STA 623 homework 4

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Problem Statement

Consider the first problem on the exam, where $X \sim Binomial(n, \theta), n = 10$. X is the number of questions correct.

$$\delta_\tau(x) = \begin{cases} 1 & x > \tau,\ hire \\ 0 & x \leq \tau,\ don't\ hire \end{cases}$$

$L(\theta, a) =$ loss function, which increases linearly with θ if $a = 0$ and increases linearly with $(1-\theta)$ if $a = 1$.

(1) Calculate expected posterior loss of δ and find the Bayes optimal τ. [analytically]

Suppose our prior for θ, $\pi(\theta)$ is $Beta(1, 1)$, then the posterior is

$$\pi(\theta|x) \propto \pi(\theta)p(x|\theta) \propto \theta^x(1-\theta)^{10-x}$$

Judging from the kernel, we know $\theta|x \sim Beta(x + 1, 11 - x)$

If we give the simple loss function as $L = I_{x \leq \tau} \times \theta + I_{x > \tau} \times (1 - \theta)$, the Bayesian expected loss can be written as

$$\begin{aligned}
\rho(a, x) &= E_{\theta|x}[L(\theta, a)] \\
&= I_{x\leq\tau}E[\theta|x] + I_{x>\tau}E[(1-\theta)|x] \\
&= I_{x\leq\tau}\frac{x+1}{12} + I_{x>\tau}\frac{11-x}{12} \\
&= I_{x\leq\tau}\frac{x+1}{12} + (1 - I_{x\leq\tau})\frac{11-x}{12} \\
&= \frac{11-x}{12} + I_{x\leq\tau}\frac{2x-10}{12}
\end{aligned}$$

When

$\tau = 0,\ \rho = 1/12, 10/12, 9/12, ..., 1/12$ for $x = 0, ..., 10$;

$\tau = 1,\ \rho = 1/12, 2/12, 9/12, ..., 1/12$ for $x = 0, ..., 10$;

$\tau = 2,\ \rho = 1/12, ..., 3/12, 8/12, ..., 1/12$ for $x = 0, ..., 10$;

$\tau = 3,\ \rho = 1/12, ..., 4/12, 7/12..., 1/12$ for $x = 0, ..., 10$;

$\tau = 4,\ \rho = 1/12, ..., 5/12, 6/12, 5/12..., 1/12$ for $x = 0, ..., 10$;

$\tau = 5,\ \rho = 1/12, ...5/12, 6/12, 5/12, ..., 1/12$ for $x = 0, ..., 10$;

$\tau = 6,\ \rho = 1/12, ..., 5/12, 6/12, 7/12, 4/12, ..., 1/12$ for $x = 0, ..., 10$;

...

So we know that the Bayes optimal τ is 4 and 5. More generally, if τ can be non-integer, the Bayes optimal $\tau \in [4, 6)$.

(2) Repeat the exercise assuming you use Monte Carlo integration and avoid calculating integrals. How do the results differ?

Suppose we have the same prior $\pi(\theta)$ and thus the same posterior $\pi(\theta|x)$, $Beta(x+1, 11-x)$. We can generate some data from our posterior distribution and plug them in the formula to estimate our expected posterior loss using Monte Carlo

```
eloss = function(tau, x) {
  n = 100000 # sample size
  r = rbeta(n, x+1, 11-x)
  l = mean((x>tau) * (1-r) + (x<=tau) * r)
  return(l)
}
Loss.mc = NULL
for(tau in 0:10) {
  loss = NULL
  for(x in 0:10) {
    loss = c(loss, eloss(tau, x))
  }
  Loss.mc = rbind(Loss.mc, loss)
}
rownames(Loss.mc) = paste('tau =', 0:10)
colnames(Loss.mc) = paste('x =', 0:10)
kable(round(Loss.mc, 4), caption = 'expected loss using MC')
```

Table 1: expected loss using MC

	x = 0	x = 1	x = 2	x = 3	x = 4	x = 5	x = 6	x = 7	x = 8	x = 9	x = 10
tau = 0	0.0831	0.8331	0.7498	0.6660	0.5831	0.5001	0.4164	0.3337	0.2505	0.1666	0.0833
tau = 1	0.0831	0.1667	0.7501	0.6671	0.5833	0.4999	0.4163	0.3337	0.2505	0.1665	0.0834
tau = 2	0.0833	0.1669	0.2500	0.6670	0.5831	0.4998	0.4165	0.3339	0.2502	0.1664	0.0833
tau = 3	0.0831	0.1663	0.2495	0.3331	0.5834	0.4995	0.4167	0.3341	0.2501	0.1665	0.0830
tau = 4	0.0829	0.1673	0.2500	0.3336	0.4166	0.5003	0.4168	0.3331	0.2497	0.1665	0.0832
tau = 5	0.0833	0.1664	0.2499	0.3336	0.4171	0.4997	0.4171	0.3334	0.2500	0.1666	0.0832
tau = 6	0.0832	0.1663	0.2501	0.3328	0.4169	0.4999	0.5833	0.3332	0.2505	0.1670	0.0834
tau = 7	0.0834	0.1670	0.2506	0.3334	0.4170	0.5004	0.5829	0.6672	0.2500	0.1666	0.0832
tau = 8	0.0834	0.1666	0.2498	0.3337	0.4166	0.4999	0.5832	0.6675	0.7504	0.1670	0.0834
tau = 9	0.0833	0.1664	0.2492	0.3336	0.4176	0.5001	0.5833	0.6667	0.7498	0.8337	0.0836
tau = 10	0.0835	0.1663	0.2501	0.3334	0.4168	0.5002	0.5836	0.6666	0.7501	0.8333	0.9170

```
Loss.theo = NULL
for(tau in 0:10) {
  loss = NULL
  for(x in 0:10) {
    l = (11-x)/12 + (x<=tau) * (2*x-10)/12
    loss = c(loss, l)
  }
  Loss.theo = rbind(Loss.theo, loss)
}
rownames(Loss.theo) = paste('tau =', 0:10)
colnames(Loss.theo) = paste('x =', 0:10)
kable(round(Loss.theo, 4), caption = 'theoretical expected loss')
```

Table 2: theoretical expected loss

	x = 0	x = 1	x = 2	x = 3	x = 4	x = 5	x = 6	x = 7	x = 8	x = 9	x = 10
tau = 0	0.0833	0.8333	0.75	0.6667	0.5833	0.5	0.4167	0.3333	0.25	0.1667	0.0833
tau = 1	0.0833	0.1667	0.75	0.6667	0.5833	0.5	0.4167	0.3333	0.25	0.1667	0.0833
tau = 2	0.0833	0.1667	0.25	0.6667	0.5833	0.5	0.4167	0.3333	0.25	0.1667	0.0833
tau = 3	0.0833	0.1667	0.25	0.3333	0.5833	0.5	0.4167	0.3333	0.25	0.1667	0.0833
tau = 4	0.0833	0.1667	0.25	0.3333	0.4167	0.5	0.4167	0.3333	0.25	0.1667	0.0833
tau = 5	0.0833	0.1667	0.25	0.3333	0.4167	0.5	0.4167	0.3333	0.25	0.1667	0.0833
tau = 6	0.0833	0.1667	0.25	0.3333	0.4167	0.5	0.5833	0.3333	0.25	0.1667	0.0833
tau = 7	0.0833	0.1667	0.25	0.3333	0.4167	0.5	0.5833	0.6667	0.25	0.1667	0.0833
tau = 8	0.0833	0.1667	0.25	0.3333	0.4167	0.5	0.5833	0.6667	0.75	0.1667	0.0833
tau = 9	0.0833	0.1667	0.25	0.3333	0.4167	0.5	0.5833	0.6667	0.75	0.8333	0.0833
tau = 10	0.0833	0.1667	0.25	0.3333	0.4167	0.5	0.5833	0.6667	0.75	0.8333	0.9167

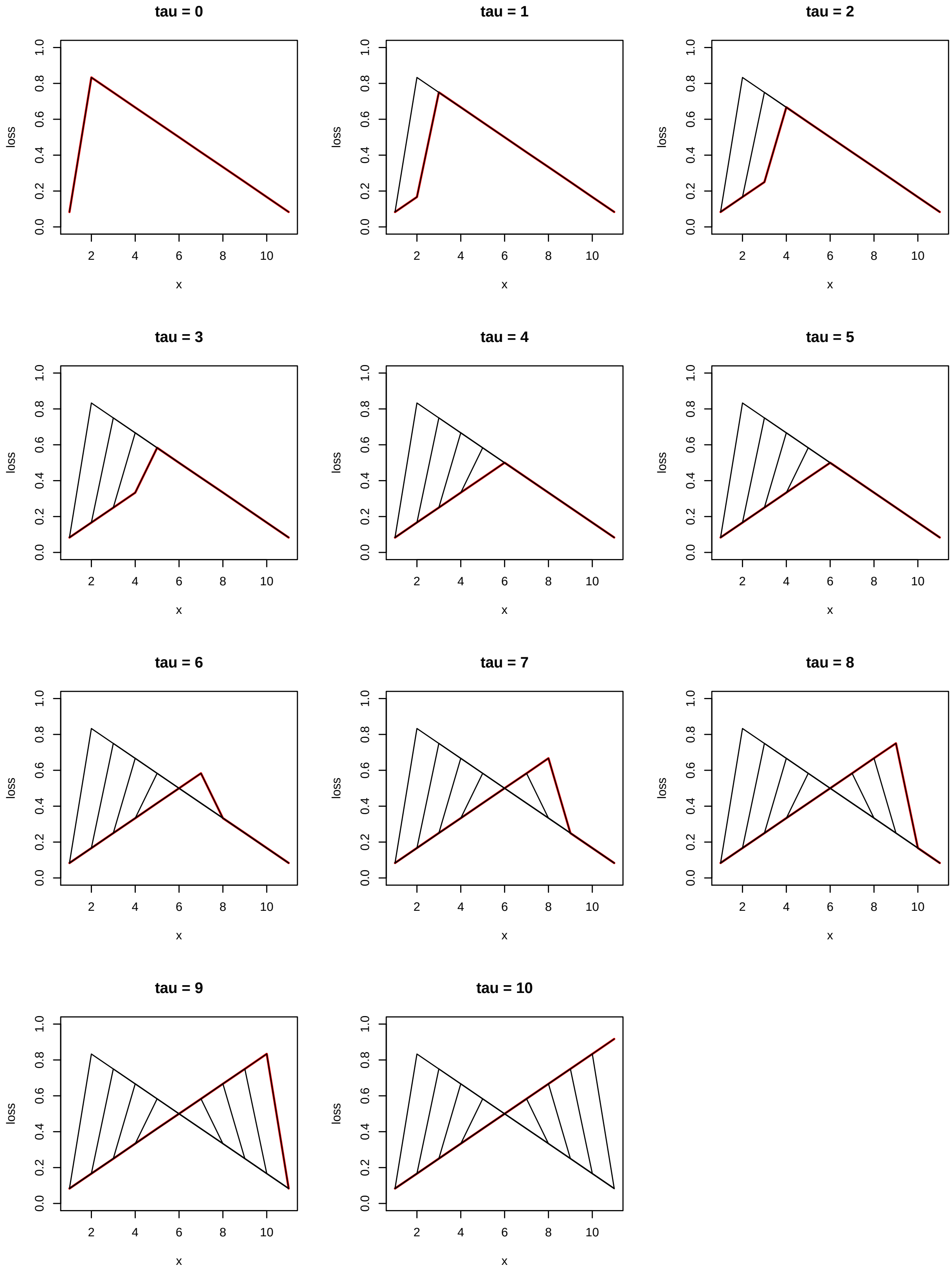
```r
kable(round(Loss.mc-Loss.theo, 4), caption = 'difference of MC method and theoretical method')
```

Table 3: difference of MC method and theoretical method

	x = 0	x = 1	x = 2	x = 3	x = 4	x = 5	x = 6	x = 7	x = 8	x = 9	x = 10
tau = 0	-2e-04	-2e-04	-2e-04	-6e-04	-2e-04	1e-04	-3e-04	4e-04	5e-04	0e+00	-1e-04
tau = 1	-2e-04	1e-04	1e-04	4e-04	0e+00	-1e-04	-4e-04	4e-04	5e-04	-2e-04	1e-04
tau = 2	0e+00	3e-04	0e+00	4e-04	-2e-04	-2e-04	-1e-04	5e-04	2e-04	-3e-04	0e+00
tau = 3	-2e-04	-3e-04	-5e-04	-2e-04	0e+00	-5e-04	0e+00	8e-04	1e-04	-2e-04	-4e-04
tau = 4	-4e-04	6e-04	0e+00	2e-04	-1e-04	3e-04	1e-04	-2e-04	-3e-04	-2e-04	-1e-04
tau = 5	0e+00	-2e-04	-1e-04	2e-04	4e-04	-3e-04	4e-04	1e-04	0e+00	-1e-04	-1e-04
tau = 6	-1e-04	-3e-04	1e-04	-5e-04	2e-04	-1e-04	0e+00	-2e-04	5e-04	3e-04	0e+00
tau = 7	1e-04	3e-04	6e-04	0e+00	3e-04	4e-04	-4e-04	5e-04	0e+00	0e+00	-2e-04
tau = 8	1e-04	-1e-04	-2e-04	3e-04	0e+00	-1e-04	-1e-04	8e-04	4e-04	4e-04	1e-04
tau = 9	0e+00	-3e-04	-8e-04	3e-04	1e-03	1e-04	-1e-04	0e+00	-2e-04	4e-04	3e-04
tau = 10	1e-04	-4e-04	1e-04	1e-04	1e-04	2e-04	3e-04	-1e-04	1e-04	-1e-04	3e-04

Comparing the expected posterior loss given different τ and x, we find that when sample size is sufficiently large, using Monte Carlo method and theoretical method are almost the same. And by drawing the plot of expected loss over different x's for different τ's, we know that the Bayes optimal τ when using MC method is also 4 and 5 (more specifically $[4, 6)$).

```r
par(mfrow = c(4,3))
for(i in 0:10) {
  plot(Loss.mc[i+1,], type = 'l', ylim = c(0, 1),
       xlab = 'x', ylab = 'loss',
       main = paste0('tau = ',i), col = 2, lwd = 2)
  for(j in 0:i) {
    lines(Loss.mc[j+1,])
  }
}
```



Besides, we can change the sample size and draw a plot displaying the convergence of $\rho = 1/12$ when

$\tau = 0, x = 0$.

```
eloss.p = function(n, tau=0, x=0) {
  r = rbeta(n, x+1, 11-x)
  l = mean((x>tau) * (1-r) + (x<=tau) * r)
  return(l)
}
par(mfrow = c(1,1))
est = NULL
s = (1:50)^3
for(n in s) {
  est = c(est, eloss.p(n))
}
plot(s, est, type = 'l', xlab = 'sample size', ylab = 'MC estimate',
     main = 'Convergence of MC estimate')
abline(h=1/12, col=2, lwd = 2, lty =2)
```