MICROECONOMIC THEORY II

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Spring 2021

BAYESIAN GAME

不完全信息博弈

payoff 服从的可能分布

- Definition: A Bayesian game consists of

  players

  - ➢ a finite set of players, denoted by I
  - ➢ a set of types $\theta_i \in \Theta_i$ (the set of signals that may be observed by player i)

    player 只知自己type，不一定知对方type → 不同type的选项都给出

  - ➢ a finite strategy set S_i. A pure strategy $s_i(\theta_i)$ is a decision rule that gives the player's strategy choice for each realization of his type.

    3个type 2个选项 有8个纯策略

    - ➠ Example: bidding function for players in an auction: $b_i(v_i)$, bid given player i's value of the item for sale.
  - ➢ a probability distribution of type, $F(\theta_1, \ldots, \theta_I)$

    给定纯策略，由type 决定最终选项

  - ➢ a utility function $\tilde{u}_i : S_i \times \Theta \to \mathbb{R}$:

$$\tilde{u}_i(s_1(\cdot), \ldots, s_I(\cdot)) = E_\theta[u_i(s_1(\theta_1), \ldots, s_I(\theta_I), \theta_i)].$$

给定策略组合 → 算期望效用

贝叶斯纳什均衡

- Definition: A (pure strategy) Bayesian Nash equilibrium for the Bayesian game $[I, S, u, \Theta, F(\cdot)]$ is a profile of decision rules $(s_1(\cdot), \ldots, s_I(\cdot))$ such that, for all i,

$$\tilde{u}_i(s_i(\cdot), s_{-i}(\cdot)) \geq \tilde{u}_i(s_i'(\cdot), s_{-i}(\cdot))$$

An example

比较期望效用

4个纯策略 ← 2个类型   1个类型 → 2个纯策略

- Boss and Tim play the game. Tim does not know the payoff of Boss.

Type I:

boss \ Tim	W	S
M	3, 2	1, 1
N	4, 3	2, 4

Type II:

boss \ Tim	W	S
M	4, 2	2, 1
N	3, 3	1, 4

monitor
NO to monitor

μ (无为而治) Type I   Type II $1-\mu$ (盯紧工人)

- How to solve the game: Harmless to split Boss into two types.

  ➢ type I (with probability μ) has a dominant strategy N;
  ➢ type II $(1-\mu)$ has dominant strategy M.

求BNE

- The equilibrium strategy of Boss: NM (I II)
- For Tim, payoff

$$W: \quad 3\mu + 2(1-\mu) = 2 + \mu$$

$$S: \quad 4\mu + (1-\mu) = 1 + 3\mu.$$

- Hence Bayesian Nash equilibrium is:
  ➢ (NM, W) if $\mu < 1/2$;
  ➢ S if $\mu \geq 1/2$.

Boss { 可能性 μ N严优; $1-\mu$ M严优 }

Boss NM   Tim W S

Find BNE

- The equilibrium strategy of Boss: *NM* 判支.
- For Tim, payoff

$$W: \quad 3\mu + 2(1-\mu) = 2 + \mu$$

$$S: \quad 4\mu + (1-\mu) = 1 + 3\mu.$$

- Hence Bayesian Nash equilibrium is:
  - ➢ (NM, W) if $\mu < 1/2$;
  - ➢ S if $\mu \geq 1/2$.

Example 2

每个player有4个
纯策略

- Two opposed armies are poised to seize an island.
- Each can choose either "attack" or "not attack."
- Each army is either "strong" or "weak" with prob. $\left(\frac{1}{2}, \frac{1}{2}\right)$. Distributions are independent, and type known only an army's general.
- Payoffs are as follows:
  - The island is worth M if captured. 对打仗一部队岛的价值
  - An army can capture the island either by attacking when its opponent does not or by attacking when its rival does if its strong and its rival is weak.
  - If two armies of equal strength both attack, neither captures the island.
  - Cost of fighting: s if strong; w if weak.
  - No cost of attacking if its rival does not.
  - $M > w > s$ and $w > M/2 > s$

2个情况占领岛
一方进攻,一方不进攻
一方strong,一方weak

岛的价值 打仗的成本 有一定的概率拿下这个岛

强
战斗力强的一方
不进攻
进攻


T-7

Analysis (1)

- Four strategies contingent upon one's types: AA (attack if strong, attack if weak), AN (attack if strong, not attack if weak), NA, NN.
- Expected payoff
  - ➢ Expected payoff for 1 from (AA, AA): 有本概率大家都强

$$\frac{1}{4}(0-s)+\frac{1}{4}(M-s)+\frac{1}{4}(0-w)+\frac{1}{4}(0-w)=\frac{M}{4}-\frac{s+w}{2}$$

  - ➢ Payoff for 1 from (AA, AN):

$$\frac{1}{4}(0-s)+\frac{1}{4}M+\frac{1}{4}(0-w)+\frac{1}{4}(M)=\frac{M}{2}-\frac{s+w}{4}$$

  - ➢ Payoff for 1 from (AN, AA):

$$\frac{1}{4}(0-s)+\frac{1}{4}(M-s)=\frac{M}{4}-\frac{s}{2}$$

Analysis

- The strategic form

player 2

	AA	AN	NA	NN
AA	$\frac{M}{4}-\frac{s+w}{2},\ \frac{M}{4}-\frac{s+w}{2}$	$\frac{M}{2}-\frac{s+w}{4},\ \frac{M}{4}-\frac{s}{2}$	$\frac{3M}{4}-\frac{s+w}{4},\ -\frac{w}{2}$	$M,\ 0$
AN	$\frac{M}{4}-\frac{s}{2},\ \frac{M}{2}-\frac{s+w}{4}$	$\frac{M-s}{4},\ \frac{M-s}{4}$	$\frac{M}{2}-\frac{s}{4},\ \frac{M-w}{4}$	$\frac{M}{2},\ 0$
NA	$-\frac{w}{2},\ \frac{3M}{4}-\frac{s+w}{4}$	$\frac{M-w}{4},\ \frac{M}{2}-\frac{s}{4}$	$\frac{M-w}{4},\ \frac{M-w}{4}$	$\frac{M}{2},\ 0$
NN	$0,\ M$	$0,\ \frac{M}{2}$	$0,\ \frac{M}{2}$	$0,\ 0$

- Two pure strategy Bayesian NE:

$$(AA, AN), \qquad (AN, AA).$$

Example 3

通用的北美/欧洲总部

分公司A/B

·若开发→则可开发出

- A large corporation has two divisions, firm A and B.
- Any independent innovation by one firm is shared fully with the other.
- The two could potentially develop a new product.
- Cost of development $c \in (0, 1)$.
- Benefit of the product to firm i privately known. 新产品带来的好处 仅i公司自己知道
- Assume that each firm i has a type θ_i,

$$\theta_i \sim U(0,1)$$

均匀分布 pdf)→0−1的均匀分布 pdf为1

相互独立

Benefit of the product is θ_i^2. 新车型给公司带来的好处

- Two firms privately observe their own type, and then simultaneously decide to develop/not.

A公司决定开发 $\theta_i^2 - c$

B公司开发，A公司得 θ_i^2

Analysis

纯策略 研发 不研发

- Let $s_i(\theta_i) = 1$ if type θ_i of firm i develops, and $s_i(\theta_i) = 0$ otherwise. 期望效用
- Expected payoff developing: $\theta_i^2 - c$;
- Expected payoff not developing:

$$\theta_i^2 Prob(s_j(\theta_j) = 1).$$

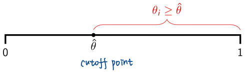
$\theta_i^2 - c > \theta_i^2 Prob(s_j(\theta_j)=1)$

开发

<

不研发

- Assume that firms each use some of cutoff strategy, develop iff θ_i large enough.
- Let $\hat{\theta}_i$ be the cutoff for firm i, so



$$s_i = 1 \quad \text{if and only if} \quad \theta_i \geq \hat{\theta}_i.$$

- As θ_i follows uniform distribution, we have:

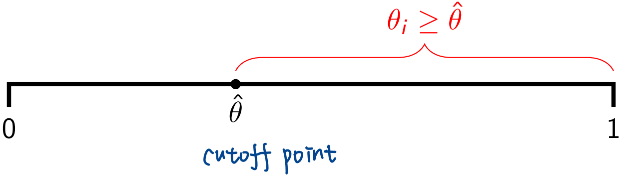
$$Prob(\theta_i \geq \hat{\theta}_i) = 1 - \hat{\theta}_i.$$

- Thus, $\hat{\theta}_i$ is determined by the following condition:

$$\hat{\theta}_i^2 - c = \hat{\theta}_i^2 \left(1 - \hat{\theta}_j\right).$$

均衡策略时 $\hat{\theta}_i = \hat{\theta}_j = \hat{\theta}$ (symmetrically)

Cut-off strategy



Analysis (2)

- In symmetric equilibrium, $\hat{\theta}_i = \hat{\theta}_j = \hat{\theta}$.
- The Bayesian NE: For $i = A, B$,

$$s_i(\theta_i) = \begin{cases} 1 & \text{if} \quad \theta_i \geq \hat{\theta} = (c)^{1/3}; \\ 0 & \textit{otherwise} \end{cases}$$

Auction 拍卖

AUCTION 拍卖

- Auctions are typically used to sell items for which there is not a market price, but only a vague idea about it, sometimes involving the personal taste of potential buyers.
- Historically, among the first known auctions there were auction for slaves and wives.
- Auction is an mechanism whereby a seller tries to sell some objects to a group of potential buyers, whose willingness to pay is unknown.
- It plays an important role in the allocation of resources in the real world, especially goods of limited supply.
- Examples: vehicle licenses in Shanghai, US treasuries bill, arts and antiques, and eBay of course.
  - ➢ The U.S. spectrum auction in April 2008 raised a total of $19.8 billion
  - ➢ Europe's frenzied 2000 and 2001 auctions reaped nearly $100 billion

Example: US treasury bond auction

"As an auction neared, the primary dealers would work the phones, polling customers to gauge their appetite for bonds. A few seconds before the clock struck one p.m. on the appointed day, the dealers phoned "runners," who stood by a wall of phones at the Federal Reserve Building downtown, waiting to scribble down orders by hand and dash to the Fed clerk's wooden box, where they jammed them inside. At the stroke of one p.m., the clerk placed his hand over the slot. That ended the auction. The government had used this antiquated system for decades."

———-The Snowball

Two important characteristics

- Uncertainty about the valuations of the bidders.
  - ➢ This uncertainty leads to the private values assumption about such valuations;
  - ➢ It is modeled as independent random variables from a common distribution.

赢者诅咒

- "Winner's curse": Because of the uncertainty of the value of the object for sale, a winner of an auction might wonder why all the other bidders' valuation were smaller than hers, and in particular whether this might have happened because of the others' more accurate information about the item's true value.

Winner's Curse in stock market

Immediately after shares of a company begin trading publicly, investors who believe that the true value of a share is higher than its current price will buy some. Increasingly optimistic investors – who believe that the shares true value exceeds the markets price will continue buying shares, which continues to increase its price. Eventually, the share reaches a certain price, specifically, the price that the most optimistic bidder is willing to pay, and stops climbing higher. This typically happens within the first few days of trading.

One example: the Snap Inc IPO. Shares were offered for around $17 each initially, but within a day of trading, the price was pushed up to $25. The share price soon dropped steeply to around $15, and remained so for several years.

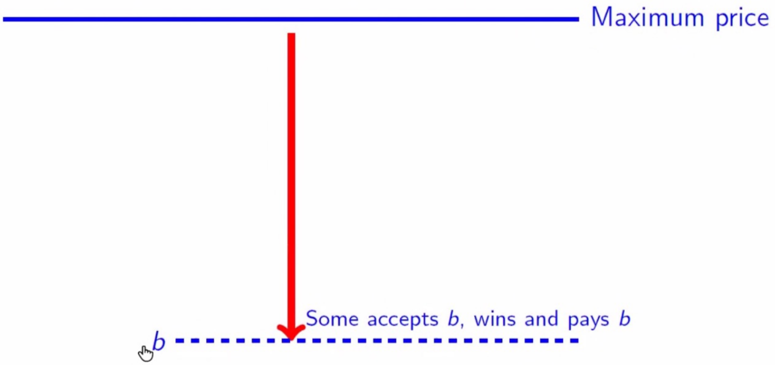
Auction usually characterized by four elements 拍卖被4个要素刻画

1. The number of goods available for sale. Whether there is one unit or multiple units.
2. The auction mechanism (i.e. the auction rules). There are several formats commonly used in single-unit auctions:
   - Dutch Auction. 价格下降式拍卖(仅一次),拍卖物仅一件,单物品拍卖
   - English auction. 价格上升式拍卖(不断举牌) 叫卖拍者 卖家看
   - First-price sealed bid auction. → 将出价放在信封中,出价highest获胜
   - Second-price sealed bid auction. F 获胜者需支付自己的bid. S 获胜者不需支付自己的bid,而是支付第2高的人的bid.

   无区别

3. Information structure 每个bidder有估价,估价间独立
   - Independent private valuation; 比如durian拍卖,即使我知道他人bid也不影响我的判断 相反
   - Common value; → 股票 拍卖物价值对all bider一致
   - Positive affiliation. 价值判断不一致,但价值并不对all bider一致. → 油田 (介于2者间)
4. Risk preferences. Risk neutrality versus risk aversion. 用来偏好的程度

Dutch auction



English auction

First-price sealed bid auction

每个bidder知道自己的v

v:有无穷多个type 风险中性

大家valuation的分布同

分布同:

- N symmetric bidders, independent value $f_i(v) = f(v)$ for all $v \in [0, 1]$

大家看上去似乎一样

[0,1]的 连续分布

纯策略.

给定一个v有相应b v→最高愿出多少钱

- Bidding function $b : [0, 1] \to \mathbb{R}_+$, r: reported value

r可与v同/不同.

- Expected payoff from reporting r if value is v:

概率·确定的payoff

$$u(r, v) = F^{N-1}(r)(v - b(r)).$$

cdf $\underbrace{\text{r比N-1个大}}$ 即据r报价b(r)

independent value

- Bidder i's problem:

$$\max_r U(r, v).$$

选取一个r使得期望效用max

Analysis

- u maximized at v, 贝叶斯纳什均衡中的bidding function

$$\frac{du(r,v)}{dr}\Big|_{r^*=v} = (N-1)F^{N-2}(v)f(v)(v-b(v)) - F^{N-1}(v)b'(v) = 0$$

- Rearranging terms

$$(N-1)F^{N-2}(v)f(v)b(v) + F^{N-1}(v)b'(v) = v(N-1)F^{N-2}(v)f(v).$$

- Integrate both sides 找出每个bid如何报价 找出C是多少、v最高愿意出价

$$F^{N-1}(v)b(v) = (N-1)\int_0^v xf(x)F^{N-2}(x)dx + C$$

- Thus we have

$$F^{N-1}(v)b(v) = (N-1)\int_0^v xf(x)F^{N-2}(x)dx + C$$

- Since $b(0) = 0$, C should be zero, 假设v是最大，其他人中最大的valuation的期望值是多少？ bidding function

条件概率分布

$$b(v) = \frac{1}{F^{N-1}(v)}\int_0^v xdF^{N-1}(x)$$

Interpretation

- In the unique symmetric equilibrium of a first-price, sealed bid auction, each bidder bids the expectation of the second highest bidder's value conditional on winning the auction
- Example: v is uniformly distributed. 均匀分布
  - ➢ In this case, $F(v) = v$ and $f(v) = 1$.
  - ➢ Replacing $F(v)$ and $f(v)$

$$b(v) = \frac{1}{v^{N-1}} \int_0^v x(N-1)x^{N-2}dx.$$

  - ➢ Integrate we have

$$\begin{aligned} b(v) &= \frac{N-1}{V^{N-1}} \left[\frac{x^N}{N} |_v \right] \\ &= \frac{N-1}{N} v \\ &= v \left[1 - \frac{1}{N} \right]. \end{aligned}$$

N个参加拍卖的人越多，
每人报价会更接近自身的valuation

Dutch auction

策略上等同

- The Dutch auction and the first-price sealed-bid auction are strategically equivalent from a game-theoretic point of view, regardless of the information structure and risk preferences.
- In a Dutch auction, each bidder needs to decide at what price he would want to claim the object, assuming that the object is unclaimed up to that point. Same in a first-price sealed bid auction.
- Symmetric equilibrium bidding

$$b(v) = \frac{1}{F^{N-1}(v)} \int_0^v x dF^{N-1}(x)$$

Second-price sealed bid auction

每个bidder应当报出自己真实的valuation

- Bidder with v chooses b_i :　　上限为v　　获胜保证 $v - \max_{j \neq i} b_j \geq 0$

$$\max_{b_i} EU(b_i, v) = Prob(b_i > \max_{j \neq i} b_j) \left(v - \max_{j \neq i} b_j \right).$$

- In the IPV case, it is a weakly dominant strategy to bid one's valuation in a second price auction.
- In equilibrium, all bidders bid their own valuation, and the object goes to the bidder with the highest valuation.
- In the case of independent private value auctions, the English auction and the Vickrey auction are strategically equivalent.

English auction

与第二价格拍卖一样会报出真实value

他人估值不会影响我的估值

在此假设下支付 = 第二价格

(风险中性 → 对卖家 → 4种同

- With independent private values, dropping out when price reaches one's value is the unique weakly dominant strategy
- With independent private values, English auction and second-price auction raises the same ex-post revenue.
- In both auction, the bidder with the highest valuation wins and pays the second highest bidder's value.
- Without independent private values, English and second-price auction would not be strategically equivalent.

Revenue equivalence

卖家风险中性(4种卖家希望期望收益等价) 前提

- Under risk-neutrality and independent private valuations, all auction formats lead to the same expected revenue to the seller.

  最后获胜的都是 expectation 最高的

  任何条件下一致

  - First-price and Dutch auction generate the same revenue under any conditions;

    v间无关联(bidder间)

  - 每个bidder如实报价 Second-price and English auction are equivalent under independent private valuations;
  - IPV与risk neutral Under the two assumptions, First-price and second-prce auctions lead to the same expected revenue.
- The equivalence does not hold if
  - common value auctions 第二价格 vs 英式 不同.
  - risk-averse traders

Revenue equivalence between first and second-price auctions

证两者对买家带来期望收益一致

第k大值

- k-order statistic of N draws from F_x

$$f_{x_k} = \frac{N!}{k!(N-k)!}[F(x)]^{N-k}[1-F(x)]^k f(x).$$

- Second-price auction revenue $b_S = v$ 有N个随机变量

每个bidder报真实价格

$$EV_S = \int_0^1 b_S f_{v_2} dv = \int_0^1 vN(N-1)F^{N-2}(v)[1-F(v)]f(v)dv.$$

N个R.V. 第二高的R.V.的分布

- First-price auction

最高 first order statistic

$$\begin{aligned} EV_F &= \int_0^1 b_F f_{v_1} dv \\ &= \int_0^1 \left[\frac{1}{F^{N-1}(v)} \int_0^v x dF^{N-1}(x)\right] NF^{N-1}(v)f(v)dv. \end{aligned}$$

Proof of equivalence

- Revenue of first-price auction: $x \le v \le 1$

$$
\begin{aligned}
EV_F &= \int_0^1 N(N-1)\left[\int_0^v xF^{N-2}(x)f(x)dx\right] f(v)dv \\
&= \int_0^1 \left[\int_x^1 f(v)dv\right] N(N-1)F^{N-2}(x)xf(x)dx \\
&= \int_0^1 N(N-1)xF^{N-2}(x)[1-F(x)]f(x)dx.
\end{aligned}
$$

- Thus, we conclude

$$EV_F = EV_S = \int_0^1 N(N-1)vF^{N-2}(v)[1-F(v)]f(v)dv.$$

不满足 IPV　1与荷 等价
2与英 不等价
1与2 不等价

不满足 R.N.　1与荷 一样
2与英 一样 (满足IPV即可)
1与2 不一样

$v-b(v)>0$

bidder 风险厌恶型 (对卖家而言) 第一更好

Mechanism design 从均衡找博弈

- In game theory we fix a game and analyze the set of possible outcomes.
- Mechanism design studies the reverse problem: we fix a set of outcomes, and try to come up with a game that has such set as the set of its equilibria.
- Mechanism design deals with defining the rules of a game so that players' actions lead to prescribed social goal
- Mechanism can be applied to the design of auctions. Typical goals are
  - ➢ Assign the item to the bidder with the highest valuation for it;
  - ➢ Giving incentive to bidders so that it is in their best interest to bid according to their true valuation

Mechanism design example: US treasury

价格发现 所有竞争(目的) (Exp 1)

- A primary dealers polls customers to gauge their appetite for bonds;
- A few seconds before 1 p.m. on the appointed day, the dealers phoned "runners," who stood by a wall of phones at the Federal Reserve Building downtown, waiting to scribble down orders by hand; and dash to the Fed clerk's wooden box, where they jammed them inside.
- At 1 p.m., the clerk placed his hand over the slot. That ended the auction.
- Only the "primary dealers" can submit bids directly to treasury;
- No individual dealer could buy more than 35% after 1980s;
- After 1990, no firm could bid more than 35% for its own account.

Example: Groves mechanism

- The problem: Should a bridge be built?

村  城

- Individual's utility from the decision

$$u_i = \theta_i x + t_i.$$

村对个人付的钱 >0/<0 均可

建桥决策1/0

- $x \in \{1, 0\}$, θ_i is i's private valuation, t_i is individual's payment to build the bridge.
- Let $c > 0$ be the cost of build the bridge.
- Efficient rule

$$x_i(\theta) = \begin{cases} 1, & \text{if } \sum_{i=1}^{I} \theta_i \geq c \\ 0 & \textit{otherwise} \end{cases}$$

How to make an optimal decision?

每个村民报告自己的$\theta_i \to \hat{\theta}_i$

- One mechanism
  - ➢ Transfer

其他报告的好处和-建桥的成本

本人报告与本人支付无关，而影响的只是能否建桥

不满足预算约束

$$t_i(\hat{\theta}) = \begin{cases} \sum_{j \neq i}^{I} \hat{\theta}_j - c, & \text{if } \sum_{j=1}^{I} \hat{\theta}_j \geq c \\ 0 & \textit{otherwise} \end{cases}$$

设 $c=1w$ 共101人，$\theta_i=100$ 我不需掏钱

如实 { 多报，支付的价格多于valuation
少报，可能建不起来

$\sum_{j \neq i}^{I} \hat{\theta}_j < c$ 可能

  - ➢ Provision decision

多报 → 我报200（其余人9800）（建桥，我付200）超出100

$$x_i(\hat{\theta}) = \begin{cases} 1, & \text{if } \sum_{i=1}^{I} \hat{\theta}_i \geq c \\ 0 & \textit{otherwise} \end{cases}$$

- It is incentive compatible to report θ_i truthfully.
- Provision is optimal!
- Violates budget-balancedness.