Multivariable Calculus

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Contents

Chapter 1

Vectors and the Geometry of Space

1.1 Three-Dimensional Coordinate Systems

We would use an ordered tuple of three numbers (x, y, z) to represent a point in three-dimensional space. The three numbers correspond to the distances along the x-axis, y-axis, and z-axis respectively.

Moreover, we can use a vector to represent a point in space. A vector $\mathbf{v}$ can be expressed as:

$$\mathbf{v} = \langle x, y, z \rangle = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are the unit vectors along the x-, y-, and z-axes respectively.

Remark. Unit vectors are vectors with a magnitude of 1. They are often used to indicate direction.

The distance, or norm, of the vector $\mathbf{v}$ from the origin can be calculated using the formula:

$$\|\mathbf{v}\|_2 = \|\mathbf{v}\| = \sqrt{x^2 + y^2 + z^2}$$

This is also known as the Euclidean norm.

As we are used to consider two-dimensional planes, we always consider the following equations as circles in two-dimensional space:

$$x^2 + y^2 = r^2$$

However, in three-dimensional space, this equation represents a cylinder extending infinitely along the z-axis. As implicitly, the equation does not restrict the value of z. Then the set of points satisfying the equation forms a cylinder.

In two-dimensional case, the set of points satisfying the equation $x^2 + y^2 = r^2$ represents a circle of radius r centered at the origin:

$$S^1 = \{(x, y) \mid x^2 + y^2 = r^2\}$$

In three-dimensional case, the set of points satisfying the equation $x^2 + y^2 = r^2$ represents a cylinder of radius r centered along the z-axis:

$$C = \{(x, y, z) \mid x^2 + y^2 = r^2, z \in \mathbb{R}\}$$

So if we want to represent a two-dimensional circle in three-dimensional space, we need to add an additional constraint on z. For example, the set of points satisfying the equations $x^2 + y^2 = r^2$ and $z = 0$ represents a circle of radius r in the xy-plane:

$$S^1 = \{(x, y, z) \mid x^2 + y^2 = r^2, z = 0\}$$

For vector operations, we have:

- Vector Addition: $\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$
- Scalar Multiplication: $c\mathbf{a} = \langle ca_1, ca_2, ca_3 \rangle$

Also, we have the dot product and cross product defined as:

- Dot Product: $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$
- Cross Product: $\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$

Moreover, the dot product can also be expressed in terms of the magnitudes of the vectors and the angle θ between them:

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\| \cos\theta$$

and the magnitude of the cross product can be expressed as:

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\| \sin\theta$$

It represents the area of the parallelogram formed by the two vectors.

If we want to project vector $\mathbf{b}$ onto vector $\mathbf{a}$, we can use the formula:

$$\text{proj}_{\mathbf{a}}\mathbf{b} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|}\right) \frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|^2}\mathbf{a}$$

The scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is given by:

$$\text{comp}_{\mathbf{a}}\mathbf{b} = \|\mathbf{b}\| \cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|}$$

For the cross product, we can use the following determinant form:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Remark. The cross product of two vectors results in a vector that is orthogonal (perpendicular) to both original vectors. The direction of the resulting vector is determined by the right-hand rule.

1.2 Lines and Planes

1.2.1 Lines

To represent a line in three-dimensional space, we can use a point and a direction vector. If we have a point $P_0(x_0, y_0, z_0)$ on the line and a direction vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, then any point $P(x, y, z)$ on the line, the vector $\overrightarrow{P_0P}$ is parallel to $\mathbf{v}$, i.e., $\overrightarrow{P_0P} = t\mathbf{v}$ for some scalar t. Then we have the parametric equations of the line as:

$$\langle x, y, z \rangle - \langle x_0, y_0, z_0 \rangle = t\langle v_1, v_2, v_3 \rangle$$

or equivalently,

$$\begin{cases} x = x_0 + tv_1 \\ y = y_0 + tv_2 \\ z = z_0 + tv_3 \end{cases}$$

which are called the *parametric equations* of the line. The t is called the *parameter* of the line.

To visualize the parametric equation of a line in 3D, consider Figure 1.1 below.

From Figure 1.1, we can also write the parametric equations as:

$$\mathbf{r}(t) = \overrightarrow{OP_0} + t\mathbf{v} = \langle x_0, y_0, z_0 \rangle + t\langle v_1, v_2, v_3 \rangle$$

which is called the *vector form* of the line.

If $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ where none of v_1, v_2, v_3 is zero, we can also express the line in *symmetric form* as:

$$\frac{x - x_0}{v_1} = \frac{y - y_0}{v_2} = \frac{z - z_0}{v_3}$$

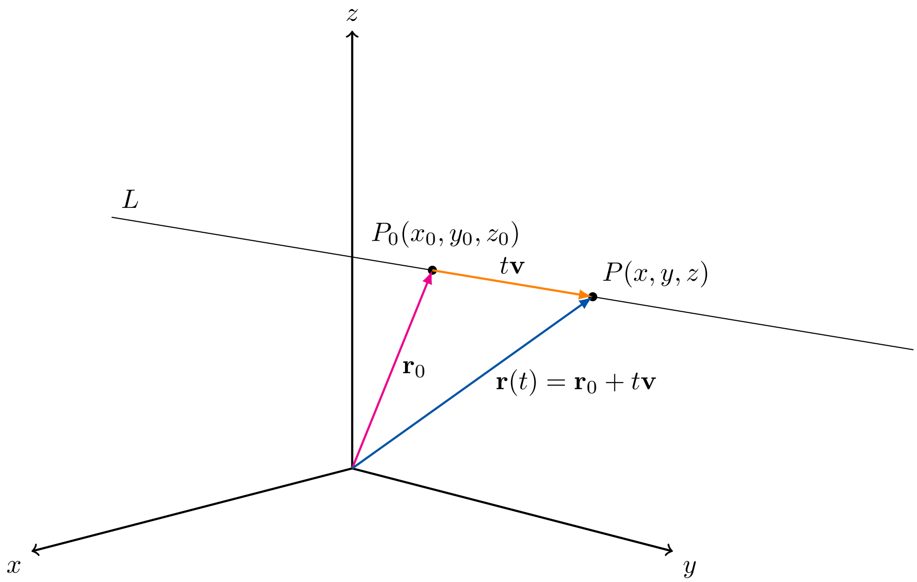


Figure 1.1: Parametric Equation of a Line in 3D

Example 1.1. *Find the parametric equations of the line that passes through the points $A(1, 2, 3)$ and $B(4, 5, 6)$. Express the line in vector form, parametric form and symmetric forms.*

Solution. In order to find the equation of the line, we need

- A point on the line: $A(1, 2, 3)$;
- A direction vector: $\mathbf{v} = \overrightarrow{AB} = \langle 4-1, 5-2, 6-3\rangle = \langle 3, 3, 3\rangle$.

Therefore, the vector form of the line is:

$$\mathbf{r}(t) = \langle 1, 2, 3\rangle + t\langle 3, 3, 3\rangle$$

The parametric form of the line is:

$$x = 1 + 3t, \quad y = 2 + 3t, \quad z = 3 + 3t.$$

The symmetric form of the line is:

$$\frac{x-1}{3} = \frac{y-2}{3} = \frac{z-3}{3}$$

$\square$

Example 1.2. *Find the parametric equations for the line passes through the point $P(0, 1, 2)$ that is perpendicular to and intersects the line*

$$x = 1 + t, \quad y = 1 - t, \quad z = 2t.$$

Solution. We can assume the point of intersection is $Q(1 + t_0, 1 - t_0, 2t_0)$. The vector $\overrightarrow{PQ}$ is perpendicular to the direction vector of the given line $\mathbf{v} = \langle 1, -1, 2\rangle$. Therefore, we have:

$$\begin{aligned} \overrightarrow{PQ} \cdot \mathbf{v} &= 0 \\ \langle (1 + t_0) - 0, (1 - t_0) - 1, 2t_0 - 2\rangle \cdot \langle 1, -1, 2\rangle &= 0 \\ t_0 &= \frac{1}{2}. \end{aligned}$$

So the direction vector of the line we want is:

$$\overrightarrow{PQ} = \left\langle 1 + \frac{1}{2}, 1 - \frac{1}{2} - 1, 2 \cdot \frac{1}{2} - 2 \right\rangle = \left\langle \frac{3}{2}, -\frac{1}{2}, -1 \right\rangle.$$

Then we take the direction vector as $\langle 3, -1, -2 \rangle$. Therefore, the parametric equations of the line is:

$$x = 3t, \quad y = 1 - t, \quad z = 2 - 2t.$$

□

There are 4 types of lines in 3D space:

- Intersecting Lines: Two lines that intersect at a single point.
- Parallel Lines: Two lines that never intersect and are always the same distance apart.
- Skew Lines: Two lines that do not intersect and are not parallel. They exist in different planes.
- Coincident Lines: Two lines that lie on top of each other, meaning they have all points in common.

Example 1.3. *Find the distance from the point P_0 to the straight line L that passes through the point P_1 with the non-zero direction vector $\mathbf{v}$.*

Solution. Let $\mathbf{r_0}$ and $\mathbf{r_1}$ be the position vectors of the points P_0 and P_1 respectively. Let the point P_2 on the line L such that $\overrightarrow{P_0P_2}$ is perpendicular to the direction vector $\mathbf{v}$. Then the distance from the point P_0 to the line L is given by the length of the vector $\overrightarrow{P_0P_2}$. We have:

$$\text{Distance} = \|\overrightarrow{P_0P_2}\| = \|\overrightarrow{P_0P_1}\| \sin\theta$$

where θ is the angle between the vectors $\overrightarrow{P_0P_1}$ and $\mathbf{v}$. Using the definition of the cross product, we have:

$$\|\overrightarrow{P_0P_1} \times \mathbf{v}\| = \|\overrightarrow{P_0P_1}\|\|\mathbf{v}\| \sin\theta.$$

Hence, the distance from the point P_0 to the line L is given by:

$$\text{Distance} = \frac{\|\overrightarrow{P_0P_1} \times \mathbf{v}\|}{\|\mathbf{v}\|} = \frac{\|(\mathbf{r_1} - \mathbf{r_0}) \times \mathbf{v}\|}{\|\mathbf{v}\|}.$$

□

Example 1.4. *Find the distance between the two lines L_1 through point P_1 parallel to direction vector $\mathbf{v_1}$ and L_2 through point P_2 parallel to direction vector $\mathbf{v_2}$.*

Solution. Consider Figure 1.2. Let $\mathbf{r_1}$ and $\mathbf{r_2}$ be the position vectors of the points P_1 and P_2 respectively. Let $\mathbf{n} = \mathbf{v_1} \times \mathbf{v_2}$ be a vector orthogonal to both direction vectors $\mathbf{v_1}$ and $\mathbf{v_2}$. Then we take the vector $\overrightarrow{P_1P_2} = \mathbf{r_2} - \mathbf{r_1}$. The distance between the two lines L_1 and L_2 is given by the length of the projection of the vector $\overrightarrow{P_1P_2}$ onto the vector $\mathbf{n}$. We have:

$$\text{Distance} = \|\text{proj}_{\mathbf{n}} \overrightarrow{P_1P_2}\| = \frac{|\overrightarrow{P_1P_2} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{|(\mathbf{r_2} - \mathbf{r_1}) \cdot (\mathbf{v_1} \times \mathbf{v_2})|}{\|\mathbf{v_1} \times \mathbf{v_2}\|}.$$

□

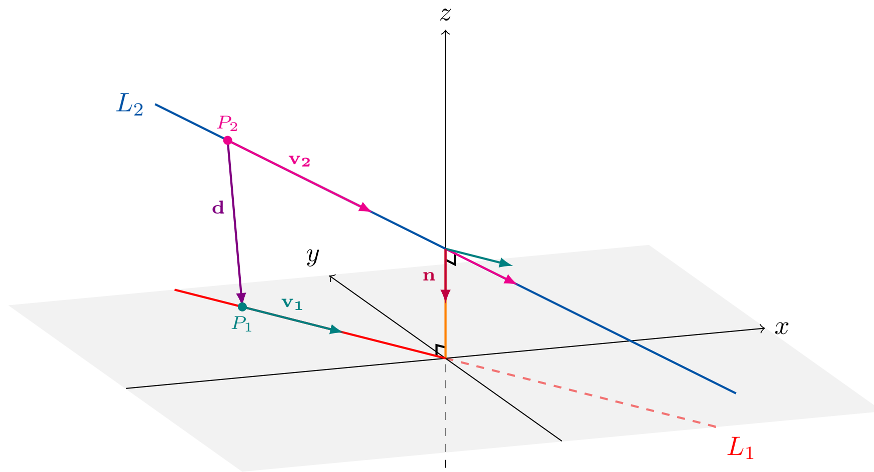


Figure 1.2: Skew Lines in 3D Space

1.2.2 Planes

A plane in three-dimensional space can be defined using a point and a normal vector. If we have a point $P_0(x_0, y_0, z_0)$ on the plane and a normal vector $\mathbf{n} = \langle A, B, C \rangle$, then any point $P(x, y, z)$ on the plane satisfies the condition that the vector $\overrightarrow{P_0P}$ is orthogonal to the normal vector $\mathbf{n}$, i.e., $\mathbf{n} \cdot \overrightarrow{P_0P} = 0$. This leads to the equation of the plane:

$$\langle A, B, C \rangle \cdot (\langle x, y, z \rangle - \langle x_0, y_0, z_0 \rangle) = 0$$

or equivalently,

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

which is called the *scalar equation* of the plane.

Expanding this, we get:

$$Ax + By + Cz = Ax_0 + By_0 + Cz_0$$

or equivalently,

$$Ax + By + Cz + D = 0$$

where $D = -(Ax_0 + By_0 + Cz_0)$ is a constant. It is called a *linear equation* in x, y and z.

To visualize the equation of a plane in 3D, consider Figure 1.3 below.

In order to find $\mathbf{n}$, we can use the cross product.

Example 1.5. *Find the equation of the plane that passes through the points:*

$$A(1, 2, 3), \quad B(4, 5, 6), \quad C(7, 8, 0).$$

Solution. In order to find the equation of the plane, we need

- A point on the plane: $A(1, 2, 3)$;
- A normal vector: $\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC}$.

First, we calculate the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$:

$$\overrightarrow{AB} = \langle 4 - 1, 5 - 2, 6 - 3 \rangle = \langle 3, 3, 3 \rangle,$$
$$\overrightarrow{AC} = \langle 7 - 1, 8 - 2, 0 - 3 \rangle = \langle 6, 6, -3 \rangle.$$

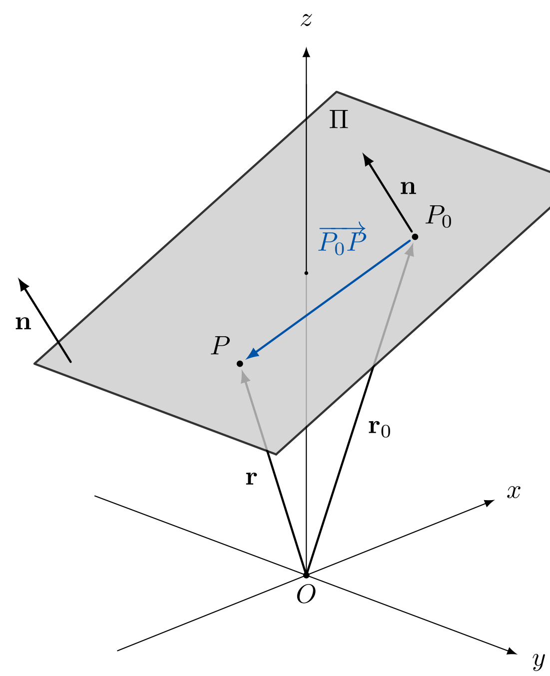


Figure 1.3: Equation of a Plane in 3D

Taking the cross product, we have:

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 3 & 3 \\ 6 & 6 & -3 \end{vmatrix} = \langle 0, 0, -9 \rangle.$$

For simplicity, we can take the normal vector as $\mathbf{n} = \langle 0, 0, 1 \rangle$. Therefore, the equation of the plane is:

$$\begin{aligned} 0(x-1) + 0(y-2) + 1(z-3) &= 0 \\ z - 3 &= 0 \\ z &= 3. \end{aligned}$$

□

If we have a point $P_1(x_1, y_1, z_1)$ not on the plane, we can calculate the distance from the point to the plane using the formula:

$$\text{Distance} = \frac{\|\mathbf{n} \cdot \mathbf{b}\|}{\|\mathbf{n}\|} = \frac{A(x_1 - x_0) + B(y_1 - y_0) + C(z_1 - z_0)}{\sqrt{A^2 + B^2 + C^2}} = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

where $\mathbf{b} = \overrightarrow{P_0P_1} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$.

Example 1.6. *Let L_1 be the line through the points $(1, 2, 6)$ and $(2, 4, 8)$. Let L_2 be the of intersection of the planes π_1 and π_2, where π_1 is the plane $x - y + 2z + 1 = 0$ and π_2 is the plane through the points $(3, 2, -1)$, $(0, 0, 1)$ and $(1, 2, 1)$. Calculate the distance between the lines L_1 and L_2.*

Solution. First, we find the direction vector of the line L_1:

$$\mathbf{v_1} = \langle 2 - 1, 4 - 2, 8 - 6 \rangle = \langle 1, 2, 2 \rangle.$$

We know that the normal vector of the plane π_1 is $\mathbf{n_1} = \langle 1, -1, 2 \rangle$. Then find two vectors on the plane π_2:

$$\begin{aligned} \overrightarrow{P_1P_2} &= \langle 0 - 3, 0 - 2, 1 - (-1) \rangle = \langle -3, -2, 2 \rangle, \\ \overrightarrow{P_1P_3} &= \langle 1 - 3, 2 - 2, 1 - (-1) \rangle = \langle -2, 0, 2 \rangle. \end{aligned}$$

Taking the cross product, we have:

$$\mathbf{n_2} = \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & -2 & 2 \\ -2 & 0 & 2 \end{vmatrix} = \langle -4, 2, -4 \rangle.$$

For simplicity, we can take the normal vector as $\mathbf{n_2} = \langle 2, -1, 2 \rangle$.

Then the direction vector of the line L_2 is perpendicular to both normal vectors of the planes π_1 and π_2. So the direction vector of the line L_2 is given by:

$$\mathbf{v_2} = \mathbf{n_1} \times \mathbf{n_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 2 \\ 2 & -1 & 2 \end{vmatrix} = \langle 0, 2, 1 \rangle.$$

Note that the point $(3, 2, -1)$ lies on two planes, so it also lies on the line L_2. Therefore, we can take the point $P_2(3, 2, -1)$ on the line L_2. We can calculate the cross product of the direction vectors:

$$\mathbf{v_1} \times \mathbf{v_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 2 \\ 0 & 2 & 1 \end{vmatrix} = \langle -2, -1, 2 \rangle.$$

Then we can calculate the distance between the two lines L_1 and L_2 using the formula:

$$\begin{aligned} \text{Distance} &= \frac{|(\mathbf{r_2} - \mathbf{r_1}) \cdot (\mathbf{v_1} \times \mathbf{v_2})|}{\|\mathbf{v_1} \times \mathbf{v_2}\|} = \frac{|\langle 3-1, 4-2, 0-6 \rangle \cdot \langle -2, -1, 2 \rangle|}{\sqrt{(-2)^2 + (-1)^2 + 2^2}} \\ &= \frac{|\langle 2, 2, -6 \rangle \cdot \langle -2, -1, 2 \rangle|}{\sqrt{9}} = \frac{|-4-2-12|}{3} = \frac{18}{3} = 6. \end{aligned}$$

□

1.3 Cylinders and Quadric Surfaces

1.3.1 Cylinders

A cylinder is a surface that consists of all lines that are parallel to a given line and pass through a given curve. The given line is called the *generatrix* of the cylinder, and the given curve is called the *directrix* of the cylinder.

Example 1.7. *Sketch the graph of the surface defined by the equation:*

$$z = x^2$$

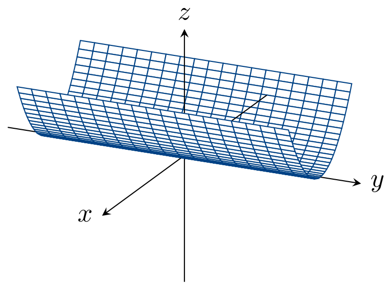
Solution. This equation represents a parabolic cylinder. For any fixed value of y, the cross-section in the xz-plane is a parabola defined by $z = x^2$. The surface extends infinitely along the y-axis, forming a cylinder-like shape. Consider the Figure 1.4a below, which illustrates the parabolic cylinder defined by the equation $z = x^2$. If we take cross-sections at different values of y, we obtain parabolas that open upwards in the xz-plane. □

Example 1.8. *Sketch the graph of the surface defined by the equation:*

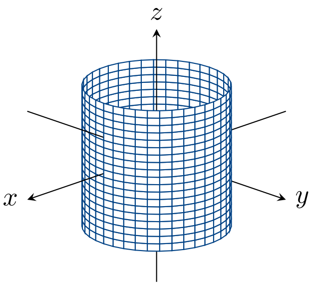
$$x^2 + y^2 = 1$$

Solution. This equation represents a circular cylinder. For any fixed value of z, the cross-section in the xy-plane is a circle defined by $x^2 + y^2 = 1$. The surface extends infinitely along the z-axis, forming a cylinder-like shape. Consider the Figure 1.4b below, which illustrates the circular cylinder defined by the equation $x^2 + y^2 = 1$. If we take cross-sections at different values of z, we obtain circles in the xy-plane. □

Figure 1.4: Cylinders in 3D Space



(a) Parabolic Cylinder of $z = x^2$



(b) Circular Cylinder of $x^2 + y^2 = 1$

1.3.2 Quadric Surfaces

A quadric surface is a surface in three-dimensional space defined by a second-degree polynomial equation in three variables x, y, and z. The general form of a quadric surface equation is:

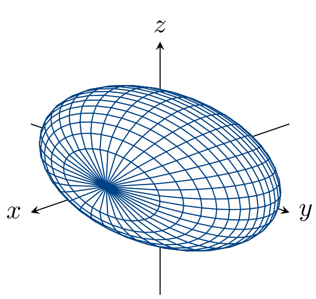
$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0.$$

By simple translation or rotations, it can be brought into one of the following forms:

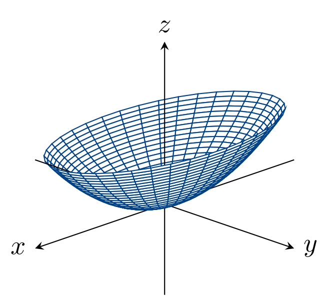
$$Ax^2 + By^2 + Cz^2 + J = 0, \quad Ax^2 + By^2 + Iz = 0$$

There are 6 kinds of quadric surfaces, as shown below:

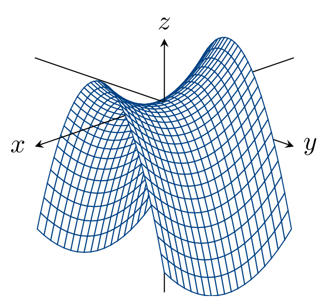
Figure 1.5: Quadric Surfaces



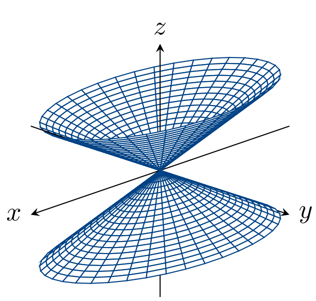
(a) Ellipsoid



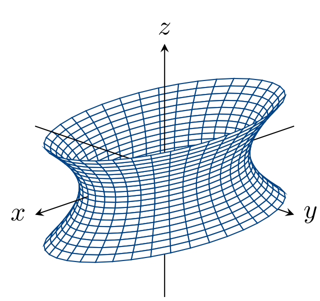
(b) Elliptic Paraboloid



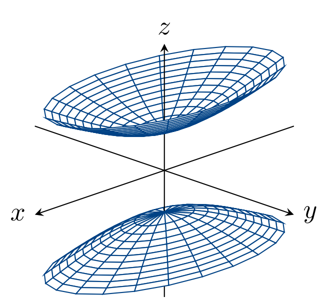
(c) Hyperbolic Paraboloid



(d) Cone



(e) Hyperboloid of One Sheet



(f) Hyperboloid of Two Sheets

1.4 Vector Functions

A vector function is a function that takes one or more variables and returns a vector. In three-dimensional space, a vector function can be represented as:

$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$$

The limit of the vector function $\mathbf{r}(t)$ as t approaches t_0 is defined as:

$$\lim_{t \to t_0} \mathbf{r}(t) = \left\langle \lim_{t \to t_0} x(t), \lim_{t \to t_0} y(t), \lim_{t \to t_0} z(t) \right\rangle$$

The derivatives of the vector function $\mathbf{r}(t)$ is defined as:

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} = \langle x'(t), y'(t), z'(t) \rangle$$

There are some properties for derivatives of vector functions:

- $\dfrac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$
- $\dfrac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$
- $\dfrac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$
- $\dfrac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$
- $\dfrac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$
- $\dfrac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$

The definite integral of vector functions $\mathbf{r}(t)$ from a to b is defined as:

$$\int_a^b \mathbf{r}(t)dt = \left\langle \int_a^b x(t)dt, \int_a^b y(t)dt, \int_a^b z(t)dt \right\rangle$$

We have the following arc length formula for a curve defined by the vector function $\mathbf{r}(t)$ from $t = a$ to $t = b$:

$$L = \int_a^b \|\mathbf{r}'(t)\|dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}dt$$

We can parametrise a curve by its arc length. The steps are as follows:
Given a curve $\mathbf{r}(t)$, compute the integral:

$$s = s(t) = \int_a^t \|\mathbf{r}'(\tau)\|d\tau$$

Then express t as a function of s, i.e., $t = t(s)$. Lastly replace all t in $\mathbf{r}(t)$ as $\mathbf{r}(t(s))$, a function in terms of s.
Note that in the arc-length parametrisation, we have $\|\tilde{\mathbf{r}}'(s)\| = 1$.

Example 1.9. *Find the arc-length parametrisation of the curve:*

$$\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle, \qquad t \in [0, 2\pi].$$

Solution. We have:

$$\|\mathbf{r}'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2} = \sqrt{2}.$$

So,

$$s = \int_0^t \sqrt{2} d\tau = \sqrt{2} t.$$

Express t in terms of s, we get $t = \frac{s}{\sqrt{2}}$. Replace all t's in $\mathbf{r}(t)$, we have the arc-length parametrisation:

$$\tilde{\mathbf{r}}(s) = \left\langle \cos\left(\frac{s}{\sqrt{2}}\right), \sin\left(\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}} \right\rangle, \qquad s \in [0, 2\pi\sqrt{2}].$$

□

Chapter 2

Partial Derivatives

2.1 Functions of Several Variables

For a function of two variables $z = f(x, y)$, the domain is a subset of the xy-plane, and the range is a subset of the z-axis. The graph of the function is a surface in three-dimensional space defined by the set of points (x, y, z) such that $z = f(x, y)$.

We can consider the "natural domain" of the function, which is the largest possible domain on $\mathbb{R}^n$ for which the function is defined for n variable functions. For example, the natural domain of the function $f(x, y) = \sqrt{9 - x^2 - y^2}$ is the disk defined by $x^2 + y^2 \leq 9$. It is to find the largest possible domain on $\mathbb{R}^2$ such that the expression under the square root is non-negative. Then the natural domain is:

$$D = \{(x, y) \mid x^2 + y^2 \leq 9\}$$

2.2 Level Sets

Instead of visualising the graph of a function of two variables in three-dimensional space, we can also visualise the function using level curves (or contour curves). A level set of a function $f : \mathbb{R}^n \to \mathbb{R}$ is a subset of the domain where the function takes on a constant value. For a function of two variables $z = f(x, y)$, the level curves are defined by the equation:

$$f(x, y) = k$$

Given $f(x, y) = x^2 + y^2$, an example of level curves is $x^2 + y^2 = 1$, which is the unit circle on $\mathbb{R}^2$ centered at the origin. The level set diagram of the two variables function consists of some representative level sets of function on $\mathbb{R}^2$. The level set diagram of the function $f(x, y) = x^2 + y^2$ is shown in Figure 2.1.

2.3 Limit and Continuity

Definition 2.1 (Limits). *The limit of a function of two variables $f(x, y)$ as (x, y) approaches (x_0, y_0) is L and we write*

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L.$$

if for every $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $0 < \|\vec{x} - \vec{x_0}\| < \delta$, it follows that $|f(\vec{x}) - L| < \epsilon$.

Example 2.1. *Show that the limit below does not exists:*

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}.$$

Solution. Let $f(x, y) = \frac{x^2-y^2}{x^2+y^2}$. We will approach the point $(0, 0)$ along two different paths: x-axis and y-axis.

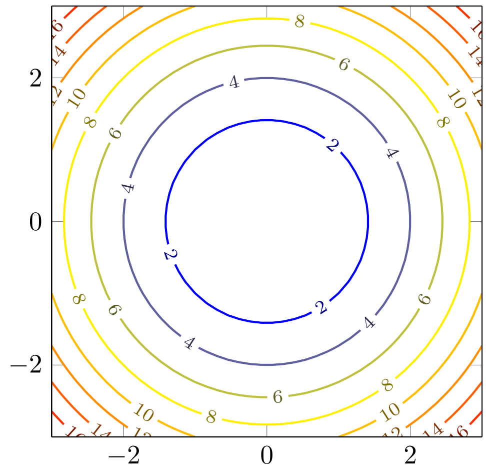


Figure 2.1: Level Sets of $f(x, y) = x^2 + y^2$

- Along the x-axis ($y = 0$):

$$f(x, 0) = \frac{x^2 - 0^2}{x^2 + 0^2} = \frac{x^2}{x^2} = 1.$$

  Thus,

$$\lim_{x \to 0} f(x, 0) = 1.$$

- Along the y-axis ($x = 0$):

$$f(0, y) = \frac{0^2 - y^2}{0^2 + y^2} = \frac{-y^2}{y^2} = -1.$$

  Thus,

$$\lim_{y \to 0} f(0, y) = -1.$$

Since the limits along the two different paths are not equal (1 and -1), the limit $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist. □

Example 2.2. *Does the limit below exist? If it exists, find the limit.*

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2}.$$

Solution. Although approaching along the x-axis and y-axis both give the limit 0, we need to check other paths to confirm the existence of the limit.

Let's approach the point $(0, 0)$ along the line $y = mx$, where m is a constant. Substituting $y = mx$ into the function, we have:

$$f(x, mx) = \frac{x(mx)}{x^2 + (mx)^2} = \frac{mx^2}{x^2 + m^2x^2} = \frac{mx^2}{x^2(1 + m^2)} = \frac{m}{1 + m^2}.$$

As $x \to 0$, the expression $\frac{m}{1+m^2}$ remains constant and depends on the value of m. Since the limit depends on the slope m of the line we choose to approach $(0, 0)$, the limit does not exist. □

Example 2.3. *Find the limit below, if it exists:*

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2 + y^2}.$$

Solution. Let $\epsilon > 0$. We need to find a $\delta > 0$ such that whenever $0 < \sqrt{x^2+y^2} < \delta$, it follows that

$$\left|\frac{3x^2y}{x^2+y^2} - 0\right| < \epsilon \Longleftrightarrow \frac{3x^2|y|}{x^2+y^2} < \epsilon.$$

Note that $x^2 \le x^2 + y^2$, so we have

$$\frac{3x^2|y|}{x^2+y^2} \le 3|y| = 3\sqrt{y^2} \le 3\sqrt{x^2+y^2}.$$

Thus, we choose $\delta = \frac{\epsilon}{3}$. Then, whenever $0 < \sqrt{x^2+y^2} < \delta$, we have

$$\frac{3x^2|y|}{x^2+y^2} \le 3\sqrt{x^2+y^2} < 3 \cdot \frac{\epsilon}{3} = \epsilon.$$

Therefore, the limit is:

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0.$$

□

We have the following properties of limits for functions of several variables:

- $\lim_{\vec{x}\to\vec{x_0}}[f(\vec{x}) + g(\vec{x})] = \lim_{\vec{x}\to\vec{x_0}} f(\vec{x}) + \lim_{\vec{x}\to\vec{x_0}} g(\vec{x})$
- $\lim_{\vec{x}\to\vec{x_0}}[cf(\vec{x})] = c\lim_{\vec{x}\to\vec{x_0}} f(\vec{x})$
- $\lim_{\vec{x}\to\vec{x_0}}[f(\vec{x})g(\vec{x})] = (\lim_{\vec{x}\to\vec{x_0}} f(\vec{x}))\,(\lim_{\vec{x}\to\vec{x_0}} g(\vec{x}))$
- $\lim_{\vec{x}\to\vec{x_0}}\left[\frac{f(\vec{x})}{g(\vec{x})}\right] = \frac{\lim_{\vec{x}\to\vec{x_0}} f(\vec{x})}{\lim_{\vec{x}\to\vec{x_0}} g(\vec{x})}$, provided that $\lim_{\vec{x}\to\vec{x_0}} g(\vec{x}) \neq 0$.
- $\lim_{\vec{x}\to\vec{x_0}}[f(\vec{x})]^q = (\lim_{\vec{x}\to\vec{x_0}} f(\vec{x}))^q$, where q is a rational number.
- $\lim_{\vec{x}\to\vec{x_0}}[f(g(\vec{x}))] = f\left(\lim_{\vec{x}\to\vec{x_0}} g(\vec{x})\right)$, provided that f is continuous at $\lim_{\vec{x}\to\vec{x_0}} g(\vec{x})$.

For the last property, functions like polynomials, exponential functions, trigonometric functions, and logarithmic functions are continuous everywhere in their domains.

If we drop the condition that $0 < \|\vec{x} - \vec{x_0}\|$, we get the definition of continuity.

Definition 2.2 (Continuity). *A function $f(x,y)$ is continuous at the point (x_0,y_0) if for every $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $\|\vec{x} - \vec{x_0}\| < \delta$, it follows that $|f(\vec{x}) - f(\vec{x_0})| < \epsilon$.*

2.4 Partial Derivatives

Definition 2.3 (Partial Derivatives). *The partial derivative of a function $f(x,y)$ with respect to x at the point (x_0,y_0) is defined as:*

$$f_x(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+h,y_0) - f(x_0,y_0)}{h}$$

Similarly, the partial derivative of $f(x,y)$ with respect to y at the point (x_0,y_0) is defined as:

$$f_y(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0,y_0+h) - f(x_0,y_0)}{h}$$

If we let (x_0,y_0) be any point in the domain of $f(x,y)$, then the partial derivatives $f_x(x_0,y_0)$ and $f_y(x_0,y_0)$ represent the rates of change of the function $f(x,y)$ in the x and y directions, respectively, at that point. We have the following notations for partial derivatives:

$$f_x = \frac{\partial f}{\partial x} = \partial_x f = D_x f, \quad f_y = \frac{\partial f}{\partial y} = \partial_y f = D_y f.$$

For higher order partial derivatives, we can interchange the order of differentiation if the function is sufficiently smooth (i.e., the mixed partial derivatives are continuous). This is known as Clairaut's theorem or Schwarz's theorem:

$$f_{xy} = f_{yx}$$

2.5 Differentiability

Definition 2.4 (Differentiability)**.** *Given a function $z = f(x, y)$. The function f is* differentiable *at (x_0, y_0) if the partial derivatives f_x and f_y exist in a neighborhood of the point (x_0, y_0) and the following equality holds:*

$$f(x, y) - L(x, y) = \epsilon_1(x, y)(x - x_0) + \epsilon_2(x, y)(y - y_0),$$

where $L(x, y)$ is the linear approximation of f at (x_0, y_0), given by this:

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0),$$

and ϵ_1 and ϵ_2 are functions such that

$$\lim_{(x,y)\to(x_0,y_0)} \epsilon_1(x, y) = 0, \qquad \lim_{(x,y)\to(x_0,y_0)} \epsilon_2(x, y) = 0,$$

2.6 The Chain Rule and Implicit Differentiation

Suppose $x = x(t)$ and $y = y(t)$ are differentiable at $t = t_0$, and $z = f(x, y)$ is a differentiable at $(x_0, y_0) = (x(t_0), y(t_0))$. Then the composite function $z = f(x(t), y(t))$ is differentiable with respect to t, and its derivative is given by:

$$\left.\frac{dz}{dt}\right|_{t=t_0} = \frac{\partial f}{\partial x}(x_0, y_0) \left.\frac{dx}{dt}\right|_{t=t_0} + \frac{\partial f}{\partial y}(x_0, y_0) \left.\frac{dy}{dt}\right|_{t=t_0} = f_x(x_0, y_0) \left.\frac{dx}{dt}\right|_{t=t_0} + f_y(x_0, y_0) \left.\frac{dy}{dt}\right|_{t=t_0}.$$

Proof. Note that from the differentiability of f, we have:

$$\begin{aligned} f(x, y) - f(x_0, y_0) &= f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + \epsilon_1(x - x_0) + \epsilon_2(y - y_0) \\ &= [f_x(x_0, y_0) + \epsilon_1](x - x_0) + [f_y(x_0, y_0) + \epsilon_2](y - y_0). \end{aligned}$$

Then we have:

$$\begin{aligned} \left.\frac{dz}{dt}\right|_{t=t_0} &= \lim_{t\to t_0} \frac{f(x(t), y(t)) - f(x_0, y_0)}{t - t_0} \\ &= \lim_{t\to t_0} \left[(f_x(x(t_0), y(t_0)) + \epsilon_1) \frac{x(t) - x(t_0)}{t - t_0} + (f_y(x(t_0), y(t_0)) + \epsilon_2) \frac{y(t) - y(t_0)}{t - t_0} \right] \\ &= f_x(x_0, y_0) \left.\frac{dx}{dt}\right|_{t=t_0} + f_y(x_0, y_0) \left.\frac{dy}{dt}\right|_{t=t_0}. \end{aligned}$$

□

More generally, if $z = f(x_1, x_2, \ldots, x_n)$ where each x_i is a function of t, then:

$$\frac{dz}{dt} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{dx_i}{dt}.$$

We can draw a tree diagram to visualise the chain rule for functions of several dependent variables with several independent variables. Two examples are shown in Figure 2.2.

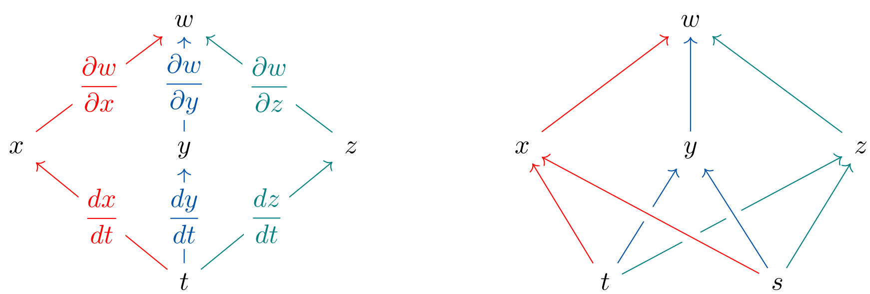
Then we can have the implicit differentiation. Suppose that $w = F(x, y)$ is differentiable and assume $F(x, y) = 0$ defines y as a differentiable function of x. Then at any point where $F_y \neq 0$, we have:

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}.$$

Proof. Let $w = F(x, y) = 0$. As y is implicitly a function of x, we can let $y = y(x)$, i.e., $F(x, y) = F(x, y(x)) = 0$. As w is a constant, then by the chain rule, we have:

$$0 = \frac{dw}{dx} = F_x + F_y \frac{dy}{dx}.$$

Rearranging the equation gives the desired result. □



(a) $w = f(x(t), y(t), z(t))$ (b) $w = f(x(t,s), y(t,s), z(t,s))$

Figure 2.2: Tree Diagrams for Chain Rule

2.7 Directional Derivatives and Gradient Vectors

Definition 2.5 (Gradient Vector)**.** *The gradient vector of a function $f(x, y)$ is defined as:*

$$\nabla f(x, y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle f_x, f_y \rangle$$

If $z = f(x, y)$ is differentiable at (x_0, y_0), then the gradient vector $\nabla f(x_0, y_0)$ is perpendicular to the level curve of f that passes through the point (x_0, y_0).

Proof. Let C be the level curve defined by $f(x, y) = k$ that passes through the point (x_0, y_0). Let $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ be a parametrisation of the curve C such that $\mathbf{r}(t_0) = (x_0, y_0)$. Then we differentiation both sides of the equation $f(x(t), y(t)) = k$ with respect to t:

$$\frac{df}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} = \langle f_x, f_y \rangle \cdot \langle x'(t), y'(t) \rangle = \nabla f(x, y) \cdot \mathbf{r}'(t) = 0.$$

Note that $\mathbf{r}'(t_0)$ is a tangent vector to the curve C at the point (x_0, y_0). Since the dot product of the gradient vector and the tangent vector is zero, it follows that the gradient vector is perpendicular to the level curve at that point. □

We have the following properties of the gradient vector:

- $\nabla(f + g) = \nabla f + \nabla g$
- $\nabla(cf) = c\nabla f$
- $\nabla(fg) = f\nabla g + g\nabla f$
- $\nabla \left(\frac{f}{g} \right) = \frac{g\nabla f - f\nabla g}{g^2}$

Definition 2.6 (Directional Derivatives)**.** *The directional derivative of a function $f(x, y)$ at the point (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle u_1, u_2 \rangle$ is defined as:*

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + hu_1, y_0 + hu_2) - f(x_0, y_0)}{h}$$

Alternatively, it can be computed using the gradient vector:

$$D_{\mathbf{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \mathbf{u} = f_x(x_0, y_0) u_1 + f_y(x_0, y_0) u_2$$

Note that the maximum and minimum values of the directional derivative occur in the direction of the gradient vector and its opposite direction, respectively, as $D_{\mathbf{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \mathbf{u}$ and $\|\mathbf{u}\| = 1$. Then we have:

$$-\|\nabla f(x_0, y_0)\| \le D_{\mathbf{u}} f(x_0, y_0) \le \|\nabla f(x_0, y_0)\|$$

Then the direction of maximum increase is called the direction of *steepest ascent*, and the direction of maximum decrease is called the direction of *steepest descent*.

Example 2.4. *Suppose that the temperature at the point (x, y, z) in space is given by*

$$T(x,y,z) = \frac{80}{1+x^2+2y^2+3z^2}.$$

In which direction does the temperature increase fastest at the point $(1, 1, -2)$? What is the maximum rate of increase?

Solution. First, we compute the gradient vector:

$$\nabla T(1,1,-2) = \frac{5}{8}\langle -1,-2,6\rangle$$

The direction of steepest ascent is in the direction of the gradient vector, i.e., $\langle -1, -2, 6\rangle$. The maximum rate of increase is the magnitude of the gradient vector:

$$\|\nabla T(1,1,-2)\| = \frac{5}{8}\sqrt{(-1)^2+(-2)^2+6^2} = \frac{5}{8}\sqrt{41}.$$

□

Example 2.5. *Find the path of the steepest ascent on the surface $f(x, y) = 20 - 4x^2 - y^2$ starting from the point $(2, -3)$.*

Solution. To find the path of steepest ascent, we need to solve the system of ordinary differential equations given by the gradient vector:

$$\nabla f(x,y) = \langle -8x, -2y\rangle.$$

Thus, we have:

$$\frac{dy}{dx} = \frac{f_y}{f_x} = \frac{-2y}{-8x} = \frac{y}{4x}.$$

Then we can separate the variables and integrate:

$$\frac{4}{y}dy = \frac{1}{x}dx \implies \ln|y|^4 = \ln|x| + C \implies y^4 = Kx,$$

where $K = e^C$ is a constant. Using the initial condition $(x, y) = (2, -3)$, we find:

$$81 = K\cdot 2 \implies K = \frac{81}{2}.$$

Therefore, the path of steepest ascent is given by:

$$y^4 = \frac{81}{2}x.$$

□

2.8 Tangent Planes and Linear Approximations

The equation of the tangent plane to the level surface $k = f(x, y, z)$ at the point (x_0, y_0, z_0) is given by:

$$\langle x-x_0, y-y_0, z-z_0\rangle \cdot \nabla f(x_0,y_0,z_0) = 0,$$

or equivalently,

$$f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0) = 0.$$

Example 2.6. *Two surfaces $x^2 + y^2 - 2 = 0$ and $x + z - 4 = 0$ intersect at a curve. Find the equation of the tangent line to the curve of intersection at the point $P_0(1, 1, 3)$.*

Solution. We first find the normal vectors of the two surfaces at the point P_0. For the first surface, we have:

$$\mathbf{n_1} = \nabla f_1(1,1,3) = \langle 2,2,0\rangle.$$

For the second surface, we have $\mathbf{n_2} = \langle 1,0,1\rangle$. Then the direction vector of the tangent line is given by the cross product of the two normal vectors:

$$\mathbf{d} = \mathbf{n_1} \times \mathbf{n_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{vmatrix} = \langle 2,-2,-2\rangle.$$

Therefore, the equation of the tangent line at the point $P_0(1,1,3)$ is given by:

$$x = 1+2t, \quad y = 1-2t, \quad z = 3-2t.$$

□

Recall that the linear approximation of a function $f(x,y)$ at the point (x_0,y_0) is given by:

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0).$$

We have the actual change in f given by:

$$\Delta f = f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0),$$

and the approximate change in f given by:

$$df = L(x_0+\Delta x, y_0+\Delta y) - L(x_0,y_0) = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y.$$

When Δx and Δy are small, df approximates Δf well.
We also have the total differential of $f(x,y,z)$ given by:

$$df = f_x dx + f_y dy + f_z dz.$$

2.9 Maximum and Minimum Values

$f(x_0,y_0)$ is a local maximum of f if there exists a neighbourhood D of (x_0,y_0) such that for all $(x,y) \in D$, we have $f(x,y) \le f(x_0,y_0)$. Similarly, for a local minimum.

If (x_0,y_0) is a local extremum of $f(x,y)$ and the partial derivatives f_x and f_y exist at (x_0,y_0), then:

$$f_x(x_0,y_0) = 0, \quad f_y(x_0,y_0) = 0.$$

Such points are called *critical points*. Note that if either f_x or f_y does not exist at (x_0,y_0), then (x_0,y_0) is also a critical point.

Proof. If f has a local extremum at (x_0,y_0), then the function $g(x) = f(x,y_0)$ has a local extremum at $x = x_0$. Hence, by single variable calculus, we have $g'(x_0) = 0$. Then we have $g'(x_0) = f_x(x_0,y_0) = 0$. Similarly, the function $h(y) = f(x_0,y)$ has a local extremum at $y = y_0$, so $h'(y_0) = f_y(x_0,y_0) = 0$. □

A differentiable function $f(x,y)$ has a saddle point at (x_0,y_0) if (x_0,y_0) is a critical point but not a local extremum, i.e., in every neighbourhood of (x_0,y_0), there exist points (x_1,y_1) and (x_2,y_2) such that $f(x_1,y_1) < f(x_0,y_0) < f(x_2,y_2)$.

To classify the critical points, we compute the second partial derivatives of f. The second derivative test uses the determinant of the Hessian matrix:

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2.$$

We have the following cases:

To find global extrema of a continuous function $f(x,y)$ on a closed and bounded region R, we follow these steps:

$\left(f_{xx}f_{yy} - f_{xy}^2\right)\big\vert_{(x_0,y_0)}$	$f_{xx}(x_0, y_0)$	(x_0, y_0) is a
$+$	$+$	local minimum
$+$	$-$	local maximum
$-$	any	saddle point
0	any	inconclusive

Table 2.1: Second Derivative Test for Functions of Two Variables

1. Find the critical points of f in the interior of R, using the second derivative test to classify them.
2. Find the maximum and minimum values of f on the boundary of R.
3. Compare all the values obtained in steps 1 and 2 to determine the global maximum and minimum.

Example 2.7. *Find the absolute maximum and minimum values of the function $f(x, y) = 2x^2 - 4x + y^2 - 4y + 1$ on the closed triangular region bounded by the lines $x = 0$, $y = 2$, and $y = 2x$ in the first quadrant.*

Solution. We first find the interior critical points by setting the first partial derivatives to zero:

$$f_x = 4x - 4 = 0 \implies x = 1, \quad f_y = 2y - 4 = 0 \implies y = 2.$$

Thus, we have one critical point at $(1, 2)$. Next, we compute the second partial derivatives:

$$f_{xx} = 4, \quad f_{yy} = 2, \quad f_{xy} = 0.$$

Then we compute the determinant of the Hessian matrix at $(1, 2)$:

$$D = f_{xx}f_{yy} - f_{xy}^2 = 4 \cdot 2 - 0^2 = 8 > 0, \quad f_{xx} = 4 > 0.$$

Therefore, $(1, 2)$ is a local minimum. Then we evaluate f at this point $f(1, 2) = -5$.

Then we check the boundary of the triangular region:

- On the line $x = 0$, we have $f(0, y) = y^2 - 4y + 1$. The endpoints are $(0, 0)$ and $(0, 2)$:
$$f(0, 0) = 1, \quad f(0, 2) = -3.$$
- On the line $y = 2$, we have $f(x, 2) = 2x^2 - 4x + 1$. The endpoints are $(0, 2)$ and $(1, 2)$:
$$f(0, 2) = -3, \quad f(1, 2) = -5.$$
- On the line $y = 2x$, we have $f(x, 2x) = 4x^2 - 4x + (2x)^2 - 4(2x) + 1 = 8x^2 - 12x + 1$. The endpoints are $(0, 0)$ and $(1, 2)$:
$$f(0, 0) = 1, \quad f(1, 2) = -5.$$

Comparing all the values, we find that the absolute maximum value is 1 at the points $(0, 0)$, and the absolute minimum value is -5 at the point $(1, 2)$. □

2.10 Lagrange Multipliers

To find the extrema of a function $f(x, y)$ subject to a constraint $g(x, y) = c$, we introduce a Lagrange multiplier λ and solve the system of equations:

$$\nabla f(x, y) = \lambda \nabla g(x, y), \quad g(x, y) = c.$$

Proof. Suppose the level curve $g(x, y) = c$ is traced out by a parametrisation $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ with $\mathbf{r}(t_0) = (x_0, y_0)$. Suppose that f has a local extremum at (x_0, y_0) subject to the constraint $g(x, y) = c$. Then we have:

$$0 = \frac{d}{dt} f(x(t), y(t))\Big|_{t=t_0} = \nabla f(x_0, y_0) \cdot \mathbf{r}'(t_0).$$

Note that $\mathbf{r}'(t_0)$ is tangent to the level curve $g(x, y) = c$ at (x_0, y_0). Since $\nabla g(x_0, y_0)$ is perpendicular to the level curve at that point, it follows that $\mathbf{r}'(t_0)$ is also perpendicular to $\nabla g(x_0, y_0)$. Therefore, both $\nabla f(x_0, y_0)$ and $\nabla g(x_0, y_0)$ are perpendicular to the same vector $\mathbf{r}'(t_0)$, which implies that they are parallel. Hence, there exists a scalar λ such that:

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0).$$

□

If we have more than one constraint, say $g_1(x, y, z) = c_1$ and $g_2(x, y, z) = c_2$, we introduce two Lagrange multipliers λ_1 and λ_2 and solve the system of equations:

$$\nabla f(x, y, z) = \lambda_1 \nabla g_1(x, y, z) + \lambda_2 \nabla g_2(x, y, z), \quad g_1(x, y, z) = c_1, \quad g_2(x, y, z) = c_2.$$

Chapter 3

Multiple Integrals

3.1 Partial Integration

We have learnt how to calculate the integration of a function in single variable. Now, we extends our knowledge to functions in several variables. One should understand that the partial integration is the reverse process of partial differentiation.

Define a function $f(x,y) : \mathbb{R}^2 \to \mathbb{R}$, we have

$$\int f dx \quad \text{and} \quad \int f dy$$

Note that the above integrals are not the same as the single variable integration since f is a function of two variables. The above integrals are called **partial integrals**. In general, we have
Given a function $f : \mathbb{R}^n \to \mathbb{R}$

$$\int f dx_1, \quad \int f dx_2, \quad \ldots, \quad \int f dx_n$$

where $x_1, x_2, \ldots, x_n$ are the variables of integration.

Example 3.1. *Given a function $f(x,y) = x^2y + 3xy^2$, find $\int f dx$ and $\int f dy$.*

Solution. Notice that when we integrate with respect to x, we treat y as a constant. So as the other way around. Thus,

$$\int x^2y + 3xy^2 dx = \frac{y}{3}x^3 + \frac{3y^2}{2}x^2 + C(y)$$
$$\int x^2y + 3xy^2 dy = \frac{x^2}{2}y^2 + xy^3 + C(x)$$

□

The integration constants $C(y)$ and $C(x)$ in this case are functions in x and y rather than just a constant number.

Example 3.2. *Given $f(x,y) = ye^{xy^2}$, find $\int f dx$ and $\int f dy$.*

Solution.

$$\int ye^{xy^2} dx = \frac{e^{xy^2}}{y} + C(y)$$
$$\int ye^{xy^2} dy = \frac{1}{2x}e^{xy^2} + C(x)$$

We can substitute $u = xy^2$, then $du = y^2dx$ and $du = 2xydy$ to compute the integrals. □

3.2 Definite integration

The concept here is similar to the single variable definite integration. We define the definite partial integral of $f(x, y)$ with respect to x from a to b as

$$\int_a^b f(x,y)dx = \int_{x=a}^{x=b} f(x,y)dx = F(b,y) - F(a,y)$$

Similarly, we may define the definite partial integral of $f(x, y)$ with respect to y from c to d as

$$\int_c^d f(x,y)dy = \int_{y=c}^{y=d} f(x,y)dy = G(x,d) - G(x,c)$$

Note that y and x are treated as constants in the above two definitions respectively.

Example 3.3. *Given* $f(x, y) = x^2y + 3xy^2$*, find* $\int_1^3 f(x, y)dx$ *and* $\int_1^3 f(x, y)dy$*.*

Solution.

$$\int_1^3 (x^2y + 3xy^2)dx = \left[\frac{y}{3}x^3 + \frac{3y^2}{2}x^2\right]_{x=1}^{x=3} = \frac{26}{3}y + 12y^2$$
$$\int_1^3 (x^2y + 3xy^2)dy = \left[\frac{x^2}{2}y^2 + xy^3\right]_{y=1}^{y=3} = 4x^2 + 26x$$

□

3.3 Double Integrals

A double integral is an extension of the single variable definite integral to functions of two variables. It is used to calculate the volume under a surface defined by a function $f(x, y)$ over a rectangular region in the xy-plane. The double integral of $f(x, y)$ over the rectangular region $R = [a, b] \times [c, d]$ is denoted as:

$$\iint_R f(x,y)dA = \int_a^b \int_c^d f(x,y)dydx$$

where dA represents a small area element on the xy-plane.

Chapter 4

Vector Calculus