Mathematics/Statistics Bootcamp

Part I: Calculus

Fan Bu[1] Kyle Burris[1]

[1]Department of Statistical Science
Duke University

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Overview

Limit and Continuity

Limit

Suppose $-\infty < a, L < +\infty$ and $f(x) : X \rightarrow Y$ is a real-valued function, then

$$\lim_{x \to a} f(x) = L$$

if for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - L| < \epsilon \text{ whenever } |x - a| < \delta.$$

Left-hand limit: $\lim_{x \to a^-} f(x) = L$ if for any $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $a - \delta < x < a$.

Right-hand limit: $\lim_{x \to a^+} f(x) = L$ if for any $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $a < x < a + \delta$.

Limit: An Example

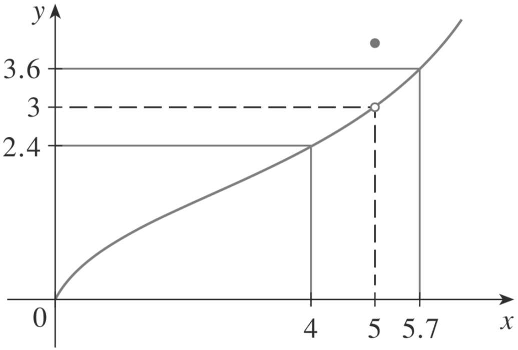


Figure: Plot of $y = f(x)$.

- What is $\lim_{x \to 5^-} f(x)$?
- What is $\lim_{x \to 5^+} f(x)$?
- What is $\lim_{x \to 5} f(x)$?

Infinite Limit/Limit at Infinity

- How to define $\lim_{x\to a} f(x) = \infty$ for $-\infty < a < +\infty$?

- How to define $\lim_{x\to\infty} f(x) = a$ for $-\infty < a < +\infty$?

Continuity

A function f is continuous at a number a if

$$\lim_{x \to a} f(x) = f(a).$$

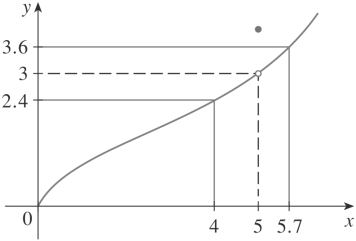
It implies 3 things:

1. $f(a)$ is defined ($a \in X$);
2. $\lim_{x \to a} f(x)$ exists;
3. $\lim_{x \to a} f(x) = f(a)$.

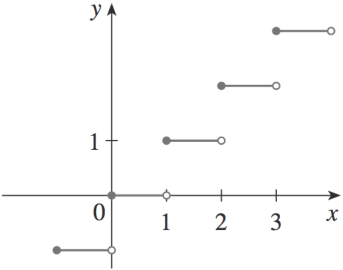
Right continuous: $\lim_{x \to a^-} f(x) = f(a)$.

Left continuous: $\lim_{x \to a^+} f(x) = f(a)$.

Continuity: Examples



This function is discontinuous at $x = 5$.



This function is discontinuous (but right continuous) at any integer x.

Other Useful Continuity Notions

- **Uniform continuity**:
  For all $\epsilon > 0$ there exists a $\delta > 0$ such that for any $x, y \in X$ with $|x - y| < \delta$ we have $|f(x) - f(y)| < \epsilon$.
  (A global property rather than a local one: as long as x, y are sufficiently close, we can guarantee that $f(x), f(y)$ are close.)

- **Lipschitz continuity**:
  There exists a constant L such that for any $x, y \in X$, we have $|f(x) - f(y)| \leq L|x - y|$.

Continuity: Exercise

Which statement(s) of the following is incorrect?

A $f(x) = x^2$, $x \in \mathbb{R}$ is not uniformly continuous, but $f(x) = x^2$, $x \in [0, 245]$ is Lipschitz continuous;

B $f(x) = x^{1/3}$, $x \in [0, 1]$ is Lipschitz continuous;

C $f(x) = x^{3/2} \sin(\frac{1}{x})$ $(x \neq 0)$, $f(0) = 0$, $x \in [0, 1]$ is uniformly continuous as well as Lipschitz continuous;

D Every continuous function defined on $[-1, 1]$ is uniformly continuous;

E A continuous function f is defined on $[-5, 5]$ and satisfies $f(-2) = -3, f(1) = 4$, then there must exist a number $x_0 \in [-2, 1]$ such that $f(x_0) = 0$.

Derivative

Definition of Derivative

The derivative of function f at $a \in X$, denoted by $f'(a)$ is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists ("differentiable").

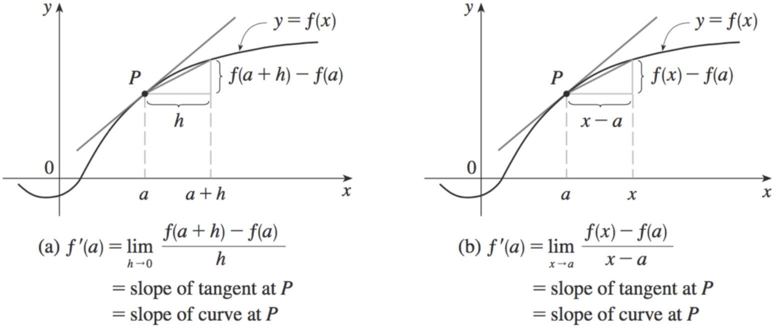


Figure: Geometric interpretations of the derivative.

Differentiation Rules

Derivatives of some common functions:

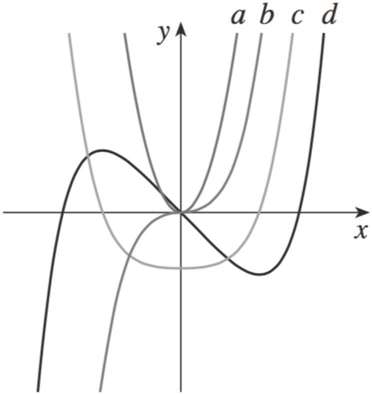
- $f(x) = const$, then $f'(x) = 0$;
- $f(x) = x^\alpha, \alpha \neq 0$, then $f'(x) = \alpha x^{\alpha-1}$;
- $(e^x)' = e^x$, $(\ln x)' = 1/x$ $(x > 0)$;
- $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$, $(\tan x)' = 1/\cos^2 x$;
- $(\sin^{-1} x)' = 1/\sqrt{1-x^2}$, $(\cos^{-1} x)' = -1/\sqrt{1-x^2}$, $(\tan^{-1} x)' = 1/1+x^2$.

If both $f(x)$ and $g(x)$ are differentiable:

- $(cf(x))' = cf'(x)$, $(f(x)+g(x))' = f'(x)+g'(x)$;
- $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$;
- $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x)-f(x)g'(x)}{g^2(x)}$ (assume $g(x) > 0$);
- The **chain rule**: if $F = f \circ g$, then $F'(x) = f'(g(x))g'(x)$.

Derivative: Exercises

1. The following figure shows the graphs of f, f', f'', and f'''. Identify each curve.



2. $f(x) = \frac{1}{\sqrt{\gamma}} \exp\left(-\frac{(x-\mu)^2}{\gamma}\right)$ where constants $\gamma > 0$ and $\mu \in \mathbb{R}$, and $x \in \mathbb{R}$. Calculate $f'(x)$ and find $x_0 \in \mathbb{R}$ such that the tangent line of $f(x)$ at x_0 is horizontal.

3. Find $\lim_{x \to 0}(1 + x)^{1/x}$.

Solution to Exercise 3

Let $f(x) = \ln x$, then

$$\begin{aligned} f'(1) &= \lim_{x\to 0} \frac{\ln(1+x) - \ln 1}{x} \\ &= \lim_{x\to 0} \frac{1}{x} \ln(1+x) \\ &= \lim_{x\to 0} \ln(1+x)^{1/x}. \end{aligned}$$

Since $f'(1) = 1$, $\lim_{x\to 0}(1+x)^{1/x} = e^1 = e$.

Minimum and Maximum

Theorem (Fermat's Theorem)

If f has a local minimum or maximum at c and $f'(c)$ exists, then $f'(c) = 0$.

Note: the converse is not true.

Theorem (The Second Derivative Test)

If f has second derivative on $(c - \epsilon_0, c + \epsilon_0)$ for a certain $\epsilon_0 > 0$, then

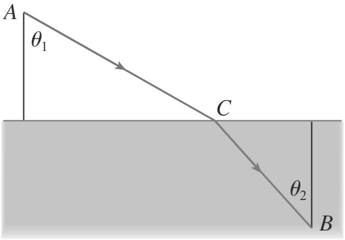
- *if $f'(c) =$ and $f'(c) > 0$, f has a local minimum at c;*
- *if $f'(c) =$ and $f'(c) < 0$, f has a local maximum at c.*

Minimum and Maximum: Example

Let v_1 be the velocity of light in air and v_2 the velocity of light in water. According to Fermat's Principle, a ray of light will travel from a point A in the air to a point B in the water by a path ACB that **minimizes** the time taken. Then

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2},$$

where θ_1 is the angle of incidence and θ_2 is the angle of refraction. This equation is known as **Snell's Law**.

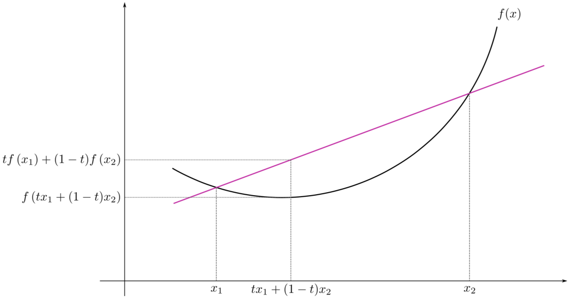


Convexity

A function defined on a convex set X, $f : X \to \mathbb{R}$ is convex if for any $x, y \in X$ and $t \in [0, 1]$,

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

Visually, a convex function has a "curve up" shape:



Convexity and Derivatives

Suppose $f(x)$ is twice differentiable on interval I, then

- f is convex on I if and only if $f'(x)$ is monotonically non-decreasing on I;
- f is convex on I if and only if $f''(x) \geq 0$ for $x \in I$ (often used to test for convexity).

A nice property of convexity:
Any local minimum of a convex function is also a global minimum; a strictly convex function has at most one global minimum.
(Therefore convexity is much desired in optimization.)

Review Exercises: Morning Session

1. Are all Lipschitz continuous functions defined on $[-1, 1]$ also differentiable on $(-1, 1)$? If not, give a counterexample.

2. Calculate $\lim_{x\to 0} \frac{\sin x}{x}$ and $\lim_{x\to 0} \frac{\tan x}{x}$.

3. Let $f(x) = \frac{1}{x}, x > 0$. For every positive integer n, find $f^{(n)}(x)$.

4. Which following functions are convex?

- $f_1(x) = |x|, x \in [-1, 1]$;
- $f_2(x) = \ln(x^2 + 1), x \in \mathbb{R}$;
- $f_3(x) = e^{-x}, x \in \mathbb{R}$.

5. $f(x) = \frac{1}{\sqrt{\gamma}} \exp\left(-\frac{(x-\mu)^2}{\gamma^2}\right)$ where constants $\gamma > 0$ and $\mu \in \mathbb{R}$, and $x \in \mathbb{R}$. Find all the global maximums of $f(x)$.

Taylor Expansion

Talor Series, by *3Blue1Brown*

Challenge Exercises: Morning Session

1. Find the Taylor expansion of $f(x) = \tan x$ around 0 and use the result to show that $\lim_{x\to 0} \frac{\tan x}{x} = 1$.

2. Calculate $\sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^x}{x!}$. Here $\lambda > 0$ is a constant and $x \in \mathbb{N}$. $(0! = 1.)$

3. Let $f(x) = x^{\alpha-1}(1-x)^{\beta-1}, x \in (0,1)$, where $\alpha, \beta > 0$ are constants. Find all the minimums and maximums (if there are any) of f.

4. $f(x)$ is a convex function on $\mathbb{R}$, $P = (x_0, f(x_0))$ is a point on $f(x)$, and l is the tangent line of the curve of $f(x)$ at P. Choose the correct statement(s).

   A l must lie above or on the curve of $f(x)$;

   B l must lie below or on the curve of $f(x)$;

   C l can intersect with $f(x)$ at multiple points.

Integrals

Properties of Definite Integrals

Let $a \leq d \leq b \in \mathbb{R}$:

- If $c \in \mathbb{R}$ is a constant, then $\int_a^b c dx = c(b-a)$;
- $\int_a^b cf(x)dx = c\int_a^b f(x)dx$;
- $\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$;
- $\int_a^d f(x)dx + \int_d^b f(x)dx = \int_a^b f(x)dx$;
- If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$;
- If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$.

The Fundamental Theorem of Calculus

If f is continuous on $[a, b]$, then:

- function $g(x) = \int_a^x f(x)dx, a \leq x \leq b$ is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$;
- $\int_a^b f(x)dx = F(b) - F(a)$, where F is any anti-derivative of f ($F' = f$).

A mini-exercise: find $\frac{d}{dx} \int_1^x \sin x^4 dx$.

Useful Rules for Integration

- **Substitution rule**: If $u = g(x)$ is continuously differentiable on $[a, b]$ and f is continuous on the range of u, then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du.$$

- **Integration by parts**: If functions u and v are both continuously differentiable on $[a, b]$, then

$$\int_a^b u(x)v'(x)dx = [u(x)v(x)]|_a^b - \int_a^b v(x)u'(x)dx.$$

Integration: Exercises

1. Calculate $\int_1^e \frac{\ln x}{x} dx$.

2. Calculate $\int_0^1 x \cos x dx$.

Improper Integrals

1. **Infinite intervals**: if $\int_a^t f(x)dx$ exists for every $t \geq a$ then

$$\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx$$

provided that this limit exists (convergent); similarly, one may define $\int_{-\infty}^a f(x)dx$, and if both $\int_a^\infty f(x)dx$ and $\int_{-\infty}^a f(x)dx$ are convergent, then $\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx$.

2. **Discontinuous integrand**: if f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)dx = \lim_{t\to b^-} \int_a^t f(x)dx$$

if this limit exists; similarly, if f is continuous on $(a, b]$ and is discontinuous at a, $\int_a^b f(x)dx = \lim_{t\to a^+} \int_t^b f(x)dx$.

Improper Integrals: Exercises

1. For what values of $p \in \mathbb{R}$ is the integral
$$\int_1^\infty \frac{1}{x^p}\,dx$$
convergent?

2. Evaluate $\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$.

Sequences and Series

Basics of Sequences

A **sequence** is a list of numbers written in a definite order. We often denote a sequence $\{a_1, a_2, a_3, \ldots\}$ by $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$.

A sequence $\{a_n\}$ has **limit** L (written as $\lim_{n\to\infty} a_n = L$, or $a_n \to L$ as $n \to \infty$) if for every $\epsilon > 0$ there is a corresponding integer N such that $|a_n - L| < \epsilon$ whenever $n > N$.

If for every $n \in \mathbb{N}$, $a_n \leq a_{n+1}$ (increasing) or $a_n \geq a_{n+1}$ (decreasing), then the sequence $\{a_n\}$ is **monotonic**. If there exists a number $M > 0$ such that $|a_n| \leq M$ for every n then the sequence $\{a_n\}$ is **bounded**.

Monotonic Sequence Theorem: Every bounded, monotonic sequence is convergent (has a limit).

Sequences: Exercises

1. Find the limit of the sequence a_n where $a_n = \left(1 + \frac{1}{n}\right)^n$, $n \in \mathbb{N}$.

2. Let $a_n = \frac{2^n}{n!}$, $n \in \mathbb{N}$. Is the sequence a_n convergent? If so, what is its limit?

Basics of Series

A **series** can be thought of as the infinite sum of a sequence a_n, written as $\sum_{n=1}^{\infty} a_n$ or $\sum a_n$. More formally, it can be defined by taking the limit of partial sums $\{s_n\}$, where $s_n = \sum_{i=1}^{n} a_i$: if $\lim_{n\to\infty} s_n$ exists then the series $\sum a_n$ is convergent, otherwise it is divergent.

If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

An important example - **the geometric series**:

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ and its sum is $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$. If $|r| \geq 1$, the geometric series is divergent.

Series: Exercises

1. Show that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent and find its sum.

2. Find $\sum_{n=0}^{\infty} x^n$ where $|x| < 1$.

Convergence of Series

Commonly used tests for convergence:

1. **The comparison test**: Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.
   (i) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent;
   (ii) If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent.
   - Video example: **famous proof that the harmonic series diverges**, by *Khan Academy*.
2. **The limit comparison test**: Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ where c is a finite positive number, then either both series converge or both diverge.
3. **The integral test** (by *Khan Academy*).
4. **The alternating series test**: If the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1} b_n$ ($b_n > 0$) satisfies (i) $b_{n+1} \leq b_n$ for all n, and (ii) $\lim_{n\to\infty} b_n = 0$ then the series is convergent.

Review Exercises: Afternoon Session

1. Evaluate the following definite integrals:
   - $\int_0^4 \frac{x}{\sqrt{x^2+9}} dx$;
   - $\int_0^1 \frac{2x}{1+x^4} dx$.

2. Evaluate the following definite integrals:
   - $\int_0^1 x \ln x dx$;
   - $\int_0^\pi e^x \sin x dx$.

3. If f is continuous on $\mathbb{R}$, show that

$$\int_a^b f(x+c)dx = \int_{a+c}^{b+c} f(x)dx.$$

4. Do the following series converge? Calculate the value of the infinite sum for each convergent series.
(a) $\sum_{n=2}^\infty 5^{n-1} \left(\frac{9}{10}\right)^n$;
(b) $\sum_{n=1}^\infty \left(\frac{3}{2}\right)^n \frac{1}{9^{n+2}}$.

5. True or false?
   - If $x_n \to 0$ as $n \to \infty$, then $\sum_{n=1}^\infty x_n$ is convergent;
   - $\sum_{n=1}^\infty x^n e^{-nx}$ is convergent for any $x > 0$;
   - $\sum_{n=2}^\infty \frac{1}{n \ln n}$ is convergent.

Multivariate Calculus

Partial Derivatives

If u is a function of n variables, $u = f(x_1, x_2, \ldots, x_n)$, its partial derivative with respect to the ith variable x_i is

$$\frac{\partial u}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, \ldots, x_{i-1}, x_i + h, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_n)}{h}.$$

(Strategy: treat all the other variables as constants and take the derivative with respect to the variable of interest.)

Suppose $u = f(x_1, x_2, \ldots, x_n)$ is defined on $\mathbb{R}^n$. If $\frac{\partial^2 u}{\partial x_i \partial x_j}$ and $\frac{\partial^2 u}{\partial x_j \partial x_i}$ are both continuous on $\mathbb{R}^n$, then $\frac{\partial^2 u}{\partial x_i \partial x_j} = \frac{\partial^2 u}{\partial x_j \partial x_i}$.

The Gradient Vector and Hessian Matrix

Suppose $f(x_1, x_2, \ldots, x_n)$ is a function of n variables such that all the partial derivatives exist, then the gradient vector of f is

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right).$$

If all the second-order partial derivatives of f also exist, the Hessian matrix of f is

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}.$$

Change of Variables

Take the two-variable case as an example:
Suppose $z = f(x, y)$ is function of x, y and $x = u(s, t)$, $y = v(s, t)$ with respect two other variables s, t, then $z = g(s, t)$ as a function of s, t, where

$$g(s, t) = f(u(s, t), v(s, t))|J|.$$

Here J is the **Jacobian** of the transformation $x = u(s, t)$, $y = v(s, t)$:

$$J = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{vmatrix} = \frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial s}.$$

Suppose that we want to integrate $f(x, y)$ over a region R. Under the transformation $x = u(s, t)$, $y = v(s, t)$ the regions becomes S and the integral becomes:

$$\iint_R f(x, y)dxdy = \iint_S f(u(s, t), v(s, t))|J|dsdt.$$

Challenge Exercises: Afternoon Session

1. Given that $\int_1^\infty \frac{1}{x} dx = \infty$ and $\sum_{n=1}^\infty \frac{1}{n} = \infty$, use two methods to show that $\sum_{n=1}^\infty \frac{1}{n^p} = \infty$ for any $p \in (0, 1)$.

2. Find $\lim_{x \to 3} \frac{x}{x-3} \int_3^x \frac{\sin t}{t} dt$.

3. The Gamma function $\Gamma(x)$ is defined for any real number $x > 0$ as $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$. Show that $\Gamma(x+1) = x\Gamma(x)$.

4. Evaluate $\int_0^\infty e^{x^2} dx$. (Hint: start with $\int_0^\infty \int_0^\infty e^{x^2+y^2} dxdy$.)

5. $f(x, y) = \frac{1}{\sqrt{2\pi y^2}} e^{-\frac{(x-s)^2}{2y^2}}$, where $s \in \mathbb{R}$ is a constant. Obtain the Hessian matrix of f.

The End