Kalman Filter

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Outline

Linear System

Model (Real Plant)

$$\begin{cases} x_{k+1} = Ax_k + Bu_k + \omega_k, \\ y_k = Cx_k + \nu_k. \end{cases}$$

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$$\begin{cases} x_{k+1} = Ax_k + Bu_k + \omega_k, \\ y_k = Cx_k + \nu_k. \end{cases}$$

where x: state; u: input; y: output (measurement); ω: process noise; ν: measurement noise.

Given

- A: State matrix
- B: Input matrix or Control matrix
- C: Output matrix or Observe matrix
- $\omega \sim N(0, Q)$
- $\nu \sim N(0, R)$

Mission

Find optimal estimate $\tilde{x}$ of x

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Define

- $\bar{x}_k$: Predict of x_k (a priori state estimate)
- $\tilde{x}_k$: Optimal estimate of x_k (a posteriori state estimate)

区分：
Predict 预测
Estimate 估计
Smooth 平滑

Kalman Filter

$$\begin{cases} \bar{x}_{k+1} = A\tilde{x}_k + Bu_k, \\ \tilde{x}_k = \bar{x}_k + \mathcal{L}(y_k - C\bar{x}_k). \end{cases}$$

where $\mathcal{L}$: Kalman Filter Gain and $\mathcal{L} \in [0, 1]$.

$$\mathcal{L} = \frac{predicted\ \ error}{predicted\ \ error + measurement\ \ error}.$$

Remark: Luenberger Observer

$$\hat{x}_{k+1} = A\hat{x}_k + Bu_k + \mathcal{L}(y_k - C\hat{x}_k)$$

Optimal Estimate

Estimate error

- $\bar{e}_k = x_k - \bar{x}_k$: a prior error

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Covariance

- $\bar{P}_k = E[\bar{e}_k \bar{e}_k^T]$
- $\tilde{P}_k = E[\tilde{e}_k \tilde{e}_k^T]$

Optimal Estimate

$\tilde{P}_k$

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$$\tilde{x}_k = \bar{x}_k + \mathcal{L}(y_k - C\bar{x}_k) = \bar{x}_k + \mathcal{L}(Cx_k + \nu_k - C\bar{x}_k)$$

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$$\tilde{x}_k - x_k = \bar{x}_k - x_k + \mathcal{L}C(x_k - \bar{x}_k) + \mathcal{L}\nu_k$$

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$$\tilde{P}_k = E[\tilde{e}_k\tilde{e}_k^T] = (I - \mathcal{L}C)\bar{P}_k(I - \mathcal{L}C)^T + \mathcal{L}R\mathcal{L}^T$$

Optimal Estimate

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$$
\begin{aligned}
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&\quad\;\; = \bar{P}_k - \mathcal{L}C\bar{P}_k - \bar{P}_kC^T\mathcal{L}^T + \mathcal{L}(C\bar{P}_kC^T + R)\mathcal{L}^T
\end{aligned}
$$

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\end{aligned}$$

$\min \tilde{P}_k$

Optimal Estimate

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$$\frac{\partial \tilde{P}_k}{\partial \mathcal{L}} = -2(\bar{P}_kC) + 2\mathcal{L}(C\bar{P}_kC^T + R) = 0$$

Optimal Estimate

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\end{aligned}
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\tilde{P}_k = \bar{P}_k - \mathcal{L}C\bar{P}_k - \bar{P}_kC^T\mathcal{L}^T + \bar{P}_kC^T(C\bar{P}_kC^T + R)^{-1}(C\bar{P}_kC^T + R)\mathcal{L}^T
$$

$$
\tilde{P}_K = (I - \mathcal{L}C)\bar{P}_k
$$

Optimal Estimate

$\bar{P}_k$

$$\bar{e}_{k+1} = x_{k+1} - \bar{x}_{k+1} = (Ax_k + Bu_k + \omega_k) - (A\tilde{x}_k + Bu_k) = A\tilde{e}_k + \omega_k$$

$\bar{P}_k$

$$\bar{e}_{k+1} = x_{k+1} - \bar{x}_{k+1} = (Ax_k + Bu_k + \omega_k) - (A\tilde{x}_k + Bu_k) = A\tilde{e}_k + \omega_k$$

$$\begin{aligned}\bar{P}_{k+1} &= E(\bar{e}_{k+1}\bar{e}_{k+1}^T) = E((A\tilde{e}_k + \omega_k(A\tilde{e}_k + \omega_k^T))) \\ &= E((A\tilde{e}_k)(A\tilde{e}_k)^T) + E(\omega_k\omega_k^T) = A\tilde{P}_K A + Q\end{aligned}$$

Summary

Takeaways

$$\begin{cases} \bar{x}_{k+1} = A\tilde{x}_k + Bu_k, \\ \tilde{x}_k = \bar{x}_k + \mathcal{L}(y_k - C\bar{x}_k), \\ \mathcal{L} = \bar{P}_k C^T (C\bar{P}_k C^T + R)^{-1}, \\ \bar{P}_{k+1} = A\tilde{P}_K A + Q, \\ \tilde{P}_K = (I - \mathcal{L}C)\bar{P}_k. \end{cases}$$

迭代式的
工程可用

最优：方差最优

Thanks for You Attention!

http://person.zju.edu.cn/ChunyueSong