Math 2131 **Name:** ________________________
Fall 2025
Final Exam
19/12/2025
Time Limit: 180 Minutes **Student ID:** ________________________

This exam contains 15 pages (including this cover page) and 7 questions. Total of points is 140.

Grade Table (for teacher use only)

Question	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
6	20	
7	20	
Total:	~~120~~ 140	

1. (20 points) True or False (`Let` $\mathbb{F}$ `be a field and` n `be a positive integer.`)

   (1) (2 points) Any two $\mathbb{F}$-linear spaces of dimension n are isomorphic.

   (2) (2 points) Any two Euclidean vector spaces of dimension n are isomorphic; any two real symplectic vector spaces of dimension $2n$ are isomorphic; and any two Hermitian vector spaces of dimension n are isomorphic.

   (3) (2 points) Any two pseudo-Euclidean vector spaces of dimension n are isomorphic.

   (4) (2 points) For any finite-dimensional $\mathbb{F}$-linear space V, the set of quadratic forms on V and the set of symmetric 2-forms on V are in one-to-one correspondence.

   (5) (2 points) If an $\mathbb{F}$-linear map is represented by two $m \times n$ matrices A and B, then $A = PBQ^{-1}$ for some $(P, Q) \in \mathrm{GL}_m(\mathbb{F}) \times \mathrm{GL}_n(\mathbb{F})$.

   (6) (2 points) If an endomorphism of an $\mathbb{F}$-linear space is represented by two $n \times n$ matrices A and B, then $A = PBP^T$ for some $P \in \mathrm{GL}_n(\mathbb{F})$.

   (7) (2 points) If a two-form on an $\mathbb{F}$-linear space is represented by two $n \times n$ matrices A and B, then $A = P^T BP$ for some $P \in \mathrm{GL}_n(\mathbb{F})$.

   (8) (2 points) The matrix representation of the inner product of a Euclidean vector space with respect to any orthonormal basis is the identity matrix.

   (9) (2 points) The matrix representation of the inner product of a Euclidean vector space with respect to any basis is a real positive matrix.

   (10) (2 points) The matrix representation of the Hermitian inner product of a Hermitian vector space with respect to any basis is a Hermitian positive matrix.

Please fill in T for True and F for False for your answer.

Question #	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
Answer	T	T	F	F	T	F	T	T	T	T

2. (20 points) Let A be a real symmetric matrix of order n. Since A is also a Hermitian matrix of order n, there exists a unitary matrix U and a real diagonal matrix D such that $A = U^\dagger DU$. Based on this fact, show that there exists an orthogonal matrix O such that $A = O^T DO$. **Hint:** `Construct` O `from` U.

Solution. By the Spectral theorem for Hermitian matrix, we know that there is an orthogonal decomposition $\mathbb{C}^n = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_k}$ where $\{\lambda_1, \cdots, \lambda_k\}$ is the set of eigenvalues of A (viewed as a Hermitian matrix) and $\{E_{\lambda_1}, \cdots, E_{\lambda_k}\}$ is the set of eigenspaces of A.

Since A and λ_i are real, each eigenspace is invariant under complex conjugation σ, that is because $Au = \lambda_i u \Leftrightarrow A\bar{u} = \lambda_i \bar{u}$.

Let $V_i = E_{\lambda_i}^\sigma = \{u \in E_{\lambda_i} \mid \bar{u} = u\}$.

Then

$$\mathbb{R}^n = (\mathbb{C}^n)^\sigma = E_{\lambda_1}^\sigma \oplus \cdots \oplus E_{\lambda_k}^\sigma = V_1 \oplus \cdots \oplus V_k.$$

Since A is symmetric, this decomposition of $\mathbb{R}^n$ is an orthogonal decomposition: for $v_i \in V_i$, $v_j \in V_j$

$$0 = v_i \cdot Av_j - Av_i \cdot v_j = (\lambda_j - \lambda_i)\, v_i \cdot v_j$$

if $i \neq j$ $\Rightarrow$ $v_i \cdot v_j = 0$, so $V_i \perp V_j$.

Now, fix an orthonormal basis B_i for each V_i, then $\mathcal{B} \overset{\text{def}}{=} \mathcal{B}_1 \sqcup \mathcal{B}_2 \sqcup \cdots \sqcup \mathcal{B}_k$ is an orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors, so we have

$$A = ODO^T$$

where D is a real diagonal matrix and O is the orthogonal matrix whose columns are the vectors in the orthonormal basis $\mathcal{B}$ of $\mathbb{R}^n$.

3. (20 points) Let A be a real symmetric matrix of order n, and let $Q : \mathbb{R}^n \to \mathbb{R}$ be the quadratic form that sends $\vec{x}$ to $\vec{x} \cdot A\vec{x}$. Since Q is continuous and the unit sphere

$$\{\vec{x} \in \mathbb{R}^n \mid \vec{x} \cdot \vec{x} = 1\}$$

is compact, it follows that Q achieves both a maximum and a minimum on the unit sphere.

(1) (10 points) Find the maximum and minimum values of Q on the sphere in terms of the eigenvalues of A.

(2) (10 points) Determine the points on the sphere at which Q achieves its maximum and minimum.

Solution: Since A is real symmetric, we have

$$A = O^T D O$$

where $D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$ is a real diagonal matrix and O is an orthogonal matrix. Then

$$Q(\vec{x}) = \vec{x}^T A \vec{x} = \vec{x}^T O^T D O \vec{x} = (O\vec{x})^T D (O\vec{x})$$

let $\vec{y} = O\vec{x}$, then $Q(\vec{x}) = \vec{y}^T D \vec{y}$ or

$$\tilde{Q}(\vec{y}) \overset{\text{def}}{=} Q(O^T\vec{y}) = \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2 .$$

(1) O^T is an automorphism of the unit sphere, so the maximum/minimum value of Q = the maximum/minimum value of $\tilde{Q}$ = the maximum/minimum of $\{\lambda_1, \cdots, \lambda_n\}$, i.e. the maximum/minimum eigenvalue of A.

(2) Q achieves the maximum/minimum value precisely on the points which are unit eigenvectors with maximum/minimum value of A.

4. (20 points) Let T be a self-adjoint operator on a Euclidean space V of dimension 3. Assume that T has rank 2.

   (1) (5 points) Show that V admits an *orthogonal* decomposition $V = V_1 \oplus V_2$ such that $\dim V_i = i$ for each i, and with respect to this decomposition, T decomposes as $T = 0 \oplus \bar{T}$.

   (2) (5 points) Prove that $\bar{T}$ is a self-adjoint operator on the Euclidean subspace V_2 of V.

   (3) (10 points) Find a formula for $\det \bar{T}$ in terms of T. **Hint:** `The Feynman diagram formula given in class should be useful.`

Solution. (1) Let $V_2 = \operatorname{Im} T$ and $V_1 = V_2^\perp$. Then V admits an orthogonal decomposition

$$V = V_1 \oplus V_2 \qquad (*)$$

To show that $T = 0 \oplus \bar{T}$ w.r.t. this decomposition, we need to verify that

(a) $Tv_1 = 0$ for all $v_1 \in V_1 \overset{\text{def}}{=} V_2^\perp$

(b) $Tv_2 \in V_2$ for all $v_2 \in V_2 \overset{\text{def}}{=} \operatorname{Im} T$

(b) is clear. To see (a), we note that

$$\langle Tv_1, Tv_1 \rangle \overset{T^t = T}{=} \langle \underbrace{v_1}_{\in V_2^\perp}, \underbrace{T^2 v_2}_{\in V_2} \rangle = 0, \text{ so } Tv_1 = 0.$$

Since $\dim V_2 = r(T) = 2$, $\dim V_1 = \underbrace{\dim V}_{=3} - \underbrace{\dim V_2}_{=2} = 1$.

In summary $\dim V_i = i$.

(2) For $u, v \in \operatorname{Im} T$, we have

$$\langle \bar{T}u, v \rangle \overset{\text{def of } \bar{T}}{=} \langle Tu, v \rangle \overset{T^t = T}{=} \langle u, Tv \rangle \overset{\text{def of } \bar{T}}{=} \langle u, \bar{T}v \rangle,$$

so $\bar{T}: V_2 \to V_2$ is self-adjoint.

(3) $\det \bar{T} = \frac{1}{2!}(\operatorname{tr} \bar{T})^2 - \frac{1}{2} \operatorname{tr} \bar{T}^2$ (Feynman diagram Formula)

Since $T = 0 \oplus \bar{T}$, we have $T^2 = 0 \oplus \bar{T}^2$. Then

$\operatorname{tr} T = \operatorname{tr} \bar{T}$, $\operatorname{tr} \bar{T}^2 = \operatorname{tr} T^2$, so

$$\det \bar{T} = \frac{1}{2!}(\operatorname{tr} T)^2 - \frac{1}{2} \operatorname{tr} T^2.$$

5. (20 points) Recall that a complex square matrix A is *normal* if $A^\dagger A = AA^\dagger$.

   (1) (15 points) Show that a complex matrix A is diagonalizable *by a unitary matrix* if and only if A is normal.

   (2) (5 points) Provide an example of a diagonalizable complex matrix that is not normal.

Solution. (1) $\Rightarrow$. Assume $A = U^\dagger D U$ where U is a unitary matrix and D is a diagonal matrix. Then $A^\dagger = U^\dagger D^\dagger U$

Since D is diagonal, $D^\dagger = \bar{D}$ is also diagonal, then $D^\dagger D = DD^\dagger$, so

$$A^\dagger A = U^\dagger D^\dagger U U^\dagger D U = \underline{U^\dagger D^\dagger D U}$$

$$AA^\dagger = U^\dagger D U U^\dagger D^\dagger U = \underline{U^\dagger D D^\dagger U}$$

$\Leftarrow$. Write $A = B + iC$ where

$$B = \frac{A + A^\dagger}{2}, \quad C = \frac{A - A^\dagger}{2i}$$

are both Hermitian matrices. Then

$$A^\dagger A = AA^\dagger \iff BC = CB.$$

so, by the simultaneous diagonalization theorem $\exists$ a unitary matrix U and a real diagonal matrices D_1 and D_2 such that

$$B = U^\dagger D_1 U, \quad C = U^\dagger D_2 U$$

Then $A = U^{\dagger}(\underbrace{D_1 + iD_2}_{\overset{\text{def}}{=} D})U$ ← a complex diagonal matrix

i.e. A is diagonalized by a unitary matrix.

(2) Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. Since A has 2 distinct eigenvalues and A is a square matrix of order 2, A must be diagonalizable. However, A is not normal:

$$A^{\dagger}A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$AA^{\dagger} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

6. (20 points) Let ω be a symplectic form on a real linear space V. Then $\dim V = 2n$ for some integer $n \geq 1$. Show that there exists a symplectic basis for the real symplectic vector space (V, ω), i.e., a basis $(x_1, y_1, \ldots, x_n, y_n)$ such that

$$\omega(x_i, x_j) = \omega(y_i, y_j) = 0, \quad \omega(x_i, y_j) = \delta_{ij}.$$

Hint: You may use the fact that any Hermitian matrix can be diagonalized by a unitary matrix.

Solution: WLOG, we may assume $V = \mathbb{R}^{2n}$ and $\omega(x, y) = x \cdot Ay$ where A is an invertible real skew-symmetric matrix of order $2n$.

By applying the spectral thm to the Hermitian matrix iA, we arrive at the eigenspace decomposition of A ← viewed as a complex matrix:

(1) $$\mathbb{C}^{2n} = \bigoplus_{j=1}^{k} \underbrace{\left(E_{i\lambda_j} \oplus \overline{E_{i\lambda_j}}\right)}_{\text{invariant under } \sigma}, \quad 0 < \lambda_1 < \cdots < \lambda_k$$

where $E_{i\lambda_j} \overset{\text{def}}{=}$ the eigenspace of A with eigenvalue $i\lambda_j$

$\overline{E_{i\lambda_j}} \overset{\text{def}}{=}$ the complex conjugation σ of $E_{i\lambda_j} \overset{\uparrow}{=}$ the eigenspace of A with eigenvalue $-i\lambda_j$. The reason is that

$$Au = i\lambda_j u \iff A\bar{u} = -i\lambda_j \bar{u}$$

Taking the fixed-point set of the complex conjugation $\sigma: u \to \bar{u}$, we arrive at the decomposition

(2) $$\mathbb{R}^{2n} = \bigoplus_{j=1}^{k} V_{\lambda_j}, \quad \text{where } V_{\lambda_j} = \left(E_{i\lambda_j} \oplus \overline{E_{i\lambda_j}}\right)^{\sigma}$$

let m_A = the multiplication by A, then viewed as a complex matrix

$$m_A = T_1 \oplus \cdots \oplus T_k \quad \text{w.r.t. decomposition (1)}$$

Since $m_A \circ \sigma = \sigma \circ m_A$, ∵ A is real

viewed as a real matrix

$$m_A = S_1 \oplus \cdots \oplus S_k \quad \text{w.r.t. decomposition (2)}$$

Since $T_j^2 = -\lambda_j^2 1_{E_{i\lambda_j} \oplus \overline{E_{i\lambda_j}}}$, we have

$$S_j^2 = -\lambda_j^2 1_{V_j}$$ ∵ $S_j^2 u = T_j^2 u$ for any $u \in V_j \subset E_{i\lambda_j} \oplus \overline{E_{i\lambda_j}}$

Since A^2 is real symmetric, its eigenspaces V_j (with eigenvalue $-\lambda_j^2$) are orthogonal. Then

∵ $-\lambda_j^2 u_i \cdot u_j = u_i \cdot A^2 u_j = A^2 u_i \cdot u_j = -\lambda_i^2 u_i \cdot u_j$

$\omega = \bigoplus_i \omega_i$ where ω_i is a symplectic form on V_{λ_i}. Indeed, for $v_i \in V_{\lambda_i}$, $v_j \in V_{\lambda_j}$, we have

$$\omega(v_i, v_j) = v_i \cdot A v_j = \underbrace{v_i}_{\text{in } V_i} \cdot \underbrace{S_j v_j}_{\text{in } V_j} = 0 \text{ if } i \neq j$$

∵ $V_i \perp V_j$

Then the proof is reduced to (You need to do a translation)

<u>Claim:</u> Let ω be a symplectic form on the real linear space with an inner product $(\,,\,)$. Suppose that $\omega(u, v) = (u, Sv)$ where $S \in \operatorname{End} V$ is skew-adjoint w.r.t. $(\,,\,)$ and $S^2 = -\lambda^2 1_V$, $\lambda > 0$

then (V, ω) has a symplectic basis.

Proof: Let $J = \frac{1}{\lambda} S$. Then $J^2 = -1_V$.

Also $(Ju, Jv) = \frac{1}{\lambda^2}(Su, Sv) = \frac{1}{\lambda^2}\omega(Su, v)$

$$= -\frac{1}{\lambda^2}\omega(v, Su) = -\frac{1}{\lambda^2}(v, S^2u) = (v, u) = (u, v)$$

So J is a complex structure that is compatible with the inner product $(\ ,\ )$. Then we can form the Hermitian inner product on the complex linear space (V, J):

$\because \frac{1}{\lambda}\omega(u,v) = \frac{1}{\lambda}(u, Sv) = (u, Jv)$

$$\langle\ ,\ \rangle = (\ ,\ ) - \sqrt{-1}\,\frac{1}{\lambda}\omega.$$

Let $(v_1, \cdots, v_n)$ be an orthogonal basis of $(V, J, \langle , \rangle)$ with $\|v_i\| = \lambda^{-1}$, then $(Jv_1, v_1, \cdots, Jv_n, v_n)$ is a symplectic basis of (V, ω). Indeed,

a Hermitian vector space

$$\omega(v_i, v_j) = \operatorname{Im} -\lambda\langle v_i, v_j\rangle = \operatorname{Im} - \delta_{ij} = 0$$

$$\omega(Jv_i, Jv_j) = (Jv_i, SJv_j) = \lambda(Jv_i, J^2v_j)$$
$$= \lambda(v_i, Jv_j) = (v_i, Sv_j) = \omega(v_i, v_j)$$
$$= 0$$

$$\omega(Jv_i, v_j) = (Jv_i, Sv_j) = \lambda(Jv_i, Jv_j) = \lambda(v_i, v_j)$$
$$= \lambda \operatorname{Re}\langle v_i, v_j\rangle = \lambda \operatorname{Re} \lambda^{-1}\delta_{ij} = \delta_{ij}$$

So we are done.

7. (20 points) Let V be a real 2-dimensional Euclidean vector space, with the inner product denoted by I. Suppose II is another symmetric 2-form on V. Let (θ_1, θ_2) be a basis of the dual space V^*. We can express the symmetric 2-forms as follows:

$$\mathrm{I} = E\theta_1^2 + 2F\theta_1\theta_2 + G\theta_2^2\,, \quad \mathrm{II} = e\theta_1^2 + 2f\theta_1\theta_2 + g\theta_2^2$$

for some real numbers E, F, G, e, f, g. Here, θ^2 refers to the symmetric tensor product of θ with itself, and $\theta_1\theta_2$ refers to the symmetric tensor product of θ_1 with θ_2.

(1) (5 points) Show that there exists a unique self-adjoint operator T on V such that

$$\mathrm{II}(u, v) = \mathrm{I}(Tu, v). \iff \mathrm{II}_\flat(u) = \mathrm{I}_\flat \circ T(u) \text{ all } u$$

all u, v

$\Updownarrow$

Hint: You may either use the idea of representation or draw commutative diagrams.

(2) (5 points) Prove that $E > 0$ and $EG - F^2 > 0$.

$\mathrm{II}_\flat = \mathrm{I}_\flat \circ T$

(3) (10 points) Compute $\det T$ and express your answer in terms of E, F, G, e, f, g.

Solution. (1) I and II yield two linear maps from V to V^*

$$\begin{tikzcd} & V \arrow[d, "\mathrm{I}_\flat", "\cong"'] \\ V \arrow[ur, "T"] \arrow[r, "\mathrm{II}_\flat"'] & V^* \end{tikzcd}$$

Being an inner product, I is non-degenerate, i.e. $\mathrm{I}_\flat$ is a linear equivalence, so $\exists!\, T \in \mathrm{End}\, V$ s.t. the triangle is commutative, i.e. $\mathrm{II}_\flat = \mathrm{I}_\flat \circ T$. Since

$\mathrm{II}(u,v) = \mathrm{I}(Tu, v)$ all u, v $\iff \mathrm{II}_\flat(u) = \mathrm{I}_\flat \circ T(u)$ all u

$\iff \mathrm{II}_\flat = \mathrm{I}_\flat \circ T$, we are done.

To see T is self-adjoint, we may pick an orthonormal basis with respect to which I is

represented by the identity matrix I_2, II is represented by a real symmetric matrix S, then I_4 is represented by I_2 and II_4 is represented matrix S provided the V^* is given the dual basis. Then relation $\mathrm{II}_4 = \mathrm{I}_4 \circ T$ says T is represented by the real symmetric matrix S w.r.t. to the chosen orthonormal basis on V, so T is self-adjoint ($\because$ $T^\dagger \leftrightarrow S^T$
$$\begin{matrix} T^\dagger & \leftrightarrow & S^T \\ \overset{?}{\|} & & \| \\ T & \leftrightarrow & S \end{matrix}$$
)

(2) Since I is positive definite, its matrix rep. for example $A = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$, is positive, then $E = \det A_{11} > 0$, $EG - F^2 = \det A_{22} > 0$

(3) In terms of matrix representation, relation $\mathrm{II}_4 = \mathrm{I}_4 \circ T$ becomes the matrix relation

$$\begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} [T]_B$$

$\swarrow$ the basis dual to the basis (θ_1, θ_2)

Then $\det T = \det [T]_B = \dfrac{eg - f^2}{EG - F^2}$.

the matrix rep. w.r.t. basis B of V.