# Information about Aggregate Variables

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# Grossman and Stiglitz (1980)

# Introduction

- How asset markets transmit information from informed traders to uninformed traders
- If information is costly, prices cannot transmit information perfectly, otherwise, no one would buy it.
- Key results: If information is costly to produce and acquire, then financial markets must be "noisy"

# Setup

- There are $Q$ traders with CARA preference

$$u(w) = -\exp(-\rho w)$$

- They have an initial endowment of consumption goods $w_0$ at $t = 0$
- The good is storable, and we normalize the safe interest rate to 1
- There is also a risky asset, with payoff at $t = 1$

$$y \sim N(\bar{y}, \frac{1}{h_y})$$

- The aggregate supply of the risky asset is

$$x \sim N(\bar{x}, \frac{1}{h_x})$$

- $x$ is not observable by anyone.
- Due to CARA, both $w_0$ and who initially owns the risky asset are irrelevant. (why?)

# Equilibrium concept

- Questions:
  - What will be the price of the asset?
  - How does it depend on $x$ and $\theta$
  - What portfolios will different traders chose
  - Who chooses to become informed?

- Noisy rational expectation equilibrium:
  - Price function $p(x, \theta)$
  - Quantities $q(p, \theta)$ for traders that observe $\theta$ and $q(p)$ for traders that don't such that
    - Traders optimize given price and their information
    - Information set includes the price
    - The market for the risky asset clears

# Demand for risky asset

- Due to CRRA-Normal, trader wants to maximize

$$\mathbb{E}[w] - \frac{\rho}{2}\mathrm{Var}\,[w]$$

- Mean-variance preference ▸ Appendix
- If trader buys $q$ unit of risky asset at a price $p$ per unit, then

$$w = w_0 + q(y - p)$$

so

$$\mathbb{E}_i(w) = w_0 + q(\mathbb{E}_i[y] - p)$$
$$\mathrm{Var}\,_i(w) = q^2\mathrm{Var}\,_i(y)$$

- Therefore, the maximization problem is

$$\max_q w_0 + q(\mathbb{E}_i[y] - p) - \frac{\rho q^2}{2}\mathrm{Var}\,_i(y)$$

- Interpretation: risk aversion governs the tradeoff between excess return and evidence

# Information of the informed traders

- By Bayesian updating ▸ Appendix

$$y \mid \theta \sim N\left(\frac{h_y \bar{y} + h_\theta \theta}{h_y + h_\theta}, \frac{1}{h_y + h_\theta}\right)$$

  - Posterior is weighted average of prior and signal
  - Weights are respective precisions (precision $=1/\text{variance}$)
  - Precision of posterior is sum of the precisions

# Learning from prices

- Uninformed traders do not see $\theta$ but they do see the price
- If the equilibrium price depends on $\theta$, it will convey infromation about $\theta$ and therefore, indirectly about $y$.
- Fixed point problem
  - How the price depends on $\theta$ determines informativeness of the price
  - Informativeness of the price determines demand by the uninformed
  - Demand by the uninformed determines the equilibrium price
- Conjecture:
$$p = \alpha + \beta(x - \bar{x}) + \gamma(\theta - \bar{y})$$
- We can prove that there exists an unique equilibrium of this form.
- Are there other non-linear equilibria? We don't know
- Let
$$z \equiv \bar{y} + \frac{p - \alpha}{\gamma} = \frac{\beta(x - \bar{x})}{\gamma} + \theta = y + \frac{\beta(x - \bar{x})}{\gamma} + \eta$$
- $z$ is just a linear transformation of the price, so observing the price is just like obverzing $z$
- $z$ is equal to $\theta$ plus normal noise, so it is also a signal of $y$ but noiser than $\theta$

# Learning from prices

- Note that
$$\operatorname{Var}\left(\frac{\beta(x-\bar{x})}{\gamma}+\eta\right)=\left(\frac{\beta}{\gamma}\right)^2\frac{1}{h_x}+\frac{1}{h_\theta}$$
so define the precision of the price by
$$h_p \equiv \frac{1}{\left(\frac{\beta}{\gamma}\right)^2\frac{1}{h_x}+\frac{1}{h_\theta}}$$
- Now Bayesian updating by uninformed traders will lead to
$$y \mid p \sim N\left(\frac{h_y\bar{y}+h_p z}{h_y+h_p},\frac{1}{h_y+h_p}\right)$$

# Market clearing and equilibrium prices

- Demands are

$$q_I = \frac{E_I y - p}{\rho \mathrm{Var}_I(y)} = \frac{\bar{y} h_y + \theta h_\theta - p(h_y + h_\theta)}{\rho}$$

  and

$$q_U = \frac{E_U y - p}{\rho \mathrm{Var}_U(y)} = \frac{\bar{y} h_y + z h_p - p(h_y + h_p)}{\rho}$$

- Market clearing condition is

$$\lambda \frac{\bar{y} h_y + \theta h_\theta - p(h_y + h_\theta)}{\rho} + (Q - \lambda) \frac{\bar{y} h_y + z h_p - p(h_y + h_p)}{\rho} = x$$

# Market clearing and equilibrium prices

- This confirms the conjecture that a linear price equilibrium exists
  - Because this is linear in $p$
- Solve by equating coefficients
- Notice one thing about how the price enters this equation
  - Standard price effects on both informed and uninformed agents
  - $z$ increasing in p (as long as $\gamma > 0$): higher price indicates that $\theta$ was high, which increases the demand of the uninformed

# Market clearing and equilibrium prices

- Solution

$$\alpha = \bar{y} - \frac{\rho \bar{x}}{\lambda(h_y + h_\theta) + (Q - x)(h_y + h_p)}$$
$$\beta = -\frac{\rho}{\lambda h_\theta}\gamma$$
$$\gamma = \frac{\lambda h_\theta + (Q - \lambda) h_p}{\lambda(h_y + h_\theta) + (Q - \lambda)(h_y + h_p)}$$

where

$$h_p = \frac{1}{\left(\frac{\rho}{\lambda h_\theta}\right)^2 \frac{1}{h_x} + \frac{1}{h_\theta}}$$

# Making $\lambda$ endogenous

- Now suppose that traders choose whether or not to become informed
- There is a cost $c$ of observing $\theta$
- Will traders be willing to pay that cost?
- What is the benefit of information?
  - You know to what extent prices are due to information or supply?
  - Choose portfolio better
- What happens if more people become informed?
  - $\gamma$ increases
  - Prices become better aligned with $\theta$
  - The informational advantage of observing $\theta$ is reduced
  - Strategic substitutability in the choice of information

# Making $\lambda$ endogenous

- Solve for $\lambda$ by looking for what values would make the informativeness of the price such that everyone is indifferent between getting the information and not getting it

$$\lambda = \frac{\rho}{h_\theta \sqrt{h_x \left[ \frac{1}{(h_u + h_x)\exp(-2\rho c) - h_u} - \frac{1}{h_\theta} \right]}}$$

- If this equation has no solution, then $\lambda = 0$: no one gets informed.
- If this equation has $\lambda > Q$, then everyone gets informed
- Otherwise, proportion informed is interior

# Informativeness of price system

- How much do you learn from observing prices?
- One way to measure it is with the $R^2$ of a regression of $\theta$ on $p$ (or equivalently on $z$)
- Recall that

$$z = \frac{\beta(x - \bar{x})}{\gamma} + \theta$$

- Therefore,

$$\begin{aligned} R^2 &= \frac{Cov(z,\theta)}{\mathrm{Var}\,(z)\mathrm{Var}\,(\theta)} \\ &= \frac{\mathrm{Var}\,(\theta)^2}{\left(\left(\frac{\beta}{\gamma}\right)^2 \mathrm{Var}\,(x) + \mathrm{Var}\,(\theta)\right)\mathrm{Var}\,(\theta)} \\ &= \frac{1}{\left(\frac{\rho}{\lambda h_\theta}\right)\frac{1}{h_x(\frac{1}{h_\theta}+\frac{1}{h_u})} + 1} \end{aligned} \tag{1}$$

# Informativeness of price system

- Informativeness is increasing in $\lambda$
- Informativeness is increasing in $h_\theta$
- Informativeness is decreasing in $h_x$
- Now take into account the endogenous $\lambda$.

$$\left(\frac{\rho}{\lambda h_\theta}\right)^2 = h_x \left[\frac{1}{(h_u + h_\theta)\exp(-2\rho c) - h_u} - \frac{1}{h_\theta}\right]$$

So the overall level of informativeness is

$$R^2 = \frac{1}{\sqrt{\left[\frac{1}{(h_u+h_\theta)\exp(-2\rho c)-h_u} - \frac{1}{h_\theta}\right]\frac{1}{\frac{1}{h_\theta}+\frac{1}{h_u}} + 1}}$$

# Informativeness of price system

- Interestingly: $h_x$ cancels out
  - More supply noise $\Rightarrow \begin{cases} \text{lower } R^2 \\ \text{higher } \lambda \Rightarrow \text{higher } R^2 \end{cases} \Rightarrow$ zero net effect
- lower $c \Rightarrow$ higher $R^2$
- lower $\rho \Rightarrow$ higher $R^2$
- As $h_\theta \longrightarrow \infty \Rightarrow \begin{cases} \lambda \longrightarrow 0 \\ R^2 \longrightarrow 1 \end{cases}$

# Nonexistence

- If
  - $h_x = \infty$
  - and

$$\exp[\rho c]\sqrt{\frac{h_y}{h_y + h_\theta}} - 1 < 0 \tag{2}$$

  then there is no equilibrium
- Prices are perfectly informative for any positive $\lambda$
  - Because there is no noise, prices perfectly reveal $\theta$
- If prices are perfectly informative, then $\lambda = 0$
- But if condition (2) holds, when no one gets informed, then the utility of an agent who did get informed would be higher than if he remained uninformed $\Rightarrow$ Contradiction!
- Noise traders necessary for model to be well defined
- Implications for efficient market hypothesis
- Hayek's argument for markets

# Some recent work based on this model

- Breon-Drish (2015). If you relax the assumption that everything has a Normal distribution, the model can lead to weird results, so everything about this model is not robust.
- Albagli et al. (2015). Add an investment decision and study the feedback between prices and investment decisions. Do everything with $\{0, 1\}$ portfolio choice and risk neutrality instead of continuous portfolio choices and CARA.
- Kurlat and Veldkamp (2015). What if the cost of the signal comes from a ratings agency that is selling the signal? When will there be a market for the signals? When will the issuer of securities pay for the signals himself? Implications for investment and welfare.
- Dávila and Parlatore (2017). Instead of noise traders, they have random hedging needs. What happens to price informativeness if trading costs rise? In a benchmark, nothing: trading costs dampen informational trading and noise in the same proportion.

# Morris and Shin (1998)

# Setup

- Utility of agent $i$

$$u_i = U(k_i, K, \theta)$$

  where

  - $k_i$: action of player $i$
  - $\theta$: exogenous random variable
  - $K = \int k_i di$: average action in the population. We could also assume that some other moment like the variance matters.

- Information:

  - Player $i$ observe signal $\omega_i$ drawn from distribution $F(\cdot \mid \theta, K, z)$

- Distribution of signals depend on

  - True fundamental $\theta$
  - (perhaps)Average action
  - (perhaps) some random variable that affects distribution of signals but not payoffs

# Examples

- Example: investment with externalities:
  - Productivity is weighted average of fundamental and other's investment
  - profit are

$$u_i = (\alpha\theta + (1-\alpha)K)\, k_i - \frac{c}{2} k_i^2$$

- Example: speculative attacks
  - $k_i \in \{0,1\}$: binary indicator of whether player $i$ shorts currency
  - $\theta$: level of reserves at the central bank
  - $K$: number of speculators who short the currency
  - Payoff for player $i$

$$U(k_i, K, \theta) = \underbrace{\mathbb{1}(k_i = 1)}_{\text{if i attacks}} [\mathbb{1}(K > \theta) - c]$$

    with $c \in (0,1)$
  - Application: bank runs, debt rollover, riots

# Equilibrium Concept

- Definition: Rational Expectation Equilibrium
  1. Strategy: $k^* : \mathbb{R}^n \longrightarrow \mathbb{R}$
  2. Mapping $K^* : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ such that
     (a) Best response:
     $$k^*(\omega) \in \arg\max_k \mathbb{E}\left[U(k, K^*(\theta, z), \theta) \mid \omega\right] \quad \forall \omega$$
     (b) Consistency:
     $$K^*(\theta, z) = \int k^*(\omega) dF(\omega \mid \theta, z)$$

# Complete information benchmark

- Best response

$$g(K, \theta) \equiv \arg\max_{k_i} U(k_i, K, \theta)$$

- Equilibrium: $K(\theta)$ solves

$$K(\theta) = g(K(\theta), \theta) \quad \forall \theta$$

- **Example**: investment-with-externalities

$$\max_{k} \left(\alpha\theta + (1-\alpha)K\right) k_i - \frac{c}{2} k_i^2$$

  - FOC: $\alpha\theta + (1-\alpha)K - ck_i = 0$
  - Fixed point

$$K = \frac{\alpha\theta + (1-\alpha)K}{c}$$

# Equilibrium

- Definition: $k^*(x,z)$ and aggregate outcome $K^*(\theta,z)$ such that
  (a) Best response
  $$k^*(x,z) \in \arg\max_k \mathbb{E}[U(k, K^*(\theta,z), \theta) \mid x, z] \quad \forall x, z$$
  (b) Consistency
  $$K^*(\theta,z) = \int k^*(x,z)\sqrt{h_x}\phi\left(\sqrt{h_x}(x_i-\theta)\right) dx$$
- Start with monotone strategies
  $$k^*(x,z) = \mathbb{1}(x \leq x^*(z))$$
  - attack only if the private signal says the regime is weak
  - Implies $K^*(\theta,z)$ decreasing in $\theta$
  - Implies there exists $\theta^*(z)$ such that
  $$K^*(\theta,z) > \theta \Leftrightarrow \theta \leq \theta^*(z)$$

# Equilibrium

- Consistency
  - Mass who attack
$$K(\theta, z) = \Pr(x_i < x^*(z) \mid \theta)$$
  - Recall that $x_i \mid \theta \sim N(\theta, \frac{1}{h_x})$ so
$$\Pr(x_i < x^*(z) \mid \theta) = \Phi(\sqrt{h_x}(x^*(z) - \theta)$$
$$K(\theta, z) = \Phi\left(\sqrt{h_x}(x^*(z) - \theta)\right)$$
  - The condition for the regime to fall is
$$K(\theta, z) > \theta$$
  - Since LHS is decreasing in $\theta$, unique $\theta^*(z)$ such that regime fails iff $\theta < \theta^*(z)$
$$\Phi\left(\sqrt{h_x}(x^*(z) - \theta)\right) = \theta^*(z)$$

# Equilibrium

- Best response
  - Player $i$'s best response is $k = 1$ if
    $$\Pr(K(\theta, z) \mid x_i, z) > c$$
  - Compute $\Pr(K(\theta, z) \mid x_i, z)$
    - Player's posterior is
      $$\theta \mid z, x_i \sim N\left(\frac{h\mu + h_z z + h_x x}{h + h_z + h_x}, \frac{1}{h + h_z + h_x}\right)$$
    - take $h \to 0$ if the prior and public signal are isomorphic, therefore
      $$\begin{aligned}\Pr(K(\theta, z) \mid x_i, z) =& \Pr(\theta < \theta^*(z) \mid x_i, z) \\ =& \Phi(\sqrt{h_x + h_z}\left(\theta^*(z) - \frac{h_z z + h_x x_i}{h_z + h_x}\right)\end{aligned}$$
    - Threshold $\theta^*(z)$ satisfies
      $$\Phi\left(\sqrt{h_x + h_z}\left(\theta^*(z) - \frac{h_z z + h_x x_i}{h_z + h_x}\right)\right) = c$$

# Equilibrium

- Fixed point
  - Solve (3) for $x^*(z)$, replace in (4) and rearrange

$$\underbrace{\frac{h_z}{\sqrt{h_x}}(z-\theta^*(z)) + \Phi^{-1}(\theta^*(z))}_{\equiv G(\theta^*(z),z)} = \sqrt{1+\frac{h_z}{h_x}}\Phi^{-1}(1-c)$$

  - Equilibrium is

$$G(\theta^*(z),z) = \sqrt{1+\frac{h_z}{h_x}}\Phi^{-1}(1-c)$$

# Equilibrium

- Study the function $G$

$$G(\theta, z) = \frac{h_z}{\sqrt{h_x}}(z - \theta) + \Phi^{-1}(\theta)$$

  - Derivative

$$\frac{\partial G}{\partial \theta} = -\frac{h_z}{\sqrt{h_x}} + \frac{1}{\phi(\Phi^{-1}(\theta))} \geq -\frac{h_z}{\sqrt{h_x}} + \sqrt{2\pi}$$

  - Therefore, if

$$-\frac{h_z}{\sqrt{h_x}} + \sqrt{2\pi} \geq 0$$

  - $G$ is increasing everywhere $\Rightarrow$ unique equilibrium
  - otherwise $\exists$ $z$ such that there are multiple equilibria
  - Relative precision of private v.s. public information

# Appendix!

# Moment generating function

- Let $X$ be a random variable following a normal distribution

$$X \sim N(\mu, \sigma^2)$$

- The moment-generating function is defined as

$$M_X(t) = \mathbb{E}[e^{tX}]$$

## Theorem

*Let $X$ be a random variable following normal distribution $X \sim N(\mu, \sigma^2)$, then, the moment-generating function of $X$ is*

$$M_X(t) = \exp\left[\mu t + \frac{1}{2}\sigma^2 t^2\right]$$

Back

# Certainty Equivalence of CARA-Normal

### Proposition

*Suppose* $U(w) = -\exp(-\rho w)$*, and* $w \sim N(\bar{w}, \sigma_w^2)$*, then*

$$\begin{aligned} \mathbb{E}\left[U(w)\right] &= -\exp\left[-\rho\left(\bar{w} - \frac{1}{2}\rho\sigma_w^2\right)\right] \\ &= U\left[\bar{w} - \frac{\rho}{2}\sigma_w^2\right] \end{aligned}$$

# Formulas for Bayesian updating with Normal signals

- Start with a Normal prior and a single signal

$$\theta \sim N(\mu, \sigma^2)$$
$$x = \theta + \varepsilon$$
$$\varepsilon \sim N(0, \sigma_\varepsilon^2)$$

- This implies $x \sim N(\mu, \sigma^2 + \sigma_\varepsilon^2)$
- Explicit densities are

$$f_\theta(\theta) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(\theta-\mu)^2}{2\sigma^2}}$$
$$f_x(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2(\sigma^2+\sigma_\varepsilon^2)}}$$
$$f_{x|\theta}(\theta, x) = \frac{1}{\sqrt{2\pi}\sigma_\varepsilon} e^{-\frac{(x-\theta)^2}{2\sigma_\varepsilon^2}}$$

# Formulas for Bayesian updating with Normal signals

- Using Bayes' rule, the conditional density of $\theta$ given $x$ is

$$
\begin{aligned}
f_{\theta|x}(\theta, x) &= \frac{f_{x|\theta}(\theta, x) f_\theta(\theta)}{f_x(x)} \\
&= \frac{\frac{1}{\sqrt{2\pi}\sigma_\varepsilon} e^{-\frac{(x-\theta)^2}{2\sigma_\varepsilon^2}} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(\theta-\mu)^2}{2\sigma^2}}}{\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2(\sigma^2+\sigma_\varepsilon^2)}}} \\
&= \frac{1}{\sqrt{2\pi}} \frac{\sqrt{\sigma^2 + \sigma_\varepsilon^2}}{\sigma_\varepsilon \sigma} e^{-\frac{1}{2}\left[\frac{(x-\theta)^2}{\sigma_\varepsilon^2} + \frac{(\theta-\mu)^2}{\sigma^2} + \frac{(x-\mu)^2}{\sigma^2+\sigma_\varepsilon^2}\right]} \\
&= \frac{1}{\sqrt{2\pi}} \sqrt{\frac{1}{\sigma_\varepsilon^2 + \frac{1}{\sigma^2}}} e^{-\frac{1}{2}\left[\theta - \frac{\sigma_\varepsilon^2 \mu + \sigma^2 x}{\sigma_\varepsilon^2 + \sigma^2}\right]^2 \left(\frac{1}{\sigma_\varepsilon^2 + \frac{1}{\sigma^2}}\right)}
\end{aligned}
$$

which follows $N\left(\theta - \frac{\sigma_\varepsilon^2 \mu + \sigma^2 x}{\sigma_\varepsilon^2 + \sigma^2}, \frac{1}{\sigma_\varepsilon^2 + \frac{1}{\sigma^2}}\right)$ Back