HOMEWORK NOVEMBER 11

1. Define the notion of the dimension of a vector space without mentioning vectors.

2. Let V and W be finite-dimensional vector space of the same dimension. Let B be a basis for V and let C be a basis for W. Let $T : V \to W$ be a bijective linear transformation. Let $T^{-1} : W \to V$ be the inverse linear transformation. Let $[T]_{C,B}$ be the matrix of T relative to B and C. Let $[T^{-1}]_{B,C}$ be the matrix of T^{-1} relative to C and B. Prove that $[T^{-1}]_{B,C}$ is the inverse matrix of $[T]_{C,B}$.

3. Let V be a vector space. Let $V^* = \mathrm{Hom}(V, \mathbb{R})$ be the dual vector space, i.e., V^* is the vector space of all linear transformations from V to $\mathbb{R}$. We have a natural linear transformation

$$\begin{aligned} E : V &\to (V^*)^* \\ v &\mapsto (h \mapsto h(v), \forall h \in V^*) \end{aligned}$$

(a) Prove that E is injective.

(b) Prove that E is surjective when V is finite-dimensional.

(c) Prove that E is not surjective when V is infinite-dimensional.

4. Let V and W be finite-dimensional vector spaces. Let $T : V \to W$ be a linear transformation. We have a dual linear transformation

$$\begin{aligned} T^* : W^* &\to V^* \\ h &\mapsto h \circ T \end{aligned}$$

Prove the following statements using definitions (instead of using matrices):

(a) T is injective if and only if T^* is surjective.

(b) T is surjective if and only if T^* is injective.

(c) $\mathrm{rank}(T) = \mathrm{rank}(T^*)$.

5. Let V be a finite-dimensional vector space. Let $\{v_1, v_2, \ldots, v_n\}$ be a basis for V. For each $i = 1, 2, \ldots, n$, we define a linear transformation $\alpha_i : V \to \mathbb{R}$ by the condition that

$$\alpha_i(v_i) = 1, \qquad \alpha_i(v_j) = 0 (j \neq i)$$

Prove that $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a basis for V^*. This basis is called the dual basis of $\{v_1, v_2 \ldots, v_n\}$.

6. Let $T : V \to W$ be a linear transformation between two finite-dimensional vector spaces. Let B be a basis for V. Let B^* be its dual basis for V^*. Let C be a basis for W. Let C^* be its dual basis for W^*. Let $[T]_{C,B}$ be the matrix of T relative to B and C. Let $[T^*]_{B^*,C^*}$ be the matrix of T^* relative to C^* and B^*. Prove that $[T^*]_{B^*,C^*}$ is the transpose of $[T]_{C,B}$.