Advanced Microeconomics I
Note 3: Consumer preference and utility

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Properties of a consumer's preference relation

- The consumer has a preference relation $\succeq$ defined on $X = \mathbb{R}^L_+$.
- $\succeq$ is rational iff it is complete and transitive.
- $\succeq$ is **monotone** if for any $x, y \in X$, $x \gg y$ implies $x \succ y$.
- $\succeq$ is **strongly monotone** if for any $x, y \in X$, $x \geq y$ and $x \neq y$ imply $x \succ y$.
- $\succeq$ is **locally nonsatiated** if for any $x \in X$ and $\epsilon > 0$, there exists $y \in X$ such that $\|y - x\| < \epsilon$ and $y \succ x$.

Proposition. *Strong monotonicity* $\mapsto$ *monotonicity* $\mapsto$ *local nonsatiation*

- Given $x \in X$,
  - the *indifference set* of x: $\{y \in X : y \sim x\}$
  - the *upper contour set* of x: $\{y \in X : y \succeq x\}$
  - the *lower contour set* of x: $\{y \in X : x \succeq y\}$
- The preference relation $\succeq$ is **convex** if for any $x \in X$, the upper contour set of x, $\{y \in X : y \succeq x\}$, is convex.
  - "diminishing marginal rate of substitution"
- $\succeq$ is **strictly convex** if for any $x, y, z \in X$ with $y \succeq x$, $z \succeq x$, and $y \neq z$, we have $\alpha y + (1 - \alpha) z \succ x$ for all $\alpha \in (0, 1)$.

Existence of utility function

$u : X \to \mathbb{R}$ is a **utility function representing** $\succeq$ on X if for any $x, y \in X$ we have: $x \succeq y$ if and only if $u(x) \geq u(y)$.

Proposition. *If $\succeq$ can be represented by a utility function, then $\succeq$ is rational.*

Can a rational preference relation always be represented by a utility function?

Consider the *lexicographic preference relation*: assume $X = R^2_+$; for any $x, y \in X$, let $x \succeq y$ if $x_1 > y_1$, or, $x_1 = y_1$ and $x_2 \geq y_2$.

Proposition. *The lexicographic preference relation $\succeq$ on $X = R^2_+$ is rational, strongly monotone and strictly convex. However, there does not exist a utility function representing $\succeq$.*

- Some mathematical facts used in the proof of non-existence:
  - A set S is *countable* if there exists a one-to-one function $f : S \to \mathbb{N}$.
  - $\mathbb{R}$ is not countable (in particular, $\mathbb{R}_+$ is not countable).
  - The set of rational numbers $\mathbb{Q}$ is countable.
  - Between any two different real numbers, there exists a rational number.

- It turns out that a *continuity* condition on preferences is essential for the existence of a utility function.
- The preference relation $\succeq$ on X is **continuous** if for any sequence $\{x^n\} \subseteq X$ with $x^n \to x$, and $y \in X$ we have (1) $x^n \succeq y$ for all n implies $x \succeq y$ (2) $y \succeq x^n$ for all n implies $y \succeq x$.
- It can be easily seen that $\succeq$ is continuous if and only if for each $x \in X$, its upper contour set and lower contour set are closed.
- The lexicographic preference relation is not continuous.

Proposition. Suppose that $\succeq$ on $X = R_+^L$ is rational, continuous and monotone. Then there exists a utility function that represents $\succeq$.

- In the proof, we need a concept of **weak monotonicity**: for any $x, y \in X$, $x \geq y$ implies $x \succeq y$.
- While strong monotonicity implies weak monotonicity (when $\succeq$ is reflexive), there is on logical relation between monotonicity and weak monotonicity.
- When $\succeq$ is continuous, monotonicity implies weak monotonicity.
  - Proof: Consider any $x, y \in X$ with $x \geq y$. Let $x^n = \left\{x + \frac{1}{n}e\right\}$, where e is a vector of 1; then $x^n \gg y$ for all n, hence $x^n \succ y$ for all n by monotonicity; since $x^n \to x$, by continuity we have $x \succeq y$.

Proof. We prove this result by explicitly constructing a utility function that represents the rational, continuous and monotone preference relation $\succeq$ on R^L_+. Let $e = (1, 1, ..., 1)^T \in \mathbb{R}^L_+$ be a vector of one. Consider any $x \in R^L_+$ and the following two sets of nonnegative real numbers

$$B = \{\alpha \geq 0 : \alpha e \succeq x\}$$

$$W = \{\alpha \geq 0 : x \succeq \alpha e\}$$

Since $x \geq 0$, by weak monotonicity $x \succeq 0$. Hence $0 \in W \neq \phi$. It is obvious that there must exist some $\bar{\alpha} > 0$ such that $\bar{\alpha} e \gg x$. Then $\bar{\alpha} e \succ x$ by monotonicity. Hence $\bar{\alpha} \in B \neq \phi$. By transitivity, $\bar{\alpha} e \succ \alpha e$ for all $\alpha \in W$. Then by monotonicity $\bar{\alpha} \geq \alpha$ for all $\alpha \in W$. In sum, we know that both B and W are non-empty, B has a lower bound, 0, and W has an upper bound, $\bar{\alpha}$. Therefore, $\inf\,(B) \in \mathbb{R}$ and $\sup\,(W) \in \mathbb{R}$.

Consider any convergent sequence $\{\alpha^n\}$ in B, with $\alpha^n \to \alpha$. Then $\alpha^n e \succeq x$ for all n and $\alpha^n e \to \alpha e$. By the continuity of $\succeq$, $\alpha e \succeq x$. Hence $\alpha \in B$ and B is closed. By a similar argument, W is also closed. So $\inf\,(B) \in B$ and $\sup\,(W) \in W$.

For any $\alpha_1 \in B$ and $\alpha_2 \in W$, by transitivity $\alpha_1 e \succeq \alpha_2 e$. Then by monotonicity we must have $\alpha_1 \geq \alpha_2$. Hence $\inf\,(B) \geq \sup\,(W)$. If the inequality is strict, then there exists some α with $\inf\,(B) > \alpha > \sup\,(W)$. This implies that $\alpha e \not\succeq x$ and $x \not\succeq \alpha e$, contradicting to completeness. Therefore, $\inf\,(B) = \sup\,(W)$. Let this common number be $u(x)$.

Notice that, since $u(x) \in B$ and $u(x) \in W$, $u(x)e \sim x$.

We have constructed a function $u(\cdot)$ and it remains to show that it represents $\succeq$. Consider any $x, y \in \mathbb{R}^L_+$ with $x \succeq y$. $u(x)e \sim x \succeq y \sim u(y)e$, so by transitivity, $u(x)e \succeq u(y)e$. Then by monotonicity we must have $u(x) \geq u(y)$. On the other hand, consider any $x, y \in \mathbb{R}^L_+$ with $u(x) \geq u(y)$. Then by weak monotonicity, $u(x)e \succeq u(y)e$. Since $u(x)e \sim x$ and $u(y)e \sim y$, by transitivity we have $x \succeq y$. □

Properties of a utility function

- More generally, it can be shown that if the preference relation is rational and continuous, then it can be represented by a *continuous* utility function.
  - If $\succeq$ is not continuous, then it may also be represented by a utility function.
  - If $\succeq$ can be represented by a continuous utility function, then $\succeq$ is continuous.
  - If $\succeq$ is continuous, it may also be represented by a utility function that is not continuous.
- Properties of preferences translate into properties of utility functions.
  - (Strong or weak) monotonicity, local nonsatiation
  - **Proposition.** *Let u represent $\succeq$. u is (strictly) quasiconcave if and only if $\succeq$ is (strictly) convex.*
- Many properties of preferences or utility functions can be seen from the shape of *indifference curves*. An *indifference curve*, corresponding to a utility level u_0, is the set $\{x \in X : u(x) = u_0\}$.
- Utility representation is not unique: if u represents $\succeq$, then any strictly increasing transformation of u still represents $\succeq$.
  - *Ordinal* properties of utility functions: invariant for any strictly increasing transformation
  - *Cardinal* properties of utility functions: not preserved under strictly increasing transformations.