For a complex matrix A, we let $A^\dagger$ denote the complex matrix $\bar{A}^T$. It is clear that $(AB)^\dagger = B^\dagger A^\dagger$. We say a complex matrix A is an Hermitian matrix if $A^\dagger = A$, a unitary matrix if $A^\dagger A = I$, and a normal matrix if $A^\dagger A = AA^\dagger$.

Recall that a Hermitian matrix A of order n is a called positive, written as $A > 0$, if the Hermitian 2-form on $\mathbb{C}^n$ with representation matrix A w.r.t. the standard bases $\vec{e}_i$ is positive-definite, i.e., the map $(\vec{x}, \vec{y}) \mapsto \overline{\vec{x}} \cdot A\vec{y}$ is an Hermitian inner product. We say A is semi-positive-definite, written as $A \geq 0$, if $\overline{\vec{x}} \cdot A\vec{x} \geq 0$ for all $\vec{x} \in \mathbb{C}^n$.

For any linear map $T\colon V \to W$ between Hermitian vector spaces, there is a unique linear map $T^\dagger\colon W \to V$ such that $\langle w, Tv\rangle = \langle T^\dagger w, v\rangle$ for all $v \in V$ and $w \in W$. Once trivialization of Hermitian vector spaces are given via orthonormal bases, if T is represented by matrix A, then $T^\dagger$ is represented by the adjoint matrix $A^\dagger$. An endomorphism T on a Hermitian vector space V is called a self-adjoint operator if $T^\dagger = T$ and is called a unitary operator if $T^\dagger T = 1_V$.

Here is the basic fact: Let T be either a Hermitian or a unitary operator on V, then T can be ''diagonalised'' by a unitary matrix, that is, the matrix representation A of T with respect to any orthonormal basis of V can diagonalised by a unitary matrix in the sense that there is a unitary matrix U and a diagonal matrix D such that $A = U^\dagger D U$. Note: the diagonal entries of D are all real if A is Hermitian and all complex numbers of modulus one if A is unitary.

From last assignment we know that the matrix representation of any symmetric (skew-symmetric resp.) 2-form on a real linear space w.r.t. any basis is a real symmetric (skew-symmetric resp.) matrix. Similarly, the matrix representation of any hermitian 2-form on a complex linear space w.r.t. any basis is a hermitian matrix.

If V is a complex linear space, then its underlying real linear space is denoted by $V_\mathbb{R}$. We let J be the endomorphism $\mathrm{i}\cdot$ on $V_\mathbb{R}$, i.e., the $\mathbb{R}$-linear map that sends $v \in V_\mathbb{R}$ to $\mathrm{i}v \in V_\mathbb{R}$.

If W be a real linear space, then its complexification is a complex linear space denoted by $W_\mathbb{C}$. By definition, $W_\mathbb{C} = W \otimes_\mathbb{R} \mathbb{C}$. We shall identify W with the real linear subspace $W \otimes 1$ of $(W_\mathbb{C})_\mathbb{R}$ this way: $w \equiv w \otimes 1$.

An endomorphism J on a real linear space W such that $J^2 = -1_W$ is called a complex structure on W. Thus, if V is a complex linear space, then $J := \mathrm{i}\cdot$ is a complex structure on the real linear space $V_\mathbb{R}$.

Recall that if V is a complex linear space, then its complex conjugate $\overline{V}$ is the complex linear space such that $V_\mathbb{R} = \overline{V}_\mathbb{R}$ and the scalar multiplication αv in $\overline{V}$ is equal to the scalar multiplication $\bar{\alpha} v$ in V.

In this assignment we assume the spectral theorem for Hermitian matrices and unitary matrices as well as the Simultaneous Diagonalization Theorem for Hermitian operators.

Let V be a f.d. complex linear space. An endomorphism σ on $V_\mathbb{R}$ is called a complex conjugation on the complex linear space V if σ satisfies these two conditions: 1) $\sigma(cu) = \bar{c}\sigma(u)$ for all vectors u and all complex numbers c, 2) $\sigma^2 = 1$. If V is a f.d. complex linear space with complex conjugation σ, we let V^σ be the set of σ-invariant vectors, i.e.,

$$V^\sigma = \{v \in V \mid \sigma(v) = v\}.$$

Note that V^σ is a real linear space.

1. Let A be a complex matrix. Show that A is diagonalised by a unitary matrix $\iff$ A is a normal matrix. Hint: Write $A = B + \mathrm{i}C$ where both B and C are hermitian matrices.

2. (1) Let V be a complex linear space and J be the endomorphism i$\cdot$ on $V_{\mathbb{R}}$. Show that $J^2 = -1_{V_{\mathbb{R}}}$.
   (2) Show that if $v = (v_1, ..., v_n)$ is a basis of the complex linear space V, then $v_{\mathbb{R}} := (v_1, Jv_1, ..., v_n, Jv_n)$ is a basis of the real linear space $V_{\mathbb{R}}$. Thus $\dim_{\mathbb{R}} V_{\mathbb{R}} = 2\dim_{\mathbb{C}} V$.
   (3) Show that if $w = (w_1, ..., w_n)$ is a basis of the real linear space W, then $w \equiv w \otimes 1$ is a basis of the complex linear space $W_{\mathbb{C}}$. Thus $\dim_{\mathbb{R}} W = \dim_{\mathbb{C}} W_{\mathbb{C}}$.
   (4) Show that a complex linear space with underlying real linear space $W \equiv$ the real linear space W together with a complex structure J on W. Also, if $V := (W, J)$ is a complex linear space, then its complex conjugate $\overline{V}$ is the complex linear space $(W, -J)$.
   (5) Let V be a complex linear space, and $J \in \operatorname{End} V_{\mathbb{R}}$ be the scalar multiplication by i on V. We prefer to write $V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ as $V \otimes_{\mathbb{R}} \mathbb{C}$ or simply as $V_{\mathbb{C}}$. Note that J extends to $J_{\mathbb{C}} := J \otimes_{\mathbb{R}} 1_{\mathbb{C}}$. Show that $J_{\mathbb{C}}$ is an endomophism of the complex linear space $V \otimes_{\mathbb{R}} \mathbb{C}$ such that $J_{\mathbb{C}}^2 = -1_{V_{\mathbb{C}}}$. Note: the complex structure on $V \otimes_{\mathbb{R}} \mathbb{C}$ comes from the complex structure on $\mathbb{C}$, not from that of V.
   (6) Show that the real linear map $V \to V \otimes_{\mathbb{R}} \mathbb{C}$ that sends v to $v \otimes_{\mathbb{R}} 1 - \mathrm{J}v \otimes_{\mathbb{R}} \mathrm{i}$ is an embedding of complex linear spaces, i.e., an injective complex linear map. Let us denote the image of this embedding as V'. Then $V \equiv V'$.
   (7) Show that the real linear map $\overline{V} \to V \otimes_{\mathbb{R}} \mathbb{C}$ that sends v to $v \otimes_{\mathbb{R}} 1 + \mathrm{J}v \otimes_{\mathbb{R}} \mathrm{i}$ is an embedding of complex linear spaces. Let us denote the image of this embedding as V''. Then $\overline{V} \equiv V''$.
   (8) Show that $V \otimes_{\mathbb{R}} \mathbb{C} = V' \oplus V'' \equiv V \oplus \overline{V}$. People usually write $V_{\mathbb{C}} = V \oplus \overline{V}$.

3. We have seen that a complex linear space is nothing but a real linear space together with a complex structure J. The goal of this exercise is to enable you to see that a real linear space is nothing but a complex linear space together with a complex conjugation.

   (1) Let W be a f.d. real linear space. Let σ_{std} be the complex conjugation on $W_{\mathbb{C}}$ that sends $w \otimes c$ to $w \otimes \bar{c}$. Show that $W \equiv W_{\mathbb{C}}^{\sigma_{std}}$ under which w in W becomes $w \otimes 1$ in $W_{\mathbb{C}}^{\sigma}$.
   (2) Let V be a f.d. complex linear space with complex conjugation σ. Show that V^{σ} is a real linear space and $V^{\sigma} \otimes \mathbb{C} \equiv V$ under which σ_{std} on $V^{\sigma} \otimes \mathbb{C}$ becomes σ on V.
   (3) Let T be an endomorphism on the real linear space W. Then $T_{\mathbb{C}} := T \otimes 1_{\mathbb{C}}$ is an endomorphism on the complex linear space $W_{\mathbb{C}}$. Show that $T_{\mathbb{C}}\,\sigma_{std} = \sigma_{std}\,T_{\mathbb{C}}$.
   (4) Show that an eigenvalue of $T_{\mathbb{C}}$ is either a real number or a complex number with its complex conjugation being also an eigenvalue. Hint: $\det(\lambda 1 - T) = \det(\lambda 1 - T_{\mathbb{C}})$.

(5) Assume that $T_{\mathbb{C}}$ is "diagonalizable", then its eigenspace decomposition of $W_{\mathbb{C}}$ must be of the form

$$W_{\mathbb{C}} = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_t} \oplus E_{\mu_1} \oplus \overline{E_{\mu_1}} \oplus \cdots \oplus E_{\mu_s} \oplus \overline{E_{\mu_t}}$$

where λ_i are real numbers, μ_i are complex numbers, E_{λ_i} is the eigenspace with eigenvalue λ_i, E_{μ_i} is the eigenspace with eigenvalue μ_i and $\overline{E_{\mu_i}}$ is the complex conjugate of E_{μ_i}, thus (why?) must be the eigenspace with eigenvalue $\overline{\mu_i}$. `By convention,` $t = 0$ `means no real eigenvalues and` $s = 0$ `means no complex numbers.`

(6) Continuing the previous sub-problem, show that W has a decomposition of the form

$$W = E_{\lambda_1}(T) \oplus \cdots \oplus E_{\lambda_t}(T) \oplus E_{\mu_1}(T) \oplus \cdots \oplus E_{\mu_s}(T)$$

with respect to which we have $T = T_{\lambda_1} \oplus \cdots \oplus T_{\lambda_t} \oplus T_{\mu_1} \oplus \cdots \oplus T_{\mu_s}$; moreover, $E_{\lambda_i}(T)$ is the eigenspace of T with eigenvalue λ_i and $E_{\mu_i}(T)$ is a real vector space with a complex structure J_i such that the characteristic polynomial of T_{μ_i} is a power of the real irreducible quadratic polynomial $x^2 - (\mu_i + \overline{\mu_i})x + |\mu_i|^2$.

4. `The conclusions here are useful and are corollaries of spectral theorem for Hermitian matrices and unitary matrices. By now I hope you feel comfortable with switching between linear maps/forms and their matrix representations. You need some help from Exercise 2. In principle you don't need Exercise 3 for help because we only deal with matrices here. This shows again that the ''cheaty method'' (i.e., the method of matrix representation) is powerful.`

Let V be an Euclidean vector space. Then its complexification $V_{\mathbb{C}}$ becomes a Hermitian vector space. One way to see it is this: The Hermitian inner product is the one such that $\langle u \otimes \alpha, v \otimes \beta \rangle := \bar{\alpha}\beta \langle u, v \rangle$ for all $u, v \in V$ and all complex numbers α and β.

(1) Show that the canonical form of a Hermitian 2-form on a complex linear space is a matrix of the form $I_p \oplus -I_q \oplus [0] \oplus \cdots \oplus [0]$. Then conclude that the canonical form of a pseudo-Hermitian inner product is a matrix of the form $I_{p,q} := I_p \oplus -I_q$.

(2) Let T be a self-adjoint operator on V. Show that $T_{\mathbb{C}} := T \otimes 1_{\mathbb{C}}$ is a self-adjoint operator on $V_{\mathbb{C}}$. `In terms of matrix representations w.r.t. orthonormal bases, it says that a real symmetric matrix is a Hermitian matrix.`

(3) Show that any real symmetric matrix A can be diagonalised by an orthogonal matrix, i.e., $A = O^T D O$ where O is an orthogonal matrix and D is a real diagonal matrix.

(4) Show that the canonical form of a symmetric 2-form on a real linear space is a matrix of the form $I_p \oplus -I_q \oplus [0] \oplus \cdots \oplus [0]$. Then conclude that the canonical form of a pseudo-inner product is a matrix of the form $I_{p,q} := I_p \oplus -I_q$.

(5) Let T be an orthogonal transformation on V. Show that $T_{\mathbb{C}} =$ is a unitary transformation on $V_{\mathbb{C}}$. `In terms of matrix representations w.r.t. orthonormal bases, it says that an orthogonal matrix is a unitary matrix.`

(6) Denote by $R(\theta)$ the 2×2 rotation matrix with rotation angle θ. By definition, a special orthogonal matrix is an orthogonal matrix whose determinant is 1. Show that any special orthogonal matrix A can be factorised this way: $A = O^T \bar{A} O$ where O is an orthogonal matrix and $\bar{A}$ is a canonical special orthogonal matrix, i.e, a matrix of the form $R(\theta_1) \oplus \cdots \oplus R(\theta_k)$ for some angles θ_i if $n = 2k$ or a matrix of the form $R(\theta_1) \oplus \cdots \oplus R(\theta_k) \oplus [1]$ for some angles θ_i if $n = 2k + 1$. `In fact` O `can be chosen to be a special orthogonal matrix.`

(7) Denote by J_2 the real skew-symmetric matrix with its $(2, 1)$-entry being 1. Let A be a real skew-symmetric matrix. Show that $A = P^T \bar{A} P$ where P is an invertible matrix and $\bar{A}$ is a real skew-symmetric matrix of the form $J_2 \oplus J_2 \oplus \cdots J_2 \oplus [0] \oplus \cdots \oplus [0]$. `Hint:` $\mathrm{i}A$ `is a Hermitian matrix.` Then conclude that the canonical form of a symplectic form ω on a real linear space V is a matrix of the form $J_2 \oplus J_2 \oplus \cdots \oplus J_2$. `In this case, the basis made of the columns of` P^T `is called a symplectic basis of the symplectic space` (V, ω).