Advanced Microeconomics I
Note 1: Individual preference and choice

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Introduction

- Individual decision making
- Suppose that X is a nonempty set of *alternatives* (the *grand set*), and an agent must choose from this set (or a subset of X).
- Two approaches to model an agent's decision making:
  - Preference-based approach
  - Choice-based approach

Preferences - binary relations

- Generally, given two sets S and T, the *Cartesian product* $S \times T$ is the set of all ordered pairs (s, t), where $s \in S$ and $t \in T$:

$$S \times T = \{(s, t) : s \in S, t \in T\}$$

- A **binary relation** B on X is a subset of $X \times X$, i.e., $B \subseteq X \times X$.
- If $(x, y) \in B$, then write xBy.
- If $(x, y) \notin B$, then write $x\bar{B}y$.

Some common properties of a binary relation

A binary relation B on X is

- **reflexive** if xBx for all $x \in X$.
- **irreflexive** if $x\bar{B}x$ for all $x \in X$.
- **symmetric** if xBy implies yBx for all $x, y \in X$.
- **asymmetric** if xBy implies $y\bar{B}x$ for all $x, y \in X$.
- **transitive** if xBy and yBz imply xBz for all $x, y, z \in X$.
- **negatively transitive** if $x\bar{B}y$ and $y\bar{B}z$ imply $x\bar{B}z$ for all $x, y, z \in X$.
- **complete** if for all $x, y \in X$, xBy or yBx.

Preferences

- There are various ways of defining /modeling preferences.
- First, consider the "P-model".
- The *primitive* of the model is a binary relation P on X, and P is interpreted as the "strictly better than" relation.
- We want to make sure that the preferences are "rational" or "consistent".
- We impose two conditions on the strict preference relation P:
  - P is asymmetric: if x is strictly better than y, then y is not strictly better than x.
  - P is negatively transitive: if x is not strictly better than y and y is not strictly better than z, then x is not strictly better than z.
- Did we require too little?

Proposition. *If P is asymmetric and negatively transitive, then*

(1) P is irreflexive.

(2) P is transitive.

(3) For any $x, y, z \in X$, xPy and $z\bar{P}y$ imply xPz; $y\bar{P}x$ and yPz imply xPz.

- Next, consider the "$\succeq$-model".
- In this case, the primitive of the model is a binary relation $\succeq$ on X, and $\succeq$ is interpreted as the "weakly better than" relation.
- We require $\succeq$ to be *complete* and *transitive*.
- It can be shown that if $\succeq$ is complete and transitive, then it is reflexive and negatively transitive.

The P-model and the $\succeq$-model are "equivalent", in the following sense.

Proposition.

(i) Given the asymmetric and negatively transitive P, define a new binary relation $\succeq'$ on X as follows: for any $x, y \in X$, $x \succeq' y$ if $y\bar{P}x$. Then $\succeq'$ is complete and transitive.

(ii) Given the complete and transitive $\succeq$, define a new binary relation P' on X as follows: for any $x, y \in X$, $xP'y$ if $x \succeq y$ and $y \not\succeq x$. Then P' is asymmetric and negatively transitive.

Proof of (i). Completeness: Consider any $x, y \in X$. If xPy, then by the asymmetry of P, we have $y\bar{P}x$. Hence by the definition of $\succeq'$, $x \succeq' y$. If $x\bar{P}y$, then by the definition of $\succeq'$, $y \succeq' x$.

Transitivity: Consider any $x, y, z \in X$ with $x \succeq' y$ and $y \succeq' z$. By the definition of $\succeq'$, $y\bar{P}x$ and $z\bar{P}y$. Then by the negative transitivity of P, $z\bar{P}x$. It follows that $x \succeq' z$.

Proof of (ii). Asymmetry is obvious.

Negative transitivity: Consider any $x, y, z \in X$ with $x\bar{P'}y$ and $y\bar{P'}z$. Suppose that $y \not\succeq x$. Then by the completeness of $\succeq$, $x \succeq y$. Hence by the construction of P', $xP'y$, contradiction. So we have $y \succeq x$. By a similar argument, it can be shown that $z \succeq y$. By the transitivity of $\succeq$, $z \succeq x$. Given the construction of P', it follows that $x\bar{P'}z$. □

- From now on, we use the $\succeq$-model.
- Define a **preference relation** on X as a binary relation $\succeq$ on X. The preference relation $\succeq$ is **rational** if it is complete and transitive.
- Given a preference relation $\succeq$ on X,
    - denote its "asymmetric component" as $\succ$: $x \succ y$ if $x \succeq y$ but $y \not\succeq x$.
    - denote its "symmetric component" as $\sim$: $x \sim y$ if $x \succeq y$ and $y \succeq x$.

- More on rationality
- Completeness: can you always compare?
    - Suppose that I offer you a trip to the moon, do you want to go to the northern part or the southern part?
- Two common sources of intransitivity:
    - Aggregation
    - The use of similarities

Choice correspondence

- Generally, given two sets S and T, a **correspondence** $f : S \to T$ is a rule that assigns a set $f(a) \subseteq T$ to every $a \in S$.
  - A *single-valued* correspondence is essentially a function.
- Let $\mathcal{D}$ be a collection of nonempty subsets of X.
  - Notice that $\mathcal{D}$ may not include *all* the subsets of X.
- $C : \mathcal{D} \to X$ is a **choice correspondence** if for every $A \in \mathcal{D}$, $C(A) \subseteq A$ and $C(A) \neq \phi$.
  - A full description of an agent's choice behavior in all possible scenarios (as defined by $\mathcal{D}$)

Weak axiom of revealed preference

- A choice correspondence C satisfies the **weak axiom of revealed preference** (WARP) if the following is true: if for some $A \in \mathcal{D}$ with $x, y \in A$ we have $x \in C(A)$ and $y \notin C(A)$, then for any $B \in \mathcal{D}$ with $x, y \in B$ we must have $y \notin C(B)$.
  - If, in some case, x is chosen over y, then y should never be chosen in the presence of x.
- An equivalent definition. C satisfies WARP if the following is true: if for some $A \in \mathcal{D}$ with $x, y \in A$ we have $x \in C(A)$, then for any $B \in \mathcal{D}$ with $x, y \in B$ and $y \in C(B)$ we must have $x \in C(B)$.
  - If, in some case, x is chosen in the presence of y, then y should never be chosen over x.
- The *richness* of the domain $\mathcal{D}$ is important.

- *Sometimes*, WARP can be decomposed into the following two conditions on a choice correspondence C.
- **Sen's property** α: given any $A, B \in \mathcal{D}$, if $x \in A \subseteq B$ and $x \in C(B)$, then $x \in C(A)$.
    - Amartya Sen's paraphrase of this: if the world champion in some game is a Pakistani, then he must also be the champion of Pakistan.
- **Sen's property** β: given any $A, B \in \mathcal{D}$, if $A \subseteq B$, $x \in C(A), y \in C(A)$ and $x \in C(B)$, then $y \in C(B)$.
    - Sen's paraphrase: if the world champion in some game is a Pakistani, then all champions (in this game) of Pakistan are also world champions.
- WARP implies Sen's properties α and β.
- If $\mathcal{D}$ includes at least all the subsets of X of size 2, then Sen's properties α and β imply WARP.
- If for any $A, B \in \mathcal{D}$ we have $A \cap B \in \mathcal{D}$, then Sen's properties α and β imply WARP.

From preference to choice correspondence

- Given a preference relation $\succeq$ on X, an *induced* correspondence is $C_{\succeq}$: for any $A \in \mathcal{D}$, $C_{\succeq}(A) = \{x \in A : x \succeq y, \forall y \in A\}$.

Proposition. *Assume that X is finite. If $\succeq$ is rational, then $C_{\succeq}$ is a well-defined choice correspondence that satisfies WARP.*

Proof. To show that $C_{\succeq}$ is a well-defined choice correspondence, it is sufficient to show that $C_{\succeq}(A)$ is nonempty for every $A \in \mathcal{D}$. Assume to the contrary, for some $A \in \mathcal{D}$, $C_{\succeq}(A) = \phi$. Consider any $x \in A$. Since $x \notin C_{\succeq}(A)$, there exists $y \in A$ such that $x \not\succeq y$. By the completeness of $\succeq$, we have $y \succeq x$ and hence $y \succ x$. That is, for every alternative in A we can find a strictly better one in A. Since A is finite, there exists a cycle that consists of $k \geq 2$ alternatives $x_1, ..., x_k \in A$ with $x_1 \succ x_2 \succ ... \succ x_{k-1} \succ x_k \succ x_1$, which contradicts to the transitivity of $\succeq$.

It remains to show that $C_{\succeq}$ satisfies WARP. Suppose not. Then there exist $x, y \in X$ and $A, B \in \mathcal{D}$ such that $x, y \in A$, $x, y \in B$, $x \in C_{\succeq}(A)$, $y \notin C_{\succeq}(A)$, and $y \in C_{\succeq}(B)$. $y \in C_{\succeq}(B)$ implies $y \succeq x$, and $x \in C_{\succeq}(A)$ implies $x \succeq z$ for all $z \in A$. By transitivity, $y \succeq z$ for all $z \in A$. It follows that $y \in C_{\succeq}(A)$, contradiction. $\square$

From choice correspondence to preference: rationalizing

A choice correspondence C can be **rationalized** if there exists a rational preference relation $\succeq$ on X such that $C = C_{\succeq}$, i.e., $C(A) = C_{\succeq}(A)$ for all $A \in \mathcal{D}$.

Proposition. *Suppose that $\mathcal{D}$ includes at least all subsets of X of size up to 3, and $|C(A)| = 1$ for all $A \in \mathcal{D}$ (i.e., C is a "choice function"). Then C can be rationalized if and only if C satisfies Sen's property α.*

Proof. "Only if" part. If C can be rationalized, then there exists rational $\succeq$ such that $C = C_{\succeq}$. From the previous discussion, we know that $C_{\succeq}$ satisfies WARP, hence Sen's property α.

"If" part. Define $\succeq$ on X as follows: for any $x, y \in X$, let $x \succeq y$ if $\{x\} = C(\{x, y\})$.

First, we show that $\succeq$ is rational. Consider any $x, y \in X$. We have $x \succeq y$ if $\{x\} = C(\{x, y\})$, $y \succeq x$ if $\{y\} = C(\{x, y\})$. So $\succeq$ is complete. Suppose that $\succeq$ is not transitive. Then there exist $x, y, z \in X$ such that $x \succeq y$, $y \succeq z$, $x \not\succeq z$ and $|\{x, y, z\}| = 3$. It follows that $C(\{x, y\}) = \{x\}$, $C(\{y, z\}) = \{y\}$ and $C(\{x, z\}) = \{z\}$. Then consider the set $\{x, y, z\} \in \mathcal{D}$. Given that C satisfies Sen's property α, we have: $C(\{x, y\}) = \{x\}$ implies $C(\{x, y, z\}) \neq \{y\}$, $C(\{y, z\}) = \{y\}$ implies $C(\{x, y, z\}) \neq \{z\}$, and $C(\{x, z\}) = \{z\}$ implies $C(\{x, y, z\}) \neq \{x\}$. That is, $C(\{x, y, z\}) = \phi$, contradiction.

It remains to show that $C = C_{\succeq}$. Consider any $A \in \mathcal{D}$ and let $C(A) = \{x\}$. For any $y \in A$, Sen's property α implies $C(\{x, y\}) = \{x\}$. So $x \succeq y$ for all $y \in A$. It follows that $x \in C_{\succeq}(A)$. Suppose that there exists $y \in C_{\succeq}(A)$ and $y \neq x$. Then clearly $y \succeq x$ and $x \succeq y$. But $y \succeq x$ implies $C(\{x, y\}) = \{y\}$, and $x \succeq y$ implies $C(\{x, y\}) = \{x\}$, contradiction. Therefore, $C_{\succeq}(A) = \{x\} = C(A)$. □

Proposition. *Suppose that $\mathcal{D}$ includes at least all subsets of X of size up to 3, and C is a choice correspondence. C can be rationalized if and only if C satisfies the weak axiom of revealed preference.*

Proof. The "only if" part can be shown in the same way as in the previous proof.

"If" part. Define $\succeq$ on X as follows: for any $x, y \in X$, let $x \succeq y$ if $x \in C(\{x, y\})$.

We first show that $\succeq$ is rational. Completeness is obvious: for any $x, y \in X$, $x \succeq y$ if $x \in C(\{x, y\})$, and $y \succeq x$ if $y \in C(\{x, y\})$. Suppose that $\succeq$ is not transitive. Then there exist $x, y, z \in X$ such that $x \succeq y$, $y \succeq z$ and $x \not\succeq z$. It follows that $x \in C(\{x, y\})$, $y \in C(\{y, z\})$, $z \in C(\{x, z\})$, and $x \notin C(\{x, z\})$. Since $z \in C(\{x, z\})$ and $x \notin C(\{x, z\})$, by WARP $x \notin C(\{x, y, z\})$. Applying WARP again, it can be seen that $x \notin C(\{x, y, z\})$ and $x \in C(\{x, y\})$ imply $y \notin C(\{x, y, z\})$, then $y \notin C(\{x, y, z\})$ and $y \in C(\{y, z\})$ imply $z \notin C(\{x, y, z\})$. Therefore, $C(\{x, y, z\}) = \phi$, contradiction.

It remains to show that $C = C_{\succeq}$. Consider any $A \in \mathcal{D}$. If $x \in C(A)$, then for any $y \in A$, WARP implies $x \in C(\{x, y\})$. So $x \succeq y$ for all $y \in A$. It follows that $x \in C_{\succeq}(A)$. That is, $C(A) \subseteq C_{\succeq}(A)$. Suppose that for some $x \in C_{\succeq}(A)$, $x \notin C(A)$. Then there exists $y \in C(A)$ and $y \neq x$. By WARP $x \notin C(\{x, y\})$. It follows that $x \not\succeq y$, contradicting to the fact that $x \in C_{\succeq}(A)$. Hence, $C_{\succeq}(A) \subseteq C(A)$. In sum, $C(A) = C_{\succeq}(A)$. □