1. On the logic set $X = \{true, false\}$, we have two binary operations: one is "OR" (denoted by $\vee$) and the other is "AND" (denoted by $\wedge$). If we use 1 to represent $true$ and 0 to represent $false$. Then

$$0 \vee 1 = 1 \vee 0 = 1 \vee 1 = 1, \quad 0 \vee 0 = 0.$$

Also,

$$0 \wedge 1 = 1 \wedge 0 = 0 \wedge 0 = 0, \quad 1 \wedge 1 = 1.$$

   (a) Show that both $\vee$ and $\wedge$ are abelian monoid structures on X.

   (b) Show that $\vee$ is distributive with respect to $\wedge$.

   (c) Show that $\wedge$ is distributive with respect to $\vee$.

   (d) Is $(X, \vee, \wedge)$ a ring? If not, can you modify $\vee$ to arrive at a new binary operation $\vee'$ so that $(X, \vee', \wedge)$ is a commutative ring with unit? If yes, is this ring a field?

2. Find a non-empty subset X of the set of real square matrix of order 2 such that

   1) the set X is closed under matrix multiplication, and

   2) there are many left-identities, but there is no two-sided identity.

3. Let $\mathbb{F}$ be a field and X be a non-empty set. Recall that $\mathbb{F}[[X]]$ is the set of $\mathbb{F}$-valued functions on X and $\mathbb{F}[X]$ is the set of finitely-supported $\mathbb{F}$-valued functions on X. Both $\mathbb{F}[[X]]$ and $\mathbb{F}[X]$ are linear spaces over the field $\mathbb{F}$.

   Let $T\colon X \to Y$ be a set map and $\mathbb{F}[T]\colon \mathbb{F}[X] \to \mathbb{F}[Y]$ be the map such that

$$\mathbb{F}[T](f)(y) = \sum_{x \in T^{-1}(y)} f(x), \quad \forall y \in Y$$

   In case $T^{-1}(y)$ is the empty set $\emptyset$, the sum is assumed to be 0. Please check that the sum above is well-defined and $\mathbb{F}[T](f)$ has a finite-support.

   (a) Show that $\mathbb{F}[1_X] = 1_{\mathbb{F}[X]}$ for all non-empty set X. Recall that 1_S denotes the identity map on the set S: $s \mapsto s$.

   (b) Show that $\mathbb{F}[TS] = \mathbb{F}[T]\mathbb{F}[S]$ for all set maps T and S such that the composition TS is defined.

   For simplicity, we may rewrite $\mathbb{F}[T]$ as T_*.

   (c) For any set map $T\colon X \to Y$, we have an induced map $T^*\colon F[[Y]] \to \mathbb{F}[[X]]$ defined via the formula $T^*f = fT$. Show that, $1_X^* = 1_{\mathbb{F}[[X]]}$ and $(TS)^* = S^*T^*$.

   (d) (optional) Can we get a natural map from $\mathbb{F}[[X]]$ to $\mathbb{F}[[Y]]$ or from $\mathbb{F}[Y]$ to $\mathbb{F}[X]$ for any set map $T\colon X \to Y$ between two infinite sets X and Y?