In this homework assignment, we let $\mathrm{Vec}_{\mathbb{F}}^{f.d.}$ denote the category of linear maps between finite dimensional linear spaces over $\mathbb{F}$ and $V_i (i = 1, 2, 3)$ and V be objects in $\mathrm{Vec}_{\mathbb{F}}^{f.d.}$. Let $m = \dim V_1$ and $n = \dim V_2$.

1. Show that there is a functor $\otimes$ from the product category $\mathrm{Vec}_{\mathbb{F}}^{f.d.} \times \mathrm{Vec}_{\mathbb{F}}^{f.d.}$ to the category $\mathrm{Vec}_{\mathbb{F}}^{f.d.}$ that sends an object (V_1, V_2) to $V_1 \otimes V_2$ and a morphism (f, g) to $f \otimes g$. Hint: please use the universal property of the tensor product.

   Show that $f \otimes g$ is bilinear, meaning it is linear in both f and g.

   Finally, show that the functors $- \otimes V$, $\mathrm{Hom}(V, -)$, and $\mathrm{Hom}(-, V)$ preserve exactness: If $A \to B \to C$ is exact, then the sequences $A \otimes V \to B \otimes V \to C \otimes V$, $\mathrm{Hom}(A, V) \leftarrow \mathrm{Hom}(B, V) \leftarrow \mathrm{Hom}(C, V)$, and $\mathrm{Hom}(V, A) \to \mathrm{Hom}(V, B) \to \mathrm{Hom}(V, C)$ are also exact.

2. (a) Show that functors $**$, $- \otimes \mathbb{F}$, $\mathbb{F} \otimes -$, $\mathrm{Hom}(\mathbb{F}, -)$, and the identity functor 1 are all naturally equivalent endofunctors on the category $\mathrm{Vec}_{\mathbb{F}}^{f.d.}$. A simpler way to record these facts is to write

   $$V^{**} \equiv V \otimes \mathbb{F} \equiv \mathbb{F} \otimes V \equiv \mathrm{Hom}(\mathbb{F}, V) \equiv V$$

   (b) Show that

   $$\mathrm{Hom}(V_1, V_2 \otimes V_3) \equiv \mathrm{Hom}(V_1, V_2) \otimes V_3.$$

   Consequently $\mathrm{Hom}(V_1, V_2) \equiv V_1^* \otimes V_2$ and $(V_1 \otimes V_2)^* \equiv V_1^* \otimes V_2^*$

   (c) $\dim(V_1 \otimes V_2) = \dim V_1 \cdot \dim V_2$, moreover, if e_i is a minimal spanning set of V_1 and f_j is a minimal spanning set of V_2, then $e_i \otimes f_j$ is a minimal spanning set of $V_1 \otimes V_2$.

   (d) Show that

   $$V_1 \otimes V_2 \equiv V_2 \otimes V_1, \quad \mathrm{End}(V) \equiv (\mathrm{End}(V))^*$$

   (e) Under the natural identification $\mathrm{End}(V) \equiv (\mathrm{End}(V))^*$, 1_V is identified with a linear map tr: $\mathrm{End}(V) \to \mathbb{F}$. Show that tr is an algebra homomorphism that sends unit 1_V to $\dim V$. (So it is not a unital algebra homomorphism unless $\dim V = 1$). This map is called the trace map.

3. Let the category $\mathrm{Vec}_{\mathbb{F}}^{f.d.}$ be denoted by $\mathcal{V}$. Denote by $\mathcal{V}^{op}$ be the opposite category of $\mathcal{V}$. Show that

   (a) $V_1 \otimes (V_2 \oplus V_3) \equiv (V_1 \otimes V_2) \oplus (V_1 \otimes V_3)$. This is a natural equivalence of two functors from $\mathcal{V} \times \mathcal{V} \times \mathcal{V}$ to $\mathcal{V}$.

   (b) $\mathrm{Hom}(V_1, V_2 \oplus V_3) \equiv \mathrm{Hom}(V_1, V_2) \oplus \mathrm{Hom}(V_1, V_3)$. This is a natural equivalence of two functors from $\mathcal{V}^{op} \times \mathcal{V} \times \mathcal{V}$ to $\mathcal{V}$.

   (c) $\mathrm{Hom}(V_1 \oplus V_2, V_3) \equiv \mathrm{Hom}(V_1, V_2) \times \mathrm{Hom}(V_2, V_3)$. This is a natural equivalence of two functors from $\mathcal{V}^{op} \times \mathcal{V}^{op} \times \mathcal{V}$ to $\mathcal{V}$.