1. Consider the two-player game whose extensive form representation (excluding payoffs) is depicted in Figure 1.

   (a) What are player 1's possible strategies? Player 2's?

   (b) Show that for any behavior strategy that player 1 might play, there is a realization equivalent mixed strategy; that is, a mixed strategy that generates the same probability distribution over the terminal nodes for any mixed strategy choice by player 2.

   (c) Show that the converse is also true: For any mixed strategy that player 1 might play, there is a realization equivalent behavior strategy.

   (d) Suppose that we change the game by merging the information sets at player 1's second round of moves (so that all four nodes are now in a single information set). Which of the two results in (b) and (c) still holds?

**Answer:**

(a) Player 1's possible (pure) strategies are:

$$S_1 = \{Lxx, Lxy, Lyx, Lyy, Mxx, Mxy, Myx, Myy, Rxx, Rxy, Ryx, Ryy\}.$$

Player 2's possible (pure) strategies are: $S_2 = \{l, r\}$.

(b) Assume that player 2 plays $l, r$ with probabilities of $s_1, s_2$ respectively. Suppose at the root, player 1 chooses $L, M, R$ with probabilities of $p_1, p_2, p_3$ respectively, where $p_1 + p_2 + p_3 = 1$; at his second information set (where the outcomes are $T_1$ through $T_4$), player 1 chooses $x, y$ with probabilities of $q_1, q_2$ resp.($q_1 + q_2 = 1$); at his third information set, player 1 chooses $x, y$ with probabilities of $r_1, r_2$ resp.($r_1 + r_2 = 1$). Now for the above behavioral strategy of player 1, there is a realization equivalent mixed strategy $\sigma_1$, which is:

$$\sigma_1(Lxx) = p_1, \sigma_1(Mxx) = p_2q_1, \sigma_1(Myx) = p_2q_2, \sigma_1(Rxx) = p_3r_1, \sigma_1(Rxy) = p_3r_2.$$

Note that $p_1 + p_2q_1 + p_2q_2 + p_3r_1 + p_3r_2 = 1$. When player 1 is using this mixed strategy, the probabilities that we reach each terminal nodes will be the same compared with his behavioral strategy stated above.

(c) Assume that player 2 plays $l, r$ with probabilities of $s_1, s_2$ respectively. Given player 1's any mixed strategy

$$\sigma_1 = (p_1, p_2, \ldots, p_{11}, p_{12}),$$

there is a realization equivalent behavioral strategy, which is:

At the root, player 1 chooses $L, M, R$ with probabilities of $p_1 + p_2 + p_3 + p_4, p_5 + p_6 + p_7 + p_8, p_9 + p_{10} + p_{11} + p_{12}$ respectively; at his second information set, player 1 chooses $x, y$ with probabilities of $(p_5+p_6)/(p_5+p_6+p_7+p_8), (p_7+p_8)/(p_5+p_6+p_7+p_8)$, resp.; at his third information set, player 1 chooses $x, y$ with probabilities of $(p_9 + p_{11})/(p_9 + p_{10} + p_{11} + p_{12}), (p_{10} + p_{12})/(p_9 + p_{10} + p_{11} + p_{12})$ resp.

(d) In this case, the game does not satisfy perfect recall, since if player 1 reaches his (only) information set after player 2 moves, he will not remember whether he chose $M$ or $R$. The result in (b) still holds, whereas the result in (c) does not always true, that is, there is a mixed strategy for player 1 that is realization equivalent to any behavioral strategy, however, there does not always exist a behavioral strategy that is realization equivalent to a mixed strategy.

2. For the game given below:

| | | Player 2 | | | |
|---|---|---|---|---|---|
| | | B1 | B2 | B3 | B4 |
| Player 1 | A1 | (0, 6) | (3, 1) | (2, 0) | (3, 7) |
| | A2 | (1, 0) | (9, 4) | (0, 12) | (1, 1) |
| | A3 | (0, 0) | (10, -1) | (2,-3) | (0, 1) |
| | A4 | (7, 3) | (0, 0) | (5, 1) | (-1, 2) |
| | A5 | (2, 8) | (-2, 1) | (3, 1) | (1, 0) |

(a) Does any player have any dominated strategies? If yes, what are they?

(b) Find all pure strategy NE of the game.

(c) Does the game have any mixed strategy NE? If yes, please find the mixed strategy NE.

**Answer:**

(a) For player 1, $A_5$ is a strictly dominated strategy, dominated by $\sigma_1 = (\frac{4}{7}A_1, 0, 0, \frac{3}{7}A_4, 0)$.
For player 2, $B_2$ is a strictly dominated strategy, dominated by $\sigma_2 = (\frac{1}{3}B_1, 0, \frac{1}{3}B_3, \frac{1}{3}B_4)$.

(b) Iterative deletion of strictly dominated strategy gives:

Player 2

| | B1 | B4 |
|---|---|---|
| A1 | (0, 6) | (3, 7) |
| A4 | (7, 3) | (-1, 2) |

Thus, the two pure strategy NE are:

$$(A_4, B_1) \qquad (A_1, B_4).$$

(c) Mixed strategy NE:

$$\left(\frac{1}{2}A_1, 0, 0, \frac{1}{2}A_4, 0; \frac{4}{11}B_1, 0, 0, \frac{7}{11}B_4\right).$$

3. For the game depicted in Figure 2:

   (a) Determine all SPNE of this game;

   **Answer:** SPNE

$$(U, al), \quad (V, ar).$$

   (b) Determine all (pure strategy) sequential equilibrium of this game.

   **Answer:** SPNE

   - Strategy: $(V, ar)$;
   - Belief: player 2 puts probability one on his left decision node; i.e., he thinks player 1 plays $V$.

4. For the game depicted in Figure 3:

   (a) Determine all pure strategy normal-form perfect equilibrium of this game;

   (b) Determine all sequential equilibrium of this game.

   **Answer:**

   (a) Note that $d$ is a weakly dominated strategy for player 2 and this rules out $(D, d, L)$. The first pure strategy normal-form perfect equilibrium:

$$(D, a, L),$$

   with $\epsilon$-perfect equilibrium

$$\sigma^k = ((\epsilon^k, 1 - \epsilon^k); (1 - \epsilon^k, \epsilon^k); (1 - \epsilon^k, \epsilon^k)$$

The second pure strategy normal-form perfect equilibrium

$$(A, a, R),$$

with its $\epsilon$-perfect equilibrium

$$\sigma^k = (((\epsilon^k)^3, 1 - (\epsilon^k)^3); (1 - \epsilon^k, \epsilon^k); (1 - \epsilon^k, \epsilon^k).$$

(b) There are two S.E.:

- First S.E.

  Strategy: $(D, a, L)$;

  Belief: $(1, 0)$.

- Second S.E.

  Strategy: $(A, a, R)$;

  Belief: $\{(\alpha, 1 - \alpha) | \alpha \leq 0.5\}$.