Advanced Econometrics II:

June 8, 2024

Assignment 2

(This assignment is due on June 14, 2024 at noon. Please submit answers to your TA on time. The total marks are indicated in each question.)

1. In a classical linear regression model $y_i = x_i'\beta + u_i$ with iid data (assuming both x_i and u_i are random and x_i is independent of u_i; $u_i \sim iid(0, \sigma^2)$, and the 4-th moment of u_i is finite), we estimate the variance of u_i by $\hat{\sigma}^2 = \frac{1}{n-k}\sum_{i=1}^{n}\hat{u}_i^2$. Please derive that:
   (a) [5 points] $\hat{\beta} - \beta = O_p(n^{-0.5})$
   (b) [10 points] $\hat{\sigma}^2 - \sigma^2 = O_p(n^{-0.5})$

2. [15 points] Suppose $\{X_t\}_{t=1}^{n}$ is generated by a classical $MA(1)$ process:
$$X_t = Z_t + \theta_1 Z_{t-1}\ Z_t \sim WN(0, \sigma^2), t = 1,2, \dots, n$$
(θ_1 and σ^2 are treated as known). If a standard $AR(1)$ process, $X_t = \phi X_{t-1} + Y_t$ is mistakenly fitted to $\{X_t\}$ (ϕ is treated as unknown). Please derive the autocorrelation function (ACF) of $\{Y_t\}$. The ACF that you derived must be a function of θ_1 and σ^2.

3. [10 points] For the classical stationary $AR(1)$ model: $y_t = \rho y_{t-1} + u_t$, where $u_t \sim iid(0, \sigma^2)$ and Ey_t^4 is finite, we have
$$\sqrt{n}(\hat{\rho} - \rho) = \frac{\frac{1}{\sqrt{n}}\sum_{t=1}^{n} y_{t-1}y_t}{\frac{1}{n}\sum_{t=1}^{n} y_{t-1}^2}$$
Please derive that the denominator has $\frac{1}{n}\sum_{t=1}^{n} y_{t-1}^2 \xrightarrow{P} E(y_{t-1}^2) = \sigma_y^2$

4. For the $VAR(2)$ model, where the estimated results of each equation are as follows:
$a_t = 0.5a_{t-1} + 0.7b_{t-1} + 0.3a_{t-2} + e_{1t}$
$b_t = -0.2a_{t-1} + 0.4b_{t-1} + e_{2t}$
   (a) [5 points] Defining y_t as a 2-element vector containing a_t and b_t, write the above estimated single-equation form into the $VAR(2)$ model form that we have learned in the class.
   (b) [5 points] What is the effect on a_{t+2} of a one-unit shock in b_t(b_t increases by one-unit)

5. [15 points] Consider a linear regression model:
$$y_t = x_{1t}\beta_1 + x_{2t}\beta_2 + u_t \equiv X_t'\beta + u_t,$$
where x_{1t} is iid with $Ex_{1t} = \mu_1 \neq 0$, $x_{2t} = x_{2,t-1} + v_t$, v_t is $iid(0, \sigma_v^2)$, u_t is $iid(0, \sigma_u^2)$ with v_t is independent of u_t, and v_t is independent of x_{1t}.

Please derive the joint distribution of $\begin{pmatrix} \sqrt{n}(\hat{\beta}_1 - \beta_1) \\ n(\hat{\beta}_2 - \beta_2) \end{pmatrix}$, where $\hat{\beta} = \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix} = (X'X)^{-1}X'Y$ is the OLS estimator of $\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$.

Hint: $\begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} + (X'X)^{-1}X'u$. Define $D_n = \begin{pmatrix} \sqrt{n} & 0 \\ 0 & n \end{pmatrix}$. Consider $\begin{pmatrix} \sqrt{n}(\hat{\beta}_1 - \beta_1) \\ n(\hat{\beta}_2 - \beta_2) \end{pmatrix} \equiv D_n\left(\begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix} - \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}\right) = D_n(X'X)^{-1}X'u = D_n(X'X)^{-1}D_nD_n^{-1}X'u = [D_n^{-1}(X'X)D_n^{-1}]^{-1}D_n^{-1}X'u$. The limiting distribution will involve integrations of Brownian motions.

6. [5 points] For the panel data, please describe how to construct the test for testing the existence of time-invariant unobservable (individual) heterogeneity?

7. [5 points] Similar as the two-way fixed effect model, we can modify the one-way random effect model by adding time specific disturbance to get the two-way random effect model in the form of:

$$y_{it} = \alpha + x_{it}'\beta + \epsilon_{it} + u_i + v_t$$

where

$$E[\epsilon_{it}|x] = E[u_i|x] = E[v_t|x] = 0$$
$$E\left[\epsilon_{it}u_j\middle|x\right] = E[\epsilon_{it}v_s|x] = E[u_iv_t|x] = 0 \text{ for all } i,t,j,s$$
$$Var[\epsilon_{it}|x] = \sigma_\epsilon^2, Cov\left[\epsilon_{it},\epsilon_{js}\middle|x\right] = 0 \text{ for all } i,t,j,s$$
$$Var[u_i|x] = \sigma_u^2, Cov\left[u_i,u_j\middle|x\right] = 0 \text{ for all } i,j$$
$$Var[v_t|x] = \sigma_v^2, Cov[v_t,v_s|x] = 0 \text{ for all } t,s$$

Please write out the full disturbance covariance matrix for a data set with n=2 and T=2

8. [10 points] For the classical random-effect model that we have learned in the class:

$$y_{it} = x_{it}'\beta + \alpha + \epsilon_{it} + u_i$$

where, x_{it}' is strictly exogenous; $Var(\epsilon_{it}) = \sigma_\epsilon^2$ and σ_ϵ^2 is unknown. We can estimate the σ_ϵ^2 by:

$$\hat{\sigma}_\epsilon^2 = \frac{1}{n(T-1)-k}\sum_{i=1}^n\sum_{t=1}^T \ddot{e}_{it}^2 \quad (*)$$

where, $\ddot{e}_{it} = e_{it} - \bar{e}_i$.

Please prove that $\hat{\sigma}_\epsilon^2$ is a consistent estimator of σ_ϵ^2.

[Hints: Please show that $\ddot{e}_{it} = \ddot{\epsilon}_{it} - \ddot{x}_{it}'(b-\beta)$ first and plug it into equation (*). Secondly, calculate $\hat{\sigma}_\epsilon^2$.]