MATH 2131

Honors in Linear and Abstract Algebra I

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Part One

1. Abstract Linear Spaces

1.1 Binary Operation

Definition 1.1.1 — Binary Operation. A *binary operation* on a set S is a mapping of the elements of the Cartesian product $S \times S$ to S.

$$\begin{aligned} f : S \times S &\to S \\ (x,y) &\mapsto f(x,y) \end{aligned}$$

■ **Example 1.1** A common example of a binary operation is addition on the set of natural numbers $\mathbb{N}$.

$$\begin{aligned} + : \mathbb{N} \times \mathbb{N} &\to \mathbb{N} \\ (x,y) &\mapsto x+y \end{aligned} \tag{1.1}$$

■

Definition 1.1.2 — Associative Operation. A binary operation $f : S \times S \to S$ is said to be *associative* if, for all $x,y,z \in S$,

$$f(x, f(y,z)) = f(f(x,y),z)$$

■ **Example 1.2** A common example of an associative (binary) operation is addition on the set of natural numbers $\mathbb{N}$. For all $x,y,z \in \mathbb{N}$, we have $x+(y+z) = (x+y)+z$. ■

Definition 1.1.3 — Identifiable Operation. A binary operation $f : S \times S \to S$ is said to be *identifiable*, or *unital*, if there exists an element $e \in S$, the *identity* or *unit element*, such that, for all $x \in S$

$$f(e,x) = x = f(x,e)$$

■ **Example 1.3** A common example of an identifiable (binary) operation is multiplication on the set of natural numbers $\mathbb{N}$. The identity element is 1, and for all $x \in \mathbb{N}$, we have $x \cdot 1 = x = 1 \cdot x$. ■

Proposition 1.1.1 The identity element of an identifiable operation is unique.

Proof. Let e_1 and e_2 be two identity elements for the operation f. Then, for any element $x \in S$, we have:

$$f(x,e_1) = x = f(e_1,x)$$
$$f(x,e_2) = x = f(e_2,x)$$

Now, consider the element e_1:

$$f(e_1,e_2) = e_1$$

But since e_2 is an identity element, we also have:

$$f(e_1,e_2) = e_2$$

Therefore, we conclude that $e_1 = e_2$, proving the uniqueness of the identity element. ■

R Two-sided identity must be unique, but one-sided identities need not be.

■ **Example 1.4** ■

Definition 1.1.4 — Inverse Operation. A binary operation $f : S \times S \to S$ is said to be *invertible* if, for every element $x \in S$, there exists an element $y \in S$, called the two-sided *inverse* of x, denoted as x^{-1}, such that

$$f(x,y) = e = f(y,x)$$

where e is the identity element of the operation.

R Invertible operation exists if inverse operation exists, i.e. there exists an identity element.

■ **Example 1.5** A common example of an invertible (binary) operation is addition on the set of integers $\mathbb{Z}$. For every integer $x \in \mathbb{Z}$, there exists an integer $y = -x$ such that:

$$x + (-x) = 0 = (-x) + x \tag{1.2}$$

where 0 is the identity element for addition. ■

Proposition 1.1.2 The inverse element of an invertible operation is unique.

Proof. Let y_1 and y_2 be two inverses of an element $x \in S$. Then, by definition of inverse, we have:

$$f(x,y_1) = e = f(y_1,x)$$
$$f(x,y_2) = e = f(y_2,x)$$

Now, consider the element y_1:

$$f(y_1,x) = e$$

But since y_2 is also an inverse of x, we can substitute e with $f(x,y_2)$:

$$f(y_1,x) = f(x,y_2) = e$$

By the associativity of the operation, we can rearrange this to:

$$y_1 = f(y_1,e) = f(y_1,f(x,y_2)) = f(f(y_1,x),y_2) = f(e,y_2) = y_2$$

Thus, the inverse element is unique. ■

Definition 1.1.5 — Commutative Operation. A binary operation $f : S \times S \to S$ is said to be *commutative* if, for all $x, y \in S$, the following holds:

$$f(x,y) = f(y,x)$$

■ **Example 1.6** A common example of a commutative operation is addition on the set of integers $\mathbb{Z}$. For all $x, y \in \mathbb{Z}$, we have:

$$x + y = y + x$$

■

Definition 1.1.6 — Distributive Operation (Harmonic Property). A binary operation $g : S \times S \to S$ is said to be *distributive* with respect to another binary operation $f : S \times S \to S$ if, for all $x, y, z \in S$, the following holds:

$$g(x, f(y,z)) = f(g(x,y), g(x,z))$$
$$g(f(y,z), x) = f(g(y,x), g(z,x))$$

■ **Example 1.7** A common example of a distributive operation is multiplication over addition on the set of integers $\mathbb{Z}$. For all $x, y, z \in \mathbb{Z}$, we have:

$$x \cdot (y + z) = x \cdot y + x \cdot z$$
$$(y + z) \cdot x = y \cdot x + z \cdot x$$

■

1.2 Groups, Rings, Fields

Definition 1.2.1 — Monoid. A *monoid* is a set M equipped with a binary operation $f : M \times M \to M$ such that the following properties hold:

1. *Closure Property:* For all $x, y \in M$, $f(x,y) \in M$.
2. *Associative Property*
3. *Identifiable Property*

We say (M, f) is a monoid, and f is the *monoid operation* on the set M. A set M with a monoid operation f is the *monoid structure*.

Definition 1.2.2 — Group. A *group* is a set G equipped with a monoid operation $f : G \times G \to G$ with the additional property that every element has an inverse, *Invertible Property*.

■ **Example 1.8** $(\mathbb{R} \setminus \{0\}, \times)$ is a group, but $(\mathbb{R}, \times)$ is not a group since 0 does not have a multiplicative inverse. ■

Definition 1.2.3 — Abelian Monoid / Group. A monoid / group (G, f) is said to be an *abelian monoid / group* if the monoid / group operation f is commutative, *Commutative Property*.

Definition 1.2.4 — Unital Ring. A (unital) ring is a set R equipped with two binary operations $f : R \times R \to R$ (addition) and $g : R \times R \to R$ (multiplication) such that the following properties hold:

1. *Additive Group:* (R, f) is an abelian group.
2. *Multiplicative Monoid:* (R, g) is a monoid.
3. *Distributive Property:* g with respect to f.

Definition 1.2.5 — Commutative Ring. A *commutative ring* is a unital ring R such that the multiplication operation $g : R \times R \to R$ is commutative.

■ **Example 1.9** $(\mathbb{Z}, +, \times)$ is a unital commutative ring. ■

Definition 1.2.6 — Field. A *field* is a unital commutative ring F such that every non-zero element has a multiplicative inverse.

■ **Example 1.10** $(\mathbb{Q}, +, \times)$, $(\mathbb{R}, +, \times)$ and $(\mathbb{C}, +, \times)$ are fields. ■

■ **Example 1.11** $(\mathbb{Z}/2\mathbb{Z}, +, \times)$ is a field, where $\mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}$, $\bar{0}$ is the set of even integers and $\bar{1}$ is the set of odd integers. It follows the additions and multiplications below:

$$\begin{array}{c|cc} + & \bar{0} & \bar{1} \\ \hline \bar{0} & \bar{0} & \bar{1} \\ \bar{1} & \bar{1} & \bar{0} \end{array} \qquad \begin{array}{c|cc} \times & \bar{0} & \bar{1} \\ \hline \bar{0} & \bar{0} & \bar{0} \\ \bar{1} & \bar{0} & \bar{1} \end{array} \tag{1.3}$$

■

1.3 Morphisms

Definition 1.3.1 — Morphisms. A *morphism* is a structure-preserving map between two algebraic structures (e.g., groups, rings, fields). Formally, let $(A, \cdot_A)$ and $(B, \cdot_B)$ be two algebraic structures. A morphism $f : A \to B$ is a set map / function such that:

$$f(x \cdot_A y) = f(x) \cdot_B f(y) \quad \forall x, y \in A$$

Definition 1.3.2 — Monoid Homomorphism. A *monoid homomorphism* is a morphism between two monoids that preserves the monoid structure. Formally, let $(M_1, \cdot_1)$ and $(M_2, \cdot_2)$ be two monoids with identity elements e_1 and e_2, respectively. A function $f : M_1 \to M_2$ is a monoid homomorphism if:

1. $f(x \cdot_1 y) = f(x) \cdot_2 f(y) \quad \forall x, y \in M_1$
2. $f(e_1) = e_2$

Definition 1.3.3 — Group Homomorphism. A *group homomorphism* is a morphism between two groups that preserves the group structure. Formally, let $(G_1, \cdot_1)$ and $(G_2, \cdot_2)$ be two groups with identity elements e_1 and e_2, respectively. A function $f : G_1 \to G_2$ is a group homomorphism if:

1. $f(x \cdot_1 y) = f(x) \cdot_2 f(y) \quad \forall x, y \in G_1$
2. $f(e_1) = e_2$
3. $f(x^{-1}) = (f(x))^{-1} \quad \forall x \in G_1$

Proposition 1.3.1 The second and third properties of a group homomorphism are consequences of the first property.

Proof. Let $f : G_1 \to G_2$ be a group homomorphism satisfying the first property. We will show that the second and third properties follow from it.

Second Property: To show that $f(e_1) = e_2$, we use the fact that e_1 is the identity element in G_1. For any element $x \in G_1$, we have:

$$f(x) = f(x \cdot_1 e_1) = f(x) \cdot_2 f(e_1)$$

Since $f(x)$ is an arbitrary element in G_2, this implies that $f(e_1)$ must be the identity element in G_2, i.e., $f(e_1) = e_2$.

Third Property: To show that $f(x^{-1}) = (f(x))^{-1}$ for all $x \in G_1$, we use the fact that x^{-1} is the inverse of x in G_1. We have:

$$e_2 = f(e_1) = f(x \cdot_1 x^{-1}) = f(x) \cdot_2 f(x^{-1})$$

This shows that $f(x^{-1})$ is the inverse of $f(x)$ in G_2, i.e., $f(x^{-1}) = (f(x))^{-1}$.

Therefore, both the second and third properties of a group homomorphism are indeed consequences of the first property. ■

R For monoid homomorphisms, the second property cannot be derived from the first property. Consider the identity element e_1 in M_1. If we apply the first property, we get $f(e_1 \cdot_1 e_1) = f(e_1) \cdot_2 f(e_1)$. This simplifies to $f(e_1) = f(e_1) \cdot_2 f(e_1)$, which does not necessarily imply that $f(e_1)$ is the identity element in M_2, i.e., $f(e_1) \neq e_2$, but $f(e_1)$ is the idempotent element in M_2. Therefore, the second property must be explicitly stated for monoid homomorphisms.

However in the case of group homomorphisms, the existence of inverses ensures that there is only one element that can be idempotent under the group operation, which is the identity element. Thus, for group homomorphisms, the second property can be derived from the first property.

Definition 1.3.4 — Ring Homomorphism. A *ring homomorphism* is a morphism between two rings that preserves both the additive and multiplicative structures. Formally, let $(R_1, +_1, \cdot_1)$ and $(R_2, +_2, \cdot_2)$ be two rings with identity elements 0_1, 1_1 and 0_2, 1_2, respectively. A function $f : R_1 \to R_2$ is a ring homomorphism if:

1. $f(x +_1 y) = f(x) +_2 f(y) \quad \forall x, y \in R_1$
2. $f(x \cdot_1 y) = f(x) \cdot_2 f(y) \quad \forall x, y \in R_1$
3. $f(1_1) = 1_2$

Definition 1.3.5 — Endomorphism. An *endomorphism* is a morphism from an algebraic structure to itself. Formally, let $(A, \cdot)$ be an algebraic structure. An endomorphism $f : A \to A$ is a set map such that:

$$f(x \cdot y) = f(x) \cdot f(y) \quad \forall x, y \in A$$

Definition 1.3.6 — Hom-set. The set of all morphisms from an algebraic structure A to another algebraic structure B is called the *hom-set*, denoted by $\mathrm{Hom}(A, B)$.

Definition 1.3.7 — Endomorphism Ring. The set of all endomorphisms of an abelian group $(A, +)$, denoted by $\mathrm{End}(A)$, forms a (non-commutative) ring under pointwise addition and composition of set maps. The addition and multiplication operations are defined as follows:

$$\begin{aligned} + : \mathrm{End}(A) \times \mathrm{End}(A) &\to \mathrm{End}(A) \\ (f, g) &\mapsto (f + g : x \mapsto f(x) + g(x)) \qquad f + g : A \to A \end{aligned}$$

$$\begin{aligned} \circ : \mathrm{End}(A) \times \mathrm{End}(A) &\to \mathrm{End}(A) \\ (f, g) &\mapsto (f \circ g : x \mapsto f(g(x))) \qquad f \circ g : A \to A \end{aligned}$$

The identity element for addition is the zero endomorphism, which maps every element to the identity element of the group.

$$\begin{aligned} 0 : A &\to A \\ x &\mapsto 0 \end{aligned}$$

The identity element for multiplication is the identity endomorphism, which maps every element to itself.

$$\begin{aligned} 1 : A &\to A \\ x &\mapsto x \end{aligned}$$

Note that all endomorphisms in $\mathrm{End}(A)$ are group homomorphisms and $\mathrm{End}(A) = \mathrm{Hom}(A, A)$.

1.4 Vector Spaces

Definition 1.4.1 — Linear Structure. A *linear structure* over a field F on a set V is a pair $(+,\cdot)$ where $(V,+)$ is an abelian group with a ring homomorphism $F \to \mathrm{End}(V)$, where $\mathrm{End}(V)$ is the endomorphism ring of the abelian group $(V,+)$.

$$\begin{aligned} \cdot &: F \to \mathrm{End}(V) \\ \alpha &\mapsto (\alpha\cdot : \vec{x} \mapsto \alpha\vec{x}) \qquad \alpha\cdot : V \to V \end{aligned}$$

The ring homomorphism is a (ring) action of the field F on the abelian group $(V,+)$, called *scalar multiplication*. The ring action can be written as a binary operation:

$$\begin{aligned} \cdot &: F \times V \to V \\ (\alpha,\vec{x}) &\mapsto \alpha\vec{x} \end{aligned}$$

Definition 1.4.2 — Linear Spaces / Vector Spaces. A linear space / vector space is a set with a linear structure over a field on the set.

Corollary 1.4.1 — Linear Spaces. A linear space over a field F is a set V equipped with two operations: vector addition $+ : V \times V \to V$ and scalar multiplication $\cdot : F \times V \to V$, satisfying the following axioms for all $\vec{u},\vec{v},\vec{w} \in V$ and $\alpha,\beta \in F$:

Axiom	Statement
1. Associativity of addition	$(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$
2. Existence of additive identity	$\exists\vec{0} \in V$ such that $\forall\vec{u} \in V$, $\vec{u}+\vec{0} = \vec{u}$
3. Existence of additive inverses	$\forall\vec{u} \in V$, $\exists -\vec{u} \in V$ such that $\vec{u}+(-\vec{u}) = \vec{0}$
4. Commutativity of addition	$\vec{u}+\vec{v} = \vec{v}+\vec{u}$
5. Distributivity of scalar multiplication with respect to vector addition	$\alpha(\vec{u}+\vec{v}) = \alpha\vec{u}+\alpha\vec{v}$
6. Distributivity of scalar multiplication with respect to field addition	$(\alpha+\beta)\cdot = \alpha\cdot+\beta\cdot$
7. Compatibility of scalar multiplication with field multiplication	$(\alpha\beta)\cdot = (\alpha\cdot)\circ(\beta\cdot)$
8. Identity element of scalar multiplication	$F \ni 1 = (1\cdot : x \mapsto x) \in \mathrm{End}(V)$

Note that the first four axioms ensure that $(V,+)$ is an abelian group, while the fifth axiom describes the endomorphism structure and the last three axioms describe the ring homomorphism.

■ **Example 1.12** F is a linear space over itself with the usual addition and multiplication operations.

$$\begin{aligned} \cdot &: F \times F \to F \\ (\alpha,\beta) &\mapsto \alpha\beta \end{aligned}$$

The first F is the field acting on the second F, which is the abelian group. ■

■ **Example 1.13** Let X be a set and F be a field. (f is a set map)

$$\begin{aligned} F[[X]] = \mathrm{Map}(X,F) &\overset{\text{def}}{=\!=} \text{the set of all } F\text{-valued functions on } X \\ &=\!= \{f : X \to F\} \end{aligned}$$

$F[[X]]$ is a linear space over F with the following operations defined pointwisely:

$$\begin{aligned} + : F[[X]] \times F[[X]] &\to F[[X]] \\ (f,g) &\mapsto (f+g : x \mapsto f(x)+g(x)) \qquad f+g : X \to F \end{aligned}$$

$$\begin{aligned} \cdot : F \times F[[X]] &\to F[[X]] \\ (\alpha, f) &\mapsto (\alpha f : x \mapsto \alpha f(x)) \qquad \alpha f : X \to F \end{aligned}$$

■

■ **Example 1.14** Let X be a set and F be a field.

$$\begin{aligned} F[X] = \mathrm{Map}_{\text{fin}}(X,F) &\overset{\text{def}}{=\!=} \text{the set of all finitely supported } F\text{-valued functions on } X \\ &=\!= \{f : X \to F \mid f \text{ is finitely supported}\} \end{aligned}$$

$F[X]$ is a linear space over F as $F[X] \subseteq F[[X]]$ and the operations are defined pointwisely as in the previous example.

$f : X \to F$ is finitely supported if the set $\{x \in X \mid f(x) \neq 0\}$ is finite or $f(x) \neq 0$ for only finitely many $x \in X$. ■

■ **Example 1.15** Let t be a formal variable. Then $F[[t]] \overset{\text{def}}{=\!=} F[[\{1,t,t^2,\cdots\}]] = \sum_{n=0}^{\infty} a_n t^n$ is the set of all formal power series in t with coefficients in F and $F[t] \overset{\text{def}}{=\!=} F[\{1,t,t^2,\cdots\}] = \sum_{n=0}^{N} a_n t^n$ is the set of all polynomials in t with coefficients in F. Both $F[[t]]$ and $F[t]$ are linear spaces over F. ■

■ **Example 1.16** Let n be a positive integer and F be a field. Then

$$F^n \overset{\text{def}}{=\!=} \left\{ \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \;\middle|\; c_i \in F \right\}$$

is the set of all *column matrices* with n entries in F. Elements in F^n are written as $\vec{x}$ and are called *column vectors*. F^n is a linear space over F with the following operations defined entrywisely:

$$\begin{aligned} + : F^n \times F^n &\to F^n \\ (\vec{a},\vec{b}) &\mapsto \vec{a}+\vec{b} = \begin{bmatrix} a_1+b_1 \\ \vdots \\ a_n+b_n \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \cdot : F \times F^n &\to F^n \\ (\alpha,\vec{a}) &\mapsto \alpha\vec{a} = \begin{bmatrix} \alpha a_1 \\ \vdots \\ \alpha a_n \end{bmatrix} \end{aligned}$$

F^n is a linear space over F automatically as F is a linear space over itself. ■

2. Linear Maps and Matrices

2.1 Linear Maps

Definition 2.1.1 — Linear Maps. Let V and W be two linear spaces over a field F. A linear map is a set map $f : V \to W$ such that for all $\vec{u}, \vec{v} \in V$ and $\alpha \in F$, the following holds:

$$f(\vec{u}+\vec{v}) = f(\vec{u}) + f(\vec{v})$$
$$f(\alpha\vec{u}) = \alpha f(\vec{u})$$

The set of all linear maps from V to W is denoted by $\mathscr{L}(V,W) = \mathrm{Hom}(V,W)$.

Definition 2.1.2 — Linear Combination. Let V be a linear space over a field F. A *linear combination* of vectors $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n \in V$ is a vector of the form:

$$\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n$$

where $\alpha_1, \alpha_2, \cdots, \alpha_n \in F$ are scalars.

Corollary 2.1.1 — Linear Maps and Linear Combinations. A set map $f : V \to W$ between two linear spaces over a field F is a linear map if and only if f respects linear combinations, i.e., for all $\vec{v}_1, \vec{v}_2 \in V$ and all scalars $\alpha_1, \alpha_2 \in F$, the following holds:

$$f(\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2) = \alpha_1 f(\vec{v}_1) + \alpha_2 f(\vec{v}_2)$$

■ **Example 2.1** Let A be an $m \times n$ matrix with entries in a field F. The map $T : F^n \to F^m$ defined by

$$T\vec{x} = T(\vec{x}) = A\vec{x}$$

where the multiplication on the right-hand side is the usual matrix multiplication, is a linear map over F. ■

Proposition 2.1.2 A linear map $T : F^n \to F^m$ is a matrix multiplication by a unique $m \times n$ matrix

A with entries in F. The matrix A is called the *standard matrix* of the linear map T.

$$\begin{aligned}\{\text{Linear maps over } F\} &\equiv \{m \times n \text{ matrices over } F\} \\ A\cdot : \vec{x} &\mapsto A\vec{x} \\ \leftarrow\!\shortmid A & \\ T &\mapsto \begin{bmatrix} T\vec{e}_1 & T\vec{e}_2 & \cdots & T\vec{e}_n \end{bmatrix}\end{aligned}$$

where the $\equiv$ sign means they are natural identifications and $\vec{e}_i \in F^n$ is the i-th standard basis vector, i.e. the column matrix with 1 in the i-th row and 0 elsewhere.

Proof. ■