In this homework assignment, we let $\mathrm{Vec}_{\mathbb{F}}^{f.d.}$ denote the category of linear maps between finite dimensional linear spaces over $\mathbb{F}$ and $V_i (i = 1, 2, 3)$ and V be objects in $\mathrm{Vec}_{\mathbb{F}}^{f.d.}$. Let $m = \dim V_1$ and $n = \dim V_2$.

1. Show that

   (a) the endo-functor $**$ on $\mathrm{Vec}_{\mathbb{F}}^{f.d.}$ that sends T to $T^{**} := (T^*)^*$ is a category isomorphism.

   (b) the set of all bilinear maps from $V_1 \times V_2$ to V_3, denoted by $\mathrm{Map}^{BL}(V_1 \times V_2, V_3)$, is a linear space.

   (c) a bilinear map $F\colon V_1 \times V_2 \to V_3$ naturally defines a linear map $F_\natural\colon V_1 \to \mathrm{Hom}(V_2, V_3)$. (where do you use the linearity of F in the 1st variable and where do you use the linearity of F in the 2nd variable?)

   (d) Show that the natural map $\natural\colon \mathrm{Map}^{BL}(V_1 \times V_2, V_3) \to \mathrm{Hom}(V_1, \mathrm{Hom}(V_2, V_3))$ that sends F to $F_\natural$ is a linear equivalence. So we write

$$\mathrm{Map}^{BL}(V_1 \times V_2, V_3) \equiv \mathrm{Hom}(V_1, \mathrm{Hom}(V_2, V_3))$$

Remark. Let $\mathcal{B}_i$ be a basis of the linear space V_i and denote by $\mathcal{B}_i^*$ the dual basis of $\mathcal{B}_i$ for the linear space V_i^*. Then a bilinear map $F\colon V_1 \times V_2 \to \mathbb{F}$ can be represented by a matrix A_F uniquely defined via the commutative diagram

$$\begin{tikzcd} V_1 \times V_2 \arrow[r, "F"] \arrow[d, "{[\,]_{\mathcal{B}_1} \times [\,]_{\mathcal{B}_2}}"'] & \mathbb{F} \\ \mathbb{F}^m \times \mathbb{F}^n \arrow[ur, "{(\vec{x},\vec{y}) \mapsto \vec{x}\cdot A_F \vec{y}}"'] & \end{tikzcd}$$

Let B_F be the matrix that represents $F_\natural$ with respect to basis $\mathcal{B}_1$ and basis $\mathcal{B}_2^*$ via the commutative diagram

$$\begin{tikzcd} V_1 \arrow[r, "F_\natural"] \arrow[d, "{[\,]_{\mathcal{B}_1}}"'] & V_2^* \arrow[d, "{[\,]_{\mathcal{B}_2^*}}"] \\ \mathbb{F}^m \arrow[r, "B_F"] & \mathbb{F}^n \end{tikzcd}$$

We say that a bilinear map $F\colon V_1 \times V_2 \to \mathbb{F}$ is non-degenerate if $F_\natural\colon V_1 \to V_2^*$ is a linear equivalence.

   (e) What is the relationship between A_F and B_F?

   (f) Show that a bilinear map $F\colon V_1 \times V_2 \to \mathbb{F}$ is non-degenerate if and only if A_F is an invertible matrix.

2. Let V be a $\mathbb{F}$-linear space of dimension $n > 0$ and V^* be its dual. Suppose that there are n elements $\underline{v}_1, \ldots, \underline{v}_n$ in V and n elements $\alpha^1, \ldots, \alpha^n$ in V^* such that $\langle \alpha^i, \underline{v}_j \rangle = \delta^i_j$. Show that $(\underline{v}_1, \ldots, \underline{v}_n)$ is a basis of V and $(\alpha^1, \ldots, \alpha^n)$ is a basis of V^*. (Each shall be called the dual basis of the other.)

3. Let $T\colon V_1 \to V_2$ be a linear map and $\mathcal{B}_i$ be a basis of linear space V_i. Denote by $\mathcal{B}_i^*$ the dual basis of $\mathcal{B}_i$ for vector space V_i^*, by $T^*\colon V_2^* \to V_1^*$ the dual map of T, i.e., the map that sends α to $\alpha \circ T$.

   (a) Show that sequence $V_1 \to V_2 \to V_3$ is exact if and only if its dual sequence $V_1^* \leftarrow V_2^* \leftarrow V_3^*$ is exact, and then conclude that T is injective if and only if T^* is surjective, and T is surjective if and only if T^* is injective.

   (b) Let A be the matrix representation of T with respect to bases $\mathcal{B}_1$, $\mathcal{B}_2$ and A^* be the matrix representation of T^* with respect to bases $\mathcal{B}_2^*$, $\mathcal{B}_1^*$. Show that A^* and A are transposes of each other.

   `Hint: It is helpful to start with commutative diagrams`

$$\begin{tikzcd} V_1 \arrow[r, "T"] \arrow[d, "{[\,]_{\mathcal{B}_1}}"'] & V_2 \arrow[d, "{[\,]_{\mathcal{B}_2}}"] \\ \mathbb{R}^n \arrow[r, "A"] & \mathbb{R}^m \end{tikzcd} \quad \texttt{and} \quad \begin{tikzcd} V_1^* \arrow[d, "{[\,]_{\mathcal{B}_1^*}}"'] & V_2^* \arrow[l, "T^*"'] \arrow[d, "{[\,]_{\mathcal{B}_2^*}}"] \\ \mathbb{R}^n & \mathbb{R}^m \arrow[l, "A^*"'] \end{tikzcd}$$

This hint shows you a general trick in linear algebra: go to the matrix representation side!

   `and then observe that the defining equation` $\langle \alpha, T(\underline{u}) \rangle = \langle T^*(\alpha), \underline{u} \rangle$ `is represented by the defining equation`

$$\vec{u} \cdot A\vec{v} = A^*\vec{u} \cdot \vec{v} \quad \texttt{or} \quad \vec{u} \cdot A\vec{v} = \vec{v} \cdot A^*\vec{u}$$

   `Take` $\vec{u} = \vec{e}_i$ `and` $\vec{v} = \vec{e}_j$`, then the identity says that the` (i, j)`-entry of` A `is equal to the` (j, i)`-entry of` A^*`.`