1. (a) Show that any matrix A can be expressed as a product BC of two matrices, where the columns of B are linearly independent and the rows of C are linearly independent. Is this decomposition $A = BC$ unique?

   (b) Let V be an n-dimensional linear space over $\mathbb{F}$, and let S be a set of linearly independent vectors in V. Prove that $|S| \leq n$ and that S can be expanded to form a minimal spanning set $\tilde{S}$ of V.

   `Hint: Your solution should be straightforward.`

2. Let V_i $(i = 1, 2)$ be subspaces of a finite-dimensional linear space V, and let S_i $(i = 1, 2)$ be subsets of a finite set S.

   (a) Show that the sequence

   $$0 \to V_1 \cap V_2 \to V_1 \to (V_1 + V_2)/V_2 \to 0$$

   is exact. Thus, we establish the natural linear isomorphism

   $$V_1/(V_1 \cap V_2) \equiv (V_1 + V_2)/V_2.$$

   and the identity

   $$\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2).$$

   (b) Show that part (a) implies

   $$|S_1 \cup S_2| = |S_1| + |S_2| - |S_1 \cap S_2|.$$

   (c) (Optional) What can you conclude if we have three linear subspaces or three subsets? You may refer to this page.