Problem Set 1 Solutions

Problem 1. Consider a new model of preferences, the *PI-model*. The primitives of this model are two binary relations, P and I, defined on X, where P is interpreted as the "strictly better than" relation, and I is interpreted as the "indifference" relation. We impose three conditions on P and I in this model: (1) for any $x \in X$, xIx and $x\bar{P}x$; (2) for any $x, y \in X$ with $x \neq y$, *exactly* one of the following three is true: xPy, yPx and xIy; (3) both P and I are transitive. Based on the construction in this model, prove the following results.

(a) I is symmetric.

(b) If xPy and yIz, then xPz. If xIy and yPz, then xPz.

(c) The PI-model is equivalent to the $\succeq$-model.

Answer. (a). Consider any $x, y \in X$ with xIy. If $y = x$, then yIx. If $y \neq x$, then, applying condition (2) to x and y, we cannot have xPy or yPx. Applying condition (2) to y and x, we have yIx.

(b). Let us denote the transitivity of P as "*PPP*", and the transitivity of I as "*III*". In this part, basically we want to show "*PIP*" and "*IPP*". First, assume to the contrary, there exist $x, y, z \in X$ such that xPy, yIz and $x\bar{P}z$. If $x = z$, then by (1), zIx. If $x \neq z$, then by (2), either zPx or zIx. In the case of zPx, since xPy, by *PPP* we have zPy, contradicting to condition (2) since $z \neq y$ and yIz. In the case of zIx, since yIz, by *III* we have yIx, contradicting to condition (2) since xPy and $x \neq y$.

The proof of *IPP* is similar. Assume to the contrary, we have xIy, yPz and $x\bar{P}z$. Then either zPx or zIx. If zPx, then by *PPP*, yPx, contradicting to (2) since $y \neq x$ and xIy. If zIx, then zIy by *III*, contradicting to (2) since yPz and $y \neq z$.

(c). *PI-model implies $\succeq$-model*. Given P and I, define $\succeq$ on X as follows. For any $x, y \in X$, let $x \succeq y$ if xPy or xIy. We first show $\succeq$ is complete. Consider any $x, y \in X$. If $x = y$, then xIy by (1). Hence $x \succeq y$. If $x \neq y$, then by (2) there are three possible cases: xPy, yPx and xIy. If xPy or xIy, then $x \succeq y$. If yPx, then $y \succeq x$. In sum, $\succeq$ is complete. It remains to show that $\succeq$ is transitive. Consider any $x, y, z \in X$ with $x \succeq y$ and $y \succeq z$. By the definition of $\succeq$, there are four possible cases: (i) xPy and

yPz, (ii) xIy and yIz, (iii) xPy and yIz, and (iv) xIy and yPz. Then, by *PPP*,*III*, *PIP* and *IPP*, we have xPz or xIz. Hence $x \succeq z$.

$\succeq$-model implies PI-model. Given $\succeq$, which is complete and transitive, define P and I on X as follows. For any $x, y \in X$, let xPy if $x \succeq y$ and $y \not\succeq x$; let xIy if $x \succeq y$ and $y \succeq x$. We need to verify conditions (1)-(3) for P and I. (1) is obvious. (2) is also easy to see, since for any $x, y \in X$ with $x \neq y$, completeness implies that exactly one of the following three is true: (i) $x \succeq y$ and $y \succeq x$, (ii) $x \succeq y$ and $y \not\succeq x$ and (iii) $y \succeq x$ and $x \not\succeq y$. It remains to show *PPP* and *III*. First, suppose xPy and yPz. xPy implies $x \succeq y$. If $z \succeq x$, then by the transitivity of $\succeq$, we have $z \succeq y$, contradicting to yPz. Hence we must have $z \not\succeq x$. By the completeness of $\succeq$, $x \succeq z$. Therefore, xPz. Finally, we show *III*. Suppose xIy and yIz. It follows from the definition of I that $x \succeq y$, $y \succeq z$, $z \succeq y$ and $y \succeq x$. Then by transitivity of $\succeq$, $x \succeq z$ and $z \succeq x$. Hence xIz.

Problem 2. Let C be a choice correspondence defined on the domain $\mathscr{D}$. Assume that for any $A, B \in \mathscr{D}$, $A \cap B \in \mathscr{D}$. Show that if C satisfies Sen's properties α and β, then C satisfies the weak axiom of revealed preference.

Answer. Assume to the contrary, C satisfies α and β, but not WARP. Then there exist $A, B \in \mathscr{D}$ such that $x, y \in A$, $x, y \in B$, $x \in C(A)$, $y \notin C(A)$ and $y \in C(B)$. Consider the set $A \cap B \in \mathscr{D}$. Clearly $x, y \in A \cap B$. Since $x \in C(A)$, by α, $x \in C(A \cap B)$. Since $y \in C(B)$, by α, $y \in C(A \cap B)$. But β is then violated as $x \in C(A)$ and $y \notin C(A)$.

Problem 3. Let $\succeq$ be a preference relation defined on a *finite* set X, and $\succ$ is the asymmetric component of $\succeq$. Notice that $\succeq$ is not assumed to be rational. We say $\succ$ is *acyclic* if there does not exist a list $(x_1, x_2, ..., x_{n-1}, x_n)$ such that $x_k \in X$ for each $k \in \{1, 2, ..., n\}$, $n \geq 2$, and $x_1 \succ x_2 \succ ... \succ x_{n-1} \succ x_n \succ x_1$. For any $A \subseteq X$, let

$$C_{\succ}(A) = \{x \in A : \text{there does not exist } \ y \in A \ \text{ such that } \ y \succ x\}$$

Prove the following results.

(a)$C_{\succ}(A) \neq \phi$ for all non-empty $A \subseteq X$ if and only if $\succ$ is acyclic.

(b) Assume $\succ$ is acyclic. $C_{\succ}$ satisfies Sen's property α, but may not satisfy property β.

Answer. *The purpose of this problem is to introduce a more general way of constructing choice correspondences from preferences. Recall that in class we defined the correspondence* $C_{\succeq}(A) = \{x \in A : x \succeq y, \forall y \in A\}$. *$C_{\succeq}(A)$ consists of the **best** (or **greatest**) elements in A. On the other hand, $C_{\succ}(A)$ consists of the **maximal** elements in A. If $\succeq$ is complete, then these two concepts coincide. But generally, $C_{\succeq}(A) \subseteq C_{\succ}(A)$. Hence $C_{\succ}$ is a well-defined choice correspondence in more circumstances. And acyclicity of $\succ$ is necessary and sufficient for it to be a well-defined choice correspondence.*

(a.) "only if" part. If $\succ$ is not acyclic, then there exists a list $(x_1, x_2, ..., x_{n-1}, x_n)$ such that $x_k \in X$ for each $k \in \{1, 2, ..., n\}$, $n \geq 2$, and $x_1 \succ x_2 \succ ... \succ x_{n-1} \succ x_n \succ x_1$. Let $A = \{x_1, x_2, ..., x_{n-1}, x_n\}$ and clearly $C_{\succ}(A) = \phi$.

"If" part. Assume to the contrary, $\succ$ is acyclic but for some non-empty $A \subseteq X$ we have $C_{\succ}(A) = \phi$. Pick some $x \in A$. Since $x \notin C_{\succ}(A)$, there exists $y \in A$ such that $y \succ x$. Continuing in this fashion, since A is finite, there exists a list $(x_1, x_2, ..., x_{n-1}, x_n)$ such that $x_k \in A$ for each $k \in \{1, 2, ..., n\}$, $n \geq 2$, and $x_1 \succ x_2 \succ ... \succ x_{n-1} \succ x_n \succ x_1$, contradicting to acyclicity.

(b). Let $x \in A \subseteq B \subseteq X$ and $x \in C_{\succ}(B)$. Since there does not exist $y \in B$ with $y \succ x$, there does not exist $y \in A$ with $y \succ x$. Hence $x \in C_{\succ}(A)$ and Sen's α is proved. To see β is not satisfied, consider the following example. $A = \{x, y\} \subseteq \{x, y, z\} = B = X$, where $|X| = 3$ and $\succeq = \{(z, y)\}$. In this case, $C_{\succ}(A) = \{x, y\}$ and $C_{\succ}(B) = \{x, z\}$. β is not satisfied.

Problem 4. Show that if a choice correspondence C (defined on the domain $\mathscr{D}$) can be rationalized, then it satisfies the *path-invariance* property: for any $B_1, B_2 \in \mathscr{D}$ such that $B_1 \cup B_2 \in \mathscr{D}$ and $C(B_1) \cup C(B_2) \in \mathscr{D}$, we have $C(B_1 \cup B_2) = C(\{C(B_1) \cup C(B_2)\})$.

Answer. Since C can be rationalized, there exists rational $\succeq$ such that $C = C_{\succeq}$. Hence, for any $B \in \mathscr{D}$, $x \in C(B)$ if and only if $x \in B$ and $x \succeq y$ for all $y \in B$.

We first show $C(B_1 \cup B_2) \subseteq C(\{C(B_1) \cup C(B_2)\})$. Consider any $x \in C(B_1 \cup B_2)$. $x \in B_1$ or $x \in B_2$. Suppose that $x \in B_1$ (the case where $x \in B_2$ can be shown similarly). Since $x \succeq y$ for all $y \in B_1 \cup B_2$, $x \succeq y$ for all $y \in B_1$. Hence $x \in C(B_1)$ and $x \in C(B_1) \cup C(B_2)$. Since $C(B_1) \cup C(B_2) \subseteq B_1 \cup B_2$, $x \succeq y$ for all $y \in C(B_1) \cup C(B_2)$. So $x \in C(\{C(B_1) \cup C(B_2)\})$.

On the other hand, consider any $x \in C(\{C(B_1) \cup C(B_2)\})$. Obviously, $x \in B_1 \cup B_2$. Pick some $y \in C(B_1)$ and some $z \in C(B_2)$, we must have $x \succeq y$ and $x \succeq z$. Since $y \in C(B_1)$ and $z \in C(B_2)$, we have $y \succeq y'$ for every $y' \in B_1$ and $z \succeq z'$ for every $z' \in B_2$. By transitivity, $x \succeq y'$ for every $y' \in B_1$ and $x \succeq z'$ for every $z' \in B_2$. It follows that $x \in C(B_1 \cup B_2)$.