Let $f\colon V \to W$ be a linear map. The kernel of f, denoted by $\ker f$, is defined to be the inverse image of $0 \in W$ under f and is a subspace of the domain V. The image of f, denoted by $\mathrm{im} f$, is defined to be the linear subspace $\{f(v) \mid v \in V\}$ of the codomain. The cokernel of f, denoted by $\mathrm{coker} f$, is defined to be the quotient space $W/\mathrm{im} f$ of the codomain by the image. The coimage of f, denoted by $\mathrm{coim} f$, is defined to be the quotient space $V/\mathrm{im} f$ of the domain by the kernel.

1. Let $f\colon V \to W$ be a linear map. Show that

   (a) f is injective $\iff$ the kernel of f is trivial (i.e., $\{0\}$).

   (b) f is surjective $\iff$ the cokernel of f is trivial.

   (c) f is isomorphism $\iff$ both kernel and cokernel of f are trivial.

   (d) f is surjective $\iff$ for any linear map $g\colon W \to Z$, $gf = 0 \implies g = 0$.

   (e) f is injective $\iff$ for any linear map $h\colon U \to V$, $fh = 0 \implies h = 0$.

   Now we assume that V and W are finite dimensional, say $V = \mathbb{F}^n$ and $W = \mathbb{F}^m$, then f is the multiplication by an $m \times n$-matrix A.

   (f) Please translate the five statements above into the corresponding statements about matrix A.

2. Let $f\colon V \to W$ be a set map between linear spaces. Show that

   (a) its graph $\Gamma_f := \{(v, f(v)) \mid v \in V\}$ is a linear subspace of the product linear space $V \times W \iff f$ is a linear map.

   (b) in case f is linear, its domain is naturally linearly equivalent to its graph: $\mathrm{domain} f \equiv \Gamma_f$.

3. We say a linear map $f\colon V \to W$ is an imbedding if the map $\bar{f}\colon V \to \mathrm{image} f$ that sends v to $f(v)$ is a linear equivalence. Show that f is an imbedding $\iff$ f is one-to-one.

   An optional exercise: We say a topological map (i.e., continuous map) $f\colon X \to Y$ is an imbedding if the map $\bar{f}\colon X \to \mathrm{image} f$ that sends x to $f(x)$ is a topological equivalence (i.e., homeomorphism). Show that f is an imbedding implies that f is one-to-one, but the converse is not true.

4. Let V be a linear subspace of W and $\sim$ be this equivalence relation on W:

   $$w \sim w' \text{ if and only if } w - w' \in V.$$

   We let W/V denote the set of equivalence classes.

   (a) Show that there is a unique linear structure on W/V such that the quotient map $q\colon W \to W/V$ is a linear map.

(b) Show that, for any linear map $\phi\colon W \to Z$ such that $\phi(v) = 0$ for any $v \in V$, there *is* a *unique* linear map $\bar{\phi}\colon W/V \to Z$ such that

$$\bar{\phi} \circ q = \phi.$$

(This statement is called the *universal property* for the quotient q.) Schematically we write

$$\begin{tikzcd} W \arrow[r, "\forall\phi"] \arrow[d, "q"'] & Z \\ W/V \arrow[ur, dotted, "\exists!\bar{\phi}"'] & \end{tikzcd}$$

Remark: W/V is called the quotient space of W by the subspace V and is also called algebraic normal space of V in W. It is a fact that $\dim W/V = \dim W - \dim V$.

(c) Let V be a linear subspace of W. Then the inclusion map $\iota\colon V \hookrightarrow W$ is a linear map with image inside V. Please formulate and prove the universal property for the inclusion map ι. Hint: One way to get it is to dualize the universal property for the quotient map $W \to W/V$. Note that $V = \ker q$.

5. Let V be a linear space and S be a spanning set for V. Show that S is a minimal spanning set for $V \iff S$ is a linearly independent set. Note that S here is not required to be finite.