STA 601 Homework 9

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10.1

Reflecting random walks: It is often useful in MCMC to have a proposal distribution which is both symmetric and has support only on a certain region. For example, if we know $\theta > 0$, we would like our proposal distribution $J(\theta_1|\theta_0)$ to have support on positive θ values. Consider the following proposal algorithm:

- sample $\tilde{\theta} \sim \text{uniform}(\theta_0 - \delta, \theta_0 + \delta)$;
- if $\tilde{\theta} < 0$, set $\theta_1 = -\tilde{\theta}$;
- if $\tilde{\theta} \geq 0$, set $\theta_1 = \tilde{\theta}$.

In other words, $\theta_1 = |\tilde{\theta}|$. Show that the above algorithm draws samples from a symmetric proposal distribution which has support on positive values of θ. It may be helpful to write out the associated proposal density $J(\theta_1|\theta_0)$ under the two conditions $\theta_0 \leq \delta$ and $\theta_0 > \delta$ separately.

Since we are getting our proposed sample $\theta_1 = |\tilde{\theta}| \geq 0$, it's straightforward that the proposal distribution has support on positive values.

For symmetry, notice that the symmetry in Metropolis algorithm is defined as $J(x|y) = J(y|x)$. According to the proposal algorithm given above, we can easily write down the proposal density as the following:

$$J(\theta_1|\theta_0) = \begin{cases} \frac{1}{\delta}, & \theta_0 \leq \delta, 0 \leq \theta_1 \leq \delta - \theta_0 \\ \frac{1}{2\delta}, & \theta_0 \leq \delta, \delta - \theta_0 \leq \theta_1 \leq \theta_0 + \delta \\ \frac{1}{2\delta}, & \theta_0 > \delta, \theta_0 - \delta \leq \theta_1 \leq \theta_0 + \delta \\ 0, & o.w. \end{cases}$$

For $\{\theta_0 \leq \delta, 0 \leq \theta_1 \leq \delta - \theta_0\}$, which is the same as $\{0 < \theta_0, 0 < \theta_1, \theta_0 + \theta_1 \leq \delta\}$, it's the triangular area constrained by vertices $(0,0),(0,\delta),(\delta,0)$.

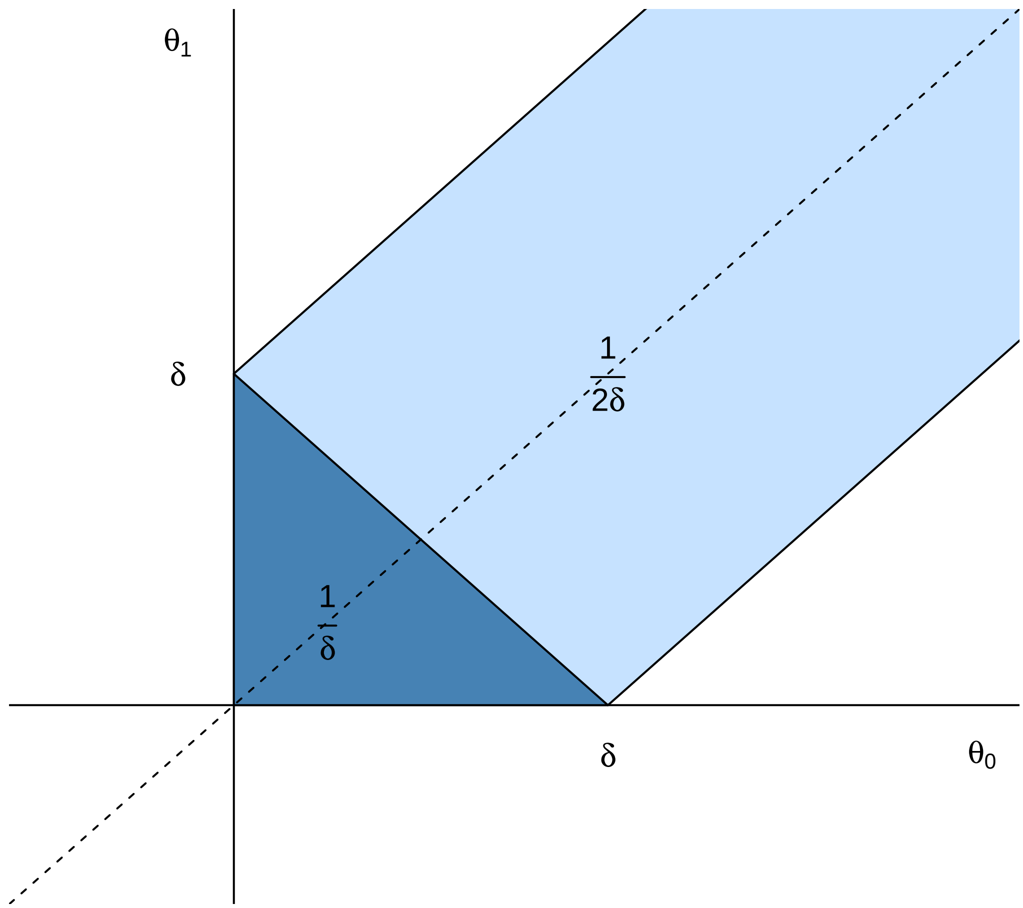
For $\{\theta_0 \leq \delta, \delta - \theta_0 \leq \theta_1 \leq \theta_0 + \delta\}$, it's the triangular area constrained by vertices $(0,\delta),(\delta,0),(0,2\delta)$.

For $\{\theta_0 > \delta, \theta_0 - \delta \leq \theta_1 \leq \theta_0 + \delta\}$, it's the area constrained by lines $\theta_0 = \delta, \theta_1 = \theta_0 - \delta, \theta_1 = \theta_0 + \delta$ (the convex area).

If we regard the density as a bivariate function, and draw the areas for each condition respectively on a θ_1 vs θ_0 coordinates, we can have the following density plot

```r
plot(NA, xlim = c(-.5,2), ylim = c(-.5, 2), axes = FALSE,
     ylab = '', xlab = '')
abline(v=0)
abline(h=0)
segments(0,1,1,0)
polygon(x = c(0,0,1),
        y = c(0,1,0),
        col = "steelblue")
segments(0,1,10,11)
```

```r
segments(1,0,11,10)
polygon(x = c(0,1,12,11),
        y = c(1,0,11,12),
        col = "slategray1")
text(0.25, 0.25, expression(frac(1,delta)))
text(1, 1, expression(frac(1,paste(2, delta))))
text(1, -0.15, expression(delta))
text(-0.15, 1, expression(delta))
text(2, -0.15, expression(theta[0]))
text(-0.15, 2, expression(theta[1]))
abline(a=0,b=1, lty = 2)
```



As is shown in the plot, we know that the support areas with probability densities are symmetric about $\theta_0 = \theta_1$ line, which means that in the θ_1 *vs* θ_0 space, the proposal distribution is completely identical if we switch θ_0 and θ_1. Therefore, we have verified that the proposal distribution $J(\theta_1|\theta_0)$ is symmetric, i.e. $J(\theta_1|\theta_0) = J(\theta_0|\theta_1)$.