

Recall that a real symmetric matrix A of order n is called **positive**, written as $A > 0$, if the symmetric 2-form on \mathbb{R}^n with representation matrix A w.r.t. the standard bases \vec{e}_i is positive-definite, i.e., the map $(\vec{x}, \vec{y}) \mapsto \vec{x} \cdot A \vec{y}$ is an inner product. We say A is **semi-positive-definite**, written as $A \geq 0$, if $\vec{x} \cdot A \vec{x} \geq 0$ for all $\vec{x} \in \mathbb{R}^n$.

Recall that a basis of V is denoted by v_i and the resulting dual basis of V^* is denoted by \hat{v}^i . If A is matrix, then A^T denotes the transpose of matrix A .

1. Let A be a real symmetric $n \times n$ -matrix and A_i be the matrix obtained from A by deleting its last $(n - i)$ rows and last $(n - i)$ columns. Show that

$$A > 0 \iff \det A_i > 0 \text{ for each } i$$

Hint: induction on n

2. Let A be a real symmetric $n \times n$ -matrix. Assume that $A \geq 0$. It is clear that $A \geq 0 \iff A + tI > 0$ for any $t > 0$. Based on this observation and the result in the previous exercise to derive a necessary and sufficient condition for $A \geq 0$.
3. Let V be a n -dimensional linear space over a field \mathbb{F} . The pairing $V^* \times V \rightarrow \mathbb{F}$ yields the multi-linear map $V^* \times V^* \times V \times V \rightarrow \mathbb{F}$ that sends $(\alpha_1, \alpha_2, v_1, v_2)$ to $\alpha_1(v_1)\alpha_2(v_2)$. Thus we have a linear map

$$(V^* \otimes V^*) \otimes (V \otimes V) \rightarrow \mathbb{F}$$

or equivalently a linear map $\iota: V^* \otimes V^* \rightarrow (V \otimes V)^*$.

- (1) Show that the linear map ι is a linear equivalence. In fact a natural one in the language of category. So we shall write $V^* \otimes V^* \equiv (V \otimes V)^*$.
- (2) The quotient linear map $V \otimes V \rightarrow S^2V$ yields the injective linear map $(S^2V)^* \rightarrow (V \otimes V)^*$. So we have the following composition map

$$(S^2V)^* \hookrightarrow (V \otimes V)^* \equiv V^* \otimes V^* \twoheadrightarrow S^2V^*.$$

Show that this natural map $(S^2V)^* \rightarrow S^2V^*$ is a linear equivalence if and only if the characteristic of the field \mathbb{F} is not 2.

- (3) Assume that the characteristic of the field \mathbb{F} is not 2, please find the inverse of the natural map in part (2).
- (4) Assume that the characteristic of the field \mathbb{F} is not 2, show that there is a natural linear equivalence $\wedge^2 V^* \rightarrow (\wedge^2 V)^*$.
4. Let ω be a 2-form on V , i.e., a bilinear map $V \times V \rightarrow \mathbb{F}$. Then $\omega_{\natural}: V \rightarrow V^*$ is the linear map that sends v to $\omega(v, -)$. Let v_i be a basis of V and the resulting dual basis of V^* be denoted by \hat{v}^i . Let A be the matrix representation of ω w.r.t. basis v_i , i.e. $A = [\omega(v_i, v_j)]$. Let A' be the matrix representation of ω_{\natural} w.r.t. bases v_i and \hat{v}^i .

- (1) Show that A' is the transpose of A . Thus ω is non-degenerate means that its any matrix representation A is invertible.

- (2) If \tilde{v}_i is another basis of V , then there is a unique invertible matrix $S = [s^i_{\tilde{i}}]$ such that $\tilde{v}_{\tilde{j}} = v_i s^i_{\tilde{j}}$. Let \tilde{A} be the matrix representation of ω w.r.t. basis \tilde{v}_i . Show that $\tilde{A} = S^T A S$.
- (3) Show that the map that sends (A, S) to $S^T A S$ is a right action of $\text{GL}_n(\mathbb{F})$ on $(V \otimes V)^*$.
- (4) The quotient map $V \otimes V \rightarrow S^2 V$ yields injective linear map $(S^2 V)^* \rightarrow (V \otimes V)^*$, that is not a surprise because any symmetric 2-form is a 2-form. Similarly, $(\wedge^2 V)^*$ is a linear subspace of $(V \otimes V)^*$ as well. Show that $\text{GL}(V)$ acts on $(V \otimes V)^*$ and leaves invariant both subspace $(S^2 V)^*$ and subspace $(\wedge^2 V)^*$.
- (5) Show that, under the identification $(V \otimes V)^* \equiv V^* \otimes V^*$, we have

$$\omega = \omega(v_i, v_j) \hat{v}^i \otimes \hat{v}^j.$$

- (6) Assume that the characteristic of the field \mathbb{F} is not 2, then we have the natural identification $(S^2 V)^* \equiv S^2 V^*$ found in the last question. Show that, if ω is a symmetric 2-form on V , then $\omega = \omega(v_i, v_j) \hat{v}^i \hat{v}^j$. Similarly, if ω is a skew-symmetric 2-form on V , then $\omega = \omega(v_i, v_j) \hat{v}^i \wedge \hat{v}^j$.