Problem Set 3

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Problem 1. An individual has Bernoulli utility function $u(\cdot)$ and initial wealth w. Let lottery L offer a payoff of G with probability p and a payoff of B with probability $1-p$.

(a) If the individual owns the lottery, what is the minimum price he would sell it for?

(b) If he does not own it, what is the maximum price he would be willing to pay for it?

(c) Are buying and selling prices equal? Give an economic interpretation for your answer. Hind conditions on the parameters of the problem under which buying and selling prices are equal.

(d) Let $G = 10$, $B = 5$, $w = 10$, and $u(x) = \sqrt{x}$. Compute the buying and selling prices for this lottery and this utility function.

Answer to Problem 1.

(a) The minimum price at which the individual would sell the lottery (denoted as R_S) is the price at which the utility of their wealth after selling the lottery is equal to the expected utility of keeping the lottery. Mathematically, this is:

$$u(w + R_S) = p \cdot u(w + G) + (1 - p) \cdot u(w + B)$$

(b) The maximum price at which the individual would buy the lottery (denoted as R_B) is the price at which the utility of their wealth after buying the lottery is equal to the utility without buying it. This can be represented as:

$$u(w - G_B) = p \cdot u(w + G - R_B) + (1 - p) \cdot u(w + B - R_B)$$

(c) Generally, the buying and selling prices are not equal due to the concept of loss aversion in behavioral economics. People tend to value a good more when they own it (endowment effect) and are generally risk-averse, meaning they would require more to give up a good than they would be willing to pay to acquire it. The buying and selling prices will be equal only under conditions of risk neutrality, where the individual's utility function is linear.

(d) Given $G = 10$, $B = 5$, $w = 10$, and $u(x) = \sqrt{x}$, we can compute the buying and selling prices. For selling, solve $\sqrt{10 + G_S} = p \cdot \sqrt{10 + 10} + (1-p) \cdot \sqrt{10 + 5}$. For buying, solve $\sqrt{10 - G_B} = p \cdot \sqrt{10 + 10 - G_B} + (1-p) \cdot \sqrt{10 + 5 - G_B}$.

Problem 2. Suppose that an individual has a Bernoulli utility function $u(x) = \sqrt{x}$.

(a) Calculate the certainty equivalent and the probability premium for a gamble $(16, 4; \frac{1}{2}, \frac{1}{2})$.

(b) Calculate the certainty equivalent and the probability premium for a gamble $(36, 16; \frac{1}{2}, \frac{1}{2})$.

(c) Compare the result in (b) with the one in (a) and interpret.

Answer to Problem 2.

For a gamble $(G_1, G_2; p, 1-p)$,

- Certainty Equivalent: $u(CE) = p \cdot u(G_1) + (1-p) \cdot u(G_2)$

- Probability Premium: $u(x) = \left[\frac{1}{2} + \pi(x, \varepsilon, u)\right] u(x + \varepsilon) + \left[\frac{1}{2} - \pi(x, \varepsilon, u)\right] u(x - \varepsilon)$

(a) $u(CE) = \frac{1}{2} \cdot \sqrt{16} + \frac{1}{2} \cdot \sqrt{4}$, $CE = 9$. $u(x) = \sqrt{16 \times \frac{1}{2} + 4 \times \frac{1}{2}} = \sqrt{10}$, $u(x + \varepsilon) + u(x - \varepsilon) = 4 + 2 = 6$, $u(x + \varepsilon) - u(x - \varepsilon) = 4 - 2 = 2$, thus $\pi(x, \varepsilon, u) = \frac{\sqrt{10} - 3}{2}$.

(b) $u(CE) = \frac{1}{2} \cdot \sqrt{36} + \frac{1}{2} \cdot \sqrt{16}$, $CE = 25$. $u(x) = \sqrt{36 \times \frac{1}{2} + 16 \times \frac{1}{2}} = \sqrt{26}$, $u(x + \varepsilon) + u(x - \varepsilon) = 6 + 4 = 10$, $u(x + \varepsilon) - u(x - \varepsilon) = 6 - 4 = 2$, thus $\pi(x, \varepsilon, u) = \frac{\sqrt{26} - 5}{2}$.

(c) The average value and the difference in certainty equivalent of the lotteries in (a)

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and (b) are both 1, but the probability premium of the lottery in (a) is higher.

Problem 3. Consider an economy with two goods, good l and a numeraire good, 2 consumers, $i = 1, 2$, and 2 firms, $j = 1, 2$. Let $x_i \in \mathbb{R}_+$ denote consumer i's consumption of good l, and $m_i \in \mathbb{R}$ her consumption of the numeraire. The utility function of each consumer i is given by:

$$u_i(m_i, x_i) = m_i + 1 - \frac{(1 - x_i)^2}{2}, \text{ if } 0 \le x_i \le 1$$
$$= m_i + 1, \text{ if } x_i > 1$$

Each firm j produces $q_j \ge 0$ units of good l using amount $c_j(q_j)$ of the numeraire where

$$c_j(q_j) = (q_j)^2$$

Each consumer has an initial endowment of 50 units of the numeraire and owns $\frac{1}{2}$ share of each firm.

(a) Derive the competitive equilibrium.

(b) Write down the set of all the efficient allocations in this economy.

Answer to Problem 3.

(a) Derive the competitive equilibrium.

- Each firm maximizes profits.

$$\max_{q_j \ge 0} p^* q_j - c_j(q_j)$$
$$\text{s.t. } p^* \le c'_j(q_j^*)$$

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- Each consumer maximizes utility subject to their budget constraint.

$$\max_{m_i, x_i} m_i + \phi_i(x_i)$$
$$\text{s.t. } m_i + p^* x_i \le w_{m_i} + \sum_{j=1}^{J} \theta \left[p^* q_j^* - c_j\left(q_j^*\right) \right]$$

When $0 \le x_1 \le 1$ and $0 \le x_2 \le 1$, we have $\phi_i(x_i) = 1 - \frac{(1-x_i)^2}{2}$. Bring $c_j(q_j) = q_j^2$, $w_{mi} = 50$, $\theta = 0.5$ and $\phi_i'(x_i) = 1 - x_i$ into competitive equilibrium conditions, we have

$$
\begin{cases}
p^* \le 2q_1^* \\
p^* \le 2q_2^* \\
1 - x_1^* \le p^* \\
1 - x_2^* \le p^* \\
x_1^* + x_2^* = q_1^* + q_2^* \\
m_i^* = 50 + \frac{1}{2} \sum_{j=1}^{J} \left(p^* q_j^* - q_j^{*2} \right) - p^* x_i^*
\end{cases}
\Rightarrow
\begin{cases}
x_1^* = \frac{1}{3} \\
x_2^* = \frac{1}{3} \\
q_1^* = \frac{1}{3} \\
q_2^* = \frac{1}{3} \\
p^* = \frac{2}{3} \\
m_1 = m_2 = 49\frac{8}{9}
\end{cases}
$$

The competitive equilibrium is $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 49\frac{8}{9}, 49\frac{8}{9})$, $p^* = \frac{2}{3}$.

When $x_i \ge 1$, $\phi_i(x_i) = 1$, thus we have $x_1 = x_2 = 1$. Now the competitive equilibrium conditions is

$$
\begin{cases}
x_1^* = 1 \\
x_2^* = 1 \\
p^* \le 2q_1^* \\
p^* \le 2q_2^* \\
x_1^* + x_2^* = q_1^* + q_2^* \\
m_i^* = 50 + \sum_{j=1}^{J} \frac{1}{2} \left(p^* q_j^* - q_j^2 \right) - p^* x_i^*
\end{cases}
\Rightarrow
\begin{cases}
x_1^* = 1 \\
x_2^* = 1 \\
q_1^* = 1 \\
q_2^* = 1 \\
p^* = 2 \\
m_1^* = m_2^* = 49
\end{cases}
$$

The competitive equilibrium is $(1, 1, 1, 1, 49, 49)$, $p^* = 2$.

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(b) Utilize the entire endowment and production without waste.

$$M_{x_i, q_j} \sum_{i=1}^{N} \phi_i(x_i) + w_m - \sum_{j=1}^{J} c_j(q_j)$$
$$\text{s.t. } x_1 \geqslant 0, q_j \geqslant 0, \sum_{i=1}^{N} x_i = \sum_{j=1}^{J} q_j$$

$$\mathcal{L} = \sum_{i=1}^{N} \phi_i(x_i) + w_m - \sum_{j=1}^{J} c_j(q_j) + \lambda \left(\sum_{i=1}^{N} x_i - \sum_{j=1}^{J} q_j \right)$$

Thus we have

$$\begin{cases} \lambda \leqslant c_j'(q_j^*) \\ \phi_i'(x_i^*) \leqslant \lambda \\ \sum_{i=1}^{N} x_i^* = \sum_{j=1}^{J} q_j^* \\ \sum_{i=1}^{N} m_i = w_m - \sum_{j=1}^{J} c_j(q_j^*) \end{cases}$$

When $0 \leq x_i \leq 1$, we have

$$\begin{cases} \lambda \leq 2q_j \\ 1 - x_i \leq \lambda \\ x_1^* + x_2^* = q_1^* + q_2^* \\ m_1^* + m_2^* = 100 - q_1^{*2} - q_2^{*2} \end{cases} \Rightarrow \begin{cases} x_1^* = x_2^* = \frac{1}{3} \\ q_1^* = q_2^* = \frac{1}{3} \\ m_1^* = m_2^* = 49\frac{8}{3} \\ \lambda = \frac{2}{3} \end{cases}$$

Now the efficient allocations is $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, 49\frac{8}{9}, 49\frac{8}{9} \right)$.

When $x_i \geq 1$, $\phi_i(x_i) = 1$, we have

$$\begin{cases} \lambda \leqslant 2q_j \\ x_1^* + x_2^* = q_1^* + q_2^* \\ m_1^* + m_2^* = 100 - q_1^{*2} - q_2^{*2} \end{cases} \Rightarrow \begin{cases} x_1^* = x_2^* = q_1^* = q_2^* = 1 \\ m_1^* = m_2^* = 49 \\ \lambda = 2 \end{cases}$$

Now the efficient allocations is $(1, 1, 1, 1, 49, 49)$.

Problem 4.

(a) Suppose that the preference relation $\succeq$ on the set of lotteries $\mathcal{L}$ is complete. Show that $\succeq$ satisfies the independence axiom if and only if the following property holds:

   For all $L_1, L_2, L_3 \in \mathcal{L}$ *and* $\alpha \in (0,1)$, $L_1 \succ L_2$ *if and only if* $\alpha L_1 + (1 - \alpha)L_3 \succ \alpha L_2 + (1 - \alpha)L_3$.

(b) Show that if $\succeq$ on $\mathcal{L}$ satisfies the independence axiom, then the following property holds:

   For all $L_1, L_2, L_3 \in \mathcal{L}$ *and* $\alpha \in (0,1)$, $L_1 \sim L_2$ *if and only if* $\alpha L_1 + (1 - \alpha)L_3 \sim \alpha L_2 + (1 - \alpha)L_3$.

Answer to Problem 4.

(a) If part: Assume $L_1 \succ L_2$. By the independence axiom, for any lottery L_3 and any $\alpha \in (0,1)$, it follows that $\alpha L_1 + (1 - \alpha)L_3 \succ \alpha L_2 + (1 - \alpha)L_3$. This is because the independence axiom states that if a lottery L_1 is preferred over L_2, then a mixed lottery that involves L_1 with probability α and any other lottery L_3 with probability $1 - \alpha$ is also preferred over a mixed lottery that involves L_2 and L_3 in the same proportions.

Only if part: Assume $\alpha L_1 + (1-\alpha)L_3 \succ \alpha L_2 + (1-\alpha)L_3$. By the given property, it must be that $L_1 \succ L_2$. This is because the property states that the preference between L_1 and L_2 is consistent with the preference between the mixed lotteries $\alpha L_1 + (1 - \alpha)L_3$ and $\alpha L_2 + (1 - \alpha)L_3$. Therefore, the preference relation satisfies the independence axiom.

(b) If part: Assume that $\alpha L_1 + (1 - \alpha)L_3 \sim \alpha L_2 + (1 - \alpha)L_3$ implies $L_1 \succ L_2$ instead of $L_1 \sim L_2$. Thus we have $\alpha L_1 + (1-\alpha)L_3 \succ \alpha L_2 + (1-\alpha)L_3$ according to (a), contradiction.

Only if part: Assume that $L_1 \sim L_2$ implies $\alpha L_1 + (1-\alpha)L_3 \succ \alpha L_2 + (1-\alpha)L_3$ instead of $\alpha L_1 + (1-\alpha)L_3 \sim \alpha L_2 + (1-\alpha)L_3$, thus we have $L_1 \succ L_2$ according to (a), contradiction.