1. Suppose Jack and Tom have the endowments $\omega_A = (6, 0)$ and $\omega_B = (0, 6)$. Their preferences are defined by a pair of utility functions. For the following cases, find the utility possibility frontier. State your answer both in precise mathematical notation and in terms of graph.

Leontief/Cobb-Douglas

$$U_J(x_{1J}, x_{2J}) = \min(x_{1J}, x_{2J}/2)$$

$$U_T(x_{1T}, x_{2T}) = x_{1T}^{1/3} x_{2T}^{2/3}.$$

Answer: The UPF is

$$U_T - (6 - U_J)^{\frac{1}{3}}(6 - 2U_J)^{\frac{2}{3}} = 0.$$

Leontief/Linear

$$U_J(x_{1J}, x_{2J}) = \min(x_{1J}, x_{2J}/2)$$

$$U_T(x_{1T}, x_{2T}) = x_{1T} + x_{2T}.$$

Answer: The UPF is

$$3U_J + U_T - 12 = 0, U_T \geq 3$$

Cobb-Douglas/Cobb-Douglas

$$U_J(x_{1J}, x_{2J}) = x_{1J}^{1/3} x_{2J}^{2/3}$$

$$U_T(x_{1T}, x_{2T}) = x_{1T}^{1/3} x_{2T}^{2/3}.$$

$MRS_J = \frac{1}{2}\frac{x_{2J}}{x_{1J}} \Rightarrow \frac{x_{2J}}{x_{1J}} = \frac{x_{2T}}{x_{1T}}$

$MRS_T = \frac{1}{2}\frac{x_{2T}}{x_{1T}}$

$x_{1J} + x_{1T} = 6 \Rightarrow x_{1J} = x_{2J}$

$x_{2J} + x_{2T} = 6 \quad x_{1T} = x_{2T} = 6 - x_{1T}$

Answer: The UPF is

$$U_J + U_T - 6 = 0.$$

$\therefore U_J = x_{1J} \Rightarrow U_J + U_T - 6 = 0$

$U_T = 6 - x_{1J}$

Cobb-Douglas/Linear

$$U_J(x_{1J}, x_{2J}) = x_{1J}^{1/3} x_{2J}^{2/3}$$

$$U_T(x_{1T}, x_{2T}) = x_{1T} + x_{2T}.$$

Answer: The UPF is

$$U_T + \frac{U_J}{\sqrt[3]{4}} + \sqrt[3]{2}U_J - 12 = 0 \ \text{ if } U_J \leq 6/\sqrt[3]{2},$$
$$U_T + \frac{U_J^3}{36} + 6 = 0 \ \text{ if } U_J \leq 6/\sqrt[3]{2}$$

Linear/Linear

$$U_J(x_{1J}, x_{2J}) = x_{1J} + x_{2J}$$

$$U_T(x_{1T}, x_{2T}) = x_{1T} + 2x_{2T}.$$

Answer: The UPF

$$2U_J + U_T - \overset{24}{\cancel{18}} = 0, \text{ if } U_J \geq 6$$
$$U_J + U_T - \overset{18}{\cancel{12}} = 0, \text{ if } U_J \cancel{\leq} 6$$

2. Suppose the economy is endowed with capital and labor, $k = 6$, $l = 6$, and can produce two outputs (b, c) with production functions given below. For each pair of production functions given, derive the transformation function $T(y)$ where $y = (y_1, y_2, y_3, y_4) = (-k, -l, b, c)$.

Leontief/Cobb-Douglas

$$b(k, l) = \min(k, l/2)$$
$$c(k, l) = k^{1/3} l^{2/3}.$$

$k_b = \frac{l_b}{2}$ $\quad l_b = 2k_b$ $\quad k_b \in [0, 3]$

$b = k_b$ $\quad k_c = 6 - k_b$ $\quad l_c = 6 - l_b = 6 - 2k_b)$

$c = (6 - k_b)^{\frac{1}{3}} (6 - 2k_b)^{\frac{2}{3}}$

$c = (6 - b)^{\frac{1}{3}} (6 - 2b)^{\frac{2}{3}}$

Answer: Transformation function

$$T(-k, -\ell, b, c) = c - (6 - b)^{1/3}(6 - 2b)^{2/3}$$

Leontief/Linear

$$b(k, l) = \min(k, l/2)$$
$$c(k, l) = k + l.$$

Answer: Transformation function

$$T(-k, -\ell, b, c) = 3b + c - 12, c \geq 3.$$

Cobb-Douglas/Cobb-Douglas

$$b(k, l) = k^{1/3} l^{2/3}$$
$$c(k, l) = k^{1/3} l^{2/3}.$$

Answer: Transformation function

$$T(-k, -\ell, b, c) = b + c - 6.$$

Cobb-Douglas/Linear

$$b(k, l) = k^{1/3} l^{2/3}$$

$$c(k, l) = k + l.$$

Answer: Transformation function

$$T(-k, -\ell, b, c) = \begin{cases} c + \frac{b}{\sqrt[3]{4}} + \sqrt[3]{2}b - 12 & \text{if } b \leq 6/\sqrt[3]{2}, \\ c + \frac{b^3}{36} + 6 & \text{if } b \leq 6/\sqrt[3]{2} \end{cases}$$

Linear/Linear

$$b(k, l) = k + l$$

$$c(k, l) = k + 2l.$$

① $l_c = 6$, $k_b < 6$

$c = k_c + 2 \times 6 = 12 + k_c$

$b = k_b + 0 = 6 - k_c$

$\Rightarrow T = b + c - 18$ $\quad b \leq 6$

Answer: Transformation function

② $k_b = 6$

$b = 6 + l_b$

$$T(-k, -\ell, b, c) = \begin{cases} b + c - 18 & \text{if } b \leq 6, \\ 2b + c - 24 & \text{if } b > 6 \end{cases}$$

$c = 0 + 2(6 - l_b) = 12 - 2l_b$

$\Rightarrow T = 2b + c - 24$ $\quad b > 6$

3. Suppose there are three people, named Bob, Jack and Tom, whose only purpose in life is to eat chocolate. Their utility functions are

$$u_B(c_B, l_B) = c_B,$$

$$u_J(c_J, l_J) = 2c_J,$$

$$u_T(c_T, l_T) = 3c_T.$$

Each of them is endowed with no chocolate and one unit of labor. There are 3 chocolate factories, whose production functions are

$$f_1(l_1) = l_1,$$

$$f_2(l_2) = 2l_2,$$

$$f_3(l_3) = 3l_3.$$

Bob owns Firm 1, Jack owns Firm 2, and Tom owns Firm 3.

(a) Determine the society's transformation function.

Answer: The society's production transformation function

$$T(-l, m) = m - 3l.$$

$T(-l, m) = m - \max_i \{f_i(l) \mid i \in \{1,2,3\}, l \in [0,3]\} = m - 3l$ $\qquad l \in [0,3]$

(b) Determine the set of Pareto efficient consumption allocations.

Answer: The set of P.E. allocations

$$X = \begin{bmatrix} l_A & l_B & l_C \\ m_A & m_B & m_C \end{bmatrix} = X = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ m_A & m_B & m_C \end{bmatrix} \middle| \, m_A + m_B + m_C = 9 \right\}, \quad y = \begin{bmatrix} -3 \\ 9 \end{bmatrix}$$

(c) Determine the set of equilibrium consumption allocation(s). For each equilibrium consumption allocation, give the corresponding production allocation and price vector.

Answer: To get the price vector, solve firm 3's profit-maximization (or cost-minimization) problems. Set price of labor $w = 1$. Only firm 3 produces

$$TC_m = \frac{wm}{3} \Longrightarrow MC = \frac{w}{3} = \frac{1}{3} = P_m.$$

Given price solving individual's utility-maximization problem yields

$$X = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 3 \end{bmatrix}, \qquad y = \begin{bmatrix} -3 \\ 9 \end{bmatrix}$$

4. Consider the following market for used cars. There are N potential sellers and M potential buyers, and $M > N$. Each seller has exactly one used car to sell and is characterized by the quality of the used car he has. Let $\theta \in [0, 1]$ index the quality of used car. If a seller of type θ sells his car for a price of P, his payoff is $u_s(P, \theta)$, and is 0 if he does not sell his car. The payoff for the buyer is $\theta - P$ if he buys a car of quality θ at price P, and is 0 if he does not buy. Information is asymmetric: Sellers know the quality of used cars but buyers do not. However, buyers know the quality of used car is uniformly distributed on [0,1].

(a) Argue that in a competitive equilibrium under asymmetric information, we must have $E[\theta|P] = P$.

(b) Find all equilibrium prices when $u_s(P, \theta) = P - \theta/2$.

(c) Find all equilibrium prices when $u_s(P, \theta) = P - \sqrt{\theta}$. Describe the equilibrium in words. In particular, which cars are traded in equilibrium?

Answer:

(a) The expected payoff for the buyer:

$$U_B = E[\theta|P] - P \quad \text{if buy a car,} \qquad 0 \qquad \textit{otherwise.}$$

- If $E[\theta|P] > P$, $U_B > 0$, all potential buyers want to buy, but given $M > N$, not enough cars to supply all buyers. Market demand exceeds supply, price has to go up to clear the market.

$u_s(P,\theta) = P - \frac{\theta}{2} \geq 0 \qquad 2P \geq \theta$

- If $E[\theta|P] < P$, $U_B < 0$, no potential buyer will buy, can't be an equilibrium.
- When $E[\theta|P] = P$, $U_B = 0$. Buyers are indifferent between buying a car or not, so market can be cleared.

(b) In this case, seller will sell if $P \geq \theta/2 \Longleftrightarrow \theta \leq 2P$.

- If $P \in (0, 1/2]$, $E[\theta|\theta \leq 2P] = P$. Buyers are indifferent, so they will buy any number of cars in the market. Market clears given the price.
  Note that when $P = 1/2$, all sellers will sell and $E[\theta|P = 0.5] = 1/2$.

if $P > \frac{1}{2}$
$E(\theta|P) = E(\theta|\theta \leq 2P)$
$= E(\theta) = \frac{1}{2} < P$

- If $P > 1/2$, can't be an equilibrium as no buyers will buy, supply exceeds demand.

(c) In this case, sellers will sell if and only if $\theta \leq P^2$. In this case, $E[\theta|\theta \leq P^2] = P^2/2$. But buyers will not buy if $P^2/2 > P$. From what we have in (a), we know that in any competitive equilibrium, it must be true that $E[\theta|P] = P$. Hence we have

$$\frac{(P^*)^2}{2} = P^* \Longrightarrow P^* = 0 \quad \text{or} \quad P^* = 2.$$

At $P^* = 0$, $E[\theta|P = 0] = 0$, no cars will be sold in equilibrium.

$P^* = 2$ can't be a competitive equilibrium either, as $E[\theta|P = 2] = 1/2$. The price $P = 2$ is too high and no one will buy. No trade takes place in this case.

HW5

Q1 (d) $u_J(x_{1J}, x_{2J}) = x_{1J}^{\frac{1}{3}} x_{2J}^{\frac{2}{3}}$

$u_T(x_{1T}, x_{2T}) = x_{1T} + x_{2T}$

① $\frac{MU_{1J}}{MU_{2J}} = \frac{MU_{1T}}{MU_{2T}} \Rightarrow x_{2J} = 2x_{1J}$ $\quad x_{1J} \in [0, 3]$

$u_J = \sqrt[3]{4}\, x_{1J}$ $\quad u_J \in (0, 3\sqrt[3]{4})$

$u_T = 6 - x_{1J} + 6 - x_{2J} = 12 - 3x_{1J}$

$u_T = 12 - 3\frac{1}{\sqrt[3]{4}} u_J$

$\Rightarrow \underline{3u_J + \sqrt[3]{4}\, u_T - 12\sqrt[3]{4} = 0}$ $\quad u_J \in [0, 3\sqrt[3]{4})$

② $x_{2J} = 6$ $\quad x_{1J} \in (3, 6]$

$u_J = 6^{\frac{2}{3}} x_{1J}^{\frac{1}{3}} \in (3\sqrt[3]{4}, 6)$ $\quad x_{1J} = u_J^3 / 36$

$u_T = 6 - x_{1J}$

$u_T = 6 - \frac{u_J^3}{36}$

$u_J^3 + 36u_T - 216 = 0$ $\quad u_J \in (3\sqrt[3]{4}, 6]$

(e) $u_J(x_{1J}, x_{2J}) = x_{1J} + x_{2J}$

$u_T(x_{1T}, x_{2T}) = x_{1T} + 2x_{2T}$

① $x_{2J} = 0$ $\quad u_J = x_{1J}$ $\quad u_T = 6 - x_{1J} + 2 \times 6 = 18 - x_{1J} = 18 - u_J$

$\Rightarrow u_T + u_J - 18 = 0$ $\quad u_J \in [0, 6]$

② $x_{2J} > 0$, $x_{1J} = 6$ $\quad u_J = 6 + x_{2J}$ $\quad x_{2J} = u_J - 6$

$u_T = 2(6 - x_{2J}) = 2(6 - u_J + 6) = 24 - 2u_J$

$u_T + 2u_J - 24 = 0$ $\quad u_J \in (6, 12]$

Q2. $\bar{k}=6$ $\bar{l}=6$

(d) $b(k,l) = k^{\frac{1}{3}} l^{\frac{2}{3}}$

$c(k,l) = k+l$

$T_b = \{ b = (-k_b, -l_b, b, 0) \mid b \le k_b^{\frac{1}{3}} l_b^{\frac{2}{3}} \}$

$T_c = \{ c = (-k_c, -l_c, 0, c) \mid c \le k_c + l_c \}$

$T(-\bar{k}, -\bar{l}, b, c) = c - \max\{ (k_c + l_c) \mid (\bar{k} - k_c)^{\frac{1}{3}} (\bar{l} - l_c)^{\frac{2}{3}} \ge b \}$

$L = k_c + l_c + \lambda [(6-k_c)^{\frac{1}{3}} (6-l_c)^{\frac{2}{3}} - b]$

foc:
$$\begin{cases} 1 - \lambda \frac{1}{3} (6-k_c)^{-\frac{2}{3}} (6-l_c)^{\frac{2}{3}} = 0 \\ 1 - \lambda \frac{2}{3} (6-k_c)^{\frac{1}{3}} (6-l_c)^{-\frac{1}{3}} = 0 \\ (6-k_c)^{\frac{1}{3}} (6-l_c)^{\frac{2}{3}} - b = 0 \end{cases}$$

$\frac{1}{3} (6-k_c)^{-\frac{2}{3}} (6-l_c)^{\frac{2}{3}} = \frac{2}{3} (6-k_c)^{\frac{1}{3}} (6-l_c)^{-\frac{1}{3}}$

$(6-l_c) = 2(6-k_c) = 12 - 2k_c = 6 - l_c$

$l_c = 2k_c - 6 \Rightarrow k_c \in [3, 6]$

$\sqrt[3]{4}(6-k_c) - b = 0$ $\quad$ $6 - k_c = \frac{b}{\sqrt[3]{4}}$ $\quad$ $k_c = 6 - \frac{b}{\sqrt[3]{4}}$

$l_c = 2k_c - 6$

① $T(-k,-l,b,c) = c - k_c - l_c = c - k_c - 2k_c + 6$

$= c - 3k_c + 6$

$\therefore T = c - 12 + \frac{3b}{\sqrt[3]{4}} \quad (3 < c \le 12,\ b \le 3\sqrt[3]{4})$

② $k_c \in [0, 3] \quad l_c = 0 \quad \Rightarrow \quad c \le 3, \quad 3\sqrt[3]{4} \le b \le 6$

$b = (6 - k_c)^{\frac{1}{3}}\, 6^{\frac{2}{3}}$

$\frac{b^3}{36} = 6 - k_c \qquad k_c = 6 - \frac{b^3}{36}$

$\therefore T = c - k_c - l_c = c + \frac{b^3}{36} - 6$

$$\therefore T = \begin{cases} c - 12 + \frac{3b}{\sqrt[3]{4}} & \text{if } b \le 3\sqrt[3]{4},\ 3 < c \le 12 \\ c + \frac{b^3}{36} - 6 & \text{if } c \le 3,\ 3\sqrt[3]{4} \le b \le 6 \end{cases}$$

$\boxed{4^{-\frac{1}{3}} + 2^{\frac{1}{3}} = \frac{3}{\sqrt[3]{4}}}$