

Kalman Filter

C. Sono

Linear System

Kalman Filter

Optimal Estimate

Conclusio

Kalman Filter

Chunyue Song csong(at)zju.edu.cn

College of Control Science & Engineering, Zhejiang University

December 20, 2019



Outline

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- 2 Kalman Filter
- **3** Optimal Estimate
- 4 Conclusion



Linear System

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Model (Real Plant)

$$\begin{cases} x_{k+1} = Ax_k + Bu_k + \omega_k, \\ y_k = Cx_k + \nu_k. \end{cases}$$



Linear System

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Model (Real Plant)

Linear System

$$\begin{cases} x_{k+1} = Ax_k + Bu_k + \omega_k, \\ y_k = Cx_k + \nu_k. \end{cases}$$

Kalman Filtei Optimal Estimate

where x: state; u: input; y: output (measurement); ω : process noise; ν : measurement noise.

Given

- A: State matrix
- B: Input matrix or Control matrix
- C: Output matrix or Observe matrix
- $\omega \sim N(0, Q)$
- $\nu \sim N(0, R)$



Mission

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Find optimal estimate \tilde{x} of x



Mission

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Estimate Conclusion

Find optimal estimate \tilde{x} of x

Define

- \bar{x}_k : Predict of x_k (a priori state estimate)
- \tilde{x}_k : Optimal estimate of x_k (a posteriori state estimate)

区分: Predict 预测

Estimate 估计 Smooth 平滑



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$$\begin{cases} \bar{x}_{k+1} = A\tilde{x}_k + Bu_k, \\ \tilde{x}_k = \bar{x}_k + \mathcal{L}(y_k - C\bar{x}_k). \end{cases}$$

where \mathcal{L} : Kalman Filter Gain and $\mathcal{L} \in [0, 1]$.

$$\mathcal{L} = \frac{predicted\ error}{predicted\ error+measurement\ error}.$$

Remark: Luenberger Observer

$$\hat{x}_{k+1} = A\hat{x}_k + Bu_k + \mathcal{L}(y_k - C\hat{x}_k)$$



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Estimate error

 $lackbox{ar{e}}_k = x_k - ar{x}_k$: a prior error



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Estimate error

 $\bar{e}_k = x_k - \bar{x}_k$: a prior error

 $\tilde{e}_k = x_k - \tilde{x}_k$: a posterior error



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Estimate error

 $\bar{e}_k = x_k - \bar{x}_k$: a prior error

 $\tilde{e}_k = x_k - \tilde{x}_k$: a posterior error



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Estimate error

- $lackbox{ar{e}}_k = x_k ar{x}_k$: a prior error
- $ilde{\mathbf{e}}_k = x_k \tilde{x}_k$: a posterior error

Covariance

- $lacksquare ar{P}_k = E[ar{e}_k ar{e}_k^T]$
- $\blacksquare \tilde{P}_k = E[\tilde{e}_k \tilde{e}_k^T]$



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 P_{k}

$$\tilde{x}_k = \bar{x}_k + \mathcal{L}(y_k - C\bar{x}_k) = \bar{x}_k + \mathcal{L}(Cx_k + \nu_k - C\bar{x}_k)$$

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 P_k

$$\tilde{x}_k = \bar{x}_k + \mathcal{L}(y_k - C\bar{x}_k) = \bar{x}_k + \mathcal{L}(Cx_k + \nu_k - C\bar{x}_k)$$

 $\tilde{x}_k - x_k = \bar{x}_k - x_k + \mathcal{L}C(x_k - \bar{x}_k) + \mathcal{L}\nu_k$

Optimal Estimate



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Optimal Estimate

$$\tilde{X}_k = \bar{X}_k + \mathcal{L}(y_k - C\bar{X}_k) = \bar{X}_k + \mathcal{L}(CX_k + \nu_k - C\bar{X}_k)$$
 $\tilde{X}_k - X_k = \bar{X}_k - X_k + \mathcal{L}C(X_k - \bar{X}_k) + \mathcal{L}\nu_k$
 $\tilde{e}_k = (I - \mathcal{L}C)\bar{e}_k - \mathcal{L}\nu_k$



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$$\begin{split} \tilde{\mathbf{X}}_k &= \bar{\mathbf{X}}_k + \mathcal{L}(\mathbf{y}_k - C\bar{\mathbf{X}}_k) = \bar{\mathbf{X}}_k + \mathcal{L}(C\mathbf{X}_k + \nu_k - C\bar{\mathbf{X}}_k) \\ \tilde{\mathbf{X}}_k - \mathbf{X}_k &= \bar{\mathbf{X}}_k - \mathbf{X}_k + \mathcal{L}C(\mathbf{X}_k - \bar{\mathbf{X}}_k) + \mathcal{L}\nu_k \\ \tilde{\mathbf{e}}_k &= (\mathbf{I} - \mathcal{L}C)\bar{\mathbf{e}}_k - \mathcal{L}\nu_k \\ \tilde{\mathbf{P}}_k &= E[\tilde{\mathbf{e}}_k\tilde{\mathbf{e}}_k^T] = (\mathbf{I} - \mathcal{L}C)\bar{\mathbf{P}}_k(\mathbf{I} - \mathcal{L}C)^T + \mathcal{L}\mathcal{R}\mathcal{L}^T \end{split}$$



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Canalysis

$$egin{aligned} & ilde{x}_k = ar{x}_k + \mathcal{L}(y_k - Car{x}_k) = ar{x}_k + \mathcal{L}(Cx_k +
u_k - Car{x}_k) \\ & ilde{x}_k - x_k = ar{x}_k - x_k + \mathcal{L}C(x_k - ar{x}_k) + \mathcal{L}
u_k \\ & ilde{e}_k = (I - \mathcal{L}C)ar{e}_k - \mathcal{L}
u_k \\ & ilde{P}_k = E[ar{e}_kar{e}_k^T] = (I - \mathcal{L}C)ar{P}_k(I - \mathcal{L}C)^T + \mathcal{L}R\mathcal{L}^T \\ & = ar{P}_k - \mathcal{L}Car{P}_k - ar{P}_kC^T\mathcal{L}^T + \mathcal{L}(Car{P}_kC^T + R)\mathcal{L}^T \end{aligned}$$



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 \tilde{P}_k

$$\begin{split} \tilde{\mathbf{x}}_k &= \bar{\mathbf{x}}_k + \mathcal{L}(\mathbf{y}_k - C\bar{\mathbf{x}}_k) = \bar{\mathbf{x}}_k + \mathcal{L}(C\mathbf{x}_k + \nu_k - C\bar{\mathbf{x}}_k) \\ \tilde{\mathbf{x}}_k - \mathbf{x}_k &= \bar{\mathbf{x}}_k - \mathbf{x}_k + \mathcal{L}C(\mathbf{x}_k - \bar{\mathbf{x}}_k) + \mathcal{L}\nu_k \\ \tilde{\mathbf{e}}_k &= (\mathbf{I} - \mathcal{L}C)\bar{\mathbf{e}}_k - \mathcal{L}\nu_k \\ \tilde{P}_k &= E[\tilde{\mathbf{e}}_k\tilde{\mathbf{e}}_k^T] = (\mathbf{I} - \mathcal{L}C)\bar{P}_k(\mathbf{I} - \mathcal{L}C)^T + \mathcal{L}R\mathcal{L}^T \\ &= \bar{P}_k - \mathcal{L}C\bar{P}_k - \bar{P}_kC^T\mathcal{L}^T + \mathcal{L}(C\bar{P}_kC^T + R)\mathcal{L}^T \end{split}$$

 $\min \tilde{P}_k$



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\tilde{P}_k

$$\begin{split} \tilde{\mathbf{x}}_k &= \bar{\mathbf{x}}_k + \mathcal{L}(\mathbf{y}_k - C\bar{\mathbf{x}}_k) = \bar{\mathbf{x}}_k + \mathcal{L}(C\mathbf{x}_k + \nu_k - C\bar{\mathbf{x}}_k) \\ \tilde{\mathbf{x}}_k - \mathbf{x}_k &= \bar{\mathbf{x}}_k - \mathbf{x}_k + \mathcal{L}C(\mathbf{x}_k - \bar{\mathbf{x}}_k) + \mathcal{L}\nu_k \\ \tilde{\mathbf{e}}_k &= (\mathbf{I} - \mathcal{L}C)\bar{\mathbf{e}}_k - \mathcal{L}\nu_k \\ \tilde{\mathbf{P}}_k &= E[\tilde{\mathbf{e}}_k\tilde{\mathbf{e}}_k^T] = (\mathbf{I} - \mathcal{L}C)\bar{\mathbf{P}}_k(\mathbf{I} - \mathcal{L}C)^T + \mathcal{L}R\mathcal{L}^T \\ &= \bar{\mathbf{P}}_k - \mathcal{L}C\bar{\mathbf{P}}_k - \bar{\mathbf{P}}_kC^T\mathcal{L}^T + \mathcal{L}(C\bar{\mathbf{P}}_kC^T + R)\mathcal{L}^T \end{split}$$

$\min \tilde{P}_k$

$$\frac{\partial \tilde{P}_k}{\partial \mathcal{L}} = -2(\bar{P}_k C) + 2\mathcal{L}(C\bar{P}_k C^T + R) = 0$$



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\tilde{P}_k

$$\begin{split} \tilde{\mathbf{x}}_k &= \bar{\mathbf{x}}_k + \mathcal{L}(\mathbf{y}_k - C\bar{\mathbf{x}}_k) = \bar{\mathbf{x}}_k + \mathcal{L}(C\mathbf{x}_k + \nu_k - C\bar{\mathbf{x}}_k) \\ \tilde{\mathbf{x}}_k - \mathbf{x}_k &= \bar{\mathbf{x}}_k - \mathbf{x}_k + \mathcal{L}C(\mathbf{x}_k - \bar{\mathbf{x}}_k) + \mathcal{L}\nu_k \\ \tilde{\mathbf{e}}_k &= (\mathbf{I} - \mathcal{L}C)\bar{\mathbf{e}}_k - \mathcal{L}\nu_k \\ \tilde{P}_k &= E[\tilde{\mathbf{e}}_k\tilde{\mathbf{e}}_k^T] = (\mathbf{I} - \mathcal{L}C)\bar{P}_k(\mathbf{I} - \mathcal{L}C)^T + \mathcal{L}R\mathcal{L}^T \\ &= \bar{P}_k - \mathcal{L}C\bar{P}_k - \bar{P}_kC^T\mathcal{L}^T + \mathcal{L}(C\bar{P}_kC^T + R)\mathcal{L}^T \end{split}$$

$\min \tilde{P}_k$

$$\begin{array}{l} \frac{\partial \tilde{P}_k}{\partial \mathcal{L}} = -2(\bar{P}_k C) + 2\mathcal{L}(C\bar{P}_k C^T + R) = 0 \\ \mathcal{L} = \bar{P}_k C^T (C\bar{P}_k C^T + R)^{-1} \end{array}$$



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\tilde{P}_k

$$\begin{split} \tilde{\mathbf{X}}_{k} &= \bar{\mathbf{X}}_{k} + \mathcal{L}(\mathbf{y}_{k} - C\bar{\mathbf{X}}_{k}) = \bar{\mathbf{X}}_{k} + \mathcal{L}(C\mathbf{X}_{k} + \nu_{k} - C\bar{\mathbf{X}}_{k}) \\ \tilde{\mathbf{X}}_{k} - \mathbf{X}_{k} &= \bar{\mathbf{X}}_{k} - \mathbf{X}_{k} + \mathcal{L}C(\mathbf{X}_{k} - \bar{\mathbf{X}}_{k}) + \mathcal{L}\nu_{k} \\ \tilde{\mathbf{e}}_{k} &= (\mathbf{I} - \mathcal{L}C)\bar{\mathbf{e}}_{k} - \mathcal{L}\nu_{k} \\ \tilde{P}_{k} &= E[\tilde{\mathbf{e}}_{k}\tilde{\mathbf{e}}_{k}^{T}] = (\mathbf{I} - \mathcal{L}C)\bar{P}_{k}(\mathbf{I} - \mathcal{L}C)^{T} + \mathcal{L}R\mathcal{L}^{T} \\ &= \bar{P}_{k} - \mathcal{L}C\bar{P}_{k} - \bar{P}_{k}C^{T}\mathcal{L}^{T} + \mathcal{L}(C\bar{P}_{k}C^{T} + R)\mathcal{L}^{T} \end{split}$$

$\min \tilde{P}_k$

$$rac{\partial ilde{P}_k}{\partial \mathcal{L}} = -2(ar{P}_k C) + 2\mathcal{L}(Car{P}_k C^T + R) = 0$$

 $\mathcal{L} = ar{P}_k C^T (Car{P}_k C^T + R)^{-1}$



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Estimate

 $\tilde{\mathbf{x}}_k = \bar{\mathbf{x}}_k + \mathcal{L}(\mathbf{y}_k - C\bar{\mathbf{x}}_k) = \bar{\mathbf{x}}_k + \mathcal{L}(C\mathbf{x}_k + \nu_k - C\bar{\mathbf{x}}_k)$ $\tilde{X}_k - X_k = \bar{X}_k - X_k + \mathcal{L}C(X_k - \bar{X}_k) + \mathcal{L}\nu_k$ $\tilde{\mathbf{e}}_{\mathbf{k}} = (\mathbf{I} - \mathcal{L}\mathbf{C})\bar{\mathbf{e}}_{\mathbf{k}} - \mathcal{L}\nu_{\mathbf{k}}$

 $\tilde{P}_k = E[\tilde{e}_k \tilde{e}_k^T] = (I - \mathcal{L}C)\bar{P}_k(I - \mathcal{L}C)^T + \mathcal{L}R\mathcal{L}^T$ $= \bar{P}_k - \mathcal{L}C\bar{P}_k - \bar{P}_kC^T\mathcal{L}^T + \mathcal{L}(C\bar{P}_kC^T + R)\mathcal{L}^T$

 $\min \tilde{P}_{k}$

 $\frac{\partial P_k}{\partial \mathcal{L}} = -2(\bar{P}_k C) + 2\mathcal{L}(C\bar{P}_k C^T + R) = 0$

 \tilde{P}_{k}

 $\tilde{P}_k = \bar{P}_k - \mathcal{L}C\bar{P}_k - \bar{P}_kC^T\mathcal{L}^T + \bar{P}_kC^T(C\bar{P}_kC^T + R)^{-1}(C\bar{P}_kC^T + R)\mathcal{L}^T$

 $\mathcal{L} = \bar{P}_{\nu} C^{T} (C\bar{P}_{\nu} C^{T} + R)^{-1}$

 $\tilde{P}_{\kappa} = (I - \mathcal{L}C)\bar{P}_{k}$

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$$\bar{P}_k$$

$$\bar{e}_{k+1} = x_{k+1} - \bar{x}_{k+1} = (Ax_k + Bu_k + \omega_k) - (A\tilde{x}_k + Bu_k) = A\tilde{e}_k + \omega_k$$



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\bar{P}_k

$$\bar{e}_{k+1} = x_{k+1} - \bar{x}_{k+1} = (Ax_k + Bu_k + \omega_k) - (A\tilde{x}_k + Bu_k) = A\tilde{e}_k + \omega_k$$

$$\begin{split} \bar{P}_{k+1} &= E(\bar{e}_{k+1}\bar{e}_{k+1}^T) = E((A\tilde{e}_k + \omega_k(A\tilde{e}_k + \omega_k^T))) \\ &= E((A\tilde{e}_k)(A\tilde{e}_k)^T) + E(\omega_k\omega_k^T) = A\tilde{P}_KA + Q \end{split}$$



Summary

Kalman Filter

Conclusion

Takeaways

$$\bar{x}_{k+1} = A\tilde{x}_k + Bu_k,$$

$$\tilde{x}_k = \bar{x}_k + \mathcal{L}(y_k - C\bar{x}_k),$$

$$\begin{cases} \mathcal{L} = \bar{P}_k C^T (C \bar{P}_k C^T + R)^{-1}, \\ \bar{P}_{k+1} = A \tilde{P}_K A + Q, \\ \tilde{P}_K = (I - \mathcal{L}C) \bar{P}_k. \end{cases}$$

$$\bar{P}_{k+1} = A\tilde{P}_K A + Q$$

$$\tilde{P}_{K} = (I - \mathcal{L}C)\bar{P}_{k}.$$

迭代式的 **工程可用**

最优: 方差最优



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Conclusion

Thanks for You Attention!



http://person.zju.edu.cn/ChunyueSong