一、1. 最优化问题：

$$\max E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, c_{t-1}, n_t)$$

$$\text{s.t.} \quad c_t + k_{t+1} - (1-\delta)k_t \le A_t k_t^{\alpha} n_t^{1-\alpha}$$

拉格朗日函数：

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \log(c_t - \eta c_{t-1}) - a \frac{n_t^{1+\gamma}}{1+\gamma} + \lambda_t \left( A_t k_t^{\alpha} n_t^{1-\alpha} - c_t - k_{t+1} + (1-\delta)k_t \right) \right\}$$

一阶条件：

$$c_t: \quad \frac{\partial \mathcal{L}}{\partial c_t} = \frac{\beta^t}{c_t - \eta c_{t-1}} - \lambda_t \beta^t + \frac{-\eta \beta^{t+1}}{c_{t+1} - \eta c_t} = 0$$

$$k_{t+1}: \quad \frac{\partial \mathcal{L}}{\partial k_{t+1}} = \beta^t(-\lambda_t) + \beta^{t+1} \cdot \lambda_{t+1} \left( \alpha A_{t+1} k_{t+1}^{\alpha-1} n_{t+1}^{1-\alpha} + (1-\delta) \right) = 0$$

$$n_t: \quad \frac{\partial \mathcal{L}}{\partial n_t} = \beta^t \left\{ (-a) n_t^{\gamma} + \lambda_t \cdot (1-\alpha) A_t k_t^{\alpha} n_t^{-\alpha} \right\} = 0$$

$$\lambda_t: \quad \frac{\partial \mathcal{L}}{\partial \lambda_t} = A_t k_t^{\alpha} n_t^{1-\alpha} - c_t - k_{t+1} + (1-\delta)k_t = 0$$

整理后得，

$$c_t: \quad \frac{1}{c_t - \eta c_{t-1}} - \frac{\eta \beta}{c_{t+1} - \eta c_t} = \lambda_t$$

$$k_{t+1}: \quad \beta \lambda_{t+1} \left( \alpha A_{t+1} k_{t+1}^{\alpha-1} n_{t+1}^{1-\alpha} + 1 - \delta \right) = \lambda_t$$

$$n_t: \quad \lambda_t (1-\alpha) A_t k_t^{\alpha} n_t^{-\alpha} = a n_t^{\gamma}.$$

$$\lambda_t: \quad c_t + k_{t+1} - (1-\delta)k_t = A_t k_t^{\alpha} n_t^{1-\alpha}.$$

2. 求稳态：

$$C:\quad \frac{1}{c-\eta c}-\frac{\eta\beta}{c-\eta c}=\lambda \quad\Rightarrow\quad \frac{1-\eta\beta}{c(1-\eta)}=\lambda$$

$$k:\quad \beta\lambda(\alpha\underbrace{k^{\alpha-1}n^{1-\alpha}}_{y/k}+1-\delta)=\lambda \quad\Rightarrow\quad \frac{k}{y}=\frac{\alpha\beta}{1-\beta(1-\delta)}$$

$$n:\quad \lambda(1-\alpha)\underbrace{Ak^{\alpha}n^{-\alpha}}_{y/n}=an^{\gamma} \quad\Rightarrow\quad \frac{1-\eta\beta}{c(1-\eta)}(1-\alpha)\cdot\frac{y}{n}=an^{\gamma}.$$

$$\lambda:\quad c+k-(1-\delta)k=\underbrace{k^{\alpha}n^{1-\alpha}}_{y} \quad\Rightarrow\quad \frac{c}{y}=1-\delta\frac{k}{y}=1-\frac{\alpha\beta\delta}{1-\beta(1-\delta)}$$

由 $y=k^{\alpha}n^{1-\alpha}\ \Rightarrow\ 1=\left(\frac{k}{y}\right)^{\alpha}\left(\frac{n}{y}\right)^{1-\alpha}\ \Rightarrow\ \frac{n}{y}=\left(\frac{\alpha\beta}{1-\beta(1-\delta)}\right)^{\frac{\alpha}{\alpha-1}}.$

由 $\frac{1-\eta\beta}{c(1-\eta)}(1-\alpha)\frac{y}{n}=an^{\gamma}$ 得

$$n=\left[\frac{(1-\eta\beta)(1-\alpha)}{a(1-\eta)}\cdot\frac{y}{c}\right]^{\frac{1}{1+\gamma}}$$

代入 $\frac{c}{y}=1-\frac{\alpha\beta\delta}{1-\beta(1-\delta)}$ 可得 $n$，

由 $\frac{n}{y}=\left(\frac{\alpha\beta}{1-\beta(1-\delta)}\right)^{\frac{\alpha}{\alpha-1}}$ 可得 $y$.

由 $\frac{c}{y}=1-\frac{\alpha\beta\delta}{1-\beta(1-\delta)}$ 可得 $c$，

由 $\frac{k}{y}=\frac{\alpha\cdot\beta}{1-\beta(1-\delta)}$ 可得 $k$.

二、1. 家庭的预算约束：

$$c_t + s_t = w_t l_t + r_t k_t$$

其中 $s_t = i_t = k_{t+1} - (1-\delta) k_t$

$$\Rightarrow \quad c_t + k_{t+1} - (1-\delta) k_t = w_t l_t + r_t k_t$$

2、拉格朗日函数：

$$\mathcal{L} = E_t \sum_{t=0}^{\infty} \beta^t \{ \gamma \ln c_t + (1-\gamma) \ln(1-l_t) + \lambda_t ( w_t l_t + r_t k_t - c_t - k_{t+1} + (1-\delta) k_t ) \}.$$

3. 一阶条件：

$$c_t: \frac{\partial \mathcal{L}}{\partial c_t} = \beta^t \{ \frac{\gamma}{c_t} - \lambda_t \} = 0$$

$$k_{t+1}: \frac{\partial \mathcal{L}}{\partial k_{t+1}} = \beta^t (-\lambda_t) + \beta^{t+1} \lambda_{t+1} ( r_{t+1} + 1 - \delta ) = 0$$

$$l_t: \frac{\partial \mathcal{L}}{\partial l_t} = \beta^t \{ -\frac{1-\gamma}{1-l_t} + \lambda_t w_t \} = 0$$

$$\lambda_t: \frac{\partial \mathcal{L}}{\partial \lambda_t} = \beta^t \{ w_t l_t + r_t k_t - c_t - k_{t+1} + (1-\delta) k_t \} = 0$$

可进一步化简 ……

4. 厂商最优化问题：

$$\max \ a_t k_t^{\alpha} l_t^{1-\alpha} - w_t l_t - r_t k_t$$

一阶条件 $\quad w_t = (1-\alpha) a_t k_t^{\alpha} l_t^{-\alpha}$

$$r_t = \alpha a_t k_t^{\alpha-1} l_t^{1-\alpha}.$$

7. 控制变量：$c_t, l_t, y_t, r_t, w_t$

状态变量：$k_t, a_t$.

三、 1. 一阶条件：

$$E_t \sum_{s=0}^{\infty} \gamma^s \Lambda_{t+s} \left[ \frac{P_{it}}{P_{t+s}} - \frac{\varepsilon}{\varepsilon-1} mc_{it+s} \right] Y_{it+s} = 0$$

由一阶条件得，

2. $$P_{it}^* = \frac{\varepsilon}{\varepsilon-1} \frac{E_t \sum_{s=0}^{\infty} \gamma^s \Lambda_{t+s} mc_{it+s} Y_{it+s}}{E_t \sum_{s=0}^{\infty} \gamma^s \Lambda_{t+s} (Y_{it+s}/P_{t+s})} = \frac{\varepsilon}{\varepsilon-1} \frac{X_{1t}}{X_{2t}}$$

$$X_{1t} = E_t \sum_{s=0}^{\infty} \gamma^s \Lambda_{t+s} mc_{it+s} Y_{it+s}$$

$$= mc_{it+s} Y_{it+s} + E_t \sum_{s=1}^{\infty} \gamma^s \Lambda_{t+s} mc_{it+s} Y_{it+s}$$

$$\cancel{= mc_{it+s} Y_{it+s} + E_t \sum_{s=0}^{\infty} \gamma^{s+1} \Lambda_{t+1+s} mc_{it+1+s} Y_{it+1+s}}$$

~~由于 $\Lambda_{t+1+s}$ =~~

$$= mc_{it+s} Y_{it+s} + E_t \sum_{s=1}^{\infty} \gamma^s \beta^s \frac{C_t}{C_{t+s}} mc_{it+s} Y_{it+s}$$

$$= mc_{it+s} Y_{it+s} + E_t \sum_{s=0}^{\infty} \gamma^{s+1} \beta^{s+1} \frac{C_t}{C_{t+1+s}} mc_{it+1+s} Y_{it+1+s}$$

$$= mc_{it+s} Y_{it+s} + E_t (\gamma\beta) \sum_{s=0}^{\infty} \cdot \gamma^s \beta^s \cdot \frac{C_t}{C_{t+1}} \cdot \frac{C_{t+1}}{C_{t+1+s}} mc_{it+1+s} Y_{it+1+s}$$

$$= mc_{it+s} Y_{it+s} + E_t (\gamma\beta) \cdot \frac{C_t}{C_{t+1}} \cdot \underbrace{\sum_{s=0}^{\infty} \gamma^s \beta^s \frac{C_{t+1}}{C_{t+1+s}} mc_{it+1+s} Y_{it+1+s}}_{X_{1t+1}}$$

$$= mc_{it+s} Y_{it+s} + \gamma\beta E_t \frac{C_t}{C_{t+1}} X_{1t+1}$$

同理， $X_{2t} = \frac{Y_{it}}{P_t} + \gamma\beta E_t \frac{C_t}{C_{t+1}} X_{2t+1}$