In this homework assignment, we let $\mathrm{Vec}_{\mathbb{F}}^{f.d.}$ denote the category of linear maps between finite dimensional linear spaces over $\mathbb{F}$ and $V_i (i = 1, 2, 3)$ and V be objects in $\mathrm{Vec}_{\mathbb{F}}^{f.d.}$. Let $m = \dim V_1$ and $n = \dim V_2$.

1. Let $f\colon M_{n\times n}(\mathbb{F}) \to \mathbb{F}$ be a function of matrix variable which is linear in each columns and skew-symmetric in columns. Show that $f(A) = \det A\, f(I)$.

2. Let V be a f.d. linear space and V^* be its dual space. The pairing $\langle , \rangle\colon V^* \times V \to \mathbb{F}$ that send (α, v) to $\alpha(v)$ can be extended to the pairing

$$\langle , \rangle : \quad \wedge^k V^* \times \wedge^k V \to \mathbb{F}.$$

By definition, this is the unque bi-linear map such that

$$\langle \alpha_1 \wedge \cdots \wedge \alpha_k, v_1 \wedge \cdots \wedge v_k \rangle = \det[\langle \alpha_i, v_j \rangle].$$

   (a) For any basis $v = (v_1, \ldots, v_n)$ of V and any ordered set $I_k = (i_1, ..., i_k)$ of k numbers with $1 \le i_1 < i_2 < \cdots < i_k \le n$, we let $v_{I^k} := v_{i_1} \wedge \cdots \wedge v_{i_k}$. Show that the set $\{v_{I^k}\}$ is a minimal spanning set for $\wedge^k V$.

   (b) For each $k \ge 1$, show that the paring is non-degenerate, i.e., the resulting map $\wedge^k V^* \to (\wedge^k V)^*$ is an isomorphsim.

   (c) Show that $\wedge^k V^* \equiv (\wedge^k V)^*$ in the sense that the two functors from $(\mathrm{Vec}_{\mathbb{F}}^{f.d.})^{op}$ to $\mathrm{Vec}_{\mathbb{F}}^{f.d.}$ are equivalent. In particular, we have $\det V^* \equiv (\det V)^*$

   (d) Denote by $\mathcal{B}_U$ the set of bases of U. Show that the diagram

$$\begin{tikzcd}
\mathcal{B}_V \arrow[rr, "\equiv"] \arrow[d] & & \mathcal{B}_{V^*} \arrow[d] \\
\mathcal{B}_{\det V} \arrow[r, "\equiv"] & \mathcal{B}_{(\det V)^*} \arrow[r, "\equiv"] & \mathcal{B}_{\det V^*}
\end{tikzcd}$$

   is commutative. Here, a vertical map always sends basis $u = (u_1, .., u_n)$ to its determinant $\det u := u_1 \wedge \cdots \wedge u_n$, the horizontal arrows map either sends a basis to its dual basis or is the isomorphism in part (b).

3. Let $\phi\colon A \to B$ be a linear map between finite dimensional linear spaces of equal dimension. Denoted by ϕ^* the dual linear map of ϕ. Since $\det \phi \in \mathrm{Hom}(\det A, \det B) \equiv (\det A)^* \otimes \det B$ and

$$\det \phi^* \in \mathrm{Hom}(\det B^*, \det A^*) \equiv (\det B^*)^* \otimes \det A^* \equiv (\det A)^* \otimes \det B,$$

it makes sense that $\det \phi^* \equiv \det \phi$. Show that this is indeed the case. This implies that $\det M^T = \det M$ for any square matrix M.

4. In class we introduced the adjoint matrix adj A for any square matrix A of order n and showed that $A\,\mathrm{adj}\,A = \mathrm{adj}\,A\,A = \det A\,I$. Assuming A is invertible, which is equivalent to say that equation $A\vec{x} = \vec{b}$ has a unique solution for any vector $\vec{b}$ in $\mathbb{F}^n$. Indeed, the unique solution is $\vec{x} = A^{-1}\vec{b}$.

(a) Show that the unique solution is $\vec{x} = \frac{1}{\det A}\mathrm{adj}\, A\,\vec{b}$.

(b) If x_i denotes the i-th entry of the solution $\vec{x}$, show that

$$x_i = \frac{\Delta_i}{\Delta}$$

where $\Delta = \det A$ and $\Delta_i = \det A_i$ with A_i being the matrix obtained from A by replacing its i-the column by the column vector $\vec{b}$. `This is called Cramer's rule`

5. Let A be a $M_{n\times n}(\mathbb{R})$-valued function of one real variable t and A' be its (entry-wise) derivative with respect to t. Show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\det A = \mathrm{tr}\,(A'\,\mathrm{adj}\,A).$$

6. (Optional exercise). Suppose that A and B are square matrices of order 3 over a field of characteristic 0. Please derive a formula for $\det(A+B)$ of the form

$$\det(A+B) = \det A + \det B + \cdots .$$

`Hint: Use the Feynman diagram formula introduced in class.`