Problem Set 3 solutions

Additional Problem 1. Consider an economy with two goods, good l and a numeraire good, 2 consumers, $i = 1,2$, and 2 firms, $j = 1,2$. Let $x_i \in \mathbb{R}_+$ denote consumer $i's$ consumption of good l, and $m_i \in \mathbb{R}$ her consumption of the numeraire. The utility function of each consumer i is given by:

$$\begin{aligned} u_i(m_i, x_i) &= m_i + 1 - \frac{(1-x_i)^2}{2}, \text{ if } 0 \le x_i \le 1 \\ &= m_i + 1, \text{ if } x_i > 1. \end{aligned}$$

Each firm j produces $q_j \ge 0$ units of good l using amount $c_j(q_j)$ of the numeraire where

$$c_j(q_j) = (q_j)^2.$$

Each consumer has an initial endowment of 50 units of the numeraire and owns $\frac{1}{2}$ share of each firm.

(a) Derive the competitive equilibrium.

(b) Write down the set of all the efficient allocations in this economy.

Answer.

(a) Let $p > 0$ be the price of good l. We first derive the demand curve for good l. Note that consuming more than 1 unit of good l cannot be utility-maximizing for each consumer. Each consumer i's marginal utility from good l is $1 - x_i$ when $x_i \in [0,1]$. Hence his demand is

$$x_i(p) = 1 - p \text{ if } p \in (0,1].$$

The aggregate demand for good l is

$$x(p) = 2 - 2p \text{ if } p \in (0,1].$$

Next we find the supply. Profit maximization implies that each firm's marginal cost is equal to the price (corner solution is not possible since $c_1'(0) = c_2'(0) = 0 < p$). That is, $p = 2q_1 = 2q_2$. Hence

$$q_1(p) = q_2(p) = \frac{p}{2}$$

and the aggregate supply for good l is

$$q(p) = p.$$

Equate supply and demand for good l:

$$2 - 2p = p \quad \Rightarrow p^* = \frac{2}{3}.$$

Given the competitive equilibrium price p^*, we have $x_i^* = \frac{1}{3}$ for each i, and $q_j^* = \frac{1}{3}$ for each j.

Each firm's profit is $\frac{2}{3} \cdot \frac{1}{3} - (\frac{1}{3})^2 = \frac{1}{9}$. Then each consumer i receives a profit of $\frac{1}{9}$ from the two firms. His expenditure on good l is $\frac{2}{9}$. Since i is also endowed with 50 units of the numeraire, we have $m_i^* = 50 + \frac{1}{9} - \frac{2}{9} = 49\frac{8}{9}$.

Therefore, the competitive equilibrium consists of the allocation $(x_1^* = \frac{1}{3}, x_2^* = \frac{1}{3}, q_1^* = \frac{1}{3}, q_2^* = \frac{1}{3}, m_1^* = 49\frac{8}{9}, m_2^* = 49\frac{8}{9})$ and the price $p^* = \frac{2}{3}$.

(b) Given that $m_1^* + m_2^* = 99\frac{7}{9}$, the set of efficient allocations is

$$\left\{(x_1, x_2, q_1, q_2, m_1, m_2) : x_1 = x_2 = q_1 = q_2 = \frac{1}{3}, m_1 + m_2 = 99\frac{7}{9}\right\}.$$

Additional Problem 2.

(a) Suppose that the preference relation $\succeq$ on the set of lotteries $\mathscr{L}$ is complete. Show that $\succeq$ satisfies the independence axiom if and only if the following property holds:

For all $L_1, L_2, L_3 \in \mathscr{L}$ *and* $\alpha \in (0,1)$, $L_1 \succ L_2$ *if and only if* $\alpha L_1 + (1-\alpha)L_3 \succ \alpha L_2 + (1-\alpha)L_3$.

(b) Show that if $\succeq$ on $\mathscr{L}$ satisfies the independence axiom, then the following property holds:

For all $L_1, L_2, L_3 \in \mathscr{L}$ *and* $\alpha \in (0,1)$, $L_1 \sim L_2$ *if and only if* $\alpha L_1 + (1-\alpha)L_3 \sim \alpha L_2 + (1-\alpha)L_3$.

Answer.
(a). "only if" part. Consider any $L_1, L_2, L_3 \in \mathscr{L}$ and $\alpha \in (0,1)$, we have

$$\begin{aligned}L_1 \nsucc L_2 &\Leftrightarrow L_2 \succeq L_1 \\ &\Leftrightarrow \alpha L_2 + (1-\alpha)L_3 \succeq \alpha L_1 + (1-\alpha)L_3 \\ &\Leftrightarrow \alpha L_1 + (1-\alpha)L_3 \nsucc \alpha L_2 + (1-\alpha)L_3\end{aligned}$$

where the second equivalence relation follows from the independence axiom, the first and the third equivalence relation follow from the completeness of $\succeq$ as well as the definition of $\succ$.

Then, clearly,

$$L_1 \nsucc L_2 \Leftrightarrow \alpha L_1 + (1-\alpha)L_3 \nsucc \alpha L_2 + (1-\alpha)L_3$$

implies

$$L_1 \succ L_2 \Leftrightarrow \alpha L_1 + (1-\alpha)L_3 \succ \alpha L_2 + (1-\alpha)L_3$$

"if" part. The proof is similar. Consider any $L_1, L_2, L_3 \in \mathscr{L}$ and $\alpha \in (0,1)$, we have

$$\begin{aligned}L_1 \nsucceq L_2 &\Leftrightarrow L_2 \succ L_1 \\ &\Leftrightarrow \alpha L_2 + (1-\alpha)L_3 \succ \alpha L_1 + (1-\alpha)L_3 \\ &\Leftrightarrow \alpha L_1 + (1-\alpha)L_3 \nsucceq \alpha L_2 + (1-\alpha)L_3\end{aligned}$$

where the second equivalence relation follows from the property stated in part (a), the first and the third equivalence relation follow from the completeness of $\succeq$ as well as the definition of $\succ$.

Then, clearly,

$$L_1 \nsucceq L_2 \Leftrightarrow \alpha L_1 + (1-\alpha)L_3 \nsucceq \alpha L_2 + (1-\alpha)L_3$$

implies

$$L_1 \succeq L_2 \Leftrightarrow \alpha L_1 + (1-\alpha)L_3 \succeq \alpha L_2 + (1-\alpha)L_3$$

(b). If $L_1 \sim L_2$, then $L_1 \succeq L_2$ and $L_2 \succeq L_1$. By the independence axiom,

$$L_1 \succeq L_2 \Rightarrow \alpha L_1 + (1-\alpha)L_3 \succeq \alpha L_2 + (1-\alpha)L_3$$

$$L_2 \succeq L_1 \Rightarrow \alpha L_2 + (1-\alpha)L_3 \succeq \alpha L_1 + (1-\alpha)L_3$$

Therefore, $\alpha L_1 + (1-\alpha)L_3 \sim \alpha L_2 + (1-\alpha)L_3$.

On the other hand, if $\alpha L_1 + (1-\alpha)L_3 \sim \alpha L_2 + (1-\alpha)L_3$, then $\alpha L_1 + (1-\alpha)L_3 \succeq \alpha L_2 + (1-\alpha)L_3$ and $\alpha L_2 + (1-\alpha)L_3 \succeq \alpha L_1 + (1-\alpha)L_3$. By the independence axiom,

$$\alpha L_1 + (1-\alpha)L_3 \succeq \alpha L_2 + (1-\alpha)L_3 \Rightarrow L_1 \succeq L_2$$

$$\alpha L_2 + (1-\alpha)L_3 \succeq \alpha L_1 + (1-\alpha)L_3 \Rightarrow L_2 \succeq L_1$$

Therefore $L_1 \sim L_2$.

Problem Set 4 Solutions

Problem 1. Solve the following game using iterative elimination of strictly dominated strategies (write down each step of elimination).

	L	C	R
T	-5,-1	2,2	3,3
M	0,10	1,0	1,-10
B	1,-3	1,2	1,1

Answer.

Step 1. Eliminate M, since it is strictly dominated by $0.1T + 0.9B$.

Step 2. Eliminate L, since it is strictly dominated by C (and R).

Step 3. Eliminate B, since it is strictly dominated by T.

Step 4. Eliminate C, since it is strictly dominated by R.

Final outcome: (T, R).

Problem 2. Find all the Nash equilibria of the following game.

	L	R
T	2,1	0,2
B	1,2	3,0

Answer. First, there does not exist any pure strategy NE. Second, for each player, his best response to each pure strategy of the other player is a unique pure strategy, so there does not exist any NE in which one player uses a pure strategy, and the other player uses a (non-degenerate) mixed strategy.

Therefore, since it is a finite game, there must exist a NE in which $\sigma_1(T) = \alpha \in (0,1)$ and $\sigma_2(L) = \beta \in (0,1)$. It follows that

$$u_1(T, \beta L + (1-\beta)R) = u_1(B, \beta L + (1-\beta)R)$$

$$u_2(L, \alpha T + (1-\alpha)B) = u_2(R, \alpha T + (1-\alpha)B)$$

These two equations imply

$$\beta \cdot 2 = \beta + (1-\beta) \cdot 3$$

$$\alpha + (1-\alpha) \cdot 2 = \alpha \cdot 2$$

Therefore

$$\alpha = \frac{2}{3}, \;\; \beta = \frac{3}{4}$$

The unique NE is

$$\left(\frac{2}{3}T + \frac{1}{3}B, \;\; \frac{3}{4}L + \frac{1}{4}R\right)$$

Problem 3 (Tragedy of the commons). Suppose there are only two farmers, Farmer 1 and Farmer 2, who can graze their goats on the village green - the Common. In the spring, both farmers simultaneously decide how many goats to own and farmer i chooses $g_i \in \mathbb{R}_+$ (ignoring the integer problem). The cost of buying and caring for one goat is $c > 0$ for both farmers. At the end of the year, the value of a goat is defined as (G: total number of goats in the village)

$$v(g_1 + g_2) = v(G) = 2c - (g_1 + g_2)^2 = 2c - G^2$$

(a) Find the best response functions for the two farmers.

(b) Find the pure strategy Nash equilibrium of this game.

(c) Find the total number of goats G^* in the NE, and the optimal number of goats G^{**} in the village (the number of goats that maximize their total profits). Compare G^* and G^{**}, what can you conclude from the comparison?

Answer.

(a). First consider farmer 1. Given g_2, he chooses g_1 to maximize his profit:

$$\underset{g_1 \geq 0}{\text{Max}} \;\; g_1[2c - (g_1 + g_2)^2 - c]$$

First order condition

$$c - 3g_1^2 - g_2^2 - 4g_1g_2 = 0$$

Then farmer 1's best response function is

$$g_1 = \frac{-2g_2 + \sqrt{g_2^2 + 3c}}{3} \tag{1}$$

Repeat the same procedure for farmer 2, we can find his best response function

$$g_2 = \frac{-2g_1 + \sqrt{g_1^2 + 3c}}{3} \tag{2}$$

(b). We can find a pair (g_1^*, g_2^*) that satisfies both equation (1) and equation (2), which is the Nash equilibrium of this game:

$$g_1^* = g_2^* = \sqrt{\frac{c}{8}}$$

(c).

$$G^* = 2\sqrt{\frac{c}{8}}$$

G^{**} is the solution to the following problem

$$\operatorname*{Max}_{G\geq 0}\ G(2c - G^2 - c)$$

Solving this problem, we get

$$G^{**} = \sqrt{\frac{c}{3}} < G^*$$

That is, if the farmers play this game non-cooperatively, they tend to overuse the common.

Problem 4 (Simultaneous Bargaining). Consider the following simple bargaining game. Players 1 and 2 have preferences over two goods, x and y. Player 1 is endowed with one unit of good x and none of good y, while player 2 is endowed with one unit of

y and none of good x. Player i, $i \in \{1,2\}$, has utility function $u_i(x_i, y_i) = \min\{x_i, y_i\}$ where x_i is i's consumption of x and y_i is his consumption of y. The "bargaining" works as follows: Each player simultaneously hands any (nonnegative) quantity of the good he possesses (up to his entire endowment) to the other player.

(a) Write down this game as a normal form game.
(b) Find all the pure strategy Nash equilibria of this game.

Answer.
(a). The normal form $[N, \{\Delta(S_i)\}_{i\in N}, (u_i)_{i\in N}]$ consists of the following three parts:
$N = \{1,2\}$.
$S_1 = S_2 = [0,1]$, where $s_i \in S_i$, $i = 1,2$, is the amount of his endowment that player i gives to the other player.
$u_1(s_1, s_2) = \min\{1 - s_1, s_2\}$, $u_2(s_1, s_2) = \min\{s_1, 1 - s_2\}$

(b). The set of pure strategy NE is

$$\{(s_1, s_2) : s_1 \geq 0, s_2 \geq 0, s_1 + s_2 \leq 1\}$$

To see this, notice that given any $s_2 \geq 0$, player 1's best response only includes all the $s_1 \geq 0$ with

$$1 - s_1 \geq s_2 \tag{3}$$

Similarly, given any $s_1 \geq 0$, player 2's best response only includes all the $s_2 \geq 0$ with

$$1 - s_2 \geq s_1 \tag{4}$$

Both inequality (3) and inequality (4) are equivalent to $s_1 + s_2 \leq 1$. Therefore, the two players are best responding to each other if and only if $s_1 + s_2 \leq 1$.

Problem 5 (Bystander effect). Alvin slips and injures himself on a sidewalk. There are $n \geq 2$ people nearby observing the accident. Alvin needs at least one of the n bystanders to call 120 for immediate medical attention. Each of the n bystanders simultaneously and independently decides whether or not to call 120. If a bystander calls, she receives a payoff of $v - c > 0$, interpreted as the difference between the benefit of knowing Alvin will be helped (v) and the cost of calling ($c > 0$). If a bystander does not call, her payoff is v if at least one of the remaining $(n-1)$ people calls for help, and her payoff is 0 if no one calls.

(a) Find all the pure strategy Nash equilibria of this game.
(b) Find the symmetric mixed strategy Nash equilibrium in which each bystander calls with the same probability. In addition, given this equilibrium, derive an expression for the probability that Alvin receives medical attention, as a function of n. Is Alvin better-off with a larger or a smaller crowd of witnesses?

Answer.
(a). There are n pure strategy Nash equilibria. In each of them, one bystander calls, the other $n-1$ bystanders do not call.
(b). Suppose that there is a symmetric mixed NE where each one calls with probability $p \in (0,1)$. Then any player is indifferent between making the call and not making the call, given the mixed strategies of others. The payoff from making the call is $v - c$. The expected payoff from not making the call is

$$(1-p)^{n-1} \cdot 0 + [1-(1-p)^{n-1}] \cdot v$$

So let

$$v - c = (1-p)^{n-1} \cdot 0 + [1-(1-p)^{n-1}] \cdot v$$

which implies

$$p = 1 - \left(\frac{c}{v}\right)^{\frac{1}{n-1}}$$

Note that p is decreasing in n. The probability that at least one bystander calls is then

$$1-(1-p)^n = 1 - \left(\frac{c}{v}\right)^{\frac{n}{n-1}}$$

It is also decreasing in n. That is, Alvin is better-off with a *smaller* crowd.

Problem 6 (Choosing locations): MWG 8.D.5

Consumers are uniformly distributed along a boardwalk that is 1 mile long. Ice-cream prices are regulated, so consumers go to the nearest vendor because they dislike walking (assume that at the regulated prices all consumers will purchase an ice cream even if they have to walk a full mile). If more than one vendor is at the same location, they split business evenly.
(a) Consider a game in which two ice-cream vendors pick their locations simultaneously. Show that there exists a unique pure strategy Nash equilibrium and that it involves both vendors locating at the midpoint of the boardwalk.
(b) Show that with three vendors, no pure strategy Nash equilibrium exists.

Answer. I have uploaded the solution manual to Chapter 8 of MWG.

Remark. *This is essentially a variant of the "linear city model" which has been widely used in social sciences. It can be reinterpreted as a basic model of electoral competition in political economy where the location of a political party signifies its choice of political position or its position on public policy. The unique NE outcome in the two-player case translates to convergence of political positions, which is indeed commonly observed in practice.*

Another variant of the linear city model can be used to model price competition with product differentiation, which provides a solution to the "Bertrand paradox". If you are interested, you can take a look at Example 12.C.2 in MWG.

Problem 7* (Early closing time?). In many cities (in the U.S.) there is a city wide 1 or 2 AM mandated closing time for bars. Although they may not tell their patrons this, many of those ordinances were proposed by groups of bar owners and whenever they come up for consideration the bar owners typically support limiting the hours in which they are allowed to operate or at least don't fight them. Explain the apparent paradox in this situation and then explain why this situation could exist using game-theoretical reasoning (*Note, there is nothing special about this being bars. Many different types of businesses turn out to be in favor of similar restrictions, the question is why would a group of businesses prefer to have their options legally restricted? Your answer should depend on the strategic nature of the situation NOT on special characteristics of running a bar).*

Answer. This is similar to the Cournot problem, or problem 3 (tragedy of the commons). The bar owners can be considered as strategic players involved in quantity competition, where "quantity" refers to opening hours. In the Nash equilibrium, the bars are closing too late, compared to the case where they can cooperate. If there are rules that restrict their opening hours a bit, then each bar can potentially enjoy a higher profit *even if they do not cooperate*. Next, we give a more formal treatment to show that each player can indeed get a higher profit *in the Nash equilibrium* when there are such restrictions.

Consider the Cournot model in the lecture note. For simplicity, assume $a = 2$ and $b = c = 1$. The two firms' best response functions are

$$q_1 = \frac{1 - q_2}{2}, \;\; q_2 = \frac{1 - q_1}{2}$$

In the NE

$$q_1^C = q_2^C = \frac{1}{3}$$

and

$$\pi_1^C = \pi_2^C = \frac{1}{9}$$

If they cooperate and operate as a monopoly, then they should produce a total quantity of

$$q^M = \frac{1}{2}$$

(instead of $\frac{2}{3}$) and their total profit would be

$$\pi^M = \frac{1}{4}$$

If they split the profit then each one is better-off compared to the NE outcome (of course whether the total profit should be equally split is a different matter: they might need to go through some bargaining process to decide how exactly the profits should be split).

Suppose that they cannot cooperate, but there is a rule that restricts each firm's quantity such that each firm cannot produce more than x, with $0 < x < \frac{1}{3}$. Then what is the NE?

Consider firm 1. If firm 1 can choose any quantity, then for any $q_2 \in [0, x]$, its profit-maximizing decision is $q_1 = \frac{1-q_2}{2}$. However, since $q_2 \leq x < \frac{1}{3}$, $\frac{1-q_2}{2} > \frac{1}{3} > x$. Notice that firm 1's profit function is strictly concave with its peak at $\frac{1-q_2}{2} > x$. Therefore, if firm 1 cannot choose any quantity larger than x, then its optimal decision is simply $q_1 = x$.

The same analysis applies to firm 2. In sum, if both firms' set of pure strategies is $[0, x]$, then each firm's best response to any pure strategy chosen by the other firm is x (in fact, x is the strictly dominant strategy for both firms). It is obvious that the only pure strategy NE is $q_1 = q_2 = x$.

Generally, if the total quantity is q, then their total profit is $\pi(q) = (2-q)q - q$. Notice that $\pi(q)$ is a strictly concave function with its peak at $q^M = \frac{1}{2}$. So as long as $2x \in [\frac{1}{2}, \frac{2}{3})$, each firm's profit is $\frac{1}{2}\pi(2x) > \frac{1}{2}\pi(\frac{2}{3}) = \pi_1^C = \pi_2^C$.

Therefore, the two firms would love to have some rules that restrict their quantities. By choosing $x = \frac{1}{4}$, such a rule can implement the cooperation outcome even if the two firms do not have any intention to cooperate. So having your options limited is not always a bad thing.

Problem Set 5 Solutions

You do not need to turn in this homework.

Problem 1 (Stackelberg). The Stackelberg model is the sequential version of Cournot. Two firms produce homogeneous product with constant unit cost c. They are competing in quantities and the market inverse demand is given by $p = a - b(q_1 + q_2)$. Assume $a > c > 0$ and $b > 0$. Suppose that firm 1 chooses q_1 first. After observing q_1, firm 2 chooses q_2. Find the SPE.

Answer. We use backward induction and look at firm 2's decision first. Note that for each q_1, there is a subgame in which the Nash equilibrium is that firm 2 chooses q_2 to maximize its profit:

$$\operatorname*{Max}_{q_2 \geq 0} \; [a - b(q_1 + q_2)]q_2 - cq_2$$

Then firm 2 chooses[1]

$$b_2(q_1) = \frac{a - c - bq_1}{2b}.$$

Next, consider the decision of firm 1

$$\operatorname*{Max}_{q_1 \geq 0} \; [a - b(q_1 + b_2(q_1))]q_1 - cq_1$$

That is,

$$\operatorname*{Max}_{q_1 \geq 0} \; [a - b(q_1 + \frac{a - c - bq_1}{2b})]q_1 - cq_1$$

Solving this problem gives

$$q_1^* = \frac{a - c}{2b}.$$

Hence, the SPE outcome is

$$q_1^* = \frac{a - c}{2b}, \;\; q_2^* = b_2(q_1^*) = \frac{a - c}{4b}.$$

[1] Strictly speaking, firm 2 chooses this amount when $q_1 \leq \frac{a-c}{b}$. When q_1 is larger than this, firm 2 chooses zero. However, this latter case will not appear in the SPE.

Notice that in the equilibrium firm 1 enjoys a higher profit than firm 2 does: there is a *first-mover advantage*.

Problem 2 (Ultimatum game). Consider the following simple sequential bargaining problem. Two players interact to decide how to divide one dollar. The game proceeds as follows: First, player 1 proposes a division $(x, 1-x)$, where $x \in [0,1]$ represents how much he wants to keep for himself; player 2 then decides whether to accept or reject. If player 2 rejects, neither player receives anything. If player 2 accepts, the money is split according to player 1's proposal.

(a) Find the pure strategy SPE.
(b) Find a NE which is not subgame perfect.

Answer. We first need to be clear about each player's strategy set. It is obvious that each of player 1's strategies is a number $x \in [0,1]$. What about player 2? In extensive form games a strategy is a complete and contingent plan, so a strategy of player 2 must specify "accept" or "reject" for every possible proposal from player 1.

(a) To find SPE, we use backward induction. Notice that each number $x \in [0,1]$ induces a subgame in which only player 2 has to make a choice. For any subgame that corresponds to $x < 1$, the unique NE is "accept". For the subgame that corresponds to $x = 1$, there are two pure strategy NE: "accept" and "reject". Suppose that player 2's strategy is "accept for any $x \in [0,1]$", then clearly player 1's best response is $x = 1$. So one SPE is that player 1 chooses $x = 1$ and player 2 chooses "accept for any $x \in [0,1]$". On the other hand, if player 2's strategy is "accept if $x < 1$, reject if $x = 1$", then player 1's best response is an empty set[2], so this strategy of player 2 cannot be a part of any SPE. Hence this game has a unique pure strategy SPE.[3]

(b) There are infinitely many NE of this game, we do not list all of them but rather give some representative examples. Player 2 can use some "threshold strategies". For instance, one type of NE is that player 1 chooses $x = a$ with $a \in (0,1)$ and player 2's

[2] Intuitively, player 1 wants to choose a number as close to 1 as possible, and he does not want to choose 1.

[3] In fact, even if mixed strategies are allowed, there is only one SPE.

strategy is "accept if $x \leq a$, reject if $x > a$". There could also be some more unreasonable NE. For instance, player 1 chooses $x = a$ with $a \in (0, 1)$ and player 2's strategy is "accept if $x = a$, reject if $x \neq a$". Both types of NE involve some non-credible threat from player 2 and they are not subgame perfect.

Problem 3 (Choosing pennies sequentially). Two players, Amy and Beth, play the following game with a jar containing 100 pennies. The players take turns; Amy goes first. Each time it is a player's turn, she chooses how many pennies to take out of the jar this time: the number chosen must be an integer x with $1 \leq x \leq 10$. The player whose move empties the jar wins.

(a) If both players play optimally, who will win the game? Does this game have a first-mover advantage? Explain your reasoning.
(b) Now suppose we change the rules so that the player whose move empties the jar loses. Does this game have a first-mover advantage? Explain your reasoning.

Answer. (a). Use backward induction to analyze the game. Starting from the end of the game: the player whose move leaves 11 pennies in the jar wins. Going backwards: the player whose move leaves 22 pennies in the jar wins. Continue the analysis, eventually the player whose move leaves 99 pennies in the jar wins. Therefore, Amy can take one penny out in the first round and she is guaranteed to win if she does the following:

- Start by choosing 1;
- no matter which number k_1 Beth chooses in her first move, choose $11 - k_1$;
- no matter which number k_2 Beth chooses in her second move, choose $11 - k_2$;
- ......
- no matter which number k_9 Beth chooses in her ninth move, choose $11 - k_9$ to empty the jar;

Hence in this game there is a first-mover advantage.

(b). The second version of the game can be solved in a similar vein. In this version, however, the first-mover Amy will always lose. The reasons follow. At the end of the game, the player who leaves 1 penny wins. Then going backwards, the player who leaves 12 pennies wins. Continue in this fashion, eventually the player who leaves 89 pennies in the jar wins, and Beth can always leave 89 in the jar in her first move.

Problem 4. Consider a homogenous good Bertrand duopoly where the two firms produce at constant marginal cost. The market demand function is given by

$$\begin{aligned} D(p) &= 1-p, \;\; 0 \leq p \leq 1, \\ &= 0, \;\; p > 1. \end{aligned}$$

Initially, both firms produce at marginal cost of 0.6. Each firm has the option of adopting a new technology that reduces its marginal cost to 0. However, adopting this technology requires the firm to incur a fixed cost F where $0 < F < \frac{1}{4}$.

Firms play a two stage game. First, they decide simultaneously whether or not to adopt the new technology. Next, after observing each other's choice in the first stage, firms set prices simultaneously (the firm with lower price gets the whole market; when their prices are the same, they split the market). Find the pure strategy subgame perfect Nash equilibria of this game.

Answer. Since each firm chooses to adopt the technology (Yes) or not to adopt the technology (No) in the first stage, we first find the NE in the following four subgames:

- (Yes, Yes) in the first stage: then in the second stage both have zero marginal cost and the unique NE is that both set prices at zero. In the NE each firm's final profit is $-F$. Note that the cost of adopting the technology is sunk cost and does not affect your analysis of the Bertrand game in the second stage.

- (Yes, No) in the first stage: then firm 1 has a lower marginal cost of 0 while firm 2 still has a marginal cost of 0.6. Given the market demand, we can find the monopoly optimal price, which is $p = \frac{1}{2}$, with the monopoly profit of $\frac{1}{4}$. Hence, the Nash equilibria are: firm 1 sets its price at $p = \frac{1}{2}$, and firm 2 sets its price at any level above $\frac{1}{2}$.

- (No, Yes) in the first stage: this is symmetric to the above case.
- (No, No) in the first stage: then the second stage NE is that both firms set their prices at the marginal cost of 0.6 and each has zero profit.

Therefore, going back to the first stage, the game is reduced to the following normal form:

	Yes	No
Yes	-F, -F	0.25-F, 0
No	0, 0.25-F	0,0

There are two pure strategy Nash equilibria here: exactly one firm adopts the new technology.