Advanced Microeconomics I
Note 7: Partial equilibrium analysis

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Introduction

- The fundamental issue of Economics: **resource allocation**.
  - This issue should be addressed from two perspectives: one *normative* and the other *positive*.
- Normative: what is a good allocation?
  - Efficiency: an allocation (or, generally, an outcome) is efficient if it is not possible to make someone better-off without hurting anyone.
    - It serves as a minimal test for the desirability of an allocation.
  - Fairness: a more controversial issue.
- Positive: investigate the *mechanisms* that allocate resources.
  - Market
    - Private ownership
    - Price plays an important role.
    - Competitive and non-competitive markets
  - Mechanisms can also be specifically designed: pricing strategies, auctions, contracts...
    - Incentives
  - There are also allocation problems without monetary transfers
    - Example: school choice, organ transplant (kidney exchange), hospital-intern matching, refugee resettlement...and more recently, COVID-19 vaccines and convalescent plasma allocation
    - Matching theory and market design

This note looks at a general allocation problem: the organization of production and the allocation of the resulting commodities among consumers.

And we focus on the (mechanism of) competitive market.

The resource allocation problem

- Consider the following **economy**.

- N consumers: $i = 1, ..., N$.
- J firms: $j = 1, ..., J$.
- L goods: $l = 1, ..., L$.

- $\omega_l \geq 0$: the initial endowment of good l in the economy.

- Each consumer i's preferences over bundles $x_i = (x_{1i}, x_{2i}, ..., x_{Li})$ are represented by u_i.
- Each firm j's production technology is summarized by its production set Y_j. $y_j = (y_{1j}, ..., y_{Lj}) \in Y_j$ represents the net output of each good from y_j.

- An **allocation** $(x_1, ..., x_N, y_1, ..., y_J)$ consists of a consumption vector x_i for each consumer i and a production vector $y_j \in Y_j$ for each firm j, such that

$$\sum_{i=1}^{N} x_{li} \leq \omega_l + \sum_{j=1}^{J} y_{lj} \;\; \forall \; l = 1, ..., L$$

Efficiency

An allocation $(x_1', ..., x_N', y_1', ..., y_J')$ **Pareto dominates** another allocation $(x_1, ..., x_N, y_1, ..., y_J)$ if

$$u_i(x_i') \geq u_i(x_i) \text{ for all } i = 1, ..., N$$

and

$$u_i(x_i') > u_i(x_i) \text{ for some } i$$

An allocation is **Pareto efficient**, or simply **efficient**, if it cannot be Pareto dominated by any other allocation.

Maximization of total utility implies efficiency, but the converse is not necessarily true.

Competitive markets

- *Private ownership*: each consumer i owns ω_{li} of good l and $\omega_l = \sum_{i=1}^{N} \omega_{li}$; consumer i owns a share $\theta_{ij} \in [0,1]$ of firm j and $\sum_{i=1}^{N} \theta_{ij} = 1$ for each j.
  - Different ownership structures correspond to different market mechanisms.
- *Complete markets*: a market exists for each of the L goods.
  - There is a price p_l for each good l.
- *Competitive*: all consumers and firms act as price takers.
- The resulting allocation of resources is given in the *competitive equilibrium*.

The allocation $(x_1^*, ..., x_N^*, y_1^*, ..., y_J^*)$ and price vector $p^* = (p_1^*, ..., p_L^*)$ constitute a **competitive equilibrium** if the following conditions are satisfied:

(i) *Profit maximization*: for each firm j, y_j^* solves

$$\text{Max } \ p^* \cdot y_j$$

$$s.t. \ \ y_j \in Y_j$$

(ii) *Utility maximization*: for each consumer i, x_i^* solves

$$\text{Max } \ u_i(x_i)$$

$$s.t. \ \ p^* \cdot x_i \leq p^* \cdot \omega_i + \sum_{j=1}^{J} \theta_{ij}(p^* \cdot y_j^*)$$

(iii) *Market clearing*: for each good $l = 1, ..., L$,

$$\sum_{i=1}^{N} x_{li}^* = \omega_l + \sum_{j=1}^{J} y_{lj}^*$$

Lemma. *If* $(x_1^*, ..., x_N^*, y_1^*, ..., y_J^*)$ *and* $p^* = (p_1^*, ..., p_L^*)$ *constitute a competitive equilibrium, then for any* $\alpha > 0$, $(x_1^*, ..., x_N^*, y_1^*, ..., y_J^*)$ *and* $\alpha p^* = (\alpha p_1^*, ..., \alpha p_L^*)$ *also constitute a competitive equilibrium*.

Lemma. *Suppose that every consumer's budget constraint is satisfied with equality. Given an allocation* $(x_1, ..., x_N, y_1, ..., y_J)$ *and a price vector* $p = (p_1, ..., p_L) \gg 0$, *if the market clearing condition is satisfied for* $L - 1$ *goods, then it is satisfied for all goods.*

Does a competitive equilibrium always exist?

If it exists, is the allocation in the equilibrium efficient?

Partial equilibrium analysis: two-good quasilinear model

- We now focus on the market for one particular good: good l.
- Two-good quasilinear model: good l and the *numeraire*
  - The numeraire represents the composite of all other goods, or the expenditure on all other goods.
- Each consumer i has a quasilinear utility function:

$$u_i(m_i, x_i) = m_i + \phi_i(x_i)$$

where m_i is the consumption of the numeraire.

- Assume that each consumer's consumption set is $\mathbb{R} \times \mathbb{R}_+$, i.e., the consumption of the numeraire is allowed to be negative.
- Assume that $\phi_i'(x_i) > 0, \phi_i''(x_i) < 0$ for all $x_i \geq 0$, $\lim_{x_i \to +\infty} \phi_i'(x_i) = 0$, and $\phi_i(0) = 0$.

- Each firm j produces good l from the numeraire, with the cost function $c_j(q_j)$: producing q_j units of good l requires $c_j(q_j)$ units of the numeraire.
- Assume that $c_j'(q_j) > 0$, $c_j''(q_j) > 0$ for all $q_j \geq 0$, $\lim_{q_j \to +\infty} c_j'(q_j) = +\infty$, and $c_j(0) = 0$.
- Assume $\omega_l = 0$, $\omega_m > 0$

- For simplicity, in this specialized two-good quasilinear model, an allocation is represented by the vector $(x_1, ..., x_N, q_1, ..., q_J, m_1, ..., m_N)$ with

$$\sum_{i=1}^{N} x_i = \sum_{j=1}^{J} q_j$$

$$\sum_{i=1}^{N} m_i + \sum_{j=1}^{J} c_j(q_j) = \omega_m$$

  - Each firm j's demand for the numeraire is given by $c_j(q_j)$.
  - We only focus on *non-wasteful* allocations.

Efficient allocations in the two-good quasilinear model

Given an allocation $(x_1, ..., x_N, q_1, ..., q_J, m_1, ..., m_N)$, the total utility is

$$\sum_{i=1}^{N} u_i(m_i, x_i) = \sum_{i=1}^{N} \phi_i(x_i) + \omega_m - \sum_{j=1}^{J} c_j(q_j)$$

Consider the following problem

$$\underset{x_i, q_j}{\text{Max}} \ \sum_{i=1}^{N} \phi_i(x_i) + \omega_m - \sum_{j=1}^{J} c_j(q_j)$$

$$s.t. \ \ x_i \geq 0, \ \ \forall i$$

$$q_j \geq 0, \ \ \forall j$$

$$\sum_{i=1}^{N} x_i = \sum_{j=1}^{J} q_j$$

This problem has a unique solution $(x_1^*, ..., x_N^*, q_1^*, ..., q_J^*)$ that satisfies the following conditions:

There exists $\lambda \in \mathbb{R}$ such that

$$\text{For each } j, \quad \lambda \leq c_j'(q_j^*), \;\text{ with equality if }\; q_j^* > 0$$

$$\text{For each } i, \quad \phi_i'(x_i^*) \leq \lambda, \;\text{ with equality if }\; x_i^* > 0$$

$$\sum_{i=1}^{N} x_i^* = \sum_{j=1}^{J} q_j^*$$

Hence, all the allocations $(x_1^*, ..., x_N^*, q_1^*, ..., q_J^*, m_1, ..., m_N)$ are efficient.

Because utilities are perfectly transferable in this two-good quasilinear model, these are also the only efficient allocations.

In sum, all the efficient allocations involve $(x_1^*, ..., x_N^*, q_1^*, ..., q_J^*)$ and they only differ in the distribution of the numeraire among consumers. They all maximize

$$\sum_{i=1}^{N} \phi_i(x_i) + \omega_m - \sum_{j=1}^{J} c_j(q_j)$$

or

$$\sum_{i=1}^{N} \phi_i(x_i) - \sum_{j=1}^{J} c_j(q_j)$$

which is the **Marshallian aggregate surplus**. It can be considered as the total utility generated from production and consumption.

Competitive equilibrium in the two-good quasilinear model

- Private ownership: ω_{mi}, θ_{ij}
- Normalize the price of the numeraire to 1.
- Suppose that the allocation $(x_1^*, ..., x_N^*, q_1^*, ..., q_J^*, m_1^*, ..., m_N^*)$ and the price of good l, p^*, constitute a competitive equilibrium.
- For each firm j, given p^*, q_j^* solves

$$\operatorname*{Max}_{q_j \geq 0} \; p^* q_j - c_j(q_j)$$

which has the necessary and sufficient first-order condition:

$$p^* \leq c_j'(q_j^*), \;\; \text{with equality if} \;\; q_j^* > 0$$

- For each consumer i, m_i^* and x_i^* must solve

$$\underset{m_i, x_i}{\text{Max}} \;\; m_i + \phi_i(x_i)$$

$$s.t. \;\; m_i + p^* x_i \leq \omega_{mi} + \sum_{j=1}^{J} \theta_{ij}(p^* q_j^* - c_j(q_j^*)) \;\text{ and }\; x_i \geq 0$$

which is equivalent to

$$\underset{x_i \geq 0}{\text{Max}} \;\; \phi_i(x_i) - p^* x_i + \left\{ \omega_{mi} + \sum_{j=1}^{J} \theta_{ij}(p^* q_j^* - c_j(q_j^*)) \right\}$$

Necessary and sufficient first-order condition:

$$\phi_i'(x_i^*) \leq p^* \;\text{ with equality if }\; x_i^* > 0$$

- Moreover,

$$m_i^* = \left\{ \omega_{mi} + \sum_{j=1}^{J} \theta_{ij}(p^* q_j^* - c_j(q_j^*)) \right\} - p^* x_i^*$$

Therefore, the allocation $(x_1^*, ..., x_N^*, q_1^*, ..., q_J^*, m_1^*, ..., m_N^*)$ and the price of good l, p^*, constitute a competitive equilibrium if and only if the following conditions are satisfied

$$\text{For each } j, \quad p^* \leq c_j'(q_j^*), \;\; \text{with equality if } \; q_j^* > 0$$

$$\text{For each } i, \quad \phi_i'(x_i^*) \leq p^*, \;\; \text{with equality if } \; x_i^* > 0$$

$$\sum_{i=1}^{N} x_i^* = \sum_{j=1}^{J} q_j^*$$

Notice that we only need to make sure that the market for good l clears.

The existence of the competitive equilibrium can be studied using the traditional Marshallian graphical analysis.

- Let $x_i(p)$ be consumer i's demand function for good l. Then $x_i(p) = [\phi_i']^{-1}(p)$ if $p \leq \phi_i'(0)$; $x_i(p) = 0$ if $p > \phi_i'(0)$.
- The aggregate demand function for good l is then $x(p) = \sum_{i=1}^{N} x_i(p)$.
- $x(p) = 0$ if $p > \text{Max}_i\ \phi_i'(0)$, and $x(p)$ is strictly decreasing if $p \leq \text{Max}_i\ \phi_i'(0)$. Moreover, it is continuous, and $\lim_{p \to 0} x(p) = +\infty$.

- Let $q_j(p)$ be firm j's supply function for good l. Then $q_j(p) = [c_j']^{-1}(p)$ if $p \geq c_j'(0)$; $q_j(p) = 0$ if $p < c_j'(0)$;
- The aggregate supply function for good l is then $q(p) = \sum_{j=1}^{J} q_j(p)$.
- $q(p) = 0$ if $p < \text{Min}_j\ c_j'(0)$, and $q(p)$ is strictly increasing if $p \geq \text{Min}_j\ c_j'(0)$. Moreover, it is continuous, and $\lim_{p \to +\infty} q(p) = +\infty$.

- Finally, assume that $\text{Max}_i\ \phi_i'(0) > \text{Min}_j\ c_j'(0)$, then clearly there exists a unique $p^* > 0$ such that $x(p^*) = q(p^*) > 0$. Therefore a competitive equilibrium exists, and it is unique.

Fundamental welfare theorems

By comparing the conditions on p.13 and p.17, it follows that the allocation in competitive equilibrium is efficient.

Moreover, any efficient allocation can be achieved in the competitive equilibrium by appropriately transferring the initial endowments among consumers (see the last equation on p.16).

We have established the fundamental welfare theorems in a partial equilibrium context.

The First Fundamental Theorem of Welfare Economics: *If the allocation $(x_1^*, ..., x_N^*, q_1^*, ..., q_J^*, m_1^*, ..., m_N^*)$ and the price of good l, p^*, constitute a competitive equilibrium, then the allocation is efficient.*

The Second Fundamental Theorem of Welfare Economics: *Given an efficient allocation $(x_1^*, ..., x_N^*, q_1^*, ..., q_J^*, m_1^*, ..., m_N^*)$, there exist transfers of the numeraire $(T_1, ..., T_N)$ with $\sum_{i=1}^N T_i = 0$, and a price of good l, $p^* > 0$, such that $(x_1^*, ..., x_N^*, q_1^*, ..., q_J^*, m_1^*, ..., m_N^*)$ and p^* constitute a competitive equilibrium reached from initial endowments $(\omega_{m1} + T_1, ..., \omega_{mN} + T_N)$.*

The key implication of the welfare theorems is that, under some mild assumptions, an allocation is efficient if and only if it can be achieved by the competitive markets.

We now proceed with some further analysis that help better understand the welfare theorems.

Let $D(\cdot) = x^{-1}(\cdot)$ be the *inverse demand function*. Given a quantity x, how to interpret $D(x)$?

If the total consumption of good l is x, then the optimal distribution of x to the consumers should (try to) equate everyone's marginal utility:

$$\text{For each consumer } i, \quad \phi_i'(x_i) = D(x) \quad \text{if} \quad \phi_i'(0) \geq D(x)$$

Hence the inverse demand function $D(\cdot)$ represents the *marginal social utility* of good l.

Similarly, let $S(\cdot) = q^{-1}(\cdot)$ be the *inverse supply function*, and it can be interpreted as the *marginal social cost* of good l. Why? If the total production is q, then the optimal distribution of q among firms should (try to) equate every firm's marginal cost:

$$\text{For each firm } j, \quad c_j'(q_j) = S(q) \quad \text{if} \quad c_j'(0) \leq S(q)$$

Therefore, the total consumption (or production) of good l in the competitive equilibrium equates the marginal social utility and marginal social cost. The distribution of consumption and production is also optimal in the competitive equilibrium.

Finally, we consider the traditional graphical analysis.

For each consumer i, given a consumption x_i of good l

$$\int_0^{x_i} \phi_i'(t)dt = \phi_i(x_i) - \phi_i(0) = \phi_i(x_i)$$

Since the inverse demand function $D(\cdot)$ is the horizontal sum of all consumers' marginal utility functions, we have

$$\int_0^{x} D(t)dt = \sum_{i=1}^{N} \phi_i(x_i)$$

when a total consumption level x is optimally distributed. Similarly, when a total production level q is optimal distributed,

$$\int_0^{q} S(t)dt = \sum_{j=1}^{J} c_j(q_j)$$

Therefore, given any allocation $(x_1, ..., x_N, q_1, ..., q_J, m_1, ..., m_N)$, if the consumption and production of good l have been optimally distributed, then the aggregate Marshallian surplus can be identified using the inverse demand and inverse supply:

$$\sum_{i=1}^{N} \phi_i(x_i) - \sum_{j=1}^{J} c_j(q_j) = \int_0^x [D(t) - S(t)]dt$$

where $x = \sum_{i=1}^{N} x_i = \sum_{j=1}^{J} q_j$.

From the graph, clearly it is maximized at the intersection of the inverse demand and inverse supply.

Conclusion

We have established the welfare theorems in the special partial equilibrium context. More generally, these theorems can be stated as follows.

The First Fundamental Theorem of Welfare Economics: If every relevant good is traded in a market at publicly known prices (there is a complete set of markets), and if consumers and firms act as price takers, then the market outcome is efficient.

The Second Fundamental Theorem of Welfare Economics: If consumers' preferences and firms' production sets are convex, every relevant good is traded in a market at publicly known prices, and if consumers and firms act as price takers, then every efficient outcome can be achieved in a competitive equilibrium with appropriate transfers of income.

- The first welfare theorem is a formal expression of Adam Smith's claim about *invisible hand*.
    - Invisible hand was introduced by Adam Smith in his book 'The Wealth of Nations': an economy can work well in a free market scenario where everyone will work for his/her own interest.
- Any inefficiencies that arise in a market must be traceable to a violation of some assumption of the first welfare theorem.
- Market equilibrium fails to be efficient - **Market failure**
- Some common sources of marker failure:
    - Market power: monopoly, Cournot
    - Externality: non-marketed "goods" or "bads"
    - Asymmetric information
        - postcontractual asymmetric information: moral hazard
        - asymmetric information at the time of contracting: adverse selection

Advanced Microeconomics I
Note 8: Choice under uncertainty

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Lotteries

- Individual decision-making under uncertainty
- Let C denote the set of all possible outcomes (alternatives, or, *consequences*).
- An economic agent's problem is not to choose an alternative, but a *risky alternative*.
- A risky alternative is represented by a **lottery**.
- We first assume that C is finite and $|C| = N$.
- A **simple lottery** is a list $L = (p_1, ..., p_N)$ with $p_n \geq 0$ for all $n = 1, ..., N$ and $\sum_{n=1}^{N} p_n = 1$.
- $\mathcal{L} = \left\{ (p_1, ..., p_N) \in \mathbb{R}_+^N : \sum_{n=1}^{N} p_n = 1 \right\}$ is the set of all the possible simple lotteries.

- A **compound lottery** is a list $(L_1, ..., L_K; \alpha_1, ..., \alpha_K)$ where each $L_k = (p_1^k, ..., p_N^k)$ is a simple lottery, $\alpha_k \geq 0$ for all k, and $\sum_{k=1}^K \alpha_k = 1$.
  - That is, a compound lottery is a probability distribution over some simple lotteries.
- Given a compound lottery $(L_1, ..., L_K; \alpha_1, ..., \alpha_K)$, the ultimate probability of outcome n is
$$\alpha_1 p_n^1 + \alpha_2 p_n^2 + ... + \alpha_K p_n^K$$
- So, in terms of the final distribution over outcomes, the compound lottery can be reduced to the simple lottery $\sum_{k=1}^K \alpha_k L_k \in \mathcal{L}$.
- Different compound lotteries may be reduced to the same simple lottery.

- Example: $N = 3$. Consider the following two compound lotteries

$$\left\{(1,0,0),(\frac{1}{4},\frac{3}{8},\frac{3}{8}),(\frac{1}{4},\frac{3}{8},\frac{3}{8});\ \ (\frac{1}{3},\frac{1}{3},\frac{1}{3})\right\}$$

$$\left\{(1,0,0),(0,\frac{1}{2},\frac{1}{2});\ \ (\frac{1}{2},\frac{1}{2})\right\}$$

They are both reduced to the simple lottery

$$(\frac{1}{2},\frac{1}{4},\frac{1}{4})$$

- We make a consequentialist assumption: the agent only cares about the final distribution over the sure outcomes C.

- We will adopt the preference-based approach and assume that the agent has some preference relation $\succeq$ defined on $\mathcal{L}$.
- In the rest of the note, we focus on two topics
  - Expected utility theory
  - Risk preferences

Expected utility theory

- Assume that $\succeq$ on $\mathcal{L}$ is *rational*.
- $\succeq$ on $\mathcal{L}$ is *continuous* if for any sequence $\{L^k\} \subseteq \mathcal{L}$ with $L^k \to L$, and $L' \in \mathcal{L}$ we have (1) $L^k \succeq L'$ for all k implies $L \succeq L'$, and (2) $L' \succeq L^k$ for all k implies $L' \succeq L$.
- If $\succeq$ is rational and continuous, then there exists a utility function $U : \mathcal{L} \to \mathbb{R}$ that represents $\succeq$: for any L and L', $L \succeq L'$ if and only if $U(L) \geq U(L')$.
- However, we want to impose more structure on U.
- We call a utility function $u : C \to \mathbb{R}$ a **Bernoulli utility function**.
    - Since we have assumed C is finite, let $u_1, ..., u_N$ denote the utilities of the sure outcomes.

The utility function $U : \mathcal{L} \to \mathbb{R}$ has an **expected utility form** if there exists a Bernoulli utility function $u : C \to \mathbb{R}$ such that for every $L = (p_1, ..., p_N) \in \mathcal{L}$,

$$U(L) = p_1 u_1 + ... + p_N u_N$$

A utility function U with the expected utility form is called a **Von Neumann-Morgenstern expected utility function**.

Proposition. *A utility function $U : \mathcal{L} \to \mathbb{R}$ has an expected utility form if and only if it satisfies*

$$U(\sum_{k=1}^{K} \alpha_k L_k) = \sum_{k=1}^{K} \alpha_k U(L_k)$$

for any compound lottery $(L_1, ..., L_K; \alpha_1, ..., \alpha_K)$.

There may not exist an expected utility function that represents a rational and continuous $\succeq$ on $\mathcal{L}$.

- $\succeq$ on $\mathcal{L}$ satisfies the **independence axiom** if for all $L_1, L_2, L_3 \in \mathcal{L}$ and $\alpha \in (0,1)$, we have

$$L_1 \succeq L_2 \text{ if and only if } \alpha L_1 + (1-\alpha)L_3 \succeq \alpha L_2 + (1-\alpha)L_3$$

- The independence axiom is at the heart of the theory of choice under uncertainty, but also controversial.
- If a preference relation $\succeq$ on $\mathcal{L}$ can be represented by an expected utility function, then $\succeq$ must satisfy rationality, continuity and the independence axiom.

Expected Utility Theorem. *Suppose that the preference relation* $\succeq$ *on the set of lotteries* $\mathcal{L}$ *satisfies rationality, continuity and the independence axiom, then there exists a Von Neumann-Morgenstern expected utility function that represents* $\succeq$*.*

Money lotteries

- In the rest of this note, we focus on *money lotteries*.
- The set of possible outcomes is $C = \mathbb{R}_+$.
- A lottery is a *cumulative distribution function* (CDF) $F : \mathbb{R}_+ \to [0, 1]$.
    - For any $x \geq 0$, $F(x)$ is the probability that the realized monetary payoff is less than or equal to x.
    - If f is the probability density function (PDF) associated with the lottery F, then $F(x) = \int_{-\infty}^{x} f(t)dt$.
- Now, $\mathcal{L}$ is the set of all the CDFs on $\mathbb{R}_+$.
- The agent has a preference relation $\succeq$ on $\mathcal{L}$.
- By a generalized version of the expected utility theorem, under similar conditions there exists an expected utility function $U : \mathcal{L} \to \mathbb{R}$ that represents $\succeq$. That is, there also exists a Bernoulli utility function $u : \mathbb{R}_+ \to \mathbb{R}$ such that for all $F \in \mathcal{L}$

$$U(F) = \int u(x)dF(x)$$

- Assume that u is continuous and strictly increasing.

Risk preferences

- Given any lottery F, let F^d denote the degenerate lottery that yields $\int x dF(x)$ with certainty.
- We first define risk preferences without utility functions.

- An agent is **risk averse** if for any $F \in \mathcal{L}$, $F^d \succeq F$.
- The agent is **strictly risk averse** if for any $F \in \mathcal{L}$ with $F \neq F^d$, $F^d \succ F$.
- The agent is **risk neutral** if for any $F \in \mathcal{L}$, $F^d \sim F$.

- Sometimes, we also say that the agent is **risk loving** if for any $F \in \mathcal{L}$, $F \succeq F^d$.

- The agent is risk averse if and only if

$$\int u(x)dF(x) \leq u\left(\int xdF(x)\right) \quad \text{for all } \ F \in \mathcal{L}$$

  - The expected utility is less than or equal to the utility of expected value.
- This inequality is actually *Jensen's inequality*, and it is a defining property of concave functions.
- Hence, (strict) risk aversion is equivalent to (strict) concavity of u.
- Being risk loving is equivalent to convexity of u.
- Being risk neutral is equivalent to linearity of u.

- Two useful concepts for analyzing risk preferences: *certainty equivalent* and *probability premium*.

- The **certainty equivalent** of a lottery F, denoted $c(F,u)$, is defined in the following equation

$$u(c(F,u)) = \int u(x)dF(x)$$

- $c(F,u)$ is the amount of money which makes the agent indifferent between the lottery F and the certain amount $c(F,u)$.
- If the agent is risk averse, then $c(F,u) \leq \int xdF(x)$.
  - $\int xdF(x) - c(F,u)$ can be interpreted as the amount of expected return the agent wants to pay to get rid of the risk.
- In fact, risk aversion is equivalent to $c(F,u) \leq \int xdF(x)$ for all $F \in \mathcal{L}$:

$$c(F,u) \leq \int xdF(x)$$

$$\Leftrightarrow u\left(c(F,u)\right) \leq u\left(\int xdF(x)\right)$$

$$\Leftrightarrow \int u(x)dF(x) \leq u\left(\int xdF(x)\right)$$

- Fix an amount of money x and a number $\epsilon \in (0, x)$. The **probability premium**, denoted by $\pi(x, \epsilon, u)$, is defined in the following equation

$$u(x) = \left(\frac{1}{2} + \pi(x, \epsilon, u)\right) u(x + \epsilon) + \left(\frac{1}{2} - \pi(x, \epsilon, u)\right) u(x - \epsilon)$$

- $\pi(x, \epsilon, u)$ is the excess in winning probability over fair odds that makes the agent indifferent between the certain amount x, and a gamble between $x + \epsilon$ and $x - \epsilon$.
- If the agent is risk averse, then $\pi(x, \epsilon, u) \geq 0$:

$$u(x) = \frac{1}{2}u(x + \epsilon) + \frac{1}{2}u(x - \epsilon) + \pi(x, \epsilon, u)\left(u(x + \epsilon) - u(x - \epsilon)\right)$$

$\pi(x, \epsilon, u) \geq 0$, since $u(x) \geq \frac{1}{2}u(x + \epsilon) + \frac{1}{2}u(x - \epsilon)$ and $u(x + \epsilon) > u(x - \epsilon)$.

- It can also be shown that $\pi(x, \epsilon, u) \geq 0$ for all x and ϵ implies risk aversion.

In sum, risk aversion can be characterized in multiple ways:

Proposition. *The following statements are equivalent*:

(i) The agent is risk averse.

(ii) u is concave.

(iii) $c(F, u) \leq \int x dF(x)$ *for all* $F \in \mathcal{L}$.

(iv) $\pi(x, \epsilon, u) \geq 0$ *for all* x *and* ϵ.

Applications

St.Petersburg paradox

- Now, I will flip a fair coin.
- I will keep flipping until the first time it comes up heads.
- If heads on flip 1, I will pay you \$2.
- If heads on flip 2, I will pay you \$4.
- Generally, if heads on flip n, I will pay you 2^n.
- How much would you be willing to pay to play this game?

Expected value of this game:

$$\sum_{n=1}^{+\infty}(0.5)^n 2^n = 1 + 1 + 1 + \ldots = +\infty$$

Will you pay infinite amounts of money to play this game?

Assume risk aversion, with (Bernoulli) utility function $u(x) = \sqrt{x}$.

Expected utility from playing the game:

$$\sum_{n=1}^{+\infty} (0.5)^n \sqrt{2^n}$$

We can find the certainty equivalent of this game:

$$\sqrt{c} = \sum_{n=1}^{+\infty} (0.5)^n \sqrt{2^n}$$

Solving this equation, we get

$$c = 5.828429$$

A simple lesson here is that an agent's decision making under uncertainty depends on his preferences, or expected utilities, not on expected values.

Two-Envelope Paradox

You are given a choice between two envelopes. You are told, reliably, that each envelope has some money in it and that one envelope contains twice as much money as the other. You don't know which has the higher amount and which has the lower. You choose one, but are given the opportunity to switch to the other. Would you switch?

Insurance I - a problem of an insurance company

- A consumer has wealth w. With probability $p \in (0,1)$ the consumer will loose L.
- Suppose the cost of full insurance is R and the consumer cannot purchase partial insurance.
- Do not buy the insurance: $(1-p)u(w) + pu(w-L)$
- Buy the insurance: $(1-p)u(w-R) + pu(w-L-R+L) = u(w-R)$
- Optimal premium R^* for the insurance company:

$$(1-p)u(w) + pu(w-L) = u(w-R^*)$$

- If the consumer is strictly risk averse

$$(1-p)w + p(w-L) > w - R^* \Rightarrow R^* > pL$$

  So the consumer is willing to pay more than the expected loss.
- The consumer sells his lottery (or risk) to the insurance company at a negative price.
- Why would the insurance company want to buy the risk from the consumer?
  - difference in risk preferences: the insurance company is usually risk-neutral or much less risk averse than each individual consumer
  - for instance, when the insurance company is risk neutral, its expected profit (or utility) $= (1-p)R^* + p(R^*-L) = R^* - pL > 0$

Insurance II - a consumer's problem

- A strictly risk averse consumer has wealth w. With probability $p \in (0,1)$ the consumer will loose L.
- The consumer can choose to buy coverage x, with a cost of px.
  - *Actuarially fair* insurance: price per dollar of coverage is equal to the probability of loss
- How much insurance coverage does the consumer want to buy?

$$\max_{x \geq 0} \ (1-p)u(w-px) + pu(w-px-L+x)$$

$$\text{FOC:} \quad -p(1-p)u'(w-px^*) + p(1-p)u'(w-px^*-L+x^*) \leq 0$$
$$\text{with equality if} \quad x^* > 0$$

If $x^* = 0$, then FOC implies

$$p(1-p)\left[u'(w-L) - u'(w)\right] \leq 0$$

which is not possible since u is strictly concave and u' is strictly decreasing. Hence

$$\begin{aligned} p(1-p)u'(w-px^*-L+x^*) &= p(1-p)u'(w-px^*) \\ u'(w-px^*-L+x^*) &= u'(w-px^*) \\ w-px^*-L+x^* &= w-px^* \\ x^* &= L \end{aligned}$$

- So a strictly risk averse consumer will always purchase full insurance if the price is actuarially fair.
- In fact, the first-order method is redundant.
- Alternatively, think about the consumer's decision in the following way.
- If the consumer chooses x, he faces the following lottery, which depends on x

$$\text{with probability } 1-p\text{, he receives } w-px$$
$$\text{with probability } p\text{, he receives } w-px-L+x$$

- The expected value of the consumer's lottery is $w-pL$, which is independent of x.
- By choosing $x=L$, the consumer receives $w-pL$ for sure.

Investment

- A risk averse agent has wealth w and wants to invest his wealth into two assets.
- For each dollar invested in the safe asset, the return is 1 dollar.
- For each dollar invested in the risky asset, the return is a random variable z with the distribution $F(z)$ and $\int z dF(z) > 1$.
- The agent's problem is to choose the optimal amount of wealth, α, to invest in the risky asset:

  $$\underset{\alpha \in [0,w]}{\text{Max}} \quad \int u(\alpha z + w - \alpha) dF(z)$$

  If at the optimal, $\alpha^* = 0$, then

  $$\int u'(\alpha^* z + w - \alpha^*)(z - 1) dF(z) > 0$$

  So we must have $\alpha^* > 0$.
- If the risk is *actuarially favorable*, then a risk averse agent will accept at least some of it.

Advanced Microeconomics I
Note 9: Normal form games

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Introduction

What is game theory?

- **Robert J. Aumann, 1985:** Briefly put, game and economic theory are concerned with the interactive behavior of rational man. . . [An] important function of game theory is the classification of interactive decision situations.
- **Roger B. Myerson, 1991:** Game theory can be defined as the study of mathematical models of conflict and cooperation between intelligent rational decision-makers. Game theory provides general mathematical techniques for analyzing situations in which two or more individuals make decisions that will influence one another's welfare.
- **Nobel Prize Citation, 1994:** Game theory is a mathematical method for analyzing strategic interaction.

- Economic applications of game theory
    - Trading processes (auctions, bargaining)
    - Labor and financial markets
    - Decision problems in organizations: performance wages, competing for promotions, competing for resources
    - International economics: countries choose tariffs and trade policies
    - Macroeconomics: effects of various government policies
- Game theory is also used in political science, psychology, philosophy, logic, computer science, biology, etc.

- **Noncooperative games**: *self-interested* and *rational* decision-makers act independently.
  - The main focus in our course.
- **Cooperative games**: situations where groups of agents can make binding agreements.
  - The primitives are groups or subgroups of agents.

- A simple classification of noncooperative games:
  - Simultaneous move games v.s. dynamic games
  - Games with complete (perfect) information v.s. games with incomplete (imperfect) information

- To describe a game, we need the following four elements:
  - Players: agents who are involved in the strategic situation
  - Rules of the game: timeline, information, possible actions
  - Outcomes: what will happen as a result of each possible combination of actions
  - Payoffs: preferences over the possible outcomes (utilities, profits, etc.)

- Example: *Matching Pennies* game.
  - Players: player 1 and player 2
  - Rules: each player simultaneously puts down a penny.
  - Outcomes: If the pennies match (both heads up or both tails up), player 1 pays \$1 to player 2; otherwise player 2 pays \$1 to player 1.
  - Payoffs
- Example: *Meeting in New York* game.
  - Players: player 1 and player 2
  - Rules: they are separated and cannot communicate. They are supposed to meet in NYC for lunch but have forgotten to specify where. They can either go to Empire State Building or Grand Central Station.
  - Outcomes: If they meet each other, they get to enjoy each other's company at lunch; otherwise they have to eat alone.
  - Payoffs: they each attach a value of \$100 to the other's company.

- The matching pennies game is a *zero-sum game* with pure conflicts of interests, while the meeting in New York game is a coordination game.
  - Noncooperative games are not limited to pure or even partial conflicts. They can also be used to study cooperation.

Norm form representation

- Two ways to model a game formally and mathematically: **Normal Form** and **Extensive Form**.
- In this note, we focus on **simultaneous move games**, and normal form representation is sufficient.

- A normal form game: $\Gamma = [N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N}]$

- $N = \{1, 2, ..., n\}$: the set of players
- S_i: the set of all possible **strategies** of player $i \in N$
- A **strategy profile**: $s = (s_1, s_2, ..., s_n)$, where $s_i \in S_i$ for each i
- $S = \prod_{i \in N} S_i$ (Cartesian product) is the set of all strategy profiles.
    - $s_{-i} = (s_1, ..., s_{i-1}, s_{i+1}, ..., s_n) \in S_{-i}$ denotes a combination of strategies of all players except i.
- Then $u_i : S \to \mathbb{R}$ is player i's utility/payoff function.
    - Player i's utility depends not only on s_i but also on s_{-i}: strategic interaction.

The Matching Pennies game can be represented as $\Gamma = [N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N}]$, where

- $N = \{\text{player 1, player 2}\}$
- $S_1 = \{H, T\}$, $S_2 = \{H, T\}$
  - four possible strategy profiles: $S_1 \times S_2 = \{(H, H), (H, T), (T, H), (T, T)\}$
- $u_1(H, H) = -1, u_1(H, T) = 1, u_1(T, H) = 1, u_1(T, T) = -1$
  $u_2(H, H) = 1, u_2(H, T) = -1, u_2(T, H) = -1, u_2(T, T) = 1$

Simple normal form games can be represented using matrices (game tables):

	H	T
H	-1,1	1, -1
T	1,-1	-1, 1

Example: *Prisoner's Dilemma* game.

	S	C
S	-2,-2	-10, -1
C	-1,-10	-5, -5

- $N = \{$row player (prisoner 1), column player (prisoner 2)$\}$
- $S_1 = \{$S (stay silent), C (confess)$\}$, $S_2 = \{$S (stay silent), C (confess)$\}$
  - Four possible strategy profiles: $S_1 \times S_2 = \{(S,S),(S,C),(C,S),(C,C)\}$
- $u_1(S,S) = -2, u_1(S,C) = -10, u_1(C,S) = -1, u_1(C,C) = -5$
  $u_2(S,S) = -2, u_2(S,C) = -1, u_2(C,S) = -10, u_2(C,C) = -5$

- Assume that the game structure $\Gamma = [N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N}]$ is *common knowledge* among all players.
  - Of course, this is much stronger than requiring complete information.
- Usually the term "common knowledge" is used loosely (what exactly common knowledge means is by no means common knowledge).
- An event E is **common knowledge** if (1) everyone knows E, (2) everyone knows that everyone knows E, and so on ad infinitum.
  - Why bother? Game theory typically involves situations where players engage in strategic thinking. It is important and fundamental to account for players' knowledge/beliefs about uncertainty (physical environment as well as opponents' knowledge/beliefs).

- Given the game $\Gamma = [N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N}]$, strictly speaking each $s_i \in S_i$ is a **pure strategy**. A player can also randomize over his pure strategies, giving rise to *mixed strategies*.
- Given player i's (finite) set of pure strategies S_i, a **mixed strategy** for player i, $\sigma_i : S_i \to [0, 1]$, assigns each of his pure strategy $s_i \in S_i$ a probability $\sigma_i(s_i)$, where $\sum_{s_i \in S_i} \sigma_i(s_i) = 1$.
- Let $\Delta(S_i)$ denote player i's set of all possible mixed strategies.
- When each player chooses a mixed strategy, we have a mixed strategy profile $\sigma = (\sigma_1, ..., \sigma_n) \in \prod_{i \in N} \Delta(S_i)$, which induces a probability distribution over pure strategy profiles: for each $s = (s_1, ..., s_n) \in S$, the probability of s is

$$\prod_{i \in N} \sigma_i(s_i) \quad \text{(assume each player randomizes on his own)}$$

- For player i, u_i can be considered as his Bernoulli utility function. Hence, given a mixed strategy profile σ, his Von Neumann-Morgenstern expected utility is

$$u_i(\sigma) = \sum_{s \in S} \left\{ \left[\prod_{j \in N} \sigma_j(s_j) \right] u_i(s) \right\}$$

$$u_i(\sigma) = \sum_{s \in S} \left\{ \left[\prod_{j \in N} \sigma_j(s_j) \right] u_i(s) \right\}$$

A player's (expected utility) from a mixed strategy profile has two other useful representations:

$$u_i(\sigma) = u_i(\sigma_i, \sigma_{-i}) = \sum_{s_i \in S_i} \{\sigma_i(s_i)\ u_i(s_i, \sigma_{-i})\}$$

$$u_i(\sigma) = u_i(\sigma_i, \sigma_{-i}) = \sum_{s_{-i} \in S_{-i}} \left\{ \prod_{j \neq i} \sigma_j(s_j)\ u_i(\sigma_i, s_{-i}) \right\}$$

If mixed strategies are allowed, we use $\Gamma = [N, \{\Delta(S_i)\}_{i \in N}, \{u_i\}_{i \in N}]$ to represent a normal form game.

Dominance

- Given a game, how would the players play the game, or what is the most likely outcome?
- We need **solution concepts**.
- We start from a non-equilibrium solution concept: **dominance**.
- If a rational player has a unique strategy that is obviously the best one, he should play this strategy.

Prisoner's Dilemma game.

	S	C
S	-2,-2	-10, -1
C	-1,-10	-5, -5

For each player, C is his best strategy, regardless what the other player chooses.

We focus on pure strategies first.

In the game $\Gamma = [N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N}]$, a strategy $s_i \in S_i$ is a **strictly dominant strategy** for player i if

$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$$

for all $s_i' \in S_i \setminus \{s_i\}$ and $s_{-i} \in S_{-i}$.

Hence in the Prisoner's Dilemma game, we predict that rational players will play (C, C): self-interested and rational behavior leads to suboptimal or Pareto inefficient outcome.

Strictly dominant strategies often do not exist. But the idea of dominance can be used to eliminate some "bad" strategies.

In the game $\Gamma = [N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N}]$, a strategy $s_i \in S_i$ is **strictly dominated** for player i if there exists another strategy $s_i' \in S_i$ such that for all $s_{-i} \in S_{-i}$

$$u_i(s_i', s_{-i}) > u_i(s_i, s_{-i})$$

In this case, we also say that s_i' **strictly dominates** s_i.

Obviously, a strategy $s_i \in S_i$ is a strictly dominant strategy for player i in the game $\Gamma = [N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N}]$ if and only if s_i strictly dominates every other strategy in S_i.

	L	R
U	1,-1	-1, 1
M	-1,-1	1, -1
D	-2,5	-3, 2

D is strictly dominated for player 1.

	L	R
U	1,-1	-1, 1
M	-1,-1	1, -1
D	0,5	0, 2

Is D strictly dominated for player 1?

There is also a weaker notion of dominance.

In the game $\Gamma = [N, \{S_i\}_{i\in N}, \{u_i\}_{i\in N}]$, a strategy $s_i' \in S_i$ **weakly dominates** $s_i \in S_i$ for player i if

$$u_i(s_i', s_{-i}) \geq u_i(s_i, s_{-i})$$

for all $s_{-i} \in S_{-i}$, and for some $s_{-i} \in S_{-i}$

$$u_i(s_i', s_{-i}) > u_i(s_i, s_{-i})$$

Then in the game $\Gamma = [N, \{S_i\}_{i\in N}, \{u_i\}_{i\in N}]$, a strategy $s_i \in S_i$ is **weakly dominated** if there exists $s_i' \in S_i$ that weakly dominates s_i. A strategy is **weakly dominant** for player i if it weakly dominates every other strategy of player i.

	L	R
U	5,1	4, 0
M	6,0	3, 1
D	6,4	4, 4

Both U and M are weakly dominated for player 1.

However, we cannot justify eliminating weakly dominated strategies.

Moreover, elimination of weakly dominated strategies might be quite problematic, which will be discussed in more details later.

Now, consider the following example.

	L	R
U	10,0	0,0
M	0,0	0,0
D	0,0	10,0

Although M is not strictly dominated by any pure strategy, it is "strictly dominated" by the mixed strategy of $\frac{1}{2}U + \frac{1}{2}D$.

From now on, we allow mixed strategies.

In the game $\Gamma = [N, \{\Delta(S_i)\}_{i \in N}, \{u_i\}_{i \in N}]$, a strategy $\sigma_i' \in \Delta(S_i)$ **strictly dominates** $\sigma_i \in \Delta(S_i)$ for player i if for all $\sigma_{-i} \in \prod_{j \neq i} \Delta(S_j)$

$$u_i(\sigma_i', \sigma_{-i}) > u_i(\sigma_i, \sigma_{-i})$$

In the game $\Gamma = [N, \{\Delta(S_i)\}_{i \in N}, \{u_i\}_{i \in N}]$, a strategy $\sigma_i \in \Delta(S_i)$ is **strictly dominated** if there exists $\sigma_i' \in \Delta(S_i)$ that strictly dominates σ_i. A strategy is **strictly dominant** for player i if it strictly dominates every other strategy of player i.

A strictly dominant strategy, if it exists, must be a pure strategy.

Allowing mixed strategies helps us eliminate more (pure) strategies. When we want to check whether a strategy is strictly dominated for player i, it is sufficient to check against pure strategies of i's opponents:

$$u_i(\sigma_i', \sigma_{-i}) > u_i(\sigma_i, \sigma_{-i}) \text{ for all } \sigma_{-i} \in \prod_{j \neq i} \Delta(S_j)$$

if and only if

$$u_i(\sigma_i', s_{-i}) > u_i(\sigma_i, s_{-i}) \text{ for all } s_{-i} \in S_{-i}$$

When a pure strategy s_i is strictly dominated, so is any mixed strategy that assigns s_i a positive probability. But a mixed strategy can also be strictly dominated even if it does not assign any positive probability to any strictly dominated pure strategy (homework question).

- A rational player will never use a strictly dominated pure strategy. So we can eliminate this strategy from the original game.
- After eliminating all the strictly dominated pure strategies for the players, some pure strategies, which are not strictly dominated before, may become strictly dominated in the reduced game. We can eliminate these strategies and continue this process until no more strategies can be eliminated: **iterative elimination of strictly dominated strategies** (ISD).
- Example:

	L	R
T	3,1	2,2
M	0,0	3,1
B	1,1	1,0

Eliminate $B \Rightarrow$ eliminate $L \Rightarrow$ eliminate $T \Rightarrow$ unique outcome (M, R)

- If ISD leads to a unique outcome, then this game is **dominance solvable** (Moulin, 1979).
- Notice that, ISD requires more than just rationality, it requires **common knowledge of rationality**.

- A very nice feature of ISD is that this procedure is *order-independent* for finite games.
  - Given a normal form game $\Gamma = [N, \{\Delta(S_i)\}_{i \in N}, \{u_i\}_{i \in N}]$, where each S_i is finite, the set of pure strategies that remains after iteratively removing strictly dominated pure strategies does not depend on the order in which the strictly dominated pure strategies are removed.
- In contrast, iterative elimination of weakly dominated strategies is not order independent.
- Example:

	L	R
U	5,1	4, 0
M	6,0	3, 1
D	6,4	4, 4

$$U \Rightarrow L \Rightarrow M \Rightarrow (D, R)$$
$$M \Rightarrow R \Rightarrow U \Rightarrow (D, L)$$

Beauty contest (John Maynard Keynes, 1936, Moulin, 1986)

Everyone simultaneously picks a number between 0 and 100. The winner of the contest is the person(s) whose number is closest to $\frac{2}{3}$ times the average of all numbers submitted.

Beauty Contest is often presented as one (simple) model of stock markets. When making an investment, some investors are not necessarily trying to identify desirable companies, but rather are attempting to identify those assets that others will consider to be desirable (and purchase these assets before their prices are driven up by the masses).

Nash Equilibrium

- Most games are not dominance solvable.
- We now turn to *Nash Equilibrium* (Nash, 1950), which is the most widely used solution concept in game theory.

  In the game $\Gamma = [N, \{\Delta(S_i)\}_{i\in N}, \{u_i\}_{i\in N}]$, a strategy profile $\sigma = (\sigma_1, ..., \sigma_n)$ is a **Nash Equilibrium** if for every player $i = 1, ..., n$

  $$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i})$$

  for all $\sigma_i' \in \Delta(S_i)$.

- Therefore, a strategy profile is a Nash equilibrium if and only if given others' strategies, no one has an incentive to deviate.

- An alternative way to describe a Nash equilibrium is to use *best responses*.
- For each player i, his **best response correspondence** is $b_i : \prod_{j \neq i} \Delta(S_j) \to \Delta(S_i)$, where for each $\sigma_{-i} \in \prod_{j \neq i} \Delta(S_j)$

$$b_i(\sigma_{-i}) = \{\sigma_i \in \Delta(S_i) : u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i}), \forall \sigma_i' \in \Delta(S_i)\}$$

- Then, a strategy profile σ is a Nash equilibrium if and only if

$$\sigma_i \in b_i(\sigma_{-i}) \text{ for all } i \in N$$

- Hence, in a Nash equilibrium, every player is best-responding to other players.

Example:

	L	C	R
U	10,0	7,9	15,8
D	10,15	5,11	12,12

Pure strategy NE can be found using cell-by-cell inspection: $(U, C), (D, L)$

Matching Pennies game.

	H	T
H	-1,1	1, -1
T	1,-1	-1, 1

There does not exist any pure strategy NE.

But there is a mixed strategy NE: $(\frac{1}{2}H + \frac{1}{2}T, \frac{1}{2}H + \frac{1}{2}T)$.

It is not surprising to see a unique NE in mixed strategies in this game. We have the same conclusion in games like rock–paper–scissors, or penalty-shooting in soccer. Mixed strategies are indeed often observed in these games (one possible exception would be the 2008 UEFA Champions League Final).

Notice that each player is indifferent between choosing H and T given that the other player chooses $\frac{1}{2}H + \frac{1}{2}T$. That is, each player randomizes to make the other player indifferent. This is a general principle behind any mixed strategy NE.

Proposition. *σ is a Nash equilibrium if and only if for any player i*

(i) $u_i(s_i, \sigma_{-i}) = u_i(s_i', \sigma_{-i})$ for all s_i and s_i' with $\sigma_i(s_i) > 0$ and $\sigma_i(s_i') > 0$.

(ii) $u_i(s_i, \sigma_{-i}) \geq u_i(s_i', \sigma_{-i})$ for all s_i and s_i' with $\sigma_i(s_i) > 0$ and $\sigma_i(s_i') = 0$.

While this result can be easily proved, it has two important implications.

First, the "only if" part provides a guideline for finding mixed strategy NE: in a player's NE strategy, he must be indifferent among all the pure strategies that are assigned a positive probability.

Second, the "if" part provides a useful test for NE. In particular, it implies that, to find a pure strategy NE, it is sufficient to consider the case where every player is only allowed to play pure strategies.

Proof. "only if" part. Assume to the contrary, for some player i, either (i) or (ii) is not true. Then there exist some $s_i^1, s_i^2 \in S_i$ such that $\sigma_i(s_i^1) > 0$ and $u_i(s_i^2, \sigma_{-i}) > u_i(s_i^1, \sigma_{-i})$. Then we can construct the following mixed strategy σ_i^* for player i:

$$\sigma_i^*(s_i^2) = \sigma_i(s_i^1) + \sigma_i(s_i^2)$$

$$\sigma_i^*(s_i^1) = 0$$

$$\sigma_i^*(s_i) = \sigma_i(s_i) \text{ if } s_i \in S_i \setminus \left\{s_i^1, s_i^2\right\}$$

Then we have

$$\begin{aligned}
u_i(\sigma_i^*, \sigma_{-i}) &= [\sigma_i(s_i^1) + \sigma_i(s_i^2)]u_i(s_i^2, \sigma_{-i}) + \sum_{s_i \in S_i \setminus \left\{s_i^1, s_i^2\right\}} \sigma_i(s_i)u_i(s_i, \sigma_{-i}) \\
&> \sigma_i(s_i^1)u_i(s_i^1, \sigma_{-i}) + \sigma_i(s_i^2)u_i(s_i^2, \sigma_{-i}) + \sum_{s_i \in S_i \setminus \left\{s_i^1, s_i^2\right\}} \sigma_i(s_i)u_i(s_i, \sigma_{-i}) \\
&= u_i(\sigma_i, \sigma_{-i})
\end{aligned}$$

This contradicts to $\sigma_i \in b_i(\sigma_{-i})$.

"if part". Consider any player i. First, it is obvious that

$$\forall \sigma_i' \in \Delta(S_i), u_i(\sigma_i', \sigma_{-i}) = \sum_{s_i \in S_i} \sigma_i'(s_i) u_i(s_i, \sigma_{-i}) \leq \text{Max}\left\{u_i(s_i, \sigma_{-i}) : s_i \in S_i\right\} \quad (1)$$

If (i) and (ii) are true, then

$$u_i(\sigma_i, \sigma_{-i}) = \text{Max}\left\{u_i(s_i, \sigma_{-i}) : s_i \in S_i\right\}$$

Hence, according to (1), $\sigma_i \in b_i(\sigma_{-i})$. That is, σ is a NE. □

Example: *Battle of sexes* game

	Soccer	Opera
Soccer	2,1	0,0
Opera	0,0	1,2

Pure strategy NE: $(S, S), (O, O)$.

What about mixed strategy NE?

If there exists a mixed strategy NE (which is not a pure strategy NE), then it must be the case that both players mix.

Suppose $(\alpha S + (1-\alpha)O, \beta S + (1-\beta)O)$ is a NE, with $\alpha \in (0,1)$ and $\beta \in (0,1)$, then by the previous proposition, player 1 is indifferent between S and O:

$$2\beta + 0(1-\beta) = 0\beta + 1(1-\beta) \Rightarrow \beta = \frac{1}{3}$$

Similarly, player 2 is also indifferent between S and O:

$$1\alpha + 0(1-\alpha) = 0\alpha + 2(1-\alpha) \Rightarrow \alpha = \frac{2}{3}$$

Hence the following is a mixed strategy NE:

$$(\frac{2}{3}S + \frac{1}{3}O, \frac{1}{3}S + \frac{2}{3}O)$$

Relationship between NE and dominance

- We briefly investigate the relationship between the two solution concepts so far. In this part, we always assume the game is finite.
- If a game is dominance solvable, the resulting pure strategy profile is a NE.
- Moreover, iterative elimination of strictly dominated strategies will not eliminate any NE.
  - If a game is dominance solvable, then the outcome is the unique NE.
  - To find NE, we can always apply ISD first.
  - Eliminating weakly dominated strategies might eliminate some NE. Example:

	L	R
T	1,1	0,0
B	0,0	0,0

  - However, the beauty contest game is weak dominance solvable and the corresponding outcome is also the unique pure strategy NE of this game.

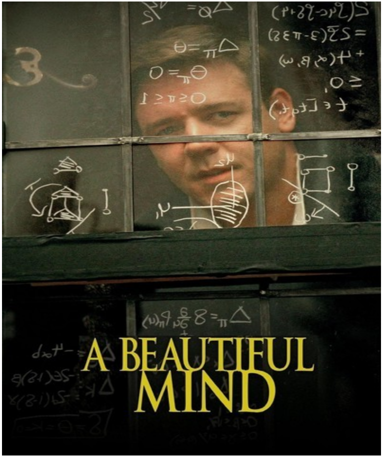
Existence of Nash equilibrium

Nash equilibrium requires that the players have correct conjectures about each other's play. This is a quite strong requirement, but a Nash equilibrium exists in a large class of games.

Theorem. *(Nash, 1950) Every finite normal form game has a Nash equilibrium.*

John Forbes Nash Jr. (June 13, 1928 – May 23, 2015) was an American mathematician who made fundamental contributions to game theory, differential geometry, and the study of partial differential equations.

He shared the 1994 Nobel Memorial Prize in Economic Sciences with game theorists Reinhard Selten and John Harsanyi.



CARNEGIE INSTITUTE OF TECHNOLOGY
SCHENLEY PARK
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DEPARTMENT OF MATHEMATICS
COLLEGE OF ENGINEERING AND SCIENCE

February 11, 1948

Professor S. Lefschetz
Department of Mathematics
Princeton University
Princeton, N. J.

Dear Professor Lefschetz:

This is to recommend Mr. John F. Nash, Jr. who has applied for entrance to the graduate college at Princeton.

Mr. Nash is nineteen years old and is graduating from Carnegie Tech in June. He is a mathematical genius.

Yours sincerely,

Richard J. Duffin

RJD:hl

Chess and go are finite games.

To prove this result, we need the following mathematical results.

Given $A \subseteq \mathbb{R}^M$ and a compact set $Y \subseteq \mathbb{R}^K$, the correspondence $f : A \to Y$ is **upper hemicontinuous** if for any two convergent sequences $x^n \to x \in A$ and $y^n \to y$ with $x^n \in A$ and $y^n \in f(x^n)$ for all n, we have $y \in f(x)$.

Kakutani's Fixed Point Theorem. *Suppose that $A \subseteq \mathbb{R}^M$ is a nonempty, convex and compact set, and $f : A \to A$ is an upper hemicontinuous correspondence such that $f(x)$ is a nonempty and convex set for every $x \in A$. Then f has a fixed point. That is, there exists $x \in A$ such that $x \in f(x)$.*

Lemma. *For each player i, $b_i(\sigma_{-i})$ is a nonempty and convex set for all $\sigma_{-i} \in \prod_{j \neq i} \Delta(S_j)$; moreover, b_i is upper hemicontinous.*

Proof. For any σ_{-i}, $b_i(\sigma_{-i})$ is the set of solutions to the problem

$$\text{Max } \ u_i(\sigma_i, \sigma_{-i})$$

$$s.t. \ \ \sigma_i \in \Delta(S_i)$$

Since u_i is continuous and $\Delta(S_i)$ is compact, $b_i(\sigma_{-i})$ is nonempty. It is also convex since u_i is quasiconcave in σ_i. Finally, to see upper hemicontinuity, consider any two sequences $\sigma^n_{-i} \to \sigma_{-i} \in \prod_{j \neq i} \Delta(S_j)$ and $\sigma_i^n \to \sigma_i$ with $\sigma^n_{-i} \in \prod_{j \neq i} \Delta(S_j)$ and $\sigma_i^n \in b_i(\sigma^n_{-i})$ for all n. For any $\sigma_i' \in \Delta(S_i)$, we have

$$u_i(\sigma_i^n, \sigma^n_{-i}) \geq u_i(\sigma_i', \sigma^n_{-i})$$

for all n. Then the continuity of u implies

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i})$$

That is, $\sigma_i \in b_i(\sigma_{-i})$. Hence b_i is upper hemicontinuous. □

Proof of the existence of NE.

Define the correspondence $b : \prod_i \Delta(S_i) \to \prod_i \Delta(S_i)$ such that

$$b(\sigma) = b_1(\sigma_{-1}) \times ... \times b_n(\sigma_{-n})$$

$\prod_i \Delta(S_i)$ is nonempty, convex and compact. Using the previous lemma, it is not difficult to see that $b(\sigma)$ is nonempty and convex for any σ, and b is upper hemicontinuous. Therefore, by the Kakutani's fixed point theorem, there exists a fixed point. That is, there exists $\sigma \in \prod_i \Delta(S_i)$ such that $\sigma \in b(\sigma)$. Clearly, σ is a Nash equilibrium. $\square$

Applications - Models of competition

- We briefly discuss some issues regarding market structures.
  - An excellent reference book in this area would be *The Theory of Industrial Organization* by Jean Tirole.
- Besides the perfectly competitive market, another market structure that does not involve strategic interaction is one with a single firm in it: the monopoly.
- The monopoly produces a single output with a cost function $c(\cdot)$, and faces its (inverse) demand curve $p(q)$.
- The monopoly's profit-maximizing problem is

$$\underset{q \geq 0}{\text{Max}} \;\; p(q)q - c(q)$$

  At an interior solution

$$p'(q^M)q^M + p(q^M) = c'(q^M)$$

- With a downward sloping demand curve, $p'(\cdot) < 0$, hence $p(q^M) > c'(q^M)$. That is, the monopoly's optimal price is above its marginal cost, and the quantity produced is below the Pareto efficient level.

- Consider a concrete case with linear demand $p(q) = a - bq$ and constant marginal cost c. Assume that $a > c > 0, b > 0$.
- Solving the monopoly's problem, we get

$$q^M = \frac{a-c}{2b}, \;\; p^M = \frac{a+c}{2}$$

- In contrast, the socially optimal (or, perfectly competitive) level of output and price are

$$q^* = \frac{a-c}{b}, \;\; p^* = c$$

- We now turn to those intermediate cases between a monopoly and a perfectly competitive market: situations of *oligopoly*. We mainly focus on the case of a *duopoly*.
- Two firms can compete in prices (Bertrand) or in quantities (Cournot).
- Quantity competition: the two firms produce the same product and simultaneously decide on their quantities q_1, q_2.
- They face the (inverse) demand curve $p(q)$ together, where $q = q_1 + q_2$.
- Cost functions: $c_1(q_1)$ and $c_2(q_2)$.
- To solve for the Nash equilibrium (q_1, q_2), we need to find each firm's best response. For firm i, given q_j, its best response q_i is the solution to the following problem

$$\underset{q_i \geq 0}{\text{Max}} \; p(q_i + q_j)q_i - c_i(q_i)$$

- For the sake of concreteness, consider the case of linear demand $p(q) = a - bq$ and constant marginal cost c, with $a > c > 0, b > 0$.
- Solving for the best response functions, we get

$$q_1 = \frac{a - c - bq_2}{2b}, \quad q_2 = \frac{a - c - bq_1}{2b}$$

  At the intersection of these two best response functions we find the NE:

$$q_1^C = q_2^C = \frac{a - c}{3b}$$

  The total quantity produced is therefore $q^C = \frac{2(a-c)}{3b}$, and the resulting market price is $p^C = \frac{a+2c}{3}$.
- A simple comparison shows that

$$p^* < p^C < p^M$$

  and

$$q^* > q^C > q^M$$

- In the Cournot NE, the two firms' profits are

$$\pi_1^C = \pi_2^C = \frac{(a-c)^2}{9b}$$

- If these two firms can collude (form a cartel), they can jointly earn

$$\pi^M = \frac{(a-c)^2}{4b}$$

- Therefore, the Cournot NE is not Pareto efficient for these two firms.
  - Each firm is producing "too much" compared to the cartel case.
  - This is because each firm is maximizing its own profit, without considering the effect of its action on the other firm (**externality**).

- Consider price competition: the Bertrand model.
- Two firms produce the same product.
- Each firm has constant marginal cost c.
- Market demand is $x(p)$: continuous and strictly decreasing
- Firms are competing in prices (p_1, p_2).
  - If one firm's price is lower, then it gets the whole market.
  - If $p_1 = p_2$, then the two firms split the market.
- Unique Nash Equilibrium: $p_1 = p_2 = c$.
  - This is a NE in weakly dominated strategies.

Applications - Auctions

- *Dutch auction (descending clock auction)*: price starts high and descends until one bidder declares he will buy, object is assigned to him, he pays his bid.
- *First price sealed bid auction*: all bidders submit bids in sealed envelopes so that each bidder does not know others' bids at the time of submission, winner is the one with the highest bid and he pays his bid.
  - Dutch auction and first price sealed bid auction are **strategically equivalent**.
- *English auction (ascending clock auction)*: bids are submitted sequentially in an ascending manner until bidding stops, the last person to submit a bid wins, he pays his bid.
- *Second price sealed bid auction (Vickrey auction)*: bidders submit sealed bids, the object is allocated to the one who submitted the highest bid, the winner pays the price of the second highest bid.
  - English auction and second price sealed bid auction are strategically equivalent.

- Consider the games under the first price (sealed bid) auction and the second price (sealed bid) auction.
  - There are two players, their valuations of the object are v_1, v_2, respectively.
  - Each player submits a bid b_i, which is a nonnegative integer.
  - If $b_1 = b_2$, each player wins the auction with probability 0.5.
  - payoff $=$ probability of winning * (payment - valuation)
  - Assume complete information: players know each other's valuation.
- Nash equilibria are not difficult to find, and there are many.
- Example: first price auction, $v_1 = 500, v_2 = 700$
  - NE: (499,500),(500,501),(501,502),...,(697,698),(698,699)
- Example: second price auction, $v_1 = 500, v_2 = 700$
  - Some NE: (b_1, b_2) with $b_1 < b_2$ and $500 \leq b_2 \leq 700$
- *Under second price auction, for each player, bidding his true valuation is his (unique) weakly dominant strategy.*

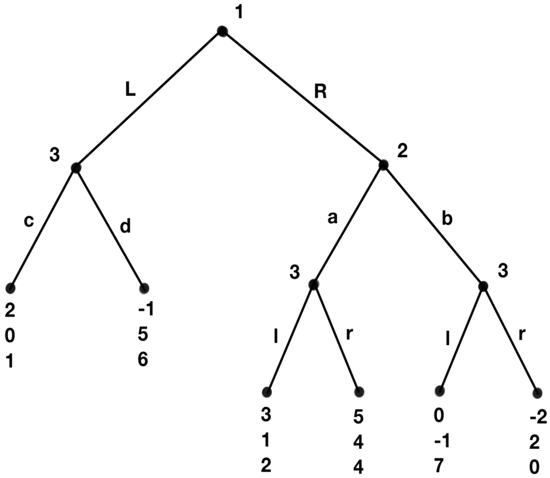
Advanced Microeconomics I
Note 10: Extensive form games

Xiang Han (SUFE)

Fall 2023

Extensive form representation

- A dynamic game generally has a richer set of rules compared to simultaneous move games.
    - Who moves when, what actions each player can take when it is his turn to move, what he knows when he moves
- To study dynamic games, we usually use the **extensive form representation**: a game tree


1
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1
2
5
4
4
0
-1
7
-2
2
0

- Extensive form: Γ_E
- A set of nodes $\mathcal{X}$, a set of possible actions $\mathcal{A}$, and a finite set of players $\{1, 2, ..., n\}$

- A function $p : \mathcal{X} \to \{\mathcal{X} \cup \phi\}$: $p(x)$ is the immediate predecessor of node x.
    - notice that $p(x)$ is unique
- There is a single node x_0 with $p(x_0) = \phi$: the initial node.
- A correspondence $s : \mathcal{X} \to \mathcal{X}$: $s(x)$ is the set of immediate successors of x.
- For any x, its set of predecessors and set of successors are disjoint.
    - No cycles are allowed.
- $T = \{x \in \mathcal{X} : s(x) = \phi\}$: terminal nodes
- $\mathcal{X} \setminus T$: decision nodes

- For each $x \in \mathcal{X}$ and $x' \in s(x)$, the branch connecting x and x' is labeled by an action $a \in \mathcal{A}$.
- For each decision node $x \in \mathcal{X} \setminus T$, $c(x)$ is the set of actions (choices) available at x.
  - For $a \in \mathcal{A}$, $a \in c(x)$ if and only if there is $x' \in s(x)$ such that x and x' are connected by a branch labeled by the action a.
  - If $x', x'' \in s(x)$ and $x' \neq x''$, then the two actions leading to x' and x'' from x cannot be the same.

- An **information set** $H \subseteq \mathcal{X} \setminus T$ is a collection of decision nodes. Each information set is labeled by one player and it represents a circumstance in which this player might be called upon to move.
- Each decision node belongs to one and only one information set. That is, all the information sets form a partition of $\mathcal{X} \setminus T$.
- If $x, x' \in H$, then $c(x) = c(x')$.
- Then, given an information set H, pick some $x \in H$ and denote $C(H) = c(x)$.
- Let $\mathcal{H}_i$ denote the collection of player i's information sets. Then $\mathcal{H} = \cup_i \mathcal{H}_i$ is the collection of all the information sets.
- Finally, each player i has a (Bernoulli) utility function $u_i : T \to \mathbb{R}$. This finishes the definition of Γ_E.

- The extensive form game Γ_E is finite if $\mathcal{X}$ is finite.
  - The game can be infinite due to infinite player set, infinite action set, or infinite time horizon.

Γ_E is an extensive form game with **perfect information** if every information set contains only one decision node. Otherwise it is an extensive form game with **imperfect information**.

However, we maintain the assumption of complete information in this note: the game structure Γ_E is common knowledge.

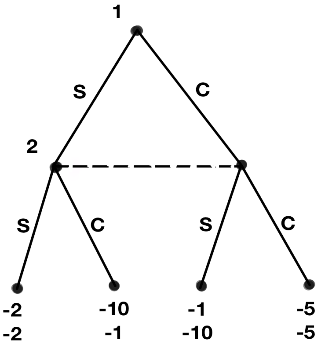
A simultaneous move game can be represented as an extensive form game with imperfect information.

Example: Prisoner's dilemma game.

Normal form representation:

	S	C
S	-2,-2	-10, -1
C	-1,-10	-5, -5

Extensive form representation:

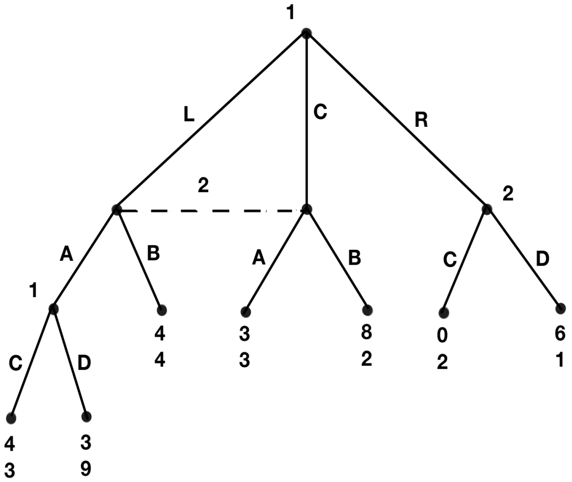


This cannot be overemphasized: *In an extensive form game, a strategy is a complete contingent plan that specifies how each player will act in every possible circumstance in which he might be called upon to move.*

Formally, in the extensive form game Γ_E, a **(pure) strategy** for player i is a function $s_i : \mathcal{H}_i \to \mathcal{A}$ such that $s_i(H) \in C(H)$ for all $H \in \mathcal{H}_i$.

Given any extensive form game, there is a unique normal form representation.

Example: find the normal form of the following game



In an extensive form game, a player can also play mixed strategies, as in normal form games. Additionally, a player has another way to randomize: he can randomize separately over the possible actions at each of his information sets. This is called a **behavior strategy**.

These two types of randomizations are equivalent in finite games with **perfect recall** (Kuhn, 1953).

A player has perfect recall if he never forgets what he once knew.

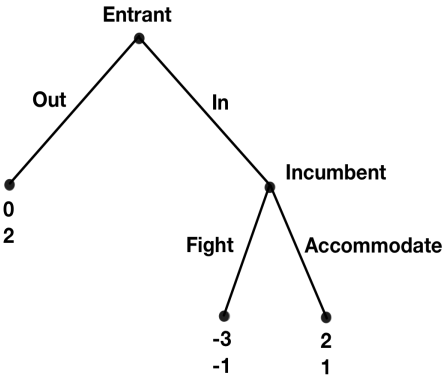
In analyzing dynamic games, we usually consider behavior strategies instead of mixed strategies.

Subgame perfection

- A natural starting point of solving a dynamic game is to find its Nash equilibria.
  - Given a dynamic game, we can (completely) represent this game using an extensive form.
  - Once the players' strategies are specified, we can derive a normal form representation of this game and apply the solution concept of Nash equilibrium.
- However, there is an important issue in this approach: the *credibility* of strategies in the NE of a dynamic game.

Example: *Entry deterrence game*

Extensive form:



Normal form:

	F	A
Out	0,2	0,2
In	-3,-1	2,1

There are two pure strategy NE: (Out, F) and (In, A), and the first one involves a *non-credible threat*.

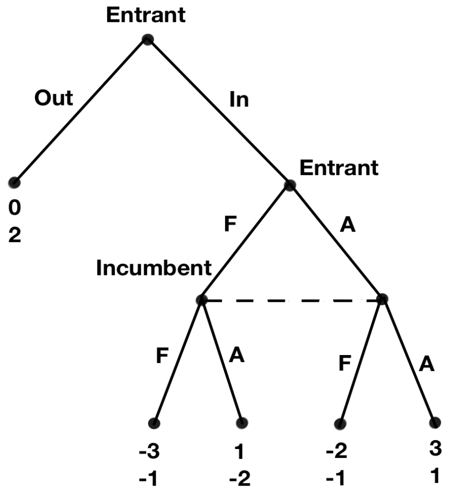
- To rule out such unreasonable equilibrium, generally in the dynamic setting we want players' equilibrium strategies to be **sequentially rational**: *the equilibrium strategies should specify optimal choice from any point in the game onward.*
- The principle of sequential rationality is first captured by a stronger solution concept called *subgame perfect Nash equilibrium*.
- Given an extensive form game Γ_E, a **subgame** is a subset of the game with the following two properties.
  - It begins with an information set containing a single decision node, contains all the decision nodes that are successors of this node, and contains only these nodes.
  - If a decision node x is in the subgame and $x, x' \in H$ for some information set H, then x' is also in the subgame.

Given an extensive form game Γ_E, a strategy profile σ is a **subgame perfect Nash equilibrium** (SPE) if it induces a Nash equilibrium in every subgame of Γ_E.

Since a game is a subgame of itself, SPE is a refinement on NE.

In the entry deterrence game, (Out, F) is not a SPE because it does not induce a NE in the subgame that starts from the incumbent's decision node (information set).

Example: Entry deterrence II



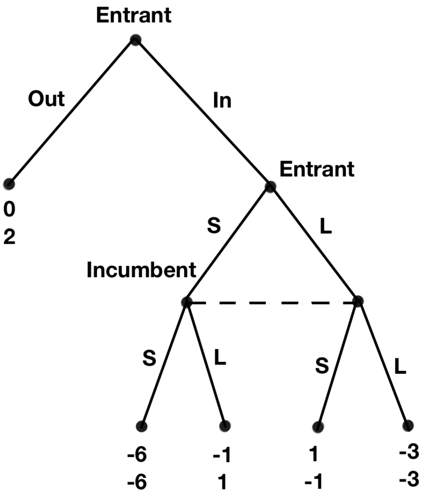
Three pure strategy NE: ((Out, A), F), ((Out, F), F), ((In, A), A)

Only the last one is SPE.

- Generally, for extensive form games, SPE can be found using the method of **backward induction**.
  - Start at the end of the game tree Γ_E, and identify the NE for all the final subgames.
  - Select one NE in each of the final subgames, and replace these final subgames using the payoffs in the selected NE.
  - Repeat this procedure for the reduced game tree, until every move in Γ_E is specified. The resulting collection of moves at each information set constitutes a SPE.

Example: find the SPE in the game on page 3 using backward induction.

Example: Entry deterrence III



Three SPE: ((In, L), S), ((Out, S), L), ((Out, $\frac{2}{9}$S+$\frac{7}{9}$L), $\frac{2}{9}$S+$\frac{7}{9}$L).

Example: the **Centipede game**

- Basic settings:
  - Two players each start with \$1 in front of them.
  - They alternate saying "stop" or "continue", starting with player 1.
  - When a player says "continue", \$1 is taken from his pile and \$2 are added to his opponent's pile.
  - When a player says "stop", the game ends and each player receives the money in his current pile.
  - The game also ends if both players' piles reach \$100.
- The unique SPE can be easily found using backward induction, and the SPE makes a sharp prediction.

Example: **sequential bargaining**

- Basic settings:
  - Player 1 and 2 are bargaining over one dollar.
  - At the beginning of $t = 1$, player 1 proposes $(s_1, 1 - s_1)$.
  - Player 2 either accepts, or rejects the offer (in which case play continues to period 2).
  - At the beginning of $t = 2$, player 2 proposes $(s_2, 1 - s_2)$.
  - Player 1 either accepts, or rejects the offer (in which case play continues to period 3).
  - At the beginning of $t = 3$, player 1 and 2 receive $(s, 1 - s)$: the *disagreement values*, with $s \in (0, 1)$.
  - Players are impatient: discount factor $\delta \in (0, 1)$.
- Agreement is immediate in the SPE.