1. Consider the following simplified version of poker. There are two players and a deck of three cards–A, K and Q of spades. Each player is randomly dealt one hand. Each player sees her own card but not that of her opponent and can either bet \$10 or "fold". If one or bother players fold, no money changes hands. If both players bet, the player with the higher card wins, so that she wins \$10 in net terms.

   A pure strategy for a player is a rule that indicates whether to bet or to fold conditional on the card held. 按照策略的定义回答，Normal form 的定义.

   (a) Informally but carefully ARGUE that some of each player's strategies are weakly dominated. (Specify which ones, that is, and why they are weakly dominated.)

   Answer: Both players have 8 strategies (a)中有6个被占优策略

   ($\underline{\text{BBB}}$, $\underline{\text{BFB}}$, BBF, BFF, $\underline{\text{FBB}}$,$\underline{\text{FFB}}$, $\underline{\text{FBF}}$, $\underline{\text{FFF}}$), where first letter stands for choice if card is A, second for choice if card is K, and third for choice if Q. B for "bet" and F for "fold".

   (b) Informally but carefully DERIVE the result of continuing the process of iterated elimination of weakly dominated strategies as far as possible. (b) 在(a)之后，可以再删除一个 只剩 BFF

   Answer: The remaining strategy is BFF.

   (c) Suppose the deck of cards is all 13 spades–A, K,. . ., 2. Each player still gets a single card, dealt at random. Answer (b) for this new situation.

   Answer: The remaining strategy is: bet if A and fold otherwise, only one strategy is not dominated.

2. For the 3-player game shown below:

Player 2

Player 1

	x	y
A	$(0, \underline{0}, \underline{3})$	$(\underline{0}, \underline{0}, \underline{3})$
B	$(-4, \underline{0}, \underline{1})$	$(-4, \underline{0}, \underline{1})$
C	$(\underline{1}, \underline{1}, -2)$	$(-4, 0, \underline{1})$
D	$(\underline{1}, \underline{1}, -2)$	$(\underline{0}, 0, 3)$

Player 3 plays L

Player 2

	x	y
A	$(0, \underline{0}, \underline{3})$	$(0, \underline{0}, \underline{3})$
B	$(3, \underline{0}, \underline{1})$	$(\underline{3}, \underline{0}, \underline{1})$
C	$(\underline{5}, \underline{4}, \underline{0})$	$(\underline{3}, 0, \underline{1})$
D	$(\underline{5}, \underline{4}, \underline{0})$	$(0, 0, \underline{3})$

Player 3 plays R

   (a) Find all pure strategy NE;

   Answer: (A, y, L), (C, x, R), (D, x, R).

   (A, y, L), (C, x, R) (D, x, R)
   (B, y, R),

B is weakly dominated by C

for P1, A is weakly dominated by D ; for P2 , y is weakly dominated by x ; for P3 , L is weakly dominated by R

(b) Among the NE from (a), identify those involving players playing dominated strategies;

Answer: (A, y, L) and (B, y, R)

(c) Show there exists NE which involves no players playing dominated strategy but nevertheless is not normal-form perfect.

$u_1(C, \sigma_{-1}^{\varepsilon}) - u_1(D, \sigma_{-1}^{\varepsilon}) = 3\eta - 7\eta\nu > 0$ for some $\nu \to 0$.

Answer: (D, x, R). Note that given the totally mixed strategies played by player 2 and 3: $(1-\eta, \eta; \nu, 1-\nu)$,

x y L R

$u_1(C, \sigma_{-1}^{\varepsilon}) = (1-\eta)\nu + (1-\eta)(1-\nu)5 - 4\eta\nu + 3\eta(1-\nu)$

$= 5 - 2\eta - 4\nu - 3\eta\nu$

$$u_1(C, \sigma_{-1}^{\epsilon}) > u_1(D, \sigma_{-1}^{\epsilon}) \Longrightarrow \sigma_1^{\epsilon}(D) < \epsilon.$$

$u_1(D, \sigma_{-1}^{\varepsilon}) = (1-\eta)[1 \cdot \nu + 5(1-\nu)] + 0 = 5 - 5\eta - 4\nu + 4\eta\nu$

3. (Mixed strategy) Show that the two-player game illustrated below has a unique equilibrium in pure strategies and that there is no additional equilibrium in mixed strategies.

	L	M	R
U	$(1, -2)$	$(-2, 1)$	$(0, 0)$
M	$(-2, 1)$	$(1, -2)$	$(0, 0)$
D	$(0, 0)$	$(0, 0)$	$(1, 1)$

Answer: Suppose there exists some mixed strategy NE $(\sigma_1, \sigma_2) = (\{p_1, p_2, p_3\}, \{q_1, q_2, q_3\})$, where p_1, p_2, p_3 is the probability assigned by player 1 to (U,M,D), and q_1, q_2, q_3 the probability by player 2 to (L,M,R).

$$\forall\ i = 1, 2, 3,\ p_i \in [0, 1],\ q_i \in [0, 1] \quad \text{and} \sum_1^3 p_i = 1,\ \sum_1^3 q_i = 1.$$

(a) case 1, $p_1, p_2, p_3 > 0$, then by Prop. 8.D.1 of MWG, $\exists\ (q_1, q_2, q_3)$ such that

$$\begin{aligned} q_1 - 2q_2 + 0 \cdot q_3 &= -2q_1 + q_2 + 0 \cdot q_3, \\ q_1 - 2q_2 + 0 \cdot q_3 &= q_3, \\ -2q_1 + q_2 &= q_3. \end{aligned}$$

The system of equations gives contraction

$q_2 > 0$

$u_1(U) = q_1 - 2q_2$

$u_1(M) = -2q_1 + q_2$

$u_1(D) = q_3$

$$q_1 = q_2;$$

$$q_1 - 2q_2 = q_3;$$

$$q_2 - 2q_1 = q_3.$$

$q_3 = q_2 - 2q_2 = -q_2 < 0$

$q_3 = q_2 - 2q_2 = -q_2$

Thus, there is no NE such that $p_1, p_2, p_3 > 0$.

$u_2(L) = p_2 - 2p_1$

$u_2(M) = -2p_2 + p_1$

$u_2(R) = p_3 = 1 - p_1 - p_2$

(c) if $P_1, P_3 > 0$, $P_2 = 0$ $q_1 - 2q_2 = 1 - q_1 - q_2 \Rightarrow 2q_1 - q_2 = 1$

$\because P_2 = 0$ $U_2(L) < 0 < U_2(M) \Rightarrow q_1 = 0$ $q_2 = -1$ Contradict

(b) case 2, $p_1, p_2 > 0,\ p_3 = 0$, similarly, $\exists\ (q_1, q_2, q_3)$ such that

$$q_1 - 2q_2 = -2q_1 + q_2 \Longrightarrow q_1 = q_2.$$

However,

$$U_1(D, \sigma_2) = q_3 \geq 0 > U_1(M, \sigma_2) = U_1(U, \sigma_2),$$

contradiction. Thus, there is no NE such that

$$p_1, p_2 > 0,\ p_3 = 0.$$

The unique pure NE (D, R)

4. Consider the following majority voting game. There are three congressmen. Their names are A, B and C. There are three possible alternatives to vote for: X, Y, and Z. The alternative that gets the majority of votes wins. In case that all alternatives receive the same vote, then each one is chosen with equal probability. The payoffs of the congressmen depend exclusively on which alternative is chosen through voting. Their payoffs are as follows:

$$U_A(X) = U_B(Y) = U_C(Z) = 2;$$
$$U_A(Y) = U_B(Z) = U_C(X) = 1;$$
$$U_A(Z) = U_B(X) = U_C(Y) = 0.$$

Determine whether there are any Nash equilibria in pure strategies for this game. (You may find more than one Nash equilibrium).

Answer:

(a) The normal-form of the game:

		B		
		X	Y	Z
A	X	2, 0, 1	2, 0, 1	2, 0, 1
	Y	2, 0, 1	1, 2, 0	1, 1, 1
	Z	2, 0, 1	1, 1, 1	0, 1, 2

C votes X

		B		
		X	Y	Z
A	X	2, 0, 1	1, 2, 0	1, 1, 1
	Y	1, 2, 0	1, 2, 0	1, 2, 0
	Z	1, 1, 1	1, 2, 0	0, 1, 2

Y

(X, X, X) (X, Y, X) (Y, Y, Y) (X, Z, Z)

(X, Y, Z) (Y, Y, Z) (Z, Z, Z)

B

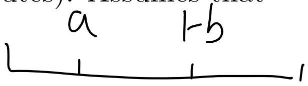
A	X	Y	Z
X	2, 0, 1	1, 1, 1	0, 1, 2
Y	1, 1, 1	1, 2, 0	0, 1, 2
Z	0, 1, 2	0, 1, 2	0, 1, 2

C votes Z

(b) Pure strategy NE

$$(X,X,X),\quad (X,Y,X),\quad ,(X,Y,Z),\quad (Y,Y,Y),\quad (Y,Y,Z),\quad (X,Z,Z),\quad (Z,Z,Z)$$

5. (Hotelling) Suppose that we have two firms located on a line of length 1. The unit costs of the good for each store is c. Consumers incur a transportation cost of tx^2 for a length of x. Consumers have unit demands, and are uniformly distributed along the line. Firm 1 is located at point $a \geq 0$ and firm 2 at point $1-b$, where $b \geq 0$ and without loss of generality, $1-a-b \geq 0$ (firms 1 is to the left of firms 2; $a = b = 0$ corresponds to maximal differentiation and $a + b = 1$ corresponds to minimal differentiation, i.e. perfect substitutes). Assumes that the market is covered and firms sell positive quantities.



(a) Let x be the distance from the left end point, i.e., 0. We first figure out the demand of the firms by finding the consumer $x(p_1, p_2)$ who is indifferent between the two firms. x is given by equating the generalized costs,

$$P_1 + t(x^2 - 2xa + a^2) = P_2 + t(1-2b+b^2-2(1-b)x+x^2)$$

$$P_1 - 2atx + ta^2 = P_2 + t(1-2b+b^2) - 2(1-b)tx$$

$$p_1 + tx^2 = p_2 + t(1-b-x)^2.$$

$$P_1 + t(x-a)^2 = P_2 + t(1-b-x)^2$$

The left-hand side is the cost for consumer x if buying from firm 1 while the right-hand side is the cost if buying from firm 2. This gives

$$x = \frac{P_2 - P_1}{2t(1-a-b)} + \frac{t(1-b)^2 - ta^2}{2t(1-a-b)}$$

$$x = \frac{p_2 - p_1}{2t(1-a-b)} + \frac{1-b+a}{2}.$$

$$x = \frac{P_2 - P_1}{2t(1-a-b)} + \frac{1-b+a}{2}$$

All consumers located left from x will purchase from firm 1, and all other consumers will buy from firm 2. Thus, the market shares for the two firms, denoted by $D_1(p_1,p_2), D_2(p_1,p_2)$ respectively, are

$$D_1(p_1,p_2) = x = \frac{p_2 - p_1}{2t(1-a-b)} + \frac{1-a-b}{2} + a,$$

$$D_2(p_1,p_2) = 1 - x = \frac{p_1 - p_2}{2t(1-a-b)} + \frac{1-a-b}{2} + b.$$

$$D_1 = \frac{b-a}{3} + \frac{1-a-b}{2} + a = \frac{2b-2a+3-3a-3b+6a}{6}$$

$$= \frac{3+a-b}{6} = \frac{1}{2} + \frac{a-b}{6}$$

$D_1 = \frac{1}{2} + \frac{a-b}{6}$

$D_2 = \frac{1}{2} + \frac{b-a}{6}$

Can you figure out the economic meanings of the two equations?

Firm 1 chooses p_1 to maximize its profit

$$(p_1 - c)D_1(p_1, p_2) = (p_1 - c)\left[\frac{p_2 - p_1}{2t(1-a-b)} + \frac{1-a-b}{2} + a\right].$$

The first order condition gives us the best response function of firm 1

$$p_1(p_2) = \frac{p_2 + c + t[(1-b)^2 - a^2]}{2}. \qquad (1)$$

Similarly, firm 2's best response function is

$$p_2(p_1) = \frac{p_1 + c + t[(1-a)^2 - b^2]}{2}. \qquad (2)$$

Solving (1) and (2) simultaneously, we obtain the NE in prices,

$$p_1^*(a,b) = c + t(1-a-b)\left(1 + \frac{a-b}{3}\right), \qquad (3)$$

$$p_2^*(a,b) = c + t(1-a-b)\left(1 + \frac{b-a}{3}\right). \qquad (4)$$

(b) See (a).

(c) Knowing that prices will be chosen in the second stage as in (a), both firms choose their location (i.e., a and b respectively) in the first stage to maximize their reduced form profit functions ($i = 1, 2$)

$$\pi_i(a,b) = [p_i^*(a,b) - c]D_i(a, b, p_1^*(a,b), p_2^*(a,b)).$$

For firm 1 to maximize $\pi_1(a,b)$ with respect to a, we need not to take the derivative

$$\frac{\partial \pi_1}{\partial p_1}\frac{\partial p_1}{\partial a},$$

which is due to the envelope theorem. Firm 1 maximizes profit with respect to price in the second period, so $\partial\pi_1/\partial p_1 = 0$. Thus we need only look at the direct effect of a on π_1. That is

$$\frac{d\pi_1}{da} = (p_i^* - c)\left(\frac{\partial D_1}{\partial a} + \frac{\partial D_1}{\partial p_2}\frac{\partial p_2}{\partial a}\right).$$

From (a), we can calculate the following

$$\frac{\partial D_1}{\partial a} = \frac{1}{2} + \frac{p_2^* - p_1^*}{2t(1-a-b)^2} = \frac{3-5a-b}{6(1-a-b)},$$

$$\frac{\partial D_1}{\partial p_2}\frac{\partial p_2}{\partial a} = \frac{1}{2t(1-a-b)} \cdot t\left(-\frac{4}{3} + \frac{2a}{3}\right) = \frac{-2+a}{3(1-a-b)}.$$

Adding up and using the assumption that $1 - a - b > 0$ and the fact that $p_1^* - c$ is positive, we reach the conclusion that

$$\frac{d\pi_1}{da} < 0.$$

Hence, firm 1 always wants to move leftward if it is to the left of firm 2, and similarly for firm 2. Therefore, the equilibrium in locations exhibit maximal differentiation.

(d) Social optimal allocation means that the choice which maximizes the total surplus of firms and consumers. Suppose there is a social planner chooses locations for the two firms. Because for given locations, and as long as the market is covered, the pricing structure does not affect the sum of consumer surplus and profits, the social planner's objective is to minimize the consumers' average transportation cost.

By symmetry, the social planner choose to locate the two firms equi-distantly on either side of the middle point, so that for equal prices a firm serves the left or the right half of the market. Given that the density of consumers is uniform and the transportation cost is quadratic, it is easy to see that the social optimal locations are $\frac{1}{4}$ and $\frac{3}{4}$.

In this case, the market outcome yields too much product differentiation in the view of social welfare.

let $X_M = \frac{1-b+a}{2}$

$$\min_{a,b\in[0,1]} \quad c + t\int_0^{X_M}(a-z)^2\,dz + t\int_{X_M}^1 t(1-b-z)^2\,dz$$

$$\Leftrightarrow \quad \min \quad \int_0^{X_M}(a-z)^2\,dz + \int_{X_M}^1 (1-b-z)^2\,dz$$

$$= \frac{2}{3}\left(\frac{1-a-b}{2}\right)^3 + \frac{a^3}{3} + \frac{b^3}{3} \triangleq T(a,b)$$

foc: $\frac{\partial T}{\partial a} = 0$

$\frac{\partial T}{\partial b} = 0$ $\quad \Rightarrow \quad a^* = b^* = \frac{1}{4}$

so firm 1 $\frac{1}{4}$ $\quad$ firm b $\frac{3}{4}$.