We assume all linear spaces are over the field $\mathbb{F}$. Recall that, for a linear space V, the following two statements are equivalent:

1. V is finitely spanned, i.e., $V = \overline{S}$ for a finite subset S of V. Note: $\overline{\emptyset} = 0$ --- the linear space consisting zero vectors only.
2. V is linearly equivalent to $\mathbb{F}^n$ for an integer $n \geq 0$.

A linear space is said to be finite-dimensional (f.d.) if any of the two statements above is true. If that is the case, the dimension of V, denoted by $\dim V$, is defined to be n in the second statement above. It is clear that $\dim V$ is the number of vectors in a minimal spanning set for V and has been shown previously to be well-defined by using matrix techniques.

By definition, the rank of the linear map $f\colon V \to W$, denoted by $r(f)$, is defined to be $\dim \operatorname{im} f$.

1. Let V_1 and V_2 be subspaces of V. Subspaces V_1 and V_2 are said to be linearly independent in the weak sense if there is only one way to write $0 = v_1 + v_2$ with $v_i \in V_i$: $0 = 0 + 0$. If this happens, the sum $V_1 + V_2$ is rewritten as $V_1 \oplus V_2$ and is called the direct sum of V_1 and V_2.

   (a) Show that, if V_1 and V_2 are finite dimensional, then $V_1 + V_2$ is finite-dimensional; moreover, $\dim(V_1+V_2) \leq \dim V_1 + \dim V_2$. In other words, the dimension function dim is a sub-additive function.
   Hint: If $V_i = \overline{S_i}$ (we can chose S_i to be minimal), then $V_1+V_2 = \overline{S_1 \cup S_2}$.

   (b) Continuing the discussion in part (a). Show that the equality sign holds $\iff$ the sum is a direct sum.

2. Consider an exact sequence

$$0 \to V_1 \xrightarrow{i_1} V \xrightarrow{j_2} V_2 \to 0$$

   for which V_2 is assumed to have a minimal spanning set. Show that

   (a) $j_2 i_1 = 0$ and $V_1 \cong \operatorname{im} i_1$.

   (b) j_2 has a right inverse. Let us fix a right inverse i_2. Since $j_2 i_2 = 1$ (i.e., 1_{V_2}), i_2 is injective, so $V_2 \cong \operatorname{im} i_2$.

   (c) $V = \operatorname{im} i_1 \oplus \operatorname{im} i_2$. I.e. any v in V can be uniquely split into the sum of two, one is of the form $i_1(v_1)$ and the other is of the form $i_2(v_2)$.

   (d) the splitting in part (c) defines two maps, one is from V to V_1 and is denoted by j_1, and the other is $j_2\colon V \to V_2$. In other words, if $v = i_1(v_1) + i_2(v_2)$, then $j_1(v) = v_1$ and $j_2(v) = v_2$.

   (e) j_1 is linear and the sequence

$$0 \leftarrow V_1 \xleftarrow{j_1} V \xleftarrow{i_2} V_2 \leftarrow 0,$$

   is exact.

(f) $j_k i_l = \delta_{kl}$, and $i_1 j_1 + i_2 j_2 = 1$ (i.e., 1_V).

(g) both V_1 and V_2 are finite-dimensional $\iff$ V is finite-dimensional. In case V is finite dimensional, we have $\dim V = \dim V_1 + \dim V_2$, thus $\dim V_i \le \dim V$.

(h) Assume that V is finite-dimensional and V_1 is a subspace of V, then $\dim V_1 \le \dim V$ and the equality sign holds $\iff$ $V_1 = V$.

(i) for any finite-dimensional linear space, none of its subspaces or quotient spaces has a bigger dimension. `In other words, taking subspace or quotient space cannot increase dimension.`

3. Let V be a linear space and $f\colon V \to W$ be a linear map between finite dimensional linear spaces.

   (a) Prove the rank–nullity theorem : $\dim V = \dim \ker f + \dim \operatorname{im} f$. `Hint:` $0 \to \ker f \to V \to \operatorname{im} f \to 0$ `is exact.`

   (b) By the universal property for the quotient map $V \to \operatorname{coim} f := V/\ker f$, we get a unique linear map $f'\colon \operatorname{coim} f \to \operatorname{im} f$. Show that, f' is a linear equivalence. `Thus` $r(f) = \dim \operatorname{coim} f$ `as well.`

   (c) The sequences $0 \to \ker f \to V \to \operatorname{coim} f \to 0$ and $0 \to \operatorname{im} f \to W \to \operatorname{coker} f \to 0$ are all exact.

4. `Let` A `be an` $m\times n$`-matrix, then the multiplication by` A `defines a linear map` $f\colon \mathbb{F}^n \to \mathbb{F}^m$. `The rank of` A`, denoted by` $r(A)$`, is defined to be the rank of the linear map` f. `Note that` $\operatorname{im} f = \operatorname{Col} A$ `--- the span of columns of` A.

   (a) Show that the rank of a matrix is unchanged under both row operations and column operations.

   `Hint: row operations do not change null space and column operations do not change column space.`

   (b) Show that $r(A+B) \le r(A) + r(B)$ provided that the matrix addition is defined here.

   (c) Show that $r(AB) \le r(A)$ and $r(AB) \le r(B)$ provided that the matrix multiplication is defined here.

   (d) let A be a square matrix of order n. Show that $r(A^n) = r(A^{n+1})$.