Multivariable Calculus

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Chapter 1

Vectors and the Geometry of Space

1.1 Three-Dimensional Coordinate Systems

We would use an ordered tuple of three numbers (x, y, z) to represent a point in three-dimensional space. The three numbers correspond to the distances along the x-axis, y-axis, and z-axis respectively.

Moreover, we can use a vector to represent a point in space. A vector $\mathbf{v}$ can be expressed as:

$$\mathbf{v} = \langle x, y, z \rangle = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are the unit vectors along the x-, y-, and z-axes respectively.

Remark. Unit vectors are vectors with a magnitude of 1. They are often used to indicate direction.

The distance, or norm, of the vector $\mathbf{v}$ from the origin can be calculated using the formula:

$$\|\mathbf{v}\|_2 = \|\mathbf{v}\| = \sqrt{x^2 + y^2 + z^2}$$

This is also known as the Euclidean norm.

As we are used to consider two-dimensional planes, we always consider the following equations as circles in two-dimensional space:

$$x^2 + y^2 = r^2$$

However, in three-dimensional space, this equation represents a cylinder extending infinitely along the z-axis. As implicitly, the equation does not restrict the value of z. Then the set of points satisfying the equation forms a cylinder.

In two-dimensional case, the set of points satisfying the equation $x^2 + y^2 = r^2$ represents a circle of radius r centered at the origin:

$$S^1 = \{(x, y) \mid x^2 + y^2 = r^2\}$$

In three-dimensional case, the set of points satisfying the equation $x^2 + y^2 = r^2$ represents a cylinder of radius r centered along the z-axis:

$$C = \{(x, y, z) \mid x^2 + y^2 = r^2, z \in \mathbb{R}\}$$

So if we want to represent a two-dimensional circle in three-dimensional space, we need to add an additional constraint on z. For example, the set of points satisfying the equations $x^2 + y^2 = r^2$ and $z = 0$ represents a circle of radius r in the xy-plane:

$$S^1 = \{(x, y, z) \mid x^2 + y^2 = r^2, z = 0\}$$

For vector operations, we have:

- Vector Addition: $\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$
- Scalar Multiplication: $c\mathbf{a} = \langle ca_1, ca_2, ca_3 \rangle$

Also, we have the dot product and cross product defined as:

- Dot Product: $\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$
- Cross Product: $\mathbf{a} \times \mathbf{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$

Moreover, the dot product can also be expressed in terms of the magnitudes of the vectors and the angle θ between them:

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos\theta$$

and the magnitude of the cross product can be expressed as:

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \|\mathbf{b}\| \sin\theta$$

It represents the area of the parallelogram formed by the two vectors.

If we want to project vector $\mathbf{b}$ onto vector $\mathbf{a}$, we can use the formula:

$$\text{proj}_{\mathbf{a}} \mathbf{b} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|} \right) \frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|^2} \mathbf{a}$$

The scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is given by:

$$\text{comp}_{\mathbf{a}} \mathbf{b} = \|\mathbf{b}\| \cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|}$$

For the cross product, we can use the following determinant form:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Remark. The cross product of two vectors results in a vector that is orthogonal (perpendicular) to both original vectors. The direction of the resulting vector is determined by the right-hand rule.

1.2 Lines and Planes

1.2.1 Lines

To represent a line in three-dimensional space, we can use a point and a direction vector. If we have a point $P_0(x_0, y_0, z_0)$ on the line and a direction vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, then any point $P(x, y, z)$ on the line, the vector $\overrightarrow{P_0P}$ is parallel to $\mathbf{v}$, i.e., $\overrightarrow{P_0P} = t\mathbf{v}$ for some scalar t. Then we have the parametric equations of the line as:

$$\langle x, y, z \rangle - \langle x_0, y_0, z_0 \rangle = t \langle v_1, v_2, v_3 \rangle$$

or equivalently,

$$\begin{cases} x = x_0 + t v_1 \\ y = y_0 + t v_2 \\ z = z_0 + t v_3 \end{cases}$$

which are called the *parametric equations* of the line. The t is called the *parameter* of the line.

To visualize the parametric equation of a line in 3D, consider Figure 1.1 below.

From Figure 1.1, we can also write the parametric equations as:

$$\mathbf{r}(t) = \overrightarrow{OP_0} + t\mathbf{v}$$

which is called the *vector form* of the line.

If $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ where none of v_1, v_2, v_3 is zero, we can also express the line in *symmetric form* as:

$$\frac{x - x_0}{v_1} = \frac{y - y_0}{v_2} = \frac{z - z_0}{v_3}$$

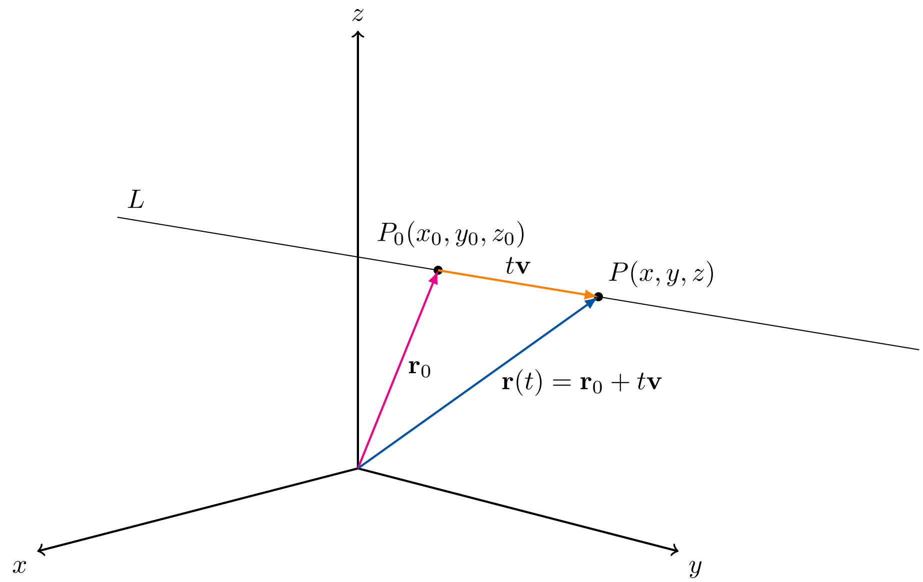


Figure 1.1: Parametric Equation of a Line in 3D

Example 1.1. *Find the parametric equations of the line that passes through the points $A(1, 2, 3)$ and $B(4, 5, 6)$. Express the line in vector form, parametric form and symmetric forms.*

Solution. In order to find the equation of the line, we need

- A point on the line: $A(1, 2, 3)$;
- A direction vector: $\mathbf{v} = \overrightarrow{AB} = \langle 4-1, 5-2, 6-3\rangle = \langle 3, 3, 3\rangle$.

Therefore, the vector form of the line is:

$$\mathbf{r}(t) = \langle 1, 2, 3\rangle + t\langle 3, 3, 3\rangle$$

The parametric form of the line is:

$$\begin{cases} x = 1 + 3t \\ y = 2 + 3t \\ z = 3 + 3t \end{cases}$$

The symmetric form of the line is:

$$\frac{x-1}{3} = \frac{y-2}{3} = \frac{z-3}{3}$$

1.2.2 Planes

A plane in three-dimensional space can be defined using a point and a normal vector. If we have a point $P_0(x_0, y_0, z_0)$ on the plane and a normal vector $\mathbf{n} = \langle A, B, C\rangle$, then any point $P(x, y, z)$ on the plane satisfies the condition that the vector $\overrightarrow{P_0P}$ is orthogonal to the normal vector $\mathbf{n}$, i.e., $\mathbf{n} \cdot \overrightarrow{P_0P} = 0$. This leads to the equation of the plane:

$$\langle A, B, C\rangle \cdot (\langle x, y, z\rangle - \langle x_0, y_0, z_0\rangle) = 0$$

or equivalently,

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

which is called the *scalar equation* of the plane.

Expanding this, we get:

$$Ax + By + Cz = Ax_0 + By_0 + Cz_0$$

or equivalently,

$$Ax + By + Cz + D = 0$$

where $D = -(Ax_0 + By_0 + Cz_0)$ is a constant. It is called a *linear equation* in x, y and z.

To visualize the equation of a plane in 3D, consider Figure 1.2 below.

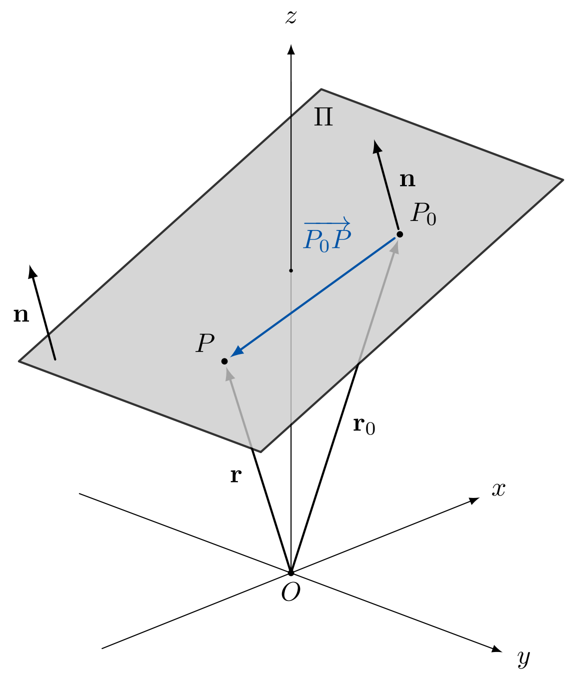


Figure 1.2: Equation of a Plane in 3D

In order to find $\mathbf{n}$, we can use the cross product.

Example 1.2. *Find the equation of the plane that passes through the points:*

$$A(1, 2, 3), \quad B(4, 5, 6), \quad C(7, 8, 0).$$

Solution. In order to find the equation of the plane, we need

- A point on the plane: $A(1, 2, 3)$;
- A normal vector: $\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC}$.

First, we calculate the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$:

$$\overrightarrow{AB} = \langle 4 - 1, 5 - 2, 6 - 3 \rangle = \langle 3, 3, 3 \rangle,$$
$$\overrightarrow{AC} = \langle 7 - 1, 8 - 2, 0 - 3 \rangle = \langle 6, 6, -3 \rangle.$$

Taking the cross product, we have:

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 3 & 3 \\ 6 & 6 & -3 \end{vmatrix} = \langle 0, 0, -9 \rangle.$$

For simplicity, we can take the normal vector as $\mathbf{n} = \langle 0, 0, 1 \rangle$. Therefore, the equation of the plane is:

$$\begin{aligned} 0(x - 1) + 0(y - 2) + 1(z - 3) &= 0 \\ z - 3 &= 0 \\ z &= 3. \end{aligned}$$

If we have a point $P_1(x_1, y_1, z_1)$ not on the plane, we can calculate the distance from the point to the plane using the formula:

$$\text{Distance} = \frac{\|\mathbf{n} \cdot \mathbf{b}\|}{\|\mathbf{n}\|} = \frac{A(x_1 - x_0) + B(y_1 - y_0) + C(z_1 - z_0)}{\sqrt{A^2 + B^2 + C^2}} = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

where $\mathbf{b} = \overrightarrow{P_0P_1} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$.

1.3 Cylinders and Quadric Surfaces

1.3.1 Cylinders

A cylinder is a surface that consists of all lines that are parallel to a given line and pass through a given curve. The given line is called the *generatrix* of the cylinder, and the given curve is called the *directrix* of the cylinder.

Example 1.3. *Sketch the graph of the surface defined by the equation:*

$$z = x^2$$

Solution. This equation represents a parabolic cylinder. For any fixed value of y, the cross-section in the xz-plane is a parabola defined by $z = x^2$. The surface extends infinitely along the y-axis, forming a cylinder-like shape. Consider the Figure 1.3 below, which illustrates the parabolic cylinder defined by the equation $z = x^2$. If we take cross-sections at different values of y, we obtain parabolas that open upwards in the xz-plane.

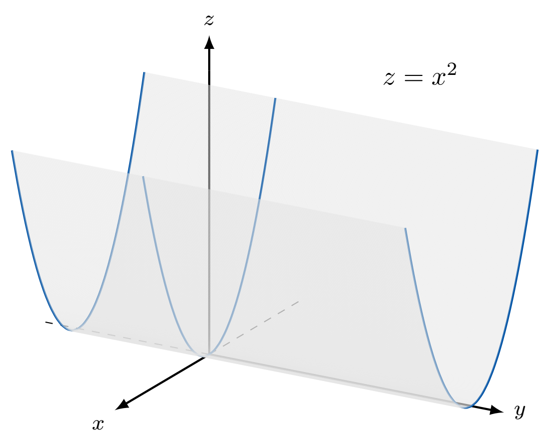


Figure 1.3: Parabolic Cylinder of $z = x^2$

Example 1.4. *Sketch the graph of the surface defined by the equation:*

$$x^2 + y^2 = 1$$

Solution. This equation represents a circular cylinder. For any fixed value of z, the cross-section in the xy-plane is a circle defined by $x^2 + y^2 = 1$. The surface extends infinitely along the z-axis, forming a cylinder-like shape. Consider the Figure 1.4 below, which illustrates the circular cylinder defined by the equation $x^2 + y^2 = 1$. If we take cross-sections at different values of z, we obtain circles in the xy-plane.

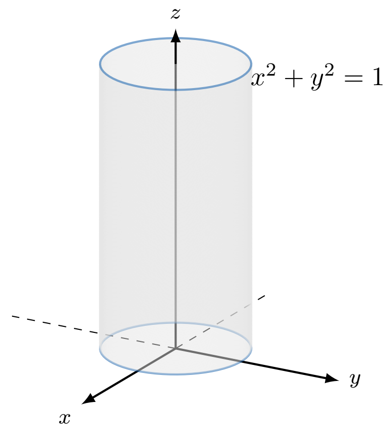


Figure 1.4: Circular Cylinder of $x^2 + y^2 = 1$

1.3.2 Quadric Surfaces

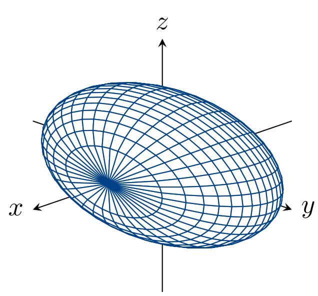
A quadric surface is a surface in three-dimensional space defined by a second-degree polynomial equation in three variables x, y, and z. The general form of a quadric surface equation is:

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0.$$

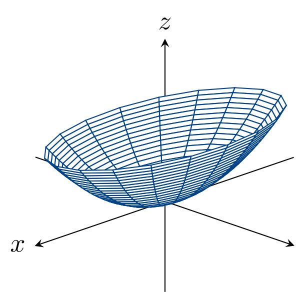
By simple translation or rotations, it can be brought into one of the following forms:

$$Ax^2 + By^2 + Cz^2 + J = 0, \quad Ax^2 + By^2 + Iz = 0$$

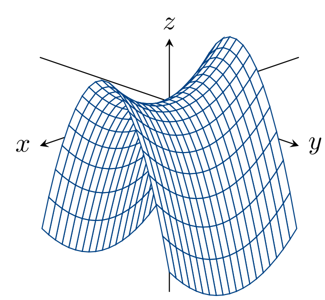
There are 6 kinds of quadric surfaces, as shown below:



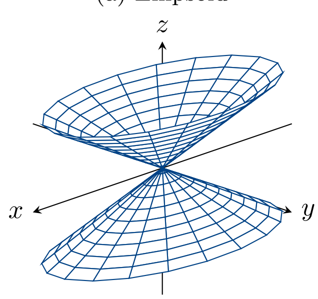
(a) Ellipsoid



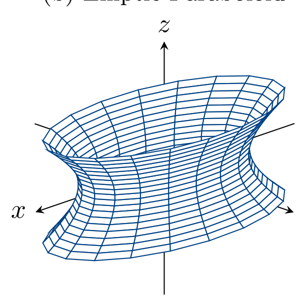
(b) Elliptic Paraboloid



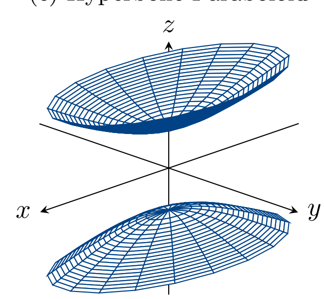
(c) Hyperbolic Paraboloid



(d) Cone



(e) Hyperboloid of One Sheet



(f) Hyperboloid of Two Sheets

1.4 Vector Functions

A vector function is a function that takes one or more variables and returns a vector. In three-dimensional space, a vector function can be represented as:

$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$$

The limit of the vector function $\mathbf{r}(t)$ as t approaches t_0 is defined as:

$$\lim_{t \to t_0} \mathbf{r}(t) = \left\langle \lim_{t \to t_0} x(t), \lim_{t \to t_0} y(t), \lim_{t \to t_0} z(t) \right\rangle$$

The derivatives of the vector function $\mathbf{r}(t)$ is defined as:

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} = \langle x'(t), y'(t), z'(t) \rangle$$

There are some properties for derivatives of vector functions:

- $\frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$
- $\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$
- $\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$
- $\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$
- $\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$
- $\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$

The definite integral of vector functions $\mathbf{r}(t)$ from a to b is defined as:

$$\int_a^b \mathbf{r}(t)dt = \left\langle \int_a^b x(t)dt, \int_a^b y(t)dt, \int_a^b z(t)dt \right\rangle$$

Arc length:

$$L = \int_a^b \|\mathbf{r}'(t)\|dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}dt$$

Arc length parametrisation:
Given a curve $\mathbf{r}(t)$, compute the integral:

$$s = s(t) = \int_a^t \|\mathbf{r}'(\tau)\|d\tau$$

Then express t as a function of s, i.e., $t = t(s)$. Finally replace all t in $\mathbf{r}(t)$ as $\mathbf{r}(t(s))$, a function in terms of s.

Example 1.5. *Find the arc-length parametrisation of the curve:*

$$\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle, \qquad t \in [0, 2\pi].$$

Solution. We have:

$$\|\mathbf{r}'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2} = \sqrt{2}.$$

So,

$$s = \int_0^t \sqrt{2} d\tau = \sqrt{2} t.$$

Express t in terms of s, we get $t = \frac{s}{\sqrt{2}}$. Replace all t's in $\mathbf{r}(t)$, we have the arc-length parametrisation:

$$\tilde{\mathbf{r}}(s) = \left\langle \cos\left(\frac{s}{\sqrt{2}}\right), \sin\left(\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}} \right\rangle, \qquad s \in [0, 2\pi\sqrt{2}].$$

Chapter 2

Partial Derivatives

Chapter 3

Multiple Integrals

3.1 Partial Integration

We have learnt how to calculate the integration of a function in single variable. Now, we extends our knowledge to functions in several variables. One should understand that the partial integration is the reverse process of partial differentiation.

Define a function $f(x, y) : \mathbb{R}^2 \to \mathbb{R}$, we have

$$\int f dx \quad \text{and} \quad \int f dy$$

Note that the above integrals are not the same as the single variable integration since f is a function of two variables. The above integrals are called **partial integrals**. In general, we have
Given a function $f : \mathbb{R}^n \to \mathbb{R}$

$$\int f dx_1, \quad \int f dx_2, \quad \ldots, \quad \int f dx_n$$

where $x_1, x_2, \ldots, x_n$ are the variables of integration.

Example 3.1. *Given a function $f(x, y) = x^2y + 3xy^2$, find $\int f dx$ and $\int f dy$.*

Solution. Notice that when we integrate with respect to x, we treat y as a constant. So as the other way around. Thus,

$$\int x^2y + 3xy^2 dx = \frac{y}{3}x^3 + \frac{3y^2}{2}x^2 + C(y)$$
$$\int x^2y + 3xy^2 dy = \frac{x^2}{2}y^2 + xy^3 + C(x)$$

The integration constants $C(y)$ and $C(x)$ in this case are functions in x and y rather than just a constant number.

Example 3.2. *Given $f(x, y) = ye^{xy^2}$, find $\int f dx$ and $\int f dy$.*

Solution.

$$\int ye^{xy^2} dx = \frac{e^{xy^2}}{y} + C(y)$$
$$\int ye^{xy^2} dy = \frac{1}{2x}e^{xy^2} + C(x)$$

We can substitute $u = xy^2$, then $du = y^2dx$ and $du = 2xydy$ to compute the integrals.

3.2 Definite integration

The concept here is similar to the single variable definite integration. We define the definite partial integral of $f(x, y)$ with respect to x from a to b as

$$\int_a^b f(x,y)dx = \int_{x=a}^{x=b} f(x,y)dx = F(b,y) - F(a,y)$$

Similarly, we may define the definite partial integral of $f(x, y)$ with respect to y from c to d as

$$\int_c^d f(x,y)dy = \int_{y=c}^{y=d} f(x,y)dy = G(x,d) - G(x,c)$$

Note that y and x are treated as constants in the above two definitions respectively.

Example 3.3. *Given* $f(x, y) = x^2y + 3xy^2$, *find* $\int_1^3 f(x, y)dx$ *and* $\int_1^3 f(x, y)dy$.

Solution.

$$\int_1^3 (x^2y + 3xy^2)dx = \left[\frac{y}{3}x^3 + \frac{3y^2}{2}x^2\right]_{x=1}^{x=3} = \frac{26}{3}y + 12y^2$$

$$\int_1^3 (x^2y + 3xy^2)dy = \left[\frac{x^2}{2}y^2 + xy^3\right]_{y=1}^{y=3} = 4x^2 + 26x$$

3.3 Double Integrals

A double integral is an extension of the single variable definite integral to functions of two variables. It is used to calculate the volume under a surface defined by a function $f(x, y)$ over a rectangular region in the xy-plane. The double integral of $f(x, y)$ over the rectangular region $R = [a, b] \times [c, d]$ is denoted as:

$$\iint_R f(x,y)dA = \int_a^b \int_c^d f(x,y)dydx$$

where dA represents a small area element on the xy-plane.

Chapter 4

Vector Calculus