Mathematics/Statistics Bootcamp

Part I: Calculus

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Overview

Limit and Continuity

Derivative
- Definition and Differentiation Rules
- Application of Derivatives

Integrals
- Definite Integrals
- Improper Integrals

Series

Multivariate Calculus

Limit and Continuity

Limit

Suppose $-\infty < a, L < +\infty$ and $f(x) : X \to Y$ is a real-valued function, then

$$\lim_{x \to a} f(x) = L$$

if for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - L| < \epsilon \text{ whenever } 0 < |x - a| < \delta.$$

(The value of $f(x)$ approaches L when x approaches a.)

Left-hand limit: $\lim_{x \to a^-} f(x) = L$ if for any $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $a - \delta < x < a$.

Right-hand limit: $\lim_{x \to a^+} f(x) = L$ if for any $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $a < x < a + \delta$.

Limit: An Example

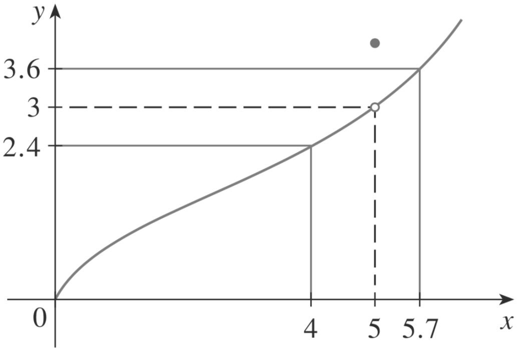


Figure: Plot of $y = f(x)$.

- What is $\lim_{x\to 5^-} f(x)$?
- What is $\lim_{x\to 5^+} f(x)$?
- What is $\lim_{x\to 5} f(x)$?

Infinite Limit/Limit at Infinity

- How to define $\lim_{x \to a} f(x) = \infty$ for $-\infty < a < +\infty$?

- How to define $\lim_{x \to \infty} f(x) = a$ for $-\infty < a < +\infty$?

Infinite Limit/Limit at Infinity

- How to define $\lim_{x \to a} f(x) = \infty$ for $-\infty < a < +\infty$?

  For any $M > 0$, there exists $\delta > 0$ such that

  $$f(x) > M \text{ whenever } 0 < |x - a| < \delta.$$

  (The value of $f(x)$ approaches ∞ when x approaches a.)

- How to define $\lim_{x \to \infty} f(x) = a$ for $-\infty < a < +\infty$?

  For any $\epsilon > 0$, there exists a $M > 0$ such that

  $$|f(x) - a| < \epsilon \text{ whenever } x > M.$$

  (The value of $f(x)$ approaches a when x approaches ∞.)

Continuity

A function f is continuous at a number a if

$$\lim_{x \to a} f(x) = f(a).$$

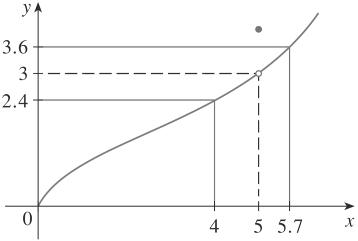
It implies 3 things:

1. $f(a)$ is defined ($a \in X$);
2. $\lim_{x \to a} f(x)$ exists;
3. $\lim_{x \to a} f(x) = f(a)$.

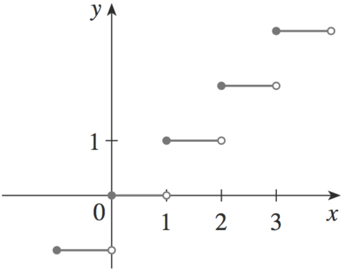
Right continuous: $\lim_{x \to a^-} f(x) = f(a)$.

Left continuous: $\lim_{x \to a^+} f(x) = f(a)$.

Continuity: Examples



This function is discontinuous at $x = 5$.



This function is discontinuous (but right continuous) at any integer x.

Derivative

Definition of Derivative

The derivative of function f at $a \in X$, denoted by $f'(a)$ is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists ("differentiable").

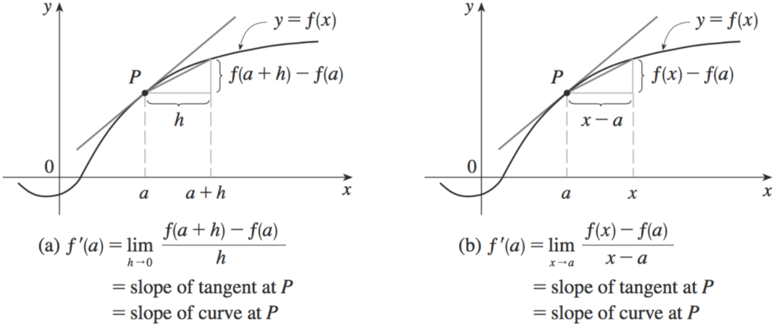


Figure: Geometric interpretations of the derivative.

Differentiation Rules

Derivatives of some common functions:

- $f(x) = const$, then $f'(x) = 0$;
- $f(x) = x^\alpha, \alpha \neq 0$, then $f'(x) = \alpha x^{\alpha-1}$;
- $(e^x)' = e^x$, $(\ln x)' = 1/x$ $(x > 0)$;
- $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$, $(\tan x)' = 1/\cos^2 x$;
- $(\sin^{-1} x)' = 1/\sqrt{1-x^2}$, $(\cos^{-1} x)' = -1/\sqrt{1-x^2}$, $(\tan^{-1} x)' = 1/1 + x^2$.

If both $f(x)$ and $g(x)$ are differentiable:

- $(cf(x))' = cf'(x)$, $(f(x) + g(x))' = f'(x) + g'(x)$;
- $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$;
- $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$ (assume $g(x) > 0$);
- The **chain rule**: if $F = f \circ g$, then $F'(x) = f'(g(x))g'(x)$.

Derivative: Exercises

1. Find the derivatives of the following functions
   - $f(x) = xe^x$;
   - $f(x) = 1 - \cos^2 x$;
   - $f(x) = \frac{\ln x}{x}$.

2. Find $\lim_{x \to 0}(1 + x)^{1/x}$.

Solution to Exercise 2

Let $f(x) = \ln x$, then

$$f'(1) = \lim_{x \to 0} \frac{\ln(1+x) - \ln 1}{x}$$
$$= \lim_{x \to 0} \frac{1}{x} \ln(1+x)$$
$$= \lim_{x \to 0} \ln(1+x)^{1/x}.$$

Since $f'(1) = 1$, $\lim_{x \to 0}(1+x)^{1/x} = e^1 = e$.

Minimum and Maximum

Theorem (Fermat's Theorem)

If f has a local minimum or maximum at c and $f'(c)$ exists, then $f'(c) = 0$.

Note: the converse is not true.

Theorem (The Second Derivative Test)

If f has second derivative on $(c - \epsilon_0, c + \epsilon_0)$ for a certain $\epsilon_0 > 0$, then

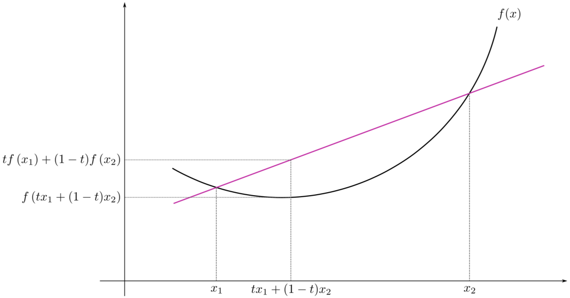
- *if $f'(c) = 0$ and $f'(c) > 0$, f has a local minimum at c;*
- *if $f'(c) = 0$ and $f'(c) < 0$, f has a local maximum at c.*

Convexity

A function defined on a convex set X, $f : X \to \mathbb{R}$ is convex if for any $x, y \in X$ and $t \in [0, 1]$,

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

Visually, a convex function has a "curve up" shape:



Convexity and Derivatives

Suppose $f(x)$ is twice differentiable on interval I, then

- f is convex on I if and only if $f'(x)$ is monotonically non-decreasing on I;
- f is convex on I if and only if $f''(x) \geq 0$ for $x \in I$ (often used to test for convexity).

A nice property of convexity:
Any local minimum of a convex function is also a global minimum;
a strictly convex function has at most one global minimum.
(Therefore convexity is much desired in optimization.)

Review Exercises: Morning Session

1. Calculate $\lim_{x\to 0} \frac{\sin x}{x}$ and $\lim_{x\to 0} \frac{\tan x}{x}$.

2. Which following functions are convex?

   A $f_1(x) = |x|, x \in [-1, 1]$;

   B $f_2(x) = \ln(x^2 + 1), x \in \mathbb{R}$;

   C $f_3(x) = e^{-x}, x \in \mathbb{R}$.

3. Let $f(x) = \frac{1}{x}, x > 0$. For every positive integer n, find $f^{(n)}(x)$.

4. $f(x) = \frac{1}{\sqrt{\gamma}} \exp\left(-\frac{(x-\mu)^2}{\gamma^2}\right)$ where constants $\gamma > 0$ and $\mu \in \mathbb{R}$, and $x \in \mathbb{R}$. Find all the global maximums of $f(x)$.

Taylor Expansion

Taylor Series, by *3Blue1Brown*

Challenge Exercises: Morning Session

Integrals

Properties of Definite Integrals

Let $a \leq d \leq b \in \mathbb{R}$:

- If $c \in \mathbb{R}$ is a constant, then $\int_a^b c\,dx = c(b-a)$;
- $\int_a^b cf(x)dx = c\int_a^b f(x)dx$;
- $\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$;
- $\int_a^d f(x)dx + \int_d^b f(x)dx = \int_a^b f(x)dx$;
- If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$;
- If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$.

The Fundamental Theorem of Calculus

If f is continuous on $[a, b]$, then:

- function $g(x) = \int_a^x f(s)ds, a \leq x \leq b$ is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$;
- $\int_a^b f(x)dx = F(b) - F(a)$, where F is any anti-derivative of f $(F' = f)$.

A mini-exercise: find $\frac{d}{dx} \int_1^x \sin t^4 dt$.

Useful Rules for Integration

- **Substitution rule**: If $u = g(x)$ is continuously differentiable on $[a, b]$ and f is continuous on the range of u, then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du.$$

- **Integration by parts**: If functions u and v are both continuously differentiable on $[a, b]$, then

$$\int_a^b u(x)v'(x)dx = [u(x)v(x)]|_a^b - \int_a^b v(x)u'(x)dx.$$

Integration: Exercises

1. Calculate $\int_1^e \frac{\ln x}{x} dx$.

2. Calculate $\int_0^\pi x \cos x dx$.

Improper Integrals

1. **Infinite intervals**: if $\int_a^t f(x)dx$ exists for every $t \geq a$ then

$$\int_a^\infty f(x)dx = \lim_{t \to \infty} \int_a^t f(x)dx$$

provided that this limit exists (convergent); similarly, one may define $\int_{-\infty}^a f(x)dx$, and if both $\int_a^\infty f(x)dx$ and $\int_{-\infty}^a f(x)dx$ are convergent, then $\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx$.

2. **Discontinuous integrand**: if f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)dx = \lim_{t \to b^-} \int_a^t f(x)dx$$

if this limit exists; similarly, if f is continuous on $(a, b]$ and is discontinuous at a, $\int_a^b f(x)dx = \lim_{t \to a^+} \int_t^b f(x)dx$.

Improper Integrals: Mini-Exercise

For what values of $p \in \mathbb{R}$ is the integral

$$\int_1^\infty \frac{1}{x^p}\,dx$$

convergent?

Series

Basics of Series

A **series** can be thought of as the infinite sum of a sequence $\{a_n\}$, written as $\sum_{n=1}^{\infty} a_n$ or $\sum a_n$. More formally, it can be defined by taking the limit of partial sums $\{s_n\}$, where $s_n = \sum_{i=1}^{n} a_i$: if $\lim_{n\to\infty} s_n$ exists then the series $\sum a_n$ is convergent, otherwise it is divergent.

If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

An important example - **the geometric series**:

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ and its sum is $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$. If $|r| \geq 1$, the geometric series is divergent.

Convergence of Series

Commonly used tests for convergence:

1. **The comparison test**: Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.
   (i) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent;
   (ii) If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent.
   - Video example: **famous proof that the harmonic series diverges**, by *Khan Academy*.
2. **The integral test** (by *Khan Academy*).

Review Exercises: Afternoon Session

1. Evaluate the following definite integrals:
   - $\int_0^4 \frac{x}{\sqrt{x^2+9}} dx$;
   - $\int_0^1 x \ln x dx$;

2. If f is continuous on $\mathbb{R}$, show that

$$\int_a^b f(x+c)dx = \int_{a+c}^{b+c} f(x)dx.$$

3. Do the following series converge? Calculate the value of the infinite sum for each convergent series.
(a) $\sum_{n=2}^{\infty} 5^{n-1} \left(\frac{9}{10}\right)^n$;
(b) $\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n \frac{1}{9^{n+2}}$.

4. True or false?
   - If $x_n \to 0$ as $n \to \infty$, then $\sum_{n=1}^{\infty} x_n$ is convergent;
   - $\sum_{n=1}^{\infty} x^n e^{-nx}$ is convergent for any $x > 0$;
   - $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ is convergent.

Multivariate Calculus

Partial Derivatives

If u is a function of n variables, $u = f(x_1, x_2, \ldots, x_n)$, its partial derivative with respect to the ith variable x_i is

$$\frac{\partial u}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, \ldots, x_{i-1}, x_i + h, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_n)}{h}.$$

(Strategy: treat all the other variables as constants and take the derivative with respect to the variable of interest.)

Suppose $u = f(x_1, x_2, \ldots, x_n)$ is defined on $\mathbb{R}^n$. If $\frac{\partial^2 u}{\partial x_i \partial x_j}$ and $\frac{\partial^2 u}{\partial x_j \partial x_i}$ are both continuous on $\mathbb{R}^n$, then $\frac{\partial^2 u}{\partial x_i \partial x_j} = \frac{\partial^2 u}{\partial x_j \partial x_i}$.

The Gradient Vector and Hessian Matrix

Suppose $f(x_1, x_2, \ldots, x_n)$ is a function of n variables such that all the partial derivatives exist, then the gradient vector of f is

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right).$$

If all the second-order partial derivatives of f also exist, the Hessian matrix of f is

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}.$$

Change of Variables

Take the two-variable case as an example:
Suppose $z = f(x, y)$ is function of x, y and $x = u(s, t)$, $y = v(s, t)$ with respect two other variables s, t, then $z = g(s, t)$ as a function of s, t, where

$$g(s, t) = f(u(s, t), v(s, t))|J|.$$

Here J is the **Jacobian** of the transformation $x = u(s, t)$, $y = v(s, t)$:

$$J = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{vmatrix} = \frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial s}.$$

Suppose that we want to integrate $f(x, y)$ over a region R. Under the transformation $x = u(s, t)$, $y = v(s, t)$ the regions becomes S and the integral becomes:

$$\iint_R f(x, y)dxdy = \iint_S f(u(s, t), v(s, t))|J|dsdt.$$

Multivariate Calculus Review Exercises

1. Let $f(x, y, z) = ye^x \ln z + z \tan z$ $(z \in (0, \frac{\pi}{2}))$ and $g(x, y, z) = x^3 y + y^3 z + z^5 + \sqrt{xy}z$.
   Compute $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial z}$, $\frac{\partial g}{\partial x}$, and $\frac{\partial g}{\partial y}$.

2. $f(x, y) = \frac{1}{\sqrt{2\pi y^2}} e^{-\frac{(x-s)^2}{2y^2}}$, where $s \in \mathbb{R}$ is a constant, and $y > 0$. Obtain the gradient vector and Hessian matrix of $g(x, y) = \ln f(x, y)$.

3. Evaluate the double integral $\int_1^3 \int_0^2 (xy + x^2 y^3) dydx$.

Challenge Exercises: Afternoon Session