MICROECONOMIC THEORY II

Bingyong Zheng

Email: by.zheng@mail.shufe.edu.cn

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What is a game

- A game is a formal representation of a situation in which a number of individuals interact in a setting of strategic interdependence.
- To describe a game, we need to know four things:
  - ➢ Players: who is involved (playing in the game)?
  - ➢ Rules: How the game is played?
  - ➢ Outcomes:
  - ➢ Payoffs:

Assumptions: rationality

- In its mildest form, rationality implies that every player is motivated by maximizing his own payoff.
- In a stricter sense, it implies that every player always maximizes his utility, thus being able to perfectly calculate the probabilistic result of every action.

Strategic-form game

- It is used to model situations in which players choose strategies without knowing the strategy choices of the other players. Also known as normal form games.
- A strategic form game is composed of
  1. Set of players: N
  2. A set of actions: A_i for each player i
  3. A payoff function: $u_i : A \rightarrow R$ for each player i
- An outcome $a = (a_1, \cdots, a_n)$ is a collection of actions, one for each player
  - ➢ Also known as an action profile or strategy profile

The capacity expansion game

- Strategic-form

Toyota

		Build	Do not build
Honda	Build	16, 16	20, 15
	Do not build	15, 20	18, 18

- Players: Toyota, Honda.
- Rules: Two firms simultaneously choose to expand or not.
  Strategies for each firm: Build, Do not build.
- 4 outcomes: (Build, Build), (Build, Do not build), (Do not build, Build), (Do not build, Do not build).
- Payoffs:
- Nash equilibrium of this game: (Build, Build).

A subjective expected-utility approach interpretation of a game

- Each player has a subjective probability distribution over all states of the world–more precisely, the probabilities that her opponents playing s_{-i} for all $s_{-i} \in \times_{j \neq i} S_j$.
- Each player acts as an expected utility maximizer, choosing a strategy that maximizes her expected payoff in the game given the probability distribution over the strategies of her opponents. This is common knowledge.
- The concept of Nash equilibrium imposes a further restriction, player's belief is consistent with the actual play of her opponents.
- In view of this interpretation, each player in a game holds the belief $p^i(s_{-i})$, and choose s_i such that:

$$s_i^* \in \arg\max_{s_i \in S_i} \sum_{s_{-i}} p^i(s_{-i}) u_i(s_i, s_{-i}).$$

Interpretation continued

- The above interpretation of subjective expected utility-maximization provides a decision theoretical foundation to the traditional definition of Nash equilibrium that each player plays optimally given the other players' equilibrium strategies.
- An important feature of the subjective expected utility approach is that it does not require randomization on the part of the players.
- Recall that the traditional interpretation of mixed strategies that assumes players explicitly randomize. The probabilistic nature of strategies now reflects the uncertainty of other players about a player's choice.
  Thinking about the traditional Chinese "Scissor-rock-cloth" game.

Main takeaway from this approach

- Belief is the KEY to how players will play the game;
- Belief will also be the most important element in solving the game, the equilibrium;
- Implication: savvy players can take advantage of their opponents by tricking them into believing something that is not true, “bluffing” in poker game, etc.

Pure strategies

- Definition: A pure strategy s_i of player i specifies the actions that a player will take under any conceivable circumstances that the player might face.
- A strategy is a complete contingency plan that says what a player will do at each of her information sets if she is called on to play there
- A player's strategy may include plans for actions that her own strategy makes irrelevant.

Interpretation of strategies

- According to Rubinstein (1991) and Reny (1992), a player's strategy can be partitioned into two parts, a *plan* that describes a rational play for *i*, and a *prediction* about *i*'s future behavior should *i* deviates from his plan.
  - A *plan* for player *i* specifies a choice for player *i* only when he is called upon to move, and does not specify what he would do at an information set of his that can not be reached according to this plan.
  - In order that others are able to specify what they would do were *i* not to follow through his plan (something *i* must know in order to evaluate the soundness of this plan in the first place), it must provide others with a *prediction* about *i*'s future behavior should *i* deviate.

Interpretation of strategies continued

- Given the SEUM approach discussed before, one natural interpretation for the specification of choices at information sets that won't be reached given a player's strategy is that they are *beliefs* of his opponents about what he would do in case he does not follow his strategy, i.e., the information sets were reached.
- The belief of his opponents is important as their choices at those information sets are based on this belief.
- Furthermore, what the player's opponents would do at those information sets rationalize his choice at at an upstream information set.
- Hence, this definition of strategy is not so odd when you interpret it as the way a player determines his strategy.

Mixed strategies

- A mixed strategy $\sigma_i \in \Delta(S_i)$ specifies probabilities to two or more pure strategies.
- For example, the traditional Chinese game, rock, scissor and cloth.

player 2

		scissor	rock	cloth
Player 1	scissor	0,0	-1, 1	1,-1
	rock	1, -1	0,0	-1, 1
	cloth	-1,1	1,-1	0,0

Find mixed strategy NE of the game

- Let the mixed strategy equilibrium be

$$(p_1, p_2, 1-p_1-p_2; q_1, q_2, 1-q_1-q_2).$$

- Recall our interpretation of strategy:
  $\sigma_1 = (p_1, p_2, 1-p_1-p_2)$ is 1's strategy, but is 2's belief.
  Player 2 thinks 1 plays Scissor with prob. p_1, rock with p_2, and cloth with $1-p_1-p_1$.
- Given player 2's belief, expected payoff from Scissor:

$$u_2(\sigma_1, S) = 0 \cdot p_1 - p_2 + (1-p_1-p_1).$$

- 2's payoff from Rock

$$u_2(\sigma_1, R) = p_1 + 0 \cdot p_2 - (1-p_1-p_2).$$

Mixed NE (2)

- 2's payoff from Cloth

$$u_2(\sigma_1, C) = -p_1 + p_2 + 0 \cdot (1 - p_1 - p_2).$$

- A pure strategy is played if and only if it gives the highest expected payoff.
- So if all 3 pure strategies of player 2 are played with positive probability, then

$$u_2(\sigma_1, S) = u_2(\sigma_1, R) = u_2(\sigma_1, C).$$

- This gives

$$p_1 = p_2 = \frac{1}{3}.$$

  Player 2 must think 1's 3 pure strategies are equally likely to play S, R and C randomly!
- Similarly, we solve player 1's problem to get

$$q_1 = q_2 = \frac{1}{3}.$$

Dominant strategy

- Dominant strategy: A strategy s_i is dominant for player i if for all $s_{-i} \in S_{-i}$ and $s_i' \in S_i / s_i$

$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i}).$$

- A strategy is a dominant strategy for a player if it is better than other strategies, no matter what the others will choose.
- In **any Nash equilibrium**, players who have a dominant strategy play the dominant strategy. Thus, it is easy to find the Nash equilibrium of games in which some of the players have a dominant strategy.
- Dominant strategies are rarity rather than norm. There is no dominant strategies in most interesting games.

Dominant strategy example

- Example: Prisoner's dilemma game

Prisoner 2

Prisoner 1		Confess	Not confess
	Confess	-5, -5	0, -10
	Not confess	-10, 0	-1, -1

Dominant strategy example

- Example: Prisoner's dilemma game

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		Confess	Not confess
Prisoner 1	Confess	-5, -5	0, -10
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- "confess" is the dominant strategy for both prisoners.

Dominant strategy example

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- "confess" is the dominant strategy for both prisoners.

Dominated strategy

- A pure strategy $s_i \in S_i$ is weakly dominated if there is another strategy $s_i' \in S_i$ such that for all $s_{-i} \in S_{-i}$,

$$u_i(s_i', s_{-i}) \geq u_i(s_i, s_{-i}),$$

with strict inequality for some s_{-i}.

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- A strategy s_i is strictly dominated for player i if there is another strategy $s_i' \in S_i$ such that for all $s_{-i} \in S_{-i}$

$$u_i\left(s_i', s_{-i}\right) > u_i\left(s_i, s_{-i}\right).$$

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- A strategy is strictly **dominated** when the player has another strategy that gives her a higher payoff no matter what the other player plays.

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- A strategy is strictly **dominated** when the player has another strategy that gives her a higher payoff no matter what the other player plays.
- A player will not play a strictly dominated strategy in Nash equilibrium.

Allowing mixed strategy

- A strategy σ_i is strictly dominated for player i if there is another strategy $\sigma_i' \in \Delta(S_i)/\sigma_i$ such that for all $\sigma_{-i} \in \Delta(S_{-i})$

$$u_i(\sigma_i, \sigma_{-i}) < u_i(\sigma_i', \sigma_{-i}).$$

Allowing mixed strategy

- A strategy σ_i is strictly dominated for player i if there is another strategy $\sigma_i' \in \Delta(S_i)/\sigma_i$ such that for all $\sigma_{-i} \in \Delta(S_{-i})$

$$u_i(\sigma_i, \sigma_{-i}) < u_i\left(\sigma_i', \sigma_{-i}\right).$$

- A pure strategy s_i is strictly dominated for player i if and only if there exists $\sigma_i \in \Delta(S_i)$ such that

$$u_i(s_i, s_{-i}) < u_i(\sigma_i, s_{-i})$$

for all s_{-i}.

Allowing mixed strategy

- A strategy σ_i is strictly dominated for player i if there is another strategy $\sigma_i' \in \Delta(S_i)/\sigma_i$ such that for all $\sigma_{-i} \in \Delta(S_{-i})$

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- A pure strategy s_i is strictly dominated for player i if and only if there exists $\sigma_i \in \Delta(S_i)$ such that

$$u_i(s_i, s_{-i}) < u_i(\sigma_i, s_{-i})$$

  for all s_{-i}.
- A mixed strategy that assigns positive probability to a pure strategy that is strictly dominated is also strictly dominated.

Dominated strategy example 1

- The game

Player 2

Player 1

	l	m	r
a	7, 5	2, 4	1, 2
b	2, 2	4, 5	0, 3
c	0, 4	7, 0	2, 1

Dominated strategy example 1

- The game

Player 2

Player 1		l	m	r
	a	7, 5	2, 4	1, 2
	b	2, 2	4, 5	0, 3
	c	0, 4	7, 0	2, 1

- While player 1's pure strategy b is not dominated by any pure strategy, it is dominated by a mixed strategy

$$\sigma_1 = \left(\frac{1}{2}a, 0b, \frac{1}{2}c\right).$$

Dominated strategy example 1

- The game

Player 2

Player 1		l	m	r
	a	7, 5	2, 4	1, 2
	b	2, 2	4, 5	0, 3
	c	0, 4	7, 0	2, 1

- While player 1's pure strategy b is not dominated by any pure strategy, it is dominated by a mixed strategy

$$\sigma_1 = (\frac{1}{2}a, 0b, \frac{1}{2}c).$$

- Player 2's r is also dominated by a mixed strategy.

Dominated strategy example 2

- Modifies capacity expansion game between Toyota and Honda

		Toyota		
		large	small	not build
Honda	large	0, 0	12, 8	18, 9
	small	8, 12	16, 16	20, 15
	not build	9, 18	15, 20	18, 18

Dominated strategy example 2

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		large	small	not build
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- Large is dominated by small.

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	large	0, 0	12, 8	18, 9
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- Large is dominated by small.
- Toyota knows that Honda will not play Large!

Dominated strategy example 2

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Honda		large	small	not build
	large	0, 0	12, 8	18, 9
	small	8, 12	16, 16	20, 15
	not build	9, 18	15, 20	18, 18

- Large is dominated by small.
- Toyota knows that Honda will not play Large!
- Honda also knows that Toyota will not play Large!

Dominated strategy example 2 (cont.)

Toyota

Honda		large	small	not build
	large	0, 0	12, 8	18, 9
	small	8, 12	16, 16	20, 15
	not build	9, 18	15, 20	18, 18

Dominated strategy example 2 (cont.)

	large	small	not build
large	0, 0	12, 8	18, 9
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Dominated strategy example 2 (cont.)

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Dominated strategy example 2 (cont.)

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Dominated strategy example 2 (cont.)

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Dominated strategy example 2 (cont.)

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Dominated strategy example 2 (cont.)

		Toyota	
		small	not build
Honda	small	16, 16	20, 15
	not build	15, 20	18, 18

Iterated Deletion of Strictly Dominated Strategies

- A rational player should never choose a strictly dominated strategies because there exists another strategies that is strictly better. a player has a dominant strategies, then all other strategies are dominated.
- If it is common knowledge that both players are rational, we can continue this process indefinitely.
- Now in situations where players do not have a chance to talk or where there is no history to rely on, Nash equilibrium may not be a good prediction. So in this case, we may prefer a solution concept that does not make strong assumptions about players knowing what each other is going to do.
- Note that Nash and IDSDS are based on different logic.
- IDSDS does not require that the players know that the equilibrium is going to be played, so it requires less coordination. However, common knowledge of rationality is itself a very strong assumption.

IDSDS application

- Consider the following game (Gibbons pp. 6) :

Player 2

Player 1		L	M	R
	U	1,0	1,2	0,1
	D	0,3	0,1	2,0

IDSDS application

- Consider the following game (Gibbons pp. 6) :

Player 2

Player 1		L	M	R
	U	1,0	1,2	0,1
	D	0,3	0,1	2,0

- Note that R is strictly dominated by M for Player 2.

IDSDS APPLICATION

- Consider the following game (Gibbons pp. 6) :

Player 2

Player 1

	L	M	R
U	1,0	1,2	0,1
D	0,3	0,1	2,0

- Note that R is strictly dominated by M for Player 2.
- Now, if Player 1 knows that Player 2 is rational, then Player 1 knows that Player 2 will never choose R. If R is eliminated, then D becomes dominated by U.
- Now, if Player 2 knows Player 1 knows that Player 2 is rational, then Player 2 knows that Player 1 will not choose D. In that case, Player 2 should choose M.

Another example

Player 2

Player 1

	L	M	R	U
A	(5, 6)	(2, 6)	(1, 5)	(0, 7)
B	(2, 10)	(1, 2)	(0, 10)	(-1, 1)
C	(4, 1)	(3, 4)	(1, 3)	(2, 0)
D	(0, 2)	(1, 3)	(3, 2)	(1, 1)

Another example

	L	M	R	U
A	(5, 6)	(2, 6)	(1, 5)	(0, 7)
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Another example

	L	M	R	U
A	(5, 6)	(2, 6)	(1, 5)	(0, 7)
C	(4, 1)	(3, 4)	(1, 3)	(2, 0)
D	(0, 2)	(1, 3)	(3, 2)	(1, 1)

Another example

	M	U
A	(2, 6)	(0, 7)
C	(3, 4)	(2, 0)
D	(1, 3)	(1, 1)

Another example

	M	U
C	(3, 4)*	(2, 0)

Nash equilibrium

- Definition: A strategy profile $s^* = (s_1^*, ..., s_n^*)$ is a **Nash Equilibrium** (NE) if for all $i \in N$ and for all $s_i' \in S_i$

$$U_i\left(s_i^*, s_{-i}^*\right) \geq U_i\left(s_i', s_{-i}^*\right).$$

- The above is the standard textbook definition of NE. According to the SEUM approach discussed above, this definition of NE includes two parts: one on strategy and one on beliefs
  1. Subjective expected-utility maximization: Each player holds the belief $p^i(s_{-i})$, and choose s_i such that:

$$s_i^* \in \arg\max_{s_i \in S_i} \sum_{s_{-i}} p^i(s_{-i}) u_i(s_i, s_{-i}).$$

  2. Beliefs coincides with opponents' equilibrium strategies:

$$\forall\ i, \quad p^i(s_{-i}) = \sigma_{-i}^*.$$

How to understand NE?

- The central concept of noncooperative game theory is Nash equilibrium. A Nash equilibrium is a profile of strategies such that for each player in the game, given the strategy chosen by the other players, the strategy is a best response for the player, that is, the strategy gives the player the highest payoff.
- Early interpretation of the concept of Nash equilibrium.
  - ➢ In most of the early literature the idea of equilibrium was that it said something about how players would play the game or about how a game theorist might recommend that they play the game.
  - ➢ However, this interpretation runs into trouble in many cases. For example, how do we interpret mixed strategy Nash equilibrium? How to motivate the refinements of Nash equilibrium?

Recent interpretation of NE

- Recently, there has been a shift to thinking of equilibria as representing not recommendations to players of how to play the game but rather the expectations of the others as to how a player will play.
- Further, if the players all have the same expectations about the play of the other players we could as well think of an outside observer having the same information about the players as they have about each other.

More on interpretation of NE

- While the first interpretation can be problematic in case of mixed strategy equilibrium, the second interpretation can accommodate mixed strategies without any trouble. In this scenario, the mixed strategy of a player does not represent a conscious randomization on the part of that player, but rather the uncertainty in the minds of the others as to how that player will act.
- Hence, the second interpretation of Nash equilibrium has become the preferred interpretation among game theorists.
- Thus the focus of the equilibrium analysis becomes, not the choices of the players, but the assessments of the players about the choices of the others.
- The basic consistency condition that we impose on the players' assessments is this: A player reasoning through the conclusions that others would draw from their assessments should not be led to revise his own assessment.

Find NE in two-player games

- Most games have finite and **ODD** number of NE.
- If both players have a dominant strategy, then playing dominant strategies is the unique Nash equilibrium in the game.
- If one player has a dominant strategy, this strategy will be this player's NE strategy. The other player's NE strategy is the best response to her opponent's dominant strategy.
- If players have dominated strategies, delete the dominated strategies from the game and work with a smaller game.
- NE must be a mutual best-response, that, given player 1 plays NE strategy, player 2 can not do better by playing some other strategies, similarly, given player 2 plays this strategy, player 1 can not do better by changing strategies. A NE is a strategy profile in which both players are playing the best-response given the other player's strategy.

Example: The game of chicken

- Strategic form

Jack

Tom		Swerve	Stay
	Swerve	0, 0	-10, 10
	Stay	10, -10	-100, -100

- Two pure strategy NE: (Swerve, Stay), (Stay, Swerve)
- One mixed strategy NE

$$((Swerve, Stay; 0.9, 0.1), (Swerve, Stay; 0.9, 0.1)).$$

Example 2 NE

- One pure strategy NE
$$(T, L).$$
- In addition, there exists two mixed NE.
- Let the mixed strategy NE be

$$(p, 1-p; q_1, q_2, 1-q_1-q_2).$$

Mixed NE of Ex. 2

- For player 1, given σ_2

$$u_1(T) = 3q_1 + 3(1 - q_1 - q_2),$$
$$u_1(B) = 3q_2.$$

- If mixed NE exists, it is necessary that $q_2 = \frac{1}{2}$!
- For player 2, given σ_1,

$$u_2(L) = 3p,$$
$$u_2(M) = 2,$$
$$u_2(R) = 3(1 - p).$$

- If $p = \frac{2}{3}$,

$$u_2(L) = u_2(M) = 2 > u_2(R)$$

- So one mixed NE:

$$\left(\frac{2}{3}, \frac{1}{3}; \frac{1}{2}, \frac{1}{2}, 0\right).$$

Mixed NE of Ex. 2 (cont.)

- If $p = \frac{1}{3}$,

$$u_2(L) < u_2(M) = 2 = u_2(R)$$

- Another mixed NE:

$$\left(\frac{1}{3}, \frac{2}{3}; 0, \frac{1}{2}, \frac{1}{2}\right).$$

- However, there is no NE in which both L and R are played with positive probabilities.
- If $u_2(L) = u_2(R)$, $p = \frac{1}{2}$, but

$$u_2(L) = u_2(R) = \frac{3}{2} < u_2(M) = 2.$$

- Given the belief 1 plays T and B with equal probabilities, player 2 should play M!

Example 3

- The game

Player 2

Player 1		A	B	C
	A	5, 8	15, 10	10, 5
	B	10, 15	20, 9	15, 0
	C	20, 20	10, 10	10, 8

Example 3 continued

- A is dominated by B; C is also dominated;
- After deleting A and C, B becomes dominated.
- Unique NE: (C, A)

Example of 3-player game

Player 2

		U	D
1	A	(1, 1,0)	(2, -2, 5)
	B	(1,-2, -1)	(0, 3, 1)

Player 3 plays L

Player 2

	U	D
A	(1, 1,-2)	(2, -2, 5)
B	(2, 2, -1)	(2, 3, 7)

Player 3 plays R

Example of 3-player game

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		U	D
1	A	(1, 1,0)	(2, -2, 5)
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Player 2

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Player 2

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Example of 3-player game

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Player 3 plays L

Player 2

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Example of 3-player game

Player 2

1		U	D
	A	(1, 1, 0)*	(2, -2, 5)
	B	(1,-2, -1)	(0, 3, 1)

Player 3 plays L

Player 2

	U	D
A	(1, 1, -2)	(2, -2, 5)
B	(2, 2, -1)	(2, 3, 7)*

Player 3 plays R

3-player game: example 2

1

	U	V	W
L	3, 0, 2	2, -1, 0	1, -2, 0
M	3, 2, 1	1, 4, -1	0, 0, -2
R	1, 1, 10	0, 2, 1	-2, 0, 3

Player 3 plays A

	U	V	W
L	2, 1, 1	3, 0, 0	2, -2, -1
M	5, 4, 2	1, 3, 4	3, 0, -2
R	1, 1, 1	0, 2, 0	-2, 0, 2

Player 3 plays B

EXAMPLE 2 (CONT.)

	U	V	W
L	2, 1, -1	3, 0, -1	2, -2, -3
M	5, 4, -1	1, 3, -2	3, 0, -4
R	1, 1, -10	0, 2, -1	-2, 0, -2

Player 3 plays C

3-PLAYER GAME: EXAMPLE 2

		U	V	W
	L	3, 0, 2	2, -1, 0	1, -2, 0
1	M	3, 2, 1	1, 4, -1	0, 0, -2
	R	1, 1, 10	0, 2, 1	-2, 0, 3

Player 3 plays A

	U	V	W
L	2, 1, 1	3, 0, 0	2, -2, -1
M	5, 4, 2	1, 3, 4	3, 0, -2
R	1, 1, 1	0, 2, 0	-2, 0, 2

Player 3 plays B

3-player game: example 2

		U	V	W
	L	3, 0, 2	2, -1, 0	1, -2, 0
1	M	3, 2, 1	1, 4, -1	0, 0, -2
	R	1, 1, 10	0, 2, 1	-2, 0, 3

Player 3 plays A

	U	V	W
L	2, 1, 1	3, 0, 0	2, -2, -1
M	5, 4, 2	1, 3, 4	3, 0, -2
R	1, 1, 1	0, 2, 0	-2, 0, 2

Player 3 plays B

3-player game: example 2

		U	V
	L	3, 0, 2	2, -1, 0
1	M	3, 2, 1	1, 4, -1

Player 3 plays A

	U	V
L	2, 1, 1	3, 0, 0
M	5, 4, 2	1, 3, 4

Player 3 plays B

3-player game: example 2

	U	V
L	3, 0, 2	2, -1, 0
1 M	3, 2, 1	1, 4, -1

Player 3 plays A

	U	V
L	2, 1, 1	3, 0, 0
M	5, 4, 2	1, 3, 4

Player 3 plays B

3-player game: example 2

		U	V
	L	3, 0, 2	2, -1, 0
1	M	3, 2, 1	1, 4, -1

Player 3 plays A

	U	V
L	2, 1, 1	3, 0, 0
M	5, 4, 2	1, 3, 4

Player 3 plays B

3-player game: example 2

		U	V
	L	3, 0, 2	2, -1, 0
1	M	3, 2, 1	1, 4, -1

Player 3 plays A

	U	V
L	2, 1, 1	3, 0, 0
M	5, 4, 2	1, 3, 4

Player 3 plays B

3-player game: example 2

		U	V
	L	3, 0, 2*	2, -1, 0
1	M	3, 2, 1	1, 4, -1

Player 3 plays A

	U	V
L	2, 1, 1	3, 0, 0
M	5, 4, 2*	1, 3, 4

Player 3 plays B

Strategic stability of NE

- Being a NE is a necessary condition for an obvious way to play the game, if an obvious way to play the game exists. But
  - ➢ Being NE is not sufficient for a strategy profile to be the obvious way to play a given game.

Strategic stability of NE

- Being a NE is a necessary condition for an obvious way to play the game, if an obvious way to play the game exists. But
  - ➢ Being NE is not sufficient for a strategy profile to be the obvious way to play a given game.
  - ➢ Not every game admits an obvious way to play the game
- Some questions to be answered:
  - ➢ How can we refine NE, the necessary condition to get the prediction of the game, an obvious way to play the game. NE can involve weakly dominated strategies, we should add to our necessary condition that the solution should be a NE in strategies that are undominated, even weakly
  - ➢ What are the means by which we are to identify "obvious way to play a game?"
  - ➢ What can one say about games that do not admit a "solution" When the game does not admit an "obvious way to play," looking at its NE can give precisely the wrong answer. The concept of NE is of no use when the game admits no "solution"

Solution concepts other than NE

- On occasion the requirement of Nash equilibrium can be too demanding at times.
- This leads to two less restrictive concepts:
  - ➢ Rationalizability;
  - ➢ correlated equilibrium.
- At other times, the requirement of NE is not strong enough to rule out multiple equilibria or implausible predictions. Two lines of research to address these difficulties:
  - ➢ Equilibrium selection: concerned with narrowing the prediction to a single prediction.
  - ➢ Refinement of NE: concerned with establishing necessary conditions for reasonable predictions.

Rationalizability

- A strategy σ_i is a best response for player i to her rivals' strategies σ_{-i} if

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i}).$$

  for all $\sigma_i' \in \Delta(S_i)$
- Strategy σ_i is never a best response if there is no σ_{-i} for which σ_i is a best response.
- The strategies in $\Delta(S_i)$ that survive iterated deletion removal of strategies that are never a best response are known as player i's rationalizable strategies.

Example

		Player 2			
		b_1	b_2	b_3	b_4
Player 1	a_1	0, 7	2, 5	7, 0	0, 1
	a_2	5, 2	3, 3	5, 2	0, 1
	a_3	7, 0	2, 5	0, 7	0, 1
	a_4	0, 0	0, -2	0, 0	10, -1

Example

		Player 2			
		b_1	b_2	b_3	b_4
	a_1	0, 7	2, 5	7, 0	0, 1
Player 1	a_2	5, 2	3, 3	5, 2	0, 1
	a_3	7, 0	2, 5	0, 7	0, 1
	a_4	0, 0	0, -2	0, 0	10, -1

- b_4 is never a best response for player 2!

Example

		Player 2		
		b_1	b_2	b_3
	a_1	0, 7	2, 5	7, 0
Player 1	a_2	5, 2	3, 3	5, 2
	a_3	7, 0	2, 5	0, 7
	a_4	0, 0	0, -2	0, 0

- a_4 is never a best response for player 2!

Example

		Player 2		
		b_1	b_2	b_3
Player 1	a_1	0, 7	2, 5	7, 0
	a_2	5, 2	3, 3	5, 2
	a_3	7, 0	2, 5	0, 7

- The set of rationalizable strategies:

$$\{a_1, a_2, a_3; b_1, b_2, b_3\}.$$

Compare IDSDS and Rationalizability

- For two-player games, rationalizable strategies are those remaining after the iterative deletion of strictly dominated strategies.
- For more than two player games, this is no longer true.
- A strictly dominated strategy is never a best response; but the reverse is not necessarily true for more than two-player game.

Example

- The game

	L	R
U	9	0
D	0	0

A

	L	R
U	0	9
D	9	0

B

	L	R
U	0	0
D	0	9

C

	L	R
U	6	0
D	0	6

D

Example

- The game

	L	R
U	9	0
D	0	0

A

	L	R
U	0	9
D	9	0

B

	L	R
U	0	0
D	0	9

C

	L	R
U	6	0
D	0	6

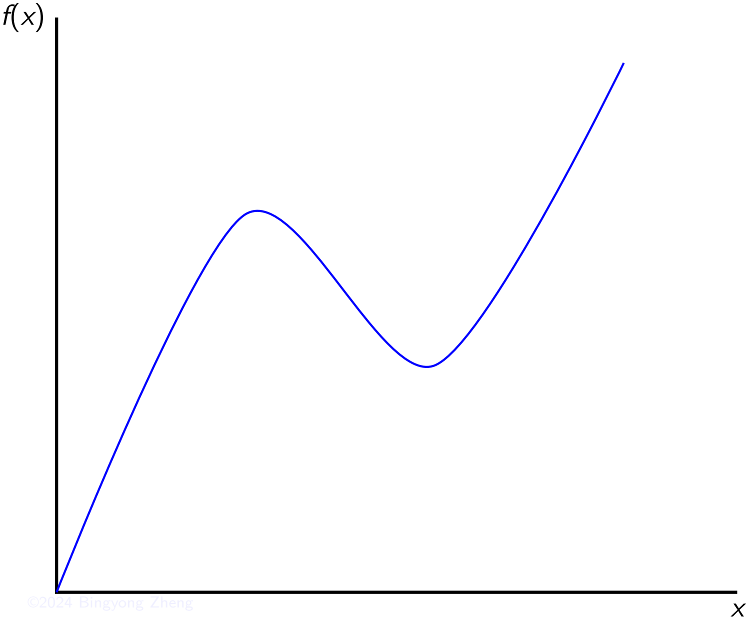
D

- In this example, D is not dominated, but never a best response for player 3.

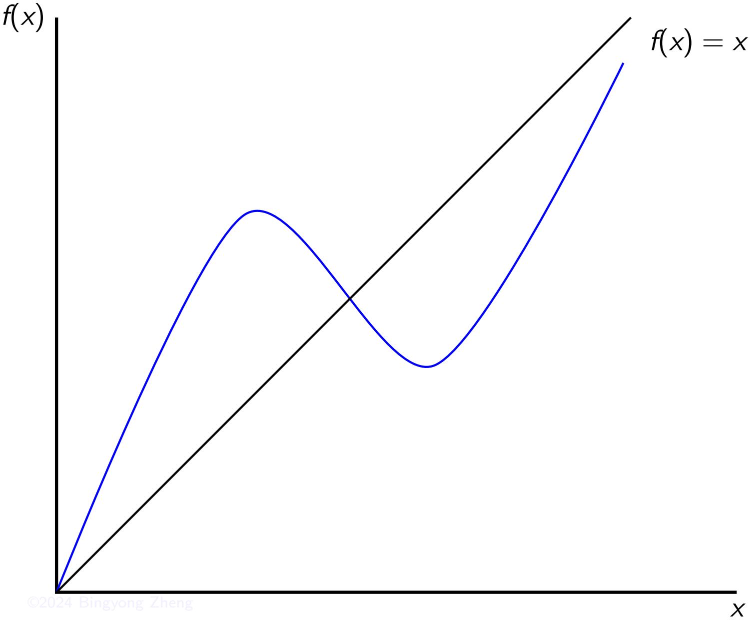
Existence of NE

- Theorem 1 (Theorem 7.2, Jehle and Reny). Every finite strategic form game has at least one NE.
- Main idea of the proof:
  - Use fixed point theorem. A continuous function $f(x)$ mapping from a convex, compact set to itself has a fixed point $x = f(x)$.
  - Two steps: construct such a continuous function; show the fixed point is NE.

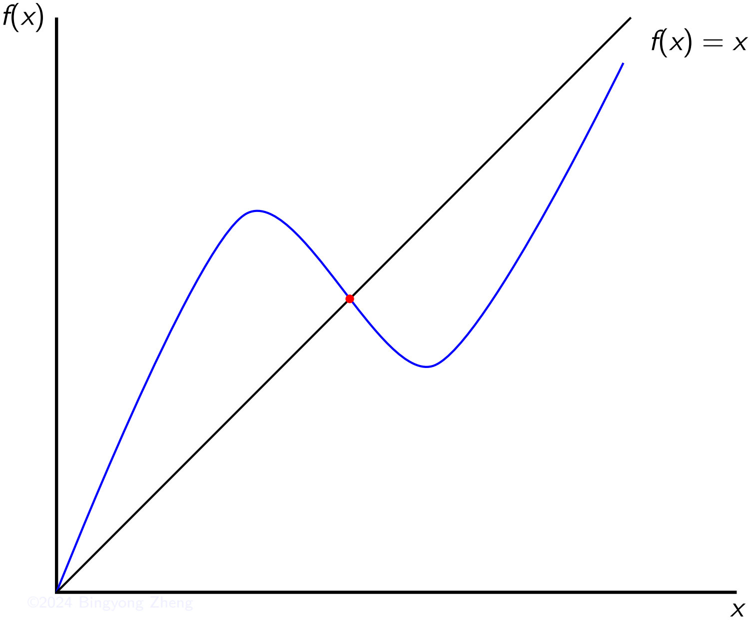
Fixed Point Theorem



Fixed Point Theorem



Fixed Point Theorem



Proof of Theorem 1

- Consider N-player game, each player has n pure strategies

$$S_i = \{1, \ldots, n\}.$$

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- Consider N-player game, each player has n pure strategies

$$S_i = \{1, \ldots, n\}.$$

- Player i's set of mixed strategy

$$M_i \equiv \{m_i = (m_{i1}, \ldots, m_{in}) | m_{ij} \in [0,1], \quad \sum_{j=1}^{n} m_{ij} = 1\}.$$

- M_i is convex and compact, so is $M = \prod_i M_i$.
- Define $f : M \to M$ as follows

$$f_{ij}(m) = \frac{m_{ij} + \max\{0, u_i(j, m_{-i}) - u_i(m)\}}{1 + \sum_{j'=1}^{n} \max\{0, u_i(j, m_{-i}) - u_i(m)\}}. \qquad \text{(Func)}$$

- For all i, j and for all m: $f_{ij} \in [0,1]$ and

$$\sum_{j=1}^{n} f_{ij}(m) = 1.$$

Proof of Theorem 1 (cont.)

- f is a continuous function.
- By Brower's Fixed Point Theorem, there exists $\widehat{m}$ such that

$$\widehat{m} = f(\widehat{m}). \tag{Fix}$$

- Next, we show $\widehat{m}$ is a NE.
- From (Func), we have

$$f_{ij}(\widehat{m}) + f_{ij}(\widehat{m}) \sum_{j'=1}^{n} \max\{0, u_i(j, \widehat{m}_{-i}) - u_i(\widehat{m})\} =$$
$$\widehat{m}_{ij} + \max\{0, u_i(j, \widehat{m}_{-i}) - u_i(\widehat{m})\}.$$

- Using the fact $\widehat{m}_{ij} = f_{ij}(\widehat{m})$:

$$\widehat{m}_{ij} \sum_{j'=1}^{n} \max\{0, u_i(j, \widehat{m}_{-i}) - u_i(\widehat{m})\} = \max\{0, u_i(j, \widehat{m}_{-i}) - u_i(\widehat{m})\}$$

Proof of Theorem 1 (cont.)

- Multiplies both sides by $u_i(j, \widehat{m}_{-i}) - u_i(\widehat{m})$ and sum for all j:

$$\sum_{j'=1}^{n} \max\{0, u_i(j, \widehat{m}_{-i}) - u_i(\widehat{m})\} \sum_{j=1}^{n} \widehat{m}_{ij}[u_i(j, \widehat{m}_{-i}) - u_i(\widehat{m})]$$

$$= \sum_{j=1}^{n} [u_i(j, \widehat{m}_{-i}) - u_i(\widehat{m})] \max\{0, u_i(j, \widehat{m}_{-i}) - u_i(\widehat{m})\}.$$

- By definition,

$$\sum_{j=1}^{n} \widehat{m}_{ij}[u_i(j, \widehat{m}_{-i}) - u_i(\widehat{m})] = u_i(\widehat{m}) - u_i(\widehat{m}) = 0.$$

- So we end up with

$$\sum_{j=1}^{n} [u_i(j, \widehat{m}_{-i}) - u_i(\widehat{m})] \max\{0, u_i(j, \widehat{m}_{-i}) - u_i(\widehat{m})\} = 0.$$

Proof of Theorem 1 (cont.)

- Since $\max\{0, u_i(j, \widehat{m}_{-i}) - u_i(\widehat{m})\} \geq 0$ for all j, we have:

$$\forall i, \ \forall \quad j : \quad u_i(j, \widehat{m}_{-i}) \leq u_i(\widehat{m}).$$

- For all i, playing mixed strategy $\widehat{m}_i$ is a best response against opponents' strategies $\widehat{m}_{-i}$!
- $\widehat{m}$ is a mixed strategy NE.

Theorem 2

- Theorem 2. NE exists if the strategy set of each player is a compact and convex subset of an Euclidean space and if the utility function of each player is continuous in the strategy profile and quasi-concave in one's own strategy.
- *Proof.*
  - ➢ Step 1: the maximizer

$$b_i(\sigma_{-i}) = arg \max_{\sigma_i \in \Delta(S_i)} u_i(\sigma_i, \sigma_{-i})$$

    is nonempty, convex-valued and upper hemicontinuous.
  - ➢ Step 2: by Kukutani's fixed point theorem, a non-empty, convex-valued upper hemicontinuous correspondence $b_i(\sigma_{-i})$ mapping from $\Delta(S)$ to itself, there must exists a fixed point.

NE may involve unreasonable beliefs

- The problem with NE is, that, many games have multiple equilibria. The natural question then arises, can we go any further and rule out any equilibria as self-enforcing assessment of the game.
- Indeed, on occasion irrational assessments by two different players might each make the other look rational.
- Consider the following game

	b_1	b_2
a_1	3, 3	0, 0
a_2	-5, -5	0, -5

  - ➢ But is (a_2, b_2) a good prediction of the game? Not likely.

Normal form perfect equilibrium

- An ϵ-*perfect equilibrium of the normal form game* is a totally mixed strategy $\sigma \equiv (\sigma_1, \ldots, \sigma_N)$, if for all i and for all $s_i, s_i' \in S_i$,

$$u_i(s_i, \sigma) > u_i(s_i', \sigma) \qquad \text{then} \qquad \sigma_i(s_i') \leq \epsilon.$$

- A *perfect equilibrium* of a normal form game is a limit ($\epsilon \to 0$) of ϵ-perfect equilibria.
- For two-player game, any NE in which no player plays dominated strategies is perfect.
- For more than two-player game, the above statement is not true. There are NE with no players playing dominated strategies that is not perfect.

NE with no weakly dominated strategy may not be perfect

- Consider the following example:

Player 2

Player 1

	a	d
A	(3, 3, 0)	(5, 5, 0)
D	(4, 4, 4)	(4, 4, 4)

3 plays L

	a	d
A	(3, 3, 0)	(2, 2, 2)
D	(1, 1, 1)	(1, 1, 1)

3 plays R

- This game has two pure NE:

$$(D, a, L), \qquad (A, a, R).$$

- But (D, a, L) is not a perfect equilibrium, even if no weakly dominated strategy is played.

Example continued

- To see (D, a, L) is not perfect, note it is the limit of totally mixed strategy profile

$$(\epsilon, 1-\epsilon;\ 1-\eta, \eta;\ 1-\nu, \nu).$$

- Given player 2's belief: $(\epsilon, 1-\epsilon)$ and $(1-\nu, \nu)$:

$$\begin{aligned} u_2(a, \sigma^\epsilon) =& (1-\nu)[3\epsilon + 4(1-\epsilon)] + \nu[3\epsilon + 1 - \epsilon] \\ =& (1-\nu)(4-\epsilon) + \nu(1+2\epsilon), \end{aligned}$$

$$\begin{aligned} u_2(d, \sigma^\epsilon) =& (1-\nu)[5\epsilon + 4(1-\epsilon)] + \nu[2\epsilon + 1 - \epsilon] \\ =& (1-\nu)(4+\epsilon) + \nu(1+\epsilon). \end{aligned}$$

- Since

$$u_2(d, \sigma^\epsilon) - u_2(a, \sigma^\epsilon) = 2\epsilon - 3\epsilon\nu,$$

which is greater than zero for small number ν. IN no ϵ-perfect equilibrium does a receive higher probability than d, indicating (D, a, L) is not a perfect equilibrium.

Another example

- The game

Player 2

3 plays l

Player 1		L	R
	T	(1, 1, 1)	(1, 0, 1)
	B	(1,1, 1)	(0, 0, 1)

3 plays r

	L	R
T	(1, 1, 0)	(0, 0, 0)
B	(0, 1, 0)	(1, 0, 0)

- What are the pure strategy NE?
- What are the perfect equilibra?

Perfect equilibrium of Example 2

- (B, L, l) is a NE with no weakly dominated strategies.
- However, for any small probabilities player 2 assigns to R and player 3 to r, the expected payoff for player 1 from T is greater than that from B.
- Thus, there exists no ϵ-perfect equilibrium in which the totally mixed strategy profile assigns more than ϵ to B.

Example 2 (continued)

- If (B, L, l) if perfect, the ϵ-perfect equilibrium must be of the form
$$(\epsilon, 1-\epsilon; 1-\eta, \eta; 1-\nu, \nu).$$
- But is this indeed an ϵ-perfect equilibrium?
- For player 1, given belief $(1-\eta, \eta; 1-\nu, \nu)$:
$$U_1(T) = (1-\nu) + \nu(1-\eta);$$
$$U_1(B) = (1-\nu)(1-\eta) + \nu\eta.$$
- Clearly,
$$U_1(T) > U_1(B).$$
- So any ϵ-perfect equilibrium must have:
$$\sigma_1(B) < \epsilon.$$

Further restriction on belief

- Perfect equilibrium does not eliminate all unreasonable outcomes in some games. Adding a dominated strategy may enlarge the set of perfect equilibria.
- Consider the following example.

	L_2	M_2
L_1	1, 1	0, 0
M_1	0, 0	0, 0

	L_2	M_2	R_2
L_1	1, 1	0, 0	-1, -2
M_1	0, 0	0, 0	0, -2
R_1	-2, -1	-2, 0	-2, -2

- (M_1, M_2) is not perfect in the first game, but perfect in the second.
- The ϵ-perfect equilibrium for (M_1, M_2) in the second game

$$(\epsilon^2, 1-\epsilon-\epsilon^2, \epsilon; \epsilon^2, 1-\epsilon-\epsilon^2, \epsilon).$$

- Is this belief reasonable?

Other solution concept: proper equilibrium

- ϵ-proper equilibrium: totally mixed strategy profile σ is ϵ-proper if for all i and for all $s_i, s_i' \in S_i$,

$$u_i(s_i, \sigma) > u_i(s_i', \sigma) \Longrightarrow \frac{\sigma_i(s_i')}{\sigma_i(s_i)} < \epsilon.$$

- A proper equilibrium is the limit of a sequence of ϵ-proper equilibria.
- A proper equilibrium is also a perfect equilibrium.
- A proper equilibrium strategy is also sequential equilibrium strategy in the corresponding extensive form game.