The Search for a Title

A Profound Subtitle

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Contents

Part One

1. Abstract Vector Spaces

1.1 Binary Operation

Definition 1.1.1 — Binary Operation. A *binary operation* on a set S is a mapping of the elements of the Cartesian product $S \times S$ to S.

$$\begin{aligned} f : S \times S &\to S \\ (x,y) &\mapsto f(x,y) \end{aligned}$$

■ **Example 1.1** A common example of a binary operation is addition on the set of natural numbers $\mathbb{N}$.

$$\begin{aligned} + : \mathbb{N} \times \mathbb{N} &\to \mathbb{N} \\ (x,y) &\mapsto x+y \end{aligned} \tag{1.1}$$

■

Definition 1.1.2 — Associative Operation. A binary operation $f : S \times S \to S$ is said to be *associative* if, for all $x,y,z \in S$, the following holds:

$$f(x, f(y,z)) = f(f(x,y),z)$$

■ **Example 1.2** A common example of an associative (binary) operation is addition on the set of natural numbers $\mathbb{N}$. For all $x,y,z \in \mathbb{N}$, we have:

$$x + (y+z) = (x+y) + z \tag{1.2}$$

■

Definition 1.1.3 — Identifiable Operation. A binary operation $f : S \times S \to S$ is said to be *identifiable*, or *unital*, if there exists an element $e \in S$, called the *identity element* or *unit element*, such that, for all $x \in S$, the following holds:

$$f(x,e) = x = f(e,x)$$

■ **Example 1.3** A common example of an identifiable (binary) operation is multiplication on the set of natural numbers $\mathbb{N}$. The identity element is 1, and for all $x \in \mathbb{N}$, we have:

$$x \cdot 1 = x = 1 \cdot x \tag{1.3}$$

■

Proposition 1.1.1 The identity element of an identifiable operation is unique.

Proof. Let e_1 and e_2 be two identity elements for the operation f. Then, for any element $x \in S$, we have:

$$f(x,e_1) = x = f(e_1,x)$$

$$f(x,e_2) = x = f(e_2,x)$$

Now, consider the element e_1:

$$f(e_1,e_2) = e_1$$

But since e_2 is an identity element, we also have:

$$f(e_1,e_2) = e_2$$

Therefore, we conclude that $e_1 = e_2$, proving the uniqueness of the identity element. ■

R Two-sided identity must be unique, but one-sided identities need not be.

■ **Example 1.4** *** To be asked ■

Definition 1.1.4 — Inverse Operation. A binary operation $f : S \times S \to S$ is said to be *invertible* if, for every element $x \in S$, there exists an element $y \in S$, called the two-sided *inverse* of x, denoted as x^{-1}, such that:

$$f(x,y) = e = f(y,x)$$

where e is the identity element of the operation.

R Invertible operation exists if inverse operation exists, i.e. there exists an identity element.

■ **Example 1.5** A common example of an invertible (binary) operation is addition on the set of integers $\mathbb{Z}$. For every integer $x \in \mathbb{Z}$, there exists an integer $y = -x$ such that:

$$x + (-x) = 0 = (-x) + x \tag{1.4}$$

where 0 is the identity element for addition. ■

Proposition 1.1.2 The inverse element of an invertible operation is unique.

Proof. Let y_1 and y_2 be two inverses of an element $x \in S$. Then, by definition of inverse, we have:

$$f(x,y_1) = e = f(y_1,x)$$

$$f(x,y_2) = e = f(y_2,x)$$

Now, consider the element y_1:

$$f(y_1,x) = e$$

But since y_2 is also an inverse of x, we can substitute e with $f(x,y_2)$:

$$f(y_1,x) = f(x,y_2) = e$$

By the associativity of the operation, we can rearrange this to:

$$y_1 = f(y_1,e) = f(y_1, f(x,y_2)) = f(f(y_1,x),y_2) = f(e,y_2) = y_2$$

Thus, the inverse element is unique. ■

Definition 1.1.5 — Commutative Operation. A binary operation $f : S \times S \to S$ is said to be *commutative* if, for all $x,y \in S$, the following holds:

$$f(x,y) = f(y,x)$$

■ **Example 1.6** A common example of a commutative operation is addition on the set of integers $\mathbb{Z}$. For all $x,y \in \mathbb{Z}$, we have:

$$x + y = y + x$$

■

Definition 1.1.6 — Distributive Operation. A binary operation $g : S \times S \to S$ is said to be *distributive* with respect to another binary operation $f : S \times S \to S$ if, for all $x,y,z \in S$, the following holds:

$$g(x, f(y,z)) = f(g(x,y), g(x,z))$$
$$g(f(y,z),x) = f(g(y,x), g(z,x))$$

■ **Example 1.7** A common example of a distributive operation is multiplication over addition on the set of integers $\mathbb{Z}$. For all $x,y,z \in \mathbb{Z}$, we have:

$$x \cdot (y+z) = x \cdot y + x \cdot z$$
$$(y+z) \cdot x = y \cdot x + z \cdot x$$

■

1.2 Groups, Rings, Fields

Definition 1.2.1 — Semigroup. A *semigroup* is a set S equipped with an associative binary operation $f : S \times S \to S$.

Definition 1.2.2 — Monoid. A *monoid* is a set M equipped with a binary operation $f : M \times M \to M$ such that the following properties hold:

1. *Closure Property:* For all $x, y \in M$, $f(x,y) \in M$.
2. *Associative Property*
3. *Identifiable Property*

We say (M, f) is a monoid, and f is the *monoid operation* on the set M. A set M with a monoid operation f is the *monoid structure*.

Definition 1.2.3 — Group. A *group* is a set G equipped with a monoid operation $f : G \times G \to G$ with the additional property that every element has an inverse, *Invertible Property*.

Definition 1.2.4 — Abelian Monoid / Group. A monoid / group (G, f) is said to be an *abelian monoid / group* if the monoid / group operation f is commutative, *Commutative Property*.

Definition 1.2.5 — Ring. A ring is a set R equipped with two binary operations $f : R \times R \to R$ (addition) and $g : R \times R \to R$ (multiplication) such that the following properties hold:

1. *Additive Group:* (R, f) is an abelian group.
2. *Multiplicative Semigroup:* (R, g) is a semigroup.
3. *Distributive Property:* g with respect to f.

Definition 1.2.6 — Unital Ring. A *unital ring* is a ring R equipped with a multiplicative identity element $1 \in R$ such that for all $x \in R$, $g(1,x) = g(x,1) = x$.

Definition 1.2.7 — Commutative Ring. A *commutative ring* is a ring R such that the multiplication operation $g : R \times R \to R$ is commutative.

■ **Example 1.8** $(\mathbb{Z}, +, \times)$ is a unital commutative ring. $(2\mathbb{Z}, +, \times)$ is a commutative ring, but not unital. ($2\mathbb{Z} := 2n \mid n \in \mathbb{Z}$) ■

Definition 1.2.8 — Field. A *field* is a unital commutative ring F such that every non-zero element has a multiplicative inverse.

■ **Example 1.9** $(\mathbb{Q}, +, \times)$, $(\mathbb{R}, +, \times)$ and $(\mathbb{C}, +, \times)$ are fields. ■

1.3 Citation

This statement requires citation [1]; this one is more specific [2, page 122].

1.4 Lists

Lists are useful to present information in a concise and/or ordered way[1].

1.4.1 Numbered List

1. The first item
2. The second item
3. The third item

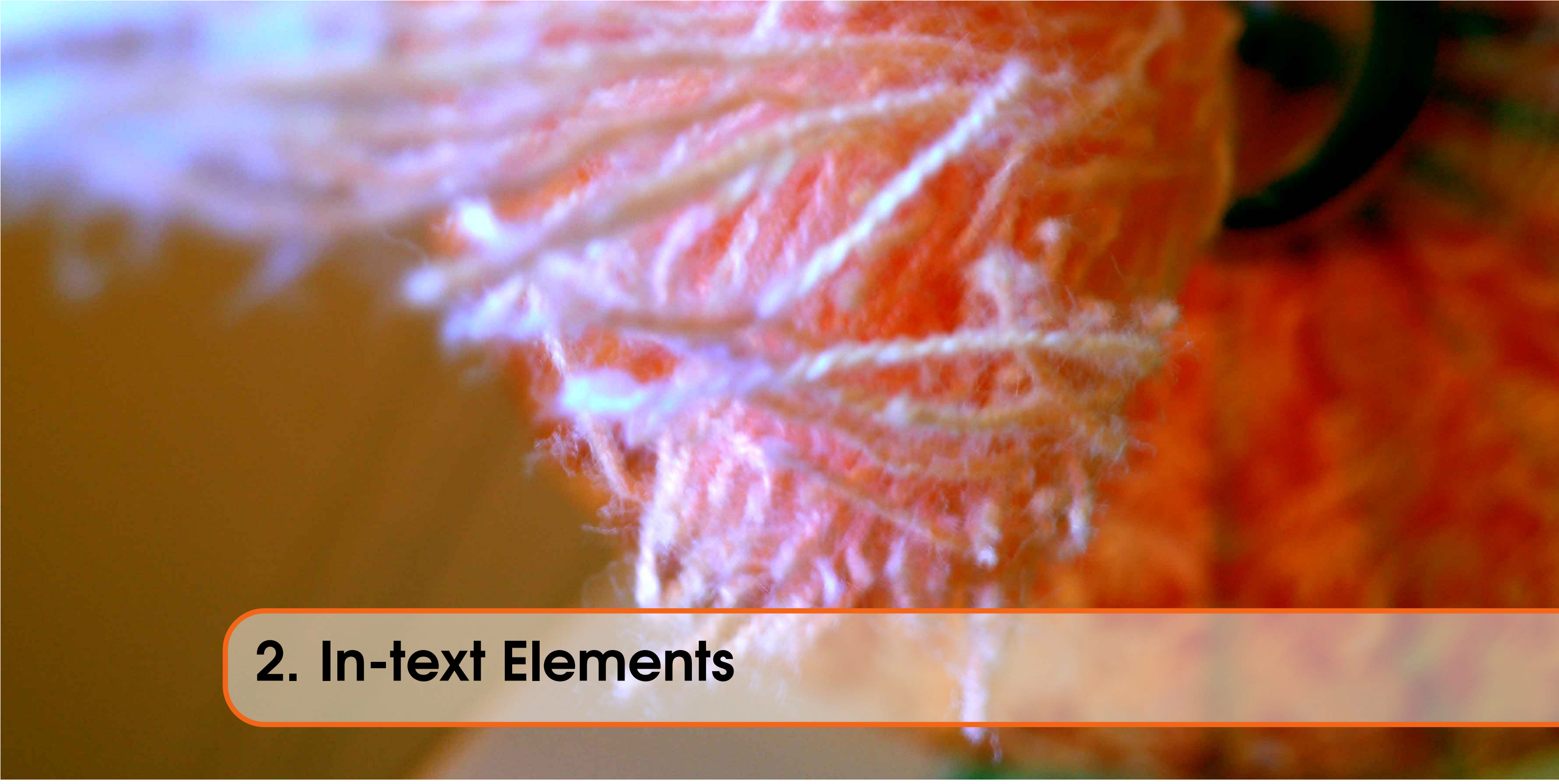
1.4.2 Bullet Points

- The first item
- The second item
- The third item

1.4.3 Descriptions and Definitions

Name Description
Word Definition
Comment Elaboration

[1] Footnote example...



2. In-text Elements

2.1 Theorems

This is an example of theorems.

2.1.1 Several equations

This is a theorem consisting of several equations.

Theorem 2.1.1 — Name of the theorem. In $E = \mathbb{R}^n$ all norms are equivalent. It has the properties:

$$\big| ||\mathbf{x}|| - ||\mathbf{y}|| \big| \leq ||\mathbf{x} - \mathbf{y}|| \tag{2.1}$$

$$||\sum_{i=1}^{n} \mathbf{x}_i|| \leq \sum_{i=1}^{n} ||\mathbf{x}_i|| \quad \text{where } n \text{ is a finite integer} \tag{2.2}$$

2.1.2 Single Line

This is a theorem consisting of just one line.

Theorem 2.1.2 A set $\mathscr{D}(G)$ in dense in $L^2(G)$, $|\cdot|_0$.

2.2 Definitions

This is an example of a definition. A definition could be mathematical or it could define a concept.

Definition 2.2.1 — Definition name. Given a vector space E, a norm on E is an application, denoted $||\cdot||$, E in $\mathbb{R}^+ = [0, +\infty[$ such that:

$$||\mathbf{x}|| = 0 \Rightarrow \mathbf{x} = \mathbf{0} \tag{2.3}$$

$$||\lambda \mathbf{x}|| = |\lambda| \cdot ||\mathbf{x}|| \tag{2.4}$$

$$||\mathbf{x} + \mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}|| \tag{2.5}$$

2.3 Notations

Notation 2.1. *Given an open subset G of $\mathbb{R}^n$, the set of functions φ are:*
1. *Bounded support G;*
2. *Infinitely differentiable;*

a vector space is denoted by $\mathscr{D}(G)$.

2.4 Remarks

This is an example of a remark.

R The concepts presented here are now in conventional employment in mathematics. Vector spaces are taken over the field $\mathbb{K}=\mathbb{R}$, however, established properties are easily extended to $\mathbb{K}=\mathbb{C}$.

2.5 Corollaries

This is an example of a corollary.

Corollary 2.5.1 — Corollary name. The concepts presented here are now in conventional employment in mathematics. Vector spaces are taken over the field $\mathbb{K}=\mathbb{R}$, however, established properties are easily extended to $\mathbb{K}=\mathbb{C}$.

2.6 Propositions

This is an example of propositions.

2.6.1 Several equations

Proposition 2.6.1 — Proposition name. It has the properties:

$$\big| ||\mathbf{x}|| - ||\mathbf{y}|| \big| \leq ||\mathbf{x}-\mathbf{y}|| \tag{2.6}$$

$$||\sum_{i=1}^{n} \mathbf{x}_i|| \leq \sum_{i=1}^{n} ||\mathbf{x}_i|| \quad \text{where } n \text{ is a finite integer} \tag{2.7}$$

2.6.2 Single Line

Proposition 2.6.2 Let $f,g \in L^2(G)$; if $\forall \varphi \in \mathscr{D}(G)$, $(f,\varphi)_0 = (g,\varphi)_0$ then $f=g$.

2.7 Examples

This is an example of examples.

2.7.1 Equation and Text

■ **Example 2.1** Let $G=\{x\in\mathbb{R}^2 : |x|<3\}$ and denoted by: $x^0=(1,1)$; consider the function:

$$f(x)=\begin{cases} e^{|x|} & \text{si } |x-x^0|\leq 1/2 \\ 0 & \text{si } |x-x^0|>1/2 \end{cases} \tag{2.8}$$

The function f has bounded support, we can take $A=\{x\in\mathbb{R}^2 : |x-x^0|\leq 1/2+\varepsilon\}$ for all $\varepsilon\in]0;5/2-\sqrt{2}[$. ■

2.7.2 Paragraph of Text

■ **Example 2.2 — Example name.** Nam dui ligula, fringilla a, euismod sodales, sollicitudin vel, wisi. Morbi auctor lorem non justo. Nam lacus libero, pretium at, lobortis vitae, ultricies et, tellus. Donec aliquet, tortor sed accumsan bibendum, erat ligula aliquet magna, vitae ornare odio metus a mi. Morbi ac orci et nisl hendrerit mollis. Suspendisse ut massa. Cras nec ante. Pellentesque a nulla. Cum sociis natoque penatibus et magnis dis parturient montes, nascetur ridiculus mus. Aliquam tincidunt urna. Nulla ullamcorper vestibulum turpis. Pellentesque cursus luctus mauris. ■

2.8 Exercises

This is an example of an exercise.

Exercise 2.1 This is a good place to ask a question to test learning progress or further cement ideas into students' minds. ■

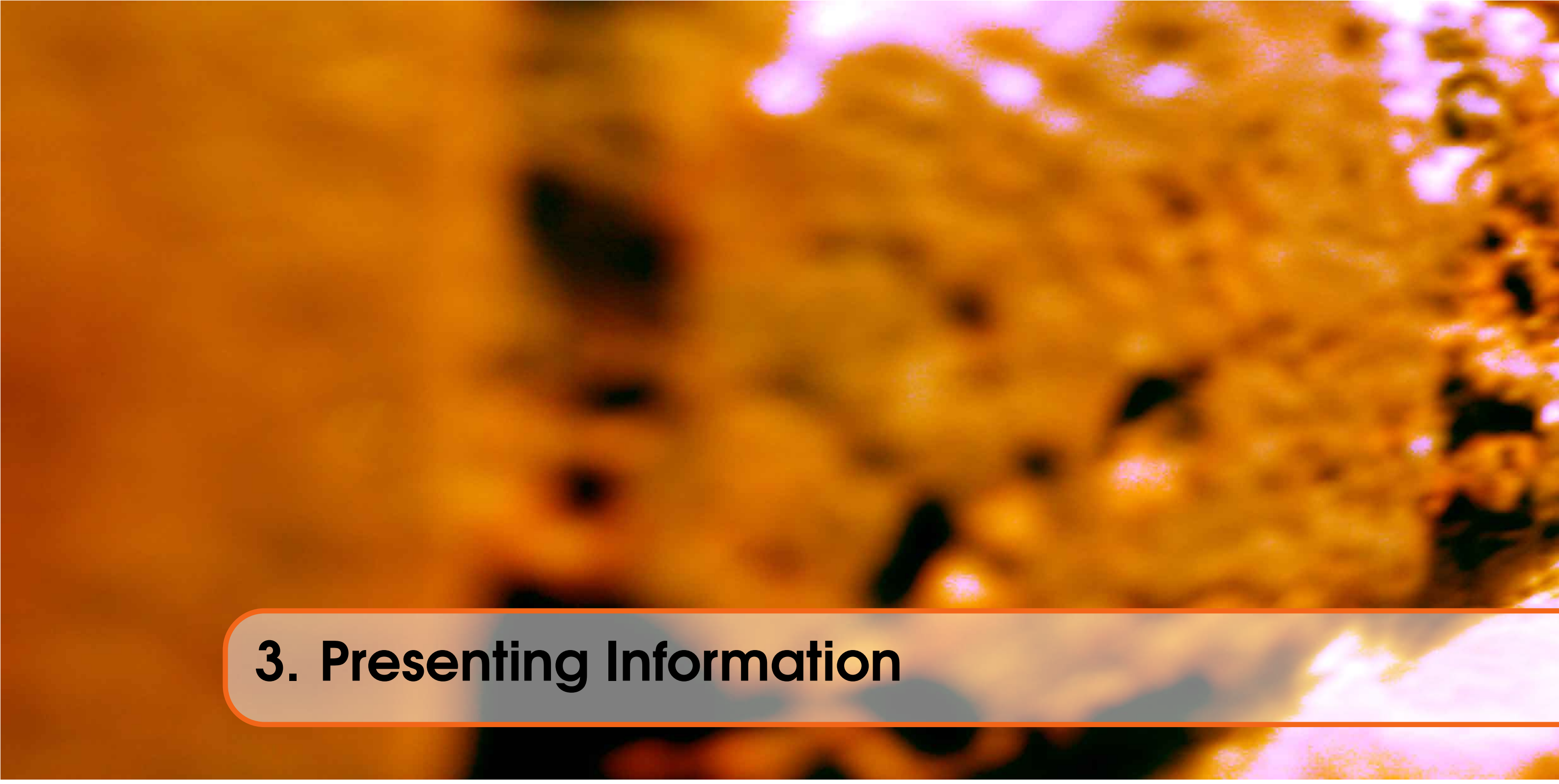
2.9 Problems

Problem 2.1 What is the average airspeed velocity of an unladen swallow?

2.10 Vocabulary

Define a word to improve a students' vocabulary.
Vocabulary 2.1 — Word. Definition of word.

Part Two



3. Presenting Information

3.1 Table

Treatments	Response 1	Response 2
Treatment 1	0.0003262	0.562
Treatment 2	0.0015681	0.910
Treatment 3	0.0009271	0.296

Table 3.1: Table caption

3.2 Figure

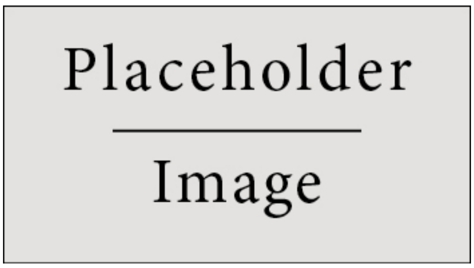
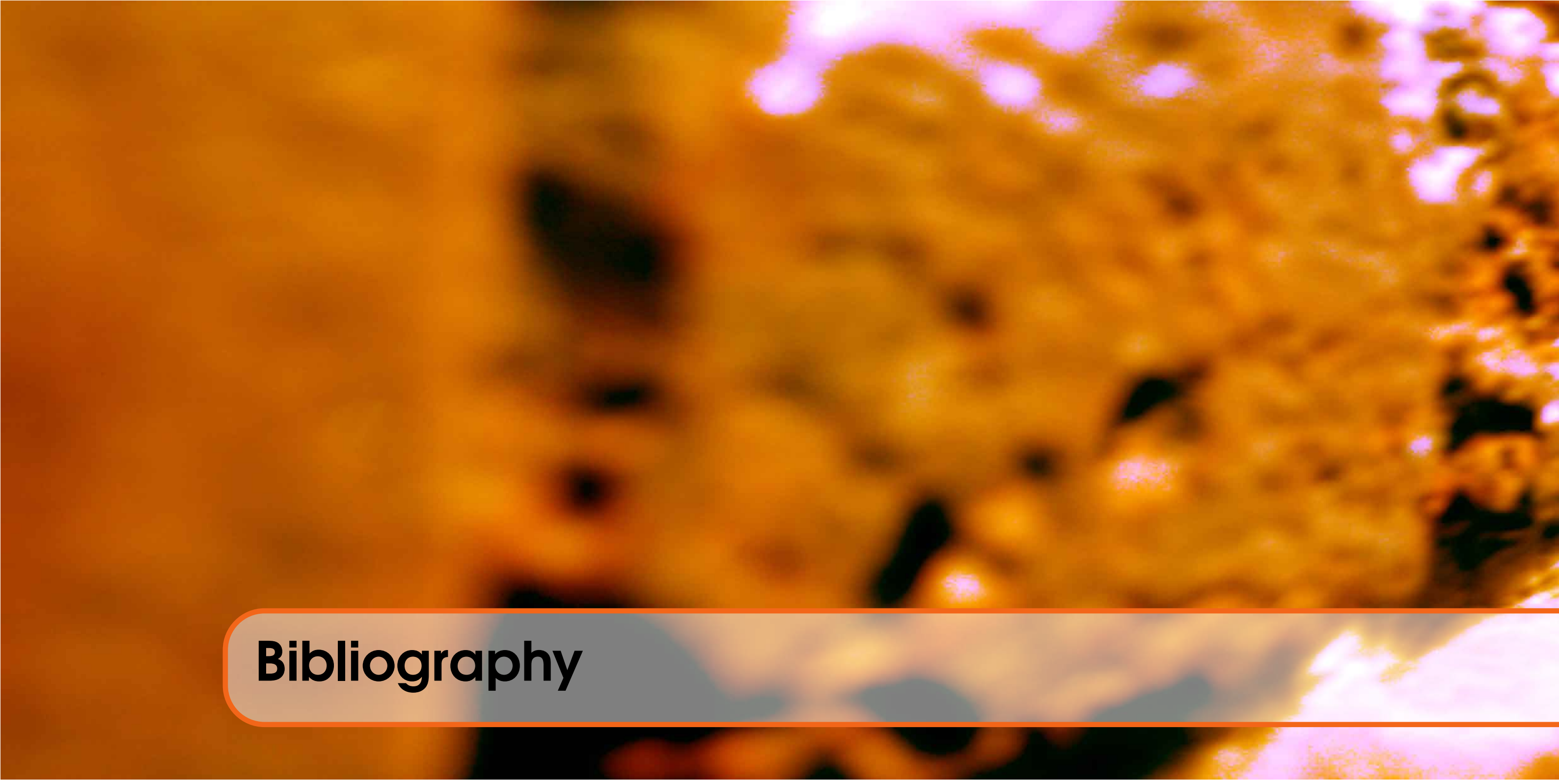


Figure 3.1: Figure caption



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