Persuasion in optimal financing

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Abstract

We examine the interplay between information disclosure and security design in a financing environment where auditing firm cash flows is costly. The optimal information structure involves disclosing a range of risk-return scenarios to induce a desired security combination consisting of convertible debt and/or performance-sensitive debt. Notably, converting debt into equity is shown to maximizes financing probability. Calibrating interest rates based on performance effectively reduces external financing costs. We also find that the investor is persuaded to audit less, which not only mitigates the financing hold-up but also enhances the entrepreneur's payoff. Intriguingly,we show that the value of the firm strictly increases with auditing costs, particularly when these costs are sufficiently high.

Key words: Bayesian persuasion, costly state verification, convertible debt, performance-sensitive debt, welfare analysis

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1 Introduction

Classic corporate finance theories consider the optimal contracting between a firm and its investors as an important approach to potentially mitigate agency conflicts. For instance, Townsend [1979] and Gale and Hellwig [1985] show that debt is the optimal contract that minimizes firms' auditing costs and induces truthful cash-flow reports. Innes [1990] shows that with some conditions debt is optimal to provide maximal incentive for insiders to exert efforts. This literature of security design focuses on how to align incentives or reduce information frictions, but it takes the information environment over which relevant parties make their investing and financing decisions as given.

In practice, firms may very well influence investors by way of disclosing predictive information related to future operations or growth prospects. For example, startups typically lay out roadmaps or experiment plans of how to roll out their business ideas when pitching to venture capitalists.[1] As a result, investors could reasonably respond to the structure of predictive information by proposing contingent plans to provide financing. In this regard, the adoption of disclosure policies and the design of optimal securities are naturally interacting and therefore need to be jointly determined.

Based on these observations, we enrich the traditional security design process in this paper to consider the possibility that firm disclosure policies can be adopted to persuade certain financing patterns. In particular, we consider a financing environment where an entrepreneur can commit to a flexible information structure that represents the business experiment plan by way of the Bayesian persuasion approach of Kamenica and Gentzkow [2011]. The results delivered by an experiment clearly provides public information of the business. One the one hand, observing the results, an investor can update the prior belief and then propose a financing contract. On the other hand, taking into account the investor's security choices, the entrepreneur can optimally design the experiment plan to obtain higher returns and/or higher chances of securing finance.

In this paper, the information friction is simply that the entrepreneur privately ob-

[1] A real-world example is that Space X began by experimenting with innovative designs for rockets and a sequence of testing flights. Its experimentation roadmap resulted in the success of Falcon 1 in 2008 which marked the first time a liquid-fueled rocket reached orbit and a significant proof of concept that helps secure contracts and crucial funding.

serves the firm cash flows if any is produced, while the investor can only observe the signal realizations (public) generated by the chosen experiment. Following Townsend [1979], the investor can also audit the firm's cash flows by incurring verification costs, which are simply dead-weight losses. The classic result implied by this costly-state-verification technology is that the standard debt is the optimal security. If these costs are high or the business has a relatively large downside, then financing can easily break down.

A relevant disclosure policy can certainly facilitate financing by bridging the information gap. But simply revealing full information is clearly not optimal, because the entrepreneur then loses the business advantage, only letting the investor take the fruits. We examine the optimal information design in this environment, as well as deriving the optimal contract given the committed disclosure. Through this exercise, we can rationalize some widely-used securities, such as convertible debt and performance-sensitive debt (PSD), that can not be easily generated in standard financing environments [2]. Moreover, we can predict joint patterns of disclosure choice and security type, highlighting the role of experimentation in persuading investors.

In our model, the entrepreneur has two primary goals in designing a business experiment: getting a higher financing probability, and obtaining larger returns. The assumption is motivated by the fact that on top of monetary payoffs, entrepreneurs normally have either empire-building preferences or reputation concerns, when pursuing their business careers.[3] The investor, by contrast, cares only about the investment returns. Given these objectives, the experiment can be designed to reveal information in two stages. The first-stage experiment has a binary result that shows two distinct paths for the business: one with large downside that incurs losses, and the other with more upside where investment generate positive returns. In the optimal disclosure, the former information environment induces the entrepreneur to prioritize financing probability in designing the second-stage

[2]For instance, the textbook of Ross et al. [2022] states "probably there is no other area of corporate finance where real-world practitioners get as confused as they do on the reasons for issuing convertible debt." Moreover, practitioners caution that PSD can exacerbate costly financial distresses, because issuers have to pay out more when cash flows deteriorate.

[3]The empire-building tendency of managers has been emphasized by Williamson [1964], Donaldson [1984] and Jensen [1986], among many others. Reputation concerns can be, for example, that managers are concerned with how their efforts affect perceived values in the labor market. Holmstrom [1999] shows that these concerns have influential effects on incentive provisions.

experiment, while the latter prioritizes monetary payoff.

If the initial signal indicates that the firm has a large chance of incurring losses, then the second-stage experiment will disclose accurate information of the positive-yield cash flows, which then induces a convertible debt as the optimal contract. This type of experiment plan and the induced contract mainly aim to enlarge the financing probability. The debt face value is simply the investment amount, and can be converted to equity if the investor desires. This way, although the firm has a large chance of incurring downside losses, the investor is compensated enough by obtaining all the positive investment surplus.

By contrast, if the initial financing signal indicates that the investment return is more likely to be positive, then the second-stage experiment will partially pool the positive-yield cash-flows to leave the entrepreneur with more monetary payoffs from the business. The disclosure strategy leads to optimal contracting in the form of PSD that effectively lowers the financing cost. Given an interest rate charge, the expected auditing costs will be lifted if some signal realization can shift the belief of positive-return cash flows toward the low-end of its distribution. Such realization can persuade the investor to lower interest rate in order to reduce auditing. Accordingly, lower chances of defaulting are designed in these signal realizations to compensate for lower investor returns, which generates a *positive correlation* between default probability and interest rate in the optimal PSD.

Our analysis uncovers several welfare effects in the security design problem, arising from the introduction of persuasion. Firstly, it mitigates the issue of information hold-up. Compared to the no-disclosure benchmark, the firm becomes more likely to secure funding. This improvement occurs because the cash-flow states with low returns and high auditing costs are disclosed to the investor, thereby preventing their financing. Second, the investor is persuaded to audit fewer positive-yield cash flows, effectively reducing dead-weight losses. These benefits accrue predominantly to the entrepreneur, while maintaining the investor's expected profit levels. Additionally, we observe a non-monotonic relationship between the firm value and auditing costs. Initially, the firm value decreases with low auditing costs, but interestingly, it begins to increase significantly with higher auditing costs. This counter-intuitive effect arises because as auditing costs escalate, fewer cash flows with positive returns are allocated to the investor, necessitating the exclusion of more cash flows with negative returns from financing.

This paper is related to the literature of security design. By introducing Bayesian persuasion, we show that convertible debt and/or PSD arise as optimal securities in different information environments to achieve distinct financing objectives. First, it is the combination of debt and equity that maximizes the chance of obtaining funds. Since equity financing is feasible only if complete information is disclosed, the debt portion is necessary to finance the negative-yield states. In addition, the equity portion leaves enough positive returns to the investor to compensate for the high default risk. In this regard, the convertibility of debt in our model is neither to lower interest rates as typically argued by practitioners,[4] nor to reduce external financing costs as argued by existing agency theories, such as Brennan and Kraus [1987] and Green [1984]. To the contrary, convertibility here bears "large financing costs" to increase likelihood of financing for purposes other than immediate monetary payoffs. In this sense, our model *predicts* that firms are more likely to issue convertible debt if their owners have larger reputation or career concerns, or if more future benefits, e.g., real options, are embedded in current investments.

Second, the variation of interest rates in PSD, correlated with debt default, lowers financing costs by reducing dead-weight losses. In this regard, our model differs from the existing literature in the rationale to issue PSD. Bhanot and Mello [2006] argues that debt with a trigger can ease the asset substitution motive of equity holders under certain conditions. In a screening model, Manso et al. [2010] shows that firms with high-growth rate may issue PSD to signal its type to the market and be separated from the low-growth firms. Adam and Streitz [2016] shows that issuing PSD can mitigate the hold-up problem in relationship lending.

The paper is also related to the theoretical literature of Bayesian persuasion where a sender designs an information structure to induce certain actions of the receiver. This paper expands the framework by introducing a follow-up stage where the state finally realizes and is privately observed by the sender (entrepreneur). In other words, we replace

[4] See finance textbooks, e.g., Ross et al. [2022], for this line of motives to issue convertible debt.

the receiver's action in the baseline persuasion model by a mechanism. In a bilateral trade setting, Roesle and Szentes [2017] which is followed up by Condorelli and Szentes [2020], studies the optimal disclosure and pricing where the buyer can design and observe a private signal about her valuation. The key difference from our paper is that the sender here commits to a public signal structure about private information.

Last but not least, our paper is related broadly to some recent analysis that incorporate information design in various financing contexts. For instance, Szydlowski [2020] studies a corporate finance setting where an entrepreneur jointly designs disclosure and financing policies. The disclosure policy in the paper simply reveals whether firm cash-flows are above a threshold or not. The optimal security in general is indeterminate. Inostroza and Tsoy [2022] studies the joint design of information and security, but with a very different setting, where the information and security are designed by the same agent who seeks funding, and information is only private to that agent. Their optimal security turns out to be pure equity. Azarmsa and Cong [2020] considers the role of persuasion in relationship finance. The sender in their model cannot commit to any information structure, and there are multiple receivers who have different access to signals.

2 Model

An entrepreneur has the ability to run a firm, but has no funds. So he tries to persuade an investor with deep pockets to finance the investment opportunity. Initiating the firm requires an investment $I > 0$ and, if invested, generates a risky cash flow (i.e., the state of the world) $x \in X \equiv \{x_1, \dots, x_N\}$, where $x_1 < x_2 < \cdots < x_N$, and $x_N > I$. The cash flow is initially known to neither party, and follows a common prior distribution $\pi \in \Delta(X)$.[5] Each cash-flow state is assumed to have positive mass, i.e., $\pi(x) > 0$ for all $x \in X$.

Besides running the firm, the entrepreneur can set up an *experiment* that indicates some plausible scenarios for the firm's cash flows. This experiment functions as a *signaling device* that persuades the investor to fund the firm. In particular, the experiment is denoted by (M, σ), where M is a measurable space, and $\sigma : X \to \Delta(M)$ is a family of

[5] We use $\Delta(X)$ to denote the set of all probability distributions on X.

distributions over M. The investor observes the entrepreneur's experiment choice (M, σ) and a signal realization in M, and then updates her belief, according to Bayes' rule, to be $\mu \in \Delta(X)$. In this sense, setting up an experiment is equivalent to choosing a distribution over $\Delta(X)$, which is denoted by $\lambda(\mu) \in \Delta(\Delta(X))$, that is *Bayes plausible*, i.e., $\int_{\mu \in \Delta(X)} \mu \, d\lambda(\mu) = \pi$. Thus, in what follows, we assume that the entrepreneur chooses a Bayes-plausible $\lambda(\mu) \in \Delta(\Delta(X))$ without loss of generality to conduct the experiment.

Observing a signal from the chosen experiment, the investor updates her belief about the cash flow and then decides whether to fund the firm or not. If she agrees to pay I, the firm will be financed and eventually generate a cash flow $x \in X$. The financing friction is that the entrepreneur privately observes the true cash flow x, potentially diverting it to himself. The investor can audit and find out the realized cash flow only by incurring a verification cost of $c(x) > 0$. In a nutshell, if we drop the experiment-design part, the model characterizes the classic financing environment of costly state verification (CSV) as in Townsend [1979].

Figure 1: Timing

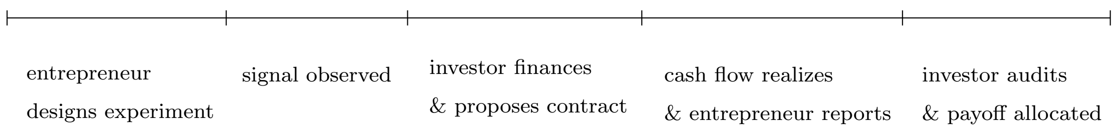

Figure 1 illustrates the timing of the events. If the firm is financed, the investor proposes a contract that specifies when to step in and audit the firm's cash flow, and how much she gets repaid. The proposed contract must guarantee that the entrepreneur has limited liability in any state of the world. Once the firm is funded and carried out, the entrepreneur gets the monetary payoff specified by the contract, on top of which he also obtains a personal benefit $B > 0$, which is non-monetary, while operating the firm.

In what follows, we first examine the optimal financing and contracting decisions given any posterior belief about the cash flow. Second, we go backward to characterize the optimal design of the experiment that induces those beliefs. Third, we analyze the welfare implications of the joint design of information and contract, and how they vary with key parameters of the model.

3 Contracting

In this section, we focus on the financing and contracting stage. After observing any signal realization from the experiment, the investor forms a posterior belief μ according to Bayes' rule. Then she makes the financing decision, denoted by $\mathbb{1}_\mu \in \{0, 1\}$, which determines whether the firm gets financed ($\mathbb{1}_\mu = 1$) or not ($\mathbb{1}_\mu = 0$). In the former case, the investor also proposes a contract Γ_μ that is contingent on both the entrepreneur's report and the auditing result if any.

By the revelation principle, we only need to consider direct contracts $\Gamma_\mu \equiv (X, a_\mu, r_\mu)$ where the entrepreneur reports cash-flows. There are two components. First, given any report $\hat{x}$, the auditing function, $a_\mu(\hat{x}) : X \to \{0, 1\}$, indicates whether the investor audits the reported cash flow ($a_\mu = 1$) or not ($a_\mu = 0$). Second, the amount of repayment to the investor r_μ, depends on the true and the reported cash-flows, and the auditing decision. In particular, if the report $\hat{x}$ is audited and found out to be x, the contract specifies the repayment amount to be $r^1_\mu(x, \hat{x})$. But if $\hat{x}$ is not audited, the amount is set to be $r^0_\mu(\hat{x})$. In sum, the repayment can be defined uniformly as $r_\mu(a_\mu, x, \hat{x}) \equiv a_\mu(\hat{x})r^1_\mu(x, \hat{x}) + (1 - a_\mu(\hat{x}))r^0_\mu(\hat{x})$. To ease notation, we drop the subscript μ in the following, implicitly referring that all contracts considered are contingent on the posterior belief.

Given a posterior belief μ, the investor maximizes her expected payoff by proposing a contract that is subject to incentive compatibility and the entrepreneur's limited liability:

$$x - r(a, x, x) \geq x - r(a, x, \hat{x}), \quad \forall x, \forall \hat{x} \neq x; \tag{IC}$$

$$r^0(x) \leq x, \; r^1(x, \hat{x}) \leq x, \quad \forall x, \forall \hat{x}. \tag{LL}$$

Let $\mathcal{M}$ collect all contracts that satisfy the incentive compatibility and limited liability conditions. From the contracting problem below, we can find the maximum payoff U_μ that the investor can gain from financing the firm.

$$U_\mu \equiv \max_{\Gamma=(X,a,r)} \mathbb{E}_\mu\big[r(x,x) - a(x)c(x)\big] \qquad \text{(In)}$$
$$s.t. \quad \Gamma \in \mathcal{M}$$

The optimal investing and auditing policies are summarized in the following result. Any optimal contract solving problem (In) consists of two regions separated by a constant (repayment threshold) $\bar{r}$ that is great than I. The region above $\bar{r}$ is characterized by no auditing and a constant investment return that is invariant to cash-flow information. The region below $\bar{r}$ where the investor intervenes to audit is shown to have information-sensitive repayments. Note that the optimal contract may not be unique in terms of the threshold $\bar{r}$. If so, our analysis focuses on the one with the lowest $\bar{r}$, which induces the largest total welfare (due to the saving of verification cost) and the entrepreneur's payoff.[6]

Proposition 1. *The investor's maximum payoff U_μ (weakly) decreases in the verification cost $c(x)$ for each x. The firm is financed, i.e., $\mathbb{1}_\mu = 1$, if and only if $U_\mu \geq I$. In that case, there exists $\bar{r} > I$ such that the optimal contract is: $a(\hat{x}) = 1$, $r^1(x, \hat{x}) = x$, for all $\hat{x} < \bar{r}$; and $a(\hat{x}) = 0$, $r^0(\hat{x}) = \bar{r}$, for all $\hat{x} \geq \bar{r}$.*

Obviously, the firm gets financed only if $U_\mu \geq I$. Otherwise, this investment opportunity will be simply forgone. When the costs are relatively high, the financing hold-up problem could occur even if $\mathbb{E}_\mu[x] > I$. In the case when the firm does get financed, there are dead-weight losses in general due to the ex-post auditing. Importantly, these qualitative features, along with the form of the optimal contract, do not depend on either the posterior belief (induced by certain signal realization) or the verification cost $c(x)$.

4 Optimal disclosure

This section goes backward to consider the firm's disclosure policy. In the early stage of the firm's development, the entrepreneur designs experiments to influence the investor's

[6]As a convention in the Bayesian persuasion literature, we assume that the entrepreneur-preferred optimal contract is chosen when it is not unique, so that the entrepreneur's indirect utility as a function of μ is upper semi-continuous.

financing and contracting decisions. If the firm gets financed upon a signal realization that induces the posterior belief μ, then Proposition 1 implies that both the optimal repayment and the auditing policies are determined by the threshold $\bar{r}_\mu$. The entrepreneur eventually gets a monetary payoff of $\max\{x - \bar{r}_\mu, 0\}$ from the firm cash-flow realization x and the personal benefit B from managing the firm. Obviously, if the firm is not financed, the entrepreneur gets zero payoff. Taking into account the investor's decisions, the entrepreneur maximizes his expected payoff and commits to an optimal disclosure policy $\lambda^*(\mu)$ that solves:

$$\max_{\lambda(\mu)} \int_{\mu\in\Delta(X)} \mathbb{E}_\mu\big\{[\max\{x - \bar{r}_\mu, 0\} + B]\mathbb{1}_\mu\big\}\, d\lambda(\mu) \qquad \text{(En)}$$
$$s.t. \quad \int_{\mu\in\Delta(X)} \mu\, d\lambda(\mu) = \pi$$

All signal realizations of an experiment will either induce a posterior belief that successfully persuades financing or inform the investor to forgo the financing opportunity. We first show an important feature that all the financing signal realizations collectively exhibit. Clearly, the firm generates positive investment surplus[7] only from the cash-flow states above I, which are referred to as the efficient states. If a signal realization does persuade financing, its posterior belief has to contain at least some measure over these states. The following proposition shows that actually the total measure of the efficient states will be included in the financing signal realizations. To facilitate notation, we denote the cash-flow distribution of any posterior belief μ conditional on being efficient as $\mu^h = \mu(x|x \geq I)$, and call it the upper truncation of μ. Similarly, the lower truncation is denoted as $\mu^l = \mu(x|x < I)$.

Proposition 2. *In any optimal disclosure policy, the upper truncation of all posterior beliefs that persuade financing constitutes a split of the measure π^h. Moreover, the expected payoff that the investor gets from the upper truncation of these beliefs locates in the interval $[\underline{u}, \bar{u}]$, where $\underline{u} \equiv U_{\pi^h} - I$ and $\bar{u} \equiv \mathbb{E}_{\pi^h}(x) - I$.*

The result implies that the upper truncation of the prior cash-flow distribution, i.e., π^h, is contained in the signal realizations that successfully persuade financing. The intuition

[7]The concept of surplus and efficiency in this paper does not take into account any personal benefit.

is straightforward. If any part of the measure π^h were not financed, then revealing it in a separate signal would be a dominant strategy. In that case, the investor would provide funding and get a non-negative net payoff from the additional surplus by auditing none of the cash flows. The entrepreneur would obtain a non-negative monetary payoff from the surplus and a positive personal benefit. So all the efficient states will be financed, generating the conditional investment surplus $\bar{u}$.

Though the investor's gain from this surplus depends on how the prior belief is split, her profit from its upper truncation π^h locates between the two boundaries $\underline{u}$ and $\bar{u}$. On the one hand, the investor can always commit to the optimal contract under the prior information π^h (no further disclosure), no matter which signal realizes. From section 3, we know the investor value from that contract is $\underline{u}$, which is the lowest investor expected payoff from any disclosure policy over π^h. On the other hand, the investor cannot get more payoff from the truncation π^h than the conditional surplus $\bar{u}$, which is simply the maximum expected payoff that the investor possibly obtains from contracting given any split of π^h.

In general, problem (En) involves a trade-off of increasing the financing probability versus getting a higher monetary payoff. To satisfy the financing constraint, any experiment design has to either shield the investor from potential net losses, or leave her with enough positive payoff. These can be achieved by disclosing separate information of either some low-yield states or some high-yield ones. To the entrepreneur, the former strategy shrinks his personal benefit due to decreased probabilities of being financed, while the latter reduces the amount of monetary payoff that he receives.

To better illustrate the optimal disclosure policy and its impact on contract offerings, we first solve two special cases of the design problem. The first case is where either the entrepreneur's personal benefit or the firm downside risk is large. The optimal disclosure turns out to be achieved through an experiment that is designed to maximize the firm's financing probability. We call this type of design the 'max-financing experiment'. In the second case, the priority of any optimal experiment is to maximize the entrepreneur's monetary payoff. This type of design is called the 'max-monetary-payoff experiment'. Then we show in general the optimal disclosure can be implemented in two stages: the first-stage signal leads to the selection of the second-stage experiment which is either the max-financing type or the max-monetary-payoff type. Essentially, the two-stage arrangement is designed to balance the entrepreneur's objectives of enlarging the financing probability and raising the monetary payoff.

4.1 Max-financing experiment

In this part, we first study the case in which the motive to enlarge financing probability dominates in the experiment design. We will provide a sufficient condition for this scenario to occur, and then characterize the resulting optimal disclosure policy.

So far we have shown that the total measure of the efficient states will be financed. How to persuade the investor to finance the inefficient states is more complex, since the investor obviously incurs losses in such states. When a cash flow x below the investment level I arises, the ex-post payoff of the investor is $y(x) \equiv x - c(x)$, given that the cash flow is in the audit region.[8] Thus, the net investor payoff will be $y(x) - I < 0$. To include any measure of this cash flow x in some financing signal realization, the experiment design may have to leave more money on the table to compensate the investor for her loss in such inefficient state, which can possibly be achieved by disclosing more information of the efficient states.

Whether such arrangement is worthwhile for the entrepreneur or not depends on the magnitude of personal benefit B. If the personal benefit is large enough and satisfies $B \geq I - y(x)$, then including some positive measure of x in a financing signal realization and sacrificing a monetary payoff of $I - y(x)$ weighted by the measure is beneficial for the entrepreneur. The set containing such cash flows is defined to be $W \equiv \{x < I : y(x) \geq I - B\}$, whose measure is denoted by $q \equiv \pi(W)$, and conditional probability by $\pi^w(x) \equiv \pi(x|W)$. The set W essentially represents the set of inefficient states that the entrepreneur would persuade the investor to finance, even if additional monetary payoffs have to be sacrificed. In other words, the entrepreneur is willing to give up more monetary payoff in exchange for higher financing probability up to the states in W, over which the

[8] Recall that Proposition 1 has proved that all inefficient states will be audited in the optimal contract following any financing signal realization.

investor's expected payoff (loss) is equal to $u \equiv \mathbb{E}_{\pi^w}[y(x)] - I$.[9]

However, the target set of cash flows W may not all be financed. Clearly, the largest profit that can be left to the investor can not exceed the expected positive investment surplus. If this amount is smaller than the expected investor loss magnitude from W then some states in this set have to be dropped from the financing signal realizations. In this case, the trade-off in the experiment design problem of (En) disappears, with the objective now being to simply maximize the financing probability. In addition, the inefficient states beyond the target set W, if any, will not be financed either, since they bring even larger losses to the investor. In sum, to save the investor from potential heavy losses, the entrepreneur can disclose the 'bad states' with the investor payoff being at the bottom so that they are not invested in.

To facilitate characterizing the disclosure policy, we relabel all the efficient states to be $I \leq x^1 < ... < x^K$, and denote the set of these states to be $S \equiv \{x^1, ..., x^K\}$ whose probability is $p \equiv \pi(S)$. Additionally, we divide the set of cash flows X into two groups, given a threshold value y of the ex-post investor payoff. In particular, we define $Y_y \equiv \{x < I : y(x) \leq y\}$ which represents the set of 'bad states', and $Z_y \equiv \{x \geq I\} \cup \{x < I : y(x) \geq y\}$. Effectively, we divide cash flows in the way that $Y_y \cup Z_y = X$, and $Y_y \cap Z_y = \{x < I : y(x) = y\}$. With these definitions, we can now summarize the properties of the disclosure policy that induces maximum financing.

Proposition 3. *If $p\bar{u} + qu \leq 0$, then any optimal disclosure maximizes the firm's financing probability. The max-financing experiment is constituted of a threshold y_f and $K + 1$ signal realizations such that*

- (a) *one signal realization induces a posterior belief, with support Y_{y_f}, under which the firm is not financed; and K signal realizations, each induces a posterior belief whose support is contained in Z_{y_f} and under which the firm is financed;*
- (b) *the upper truncation of each financing belief has measure at only one state;*
- (c) *the investor expects zero payoff from each signal realization.*

[9] See Lemma A.2 in the Appendix for a proof of this result.

When the firm downside risk is large (p small) or the entrepreneur's personal benefit is large (which implies that q is large by the definition of the set W), Proposition 3 shows that the objective of designing the experiment shrinks to simply maximize the financing probability of the firm. This is because the expected positive investment surplus $p\bar{u}$, in this case, is not even enough to compensate the potential expected loss qu from financing the whole cash-flow set W. Then it is optimal to leave all the surplus to the investor. This way, the investor gets the highest possible profit, and can be persuaded to finance as many inefficient states in the target set W as possible.

The result also shows that this type of allocation can be achieved by revealing accurate information regarding the efficient states. If some of them are pooled together, the investor has to either lower the contract repayment or audit more often for the additive measure, both of which reduce profits. As a result, all efficient states will be fully separated and mixed with the inefficient ones to achieve the max-financing goal. In the optimal disclosure, each financing signal realization reveals one and only one of the efficient states.

In addition, the optimal disclosure informs the investor to avoid providing funds when the no-financing signal realizes with positive probability. There is no need to further differentiate the contained states here which are designed to induce the same investor decision. Hence, the simplest experiment contains one no-financing signal realization and K other realizations that successfully persuade financing. The former induces a posterior belief whose support is Y_{y_f}, while the latter all induce posterior beliefs with support contained in Z_{y_f}. The efficient cash flows are all contained in the financing support Z_{y_f}. But the inefficient ones are separated to two groups by a cutoff value y_f. The states under which the investor's payoff with auditing $y(x) \geq y_f$ are contained in Z_{y_f}, while those satisfying $y(x) \leq y_f$ are contained in the no-financing signal's support Y_{y_f}.[10]

Beside the mass of one efficient state, each financing signal realization also contains some inefficient states included in the overall financing measure. Although the way in which they mix up is not unique, each financing belief must induce zero expected payoff

[10]In the proof of Proposition 3 in the Appendix, we show that the posterior belief contains all the mass of the cash-flow states $x \in Y_{y_f}$ that satisfies $y(x) < y_f$ from the prior measure π, and possibly partial mass of the states $x \in Y_{y_f}$ satisfying $y(x) = y_f$.

for the investor. Otherwise, some no-financing measure would be mixed in to enhance the financing probability. A simple and straightforward optimal design consists of $K+1$ signal realizations in total, among which one induces the no-financing decision and the rest K financing. Then each financing signal realization j induces a posterior belief that contains the total mass of x^j and some fraction of the measure over the chosen inefficient states. We postpone the details of that discussion to section 4.3.

4.2 Max-monetary-payoff experiment

In this part, we consider the other special case where the entrepreneur prioritizes his monetary payoff in designing the experiment. When the entrepreneur's personal benefit B or the firm's downside risk is small, the incentive to reveal more accurate information in exchange for larger financing probability disappears. The trade-off associated with the experiment design leans toward maximizing the entrepreneur's monetary payoff first. The following result shows a sufficient condition under which this scenario arises.

Lemma 1. *If $p\underline{u} + qu \geq 0$, then any optimal disclosure maximizes the entrepreneur's monetary payoff.*

Recall that the candidate set of cash-flows to be financed among the inefficient states is W. The entrepreneur would not persuade the investor to finance more inefficient states beyond this set if he has to give up more monetary payoff. The investor's expected losses from these states qu is relatively limited when the parameter B is small. If the magnitude of this loss is smaller than the smallest investor expected payoff from the efficient states $p\underline{u}$, then the target set of W will be contained in the financing measure, since the total investor payoff is non-negative. In this case, the primary objective of designing the experiment is to maximize the entrepreneur's expected monetary payoff.

We know from proposition 1 that all inefficient cash-flows that are financed will be eventually audited and transferred to the investor. In addition, given any financing belief μ, the entrepreneur gets monetary payoffs only from the states above the repayment threshold $\bar{r}_\mu$, which is larger than the investment level I. These payoff features imply that the optimal design here focuses on the truncated measure of μ over the efficient states when maximizing the entrepreneur's monetary payoff. In particular, the following Lemma shows that the entrepreneur's expected monetary payoff keeps constant when we split the belief μ to its upper and lower truncation.

Lemma 2. *Given any financing belief μ (i.e., $\mathbb{1}_\mu = 1$) with $\mu(\{x < I\}) > 0$, the repayment threshold induced by μ is the same as that induced by its upper truncation μ^h. Moreover, the entrepreneur gets the same expected monetary payoff from μ as from the split of μ^h and μ^l.*

When designing the contract given any posterior belief, the investor has to incur more verification costs if she raises the repayment threshold to boost the investment return. Truncating the belief μ to μ^h does not change the relative measure of the efficient states. Then the investor's trade-off of larger verification cost versus higher return in choosing the repayment threshold among the efficient states is the same for μ and μ^h. Since the investor faces the same marginal decision given the two beliefs, the optimal repayment thresholds will be the same, or $\bar{r}_\mu = \bar{r}_{\mu^h}$. This fact further implies that while truncating the belief μ to μ^h, the entrepreneur gets monetary payoff from the same set of states, i,e., $\{x \geq \bar{r}_\mu\}$. Therefore, the expected monetary payoff for the entrepreneur does not vary when splitting any belief μ to its truncation μ^h and μ^l.

Lemma 2 implies that for any experiment design of the prior belief π, we can further truncate each posterior belief without affecting the total expected monetary payoff for the entrepreneur. Now the sequence of these upper truncating measures must constitute a split of π^h, by the result of Proposition 2. Further, these upper truncating measures must be induced by an optimal disclosure of π^h, if the original experiment indeed maximizes the monetary payoff of the entrepreneur, because he gets no monetary payoff from the lower truncation π^l of the prior belief. Hence, given the result, we can search for the desired experiment design here in two separate steps. We first construct an optimal disclosure over the efficient measure π^h. Then, we characterize which inefficient states to be mixed in to form the financing beliefs.

Now let us consider the first step, that is, how to design an experiment over the efficient measure π^h taking into account the subsequent investor choice of the financing contract. This task turns out to be more complex than the second step in searching for the solution. Obviously, any posterior belief out of this experiment will induce financing, since its support contains only states above I. In the contracting stage, the investor can benefit from raising the repayment threshold to get higher returns. But she may also incur more costs to audit additional cash-flow reports. In light of this trade-off, the entrepreneur can potentially persuade the investor to lower repayment thresholds.

To get the optimal disclosure, we can conduct a sequence of binary splits starting from the prior belief. The basic idea of the design is to separate out a posterior belief in each split that i) induces the lowest cash flow in its support as the repayment threshold; and ii) utilizes as little measure as possible of the relatively lower states. The first criterion guarantees the largest entrepreneur payoff for the belief picked in the current split. The second criterion is to leave more measure of relatively lower states to the residual belief, possibly inducing lower repayment thresholds in future rounds of splits. The Lemma below illustrates how to find the candidates of such posterior beliefs that we can separate out in each optimal split.

Lemma 3. *For any nonempty subset $S' \subseteq S$, there exists a unique belief $\mu^{S'} \in \Delta(S')$ such that: $Supp(\mu^{S'}) = S'$; and it is indifferent for the investor to choose any element of S' as the optimal repayment threshold.*

The result shows that we can find a posterior belief on any set of efficient cash flows that makes it indifferent for the investor to choose any contained state as the repayment threshold. For instance, let us consider the largest set S. Setting the repayment threshold $\bar{r}$ to be x^1, the investor obviously obtains a constant payoff x^1 no matter which cash flow realizes. By raising $\bar{r}$ to be x^2, the investor gets the expected payoff $x^2 - \mu(x^1)[x^2 - x^1 + c(x^1)]$. The repayment becomes higher, so does the verification costs. To persuade the investor not to raise $\bar{r}$ to x^2, the entrepreneur needs to design the probability $\mu(x^1)$ to be larger than some critical value, which is the indifference level $\mu^S(x^1)$ specified in Lemma 3. Similarly, persuading the investor not to raise $\bar{r}$ from x^1 to x^3 implies that the total expected cost of auditing the states x^1 and x^2 cannot be too small if $\bar{r} = x^3$. In other words, the persuasion requires a joint minimum condition for $\mu(x^1)$ and $\mu(x^2)$, which determines the indifference probability $\mu^S(x^2)$, given that $\mu(x^1)$ is set to be $\mu^S(x^1)$. Similarly, persuading the investor not to further raise $\bar{r}$ implies the rest probabilities in the belief μ^S. In sum, the indifference probabilities are pinned down upward from the bottom, containing the smallest measure in each comparison to make the investor just indifferent of whether to raise the threshold or not.

Given these candidates of posterior beliefs, we now illustrate a general procedure to obtain an optimal disclosure. The motive of persuasion here is clearly to reduce repayment thresholds. Whether to continue splitting or not depends on the investor's decision. Given any belief, if the investor chooses the repayment threshold to be the lowest cash flow in the current support, then the entrepreneur will not persuade any deviation, since his expected payoff has already been maximized. In this case, no further split of the belief is needed. By contrast, if the chosen repayment threshold by the investor is larger than the minimum of the current support, then the entrepreneur does have incentive to continue splitting, persuading the investor to lower the threshold.

Proposition 4. *It is optimal to split the truncated prior π^h to a sequence of indifference beliefs and a residual through the following iteration.*

1. *Let $\hat{\mu}_0 = \pi^h$, $S_0 = S$, and $i = 1$.*

2. *If $\bar{r}_{\hat{\mu}_{i-1}} = \min\{S_{i-1}\}$, then no further split is needed, and let $J = i - 1$. Otherwise, let $\xi_i = \min_{x \in S_{i-1}} \frac{\hat{\mu}_{i-1}(x)}{\mu^{S_{i-1}}(x)}$, and split $\hat{\mu}_{i-1}$ to the two beliefs $\mu^{S_{i-1}}$ and $\hat{\mu}_i$ such that $\hat{\mu}_{i-1} = \xi_i \mu^{S_{i-1}} + (1 - \xi_i)\hat{\mu}_i$. It can be proved that $\bar{r}_{\hat{\mu}_i} = \bar{r}_{\pi^h}$.*

3. *Let $S_i = supp(\hat{\mu}_i)$. Then, repeat step 2 after setting i to be $i + 1$.*

As a result, we have $\pi^h = \sum_{i=1}^{J} \alpha_i \mu^{S_{i-1}} + \alpha_{J+1}\hat{\mu}_J$, where $\alpha_i = \xi_i \prod_{j=1}^{i-1}(1 - \xi_j)$ for $i = 1, , , J$, $\alpha_{J+1} = 1 - \sum_{i=1}^{J} \alpha_i$.

Proposition 4 illustrates how we can obtain an optimal disclosure of the prior π^h. If it is necessary to disclose more information than the prior, i.e., $\bar{r}_{\pi^h} > x^1$, then the process begins with splitting π^h to the indifference belief of all the efficient states μ^{S_0}. Its loading ξ_1 is maximized to the extend that the mass of some cash-flow state is just exhausted. Note that it is possible that multiple states are exhausted at the same time. The belief $\hat{\mu}_1$ is the residual after the first split. Excluding those exhausted states leaves us with the support of $\hat{\mu}_1$, denoted as S_1. This set contains the cash flows for the second round split. Then we repeat this process until step J when the investor's choice of repayment threshold given the residual belief becomes the minimum of its support. If $J > 1$ or at least one split is needed, the proposed procedure results in posterior beliefs $\mu^{S_0}, ..., \mu^{S_{J-1}}$, and $\hat{\mu}_J$, whose supports are a sequence of shrinking subsets $S_0 \supsetneq S_1 \supsetneq \cdots \supsetneq S_J$. The repayment thresholds for these posterior beliefs are the minimum of their supports, and therefore are (weakly) increasing to reach the prior threshold $\bar{r}_{\pi^h}$.

The entrepreneur can successfully persuade the investor to lower the choice of repayment threshold in each round of belief split. Since the posterior belief that is separated out in each step induces indifferent threshold choices among the available cash-flow states, it is optimal for the investor to choose the lowest one that benefits the entrepreneur most. Additionally, the result also indicates that the residual belief in each step $\hat{\mu}_i$ induces the same repayment threshold as the prior π^h. Hence, each binary belief split strictly increases the entrepreneur payoff in the separated-out component, without reducing his payoff in the residual part. Then, iterating the process until there is no room to improve clearly makes the entrepreneur strictly better off.

It is worth noting that the proposed disclosure policy possibly contains redundant signal realizations in terms of payoffs. It is possible that multiple indifference beliefs in Proposition 4 induce the investor to choose the same repayment threshold; that is, their supports share the same minimum element. Since the entrepreneur only gets positive payoffs above the repayment level, combining these measures into one posterior belief does not impact his expected payoff. The result below provides such an simplification.

Corollary 1. *Suppose $\bar{r}_{\pi^h} = x^t$. An optimal disclosure $\{\mu_j^m, \lambda_j^m\}_{j=1}^t$ of π^h that satisfies $\bar{r}_{\mu_j^m} = x^j$ can be constructed from Proposition 4 as: $\lambda_t^m = \alpha_{J+1}, \mu_t^m = \hat{\mu}_J$; for $1 \le j \le t-1$, $\lambda_j^m = \sum_{i=1}^J \alpha_i \mathbb{1}_{ij}$, $\mu_j^m = \frac{\sum_{i=1}^J \mathbb{1}_{ij}\alpha_i \mu^{S_{i-1}}}{\lambda_j^m}$ if $\lambda_j^m > 0$, where $\mathbb{1}_{ij} = 1$ if $\min\{S_{i-1}\} = x^j$, otherwise $\mathbb{1}_{ij} = 0$. The investor's total expected payoff is $\sum_{j=1}^t \lambda_j^m U_{\mu_j^m} - I = \underline{u}$.*

Corollary 1 shows that whenever the repayment threshold is strictly larger than x^1 given the prior π^h, there always exists some disclosure design that makes the entrepreneur better off. In general, if the original threshold is x^t given π^h, then at most t signals are needed in the optimal disclosure of Corollary 1. The jth signal realization $(j = 1, ..., t)$ induces a posterior belief μ_j^m that persuades the investor to write a contract that requires a repayment of x^j. Moreover, this repayment level is the lowest cash flow possible when the posterior μ_j^m arises, which means that the investor never verifies the efficient states.

The construction of this disclosure policy essentially combines the adjacent steps in the procedure of Proposition 4 that induce the same repayment choice. In this regard, the disclosure policy here can be generated through a similar iteration with modified 'larger steps' that suppress redundant information originally derived. The obtained posterior μ_j^m in the jth step now has a clear economic meaning. It is the posterior that persuades the investor to pick the lowest repayment x^j with the largest probability given the starting belief in the jth round of the new iteration. This is because what we suppress in the new iteration are a bunch of indifference beliefs that together exhaust the mass of their lowest support x^j. If we combine more measure of higher cash flow to μ_j^m linearly, then the investor would strictly prefer a higher repayment than x^j. In this sense, the value $\frac{\lambda_j^m}{1-\sum_{j'=1}^{j-1}\lambda_{j'}^m}$, represents the largest probability that the investor can be persuaded to pick the lowest possible repayment given the starting belief in the jth step.

Corollary 1 also shows that the disclosure policy stated above does not alter the investor's expected payoff. Given the belief π^h, the investor would optimally set $\bar{r}_{\pi^h} = x^t$ and obtain the total payoff of $\underline{u}$ from investing and contracting, if no information is disclosed. When the disclosure policy of Corollary 1 is in place, the investor gets payoff x^j if the jth signal realization arises. And this payoff would not vary even if the investor sets the repayment threshold to be x^t in this case. The reason is that by construction each signal realization is composed of a bunch of indifference beliefs all of which contain positive measure of the cash flow x^t. In other words, setting a constant x^t across all signal realizations is also optimal for the investor. Hence, in expectation, the investor gets the total payoff of $\underline{u}$, as if there were no further splitting of the prior π^h.

Based on the analysis of the upper truncation π^h, we can now get an optimal disclosure of the prior belief π and the induced allocations in the case where the entrepreneur maximizes monetary payoff.

Proposition 5. *Assume that $p\underline{u} + qu \geq 0$ and $\bar{r}_{\pi^h} = x^t$. The max-monetary-payoff experiment is constituted of a threshold y_m and $t+1$ signal realizations such that*

(a) *one signal realization induces a posterior belief, with support Y_{y_m}, under which the* *firm is not financed; and t signal realizations, each induces a posterior belief whose support is contained in Z_{y_m} and under which the firm is financed;*

(b) *the upper truncation of the jth financing belief is μ_j^m;*

(c) *the investor expects zero payoff from each signal realization if $\mathbb{1}_\pi = 0$; otherwise, the no-financing signal realization never arises, i.e., $y_m = \arg\min_{x<I} y(x)$.*

4.3 General disclosure with two-stage

With the analyses of the previous two special scenarios, we are ready to consider the general case of the experiment design by taking into account the trade-off between financing probability and monetary payoff. Increasing the financing probability requires more inefficient states to be included in the financing signal realizations, which reduces the investor's payoff. To persuade financing, the entrepreneur may have to leave more monetary payoff to the investor. This can be done by revealing more accurate information of the efficient states. The following result shows that a natural way to conduct an optimal experiment is to disclose information in two stages. The first-stage realization selects the experimentation type, while the second-stage discloses more detailed information that eventually induces the desired financing and contracting decisions.

Theorem 1. *The following two-stage experiment is optimal.*

(a) *In the first stage, a binary signal splits the prior by $\pi = \beta\mu_f + (1-\beta)\mu_m$, where μ_f and μ_m both have support X and upper truncation π^h, and*

$$\beta = \frac{p}{p+q}(1+\frac{\bar{u}}{-u}) \cdot \min\left\{\frac{-\min\{p\underline{u}+qu, 0\}}{p(\bar{u}-\underline{u})}, 1\right\}.$$

(b) *In the second stage, it is optimal to design a max-financing experiment under the intermediate belief μ_f, and a max-monetary-payoff experiment under μ_m.*

(c) *There exists a common investor ex-post payoff y across the second stages that separates the no-financing-realization support Y_y and the supports of financing realizations which are contained in Z_y.*

The proposed experiment reflects that there are two channels in this design that can help relieve the information hold-up in the firm financing environment. On the one hand, the experiment can reveal to the investor whether her ex-post payoff would be at the bottom of its range or not. By teasing out enough states that incur large investor losses, the entrepreneur can possibly persuade financing. On the other hand, the experiment can also reveal to the investor more accurate information of the efficient states. That way, the investor can acquire larger fraction of the positive investment surplus to compensate for potential losses. The former channel implies that the signal realizations will either induce financing or not. Forgoing the set of cash flows Y_y reduces the financing probability and results in a lower personal benefit for the entrepreneur. The latter channel leads to the randomization in the first-stage of the experiment. Revealing complete information of efficient cash flows with probability β reduces the entrepreneur monetary payoff. The overall disclosure strategy essentially trades off larger financing probability (smaller y) versus higher monetary payoff (smaller β) in determining the optimal experiment.

4.3.1 Financing

Theorem 1 shows a common financing feature of any optimal disclosure design. That is, one no-financing realization is sufficient to inform the investor that her ex-post total payoff $y(x)$ would be below the cutoff y and therefore no need to invest. This design relieves the information hold-up problem in the firm financing environment by teasing out the largest investor losses. Funding can be rationally induced when the no-financing realization does not arise. Note that the excluded states from financing in this design may not be those with lowest cash-flows, because it is the net investor payoff that determines the financing pattern. The value $y(x)$ could be non-monotone due to the possibility that the auditing cost $c(x)$ potentially varies across cash-flows. Otherwise, we get the following financing patterns in terms of cash-flows.[11]

Corollary 2. *Suppose $y(x)$ is monotone. Then there exists a threshold cash-flow state $\hat{x} \leq I$ and $y(\hat{x}) = y$ such that the no-financing support $Y_y = [x_1, \hat{x}]$ if $y(x)$ is increasing, while $Y_y = [\hat{x}, I)$ if $y(x)$ is decreasing.*

[11] The proof of Corollary 2 follows directly from Theorem 1, and thus is omitted.

When the investor's net payoff with auditing $y(x)$ is monotone, it is straightforward to obtain that the financing decisions exhibit a threshold property. In particular, when $y(x)$ is increasing, the no-financing signal reveals the bottom cash-flow states below $\hat{x}$, while other signals reveal those above $\hat{x}$. By contrast, when $y(x)$ is decreasing, any financing signal reveals the cash-flow states to be either below $\hat{x}$ or above I, while the no-financing signal reveals the middle states between $\hat{x}$ and I. In other words, the top and bottom states are financed, but the middle ones are not. This can be the case when the verification cost is increasing with cash flow, and the rate is higher than that of the cash-flow increase itself.

4.3.2 Experiment choice

The result of Theorem 1 shows that there are three possible scenarios of any optimal disclosure: a max-financing experiment ($\beta = 1$), a max-monetary-payoff experiment ($\beta = 0$), or some randomization of these two pure types ($\beta \in (0, 1)$). The first two scenarios, which have been illustrated before, correspond to a degenerate first-stage in the general design of the experiment. Recall that the measure π^w is the targeted inefficient states to be financed, and the expected investor loss from this measure is qu. The expected investor gain from the efficient measure π^h can be any level in the range of $[p\underline{u}, p\bar{u}]$. If the magnitude of expected investor loss $-qu$ is out of this range, then any optimal disclosure boils down to have only one objective, either max-financing or max-monetary-payoff. Otherwise, optimal is achieved by randomizing experiments that trades off the two goals.

In the case where the magnitude of the expected investor loss $-qu$ is larger than the expected profit upper bound $p\bar{u}$, all the positive investment surplus are left to the investor to enhance the financing probability. In fact, some measure of the candidate cash-flow set W have to be foregone, in order to further reduce the investor loss. By contrast, in the case where the magnitude of the expected investor loss $-qu$ is smaller than the expected profit lower bound $p\underline{u}$, the entrepreneur does not need to leave extra money on the table to finance the targeted inefficient measure π^w. Hence, the max-monetary-payoff experiment is adopted to let the entrepreneur acquire the largest possible monetary payoff. Actually, it is optimal to finance states with larger investor losses than the candidate set W to increase the entrepreneur's private benefit.

In the interior case, the optimal design involves the binary disclosure in the first stage that reveals intermediate information and induces distinct second-stage experiments. Now the combined measure of inefficient states that get financed is exactly π^w. The first stage randomization $\beta \in (0,1)$ is chosen to leave an expected investor profit from π^h that is equal to exactly the expected loss magnitude of $-qu$. In other words, the probability of adopting each second-stage experiment is pinned down by the ex-ante zero-profit condition of the investor. The intermediate belief μ_f weights more on the downside risk compared with the prior, rationalizing the max-financing experiment in the second stage. By contrast, the other intermediate belief μ_m shrinks the downside risk and makes agents believe the upside or the efficient cash flows are more plausible, resulting in the adoption of the max-monetary-payoff experiment. We show in the Appendix how these intermediate beliefs are constructed explicitly given the randomization choice β.

Corollary 3. *If the firm is funded under the prior belief ($\mathbb{1}_\pi = 1$), then the first stage is degenerate with $\beta = 0$ and $\mu_m = \pi$. The max-monetary-payoff experiment is optimal, and it contains only financing signal realizations ($Y_y = \emptyset$).*

If the investor provides funding for the firm under the prior cash-flow distribution π (i.e., $\mathbb{1}_\pi = 1$), the trade-off between financing and the entrepreneur monetary payoff would not arise, due to relatively low auditing costs.[12] In this case, the largest expected investor loss, i.e., the loss from the lower truncation of the prior π^l, is no more than the lower bound of her expected payoff from the efficient measure π^h. There is no need to reveal either some bottom inefficient states or some efficient states to persuade the investor. Since the financing probability is already one, the max-monetary-payoff experiment is adopted to obtain the largest monetary payoff for the entrepreneur.

4.3.3 Audit and repayment

Though Proposition 3 and 5 characterize the key allocation and information properties of each second-stage in the general experiment proposed by Theorem 1, their financing beliefs have not been explicitly constructed. Now we propose a design and illustrate the

[12]Corollary 3 is directly implied by Proposition 5.

resulting audit and repayment decisions. In each second-stage, the financing realizations collectively induce a measure in $\Delta(Z_y)$, which is the conditional distributions of the intermediate belief (μ_f or μ_m) on the financing support Z_y. Since designing the financing signal realizations in each second-stage follows the same approach, we ignore the impact of intermediate information here. Instead, we illustrate the construction of the financing beliefs in any second stage, supposing that their collective measure has been obtained as $\pi_1 \in \Delta(Z_y)$.

Part (b) of Theorem 1 indicates that the financing measure's upper truncation is π^h, no matter which second stage is at play. On the one hand, the entrepreneur may find it worthwhile to further split π^h as shown in sections 4.1 and 4.2, in order to induce the entrepreneur's desired audit and repayment policies. On the other hand, there is no necessity to future split the lower truncation of π_1, denoted as π_1^l, since the entrepreneur only obtains the personal benefit from this part of the financing measure. The following result shows how π_1 is optimally split based on previous analysis.

Proposition 6. *Let $\{\mu_j^h, \lambda_j^h\}_{j=1}^n$ be the split of π^h in some second stage of the optimal experiment. There exists coefficients $\{\eta_j\}_{j=1}^n \in (0,1)$ such that $\{\mu_j = \eta_j \mu_j^h + (1-\eta_j)\pi_1^l\}_{j=1}^n$ and $\{\lambda_j = \frac{\pi_1(S)\lambda_j^h}{\eta_j}\}_{j=1}^n$ form an optimal split of the financing measure π_1.*

The result proposes a simple way to construct the posterior financing beliefs. First, we can consider how to split π^h. In the max-financing experiment, where the investor expects to get $\bar{u}$ from this measure, the split is simply that $\mu_j^h(x) = \mathbb{1}_{\{x=x^j\}}$ and $\lambda_j^h = \pi^h(x^j)$. In the max-monetary-payoff experiment, where $\underline{u}$ is expected by the investor, the split of π^h is given by Corollary 1, i.e., $\mu_j^h = \mu_j^m$ and $\lambda_j^h = \lambda_j^m$. Second, we can treat the lower truncation π_1^l as a whole and mix it with each belief μ_j^h to arrive at the jth financing belief μ_j. The loading η_j on the belief μ_j^h is chosen to guarantee that the firm is indeed financed under each constructed belief μ_j. In total, the number of financing beliefs n is either equal to K in the max-financing experiment, or t in the max-monetary-payoff case given that $\bar{r}_{\pi^h} = x^t$.

Corollary 4. *Given the disclosure $\{\mu_j, \lambda_j\}_{j=1}^n$, the firm is always financed, and the investor audits all the cash flows $x < I$, but not the ones $x \geq I$. Moreover, the repayment threshold induced by the jth signal realization is set to be $\bar{r}_{\mu_j} = \min\{x \geq I : \mu_j(x) > 0\}$.*

The Corollary shows the allocations and optimal contract under the financing disclosure policy obtained in Proposition 6.[13] Regarding the contracted repayments, mixing up the lower truncation π_1^l with the belief μ_j^h has no impact on the optimal repayment threshold. This is because truncating any belief does not change the marginal trade-off in contracting. So the disclosure enables the entrepreneur to obtain personal benefit from the measure π_1^l without lowering his monetary payoff. Regarding the auditing decisions, the investor will simply pay verification costs to audit any cash-flow below I, but never audit any cash flow above I, no matter which realization occurs. If any efficient cash flow was audited, then the entrepreneur could always further split some posterior belief to increase the weights of the audited states in certain signal realizations, reducing the investor's incentive to commit auditing when such realizations arise.

5 Firm value and welfare

The auditing technology that verifies firm cash-flows ex-post is an important mechanism facilitating financing. But it comes with the dead-weight loss of verification costs. The information disclosure is an alternative through which the entrepreneur sends out relevant information ex-ante that helps facilitate financing, as well as reducing the dead-weight loss. However, the information structure here is strategically designed by the entrepreneur. This section analyzes how the disclosure policy impacts the firm value, and the expected payoffs of the entrepreneur and the investor, respectively. We also consider how this welfare analysis varies with key parameters such as the verification costs and the private benefit of managing the project.

Each signal realization sent from the experiment determines the financing decision, the monetary payoffs, and the personal benefit. Given any belief μ induced by some signal realization, the total surplus can be divided into two parts. That is, the entrepreneur value $V_\mu^E = \mathbb{E}_\mu\{[\max(x - \bar{r}_\mu, 0) + B]\mathbb{1}_\mu\}$, and the investor value $V_\mu^I = \mathbb{E}_\mu\{[\min(x, \bar{r}_\mu) - a_\mu(x)c(x) - I]\mathbb{1}_\mu\}$. Dropping the personal benefit from the total surplus, we can also construct the firm value as $V_\mu^F = \mathbb{E}_\mu\{[x - a_\mu(x)c(x) - I]\mathbb{1}_\mu\}$. If we replace the belief μ by the prior π, then the valuations V_π^i, where $i \in \{F, I, E\}$, conveniently represents those

[13] The proof of Corollary 4 follows directly from Theorem 1.

in the benchmark case of no disclosure. In addition, according to the optimal disclosure derived in Theorem 1, we can obtain the expected valuations, for the entrepreneur, the investor, and the firm, before any signal realization arises as $\overline{V}^i$. The following Lemma shows how these values change if information can be committed before the firm financing.

Lemma 4. *Compared with the no-disclosure benchmark, while the firm and the entrepreneur are both better off, the investor is indifferent. That is,* $\overline{V}^F \geq V_\pi^F$, $\overline{V}^E \geq V_\pi^E$, *and* $\overline{V}^I = V_\pi^I$.

Lemma 4 shows that the information disclosure raises the firm value by facilitating financing through two channels. First, some inefficient cash flows may be excluded from financing. Second, the auditing costs of efficient states, in any, are all saved post financing. Moreover, the enlarged firm value is acquired solely by the entrepreneur, leaving the investor with the same expected payoff as if no information was disclosed. With disclosure, the entrepreneur may also gain higher personal benefit from the enlarged financing probability which is at least p, i.e., the measure of efficient cash flows.

On the one hand, the entrepreneur has the information advantage who can always stick to no-disclosure, implying the committed disclosure cannot reduce his expected value from the benchmark. On the other hand, the investor can always ignore the disclosed information when she makes the financing and contracting decisions, implying that her expected payoff cannot be reduced either, compared to the benchmark. Through deriving the optimal disclosure explicitly, we actually show that the entrepreneur could take all the enlarged firm value by inducing desired investor decisions.

Now we examine how verification costs affect the valuations in equilibrium. The cost parameters essentially impact valuations through disclosure and contracting policies. We consider two scenarios: first, only the cost of auditing certain cash flow changes; and second, the costs of auditing all cash flows are the same and they change simultaneously.

Proposition 7. *As the verification costs vary, the ex-ante valuations exhibit the following properties.*

(a) If $x \geq \bar{r}_\pi$, *then* $\frac{\partial \overline{V}^i}{\partial c(x)} = 0$, *where* $i \in \{F, I, E\}$. *If* $x < I$, *then* $\frac{\partial \overline{V}^i}{\partial c(x)} \leq 0$. *If* $I \leq x < \bar{r}_\pi$, *then* $\frac{\partial \overline{V}^F}{\partial c(x)} \geq 0$, $\frac{\partial \overline{V}^I}{\partial c(x)} \leq 0$, *and* $\frac{\partial \overline{V}^E}{\partial c(x)} \geq 0$.

(b) *Suppose* $c(x) = c$ *for all* $x \in X$*. There exists* $0 < \underline{c} \leq \bar{c}$ *such that: i) when* $c < \underline{c}$*, the firm value* $\overline{V}^F(c)$ *strictly decreases in* c*; ii) when* $c > \bar{c}$*, the firm value* $\overline{V}^F(c)$ *strictly increases and* $\lim_{c\to\infty} \overline{V}^F(c) = pV^F_{\pi^h} > V^F_\pi(0)$*; iii) when* $c \in [\underline{c}, \bar{c}]$*,* $\overline{V}^F(c)$ *is non-monotone.*

Part (a) of Proposition 7 shows the impact of a particular verification cost, which is heterogeneous across cash flows. First, the costs of states higher than the benchmark repayment threshold $\bar{r}_\pi$ has no equilibrium effect. The result of Corollary 1 shows that the repayment threshold corresponding to any optimal signal is no larger than $\bar{r}_\pi$, which further implies that the cash flows above this level is never audited. So the verification costs of such states have no effect.

Second, the costs of states below the investment I has no or negative effect on valuations. Such costs are essentially dead-weight losses if the corresponding states are included in some financing signals. So they have non-positive impact on the firm and the investor values. The entrepreneur is also worse off by higher such costs, because he may need to either exclude more inefficient cash flows from financing, or reveal more efficient-cash-flow information to compensate for the investor losses.

Third, although the middle cash flows, above I and below $\bar{r}_\pi$, are not audited, the verification costs for these states does affect the information design and therefore may impact equilibrium valuations. In the max-monetary-payoff disclosure, a marginally larger $c(x)$ reduces the investor's conditional profit from the efficient states, leading to a lower $\underline{u}$. In equilibrium, if maximizing monetary payoff is the dominant concern in the experiment design (i.e., $\beta = 0$), then the entrepreneur can adopt this type of disclosure to get higher payoff by squeezing profits from the investor. Further, if the financing probability is smaller than one in equilibrium, then the firm value will actually increase, since additional measure of inefficient states need to be excluded from financing to compensate the investor. On the contrary, in the max-financing disclosure, higher such costs does not affect equilibrium strategies and values.

Part (b) of the proposition highlights the difference between our model and the no-disclosure benchmark. The verification costs in the benchmark are simply dead-weight losses, causing financing to break down when they are sufficiently large. However, the verification costs in our model have two contravening forces. On the one hand, higher costs to audit cash flows do reduce efficiency, if the financing probability is not shrank in equilibrium. On the other hand, higher verification costs may also push the entrepreneur to disclose more inefficient states, which reduces the financing probability and improves efficiency. These two competing forces drive the firm value to be non-monotone in general as auditing becomes more costly.

When the verification cost is sufficiently small ($c < \underline{c}$), the financing probability is always one as c goes up. If the investor payoff is positive, then she simply absorbs the higher auditing costs. Otherwise, the investor will be compensated by getting more profits from the better disclosed efficient cash flows. Either way, the firm value becomes smaller due to the increased auditing inefficiency. When the verification cost is sufficiently large ($c > \bar{c}$), the max-monetary-payoff experiment is optimal ($\beta = 0$) in equilibrium. In this case, the entrepreneur is not willing to give up any monetary payoff in exchange for more financing. As the cost c goes up, the entrepreneur can squeeze more monetary payoff from the efficient measure π^h, since the investor's conditional payoff from this measure $\underline{u}$ becomes smaller. This causes more inefficient-cash-flow measure to be disclosed and excluded from financing, which strictly boosts the firm value. Though the financing signals always contain some inefficient cash flows, their measure approaches zero as c keeps increasing, implying the firm value approaches its upper bound $pV^F_{\pi^h} = p\mathbb{E}(x - I|x \geq I)$. Note that this upper bound is strictly larger than the benchmark firm value with no verification cost which is $V^F_\pi(0) = \mathbb{E}(x-I)$. Hence, when the verification cost is sufficiently large, the experiment signals to the investor to not finance most of the inefficient states, resulting in higher expected firm value than the largest possible one in the no-disclosure benchmark.

6 Implementation

When the disclosure policy is committed, the investor can propose a financing plan that is contingent on the signal realizations. Any optimal contract that is derived previously given a posterior belief is what this financing plan specifies contingent on the associated signal realization. In this section, we consider how to implement the optimal financing plan by means of securities transacted in practice.

The optimal contracts in our model have two common features, no matter in which experiment and under which posterior belief. First, all the cash flows below the investment level I are always audited and eventually repaid to the investor. Second, when the cash flows are not audited, the investor payoff is varying across signal realizations. The former implies that the firm is financed by a debt security (or a loan) with the face value I, and defaults when its revenue falls short. The latter implies that when the firm debt is not defaulted, it can either be converted to some revenue-sensitive security or have a performance-linked rate. To better illustrate our implementation result, we now define the following two securities that potentially capture the payoff features in our model.

Convertible debt: it is a debt security that grants the investor an option to convert her stake to equity at a predetermined conversion ratio when the firm's performance reaches certain milestone.

Performance-sensitive debt (PSD): it is a debt security whose interest rate is sensitive to some performance measure of the borrower and potentially adjusts based on the associated default risk.

Proposition 8. *Conditional on the no-financing realizations not occurring, it is optimal to finance the firm by a debt with face value I that has the following contingent terms.*

1. *In the max-financing experiment, the debt can be converted to the whole equity of the firm when its cash-flow is at least I.*

2. *In the max-monetary-payoff experiment, the debt interest rate is sensitive to performance, and is set to $R_{\mu_j} = \bar{r}_{\mu_j}/I - 1$ when the jth financing-realization arises.*

Two types of securities, convertible debt and PSD, are used in our model to achieve different financing objectives while reducing the dead-weight losses to the largest extent. The debt feature of both securities are driven by the fact that all inefficient cash flows must be audited. As shown in Corollary 4, the inefficient cash flows would not be invested if they were disclosed separately, implying that the debt face value is at least I.

In the max-financing experiment, the convertible debt is used to leave all the upside and positive returns to the investor, so as to maximize the financing probability. Disclosing complete information of efficient cash flows makes it possible to implement the plan to convert debt to equity. In the max-monetary-payoff experiment, the key financing feature is that the debt interest rate varies with information as well as firm performance. Compared with the prior belief, the optimal disclosure shifts the efficient-cash-flow distribution toward its low-end levels, raising the expected verification costs. As a result, it is optimal for the investor to lower interest payments in order to audit less. The following result shows the specific features of PSD in our model.

Corollary 5. *When $\mathbb{1}_\pi = 0$ and the optimal financing is implemented by PSD, the debt interest rate is positively correlated with its default probability. Moreover, the interest rate R_{μ_j} strictly increases in j, with their upper bound being the no-disclosure rate of R_π.*

Clearly, proper disclosure shifts information and the associated debt contract terms across signal realizations, making it possbiel to finance with PSD. In particular, Corollary 5 shows that the optimal disclosure helps lower financing costs, i.e., the interest rate associated with each financing-realization is strictly lower than the no-disclosure rate. This is done by reducing dead-weight losses of auditing cash flows. In addition, the debt will be less likely to default, compensating the investor for lower returns. We can easily quantify the debt default probability as $\text{Prob}(x < I)$. Then as the financing-realization j increases, the interest rate R_{μ_j} goes up, so does the probability of default.

7 Conclusion

We consider the possibility that an entrepreneur can influence investors by disclosing flexible information by way of Bayesian persuasion. The investor can propose optimal financing contracts contingent on each signal realization. In equilibrium, the entrepreneur commits to an optimal experiment plan that reveals business information in two stages. The first stage shows a binary path that indicates the experiment is to prioritize either the financing probability or the entrepreneur's monetary payoff. In the max-financing experiment where the firm is more likely to incur losses , all positive-yield cash flows are fully disclosed to enlarge the financing probability, resulting in an convertible debt being the optimal security. In the max-monetary-payoff experiment, the high-yield cash flows will be partially pooled to leave the entrepreneur with more returns from the business, leading to optimal contracting in the form of PSD that lowers the financing cost. Compared with the no-disclosure benchmark, The information hold-up is relieved. While the firm and the entrepreneur are both better off, the investor is indifferent. We observe a non-monotonic relationship between the firm value and auditing costs. Initially, the firm value decreases with low auditing costs, but interestingly, it begins to increase significantly with higher auditing costs. This counter-intuitive effect arises because as auditing costs escalate, fewer cash flows with positive returns are allocated to the investor, necessitating the exclusion of more cash flows with negative returns from financing.

A Appendix

Proof of Proposition 1. Let $\Gamma = (X, a, r)$ be the optimal contract given μ. Define the auditing region as $X^1 = \{x \in X \mid a(x) = 1\}$, and the non-auditing region as $X^0 = \{x \in X \mid a(x) = 0\}$. Due to the IC and LL conditions, the repayment for the non-auditing region must be unique and below the minimum cash flow within that region. If the entrepreneur truthfully reports a cash flow belonging to the auditing region, the repayment must be smaller than both the cash flow (due to the LL condition) and the repayment for the non-auditing region (to discourage the entrepreneur from misreporting a non-auditing cash flow). If a misreport is detected, it is without loss of generality to set the repayment equal to the true cash flow in order to prevent such deviation. Thus, we can characterize the features of all contracts satisfying the IC and LL conditions.

Lemma A.1. *Any $\Gamma \in \mathcal{M}$ satisfies:*

1. *For all $x \in X^0$, $r^0(x) \equiv \bar{r} \leq \min X^0$;*

2. *For all $x \in X^1$, $r^1(x, x) \leq \min\{x, \bar{r}\}$, and $r^1(x, \hat{x}) = x$ if $\hat{x} \in X^1$ and $x \neq \hat{x}$.*

To maximize the investor's expected payoff, any report of $x \geq \bar{r}$ should not be audited, and the repayment for the non-auditing region must be equal to $\min X^0$; on the other hand, any truthful report of $x < \bar{r}$ should be audited and the cash flow must be completely transferred to the investor. Thus, the optimal contract is determined by a threshold $\bar{r}$ such that: $a(\hat{x}) = 0$, $r^0(\hat{x}) = \bar{r}$, for all $\hat{x} \geq \bar{r}$; $a(\hat{x}) = 1$, $r^1(x, \hat{x}) = x$, for all $\hat{x} < \bar{r}$; and

$$\bar{r} := \arg\max_r U_\mu(r) = \arg\max_r \sum_{x<r} \mu(x)\big(x - c(x)\big) + r \sum_{x \geq r} \mu(x).$$

If the firm is financed, i.e., $U_\mu \geq I$, we must have $\bar{r} > I$.

Now consider two verification cost functions $c_0(x)$ and $c_1(x)$ such that $c_0(x) = c_1(x)$ for all $x \neq \tilde{x}$ and $c_0(\tilde{x}) < c_1(\tilde{x})$. When the cost function is c_i, we define $U_\mu^{c_i}(r)$ as the investor's payoff with threshold r, and let $\bar{r}(c_i)$ be the optimal threshold and $U_\mu^{c_i}$ the corresponding maximum payoff. If $\bar{r}(c_0) = \bar{r}(c_1)$, we have $U_\mu^{c_0} = U_\mu^{c_1}$ (if $\tilde{x} \geq \bar{r}(c_0)$) or $U_\mu^{c_0} > U_\mu^{c_1}$ (if $\tilde{x} < \bar{r}(c_0)$). If $\bar{r}(c_0) \neq \bar{r}(c_1)$, we cannot have $\bar{r}(c_1) \geq \tilde{x}$; otherwise, we would have $U_\mu^{c_0}(\bar{r}(c_1)) = U_\mu^{c_1}(\bar{r}(c_1)) + \mu(\tilde{x})\big(c_1(\tilde{x}) - c_0(\tilde{x})\big) > U_\mu^{c_1}(\bar{r}(c_0)) + \mu(\tilde{x})\big(c_1(\tilde{x}) -$

$c_0(\tilde{x})\big) \geq U_\mu^{c_0}(\bar{r}(c_0))$, contradicting $U_\mu^{c_0}(\bar{r}(c_1)) < U_\mu^{c_0}(\bar{r}(c_0))$. Then, we have $\bar{r}(c_1) < \tilde{x}$ if $\bar{r}(c_0) \neq \bar{r}(c_1)$. It follows that $U_\mu^{c_1} = U_\mu^{c_1}(\bar{r}(c_1)) = U_\mu^{c_0}(\bar{r}(c_1)) < U_\mu^{c_0}(\bar{r}(c_0)) = U_\mu^{c_0}$. □

Proof of Proposition 2. Let $\lambda(\mu)$ be an optimal disclosure policy. Suppose that there exists some $\mu_0 \in \Delta(X)$ such that $\mathbb{1}_{\mu_0} = 0$, $\lambda(\mu_0) > 0$ and $\sum_{x \geq I} \mu_0(x) > 0$. We construct a different disclosure policy $\lambda'(\mu)$ satisfying: (i) $\lambda'(\mu) = \lambda(\mu)$ for all $\mu \in \{\mu \in \Delta(X) \mid \lambda(\mu) > 0, \mu \neq \mu_0\}$; (ii) $\lambda'(\mu_0^h) = \lambda(\mu_0) \sum_{x \geq I} \mu_0(x)$, $\lambda'(\mu_0^l) = \lambda(\mu_0) \sum_{x < I} \mu_0(x)$; (iii) $\lambda'(\mu) = 0$, otherwise. This modification is profitable for the entrepreneur, because now he obtains strictly more personable benefit (as well as a non-negative monetary payoff) from the signal realization μ_0^h, contradicting that $\lambda(\mu)$ is optimal. Thus, for all $x \geq I$, we have $\int_{\mu \in \Delta(X)} \mu(x) \mathbb{1}_\mu d\lambda(\mu) = \pi(x)$, which implies

$$\int_{\mu:\mathbb{1}_\mu=1} \mu^h(x) \cdot \left(\frac{\frac{\sum_{x' \geq I} \mu(x')}{\int_{\mu':\mathbb{1}_{\mu'}=1} d\lambda(\mu')}}{\int_{\mu:\mathbb{1}_\mu=1} \frac{\sum_{x' \geq I} \mu(x')}{\int_{\mu':\mathbb{1}_{\mu'}=1} d\lambda(\mu')} d\lambda(\mu)} \right) d\lambda(\mu) = \pi^h(x).$$

For whatever spit of the measure π^h, the investor can always ignore the disclosure policy and choose the contract with threshold $\bar{r}_{\pi^h} = \arg\max_r U_{\pi^h}(r)$, which secures a payoff of $U_{\pi^h} - I$ for her. On the other hand, due to the limited liability condition, the investor's repayment cannot exceed the cash flow, and thus, her expected payoff given any split of π^h is bounded above by $\mathbb{E}_{\pi^h}(x) - I$. □

Proof of Proposition 3. Let $\lambda(\mu)$ be an optimal disclosure policy. Define $\delta_x \in \Delta(X)$ as the degenerate distribution that assigns all the measure to state x. First, we prove:

Lemma A.2. *If the no-financing posterior belief, denoted by μ_0, satisfies $\mu_0(\tilde{x}) > 0$ for some $\tilde{x} \in W$, then the upper truncation of any financing posterior belief is degenerate.*

Proof of the lemma. Since a positive measure of inefficient states is not financed, the investor gets zero payoff for all financing posterior beliefs; otherwise, the entrepreneur will be strictly better off by pooling some measure of those inefficient states with the financing posterior belief which generates a positive payoff.

Suppose that μ is a financing posterior belief whose upper truncation, μ^h, is not degenerate. Then we must have $U_{\mu^h} < \mathbb{E}_{\mu^h}(x)$, since the investor has to either pay the verification cost or leave some information rent to the entrepreneur. Now we modify $\lambda(\mu)$ as follows. We split μ_0 to $\{\delta_{\tilde{x}}, \mu'_{nf}\}$ according to $\mu_0 = \frac{\varepsilon}{\lambda(\mu_0)}\delta_{\tilde{x}} + (1 - \frac{\varepsilon}{\lambda(\mu_0)})\mu'_0$, where $\varepsilon > 0$ is sufficiently small. Effectively, the total measure of state $\tilde{x}$ that is separate from the no-financing signal is $\lambda(\mu_0) \cdot \frac{\varepsilon}{\lambda(\mu_0)} = \varepsilon$. If it is merged with some financing posterior belief, it will generate an expected loss of $\varepsilon\big(I - y(\tilde{x})\big)$ to the investor. To compensate for such loss, we split a proportion ε_1 of μ to several posterior beliefs; each pools a single efficient state with the inefficient states, so that the investor can extract the entire efficient cash flows and earn zero profit for each posterior belief. Thus, ε_1 satisfies

$$\varepsilon_1 \lambda(\mu)\Big(\sum_{x \geq I} \mu(x)\Big)\big(\mathbb{E}_{\mu^h}(x) - U_{\mu^h}\big) = \varepsilon\big(I - y(\tilde{x})\big);$$

and we get a new disclosure policy, λ', which differs from λ in the following domain: $\lambda'(\mu_0) = 0$, $\lambda'(\mu'_0) = \lambda(\mu_0) - \varepsilon$, $\lambda'(\mu) = (1 - \varepsilon_1)\lambda(\mu)$, and for each $x \geq I$,

$$\lambda'(\mu_x) = \varepsilon_1 \lambda(\mu)\mu(x) + \frac{\mu(x)(x - I)}{\sum_{x' \geq I} \mu(x')(x' - I)}\Big(\varepsilon_1 \lambda(\mu) \sum_{x' < I} \mu(x') + \varepsilon\Big),$$

where

$$\mu_x = \frac{1}{\lambda'(\mu_x)}\Big[\varepsilon_1 \lambda(\mu)\mu(x) \cdot \delta_x + \frac{\mu(x)(x - I)}{\sum_{x' \geq I} \mu(x')(x' - I)}\Big(\varepsilon_1 \lambda(\mu) \sum_{x' < I} \mu(x') \cdot \mu^l + \varepsilon \cdot \delta_{\tilde{x}}\Big)\Big].$$

It follows that the entrepreneur's expected payoff changes by $-\varepsilon\big(I - y(\tilde{x})\big) + \varepsilon B > 0$ (since $\tilde{x} \in W$), contradicting that $\lambda(\mu)$ is optimal. Thus, we establish the lemma; moreover, the above modification is not profitable for inefficient states outside of W. □

To prove the proposition, we relabel the inefficient states to be $x^{-(N-K)}, \ldots, x^{-1}$ such that $y(x^{-(N-K)}) \leq \cdots \leq y(x^{-1})$. Since $p\bar{u} + qu \leq 0$, the support of any financing posterior belief must contain a single efficient state, together with a subset of inefficient states in W to make the investor receiver zero profit (due to Lemma A.2). Moreover, there exist a unique pair of $\tau^* \in \{1, \ldots, N - K\}$ and $\rho^* \in (0, 1]$ such that

$$\sum_{x \geq I} \pi(x)(x - I) + \sum_{i=1}^{\tau^* - 1} \pi(x^{-i})\big(y(x^{-i}) - I\big) + \rho^* \pi(x^{-\tau^*})\big(y(x^{-\tau^*}) - I\big) = 0.$$

Clearly, the way to pool each efficient state with the inefficient states $\{x^{-1}, \dots, x^{-\tau^*}\}$ is not unique. Here we provide a construction of it. The optimal disclosure policy, λ^*, splits π to $\{\mu_0, \mu_1, \dots, \mu_K\}$, satisfying: $\lambda^*(\mu_0) = \sum_{i=\tau^*+1}^{N-K} \pi(x^{-i}) + (1-\rho^*)\pi(x^{-\tau^*})$, where $\mu_0(x^{-i}) = \frac{\pi(x^{-i})}{\lambda^*(\mu_0)}$ for $i = \tau^*+1, \dots, N-K$, $\mu_0(x^{-\tau^*}) = \frac{(1-\rho^*)\pi(x^{-\tau^*})}{\lambda^*(\mu_0)}$, and $\mu_0(x) = 0$ for the other states; while for each $j \in \{1, \dots, K\}$,

$$\lambda^*(\mu_j) = \pi(x^j) + \frac{\pi(x^j)(x^j - I)}{\sum_{i=1}^K \pi(x^i)(x^i - I)} \Big(\sum_{i=1}^{\tau^*-1} \pi(x^{-i}) + \rho^* \pi(x^{-\tau^*}) \Big),$$

where μ_j is defined by $\mu_j(x^j) = \frac{\pi(x^j)}{\lambda^*(\mu_j)}$, $\mu_j(x^{-i}) = \frac{\pi(x^{-i})}{\lambda^*(\mu_j)}$ for $i = 1, \dots, \tau^*-1$, $\mu_j(x^{-\tau^*}) = \frac{\rho^*\pi(x^{-\tau^*})}{\lambda^*(\mu_j)}$, and $\mu_j(x) = 0$ for the remaining states. Clearly, the threshold y_f in the proposition is given by $y(x^{-\tau^*})$. $\square$

Proof of Lemma 1. We have proved in Proposition 2 that the investor's expected payoff from the efficient states is always above $p\underline{u}$, and in Lemma A.2 that it is not profitable for the entrepreneur to sacrifice the monetary payoff for pooling the inefficient states outside of W with the financing posterior beliefs. Since we have $p\underline{u} + qu \geq 0$, all inefficient states in W can be pooled with the financing posterior beliefs, regardless of how to split π^h. Thus, the entrepreneur no longer needs to take the personal benefit into consideration when solving the optimal way to split π^h. $\square$

Proof of Lemma 2. Because $\mathbb{1}_\mu = 1$, $U_\mu \geq I$. Take any $r(\neq \bar{r}_\mu) \geq I$, because

$$\begin{aligned}
&\Big[\sum_{x<\bar{r}_\mu} \mu^h(x)(x - c(x)) + \bar{r}_\mu \sum_{x\geq\bar{r}_\mu} \mu^h(x)\Big] - \Big[\sum_{x<r} \mu^h(x)(x - c(x)) + r \sum_{x\geq r} \mu^h(x)\Big] \\
=&\frac{\sum_{I\leq x<\bar{r}_\mu} \mu(x)(x - c(x)) + \bar{r}_\mu \sum_{x\geq\bar{r}_\mu} \mu(x)}{\sum_{x'\geq I} \mu(x')} - \frac{\sum_{I\leq x<r} \mu(x)(x - c(x)) + r \sum_{x\geq r} \mu(x)}{\sum_{x'\geq I} \mu(x')} \\
=&\frac{\sum_{x<\bar{r}_\mu} \mu(x)(x - c(x)) + \bar{r}_\mu \sum_{x\geq\bar{r}_\mu} \mu(x)}{\sum_{x'\geq I} \mu(x')} - \frac{\sum_{x<r} \mu(x)(x - c(x)) + r \sum_{x\geq r} \mu(x)}{\sum_{x'\geq I} \mu(x')} \geq 0,
\end{aligned}$$

and $\sum_{x<\bar{r}_\mu} \mu^h(x)(x - c(x)) + \bar{r}_\mu \sum_{x\geq\bar{r}_\mu} \mu^h(x) \geq I$, we have $\bar{r}_{\mu^h} = \bar{r}_\mu$. Since $\mathbb{1}_{\mu^l} = 0$, the entrepreneur's expected payoff stays the same, which is equal to $\sum_{x \geq \bar{r}_\mu} \mu(x)\big(x - \bar{r}_\mu\big)$. $\square$

Proof of Lemma 3. Recall that $S = \{x^1, \ldots, x^K\}$ where $x^1 < \cdots < x^K$. Assume that $S' = \{x^{t_1}, \ldots, x^{t_k}\} \subseteq S$, where $1 \leq t_1 < \ldots < t_k \leq K$ and $k \leq K$. Given $\mu^{S'} \in \Delta(S')$, Investor's expected payoff induced by the direct contract with threshold $\bar{r} = x^{t_\tau}$ is

$$\begin{aligned} U_{\mu^{S'}}(x^{t_\tau}) &= \sum_{i=1}^{\tau-1} \mu^{S'}(x^{t_i})(x^{t_i} - c_{t_i}) + x^{t_\tau} \sum_{i=\tau}^{k} \mu^{S'}(x^{t_i}) \\ &= x^{t_\tau} - \sum_{i=1}^{\tau-1} \mu^{S'}(x^{t_i})(x^{t_\tau} - x^{t_i} + c_{t_i}), \end{aligned}$$

for $\tau = 1, \ldots, k$, where c_{t_i} represents $c(x^{t_i})$ for each i. Then, from Investor's indifference conditions, we get a system of linear equations:

$$\begin{pmatrix} x^{t_2} - x^{t_1} + c_{t_1} & 0 & \ldots & 0 \\ x^{t_3} - x^{t_1} + c_{t_1} & x^{t_3} - x^{t_2} + c_{t_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x^{t_k} - x^{t_1} + c_{t_1} & x^{t_k} - x^{t_2} + c_{t_2} & \ldots & x^{t_k} - x^{t_{k-1}} + c_{t_{k-1}} \end{pmatrix} \begin{pmatrix} \mu^{S'}(x^{t_1}) \\ \mu^{S'}(x^{t_2}) \\ \vdots \\ \mu^{S'}(x^{t_{k-1}}) \end{pmatrix} = \begin{pmatrix} x^{t_2} - x^{t_1} \\ x^{t_3} - x^{t_1} \\ \vdots \\ x^{t_k} - x^{t_1} \end{pmatrix},$$

which has a unique solution:

$$\mu^{S'}(x^{t_\tau}) = \frac{\big(\prod_{i=1}^{\tau-1} c_{t_i}\big)(x^{t_{\tau+1}} - x^{t_\tau})}{\prod_{i=1}^{\tau}(x^{t_{i+1}} - x^{t_i} + c_{t_i})}, \tau = 1, \ldots, k-1; \quad \mu^{S'}(x^{t_k}) = \frac{\big(\prod_{i=1}^{k-1} c_{t_i}\big)}{\prod_{i=1}^{k-1}(x^{t_{i+1}} - x^{t_i} + c_{t_i})}.$$

Obviously, $\mu^{S'}(x^{t_\tau}) > 0$ for $\tau = 1, \ldots, k$. $\square$

Proof of Proposition 4. We first ignore the stopping rule in step 2; that is, even if we get $\bar{r}_{\hat{\mu}_{i-1}} = \min\{S_{i-1}\}$ for some i, we continue to split $\hat{\mu}_{i-1}$. The definition of ξ_i implies that $\arg\min_{x \in S_{i-1}} \frac{\hat{\mu}_{i-1}(x)}{\mu^{S_{i-1}}(x)}$ is included in S_{i-1} but not in S_i. Thus, we have $S_0 \supsetneq S_1 \supsetneq \cdots \supsetneq S_J$, where S_J is a singleton set or the last residual is already an indifference belief, which means that the iteration stops after finitely many steps.

To prove that this algorithm gives the optimal disclosure policy, we need to establish a series of lemmas. Recall that $U_\mu(r)$ is the investor's expected payoff by choosing a contract with threshold r if the posterior belief is $\mu \in \Delta(S)$. Let $U_E(r, \mu)$ denote the entrepreneur's expected payoff when the posterior belief is $\mu \in \Delta(S)$ and the investor chooses a contract with threshold r; meanwhile, define $V_E(\mu) = U_E(\bar{r}_\mu, \mu)$, and let $\overline{V}_E$ be the smallest concave function everywhere above V_E.

Lemma A.3. *It is without loss of generality to assume that the optimal disclosure policy λ^* can only put strictly positive weights on $\{\mu^{S'} \mid \emptyset \neq S' \subseteq S\}$. If S' is a singleton set, say $S' = \{x\}$, $\mu^{S'}$ stands for the degenerate distribution which puts all weights on x.*

Proof of the lemma. It suffices to show that the split in each step of Proposition 4 makes the entrepreneur (weakly) better off.

First, we show that $\bar{r}_{\hat{\mu}_{i-1}} = \bar{r}_{\hat{\mu}_i}$. Pick any distinct pair $x', x'' \in S_{i-1}$, and we assume without loss of generality that $U_{\hat{\mu}_i}(x') \leq U_{\hat{\mu}_i}(x'')$. Because $U_\mu(r)$ is linear with respect to μ, the subset $\mathcal{P}_{x'' \succcurlyeq x'} := \{\mu \in \Delta(S_{i-1}) \mid U_\mu(x') \leq U_\mu(x'')\}$ is convex. Notice that $U_{\mu^{S_{i-1}}}(x') = U_{\mu^{S_{i-1}}}(x'')$, then $\mu^{S_{i-1}}$ and $\hat{\mu}_i$ are both in $\mathcal{P}_{x'' \succcurlyeq x'}$. Since $\hat{\mu}_{i-1}$ is a linear combination of $\mu^{S_{i-1}}$ and $\hat{\mu}_i$, $\hat{\mu}_{i-1}$ belongs to $\mathcal{P}_{x'' \succcurlyeq x'}$ as well, implying that $U_{\hat{\mu}_{i-1}}(x') \leq U_{\hat{\mu}_{i-1}}(x'')$. Thus, $\bar{r}_{\hat{\mu}_{i-1}} = \bar{r}_{\hat{\mu}_i}$ for each i, and it follows that $\bar{r}_{\hat{\mu}_i} = \bar{r}_{\pi^h}$.

If $\bar{r}_{\hat{\mu}_{i-1}} = \min\{S_{i-1}\}$, then step (2) does not change the entrepreneur's expected payoff, since $\min\{S_{i-1}\}$ is always the threshold chosen by the investor. If $\bar{r}_{\hat{\mu}_{i-1}} > \min\{S_{i-1}\}$, then the entrepreneur is strictly better off, because

$$
\begin{aligned}
V_E(\hat{\mu}_{i-1}) = U_E(\bar{r}_{\hat{\mu}_{i-1}}, \hat{\mu}_{i-1}) &= \xi_i U_E(\bar{r}_{\hat{\mu}_{i-1}}, \mu^{S_{i-1}}) + (1 - \xi_i) U_E(\bar{r}_{\hat{\mu}_{i-1}}, \hat{\mu}_i) \\
&= \xi_i U_E(\bar{r}_{\hat{\mu}_{i-1}}, \mu^{S_{i-1}}) + (1 - \xi_i) U_E(\bar{r}_{\hat{\mu}_i}, \hat{\mu}_i) \\
&< \xi_i V_E(\mu^{S_{i-1}}) + (1 - \xi_i) V_E(\hat{\mu}_i).
\end{aligned}
$$

□

Lemma A.4. $\overline{V}_E = V_E$ *over* $\{\mu^{S'} \mid \emptyset \neq S' \subseteq S\}$.

Proof. Suppose $\overline{V}_E(\mu^{S'}) > V_E(\mu^{S'})$ for some non-empty subset $S' \subseteq S$, then there exists $\lambda \in \Delta(\Delta(S))$ such that $\int_\mu \mu d\lambda(\mu) = \mu^{S'}$ and $\int_\mu V_E(\mu) d\lambda(\mu) > V_E(\mu^{S'})$. Since λ is a mean-preserving spread of $\mu^{S'}$, for any $\mu \in \Delta(S)$ such that $\lambda(\mu) > 0$, we must have $\text{supp}(\mu) \subseteq S'$, which implies $\bar{r}_\mu \geq \min \text{supp}(\mu) \geq \min S' = \bar{r}_{\mu^{S'}}$. It follows that $V_E(\mu^{S'}) = U_E(\bar{r}_{\mu^{S'}}, \mu^{S'}) = \int_\mu U_E(\min S', \mu) d\lambda(\mu) \geq \int_\mu U_E(\bar{r}_\mu, \mu) d\lambda(\mu) = \int_\mu V_E(\mu) d\lambda(\mu)$, which forms a contradiction. □

Lemma A.5. *For each $k \in \{1, \ldots, K\}$, all points in $\big\{\big(\mu^{S'}, \overline{V}_E(\mu^{S'})\big) \mid \forall S'$ such that $x^k \in S' \subseteq S\big\}$ are on a hyperplane in the space $\Delta(S) \times \mathbb{R}$.*

Proof. Fix arbitrary k, and pick any S' such that $x^k \in S' \subseteq S$, then we have

$$\begin{aligned}
\overline{V}_E(\mu^{S'}) = V_E(\mu^{S'}) = U_E(\min S', \mu^{S'}) &= \sum_{x \in S} \mu^{S'}(x)(x - \min S') \\
&= \sum_{x \in S} \mu^{S'}(x)x - \sum_{x \in S} \mu^{S'}(x) \min S' \\
&= \sum_{x \in S} \mu^{S'}(x)x - \Big[\sum_{x < \min S'} \mu^{S'}(x)\big(x - c(x)\big) + \min S' \sum_{x \geq \min S'} \mu^{S'}(x)\Big] \\
&= \sum_{x \in S} \mu^{S'}(x)x - \Big[\sum_{x < x^k} \mu^{S'}(x)\big(x - c(x)\big) + x^k \sum_{x \geq x^k} \mu^{S'}(x)\Big] \\
&= \sum_{x < x^k} \mu^{S'}(x)c(x) + \sum_{x \geq x^k} \mu^{S'}(x)(x - x^k),
\end{aligned}$$

which is linear with respect to $\mu^{S'}$. $\square$

Lemma A.6. *For each $k \in \{1, \ldots, K\}$, if $\mu = \sum_{x^k \in S' \subseteq S} \lambda(\mu^{S'})\mu^{S'}$ such that $\sum_{x^k \in S' \subseteq S} \lambda(\mu^{S'}) = 1$ and $\lambda(\mu^{S'}) \geq 0$ for each S', then we have $\overline{V}_E(\mu) = \sum_{x^k \in S' \subseteq S} \lambda(\mu^{S'}) V_E(\mu^{S'})$.*

Proof. By Lemma A.3, it suffices to show that, for any $S' \neq \emptyset$ such that $x^k \notin S'$, the point $\big(\mu^{S'}, \overline{V}_E(\mu^{S'})\big)$ is on or below the hyperplane in Lemma A.5, that is,

$$\Big\{\big(\mu, \overline{V}_E(\mu)\big) \mid \mu \in \Delta(S), \overline{V}_E(\mu) = \sum_{x < x^k} \mu(x)c(x) + \sum_{x \geq x^k} \mu(x)(x - x^k)\Big\}.$$

Suppose $\big(\mu^{S'}, \overline{V}_E(\mu^{S'})\big)$ is strictly above that hyperplane. Since $\big(\mu^{S' \cup \{x^k\}}, V_E(\mu^{S' \cup \{x^k\}})\big)$ is on that hyperplane, and $\mu^{S' \cup \{x^k\}}$ is a convex combination of $\{\mu^{S'}\} \cup \{\mu^{\{S' \setminus \{x\}\} \cup \{x^k\}} \mid \forall x \in S'\}$, we have $\overline{V}_E(\mu^{S' \cup \{x^k\}}) > V_E(\mu^{S' \cup \{x^k\}})$, which contradicts Lemma A.4. $\square$

Now we are ready to prove the optimality of the disclosure policy given by the above algorithm (ignoring the stopping rule in step 2 of Proposition 4); that is,

$$\pi^h = \sum_{j=1}^{J} \Big(\xi_j \prod_{i=1}^{j-1}(1 - \xi_i)\Big)\mu^{S_{j-1}} + \prod_{i=1}^{J}(1 - \xi_i)\mu^{S_J}.$$

Since $\bar{r}_{\hat{\mu}_i} = \bar{r}_{\pi^h}$ for each i (by Lemma A.3), we have $\bar{r}_{\pi^h} \in S_i$ for $i = 0, 1, \ldots, J$, which means that π^h can be written as the convex combination of $\{\mu^{S'} \mid \forall S' \text{ s.t. } \bar{r}_{\pi^h} \in S' \subseteq S\}$. Then, by Lemma A.6, the above disclosure policy indeed achieves $\overline{V}_E(\pi^h)$.

The remaining task is to show that if we stop when $\bar{r}_{\hat{\mu}_{J_0}} = \min\{S_{J_0}\}$, the resulting disclosure policy is also optimal. Suppose we continue to split $\hat{\mu}_{J_0}$ according to the above algorithm and get $\hat{\mu}_{J_0+1}, \ldots, \hat{\mu}_J$. By Lemma A.3, we have $\bar{r}_{\hat{\mu}_i} = \min\{S_{J_0}\}$ for $i = J_0 + 1, \ldots, J$, which implies $\min\{S_i\} = \min\{S_{J_0}\}$. Thus, the additional split will not further reduce the threshold of the contract, and the following lemma shows that it is without loss of generality to merge those posterior beliefs.

Lemma A.7. *Let λ^* be the optimal disclosure policy obtained by the algorithm, if there exist $S', S'' \in 2^S \setminus \emptyset$ such that $\lambda^*(\mu^{S'}) > 0, \lambda^*(\mu^{S''}) > 0$ and $\min S' = \min S''$, then the entrepreneur's expected payoff is unchanged if we merge $\mu^{S'}$ and $\mu^{S''}$.*

Proof of the lemma. Let $\mu = \frac{\lambda^*(\mu^{S'})}{\lambda^*(\mu^{S'})+\lambda^*(\mu^{S''})}\mu^{S'} + \frac{\lambda^*(\mu^{S''})}{\lambda^*(\mu^{S'})+\lambda^*(\mu^{S''})}\mu^{S''}$. Suppose $\bar{r}_\mu > \min S'$, and then we have

$$
\begin{aligned}
U_\mu(\min S') < U_\mu(\bar{r}_\mu) &= \frac{\lambda^*(\mu^{S'})}{\lambda^*(\mu^{S'}) + \lambda^*(\mu^{S''})} U_{\mu^{S'}}(\bar{r}_\mu) + \frac{\lambda^*(\mu^{S''})}{\lambda^*(\mu^{S'}) + \lambda^*(\mu^{S''})} U_{\mu^{S''}}(\bar{r}_\mu) \\
&\leq \frac{\lambda^*(\mu^{S'})}{\lambda^*(\mu^{S'}) + \lambda^*(\mu^{S''})} U_{\mu^{S'}}(\min S') + \frac{\lambda^*(\mu^{S''})}{\lambda^*(\mu^{S'}) + \lambda^*(\mu^{S''})} U_{\mu^{S''}}(\min S') = U_\mu(\min S'),
\end{aligned}
$$

which forms a contradiction. Thus, $\bar{r}_\mu = \min S'$. Notice that $(\lambda^*(\mu^{S'}) + \lambda^*(\mu^{S''}))V_E(\mu) = \lambda^*(\mu^{S'})U_E(\min S', \mu^{S'}) + \lambda^*(\mu^{S''})U_E(\min S'', \mu^{S''}) = \lambda^*(\mu^{S'})V_E(\mu^{S'}) + \lambda^*(\mu^{S''})V_E(\mu^{S''})$, the entrepreneur's expected payoff stays unchanged. $\square$

We conclude that the algorithm in Proposition 4 gives an optimal split of π^h. $\square$

Proof of Corollary 1. From Proposition 4, we have $S_0 \supsetneq S_1 \supsetneq \cdots \supsetneq S_J \ni \bar{r}_{\pi^h} = x^t$, and $x^1 = \bar{r}_{\mu^{S_0}} \leq \cdots \leq \bar{r}_{\mu^{S_{J-1}}} \leq \bar{r}_{\hat{\mu}_J} = x^t$. By Lemma A.7, we can merge the posterior beliefs that induce the same threshold, and thus it is without loss of generality to consider an optimal disclosure policy which has at most t posterior beliefs, each inducing a distinct threshold. Since x^t is included in the supports of all indifference beliefs, the investor's expected payoff is the same as always setting x^t to be the threshold, which is $\underline{u}$. $\square$

Proof of Proposition 5. Given the optimal split of π^h in Corollary 1, it suffices to show how to pool the inefficient states with the financing posterior beliefs. Recall that we relabel the inefficient states to be $x^{-(N-K)}, \dots, x^{-1}$ such that $y(x^{-(N-K)}) \leq \cdots \leq y(x^{-1})$. Since $p\underline{u} + qu \geq 0$, the entire measure of the inefficient states in W can be properly pooled with the financing posterior beliefs. The question is whether the entrepreneur can continue to pool the inefficient states outside of W with the financing posterior beliefs to gain more personal benefit. Particularly, we distinguish two situations.

When $\mathbb{1}_\pi = 0$, the investor will not finance if all inefficient states are pooled with the split of π^h. On the other hand, as long as the investor obtains strictly positive payoff under some financing posterior belief, the entrepreneur can pool it with a positive measure of inefficient states such that the investor still gets non-negative payoff. From the proof of Lemma 2, we know that this will not change the optimal threshold, thus the entrepreneur's monetary payoff is unaffected but his personal benefit is strictly increased. As in Proposition 3, the investor obtains zero payoff under any financing posterior belief; more precisely, we choose $\tau^* \in \{1, \dots, N-K\}$ and $\rho^* \in (0, 1]$ satisfying [14]

$$p(U_{\pi^h} - I) + \sum_{i=1}^{\tau^*-1} \pi(x^{-i})\big(y(x^{-i}) - I\big) + \rho^* \pi(x^{-\tau^*})\big(y(x^{-\tau^*}) - I\big) = 0,$$

and the optimal disclosure policy, λ^*, splits π to $\{\mu_0, \mu_1, \dots, \mu_t\}$, satisfying: $\lambda^*(\mu_0) = \sum_{i=\tau^*+1}^{N-K} \pi(x^{-i}) + (1-\rho^*)\pi(x^{-\tau^*})$, where $\mu_0(x^{-i}) = \frac{\pi(x^{-i})}{\lambda^*(\mu_0)}$ for $i = \tau^* + 1, \dots, N-K$, $\mu_0(x^{-\tau^*}) = \frac{(1-\rho^*)\pi(x^{-\tau^*})}{\lambda^*(\mu_0)}$, and $\mu_0(x) = 0$ for the other states; while for each $j \in \{1, \dots, t\}$,

$$\lambda^*(\mu_j) = p\lambda_j^m + \frac{\lambda_j^m (U_{\mu_j^m} - I)}{\sum_{i=1}^t \lambda_i^m (U_{\mu_i^m} - I)} \Big(\sum_{i=1}^{\tau^*-1} \pi(x^{-i}) + \rho^* \pi(x^{-\tau^*}) \Big),$$

where μ_j is defined by $\mu_j(x^i) = \frac{p\lambda_j^m \mu_j^m(x^i)}{\lambda^*(\mu_j)}$ for $i = 1, \dots, K$, $\mu_j(x^{-i}) = \frac{\pi(x^{-i})}{\lambda^*(\mu_j)}$ for $i = 1, \dots, \tau^* - 1$, $\mu_j(x^{-\tau^*}) = \frac{\rho^* \pi(x^{-\tau^*})}{\lambda^*(\mu_j)}$, and $\mu_j(x) = 0$ for the remaining states. Clearly, the threshold y_m in the proposition is given by $y(x^{-\tau^*})$.

When $\mathbb{1}_\pi = 1$, the entire measure of inefficient states can be pooled with the split of π^h in Corollary 1. Clearly, there are infinite ways to construct the optimal disclosure

[14] Since all states in W are included in the financing posterior beliefs, we have $W \subseteq \{x^{-1}, \dots, x^{-\tau^*}\}$.

policy. Here is one solution: λ^* splits π to $\{\mu_1, \ldots, \mu_t\}$, satisfying: for each $j \in \{1, \ldots, t\}$, $\lambda^*(\mu_j) = p\lambda_j^m + \frac{1}{\underline{u}}\lambda_j^m(U_{\mu_j^m} - I)\sum_{i=1}^{N-K} \pi(x^{-i})$, where μ_j is defined by $\mu_j(x^i) = \frac{p\lambda_j^m \mu_j^m(x^i)}{\lambda^*(\mu_j)}$ for $i = 1, \ldots, K$, $\mu_j(x^{-i}) = \frac{\pi(x^{-i})}{\lambda^*(\mu_j)}$ for $i = 1, \ldots, N-K$. Accordingly, the threshold y_m in the proposition is given by $\arg\min_{x<I} y(x) = y(x^{-(N-K)})$. □

Proof of Theorem 1. If $p\bar{u} + qu \leq 0$, by definition, we have $\beta = 1$, and the optimal experiment is given by Proposition 3. If $p\underline{u} + qu \geq 0$, we have $\beta = 0$, and the optimal experiment is given by Proposition 5.

If $p\underline{u} < -qu < p\bar{u}$, the investor's expected payoff given the optimal split of π^h defined by Corollary 1 is not large enough to cover her loss under the states in W, suggesting that it is profitable for the entrepreneur to sacrifice certain amount of monetary payoff for personal benefits. On the other hand, since such loss is not too large, the entire measure of W can be included in the financing posterior beliefs. The optimal experiment must also satisfy that no inefficient states outside of W are financed, and that the investor obtains zero profit given any financing signal realization; otherwise the entrepreneur can be strictly better off by decreasing the monetary payoff that he sacrifices. Clearly, the optimal experiment is not unique. Here is one construction of it.

From $p\underline{u} < -qu < p\bar{u}$, we have $\tilde{\beta} := \min\left\{\frac{-\min\{p\underline{u}+qu,0\}}{p(\bar{u}-\underline{u})}, 1\right\} = \frac{-(p\underline{u}+qu)}{p(\bar{u}-\underline{u})} \in (0,1)$, that is, $\tilde{\beta}(p\bar{u} + qu) + (1 - \tilde{\beta})(p\underline{u} + qu) = 0$. This equation suggests that the fraction of efficient states which are fully revealed is equal to $\tilde{\beta}$, and the remaining efficient states are assigned the optimal split of π^h; meanwhile, the states in W are properly mixed with the efficient states such that the former becomes a max-financing experiment and the latter becomes a max-monetary-payoff experiment. Define a two-stage experiment as follows. Let $\mu_f = \frac{(-u)}{-u+\bar{u}} \cdot \pi^h + \frac{\bar{u}}{-u+\bar{u}} \cdot \pi^w$, and $\mu_m = \frac{(-u)}{-u+\underline{u}} \cdot \pi^h + \frac{\underline{u}}{-u+\underline{u}} \cdot \pi^w$, and then we have

$$\pi = \tilde{\beta}\frac{p}{p+q}(1 + \frac{\bar{u}}{-u}) \cdot \mu_f + (1 - \tilde{\beta})\frac{p}{p+q}(1 + \frac{\underline{u}}{-u}) \cdot \mu_m = \beta\mu_f + (1 - \beta)\mu_m.$$

Notice that, given μ_f, the investor's expected loss from the states in W, which is $\frac{\bar{u}}{-u+\bar{u}} \cdot (-u)$, is equal to her maximum gains from the efficient states, which is $\frac{(-u)}{-u+\bar{u}} \cdot \bar{u}$. Then, by Proposition 3, the max-financing experiment is optimal in the second stage under μ_f. Similarly, in the second stage under μ_m, because the investor's expected loss from the states in W is equal to her minimum gains from the efficient states, the max-monetary-payoff experiment is optimal by Proposition 5. Additionally, the optimal experiments under both μ_f and μ_m share a common ex-post payoff y defined in part (c), which is equal to $\min_{x \in W} y(x)$.

The last step is to show that the above two-stage experiment is optimal. Because the investor always obtains zero payoff in the optimal experiment when $p\underline{u} < -qu < p\bar{u}$, the social welfare is equal to the entrepreneur's payoff. Moreover, the above two-stage experiment achieves the efficient outcome: all the efficient states are financed without being verified, and all the inefficient states that are social efficient (i.e., any $x < I$ such that $x - c(x) - I + B \geq 0$) are also financed. Thus, there does not exist any experiment that is strictly better for the entrepreneur. $\square$

Proof of Proposition 6. For $j = 1, \ldots, n$, define $\lambda_j = \pi_1(S)\lambda_j^h + \big(1 - \pi_1(S)\big)\frac{\lambda_j^h\left(U_{\mu_j^h} - I\right)}{\sum_{i=1}^n \lambda_i^h\left(U_{\mu_i^h} - I\right)}$, and $\eta_j = \frac{\pi_1(S)\lambda_j^h}{\lambda_j}$. Then, the investor's expected payoff under each μ_j is equal to

$$
\begin{aligned}
&\frac{\pi_1(S)\lambda_j^h}{\lambda_j}\big(U_{\mu_j^h} - I\big) + \frac{1 - \pi_1(S)}{\lambda_j}\frac{\lambda_j^h\big(U_{\mu_j^h} - I\big)}{\sum_{i=1}^n \lambda_i^h\big(U_{\mu_i^h} - I\big)}\sum_{x<I}\pi_1^l\big(x - c(x)\big) \\
&= \frac{\lambda_j^h\big(U_{\mu_j^h} - I\big)}{\lambda_j\sum_{i=1}^n \lambda_i^h\big(U_{\mu_i^h} - I\big)}\left[\pi_1(S)\sum_{i=1}^n \lambda_i^h\big(U_{\mu_i^h} - I\big) + \big(1 - \pi_1(S)\big)\sum_{x<I}\pi_1^l\big(x - c(x)\big)\right] \geq 0,
\end{aligned}
$$

where the terms in the brackets represent the investor's ex-ante payoff given π_1, which is non-negative since it is the expectation of all financing posterior beliefs. $\square$

Proof of Lemma 4. The entrepreneur is (weakly) better off with optimal disclosure, i.e., $\overline{V}^E \geq V_\pi^E$, because he can always disclose no information.

By Theorem 1, if $\mathbb{1}_\pi = 0$, then $\overline{V}^I = V_\pi^I = 0$. Since the investor's payoff is always zero, the firm value is equal to the entrepreneur's monetary payoff, and thus, we have $\overline{V}^F = \mathbb{E}_\mu\{[\max(x - \bar{r}_\mu, 0)]\mathbb{1}_\mu\} \geq 0 = V_\pi^F$.

If $\mathbb{1}_\pi = 1$, then $\overline{V}^I = V_\pi^I = p\underline{u} + \sum_{x<I}\big(x - c(x)\big)$. Since the firm is always financed, the entrepreneur's personal benefit stays the same, implying that his monetary payoff is (weakly) higher with optimal disclosure. Given that the investor's payoff stays unchanged, we have $\overline{V}^F \geq V_\pi^F$. $\square$

Proof of Proposition 7. First, we prove part (a). Since any $x \geq \bar{r}_\pi$ is never audited in the optimal experiment, its verification cost does not appear in neither the investor's nor the entrepreneur's payoffs. Thus, $c(x)$ has no effect, that is, $\frac{\partial \overline{V}^i}{\partial c(x)} = 0$ for $i \in \{F, I, E\}$.

If $x < I$, its verification cost is relevant only when it is included in the financing posterior belief. However, an increase in $c(x)$ will reduce the investor's payoff and increase the dead-weight loss; moreover, the entrepreneur may have to either exclude more measure of inefficient states or sacrifice more monetary payoff. Thus, $\frac{\partial \overline{V}^i}{\partial c(x)} \leq 0$ for $i \in \{F, I, E\}$.

If $I \leq x < \bar{r}_\pi$, we distinguish four cases. When $\mathbb{1}_\pi = 1$, the max-monetary-payoff experiment is optimal. By Lemma A.5 and Lemma A.6, we know that $(\pi^h, \overline{V}^E(\pi^h))$ is the convex combination of the points in $\left\{\left(\mu^{S'}, \overline{V}^E_{\mu^{S'}}\right) \mid \bar{r}_{\pi^h} \in S' \subseteq S\right\}$, which all lie on the hyperplane $\overline{V}^E(\mu) = \sum_{x<\bar{r}_{\pi^h}} \mu(x)c(x) + \sum_{x \geq \bar{r}_{\pi^h}} \mu(x)\big(x - \bar{r}_{\pi^h}\big)$. Clearly, the investor's monetary payoff is increasing in $c(x)$. Notice that his personal benefit is constant, and thus we have $\frac{\partial \overline{V}^E}{\partial c(x)} > 0$. Because x is not audited and the firm is always financed, we have $\frac{\partial \overline{V}^F}{\partial c(x)} = 0$. Since $\overline{V}^I = \sum_{x<\bar{r}_\pi} \pi(x)\big(x - c(x)\big) + \sum_{x \geq \bar{r}_\pi} \pi(x)\bar{r}_\pi - I$, we have $\frac{\partial \overline{V}^I}{\partial c(x)} < 0$. When $\mathbb{1}_\pi = 0$ and $p\underline{u} + qu \geq 0$, the max-monetary-payoff experiment is still optimal. Since the investor's payoff is always zero, we have $\frac{\partial \overline{V}^I}{\partial c(x)} = 0$; meanwhile, the zero-payoff condition implies that an increase in $c(x)$ will exclude some measure of the marginal inefficient states from the financing posterior beliefs, which increases the firm value, that is, $\frac{\partial \overline{V}^F}{\partial c(x)} > 0$. Notice that the entrepreneur's payoff is equal to the social welfare, which is also increased since the excluded states satisfy $x - c(x) - I + B < 0$, and thus we have $\frac{\partial \overline{V}^E}{\partial c(x)} > 0$. When $p\underline{u} < -qu < p\bar{u}$, the two-stage experiment in Theorem 1 is optimal. We have $\frac{\partial \overline{V}^I}{\partial c(x)} = 0$ since the investor's payoff is always zero. And we have $\frac{\partial \overline{V}^F}{\partial c(x)} = 0$ since the states in $W \cup S$ are always financed and the states in S are never audited. Because the entrepreneur's payoff is equal to the social welfare, which is unaffected, we have $\frac{\partial \overline{V}^E}{\partial c(x)} = 0$. When $p\bar{u} + qu \leq 0$, the max-financing experiment is optimal, and we have $\frac{\partial \overline{V}^i}{\partial c(x)} = 0$ for $i \in \{F, I, E\}$ because nothing changes with $c(x)$.

Next, we prove part (b). Choose $\underline{c}$ such that $\bar{r}_\pi \sum_{x \geq \bar{r}_\pi} \pi(x) + \sum_{x<\bar{r}_\pi} \pi(x)(x - \underline{c}) = 0$; that is, $\underline{c}$ is the maximum c such that $\mathbb{1}_\pi = 1$. When $c < \underline{c}$, the max-monetary-payoff experiment (without the no-financing signal) is optimal, and thus an increase in c simply raises the dead-weight loss caused by verification. Then we have $\frac{\partial \overline{V}^F(c)}{\partial c} < 0$.

Define $\bar{c} = \max\left\{\underline{c}, x^{-1} - I + B\right\}$, that is, $\bar{c}$ is the minimum $c \geq \underline{c}$ such that $W = \emptyset$. When $c > \bar{c}$, the max-monetary-payoff experiment (with a no-financing signal) is optimal, and the investor's payoff is always zero. Then, an increase in c will exclude some measure of the inefficient states from the financing posterior beliefs, which increases the firm value, that is, $\frac{\partial \overline{V}^F(c)}{\partial c} > 0$. As c goes to infinity, the measure of inefficient states that are included in the financing posterior beliefs shrinks to zero, and thus, we have $\lim_{c\to\infty} \overline{V}^F(c) = pV^F_{\pi^h}$. Notice that a positive measure of inefficient states is included in the financing posterior beliefs when $c = 0$, which induces welfare loss, and we conclude $pV^F_{\pi^h} > V^F_{\pi}(0)$.

When $\underline{c} \leq c \leq \bar{c}$, we choose $\tau^*_c \in \{1, \ldots, N-K\}$ and $\rho^*_c \in (0, 1]$ satisfying

$$p(U_{\pi^h} - I) + \sum_{i=1}^{\tau^*_c - 1} \pi(x^{-i})\big(y(x^{-i}) - I\big) + \rho^*_c \pi(x^{-\tau^*})\big(y(x^{-\tau^*}) - I\big) = 0.$$

If $x^{-\tau^*_c} \notin W$, the max-monetary-payoff experiment (with a no-financing signal) is optimal, and the investor's payoff is always zero. As in the case $c > \bar{c}$, the firm value increases with c, that is, $\frac{\partial \overline{V}^F(c)}{\partial c} > 0$. If $x^{-\tau^*_c} \in W$ and $p\bar{u} + qu \geq 0$, the two-stage experiment is optimal, and then all states in W are financed. Thus, an increase in c will only raise the verification cost, which reduces the firm value; that is, $\frac{\partial \overline{V}^F(c)}{\partial c} < 0$. If $x^{-\tau^*_c} \in W$ and $p\bar{u} + qu < 0$, the max-financing experiment is optimal. Then the firm value, which is the sum of the investor's payoff and the entrepreneur's monetary payoff, is equal to zero. Thus, we have $\frac{\partial \overline{V}^F(c)}{\partial c} = 0$. Because W and $\{x^{-1}, \ldots, x^{-\tau^*_c}\}$ both shrink as c increases, how the firm value varies with c depends on which of the above three situations occurs, and thus we get a non-monotone pattern for $\underline{c} \leq c \leq \bar{c}$. $\square$

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