z = f(x, y)$, then f is differentiable at (x_0, y_0) if the linear approximation:

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

approximates $f(x, y)$ well near the point (x_0, y_0). In other words, the function $f(x, y)$ can be approximated by its tangent plane near the point (x_0, y_0).

Differential of f:

$$df = f_x(x, y)dx + f_y(x, y)dy$$

2.6 The Chain Rule

Suppose $z = f(x, y)$, where $x = g(t)$ and $y = h(t)$ are functions of t. Then the derivative of z with respect to t is given by:

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

2.7 Directional Derivatives and Gradient Vectors

Definition 2.4 (Gradient Vector)**.** *The gradient vector of a function $f(x, y)$ is defined as:*

$$\nabla f(x, y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle f_x, f_y \rangle$$

Definition 2.5 (Directional Derivatives)**.** *The directional derivative of a function $f(x, y)$ at the point (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle u_1, u_2 \rangle$ is defined as:*

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + hu_1, y_0 + hu_2) - f(x_0, y_0)}{h}$$

Alternatively, it can be computed using the gradient vector:

$$D_{\mathbf{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \mathbf{u} = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2$$

2.8 Maximum and Minimum Values

We first find the critical points of the function by solving the system of equations:

$$f_x(x, y) = 0, \quad f_y(x, y) = 0.$$

Next, we use the second derivative test to classify the critical points. We compute the second partial derivatives:

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - (f_{xy})^2.$$

We have the following cases:

- If $D > 0$ and $f_{xx} > 0$, then f has a local minimum at the critical point.
- If $D > 0$ and $f_{xx} < 0$, then f has a local maximum at the critical point.
- If $D < 0$, then f has a saddle point at the critical point.
- If $D = 0$, the test is inconclusive.

2.9 Lagrange Multipliers

To find the extrema of a function $f(x, y)$ subject to a constraint $g(x, y) = c$, we introduce a Lagrange multiplier λ and solve the system of equations:

$$\nabla f(x, y) = \lambda \nabla g(x, y), \quad g(x, y) = c.$$

If we have more than one constraint, say $g_1(x, y, z) = c_1$ and $g_2(x, y, z) = c_2$, we introduce two Lagrange multipliers λ_1 and λ_2 and solve the system of equations:

$$\nabla f(x, y, z) = \lambda_1 \nabla g_1(x, y, z) + \lambda_2 \nabla g_2(x, y, z), \quad g_1(x, y, z) = c_1, \quad g_2(x, y, z) = c_2.$$

Chapter 3

Multiple Integrals

Chapter 4

Vector Calculus