In this homework assignment, we let A be an order-n square matrix over field $\mathbb{F}$, and $P_A(\lambda) := \det(\lambda I - A)$. Then $P_A(\lambda)$ is a polynomial in variable λ, over the field $\mathbb{F}$, and with degree n. We call $P_A(\lambda)$ the **characteristic polynomial** of the square matrix A. By definition, roots of $P_A(\lambda)$ are called **eigenvalues** of A, so the set of roots of $P_A(\lambda)$, denoted by $\sigma(A)$, is called the set of eigenvalues of A. Note that $\sigma(A)$ can be an empty set, for example, if A is the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and the field is $\mathbb{R}$, then $\sigma(A) = \emptyset$. It is clear that $\sigma(A)$ has at most n elements, i.e., A has at most n eigenvalues.

If λ_i is an eigenvalue of A, then the nontrivial subspace $E_{\lambda_i}(A) := \mathrm{Nul}\,(\lambda_i I - A)$ is called the **eigenspace** of A with eigenvalue λ_i. ($E_{\lambda_i}(A)$ must be nontrivial, otherwise we would have $\det(\lambda_i I - A) \neq 0$, a contradiction.) Any NONZERO vector $\vec{v}$ in $E_{\lambda_i}(A)$, i.e. any vector $\vec{v}$ with $A\vec{v} = \lambda_i \vec{v}$, is called an **eigenvector** of A with eigenvalue λ_i.

We say that A is diagonalizable if there is an invertible matrix P (written as $[\vec{v}_1, ..., \vec{v}_n]$) and a diagonal matrix D (with diagonal entries d_{ii} denoted by λ_i) such that $AP = PD$, i.e., $A\vec{v}_i = \lambda_i \vec{v}_i$ for each i.

1. (a) Show that A is diagonalizable $\iff$ $\mathbb{F}^n$ has a basis consisting of eigenvectors of A.
   (b) Show that eigenspaces are linearly independent, i.e., they are all nontrivial, and there is only one way to write $\vec{0}$ as a finite sum of vectors, one from each of them.
   (c) Let $\sigma(A) = \{\lambda_1, ..., \lambda_k\}$. Show that A is diagonalizable $\iff$ $\mathbb{F}^n = \oplus_i E_{\lambda_i}(A)$ $\iff$ $n = \sum_i \dim E_{\lambda_i}(A)$.
   (d) Show that A is diagonalizable if $|\sigma(A)| = n$, i.e. A has n distinct eigenvalues.
   (e) Find a non-diagonalizble square matrix A of order 2.

2. The goal of this exercise is to give a road map for a sketchy proof of Cayley-Hamil Theorem that I mentioned in class.

   Let $f \in \mathbb{F}[x]$ be a monic polynomial of degree $n \geq 1$. Over the algebraic closure $\bar{\mathbb{F}}$ of $\mathbb{F}$, we can factorize f as $(x - x_1) \cdots (x - x_n)$ with $x_i \in \bar{\mathbb{F}}$. The discriminant of f, denoted by $\mathrm{Disc}(f)$, is defined to be $\prod_{i<j}(x_i - x_j)^2$. Being symmetric in x_1, ..., x_n, $\mathrm{Disc}(f)$ must be a polynomial in the coefficients of f.

   (a) Show that the discriminant of the quadratic polynomial $x^2 + bx + c$ is $b^2 - 4c$. How about the discriminant of the cubic polynomial $x^3 + px + q$?
   (b) Show that f has n-ditinct roots in $\bar{\mathbb{F}}$ $\iff$ $\mathrm{Disc}(f) \neq 0$.
   (c) Show that $P_A(\lambda)|_{\lambda=A} = 0$ $\iff$ $P_A(\lambda)|_{\lambda=A} = 0$ when A is viewed as a square matrix over $\bar{\mathbb{F}}$. Thus, to prove $P_A(\lambda)|_{\lambda=A} = 0$, WLOG, we shall assume in the following that the field $\mathbb{F}$ is algebraically closed.
   (d) Show that $P_A(\lambda)|_{\lambda=A} = 0$ if A is a diagonal matrix.
   (e) Show that $P_A(\lambda)|_{\lambda=A} = 0$ if A is a diagonalizable matrix.
   (f) Show that $P_A(\lambda)|_{\lambda=A} = 0$ if $\mathrm{Disc}(P_A) \neq 0$.
   (g) Show that the map $A \mapsto \mathrm{Disc}(P_A)$ is a polynomial map f from the affine space $\mathrm{End}(\mathbb{F}^n)$ to $\mathbb{F}$. Note that $\mathrm{End}(\mathbb{F}^n)$ is isomorphic to the affine space $\mathbb{A}_{\mathbb{F}}^{n^2}$ of dimension n^2.

(h) Facts: 1) In Zariski Topology, any finite dimensional affine space over an algebraically closed field is an irreducible topological space; 2) any non-empty open set U in an irreducible topological space X must be dense, i.e., X is equal to the topological closure $\bar{U}$ of U. Assume these facts and let f be the polynomial f in part (g) and f_{ij} be the polynomial map from affine space $\mathrm{End}(\mathbb{F}^n)$ to $\mathbb{F}$ that sends A to the (i,j)-entry of the matrix $P_A(\lambda)|_{\lambda=A}$. Show that $[f_{ij}(A)] = P_A(\lambda)|_{\lambda=A} = 0$ for all A on the non-empty Zariski open set $\{f \neq 0\}$ of $\mathrm{End}(\mathbb{F}^n)$.

(i) Show that all $P_A(\lambda)|_{\lambda=A} = 0$ for all A in the affine space $\mathrm{End}(\mathbb{F}^n)$. Hint: U is a subset of the closed set $\cap_{(i,j)}\{g_{ij} = 0\}$. Taking closure, we have

$$\mathrm{End}(\mathbb{F}^n) = \bar{U} \subseteq \cap_{(i,j)}\{g_{ij} = 0\} \subseteq \mathrm{End}(\mathbb{F}^n).$$

3. The goal of this exercise is to outline an elementary proof of Jordan Canonical Form.

Assume the field $\mathbb{F}$ is algebraically closed and $\sigma(A) = \{\lambda_1, ..., \lambda_k\}$. Then $P_A(\lambda) = \prod_{i=1}^k (\lambda - \lambda_i)^{n_i}$ where each integer n_i is positive. Cayley-Hamilton Theorem says that

$$\prod_{i=1}^k (A - \lambda_i I)^{n_i} = 0.$$

(a) Show that, for each i, we have a sequnce

$$\mathrm{Nul}\,(A - \lambda_i I) \subseteq \mathrm{Nul}\,(A - \lambda_i I)^2 \subseteq \cdots$$

that will eventually stabilize.

The **generalized eigenspace** of A with eigenvalue λ_i, denoted by $\tilde{E}_{\lambda_i}(A)$ is defined to be the increasing union $\cup_{k\geq 1}\mathrm{Nul}\,(A-\lambda_i I)^k$. By definition, any nonzero vector v in $\tilde{E}_{\lambda_i}(A)$ is called a **generalized eigenvector** of A with eigenvalue λ_i. Let v be a generalized eigenvector of A with eigenvalue λ.

(b) Show that there is an integer $m \geq 0$ such that $v_m := (A - \lambda I)^m v$ is an eigenvector of A with eigenvalue λ.

(c) Show that v is never a generalized eigenvector of A with eigenvalue $\mu \neq \lambda$.

(d) Show that, for any $k \geq 0$ and scalar $\mu \neq \lambda$, $(A - \mu I)^k v$ is always a generalized vector of A with eigenvalue λ. Consequently, $(A - \mu I)^k$ maps $\tilde{E}_\lambda(A)$ isomorphically onto $\tilde{E}_\lambda(A)$.

(e) Show that the algebraic multiplicity of the eigenvalue λ_i is bigger than or equal to the geometric multiplicity (i.e., $\dim \tilde{E}_\lambda(A)$) of the eigenvalue λ_i

(f) Show that the generalized eigenspaces of A are linearly independent and their direct sum is the entire linear space $\mathbb{F}^n$.

(g) Show that, with respect to the decomposition $\mathbb{F}^n = \tilde{E}_{\lambda_1}(A) \oplus \cdots \oplus \tilde{E}_{\lambda_k}(A)$, we have the decomposition $A = (\lambda_1 I_{n_1} + N_1) \oplus \cdots \oplus (\lambda_k I_{n_k} + N_k)$ where N_i is a nilpotent matrix of order n_i.

The question has been reduced to finding the canonical form of nilpotent matrices.

(h)

(i)