Let $\mathbb{F}$ be a field. Recall that a column vector in $\mathbb{F}^n$ is denoted by $\vec{u} = \begin{bmatrix} u^1 \\ \vdots \\ u^n \end{bmatrix}$ and $\vec{e}_i \in \mathbb{F}^n$ is the column vector whose i-th entry is 1 and other entries are zero. Let $(\mathbb{F}^m)^*$ denote the linear space of row vectors with m entries in $\mathbb{F}$. A row vector in $(\mathbb{F}^m)^*$ shall be denoted by $\mu = (\mu_1, \cdots, \mu_m)$ and $\hat{e}^i \in (\mathbb{F}^m)^*$ denote the transpose of $e_i \in \mathbb{F}^m$. We write an $m\times n$-matrix A as $[\vec{a}_1, \cdots, \vec{a}_n]$ with $\vec{a}_j$ being the j-th column of A or as $\begin{bmatrix} \hat{\alpha}^1 \\ \vdots \\ \hat{\alpha}^m \end{bmatrix}$ with $\hat{\alpha}^i$ being the i-th row of A. The (i,j)-entry of A is denoted by $a^i{}_j$. Recall that the j-th column of A is the column matrix $A\vec{e}_j$.

1. Show that
   (a) for any $m \times n$-matrix A, the map $(\mathbb{F}^m)^* \to (\mathbb{F}^n)^*$ that sends α to αA is a linear map.
   (b) any linear map ϕ: $(\mathbb{F}^m)^* \to (\mathbb{F}^n)^*$ is of the form $\phi(\alpha) = \alpha A$ for a unique matrix A.
   (c) the i-th row of A is the row matrix $\hat{e}^i A$.
   (d) $a^i{}_j = \hat{e}^i A \vec{e}_j$.
   (e) $A = \sum_{1\le i\le m, 1\le j\le n} a^i{}_j E_i^j$ where $E_i^j = e_i \hat{e}^j$

In class we have shown that an elementary row operation on a column matrix in $\mathbb{F}^m$ is an invertible linear map from $\mathbb{F}^m$ to itself (we call it an automorphism of the linear space $\mathbb{F}^m$), thus it is the multiplication from left by an invertible matrix E of order m. Then, if $A = [\vec{a}_1, \cdots, \vec{a}_n] \in M_{m\times n}(\mathbb{F})$, this elementary row operation on A is the simultaneous elementary row operation on all columns of A, thus it turns A into $[E\vec{a}_1, \ldots, E\vec{a}_n]$ which is EA, so it defines an automorphism of the linear space $M_{m\times n}(\mathbb{F})$.

Dually, an elementary column operation on a row matrix in $(\mathbb{F}^n)^*$ is an automorphism of the linear space $(\mathbb{F}^n)^*$, thus it is the multiplication from right by an invertible matrix F of order n. Then, if $A \in M_{m\times n}(\mathbb{F})$, this elementary column operation on A is the simultaneous elementary column operation on all rows of A, thus it turns A into $\begin{bmatrix} \hat{\alpha}^1 F \\ \vdots \\ \hat{\alpha}^m F \end{bmatrix}$ which is AF, so it defines an automorphism of the linear space $M_{m\times n}(\mathbb{F})$.

2. Show that an elementary matrix E that corresponds to an elementary row operation is also an elementary matrix F that corresponds to an elementary column operation. Prove by induction that any matrix can be turned into a matrix of the block form $\begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$ by finitely many elementary row or column operations. Here, I_r denotes the identity matrix of order r and matrices O denote zero matrices.

3. Let $r \le s \le n$ be non-negative integers. Denote by A_r the square matrix of order n of the block form $\begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$. Show that, if there are invertible matrices P and Q such that $PA_rQ^{-1} = A_s$, then $r = s$.

4. Let $T\colon V \to W$ be a linear map between finite dimensional linear spaces over $\mathbb{F}$. Show that T is injective $\iff$ T has a left inverse, and dually T is surjective $\iff$ T has a right inverse.

5. Conitnuing Exercise 3 in Assignment 1, please show that both T_* and T^* are linear maps. Please also show that, if T is a bijection, then both T_* and T^* are linear equivalences.

6. Let V be an n-dimensional linear space, and $S = (v_1, \ldots, v_k)$ be an ordered set of k vectors in V. Let $\phi_S\colon \mathbb{F}^k \to V$ be the linear map that sends $\vec{x} \in \mathbb{F}^k$ to $x^1v_2 + \cdots + x^kv_k$. Show that

   (a) S is a linearly independent set $\iff$ ϕ_S is injective.

   (b) S is a spanning set for V $\iff$ ϕ_S is surjective.

   (c) S is a minimal spanning set for V $\iff$ ϕ_S is invertible. Note: a minimal ordered spanning set is called a basis.

   `In case` S `is a basis, the inverse` ϕ_S^{-1} `is written as` $[\,]_S$ `and is called the coordinate map with respect to basis` S`. The coordinate map` $[\,]_S$ `is also called the trivialisation of` V `with respect to basis` S`.`