Math 2131 **Name:** ______________________
Fall 2025
Midterm Exam
24/10/2025
Time Limit: 80 Minutes **Student ID:** ______________________

This exam contains 12 pages (including this cover page) and 5 questions. Total of points is 100. You would be rewarded if you do things naturally.

Grade Table (for teacher use only)

Question	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

A solution to the Midterm exam

Note: In part (b) of Q2, we need to assume that $A \neq 0$ (i.e. $r \neq 0$) (because no I_0).

The staments in Q2 are translated from the following statements: For any linear map f between f.d. linear spaces, One can find linear maps g, h, g', h' with g, h' surjective, and h, g' injective such that $f = gh = g'h'$.

1. (20 points) Let S be a set of k vectors in an n-dim. linear space over $\mathbb{F}$. In class and a homework assignment as well, we introduce the linear map $\phi_S\colon \mathbb{F}^k \to V$. Arguing in terms of linear maps, show that

   (a) (10 points) S cannot be a linearly independent set if $k > n$. `Too many cannot be (linearly) independent`

   (b) (10 points) S cannot be a (linearly) spanning set for V if $k < n$. `Too few cannot span`

Proof.

It is equivalent to prove the contrapositive statements.

(a) Assume S is linearly ind., then $\ker \phi_S = 0$. From the exact sequence

$$0 \to \ker \phi_S \to \mathbb{F}^k \to \operatorname{Im} \phi_S \to 0$$

we have

$$(1)\quad k = \dim \mathbb{F}^k = \underbrace{\dim \phi_S}_{=0} + \dim \operatorname{Im} \phi_S = \dim \operatorname{Im} \phi_S$$

From the exact sequence

$$0 \to \operatorname{Im} \phi_S \to V \to \operatorname{Coker} \phi_S \to 0$$

we have $\underbrace{\underline{\dim V}}_{= n} = \dim \operatorname{Im} \phi_S + \dim \operatorname{Coker} \phi_S$

$\geq \dim \operatorname{Im} \phi_S \quad (2)$

(1) + (2) $\Longrightarrow k \leq n$.

(b) The dual version with a dual proof.

2. (20 points) Let A be a matrix over a field $\mathbb{F}$.

   (a) (10 points) Show that $A = BC$ with B be a matrix with independent rows and C be a matrix with independent columns.

   (b) (10 points) Show that $A = B'C'$ with B' be a matrix with independent columns and C' be a matrix with independent rows.

Solution.

Assume A is $m \times n$. Since

$$0 \to \operatorname{Nul} A \to \mathbb{F}^n \to \operatorname{Col} A \to 0$$

is exact, we have an isomorphism

$$\mathbb{F}^n \xrightarrow{\cong} \operatorname{Nul} A \oplus \operatorname{Im} A$$

Then

$$\begin{tikzcd}
 & & \mathbb{F}^{n(A)+m} \arrow[dd, bend left=80, "B \text{ has ind. row}"] \\
\mathbb{F}^n \arrow[urr, bend left, "C \text{ has ind. col.}"] \arrow[r, "\cong"] \arrow[dr] \arrow[drr, "A"] \arrow[ddr, "C' \text{ has ind. row}"'] & \operatorname{Nul} A \oplus \operatorname{Im} A \arrow[r, hook] \arrow[d, two heads] & \operatorname{Nul} A \oplus \mathbb{F}^m \arrow[u, "\cong"'] \arrow[d, two heads] \\
 & \operatorname{Im} A \arrow[r, hook] \arrow[d, "\cong"] & \mathbb{F}^m \\
 & \mathbb{F}^r \arrow[ur, "B' \text{ has ind. col.}"'] &
\end{tikzcd}$$

Here $r = \operatorname{rank} A$, $n(A) = \dim \operatorname{Nul} A$.

Cheaty Method (i.e. the ideal of representation)

WLOG we may assume that

$$A = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

Then

$$A = \underbrace{\begin{bmatrix} I_r \\ 0 \end{bmatrix}}_{B'} \underbrace{\begin{bmatrix} I_r & 0 \end{bmatrix}}_{C'}$$

Also

$$A = \underbrace{\begin{bmatrix} I_r & 0 & 0 \\ 0 & 0 & I \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} I_r & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix}}_{C}$$

<u>Remark:</u> All are fine for $r \geq 1$.

For case $r = 0$, part (a) is still fine but part (b) is not fine unless we use a convention

3. (20 points) Let A, B, C be three matrices such that the product ABC makes sense. Please prove the following inequality about rank:

$$r(B) + r(ABC) \geq r(AB) + r(BC).$$

`Hint: start with a commutative square.`

Proof.

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\mathrm{col}\,BC \arrow[r, two heads] \arrow[d] & \mathrm{col}\,ABC \arrow[d] \\
\mathrm{col}\,B \arrow[r, two heads] \arrow[d] \arrow[dr, bend left, "\phi"] & \mathrm{col}\,AB \arrow[d] \\
\mathrm{col}\,B/\mathrm{col}\,BC \arrow[r, dotted, "\exists!\,\overline{\phi}"] \arrow[d] & \mathrm{col}\,AB/\mathrm{col}\,ABC \arrow[d] \\
0 & 0
\end{tikzcd}$$

columns are exact.

diagrams are commutative.

The unique map ϕ exists because of the universal property of quotient space.

ϕ is surjective, so $\overline{\phi}$ surjective $\Longrightarrow$

" composition of surjective maps

$$\dim \operatorname{col} B / \operatorname{col} BC = \dim \ker \bar{\phi} + \dim \operatorname{col} AB / \operatorname{col} ABC$$

$$\geq \dim \operatorname{col} AB / \operatorname{col} ABC$$

$$\dim \operatorname{col} B / \operatorname{col} BC = \dim \operatorname{col} B - \dim \operatorname{col} BC$$

$$\dim \operatorname{col} AB / \operatorname{col} ABC = \dim \operatorname{col} AB - \dim \operatorname{col} ABC$$

So $\quad r(B) - r(BC) \geq r(AB) - r(ABC)$

or

$$r(B) + r(ABC) \geq r(AB) + r(BC).$$

4. (20 points) Let V be a finite dimensional linear space. For the quotient algebra homomorphism $T^{\cdot}V \to S^{\cdot}V$, its k-th component is a surjective linear map $V^{\otimes k} \to S^k V$. Similarity, we have a surjective linear map $V^{\otimes k} \to \wedge^k V$. By the universal property for product, we have a unique linear map

$$\phi_k : V^{\otimes k} \to S^k V \times \wedge^k V.$$

Show that ϕ_k is a linear equivalence if $k = 2$. What about $k > 2$?

Solution. Let $n = \dim V$.

$\dim S^2V = \frac{(n+1)n}{2}$, $\dim \wedge^2 V = \frac{(n-1)n}{2}$

So $\dim S^2V \times \wedge^2 V = n^2$.

Since $\dim V^{\otimes 2} = n^2 = \dim S^2V \times \wedge^2 V$

it suffices to show that ϕ_2 is injective. For that, we fix a basis $v = (v_1, \dots, v_n)$ of V. Then any $x \in V^{\otimes 2}$ is of the form

$$x = \sum c_{ij} v_i \otimes v_j$$

thus

$$\phi_2(x) = \left(\sum c_{ij} v_i v_j, \sum c_{ij} v_i \wedge v_j\right).$$

Assume $x \in \ker \phi_2$, then

$\sum c_{ij} v_i v_j = 0$ and $\sum c_{ij} v_i \wedge v_j = 0$

It is clear that

$S \overset{\text{def}}{=} \{ v_i v_j \mid 1 \leq i \leq j \leq n \}$ is

a spanning set of $S^2 V$. since

$\dim S^2 V = \frac{(n+1)n}{2} = |S|$, S is

a minimal spanning set. Therefore

$\sum c_{ij} v_i v_j = 0 \Rightarrow c_{ij} + c_{ji} = 0$. (1)

Similary

$\sum c_{ij} v_i \wedge v_j = 0 \Rightarrow c_{ij} - c_{ji} = 0$ (2)

(1) + (2) $\Rightarrow c_{ij} = 0$. So

$x = 0$

Thus ϕ_2 is injective. Then done with the proof.

5. (20 points) Please compute the determinant of matrix

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & \vdots & \vdots \\ x_1^{n-2} & x_2^{n-2} & \cdots & x_n^{n-2} \\ x_1^n & x_2^n & \cdots & x_n^n \end{bmatrix}.$$

Solution. This determinant is a polynomial in $x_1, \cdots, x_n$ and its degree is

$$0 + 1 + \cdots + (n-2) + n = \frac{(n-1)n}{2} + 1.$$

On the other hand, this polynomial is divisible by the Vandermonde polynomial

$$\Delta(x_1, \cdots, x_n) \stackrel{\text{def}}{=} \prod_{n \geq j > i \geq 1} (x_j - x_i)$$

which has degree $= \frac{n^2 - n}{2} = \frac{(n-1)n}{2}$, one less than the degree of $\det A$.

So $\det A = P_1 \Delta$ where P_1 is a polynomial of degree 1. Since $\det A$ and Δ changes sign whenever two variables are swapped, P_1 is a symmetric deg 1 polynomial, So $p_1 = c_n(x_1 + \cdots + x_n)$ for a constant c_n which might depends on n.

To find c_n, we need to find the leading coefficient of x_n on the both sides of equality

$$\det A = c_n(x_1 + \cdots + x_n)\,\Delta(x_1, \ldots, x_n)$$

Lap. exp. along the last column

$$\det A = \Delta(x_1, \ldots, x_{n-1})\, x_n^n + \text{lower order term in } x_n$$

$$c_n(x_1 + \cdots + x_n)\,\Delta(x_1, \ldots, x_n) = x_n^{n-1}\,\Delta(x_1, \ldots, x_{n-1}) + \text{lower order term in } x_n$$

$$\Delta(x_1, \ldots, x_{n-1})\, x_n^n = c_n x_n^n\, \Delta(x_1, \ldots, x_{n-1})$$

$$\Rightarrow c_n = 1$$

Thus

$$\boxed{\det A = (x_1 + \cdots + x_n) \prod_{n \geq j > i \geq 1} (x_j - x_i)}$$