

Recall that a real symmetric matrix  $A$  of order  $n$  is called **positive**, written as  $A > 0$ , if the symmetric 2-form on  $\mathbb{R}^n$  with representation matrix  $A$  w.r.t. the standard bases  $\vec{e}_i$  is positive-definite, i.e., the map  $(\vec{x}, \vec{y}) \mapsto \vec{x} \cdot A \vec{y}$  is an inner product. We say  $A$  is **semi-positive-definite**, written as  $A \geq 0$ , if  $\vec{x} \cdot A \vec{x} \geq 0$  for all  $\vec{x} \in \mathbb{R}^n$ .

Recall that a basis of  $V$  is denoted by  $v_i$  and the resulting dual basis of  $V^*$  is denoted by  $\hat{v}^i$ . If  $A$  is matrix, then  $A^T$  denotes the transpose of matrix  $A$ .

1. Let  $A$  be a real symmetric  $n \times n$ -matrix and  $A_i$  be the matrix obtained from  $A$  by deleting its last  $(n - i)$  rows and last  $(n - i)$  columns. Show that

$$A > 0 \iff \det A_i > 0 \text{ for each } i$$

Hint: induction on  $n$

2. Let  $A$  be a real symmetric  $n \times n$ -matrix. Assume that  $A \geq 0$ . It is clear that  $A \geq 0 \iff A + tI > 0$  for any  $t > 0$ . Based on this observation and the result in the previous exercise to derive a necessary and sufficient condition for  $A \geq 0$ .
3. Let  $V$  be a  $n$ -dimensional linear space over a field  $\mathbb{F}$ . The pairing  $V^* \times V \rightarrow \mathbb{F}$  yields the multi-linear map  $V^* \times V^* \times V \times V \rightarrow \mathbb{F}$  that sends  $(\alpha_1, \alpha_2, v_1, v_2)$  to  $\alpha_1(v_1)\alpha_2(v_2)$ . Thus we have a linear map

$$(V^* \otimes V^*) \otimes (V \otimes V) \rightarrow \mathbb{F}$$

or equivalently a linear map  $\iota: V^* \otimes V^* \rightarrow (V \otimes V)^*$ .

- Show that the linear map  $\iota$  is a linear equivalence. In fact a natural one in the language of category. So we shall write  $V^* \otimes V^* \equiv (V \otimes V)^*$ .
- The quotient linear map  $V \otimes V \rightarrow S^2 V$  yields the injective linear map  $(S^2 V)^* \rightarrow (V \otimes V)^*$ . So we have the following composition map

$$(S^2 V)^* \hookrightarrow (V \otimes V)^* \equiv V^* \otimes V^* \twoheadrightarrow S^2 V^*.$$

Show that this natural map  $(S^2 V)^* \rightarrow S^2 V^*$  is a linear equivalence if and only if the characteristic of the field  $\mathbb{F}$  is not 2.

- Assume that the characteristic of the field  $\mathbb{F}$  is not 2, please find the inverse of the natural map in part (2).
- Assume that the characteristic of the field  $\mathbb{F}$  is not 2, show that there is a natural linear equivalence  $\wedge^2 V^* \rightarrow (\wedge^2 V)^*$ .
- Let  $\omega$  be a 2-form on  $V$ , i.e., a bilinear map  $V \times V \rightarrow \mathbb{F}$ . Then  $\omega_{\natural}: V \rightarrow V^*$  is the linear map that sends  $v$  to  $\omega(v, -)$ . Let  $v_i$  be a basis of  $V$  and the resulting dual basis of  $V^*$  be denoted by  $\hat{v}^i$ . Let  $A$  be the matrix representation of  $\omega$  w.r.t. basis  $v_i$ , i.e.  $A = [\omega(v_i, v_j)]$ . Let  $A'$  be the matrix representation of  $\omega_{\natural}$  w.r.t. bases  $v_i$  and  $\hat{v}^i$ .
  - Show that  $A'$  is the transpose of  $A$ . Thus  $\omega$  is non-degenerate means that its any matrix representation  $A$  is invertible.

- (2) If  $\tilde{v}_{\tilde{i}}$  is another basis of  $V$ , then there is a unique invertible matrix  $S = [s^i_{\tilde{j}}]$  such that  $\tilde{v}_{\tilde{j}} = v_i s^i_{\tilde{j}}$ . Let  $\tilde{A}$  be the matrix representation of  $\omega$  w.r.t. basis  $\tilde{v}_{\tilde{i}}$ . Show that  $\tilde{A} = S^T A S$ .
- (3) Show that the map that sends  $(A, S)$  to  $S^T A S$  is a right action of  $\mathrm{GL}_n(\mathbb{F})$  on  $(V \otimes V)^*$ .
- (4) The quotient map  $V \otimes V \rightarrow S^2 V$  yields injective linear map  $(S^2 V)^* \rightarrow (V \otimes V)^*$ , that is not a surprise because any symmetric 2-form is a 2-form. Similarly,  $(\wedge^2 V)^*$  is a linear subspace of  $(V \otimes V)^*$  as well. Show that  $\mathrm{GL}(V)$  acts on  $(V \otimes V)^*$  and leaves invariant both subspace  $(S^2 V)^*$  and subspace  $(\wedge^2 V)^*$ .
- (5) Show that, under the identification  $(V \otimes V)^* \equiv V^* \otimes V^*$ , we have

$$\omega = \omega(v_i, v_j) \hat{v}^i \otimes \hat{v}^j.$$

- (6) Assume that the characteristic of the field  $\mathbb{F}$  is not 2, then we have the natural identification  $(S^2 V)^* \equiv S^2 V^*$  found in the last question. Show that, if  $\omega$  is a symmetric 2-form on  $V$ , then  $\omega = \omega(v_i, v_j) \hat{v}^i \hat{v}^j$ . Similarly, if  $\omega$  is a skew-symmetric 2-form on  $V$ , then  $\omega = \omega(v_i, v_j) \hat{v}^i \wedge \hat{v}^j$ .