MICROECONOMIC THEORY II

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Chapter 1

Strategic form game

1.1 Introduction to game theory

1.1.1 What is Game theory?

Game theory is the branch of microeconomics concerned with the analysis of optimal decision making in competitive situations in which the actions of each decision maker have significant impact on the fortune of the others. It helps us answer many questions traditional economic theory can not, and is widely used in social science and economics.

One limitation of neoclassical economics is the lack of analysis on the interrelationship between economic agents. A firm is abstracted as a production function that combines inputs to produce outputs. There is no role played by one of the most important actor of the market economy, people. Neither is there any tools that can be used to analyze the interaction between people in the the market place or in an organization. While this may not be a serious issue in many goods markets, it faces serious challenges in others, in particular, the financial markets. As the famous hedge fund manager George Soros pointed out,[1]

"The shape of the supply and demand curves cannot be taken as independently given, because both of them incorporate the participants' expectations about events that are shaped by their own expectations. Nowhere is the role of expectations more clearly visible than in financial markets. Buy and sell decisions are based on expectations about future prices, and future prices, in turn, are contingent on present buy and sell decisions. To speak of supply and demand as if they were determined by forces that are independent of the market participants' expectations is quite

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[1] The Alchemy of Finance

misleading. The situation is not quite so clear-cut in the case of commodities, where supply is largely dependent on production and demand on consumption."

In a similar vein, in explaining the collapse of Long-term Capital Management, an hedge fund with Nobel-prize winners Robert Merton and Myron Scholes among its founders, Professor Bookstabber wrote,[2]

"The problem was that their models assumed they were in a 'game against nature' where their decisions did not alter the playing field. In a normal market environment, with small players, this is a reasonable assumption. But as the largest player in a world of looming illiquidity, this worldview was naive at best. Their actions did change the game, because the decisions of other traders would change depending on the actions LTCM took, or was perceived to take. The partners looked at their risk as if they were playing a game of roulette, where the possible outcomes were unaffected by what was bet and how much was bet. The market turned out to be more like a game of poker, where the outcomes depended on the behavior of the other players, and whose behavior in turn would change in response to their opponents'."

Game theory provides a tool to address situations in which the outcome of a person's decisions depends not only on his own choices, but also on the actions of other people he interacts with. It explicitly considers how interaction among people affects each other's welfare and how consideration of other people determine the choice of each individual. As such, it is not only an important tool for economists, it also widely applied in sociology, political science, management, investment, etc.

Some investors have used game theory to quantify human behavior and predict the outcome of events. In an article in *Business Insider*, Professor Aaron Brown expounds upon how a savvy trader can use game theory to make a profit in the stock market.[3]

1.1.2 Game representation of strategic situation

What is a game: A game is a formal representation of a situation in which a number of individuals interact in a setting of strategic interdependence. To describe a game, we need to know four things:

1. Players: who is involved (playing in the game)?

2. Rules: How the game is played?

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[2] A demon of our own design

[3] "Here's How You Can Use Game Theory To Make Money In The Markets," *Business Insider*, Feb. 17, 2012. A brief introduction of his insights is also available at Cornell's website: https://blogs.cornell.edu/info2040/2012/09/18/exploiting-game-theory-for-profit-in-the-stock-market/.

3. Outcomes:

4. Payoffs:

1.1.3 Rationality and common knowledge

1. Rationality Assumption:

2. Common knowledge: A standard assumption is that the game (players, strategies, and payoff functions) is **common knowledge** among players. Common knowledge is an important concept in game theory. A fact is common knowledge among players if each player knows the fact, and each player knows everyone else knows, and each knows everyone else knows everyone else knows, and so on. For example, a handshake is common knowledge between the two persons involved. When I shake hand with you, I know you know I know you know,....., that we shake hand. Neither person can convince the other that she does not know that they shake hand. So, perhaps it is not entirely random that we sometimes use a handshake to signal an agreement or a deal.

3. What if we don't have common knowledge?

4. **Example 1:** Muddy children puzzle

   n children playing together. Each child wants to keep clean, but each would love to see the others get dirty. Now it happens during their play that some of the children, say k ($k > 1$) of them, get mud on their foreheads. Each can see the mud on others but no on his own forehead. No one says a thing. Along comes the father, who says, "At least one of you has mud on your forehead." The father then asks the following question, over and over:"Does any of you know whether you have mud on your own forehead?" Assuming that all the children are perceptive, intelligent, truthful, and that they answer simultaneously, what will happen?

5. **Example 2:** The general's problem

   Two divisions of an army, each commanded by a general, camped on two hilltops overlooking a valley the enemy stays. If both divisions attack the enemy simultaneously they will win the battle, while if only one division attacks it will be defeated. Neither general will attack unless he is absolutely sure that the other will attack with him: a general will not attack if he receives no messages. The general of the first division wishes to coordinate a simultaneously attack (at some time the next day). They can communicate only by means of messengers. Normally, it takes a messenger one hour to get from on encampment to the other and on this particular night, everything goes smoothly. How long will it takes them to coordinate an attack?

1.2 Strategic form game

1.2.1 A subjective expected-utility maximization approach

1. A subjective expected-utility approach interpretation of a game:

   (a) Each player has a subjective probability distribution over all states of the world–more precisely, the probabilities that her opponents playing s_{-i} for all $s_{-i} \in \times_{j \neq i} S_j$.

   (b) Each player acts as an expected utility maximizer, choosing a strategy that maximizes her expected payoff in the game given the probability distribution over the strategies of her opponents. This is common knowledge.

   (c) The concept of Nash equilibrium imposes a further restriction, player's belief is consistent with the actual play of her opponents.

   (d) In view of this interpretation, each player in a game holds the belief $p^i(s_{-i})$, and choose s_i such that:

$$s_i^* \in \arg\max_{s_i \in S_i} \sum_{s_{-i}} p^i(s_{-i}) u_i(s_i, s_{-i}).$$

2. The above interpretation of subjective expected utility-maximization provides a decision theoretical foundation to the traditional definition of Nash equilibrium that each player plays optimally given the other players' equilibrium strategies.

3. An important feature of the subjective expected utility approach is that it does not require randomization on the part of the players. Recall that the traditional interpretation of mixed strategies that assumes players explicitly randomize. The probabilistic nature of strategies now reflects the uncertainty of other players about a player's choice. Thinking about the traditional Chinese "Scissor-rock-cloth" game.

1.2.2 Belief is the key

How players will play crucially depend on their beliefs, that is, their expectations how other players will play. Thus, in any game, the key is model players' belief. In financial market, as well as in poker games, savvy investors frequently try to affect the beliefs of other participants of the market through various manipulations to their own advantages.

Example: Tiger Management's bet again Yen[4]

In the first half of 1998, as Japanese capital flooded abroad, Tiger Management, the hedge fund of Julian Robertson expected that the yen would head down. Robertson left his investors know that he would short Japan's currency. Unfortunately for him, his bet went against him after Russia defaulted in August.

The problem for Robertson is: everyone in the Street knew his positions. The moment that Robertson sent out his July letter, every trader knew he was short Japan's currency; and the more the yen rose, the more they expected him to be forced to staunch his losses by buying back yen and closing his position.

On October 7 the yen jumped especially sharply, and traders sensed that Robertson would crack. They drove the yen up still more, calculating that Tiger's compelled exit from its trade would deliver yen holders a handsome profit. By around 10:00 A.M. on October 8, 1998, Japan's currency had appreciated drastically, trading at 130 to the dollar the previous morning and was now trading at 114. More than \$2 *billion of Tiger's equity had gone up in smoke.*

Robertson convened a crisis council of his top lieutenants. Rather than closing out its yen short, as the market expected, they concluded that Tiger should demonstrate its fearlessness by adding to its bet against Japan's currency.

Tiger's trader called a dealer at one of the big banks. He asked for a two-way price on dollar-yen, not wanting to give away whether he was buying or selling. The expectation that Tiger would soon be forced to buy yen by the billion had scared potential sellers to the sidelines; who wanted to shed yen when Tiger was about to force their price up? Because of the dearth of sellers, the market had dried up; there were no trades and no prices.

The bank would sell Tiger dollars using an exchange rate of 113.5 yen to the dollar; it would buy dollars back using an exchange rate of 111.5 yen to the dollar. The two-yen gap between the quotes was astronomical–maybe forty times the spread the market in a normal market. Tiger sent

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[4] Chapter 10, More Money than God, Hedge Fund and the Making of New Elites

a buy order for dollar (selling yen), as opposed to market expectation.

In that instant, the bank that took the order knew that Tiger was not going to be squeezed out of its trade. Julian Robertson and his Tigers still had the will to fight! Only a fool would trade against them! Seconds later the dearth of sellers came to an abrupt end: The bank's proprietary traders began dumping yen, and the dumps communicated the sea change to every currency desk on Wall Street. The yen started falling as quickly as it had risen earlier in the day.

1.2.3 Strategical form game

The capacity expansion game between Honda and Toyota

Toyota

		Build	Do not build
Honda	Build	$\underline{16}$, $\underline{16}$	$\underline{20}$, 15
	Do not build	15, $\underline{20}$	18, 18

1. Players: Toyota, Honda.
2. Rules: Two firms simultaneously choose to expand or not.

   Strategies for each firm: Build, Do not build.
3. 4 outcomes: (Build, Build), (Build, Do not build), (Do not build, Build), (Do not build, Do not build).
4. Payoffs:
5. Nash equilibrium of this game: (Build, Build).

1.2.4 Strategies

1. Definition: A pure strategy s_i of player i specifies the actions that a player will take under any conceivable circumstances that the player might face.
2. A strategy is a complete contingency plan that says what a player will do at each of her information sets if she is called on to play there
3. A player's strategy may include plans for actions that her own strategy makes irrelevant.

4. According to Rubinstein (1991) and Reny (1992), a player's strategy can be partitioned into two parts, a *plan* that describes a rational play for i, and a *prediction* about i's future behavior should i deviates from his plan.

    - A *plan* for player i specifies a choice for player i only when he is called upon to move, and does not specify what he would do at an information set of his that can not be reached according to this plan.
    - In order that others are able to specify what they would do were i not to follow through his plan (something i must know in order to evaluate the soundness of this plan in the first place), it must provide others with a *prediction* about i's future behavior should i deviate.

5. Given the SEUM approach discussed before, one natural interpretation for the specification of choices at information sets that won't be reached given a player's strategy is that they are *beliefs* of his opponents about what he would do in case he does not follow his strategy, i.e., the information sets were reached. The belief of his opponents is important as their choices at those information sets are based on this belief. Furthermore, what the player's opponents would do at those information sets rationalize his choice at at an upstream information set. Hence, this definition of strategy is not so odd when you interpret it as the way a player determines his strategy.

6. Pure strategy and mixed strategy

    A mixed strategy $\sigma_i \in \Delta(S_i)$ specifies probabilities to two or more pure strategies. For example, the traditional Chinese game, rock, scissor and cloth.

player 2

		scissor	rock	cloth
Player 1	scissor	0,0	-1, 1	1,-1
	rock	1, -1	0,0	-1, 1
	cloth	-1,1	1,-1	0,0

Example 2. A game of "match the coin": Both players, Tom and Jack, choose whether to place the coin Head up or Tail up. Jack wins if two "Head" or two "Tail" appear, and loses otherwise.

Jack

		Head	Tail
Tom	Head	-1, 1	1, -1
	Tail	1, -1	-1, 1

1.2.5 Dominant strategy

1. Dominant strategy: A strategy s_i is dominant for player i if for all $s_{-i} \in S_{-i}$ and $s_i' \in S_i/s_i$

   $$u_i\left(s_i, s_{-i}\right) > u_i\left(s_i', s_{-i}\right).$$

   A strategy is a dominant strategy for a player if it is better than other strategies, no matter what the others will choose.

   In **any Nash equilibrium**, players who have a dominant strategy play the dominant strategy. Thus, it is easy to find the Nash equilibrium of games in which some of the players have a dominant strategy.

2. “Confess” is a dominant strategy for prisoner 1 and prisoner 2.

Prisoner 1

		Confess	Not confess
Prisoner 2	Confess	$\underline{-5}$, $\underline{-5}$	$\underline{0}$, -10
	Not confess	-10, $\underline{0}$	-1, -1

3. Dominant strategies are rarity rather than norm. There is no dominant strategies in most interesting games.

1.2.6 Dominated strategy

1. A pure strategy $s_i \in S_i$ is weakly dominated if there is another strategy $s_i' \in S_i$ such that for all $s_{-i} \in S_{-i}$,

   $$u_i(s_i', s_{-i}) \geq u_i(s_i, s_{-i}),$$

   with strict inequality for some s_{-i}.

A strategy s_i is strictly dominated for player i if there is another strategy $s'_i \in S_i$ such that for all $s_{-i} \in S_{-i}$

$$u_i\left(s'_i, s_{-i}\right) > u_i\left(s_i, s_{-i}\right).$$

A strategy is strictly **dominated** when the player has another strategy that gives her a higher payoff no matter what the other player plays.

2. Allowing mixed strategy:

   - A strategy σ_i is strictly dominated for player i if there is another strategy $\sigma'_i \in \Delta(S_i)/\sigma_i$ such that for all $\sigma_{-i} \in \Delta(S_{-i})$

     $$u_i\left(\sigma_i, \sigma_{-i}\right) < u_i\left(\sigma'_i, \sigma_{-i}\right).$$

   - A pure strategy s_i is strictly dominated for player i if and only if there exists $\sigma_i \in \Delta(S_i)$ such that

     $$u_i\left(s_i, s_{-i}\right) < u_i\left(\sigma_i, s_{-i}\right)$$

     for all s_{-i}.

3. A mixed strategy that assigns positive probability to a pure strategy that is strictly dominated is also strictly dominated.

4. Example:

   Player 2

		l	m	r
Player 1	a	7, 5	2, 4	1, 2
	b	2, 2	4, 5	0, 3
	c	0, 4	7, 0	2, 1

   While player 1's pure strategy b is not dominated by any pure strategy, it is dominated by a mixed strategy

   $$\sigma_1 = (\frac{1}{2}a, 0b, \frac{1}{2}c).$$

   Player 2's r is also dominated by a mixed strategy.

5. Modifies capacity expansion game between Toyota and Honda

Toyota

		~~large~~	small	not build
Honda	~~large~~	~~0, 0~~	~~12, 8~~	~~18, 9~~
	small	~~8, 12~~	$\underline{16}$, $\underline{16}$	$\underline{20}$, 15
	not build	$\underline{9}$, $\underline{18}$	15, $\underline{20}$	18, 18

"large" is a dominated strategy for both firm as it can do better by choosing "small", regardless of what the other firm is going to do.

6. A player will not play a strictly dominated strategy in Nash equilibrium.

1.2.7 Nash equilibrium

Definition and interpretation

1. Definition: A strategy profile $s^* = (s_1^*, ..., s_n^*)$ is a **Nash Equilibrium** (NE) if for all $i \in N$ and for all $s_i' \in S_i$
$$U_i\left(s_i^*, s_{-i}^*\right) \geq U_i\left(s_i', s_{-i}^*\right).$$

2. The above is the standard textbook definition of NE. According to the SEUM approach discussed above, this definition of NE includes two parts: one on strategy and one on beliefs
   (a) Subjective expected-utility maximization: Each player holds the belief $p^i(s_{-i})$, and choose s_i such that:
   $$s_i^* \in \arg\max_{s_i \in S_i} \sum_{s_{-i}} p^i(s_{-i}) u_i(s_i, s_{-i}).$$
   (b) Beliefs coincides with opponents' equilibrium strategies:
   $$\forall\ i, \quad p^i(s_{-i}) = \sigma_{-i}^*.$$

3. The central concept of noncooperative game theory is Nash equilibrium. A Nash equilibrium is a profile of strategies such that for each player in the game, given the strategy chosen by the other players, the strategy is a best response for the player, that is, the strategy gives the player the highest payoff.

4. How to interpret the concept of Nash equilibrium.

(a) In most of the early literature the idea of equilibrium was that it said something about how players would play the game or about how a game theorist might recommend that they play the game. However, this interpretation runs into trouble in many cases. For example, how do we interpret mixed strategy Nash equilibrium? How to motivate the refinements of Nash equilibrium?

(b) Recently, there has been a shift to thinking of equilibria as representing not recommendations to players of how to play the game but rather the expectations of the others as to how a player will play. Further, if the players all have the same expectations about the play of the other players we could as well think of an outside observer having the same information about the players as they have about each other.

5. While the first interpretation of the equilibrium can be problematic in case of mixed strategy equilibrium, the second interpretation can accommodate mixed strategies without any trouble. In this scenario, the mixed strategy of a player does not represent a conscious randomization on the part of that player, but rather the uncertainty in the minds of the others as to how that player will act. Hence, the second interpretation of Nash equilibrium has become the preferred interpretation among game theorists.

   Thus the focus of the equilibrium analysis becomes, not the choices of the players, but the assessments of the players about the choices of the others. The basic consistency condition that we impose on the players' assessments is this: A player reasoning through the conclusions that others would draw from their assessments should not be led to revise his own assessment.

Find NE in two-player games

1. Most games have finite and **ODD** number of NE.

   - Example 1: The game of chicken

Jack

		Swerve	Stay
Tom	Swerve	0, 0	$\underline{-10}$, $\underline{10}$
	Stay	$\underline{10}$, $\underline{-10}$	-100, -100

Two pure strategy NE: (Swerve, Stay), (Stay, Swerve)

One mixed strategy NE

$$((Swerve, Stay; 0.9, 0.1), (Swerve, Stay; 0.9, 0.1)).$$

- Example 2:

	L	M	R
T	$\underline{3}$, $\underline{3}$	0, 2	$\underline{3}$, 0
B	0, 0	$\underline{3}$, 2	0, $\underline{3}$

2. If both players have a dominant strategy, then playing dominant strategies is the unique Nash equilibrium in the game.

3. If one player has a dominant strategy, this strategy will be this player's NE strategy. The other player's NE strategy is the best response to her opponent's dominant strategy.

4. If players have dominated strategies, delete the dominated strategies from the game and work with a smaller game.

   - Example 1.

Player 2

		A	B	C
Player 1	A	5, 8	15, $\underline{10}$	10, 5
	B	10, $\underline{15}$	$\underline{20}$, 9	$\underline{15}$, 0
	C	$\underline{20}$, $\underline{20}$	10, 10	10, 8

   - Example 2:

Player 2

		l	m	r
Player 1	a	$\underline{7}$, $\underline{5}$	2, 4	1, 2
	b	2, 2	4, $\underline{5}$	0, 3
	c	0, $\underline{4}$	$\underline{7}$, 0	$\underline{2}$, 1

5. NE must be a mutual best-response, that, given player 1 plays NE strategy, player 2 can not do better by playing some other strategies, similarly, given player 2 plays this strategy, player 1 can not do better by changing strategies. A NE is a strategy profile in which both players are playing the best-response given the other player's strategy.

Three player game

1. A simple example

Player 2

		U	D
1	A	(1, 1,0)	(2, -2, 5)
	B	(1,-2, -1)	(0, 3, 1)

Player 3 plays L

Player 2

	U	D
A	(1, 1,-2)	(2, -2, 5)
B	(2, 2, -1)	(2, 3, 7)

Player 3 plays R

2. Another example

		U	V	W
	L	3, 0, 2	2, -1, 0	1, -2, 0
1	M	3, 2, 1	1, 4, -1	0, 0, -2
	R	1, 1, 10	0, 2, 1	-2, 0, 3

Player 3 plays A

	U	V	W
L	2, 1, 1	3, 0, 0	2, -2, -1
M	5, 4, 2	1, 3, 4	3, 0, -2
R	1, 1, 1	0, 2, 0	-2, 0, 2

Player 3 plays B

	U	V	W
L	2, 1, -1	3, 0, -1	2, -2, -3
M	5, 4, -1	1, 3, -2	3, 0, -4
R	1, 1, -10	0, 2, -1	-2, 0, -2

Player 3 plays C

Player 1's dominated strategy R, player 2's dominated strategy W. Player 3's dominated strategy C. The pure strategy NE in this game (L, U, A), (M, U, B).

Strategic Stability of NE

1. Being a NE is a necessary condition for an obvious way to play the game, if an obvious way to play the game exists. But

    - Being NE is not sufficient for a strategy profile to be the obvious way to play a given game.
    - Not every game admits an obvious way to play the game

2. Some questions to be answered:

- How can we refine NE, the necessary condition to get the prediction of the game, an obvious way to play the game.

  NE can involve weakly dominated strategies, we should add to our necessary condition that the solution should be a NE in strategies that are undominated, even weakly

- What are the means by which we are to identify "obvious way to play a game?"

- What can one say about games that do not admit a "solution"

  When the game does not admit an "obvious way to play," looking at its NE can give precisely the wrong answer. The concept of NE is of no use when the game admits no "solution"

3. On occasion the requirement of Nash equilibrium can be too demanding at times. This leads to two less restrictive concepts: rationalizability, and correlated equilibrium. Some new solution concepts are thus advanced: rationalizability, correlated equilibrium

4. Rationalizability (Pearce 1984 Econometrica)

   - Rationalizable strategies:

   - A strategy σ_i is a best response for player i to her rivals' strategies σ_{-i} if

     $$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i}).$$

     for all $\sigma_i' \in \Delta(S_i)$

   - Strategy σ_i is never a best response if there is no σ_{-i} for which σ_i is a best response.

   - The strategies in $\Delta(S_i)$ that survive iterated deletion removal of strategies that are never a best response are known as player i's rationalizable strategies.

Player 2

		b_1	b_2	b_3	b_4
	a_1	0, 7	2, 5	7, 0	0, 1
Player 1	a_2	5, 2	3, 3	5, 2	0, 1
	a_3	7, 0	2, 5	0, 7	0, 1
	a_4	0, 0	0, -2	0, 0	10, -1

5. At other times, the requirement of NE is not strong enough to rule out multiple equilibria or implausible predictions. Two lines of research to address these difficulties:

- Equilibrium selection: concerned with narrowing the prediction to a single prediction. See *A General Theory of Equilibrium Selection in Games*/Harsany and Selten, 1988.
- Refinement of NE: concerned with establishing necessary conditions for reasonable predictions.

1.2.8 Iterated Deletion of Strictly Dominated Strategies

1. Now in situations where players do not have a chance to talk or where there is no history to rely on, Nash equilibrium may not be a good prediction. So in this case, we may prefer a solution concept that does not make strong assumptions about players knowing what each other is going to do.

2. A rational player should never choose a strictly dominated strategies because there exists another strategies that is strictly better. Note that if a player has a dominant strategies, then all other strategies are dominated.

3. Consider the following game (Gibbons pp. 6) :

Player 2

		L	M	R
Player 1	U	1,0	1,2	0,1
	D	0,3	0,1	2,0

4. Note that R is strictly dominated by M for Player 2. Now, if Player 1 knows that Player 2 is rational, then Player 1 knows that Player 2 will never choose R. If R is eliminated, then D becomes dominated by U. Now, if Player 2 knows Player 1 knows that Player 2 is rational, then Player 2 knows that Player 1 will not choose D. In that case, Player 2 should choose M.

5. If it is common knowledge that both players are rational, we can continue this process indefinitely.

   - Example:

Player 2

		L	M	R	U
Player 1	A	(5, 6)	(2, 6)	(1, 5)	(0, 7)
	B	(2, 10)	(1, 2)	(0, 10)	(-1, 1)
	C	(4, 1)	(3, 4)	(1, 3)	(2, 0)
	D	(0, 2)	(1, 3)	(3, 2)	(1, 1)

- Iterated deletion of strictly dominated strategies yields a unique NE in this game: (C, M)

6. Note that Nash and IDSDS are based on different logic. IDSDS does not require that the players know that the equilibrium is going to be played, so it requires less coordination. However, common knowledge of rationality is itself a very strong assumption.

7. For two-player games, rationalizable strategies are those remaining after the iterative deletion of strictly dominated strategies.

8. For more than two player games, this is no longer true.

   See, for example.

	L	R
U	9	0
D	0	0

A

	L	R
U	0	9
D	9	0

B

	L	R
U	0	0
D	0	9

C

	L	R
U	6	0
D	0	6

D

In this example, D is not dominated, but never a best response for player 3.

1.2.9 Existence of NE

1. Theorem 1 (Theorem 7.2, Jehle and Reny). Every finite strategic form game has at least one NE.

   Proof. See Jehle and Reny pp. 278.

2. Theorem 2. NE exists if the strategy set of each player is a compact and convex subset of an Euclidean space and if the utility function of each player is continuous in the strategy profile and quasi-concave in one's own strategy.

   Proof. Step 1: the maximizer

   $$b_i(\sigma_{-i}) = arg \max_{\sigma_i \in \Delta(S_i)} \; u_i(\sigma_i, \sigma_{-i})$$

   is nonempty, convex-valued and upper hemicontinuous.

   Step 2: by Kukutani's fixed point theorem, a non-empty, convex-valued upper hemicontinuous correspondence $b_i(\sigma_{-i})$ mapping from $\Delta(S)$ to itself, there must exists a fixed point.

1.3 Normal-form perfect equilibrium

1. The problem with NE is, that, many games have multiple equilibria. The natural question then arises, can we go any further and rule out any equilibria as self-enforcing assessment of the game. Indeed, on occastion irrational assessments by two different players might each make the other look rational.

2. As an example, consider the following game

	b_1	b_2
a_1	3, 3	0, 0
a_2	-5, -5	0, -5

But is (a_2, b_2) a good prediction of the game? Not likely.

3. An *ϵ-perfect equilibrium of the normal form game* is a totally mixed strategy $\sigma \equiv (\sigma_1, \dots, \sigma_N)$, if for all i and for all $s_i, s_i' \in S_i$,

$$u_i(s_i, \sigma) > u_i(s_i', \sigma) \qquad \text{then} \qquad \sigma_i(s_i') \le \epsilon.$$

4. A *perfect equilibrium* of a normal form game is a limit ($\epsilon \to 0$) of ϵ-perfect equilibria.

5. For two-player game, any NE in which no player plays dominated strategies is perfect.

6. For more than two-player game, the above statement is not true. There are NE with no players playing dominated strategies that is not perfect.

7. To see this, consider the following example:

Player 2

		a	d
Player 1	A	(3, 3, 0)	(5, 5, 0)
	D	(4, 4, 4)	(4, 4, 4)

Player 3: L

Player 2

	a	d
A	(3, 3, 0)	(2, 2, 2)
D	(1, 1, 1)	(1, 1, 1)

R

- This game has two pure NE:

$$(D, a, L), \qquad (A, a, R).$$

- But (D, a, L) is not a perfect equilibrium, even if no weakly dominated strategy is played.
- To see (D, a, L) is not perfect, note that to be a perfect equilibrium, the corresponding totally mixed strategy profile has to take the form of

$$(\epsilon, 1-\epsilon;\ 1-\eta, \eta;\ 1-\nu, \nu).$$

- Given player 2's belief: $(\epsilon, 1-\epsilon)$ and $(1-\nu, \nu)$:

$$\begin{aligned} u_2(a, \sigma^\epsilon) =& (1-\nu)[3\epsilon + 4(1-\epsilon)] + \nu[3\epsilon + 1 - \epsilon] \\ =& (1-\nu)(4-\epsilon) + \nu(1+2\epsilon), \end{aligned}$$

$$\begin{aligned} u_2(d,\sigma^{\epsilon}) =& (1-\nu)[5\epsilon+4(1-\epsilon)]+\nu[2\epsilon+1-\epsilon] \\ =& (1-\nu)(4+\epsilon)+\nu(1+\epsilon). \end{aligned}$$

Since

$$u_2(d,\sigma^{\epsilon})-u_2(a,\sigma^{\epsilon})=2\epsilon-3\epsilon\nu,$$

which is greater than zero for small number ν, there is no ϵ-perfect equilibrium in which a receives higher probability than d, indicating (D,a,L) is not a perfect equilibrium.

8. Another example:

Player 2

Player 1

	L	R
T	(1, 1, 1)	(1, 0, 1)
B	(1,1, 1)	(0, 0, 1)

Player 3 plays l

Player 2

	L	R
T	(1, 1, 0)	(0, 0, 0)
B	(0, 1, 0)	(1, 0, 0)

Player 3 plays r

- (B,L,l) is a NE with no weakly dominated strategies.
- However, for any small probabilities player 2 assigns to R and player 3 to r, the expected payoff for player 1 from T is greater than that from B. Thus, there exists no ϵ-perfect equilibrium in which the totally mixed strategy profile assigns more than ϵ to B.

9. Perfect equilibrium does not eliminate all unreasonable outcomes in some games. Adding a dominated strategy may enlarge the set of perfect equilibria.

10. Consider the following example.

	L_2	M_2
L_1	1, 1	0, 0
M_1	0, 0	0, 0

	L_2	M_2	R_2
L_1	1, 1	0, 0	-1, -2
M_1	0, 0	0, 0	0, -2
R_1	-2, -1	-2, 0	-2, -2

11. Solution concept that is free of the drawback: proper equilibrium

- ϵ-proper equilibrium: totally mixed strategy profile σ is ϵ-proper if for all i and for all $s_i, s_i' \in S_i$,

$$u_i(s_i,\sigma)>u_i(s_i',\sigma)\Longrightarrow\frac{\sigma_i(s_i)}{\sigma_i(s_i')}<\epsilon.$$

- A proper equilibrium is the limit of a sequence of ϵ-proper equilibria.

- A property equilibrium is normal form perfect, but the reverse is not true.

Chapter 2

Extensive form game

2.1 Introducing Extensive form games

1. Strategic form games describe a game by its strategies—complete contingent plans of how to react in each possible scenario—and play down the temporal aspect of the situation—who moves first, who moves second, etc. It is like a computer chess program. Once each player submit the programs, the computer will take over and decide which side will win. You don't get to see the actual step-by-step plays.

2. Extensive-form games explicitly describe how the game is played through time, including details about who moves first, who moves second, and so on. In this sense, extensive form game provides more information than the strategic form.

3. Definition: An extensive form game Γ_E contains the following information:

   - Set of nodes $\mathscr{X}$, set of actions $\mathscr{A}$ and set of players $\{1, \ldots, I\}$.
   - The order of moves—i.e., who moves when
     - ➢ Predecessor function: $p : \mathscr{X} \to \mathscr{X} \cup \varnothing$; $p(x_0) = \varnothing$
     - ➢ Successor function: $s(x) = p^{-1}(x)$; $T = \{x \in \mathscr{X} : s(x) = \varnothing\}$
     - ➢ $\alpha : \mathscr{X} \backslash \{x_0\} \to \mathscr{A}$ giving the action that leads to x from $p(x)$

     $$c(x) = \{\alpha \in \mathscr{A} : a = \alpha(x'), x' \in s(x)\}$$

   - Information sets: $H : \mathscr{X} \to \mathscr{H}$,

     $$c(x) = c(x') \quad \text{if} \;\; H(x) = H(x')$$

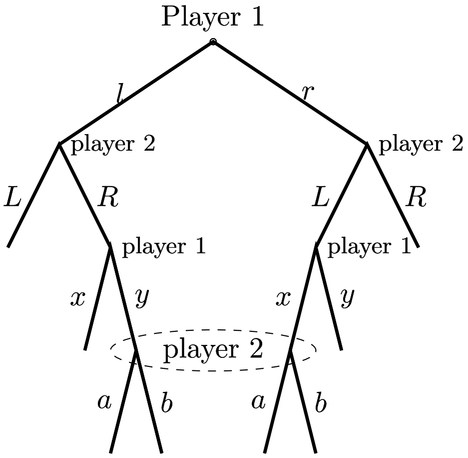
- The probability distribution over any exogenous events:

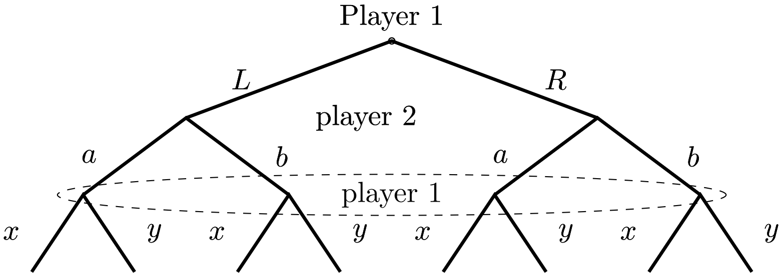
$$\rho: \ \mathscr{H}_0 \times \mathscr{A} \to [0,1], \quad \rho(H,a) = 0 \quad \text{if } a \notin C(H) \text{ and}$$
$$\sum_{a \in C(H)} \rho(H,a) = 1$$

- The players' payoffs as a function of the moves that were made.

4. We say a game is of *perfect recall* if "no player ever forgets any information he once knew, and all players know the actions they have chosen previously. More formally, if x and x' belongs to the same information set of player i, then it must be that 1) the sequence of moves that leads to x and the sequence of moves that leads to x' must pass through the same sequence of information sets for player i, and 2) in each of the information set of players i that leads to x and x', the same action must be chosen by player i.

5. Examples of games of imperfect recall,

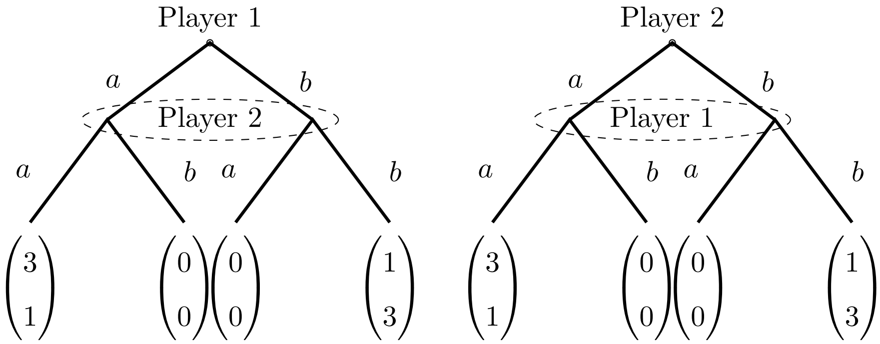




6. One problem with this way of writing down a game is that there is no natural way to express a simultaneous move. Example: Battle of Sexes. When we write down this game in extensive

form, we write it as if someone moves first, and the second player does not observe the move of the first mover. This maintain the same information structure as the simultaneous game but change the sequence of moves. The point is: although the extensive form tells us more about the sequence of moves, it is not a completely accurate description (when the game involves simultaneous moves).

	a	b
a	3,1	0,0
b	0,0	1,3



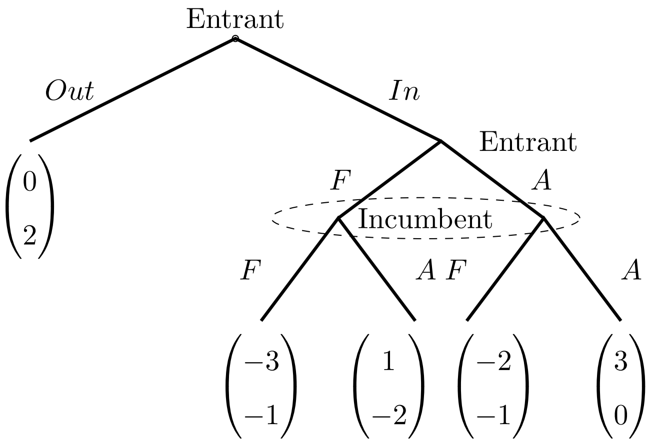
7. A pure strategy of an extensive form game for a player i prescribes an action at each information set of player i.

8. Example: In the game given below:

    - Firm I has two pure strategies: fight, accommodate

    - Firm E has four pure strategies:

      Out and Fight if In (OF), Out and Accommodate if In (OA), In and Fight if In (IF), In and Accommodate if In (IA).

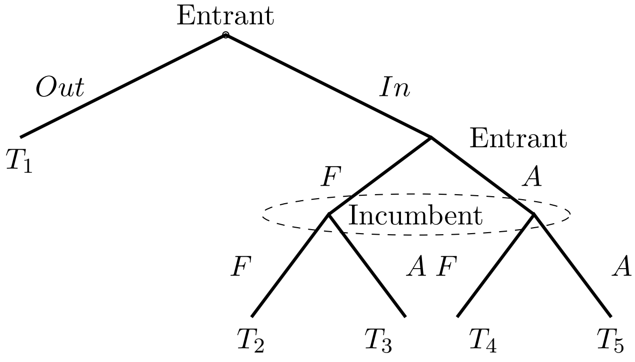


- A strategy is not really a plan of actions, as it requires a player to specify his actions at information sets that are impossible to be reached if he carries out his plan. So, if a player wants to let an agent to play the game for him, there is no need to tell the agent what to do in those information sets. In fact, two strategies that are different only in information sets ruled out by own strategy are *strategically equivalent* in the sense they always lead to the same payoffs.
- One way to interpret this is that a strategy for player i actually includes two parts: a rational plan for player i at information set that he may be called upon to play; and a prediction about i's future behavior should she deviates from her plan.
- Player i's PLAN: The rational plan specifies player i's choice at her information sets that could be reached given the plan how she would play the game.
- Other players' BELIEF: The prediction of what player i would do at information sets that can't be reached given her own plan is important for other players to specify what they would do should player i deviate from her plan. In addition, to know the belief of other players about her play at those information set and how they would respond help rationalize player i's plan in the first place (her choices at information sets that could be reached given her PLAN).

9. There are two ways to define mixed strategies.

- A *mixed strategy* σ_i for player i assigns to each pure strategy $s_i \in S_i$ a probability $\sigma_i(s_i) \geq 0$ that it will be played, where $\sum_{s_i \in S_i} \sigma_i(s_i) = 1$

- A *behavioral strategy* for player i prescribes, for every information set H and action a_i a probability $\lambda_i(a_i, H)$, with $\sum_{a_i} \lambda_i(a_i, H) = 1$ for all H.
- Mixed strategies and behavior strategies are strategically equivalent in games of perfect recall. This implies that we can use either one of the two at our convenience. We typically use *behavior* strategies for *extensive* form game, and *mixed* strategies for *strategical* form game.
- Example: Given the extensive form game



➢ For any mixed strategy of Firm E (OF,OA,IF,IA; p_1, p_2, p_3, p_4), there exists a unique behavior strategy such that the probability of reaching terminal nodes T_1, ..., T_5 is the same.

To see this, let the mixed strategy for Firm I be (F, A; σ, $1-\sigma$), we show there is a unique behavior strategy for Firm E that assigns q and $1-q$ to "Out" and "In" at the first information set, and assigns r and $1-r$ to "F" and "A" at the second information set.

Given the mixed strategy of Firm E and Firm I:

$$Pr(T_1) = p_1 + p_2, \quad Pr(T_2) = p_3\sigma, \quad Pr(T_3) = p_3(1-\sigma)$$

$$Pr(T_4) = p_4\sigma, \quad Pr(T_5) = p_4(1-\sigma).$$

Hence we have the unique behavior strategy

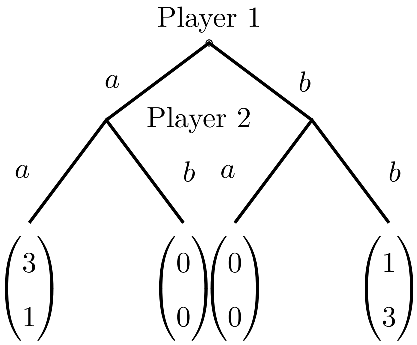
$$q = p_1 + p_2, \quad r = \frac{p_3}{1-(p_1+p_2)}.$$

➢ On the other hand, given a behavior strategy $(q, 1-q)$ at the first information set and $(r, 1-r)$ at the second information set, the unique mixed strategy that is equivalent to the behavior strategy is:

$$p_1 = qr, \quad p_2 = q(1-r), \quad p_3 = (1-q)r, \quad p_4 = (1-q)(1-r).$$

10. Nash Equilibrium is defined in the usual way.

11. The sequential battle of sexes game:



12. Obviously every Nash equilibrium of the extensive form game is a Nash equilibrium in the strategic form game, and vice versa. The strategic form however does not capture all the information, namely, the order of moves, contained in an extensive form game. Two extensive form games may have the same strategic form. For, example, the game above may also be a 2x4 simultaneous-move game.

2.2 Sequential rationality

1. The principle of sequential rationality: a player's strategy should specify optimal actions *at every point in the game tree.*

2. Backward induction ensures that a player's strategies specify optimal behavior at every decision node of the game.

2.2.1 Backward induction

1. There are three Nash equilibria in SBoS: (a,aa), (a,ab), (b,bb).

2. In the last equilibrium, player 2 threatens to choose b when player 1 chooses a, even. when doing so harms both players. Believing the threat, player 1 chooses b, leading to the outcome (b,b). (This is a Nash equilibrium because Player 2 will not be called upon to carry out the threat.)

3. Most people will think this type of threat is not credible. If player 1 calls player 2's bluff by choosing a, it is not in the interest of player 2 to actually carry the threat. The concept of

Nash equilibrium does not distinguish whether a threat is credible because as long as a threat is effective, it has no payoff consequences.

4. In games of perfect information, we can formalize the idea that each player should act rationally in decisions nodes off the equilibrium path by the procedure of backward induction.

5. Definition: An extensive form game is of perfect information if every information set is a singleton (which means there is no exogenous uncertainty and each player also all the moves up to that point).

6. Backward Induction: Start with the decision nodes in the final stage (those whose successors are all terminal nodes) At each of these nodes, selects one of the best alternatives for the player who is making the decision and eliminates the rest. Repeat the same procedure until the initial node is reached. The resulting payoff profile is called a backward induction solution. Backward induction solutions are all Nash equilibrium, but the converse is false. The solution is unique if no player is ever indifferent between two actions.

7. Subgame perfect NE

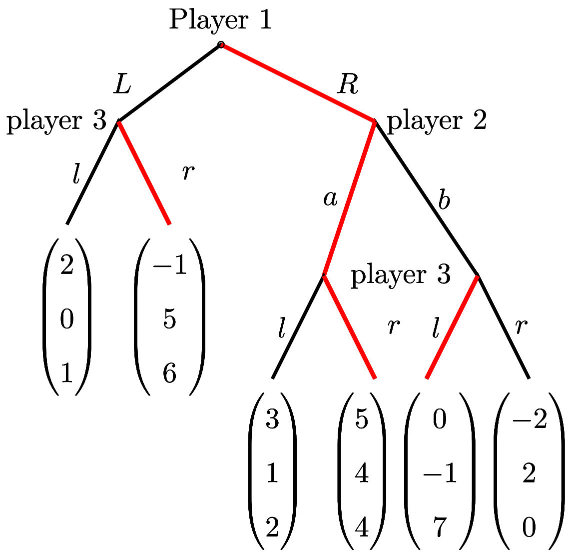
   - Subgame: The portion of the game tree that follows a decision node x is a *subgame* if it constitutes a well-defined extensive form game. That is, if (1) the information set that contains x is a singleton; and (2) if x belongs to the subgame, then every x' in the same information set as x must also belong to the subgame.

   - Subgame Perfection: A Nash equilibrium is subgame perfect if it prescribes a Nash equilibrium in every subgame.

   - Subgame perfection generalizes the idea of backward induction to games of imperfect information. The backward induction solution is always subgame perfect.

   - The way to find subgame perfect equilibrium is similar to backward induction: starting from the subgame near the end and work backward.

   - An application: consider the game below.



Player 3

	lll	llr	lrl	lrr	rrl	rlr	rll	rrr
L	2, 0, 1	2, 0, 1	2, 0, 1	2, 0, 1	-1, 5, 6	-1, 5, 6	-1, 5, 6	-1, 5, 6
R	3, 1, 2	3, 1, 2	5, 4, 4	5, 4, 4	5, 4, 4	3, 1, 2	3, 1, 2	5, 4, 4

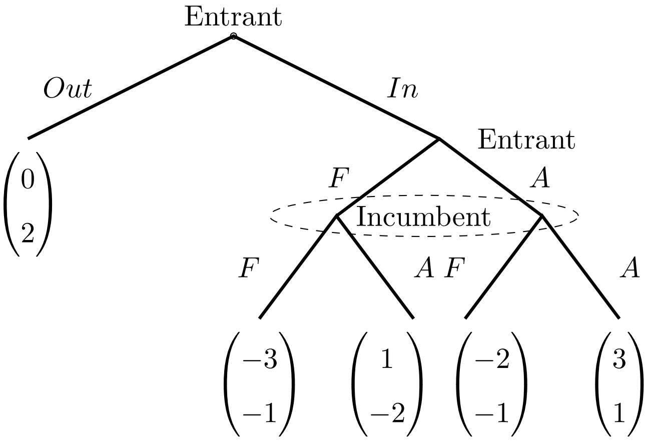
Player 2 plays a

Player 3

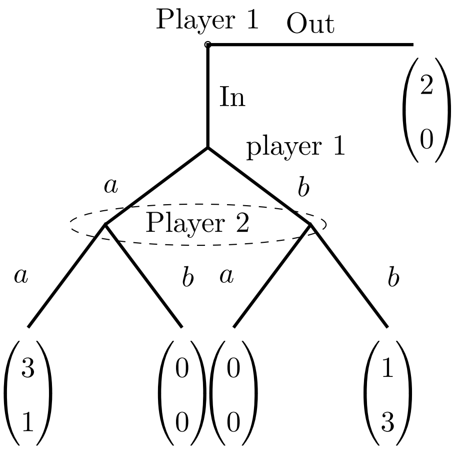
	lll	llr	lrl	lrr	rrl	rlr	rll	rrr
L	2, 0, 1	2, 0, 1	2, 0, 1	2, 0, 1	-1, 5, 6	-1, 5, 6	-1, 5, 6	-1, 5, 6
R	0, -1, 7	-2, 2, 0	0, -1, 7	-2, 2, 0	0, -1, 7	-2, 2, 0	0, -1, 7	-2, 2, 0

Player 2 plays b

- Example 2



- Example 3



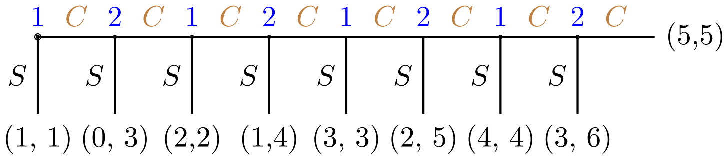
8. Existence of SPNE: Every finite extensive form game of perfect information has a pure strategy SPNE.

9. Intuitively, backward induction rules out incredible threats. The backward induction solution of SBoS is (a,ab).

10. This example illustrates the value of commitment in strategic situations. Note that here the second mover is harmed by his own rationality—he will be better off if he can convince the first mover that he is irrational. That's one reason why young children often get what they want from parents.

11. Backward induction in some sense relies on the common knowledge of rationality at every decision node. But it is problematic to maintain the assumption of rationality off the equilibrium path. According to backward induction logic, a rational player should not deviate in the first place. There is no completely satisfactory solution to this problem.

    Example: Centipede game



- The unique SPNE is for 1 & 2 to choose "S" , which follows from Iterated deletion of weakly dominated strategies. But this SPNE is rather doubtful.

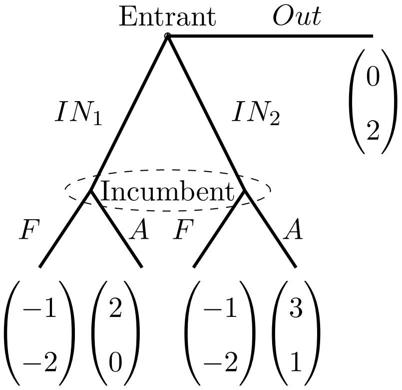
- A real world example of the Centipede game may be the one played by the WallStreetBets investors in their fight against Melwin Capital that pushes the price of GameStop from 20 to more than 400 dollars in a short period of time in Jan. 2021. Of course, in this case, there are hundreds of thousands of players, not two!

2.3 Sequential equilibrium

1. Definition: An extensive form game is of imperfect information if not all information sets are singletons.

2. Backward induction may not work in games of imperfect information.

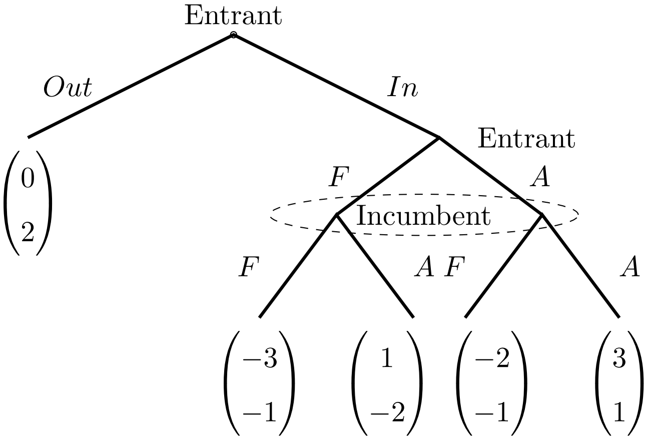
   Example 1: Entrant-incumbent game 1

   In view of this problem, a natural solution is to require each player to make optimal choices at every information set. This solves the problem in the above example.

3. But is this enough? Consider the following example

   - Example 2: Entrant-incumbent game 2

   - The NE (OA, F) clearly satisifes the condition that every player makes optimal choice at every information set, if Incumbent believes Entrant chooses F at his second information set, the off-equilibrium path information set.

- Thus, there needs to be some reasonable restrictions on players' off-equilibrium path belief as well!

4. Definition: A system of beliefs μ in an extensive form game Γ_E is a specification of probability $\mu(x) \in [0,1]$ for each decision node x in Γ_E such that for all information set $h \in \mathbf{H}$,

$$\sum_{x \in h} \mu(x) = 1.$$

5. Definition: A strategy profile $\sigma = (\sigma_1, \ldots, \sigma_I)$ is sequentially rational *at information set* h given belief μ if for player i who moves at information set h,

$$E[U_i|\mu, \sigma_i, \sigma_{-i}] \geq E[U_i|\mu, \sigma_i', \sigma_{-i}]$$

   for all σ_i' of player i.

   A strategy profile σ is sequentially rational given belief μ if this condition is satisfied for all information sets h.

6. A strategy profile and system of beliefs (σ, μ) is a sequential equilibrium of Γ_E if

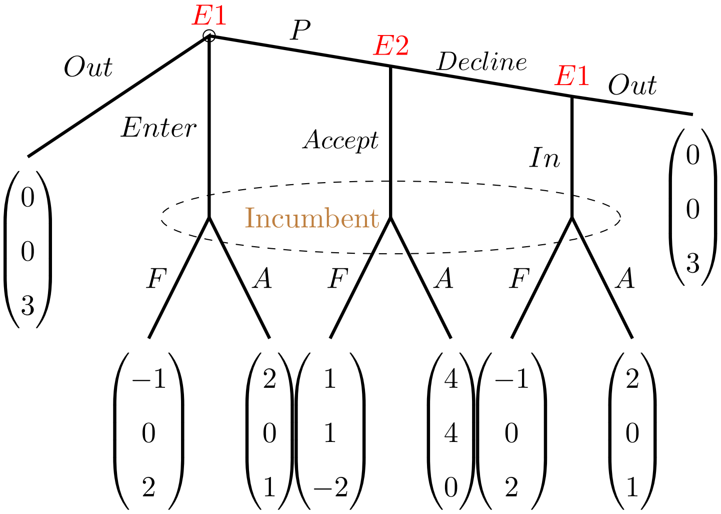
   i. The strategy profile σ is sequentially rational given μ.

   ii. There exists completely mixed strategies $\{\sigma^k\}_{k=1}^{\infty}$ with $\lim_{k\to\infty} \sigma^k = \sigma$, such that $\mu = \lim_{k\to\infty} \mu^k$, where μ^k is derived from σ^k using Bayes rule.

7. Interpretation of the definition:

   - The concept of sequential equilibrium captures the intuition of backward induction - - each player believes the other players are rational and thus will play optimally in any continuation of the game - - by defining an equilibrium to be a pair consisting of a behavioral strategy and a system of beliefs.
   - The behavior strategy is *sequentially rational* with the system of beliefs, namely that at every information set at which a player moves, the player's behavioral strategy maximizes his conditional payoff, given his belief at that information set and the strategies of the other players.
   - The system of belief is *consistent* with the behavioral strategy, that is, it is the limit of a sequence of beliefs each being the actual conditional distribution on nodes of the various information sets induced by a sequence of totally mixed behavioral strategies converging to the given behavioral strategy.

8. Entrant incumbent game 3



Firm E2

Firm E1

	Accept	Decline
OI	0, 0, 3	0, 0, 3
OO	0, 0, 3	0, 0, 3
EI	-1, 0, 2	-1, 0, 2
EO	-1, 0, 2	-1, 0, 2
PI	1, 1, -2	-1, 0, 2
PO	1, 1, -2	0, 0, 3

firm I fight

Firm E2

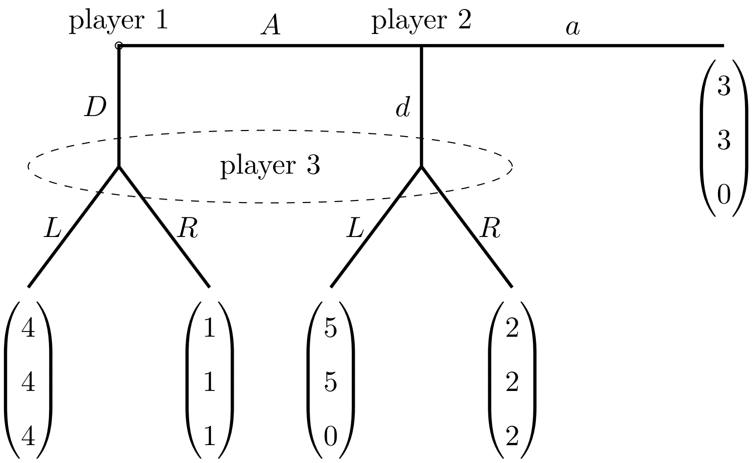
Firm E1

	Accept	Decline
OI	0, 0, 3	0, 0, 3
OO	0, 0, 3	0, 0, 3
EI	2, 0, 1	2, 0, 1
EO	2, 0, 1	2, 0, 1
PI	4, 4, 0	2, 0, 1
PO	4, 4, 0	0, 0, 3

I accommodate

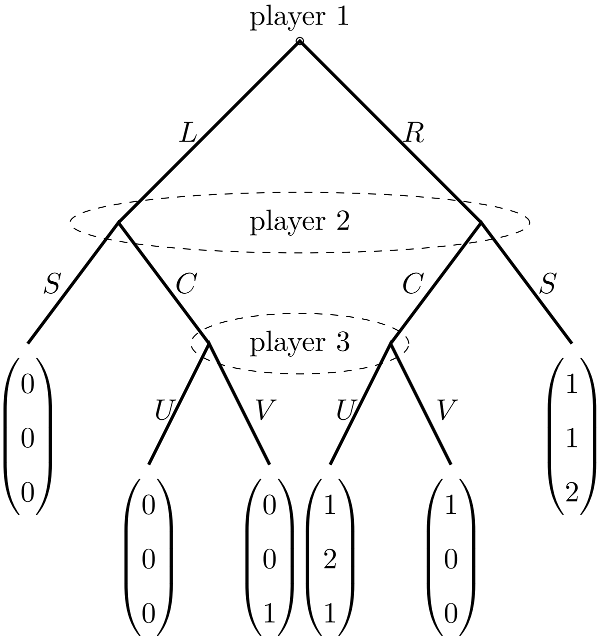
While this game has several pure strategy NE, there is only one SE.

9. Selten's Horse



Two pure NE: (D, a, L), (A, a, R).

10. Example 5.



Two pure NE: (R, C, U), (R, S, V).

11. Implications of the two conditions imposed by sequential equilibrium:

    (a) On behavior: In NO circumstances should a player makes a choices that is dominated by other choices. Therefore, the strategy should specify optimal choice at every information set given the beliefs about what has happened previously, thus the probability distribution over different decision nodes at the information set, as well as what the other players are playing.

(b) On belief (off-the equilibrium-path behavior by the other players):

- Simply put, belief about what has happened thus which decision note one faces should be consistent with sequential rationality on the part of opponents;

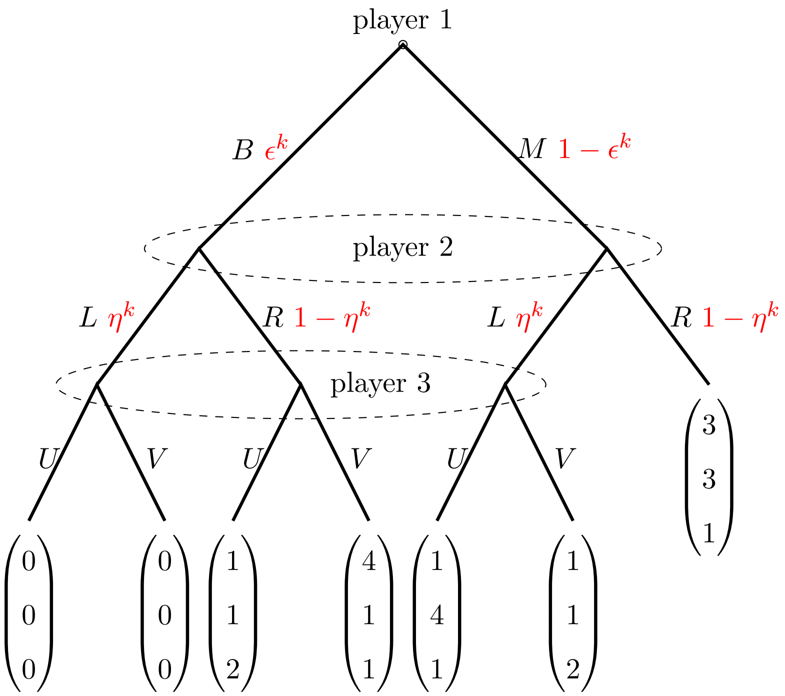
- At EVERY information sets when an opponent played the game, one should think that she has played her best response.

  As a direct consequence, at every information set, if the player has a dominant choice, one that is better than the rest of choices regardless of what the choices of other players, then her opponent's belief should put probability one to the dominant choice, and zero to the rest of choices.

12. The concept of sequential equilibrium is a strengthening of the concept of subgame perfection. Any sequential equilibrium is necessarily subgame perfect, but the converse is not true. The difference of the two, of course, only lies in imperfect information game.

13. Consistency requirement: There are sequential equilibrium in which consistency may impose restrictions on the possible sequences of totally mixed strategy, and in turn also on the possible belief players may have off-the-equilibrium path.

    Example 228.2 of Osborne and Rubinstein 1995.



(a) The NE of this game is $(M, R, (\alpha, 1-\alpha) | \alpha \in [1/3, 2/3])$.

(b) S.E of this game:

- Strategy σ

$$\left\{ M; R; (\alpha, 1-\alpha) | \alpha \in \left[\frac{1}{3}, \frac{2}{3}\right] \right\};$$

- Belief μ

  Player 2's belief: $(0, 1)$

  Player 3's belief: $(0, 0.5, 0.5)$.

- To derive player 2's belief: let the totally mixed strategy profit σ^k be

$$(\epsilon^k, 1-\epsilon^k; \eta^k, 1-\eta^k; \alpha, 1-\alpha)$$

  such that

$$\lim_{k\to\infty} \epsilon^k = 0, \qquad \lim_{k\to\infty} \eta^k = 0.$$

  Denote the 2 decision nodes, in the order of left, right, as z_L, z_R.

$$\mu^k(z_L) = \frac{\epsilon^k}{1} = \epsilon^k$$

$$\mu^k(z_R) = \frac{1-\epsilon^k}{1} = 1-\epsilon^k$$

Taking limit we have

$$\mu(z_L) = \lim_{k\to\infty} \mu^k(z_L) = 0$$

$$\mu(x_M) = \lim_{k\to\infty} \mu^k(z_R) = 1.$$

- To derive player 3's belief: let the totally mixed strategy profit σ^k be

$$(\epsilon^k, 1-\epsilon^k; \eta^k, 1-\eta^k; \alpha, 1-\alpha).$$

Denote the 3 decision nodes, in the order of left, middle and right, as x_L, x_M, x_R.

$$\mu^k(x_L) = \frac{\eta^k\epsilon^k}{\eta^k\epsilon^k + \epsilon^k(1-\eta^k) + (1-\epsilon^k)\eta^k} = \frac{\eta^k\epsilon^k}{\epsilon^k + (1-\epsilon^k)\eta^k}$$

$$\mu^k(x_M) = \frac{(1-\eta^k)\epsilon^k}{\eta^k\epsilon^k + \epsilon^k(1-\eta^k) + (1-\epsilon^k)\eta^k} = \frac{(1-\eta^k)\epsilon^k}{\epsilon^k + (1-\epsilon^k)\eta^k}$$

$$\mu^k(x_R) = \frac{(1-\epsilon^k)\eta^k}{\eta^k\epsilon^k + \epsilon^k(1-\eta^k) + (1-\epsilon^k)\eta^k} = \frac{\eta^k(1-\epsilon^k)}{\epsilon^k + (1-\epsilon^k)\eta^k}$$

Taking limit we have

$$\mu(x_L) = \lim_{k\to\infty} \mu^k(x_L) = 0$$

$$\mu(x_M) = \lim_{k\to\infty} \mu^k(x_M) = \frac{1}{2}.$$

$$\mu(x_R) = \lim_{k\to\infty} \mu^k(x_L) = \frac{1}{2}.$$

(c) While in general, the totally mixed strategy for player 1 could be $(1-\epsilon, \epsilon)$ and for player 2 could be $(1-\eta, \eta)$, for consistency, it nevertheless must be true that $\eta(1-\epsilon) = \epsilon(1-\eta)$, so that player 3 assigns equal probability to the upper and the middle decision nodes. Only in this case will player 3 be indifferent between the two pure strategies U and V, which makes the mixed strategy best response motivated.

Hence, it must be true that $\epsilon = \eta$.

(d) The belief for player 3, while consistent with the totally mixed strategy, is not structurally consistent.

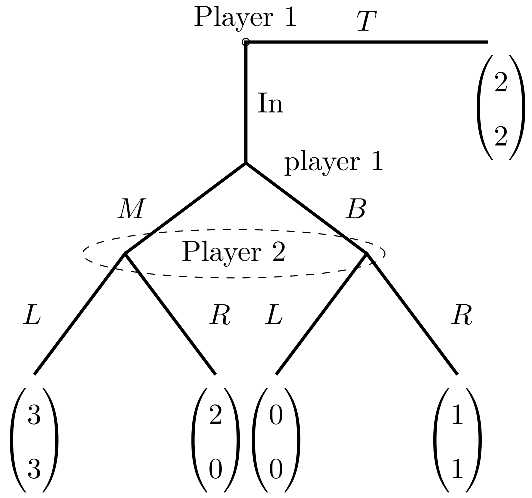
(e) *Structural consistency*: A belief system μ is structurally consistent if for each information set h, there exists some strategy profile σ such that for all $x \in h$,

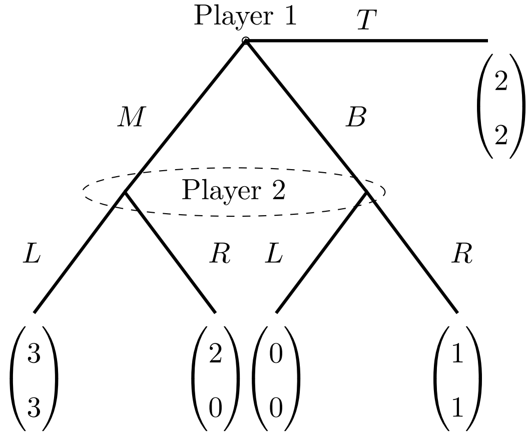
$$\mu(x) = \frac{prob(x|\sigma)}{prob(h|\sigma)}.$$

2.3.1 Deficiencies of sequential equilibrium

1. Conformity with backward induction, while being necessary, is not sufficient for strategic stability.

2. A basic flaw in the concept of "sequential equilibrium": it depends on all the arbitrary details with which the game tree is drawn.





3. Sequential equilibrium may involve players playing dominated strategies.

4. The main problems: the requirement of consistency on belief allows unreasonable beliefs.

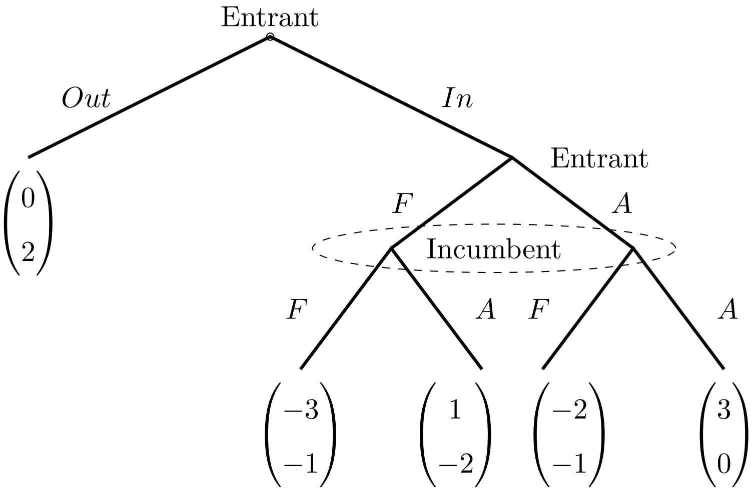
    - Requirement on strategies: OK, players should make optimal choice at any point in the game tree given the belief.

- Requirement on beliefs: only requirement the belief to come from a sequence of totally mixed strategies. But some sequence of totally mixed strategies may not make sense at all, thereby leading to unreasonable belief in sequential equilibrium.

2.4 Agent norm form perfect equilibrium

1. The *agent normal form* of an extensive form game is the normal form of the game between agents, obtained by letting each information set be manned by a different agent, and by giving any agent of the same player that player's payoff.

2. Example: Entrant-incumbent game.



The Agent normal-form:

	F	A
Out	0, 2	0, 2
In	-3, -1	1, -2

E_2 plays F

	F	A
Out	0, 2	0, 2
In	-2, -1	3, 0

E_2 plays A

2.5 Perfect Bayesian equilibrium

1. A profile of strategies and system of beliefs (σ, μ) is Perfect Bayesian equilibrium if

   i. σ is sequentially rational given μ;

ii. The system of belief μ is obtained using Bayes rule **whenever possible**.

2. Perfect Bayesian equilibrium is also weak PBE as it requires Bayesian updating for all information set that is reached with positive probability under equilibrium strategy σ; but it also requires:

   iii. If an information set I is reached with zero probability under σ (off the equilibrium path), the belief at I is derived from σ using Bayes' rule, if possible.

3. The restriction (iii) comes from Gibbons (1992). But, this restriction is evidently vague. One can interpret it as follows:

   If an information set I is reached with zero probability under σ (off the equilibrium path), the belief at I is derived, using Bayes' rule, from the beliefs at the information sets that precede I and players' continuation strategies as specified by σ; if possible.

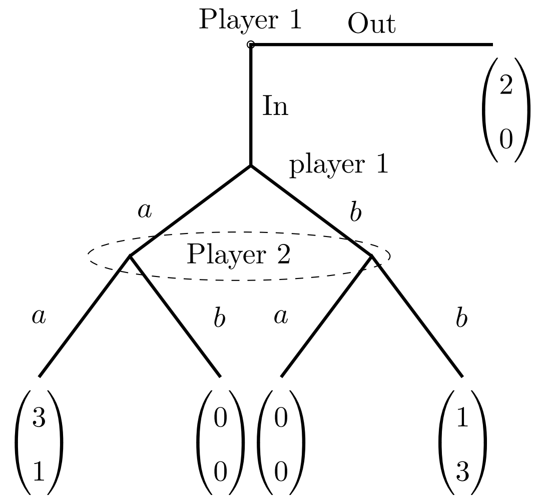
4. PBE is a weak PBE, and also implies subgame perfection.

5. Sequential equilibrium is equivalent to a PBE in a general class of games. However, for some games, sequential equilibrium imposes more restrictions on off-the-equilibrium beliefs. Sequential equilibrium requires the beliefs of players at information sets not reached in the equilibrium to be derived from the SAME sequence of mixed strategies. PBE imposes no such restrictions on off-the-equilibrium beliefs.

2.6 Forward induction

1. While SPNE and sequential equilibrium concept can help to rule out noncredible threat in extensive form game, a large range of off-equilibrium behavior can be justified by picking off-equilibrium-path behavior appropriately.

2. Outside option game



- Two pure NE. Both are also SPNE.

$$(Ia, a), \quad (Ob, b).$$

- Two pure PBE,

  First one: strategy profile is (Ia, a); belief (1,0), that is, at her information set, player 2 believes 1 has played a.

  Second one: strategy profile is (Ob, b); belief (0, 1), that is, at her information set, player 2 believes 1 has played b.

3. Furthermore, PBE as well as sequential equilibrium can be sensitive to what may seem like irrelevant changes in the extensive form game.

4. The key underlying forward induction is that players maintain the assumption that their opponents have maximized their utility in the past as long as the assumption is tenable, even if unexpected is observed. That is, while finding himself off the equilibrium path, he should not interpret it as a result of unintentional mistake by his opponents as long as the deviations by his opponents are *rationalizable*.

5. A crucial consequence of forward induction is: a subgame can not be treated as a game on its own. In other words, a forward induction of a subgame need not be part of the solution of the whole game. This is *different* from backward induction. REMEMBER: A backward induction solution of a subgame is part of the subgame perfect NE of the whole game.

6. In forward induction, how a subgame is reached conveys information about intended play in the subgame.

7. One can use reduced strategic form game for forward induction, rather than the extensive form game used in backward induction.

8. For a large class of generic games, forward induction and iterated deletion of weakly dominated strategies yield the same set of solutions.

2.7 Sequential Bargaining

1. Split the dollar game: Two players divide a dollar between themselves. Let x_i denote the share of player i, $i = 1, 2$. The set of agreements is

$$X = \{(x_1, x_2) : x_i \geq 0, i = 1, 2 \text{ and } x_1 + x_2 = 1\}.$$

2. The game last for T periods. In period 1, player 1 makes an offer to player 2. If player 2 accepts, then they split the dollar according to the offer. If player 2 rejects, then they move to period 2. In period 2, they exchange roles with player 2 making an offer and player 1 decides whether to accept. In general, player 1 makes offer in odd periods; player 2 makes offers in even periods. The game continues until an agreement is reached or after the end of period T.

3. If an agreement (x_1, x_2) is reached in period t, then player i receives payoff

$$u_i(x_i, t) = \delta_i^t x_i$$

where $\delta_i < 1$ is player i's discount factor. If no agreement is reached after $T - 1$ periods, then a settlement $(s, 1 - s)$ is enforced in period T.

4. This is commonly known as the Rubinstein bargaining game.

5. For finite T, we can solve the game by backward induction.

6. First, suppose $T = 3$. In period $t = 2$, Player 1 can obtain s in the next period by rejecting player 2's present offer. Thus, player 1 will reject any offer if and only if it is strictly worse than $(\delta_1 s, 1 - \delta_1 s)$. (We will assume that a player accepts whenever he is indifferent between accepting and rejecting. All payoffs are evaluated from the current period.). In period 1,

player 2 knows that he can obtain $1-\delta_1 s$ in the next period. Hence, by the same reasoning, he will accept the present offer iff

$$x_2 \geq \delta_2 (1-\delta_1 s) = \delta_2 - \delta_1\delta_2 s.$$

Hence, in equilibrium player 1 proposes $(1-\delta_2+\delta_1\delta_2 s, \delta_2-\delta_1\delta_2 s)$ in $T=1$ and player 2 accepts.

7. The case of $T=5$ is equivalent to $T=3$ with the breakdown's payoff equal to

$$(1-\delta_2+\delta_1\delta_2 s, \delta_2-\delta_1\delta_2 s).$$

Substituting the new breakdown payoff into the equilibrium for $T=3$ gives the first period offer:

$$\begin{aligned} x_1 &= 1-\delta_2+\delta_1\delta_2(1-\delta_2-\delta_1\delta_2 s) \\ &= (1-\delta_2)(1+\delta_1\delta_2)+(\delta_1\delta_2)^2 s, \end{aligned}$$

and

$$x_2 = 1-(1-\delta_2)(1+\delta_1\delta_2)-(\delta_1\delta_2)^2 s.$$

8. In general, when $T=2n+1$, we have player 1's equilibrium share

$$x_1^*(2n+1) = (1-\delta_2)\sum_{i=1}^{n}(\delta_1\delta_2)^{i-1}+(\delta_1\delta_2)^n s.$$

9. When $T=2n+2$, we know that if the game proceeds to period 2, player 2 will obtain

$$(1-\delta_1)\sum_{i=1}^{n}(\delta_1\delta_2)^{i-1}+(\delta_1\delta_2)^n s.$$

So, in this case, player 1 offers in period 1

$$x_1^*(2n+2) = 1-\delta_2(1-\delta_1)\sum_{i=1}^{n}(\delta_1\delta_2)^{i-1}-\delta_2(\delta_1\delta_2)^n s.$$

10. As T goes to infinity,

$$\begin{aligned} \lim_T x_1^*(T) &\equiv x_1^* = \frac{1-\delta_2}{1-\delta_1\delta_2}; \\ \lim_T x_2^*(T) &\equiv x_2^* = \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}. \end{aligned}$$

Note that the limit is the same whether T is odd or even.

11. There are several things that are remarkable about this result. First, there is no delay. Agreement is reached immediately. Second, the breakdown share is irrelevant; the division is entirely driven by the discounts factor. A player's share increases with his discount factor and decreases with his opponent's discount factor. Third, there is a first-mover advantage even though there are many periods of negotiation.

12. Let $(y_1^*(T), y_2^*(T))$ denote the equilibrium division when we interchange the roles of the players and let player 2 proposes in the first period. When T goes to infinity, the equilibrium share will become

$$\lim_T y_1^*(T) \equiv y_1^* = \frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2};$$
$$\lim_T y_2^*(T) \equiv y_2^* = \frac{1-\delta_1}{1-\delta_1\delta_2}.$$

Note that

$$x_2^* = \delta_2 y_2^* \text{ and } y_1^* = \delta_1 x_1^*.$$

13. When $T = \infty$, we can no longer solve the game by backward induction (since there is no final period). In this case, we need to use an extra trick to final the equilibrium.

14. **The one-step deviation proof principle:** In any perfect information extensive-form game with either finite horizon or discounting, a strategy profile is a subgame perfect equilibrium if and only if no player can be better off in any subgame (including those not reached by the original equilibrium strategies) by deviating in only one information set in the subgame.

15. The principle is extremely useful in games with an infinite horizon, such as the current bargaining game or infinitely repeated games (which we will cover next week). In these games, since the players have an infinite number of strategies, it is hard to show that any particular strategy is a best response. The one-step deviation proof principle says that we need only to show that at every decision node a player will not deviate in that decision node and follow the equilibrium strategy in the future.

16. It is obvious that SPNE implies one-step deviation proof principle. The formal proof is tedious, but the basic idea is quite simple. If a strategy profile is not subgame perfect then some player can deviate and obtain a strictly higher payoff, say by $\varepsilon > 0$, in some subgame. Since future payoffs are discounted, the player must be better off by deviating only in the first T periods for some finite T. Now look at the last deviation. If it makes the player better

off, then the strategy is not 1SDP. If it does not makes the player better off, then the player will still be better off without the last deviation. The same argument can be repeated until we find a single beneficial deviation.

17. Note that the principle only works for subgame perfect equilibrium in perfect information games. It is not true for Nash equilibrium, and it is not true for SPNE in games of imperfect information.

18. Theorem: In the Rubinstein bargaining game with infinite horizon, there is a unique subgame perfect equilibrium where in every odd period, player 1 proposes (x_1^*, x_2^*) and player 2 accepts any $x_2 \geq x_2^*$, and in every even period player 2 proposes (y_1^*, y_2^*) and player 1 accepts any $y_1 \geq y_1^*$.

19. The theorem says that in each period the players will behave as if in a extremely long finite horizon game.

20. Note that the game is *stationary* the subgame starting from any period t looks exactly like the original game. This is an extremely important property because it implies that if a strategy profile is an equilibrium in period t, it will be an equilibrium in the next period as well.

21. Proof: To show that the strategy profile is subgame perfect, we need to show that no player can gain by deviating once immediately and follow the equilibrium strategy in the future. In all odd periods, player 1 obviously would not gain by proposing proposing $x_2 > x_2^*$. If player 1 proposes $x_2 < x_2^*$, then player 1 will rejects the offer and player 1 will obtain

$$y_1^* = \delta_1 x_1^* < x_1^*$$

in the next period, making him worse off. If player 2 rejects x_2^*, then he will obtain y_2^* in the next period. Hence it is a best response to accept any $x_2 \geq \delta_2 y_2^*$.The case for even periods is similar.

22. To show that the equilibrium is unique. Let $\overline{x}_1$ and $\underline{x}_1$ denote the max and min SPNE payoff for player 1 when player 1 is the proposer. Let $\overline{y}_2$ and $\underline{y}_2$ denote the max and min SPNE payoff for player 2 when player 2 is the proposer. Since player 2 can get at least $\underline{y}_2$ in the next period by rejecting player 1's offer, in any subgame perfect equilibrium, player 1 must offer player 2 at least $\delta_2 \underline{y}_2$. Hence,

$$\overline{x}_1 \leq 1 - \delta_2 \underline{y}_2.$$

On the other hand, player 1 can get at most $\overline{x}_1$ in the next period by rejecting player 2's offer. It would not be an equilibrium for player 2 to offer more than $1-\delta_1\overline{x}_1$. Hence,

$$\underline{y}_2 \geq 1-\delta_1\overline{x}_1.$$

Combining the two equations, we have

$$\begin{aligned}\overline{x}_1 &\leq 1-\delta_2\left(1-\delta_1\overline{x}_1\right)\\ &= 1-\delta_2-\delta_1\delta_2\overline{x}_1,\end{aligned}$$

which means that

$$\overline{x}_1 \leq \frac{1-\delta_2}{1-\delta_1\delta_2} = x_1^*$$

Interchanging the roles of the players, the same argument implies that

$$\underline{x}_1 \geq 1-\delta_2\overline{y}_2.$$

and

$$\overline{y}_2 \leq 1-\delta_1\underline{x}_1.$$

Combining the equations mean that

$$\begin{aligned}\underline{x}_1 &\geq 1-\delta_2\left(1-\delta_1\underline{x}_1\right)\\ &= 1-\delta_2-\delta_1\delta_2\underline{x}_1,\end{aligned}$$

which means that

$$\underline{x}_1 \geq \frac{1-\delta_2}{1-\delta_1\delta_2} = x_1^*.$$

Since by supposition $\overline{x}_1 \geq \underline{x}_1$, it follows that $\overline{x}_1 = \underline{x}_1 = x_1^*$. By the same logic, we can show that $\overline{y}_2 = \underline{y}_2 = y_2^*$. This shows that the subgame perfect equilibrium payoff is unique. Given this, it is obvious that the equilibrium itself must also be unique.

Chapter 3

Static games of incomplete information

3.1 Bayesian game

1. Definition: A Bayesian game consists of

   - a finite set of players, denoted by I
   - a set of types $\theta_i \in \Theta_i$ (the set of signals that may be observed by player i)
   - a finite strategy set S_i. A pure strategy $s_i(\theta_i)$ is a decision rule that gives the player's strategy choice for each realization of his type.

     Example: bidding function for players in an auction: $b_i(v_i)$, bid given player i's value of the item for sale.
   - a probability distribution of type, $F(\theta_1, \ldots, \theta_I)$
   - a utility function $\tilde{u}_i : S_i \times \mathit{\Theta} \to \mathbb{R}$:

$$\tilde{u}_i(s_1(\cdot), \ldots, s_I(\cdot)) = E_\theta[u_i(s_1(\theta_1), \ldots, s_I(\theta_I), \theta_i)].$$

2. Definition: A (pure strategy) Bayesian Nash equilibrium for the Bayesian game $[I, S, u, \mathit{\Theta}, F(\cdot)]$ is a profile of decision rules $(s_1(\cdot), \ldots, S_I(\cdot))$ such that, for all i,

$$\tilde{u}_i(s_i(\cdot), s_{-i}(\cdot)) \geq \tilde{u}_i(s_i'(\cdot), s_{-i}(\cdot))$$

   for all $s_i'(\cdot)$

3. Example 1. Boss and Tim play the game. Tim does not know the payoff of Boss.

Tim

		W	S
boss	M	3, 2	1, 1
	N	4, 3	2, 4

Type I

Tim

		W	S
boss	M	4, 2	2, 1
	N	3, 3	1, 4

Type II

How to solve the game: Harmless to split Boss into two types. I (with probability μ) has a dominant strategy N; II $(1-\mu)$ has dominant strategy M. For Tim, payoff

$$W: \quad 3\mu + 2(1-\mu) = 2+\mu$$

$$S: \quad 4\mu + (1-\mu) = 1+3\mu.$$

Hence Bayesian Nash equilibrium is: (NM, W if $\mu < 1/2$, S if $\mu \geq 1/2$).

4. Example 2: Two opposed armies are poised to seize an island. Each army's general can choose either "attack" or "not attack." In addition, each army is either "strong" or "weak" with equal probability (the draw for each army are independent), and an army's type is know only to its general. Payoffs are as follows: The island is worth M if captured. An army can capture the island either by attacking when its opponent does not or by attacking when its rival does if its strong and its rival is weak. If two armies of equal strength both attack, neither captures the island. An army also has a cost of fighting, which is s if its is strong and w if it is weak. There is no cost of attacking if its rival does not. We assume that $M > w > s$ and $w > M/2 > s$

Four strategies contingent upon one's types: AA (attack if strong, attack if weak), AN (attack if strong, not attack if weak), NA, NN.

- Given the probability of $(0.5, 0.5)$ of being strong and weak, the expected payoff for player 1 associated with (AA, AA) can be calculated as follows

$$\frac{1}{4}(0-s) + \frac{1}{4}(M-s) + \frac{1}{4}(0-w) + \frac{1}{4}(0-w) = \frac{M}{4} - \frac{s+w}{2}$$

- The expected payoff for player 1 associated with (AA, AN) equals

$$\frac{1}{4}(0-s) + \frac{1}{4}M + \frac{1}{4}(0-w) + \frac{1}{4}(M) = \frac{M}{2} - \frac{s+w}{4}$$

- The expected payoff for player 1 associated with (AN, AA) equals

$$\frac{1}{4}(0-s) + \frac{1}{4}(M-s) = \frac{M}{4} - \frac{s}{2}$$

Similarly we can get the payoff associated with other strategies profiles. Thus we can write down the normal form

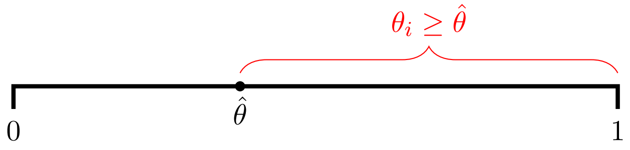
player 2

		AA	AN	NA	NN
Player 1	AA	$\frac{M}{4} - \frac{s+w}{2},\ \frac{M}{4} - \frac{s+w}{2}$	$\frac{M}{2} - \frac{s+w}{4},\ \frac{M}{4} - \frac{s}{2}$	$\frac{3M}{4} - \frac{s+w}{4},\ -\frac{w}{2}$	$M,\ 0$
	AN	$\frac{M}{4} - \frac{s}{2},\ \frac{M}{2} - \frac{s+w}{4}$	$\frac{M-s}{4},\ \frac{M-s}{4}$	$\frac{M}{2} - \frac{s}{4},\ \frac{M-w}{4}$	$\frac{M}{2},\ 0$
	NA	$-\frac{w}{2},\ \frac{3M}{4} - \frac{s+w}{4}$	$\frac{M-w}{4},\ \frac{M}{2} - \frac{s}{4}$	$\frac{M-w}{4},\ \frac{M-w}{4}$	$\frac{M}{2},\ 0$
	NN	$0,\ M$	$0,\ \frac{M}{2}$	$0,\ \frac{M}{2}$	$0,\ 0$

Two pure strategy Bayesian NE (AA, AN) and (AN, AA).

5. Example 3: A large corporation has two divisions, for simplicity call them firm A and firm B. The rules of the corporation are that any independent innovation by one firm is shared fully with the other. Suppose that the two firm could potentially develop a new product. To develop the new product a firm incurs a cost $c \in (0,1)$. The benefit of the product to firm i is observed only by the firm itself. Assume that each firm i has a type θ_i that is independently drawn form a uniform distribution on $[0,1]$, and the benefit from the new product is θ_i^2. The time is as follows: The two firms privately observe their own type, and then simultaneously choose either to develop the product or not.

   - Let $s_i(\theta_i) = 1$ if type θ_i of firm i develops, and $s_i(\theta_i) = 0$ otherwise.
   - Expected payoff developing: $\theta_i^2 - c$;
   - Expected payoff not developing:
     $$\theta_i^2 Prob(s_j(\theta_j) = 1).$$
   - It's reasonable to assume that firms each use some of cutoff strategy, that is, develop if and only if the benefit is large enough. Let $\hat{\theta}_i$ be the cutoff for firm i, so
     $$s_i = 1 \quad \text{if and only if} \quad \theta_i \geq \hat{\theta}_i.$$
   - As θ_i follows uniform distribution, we have:
     $$Prob(\theta_i \geq \hat{\theta}_i) = 1 - \hat{\theta}_i.$$



- Thus, θ_i is determined by the following condition:

$$\hat{\theta}_i^2 - c = \hat{\theta}_i \left(1 - \hat{\theta}_j\right).$$

- In symmetric equilibrium, $\hat{\theta}_i = \hat{\theta}_j = \hat{\theta}$.
- The Bayesian NE: For $i = A, B$,

$$s_i(\theta_i) = \begin{cases} 1 & \text{if} \quad \theta_i \geq \hat{\theta} = (c)^{1/3}; \\ 0 & otherwise \end{cases}$$

3.2 Applications: Auction

1. Auctions are typically used to sell items for which there is not a market price, but only a vague idea about it, sometimes involving the personal taste of potential buyers. Historically, among the first known auctions there were auction for slaves and wives.

2. Auction is an mechanism whereby a seller tries to sell some objects to a group of potential buyers, whose willingness to pay is unknown. It plays an important role in the allocation of resources in the real world, especially goods of limited supply.

   Examples: vehicle licenses in Shanghai, US treasuries bill, arts and antiques, and eBay of course. It is no wonder that auction theory has been one of the most important area in economic theory in last twenty years.

   (a) The U.S. spectrum auction in April 2008 raised a total of \$19.8 billion

   (b) Europe's frenzied 2000 and 2001 auctions reaped nearly \$100 billion

3. Two important characteristics of auctions are:

   (a) Uncertainty about the valuations of the bidders. This uncertainty leads to the private values assumption about such valuations, which are modeled as independent random variables from a common distribution.

   (b) "Winner's curse": Because of the uncertainty of the value of the object for sale, a winner of an auction might wonder why all the other bidders' valuation were smaller than hers, and in particular whether this might have happened because of the others' more accurate information about the item's true value.

4. Roughly speaking, an auction is characterized by four elements:

    (a) The number of goods available for sale. Whether there is one unit or multiple units.

    (b) The auction mechanism (i.e. the auction rules). There are several formats commonly used in single-unit auctions:

        i. Dutch Auction. The auctioneer sets a max. price. As the auction proceeds, the price starts declining. A bidder can stop the auction anytime, claim the object, and pay the ongoing price.

        Dutch auctions are more commonly used to sell homogenous goods, such as cut flowers in 17th century Holland or government bonds today. It's also used in share buybacks. For example, when GEICO, the insurance company that is part of Buffett's Berkshire, was still a public company, it frequently used Dutch auction tenders to buy back its own shares.[1]

        ii. English Auction. It is also known as the ascending-bid auction. It is the most common format and is sort of the opposite to the Dutch auction. Here, bidding starts at the price floor. Bidders compete by submitting ascending bids until all bidders but one drops out. The last remaining bidder wins the object and pay the ongoing price. The eBay auction is a variation of the English auction, in which there is a fixed deadline. The seller can also name a sell-price in a eBay auction. Any bidder can claim the object by paying the sell-price.

        iii. First-price sealed-bid auction. This is also very common, especially in procurement auctions. Each bidder submits a sealed bid (meaning that the bid is not revealed to other bidders. The highest bidder wins and pays her own bid.

        iv. Second-price sealed-bid auction. It is also referred to as the Vickrey auction in the literature. (Vickrey won the Nobel price mainly for his early contribution to auction theory.) The mechanism is like the first-price sealed-bid auction except that the highest bidder pays the second-highest bid. Vickrey auction is seldom used in practice but is common in theoretical work because of its simplicity and its closeness to the English auction.

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[1] See "Going Dutch" in *The Economist* December 17, 2022 issue for how a Chicago artist used Dutch auction to sell his painting.

(c) Information structure—The way different bidders' valuations are correlated. We typically assume that each bidder observes a private signal about how much the object is worth to him, and a bidder signal is not observed by other bidders and the seller. (If the seller knows how much a bidder is willing to pay, there is no need to run an auction.) An auction is of **independent private value** if each bidder's valuation depend only on his own signal, and each bidder's own signal tells him nothing about another bidder's signal. Examples: vehicle license (assuming the person is not thinking about resale). An auction is of **common value** all bidders' have the same valuation, which depends on all signals. For example, bidding for a business license. In this case, each bidder may have done some prior research on the size and profitability of the market. Finally, an auction is of **positive affiliation** if the valuations are "positively correlated." (Affiliation is a technical term referring to a condition stronger than correlation.)

(d) Risk preferences. Risk neutrality versus risk aversion.

3.2.1 First-price auction

1. N symmetric bidders, independent value $f_i(v) = f(v)$ for all $v \in [0,1]$

2. Bidding function $b : [0,1] \to \mathbb{R}_+$, r: reported value

3. Expected payoff from reporting r,

$$u(r,v) = F^{N-1}(r)(v - b(r)).$$

4. u maximized at v,

$$\frac{du(r,v)}{dr}|_{r^*=v} = (N-1)F^{N-2}(v)f(v)(v-b(v)) - F^{N-1}(v)b'(v) = 0 \Longrightarrow$$

$$(N-1)F^{N-2}(v)f(v)b(v) + F^{N-1}(v)b'(v) = v(N-1)F^{N-2}(v)f(v).$$

So we have

$$F^{N-1}(v)b(v) = (N-1)\int_0^v xf(x)F^{N-2}(x)dx + C$$

To see this, note that:

From calculus, the **Product Rule** tells us that

$$\frac{d}{dv}[F^{N-1}(v)b(v)] = (N-1)F^{N-2}(v)f(v)b(v) + F^{N-1}(v)b'(v) \Longrightarrow$$

$$F^{N-1}(v)b(v) = \int [(N-1)F^{N-2}(v)f(v)b(v) + F^{N-1}(v)b'(v)]dv.$$

And the **Leibnitz integarl rule** indicates

$$\frac{d}{dv}\left[\int_0^v (N-1)xf(x)F^{N-2}(x)dx + C\right] = v(N-1)F^{N-2}(v)f(v) \Longrightarrow$$

$$\int_0^v (N-1)xf(x)F^{N-2}(x)dx + C = \int v(N-1)F^{N-2}(v)f(v)dv$$

5. Some calculus

   (a) **The Product rule**:

   $$\frac{d}{dx}[A(x)B(x)] = B(x)\frac{dA(x)}{dx} + A(x)\frac{dA(x)}{dx}.$$

   (b) **The Leibnitz integral rule**:

   $$\frac{d}{dx}\int_{A(x)}^{B(x)} f(x,t)dt = f(x,B(x))B'(x) - f(x,A(x))A'(x) + \int_{A(x)}^{B(x)} \frac{\partial f(x,t)}{\partial x}dt.$$

6. Thus we have

   $$F^{N-1}(v)b(v) = (N-1)\int_0^v xf(x)F^{N-2}(x)dx + C$$

7. If a bidder's valuation of the object is zero, the bidder obviously should bid just zero. Thus, C should be zero,

   $$b(v) = \frac{1}{F^{N-1}(v)}\int_0^v xdF^{N-1}(x)$$

8. In the unique symmetric equilibrium of a first-price, sealed bid auction, each bidder bids the expectation of the second highest bidder's value conditional on winning the auction

9. Example: v is uniformly distributed. In this case, $F(v) = v$ and $f(v) = 1$.

   $$\begin{aligned} b(v) &= \frac{1}{v^{N-1}}\int_0^v x(N-1)x^{N-2}dx \\ &= \frac{N-1}{V^{N-1}}\left[\frac{x^N}{N}|_v\right] \\ &= \frac{N-1}{N}v \\ &= v\left[1 - \frac{1}{N}\right]. \end{aligned}$$

3.2.2 Dutch auction

1. The Dutch auction and the first-price sealed-bid auction are strategically equivalent from a game-theoretic point of view, regardless of the information structure and risk preferences. In a Dutch auction, each bidder needs to decide at what price he would want to claim the object, assuming that the object is unclaimed up to that point. Same in a first-price sealed bid auction. When a bidder in a first-price sealed-bid auction think about whether to bid, say, \$9 or \$10, he is trading off winning with a lower bid against the worry someone may bid between \$9 and \$10. Exactly the same consideration goes on in the Dutch auction. Interestingly, empirically, the Dutch auction seems to generate higher revenues. Possibly, bidders can't resist the temptation to claim an object right away in a Dutch auction.

2. Symmetric equilibrium bidding

$$b(v) = \frac{1}{F^{N-1}(v)} \int_0^v x dF^{N-1}(x)$$

Second-price auction

1. In the IPV case, it is a weakly dominant strategy to bid one's valuation in a second price auction. This is obvious when we think in terms of the English auction. Of course, one should drop out when the current bid is equal to one's valuation. If one drops out after that, he may have to pay a higher price than what the object is worth to him. On the other hand, if one drops out too early, he may give up profitable opportunities. Thus, in equilibrium, all bidders bid their own valuation, and the object goes to the bidder with the highest valuation.

2. In the case of independent private value auctions, the English auction and the Vickrey auction are strategically equivalent. In an English auction, each bidder i needs to decide when to drop out of the bidding process. Call this value b_i. Note that the bidding process ends when there is only one bidder left, so the winner is effectively paying the second highest b_i (the winner has the highest), as in the case of a Vickrey auction. The equivalence holds only in the case of independent private value because in the other cases (common value or positive affiliation), a bidder's bidding strategy may depend on the bids of the other bidders in an English auction. In the IPV case, these bids has no information value.

3.2.3 English auction

1. With independent private values, dropping out when price reaches one's value is the unique weakly dominant strategy
2. With independent private values, English auction and second-price auction raises the same ex-post revenue. In both auction, the bidder with the highest valuation wins and pays the second highest bidder's value.

Revenue equivalence

1. When both bidders and sellers are risk-neutral and the bidders have independent private valuation for the item on sale, then all auction formats leads to the same expected revenue to the seller.
2. For our purpose, we only need to show expected revenue is the same for first-price and second-price sealed bid auction.
3. Recall that the k-order statistic of N draws from F_x
$$f_{x_k} = \frac{N!}{k!(N-k)!}[F(x)]^{N-k}[1-F(x)]^k f(x).$$
4. Second-price auction revenue
$$EV_S = \int_0^1 b_S f_{v_2} dv = \int_0^1 vN(N-1)F^{N-2}(v)[1-F(v)]f(v)dv.$$
5. First-price auction
$$\begin{aligned} EV_F &= \int_0^1 b_F f_{v_1} dv \\ &= \int_0^1 \left[\frac{1}{F^{N-1}(v)} \int_0^v x dF^{N-1}(x)\right] NF^{N-1}(v)f(v)dv. \end{aligned}$$
6. Revenue of first-price auction:
$$\begin{aligned} EV_F &= \int_0^1 N(N-1)\left[\int_0^v xF^{N-2}(x)f(x)dx\right] f(v)dv \\ &= \int_0^1 \left[\int_x^1 f(v)dv\right] N(N-1)F^{N-2}(x)xf(x)dx \\ &= \int_0^1 N(N-1)xF^{N-2}(x)[1-F(x)]f(x)dx. \end{aligned}$$

7. Thus, we conclude

$$EV_F = EV_S = \int_0^1 N(N-1)vF^{N-2}(v)[1-F(v)]f(v)dv.$$

8. The equivalence does not hold if

   - common value auctions
   - risk-averse traders

3.3 Mechanism design

1. In game theory we fix a game and analyze the set of possible outcomes. However, if we fix a set of outcomes, which we might think of as describing a socially desirable goal, and try to come up with a game that has such set as the set of its equilibria, then we are studying the inverse problem, *mechanism design*

2. Mechanism design deals with defining the rules of a game so that players' actions lead to prescribed social goal

3. Mechanism can be applied to the design of auctions. Typical goals are

   - Assign the item to the bidder with the highest valuation for it;
   - Giving incentive to bidders so that it is in their best interest to bid according to their true valuation

4. Definition: A direct selling mechanism is a collection of N probability assignment functions, $p_1(v_1, \ldots, v_N)$, …, $p_N(v_1, \ldots, v_N)$ and N cost functions $c_1(v_1, \ldots, v_N)$, …, $c_N(v_1, \ldots, v_N)$. For each i and all $(v_1, \ldots, v_N)$, $p_i(v_1, \ldots, v_N) \in [0, 1]$ denotes the probability that bidder i receives the object and $c_i(v_1, \ldots, v_N) \in \mathbb{R}$ denotes the payment bidder i must make to the seller.

5. Example: First-price auction as a direct selling mechanism

$$p_i(v_1, \ldots, v_N) = \begin{cases} 1, & \text{if } v_i > v_j \text{ for all } j \neq i \\ 0 & otherwise \end{cases}$$

$$c_i(v_1, \ldots, v_N) = \begin{cases} b(v_i), & \text{if } v_i > v_j \text{ for all } j \neq i \\ 0 & otherwise \end{cases}$$

6. Incentive compatibility: A mechanism is incentive compatible if agents (bidders for auction) report their private information truthfully in equilibrium.

   A direct selling mechanism is incentive-compatible if and only if for every v_i, $u_i(r_i, v_i)$ is maximized in r_i at $r_i = v_i$,

   $$u_i(v_i, v_i) \geq u_i(r_i, v_i) \quad \forall r_i$$

3.3.1 Groves mechanism: example

1. Should a bridge be built? Individual's utility from the decision

   $$u_i = \theta_i x + t_i$$

   $x \in \{1, 0\}$ denotes the decision "build" or "not to build" the bridge, θ_i is i's private valuation or willingness to pay for the public good, t_i is individual's payment to build the bridge.

   Let $c > 0$ be the cost of build the bridge.

2. Efficient rule

   $$x_i(\theta) = \begin{cases} 1, & \text{if } \sum_{i=1}^{I} \theta_i \geq c \\ 0 & otherwise \end{cases}$$

3. How to make an optimal decision?

   One mechanism

   $$t_i(\hat{\theta}) = \begin{cases} \sum_{j \neq i}^{I} \hat{\theta}_j - c, & \text{if } \sum_{j=1}^{I} \hat{\theta}_j \geq c \\ 0 & otherwise \end{cases}$$

   and

   $$x_i(\hat{\theta}) = \begin{cases} 1, & \text{if } \sum_{i=1}^{I} \hat{\theta}_i \geq c \\ 0 & otherwise \end{cases}$$

Chapter 4

General equilibrium: exchange economy

Equilibrium is the product of an axiomatic system. Economic theory is constructed like logic or mathematics: it is based on certain postulates and all of its conclusions are derived from them by logical manipulation.The crowning achievement of the axiomatic approach is the theory of perfect Competition. Although it was first propounded nearly two hundred years ago, it has never been superseded; only the method of analysis has been refined. The theory holds that under certain specified circumstances the unrestrained pursuit of self-interest leads to the optimum allocation of resources. The equilibrium point is reached when each firm produces at a level where its marginal; cost equals the market price and each consumer buys an amount whose marginal "utility" equals the market price. Analysis shows that the equilibrium position maximizes the benefit of all participants, provided no individual buyer or seller can influence market prices. It is this line of argument that has served as the theoretical underpinning for the laissez-faire policies of the nineteenth century, and it is also the basis of the current belief in the "magic of the marketplace."(G. Soros, The Alchemy of Finance, pp. 28)

4.1 Exchange economy

4.1.1 Model

1. Exogenous variables

    - Consumers $i \in \{1, 2, \ldots, I\}$

- Goods $l \in \{1, 2, \ldots, L\}$
- Endowments

$$\omega = [\omega_1, \ldots, \omega_I] = \begin{bmatrix} \omega_{11} \cdots & \cdots \omega_{1I} \\ . & . \\ . & . \\ \omega_{L1} \cdots & \cdots \omega_{LI} \end{bmatrix}_{L \times I}$$

- Preferences

$$\{\succeq\}_{i=1}^{I} = \{\succeq_1, \succeq_2, \ldots, \succeq_I\}$$

2. Endogenous

- Consumption

$$X = [X_1, \ldots, X_I] = \begin{bmatrix} x_{11} \cdots & \cdots x_{1I} \\ . & . \\ . & . \\ x_{L1} \cdots & \cdots x_{LI} \end{bmatrix}_{L \times I}$$

Consumption allocation, X, consumption bundle for agent i, X_i

- Prices

3. Implicit assumptions:

- private ownership

Definition: Every portion of every good is owned by exactly one person and that person has the exclusive right to use it in consumption and exchange (or production).

- No externality

4.1.2 Some basic stuff

1. Pareto efficiency

- Definition: X is feasible if $\sum_i X_i \leq \sum_i \omega_i$
- Definition: A feasible allocation X is (Pareto) efficient if there exists no feasible allocation X' such that $\forall i$

$$X_i' \succeq_i X_i$$

and $\exists i$, $X_i' \succ_i X_i$

2. Walrasian equilibrium

   Definition: (X^*, P) is an equilibrium if

   - $\forall i$, X_i^* is maximal for $\succeq_i$ in $\{X_i | PX_i^* \leq P\omega_i\}$;
   - $\sum_i X_i^* = \sum_i \omega_i$.

3. Definitions

   - $\succeq$ on X is monotonic if $x \in X$ and $y \gg x$ implies $y \succ x$. It is strongly monotonic if $y \geq x$ and $y \neq x$ imply that $y \succ x$.
   - $\succeq$ is convex if $\forall x \in X$,

     $$y \succeq x, z \succeq x \Longrightarrow \ \forall\ \alpha \in [0,1], \quad \alpha y + (1-\alpha) z \succeq x.$$

     $\succeq$ is strictly convex if $\forall\ x \in X$,

     $$y \succeq x, z \succeq x,\ y \neq z \Longrightarrow \ \forall\ \alpha \in (0,1), \quad \alpha y + (1-\alpha) z \succ x.$$

4. Theorem: Suppose $X^* \gg 0$, and that $\forall i$, $\succeq_i$ is represented by a concave u_i which is twice continuously differentiable and strongly monotonic around X_i^*. Then the following are equivalent

   - X^* is (Pareto) efficient;
   - $\forall i$,

     $$X^* \in arg\max\{u_i(X_i) | X \geq 0, \sum_i X_i \leq \sum_i \omega_i, (\forall h \neq i)\ u_h(X_h) \geq u_h(X_h^*)\}$$

   - $\exists q = (q_1, \ldots, q_L) \in \mathbb{R}^L_{++}$, shadow prices
     $\exists (s_1, \ldots, S_I) \in \mathbb{R}^I_{++}$
     $\forall i$, $s_i D u_i(X_i^*) = q$ and $\sum_i X_i = \sum_i \omega_i$.

     Social planner's problem:

     $$\max_X \sum_{i=1}^{I} s_i u_i(X_i)\ s.t.\ constraint.$$

     This is equivalent to the optimization problem: $(\forall i)$

     $$\max_X u_i(X_i) \qquad s.t.$$

     $$\sum_i X_i = \sum_i \omega_i$$

     $$\forall h \neq i \quad u_h(X_h) \geq u_h(X_h^*)$$

- $\forall i \in \{1, 2, \ldots, I-1\}$ and $\forall l \in \{1, 2, \ldots, L-1\}$,

$$MRS_i^{l,l+1} = MRS_{i+1}^{l,l+1}$$

$$\sum_i X_i = \sum_i \omega_i.$$

4.1.3 Pareto efficient allocation examples

Two-consumer example 1

1. Two typical consumers: Consumer A has 7 units of goods x_1 and 3 units of good x_1, and consumer B has 3 units of x_1 and 7 units of x_2. They both have same utility function

$$U_A(x_{1A}, x_{2A}) = (x_{1A}x_{2A})^{1/2} \qquad U_B(x_{1B}, x_{2B}) = (x_{1B}x_{2B})^{1/2}.$$

2. Feasible allocation: any points in the Edgeworth box such that

$$x_{1A} + x_{1B} \leq 10,$$

$$x_{2B} + x_{2B} \leq 10.$$

3. Pareto efficient allocation: An allocation is Pareto efficient if there is no feasible allocation that can make one agent better off without hurting others.

4. Contract curve gives all efficient allocation in the Edgeworth box.

Linear preferences: contract curve

1. Preferences

$$U_A = x_{1A} + 2x_{2A}, \qquad U_B = 2x_{1B} + x_{2B}$$

2. Initial endowment

$$\omega^A = (7, 3), \qquad \omega^B = (3, 7).$$

Efficient allocations: Leontief preferences

1. Preferences

$$U_A = \min(2x_{1A}, x_{2A}), \qquad U_B = \min(2x_{1B}, x_{2B}).$$

2. Initial endowment

$$\omega^A = (7, 3), \qquad \omega^B = (3, 7).$$

4.1.4 Social planner's problem

1. Take $i = 2$, the objective function

$$\max_X u_2(X_2)$$

with $L \times I$ unknows, subject to

$$\sum_i X_i = \sum_i \omega_i$$

with L constraints and

$$\forall h \neq 2 \quad u_h(X_h) \geq u_h(X_h^*)$$

with $I - 1$ constraints

2. The Lagrangian

$$\mathcal{L} = u_2(X_2) + \sum_{l=1}^{L} q_l \left[\sum_{i=1}^{I} \omega_{li} - \sum_{i=1}^{I} x_{li} \right] + \sum_{i \neq 2} s_i \left[u_i(X_i) - u_i(X_i^*) \right]$$

First-order condition gives

$$\forall i,\ s_i D u_i(X_i^*) = q$$

and

$$\sum_i X_i = \sum_i \omega_i.$$

3. Suppose $L = 2, I = 3$

$$X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix}_{2\times 3}$$

FOC yields

$$D_{x_{11}}\mathcal{L} = -q_1 + s_1 \frac{\partial u_1}{\partial x_{11}} = 0$$

$$D_{x_{21}}\mathcal{L} = -q_2 + s_1 \frac{\partial u_1}{\partial x_{21}} = 0$$

$$D_{x_{12}}\mathcal{L} = -q_1 + \frac{\partial u_2}{\partial x_{12}} = 0$$

$$D_{x_{22}}\mathcal{L} = -q_2 + \frac{\partial u_2}{\partial x_{22}} = 0$$

$$D_{x_{13}}\mathcal{L} = -q_1 + s_3 \frac{\partial u_3}{\partial x_{13}} = 0$$

$$D_{x_{23}}\mathcal{L} = -q_2 + s_3 \frac{\partial u_3}{\partial x_{23}} = 0$$

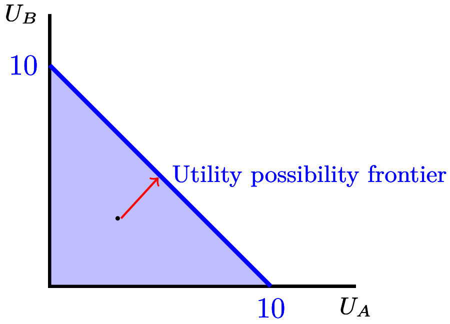
Set $s_2 = 1$, we have

$$s_i D u_i = q$$

4. Utility possibility frontier: A curve that connects all the possible combinations of utilities that could arise at the various economically efficient allocations of goods and inputs.

5. Find the UPF: identify all PE allocations.

6. UPF for the example:



4.1.5 Competitive equilibrium

Simple example

1. Two typical consumers: Consumer A has 7 units of goods x_1 and 3 units of good x_1, and consumer B has 3 units of x_1 and 7 units of x_2. They both have same utility function

$$U_A(x_{1A}, x_{2A}) = (x_{1A}x_{2A})^{1/2} \qquad U_B(x_{1B}, x_{2B}) = (x_{1B}x_{2B})^{1/2}.$$

2. Feasible allocation: any points in the Edgeworth box such that

$$x_{1A} + x_{1B} \leq 10,$$

$$x_{2B} + x_{2B} \leq 10.$$

3. Utility maximizing for consumers A and B,

$$x_{1A} = \frac{m_A}{2P_1},\ x_{2A} = \frac{m_A}{2P_2} \text{ where } \ m_A = 7P_1 + 3P_2$$

$$x_{1B} = \frac{m_B}{2P_1},\ x_{2B} = \frac{m_B}{2P_2} \text{ where } \ m_B = 3P_2 + 7P_2.$$

4. Plugging m_A, m_b into the allocations yields

$$x_{1A} = \frac{7P_1 + 3P_2}{2P_1},\ x_{2A} = \frac{7P_1 + 3P_2}{2P_2},$$

$$x_{1B} = \frac{3P_1 + 7P_2}{2P_1},\ x_{2B} = \frac{3P_1 + 7P_2}{2P_2}.$$

5. In equilibrium, market clears

$$5 + \frac{5P_2}{P_1} = 10, \quad 5 + \frac{5P_1}{P_2} = 10.$$

6. So the competitive equilibrium (let $P_1 = 1$)

$$P_1 = 1,\ P_2 = 1,\ ,x_{1A} = x_{2A} = 5,\ x_{1B} = x_{2B} = 5.$$

7. Compare PE and equilibrium allocation

P.E. allocations	C.E. allocations
1. Exchange efficiency: $MRS^A_{1,2} = MRS^B_{1,2}$	1. Utility-maximization (a) $MRS^i_{1,2} = \frac{P_1}{P_2} \Longrightarrow$ $MRS^A_{1,2} = MRS^B_{1,2}$ (b) $Px_i = P\omega_i$
2. No resources wasted $\forall l:\ x_{lA} + x_{lB} = \omega_{lA} + \omega_{lB}$	2. Market clears $\forall l:\ x_{lA} + x_{lB} = \omega_{lA} + \omega_{lB}$

Main result

1. Theorem: Suppose $X^* \gg 0$ and that $\forall i$, $\succeq_i$ is represented by a concave u_i which is twice continuously differentiable and strongly monotonic around X^*_i, the following are equivalent
   - (X^*, P) is an equilibrium;
   - $(\exists\ \lambda_1, \ldots, \lambda_I) \in \mathbb{R}^I_{++}$:
     - ➢ $\forall i$, $Du_i(X^*_i) = \lambda_i P$;
     - ➢ $\sum_i X^*_i = \sum_i \omega_i$;
     - ➢ $PX^*_i = P\omega_i$.
2. First Welfare Theorem: Suppose $\succeq_i$ is locally nonsatiated. Then every equilibrium allocation is efficient. (market is good)
   - Implications of FWT: A private market, with each consumer seeking to maximize his or her own welfare, will result in an allocation that achieves Pareto efficiency. The competitive market ensures Pareto efficient outcomes. The competitive market economizes on the information that any one consumer needs to possess. The only thing that any one

consumer needs to know to make consumption decisions are the prices of the goods he or she is considering consuming. Consumers do not need to know anything about how the goods are produced, or who owns what goods, etc.

If the markets function well enough to determine the competitive price, we are guaranteed an efficient outcome. The fact that competitive markets economize on information in this way is a strong argument in favor of market as a mechanism to allocation resources in an economy.

- Local non-satiation: For any consumption bundle x, for all ε, there exists another consumption bundle y with $||x-y|| < \varepsilon$ such that $y \succ x$.
- Monotonicity implies local nonsatiation.

3. Lemma: Suppose $\succeq_i$ is locally nonsatiated, X_i^* is maximal for $\succeq_i$ in $\{X_i | PX_i \leq P\omega_i\}$. If $X_i \succeq_i X_i^*$, then $PX_i \geq PX_i^*$. If $X_i \succ_i X_i^*$, then $PX_i > PX_i^*$.

4. Proof of the FWT: Suppose (X^*, P) is an equilibrium, but it is not efficient. Then there must exist X' that satisfy the constraint

$$\sum_i X_i' \leq \sum_i \omega_i \Longrightarrow \sum_i PX_i' \leq \sum_i P\omega_i$$

and in addition,

$$\forall i,\ \ X_i' \succeq X_i^* \Longrightarrow PX_i' \geq PX_i^*$$

and

$$\exists i,\ \ X_i' \succ X_i^* \Longrightarrow PX_i' > PX_i^*.$$

However, the last two conditions would imply that

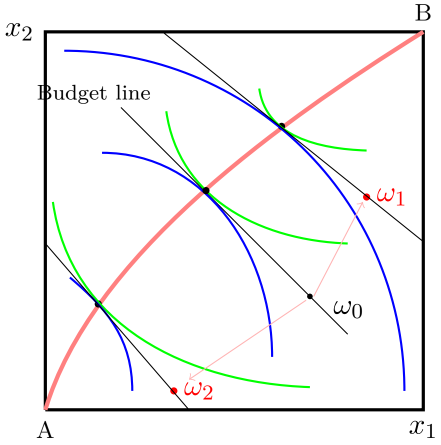
$$\sum_i PX_i' > \sum_i P\omega_i$$

and contradicts the constraint.

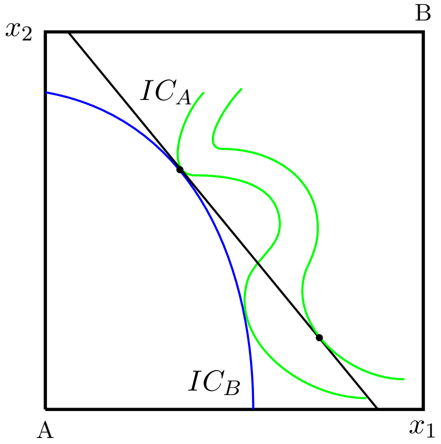
5. Second Welfare Theorem: Suppose X^* is an efficient allocation and that an equilibrium exists from X^*. Then X^* is an equilibrium allocation.

   - Under certain conditions, every Pareto efficient allocation can be achieved as a competitive equilibrium.

- One condition: if preferences are convex, then every efficient allocations can be achieved as an equilibrium.
- The problems of distribution and efficiency can be separated: redistribute endowments of goods to determine how much wealth consumers have; use prices to indicate the relative scarcity of goods.
- To achieve efficiency, each consumer must face the true social cost of his or her actions and to make choices that reflect those cost. In competitive market, this is achieved through consumers' marginal decision of whether to consume more or less of some goods given the price, which measures the relative scarcity of the goods.
- To achieve distribution goal, all that is needed is to transfer the purchasing power of the endowment.
- Graphical illustration



- Example of non-convex preferences



4.1.6 Core

1. A coalition $S \subseteq \{1, \ldots, I\}$ blocks an allocation X if $\exists\ X'$ such that

   - $(\forall\ i \in S)$, $X'_i \succeq X_i$ and
     $(\exists\ i \in S)$, $X'_i \succ X_i$;
   - $$\sum_{i \in S} X'_i \leq \sum_{i \in S} \omega_i.$$

2. The *Core* is the set of unblocked allocations.

3. Observation:

   - An allocation X is unblocked by $S = \{1, \ldots, I\}$ (coalition of the whole) iff X is efficient;
   - Inefficient allocations are blocked by $S = \{1, \ldots, I\}$;
   - Equilibrium must be in the core.

4. Some examples

   Three individual exchange economy

   $$U^A = x^{1/2}y^{1/2}, \quad U^B = 2x^{1/2}y^{1/2}, \quad U^C = \min(x, y).$$

   $$\omega = \begin{bmatrix} 5 & 9 & 1 \\ 5 & 1 & 9 \end{bmatrix}$$

   Are the following 3 allocations in the core? If not, find a blocking coalition that will block it.

   $$X = \begin{bmatrix} 7 & 6 & 2 \\ 4 & 3 & 8 \end{bmatrix} \qquad X = \begin{bmatrix} 7 & 4 & 4 \\ 7 & 4 & 4 \end{bmatrix} \qquad X = \begin{bmatrix} 4 & 6 & 5 \\ 4 & 6 & 5 \end{bmatrix}$$

5. Theorem: Suppose each $\succeq_i$ is locally nonsatiated. Then every equilibrium is in the core.

   Proof. Suppose (X^*) is an equilibrium allocation and that S and X_S is such that

   $$\forall\ i \in S,\ X'_i \succeq_i X^*_i$$

   $$\exists\ i \in S,\ X'_i \succ X^*_i$$

   and

   $$\sum_{i \in S} X'_i \leq \sum_{i \in S} \omega_i.$$

Following the first two conditions we have

$$\forall\ i,\ \ X_i' \succeq_i X_i^* \Longrightarrow PX_i' \geq P\omega_i$$

$$\exists\ i,\ \ X_i' \succ X_i^* \Longrightarrow PX_i' > P\omega_i.$$

This implies

$$\sum_{i\in S} PX_i' > \sum_{i\in S} P\omega_i.$$

But this contradicts the condition

$$\sum_{i\in S} X_i' \leq \sum_{i\in S} \omega_i \Longrightarrow \sum_{i\in S} PX_i' \leq \sum_{i\in S} P\omega_i.$$

6. Efficiency, core and equilibrium:

   $\{$Equilibrium allocations from $\omega\}\subseteq$ $\{$Core from $\omega\}\subseteq\{$Efficient allocation from $\omega\}$

4.1.7 Excess demand

1. Excess demand:

$$Z_i(P) = X_i(P, \omega_i) - \omega_i$$

$$Z(P) = \sum_i Z_i(P).$$

   Here $X_i(P, \omega_i)$ is the maximal for $\succeq_i$ in $\{X_i | PX_i = P\omega_i\}$

2. Example 1.

$$U_A = x_{1A}x_{2A} \qquad \omega_A = (4, 1)$$

$$U_B = x_{1B}x_{2B} \qquad \omega_B = (1, 4).$$

   In this case,

$$x_{1A}(P_1, P_2, P_1\omega_1 + P_2\omega_2) = \frac{4P_1 + P_2}{2P_1} \Longrightarrow$$

$$Z_{1A}(P) = \tfrac{4P_1+P_2}{2P_1} - 4 = \tfrac{P_2}{2P_1} - 2 = \begin{cases} > 0 & if\ \ P_2 > 4P_1 \\ = 0 & if\ \ P_2 = 4P_1 \\ < 0 & if\ \ P_2 < 4P_1. \end{cases}$$

$$x_{2A}(P_1, P_2, P_1\omega_1 + P_2\omega_2) = \frac{2P_1}{P_2} + \frac{1}{2} \Longrightarrow$$

$$Z_{2A}(P) = \frac{2P_1}{P_2} - \frac{1}{2}.$$

$$Z_{1B}(P) = \frac{2P_2}{P_1} - \frac{1}{2}$$

$$Z_{2B}(P) = \frac{P_1}{2P_2} - 2.$$

$$Z(P) = \begin{bmatrix} \frac{5P_2}{2P_1} - \frac{5}{2} \\ \frac{5P_1}{2P_2} - \frac{5}{2} \end{bmatrix}$$

$$PZ(P) = 0.$$

3. Example 2. Robinson-Crusoe economy

$$U = x_1^{1/2} + x_2^{1/2}, \qquad \omega = (1, 1).$$

$$Z(P) = \begin{bmatrix} \frac{1+P_2/P_1}{1+P_1/P_2} - 1 \\ \frac{1+P_1/P_2}{1+P_2/P_1} - 1 \end{bmatrix} = \begin{bmatrix} \frac{P_2}{P_1} - 1 \\ \frac{P_1}{P_2} - 1 \end{bmatrix}$$

$$PZ(P) = 0.$$

4. Example 3.

$$u_A = \min\{x_{1A}, x_{2A}\}, \qquad u_B = \min\{x_{1B}, x_{2B}\}$$

$$\omega_A = (4, 1), \qquad \omega_B = (1, 4).$$

$$Z(P) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

5. Example 4.

$$u_A = x_{1A}^{1/2} + x_{2A}^{1/2}, \qquad u_B = x_{1B}.$$

$$\omega_A = (0, 1), \qquad \omega_B = (1, 0).$$

$$Z(P) = \begin{bmatrix} \frac{P_2^2}{P_1P_2+P_1^2} \\ \frac{P_1^2}{P_1P_2+P_1^2} - 1 \end{bmatrix}$$

4.1.8 Existence of Walrasian equilibrium

1. Definition: A vector $P^* \in \mathbb{R}_{++}$ is called a Walrasian equilibrium if $Z(P^*) = 0$.

2. Proposition 17B.2 (MWG): If $\forall i,\ X_i \in \mathbb{R}^L_+$, $\succeq_i$ is continuous, strictly convex and strongly monotonic, $\sum_i \omega_i \gg 0$, then there exists $Z : \mathbb{R}^L_+ \to \mathbb{R}$

   - Z is homogeneous of degree zero;
   - Walras's Law: $(\forall\ P \in \mathbb{R}^L_+)$, $PZ(P) = 0$;
   - $(\exists\ s \in \mathbb{R}_+)$, $(\forall\ l)$ $(\forall\ P)$, $Z_l(P) > -s$;
   - $(\forall (P^n)_{n=1}^{\infty} \to P \neq 0)$ where $P_l = 0$ for some l,

   $$\max_l Z_l(P) \to +\infty.$$

3. Proposition 17C.1 (MWG): A Walrasian equilibrium exists in any pure exchange economy in which $\sum_i \omega_i \gg 0$ and $\forall i$, $X_i \in \mathbb{R}^L_+$, $\succeq_i$ is continuous, strictly convex and strongly monotonic.

4. Gross substitute property: Z satisfied gross substitute property if $(\forall\ P)$ $(\forall\ l)$ and $(\forall\ j \neq l)$

   $$\frac{\partial Z_l(P)}{\partial P_j} > 0.$$

5. Theorem 17F.3. Suppose Z has the gross substitute property and is homogeneous of degree 0. If P' and P^0 are such that $Z(P') = Z(P^0) = 0$, then

   $$\exists\ \lambda > 0,\ \ P' = \lambda P^0.$$

   If the gross substitute property is satisfied, equilibrium is unique.

6. Proof of Proposition 17c.1: By proposition 17B.2, when the economy is continuous, strictly convex and strongly monotonic, we have

   - $Z(\cdot)$ is continuous;
   - Walras law;
   - $(\forall (P^n)_{n=1}^{\infty} \to P \neq 0)$ where $P_l = 0$ for some l,

   $$\max_l Z_l(P) \to +\infty.$$

Now define $\bar{Z}_l(P) = \min(Z_l(P), 1)$ for P, and let $\bar{Z}(P) = (\bar{Z}_1(P), \ldots, \bar{Z}_L(P))$. Note that for all $P \gg 0$

$$P\bar{Z}(P) \leq PZ(P) = 0.$$

Fix $\varepsilon > 0$, let

$$S_\varepsilon = \{P | \sum_{l=1}^{L} P_l = 1, \quad \forall l, \; P_l \geq \frac{\varepsilon}{2L+1}\}.$$

Note that S_ε is compact, convex and nonempty.

Next for all l and for every $P \in S_\varepsilon$, define

$$f_l(P) = \frac{\varepsilon + P_l + \max(0, \bar{Z}_l(P))}{L\varepsilon + 1 + \sum_{k=1}^{L} \max(0, \bar{Z}_k(P))}.$$

We can see $\sum f_l(P) = 1$. Also note that

$$f_l(P) \geq \frac{\varepsilon}{2L+1}.$$

This is true as

$$\begin{aligned} f_l(P) &\geq \frac{\varepsilon + P_l + \max(0, \bar{Z}_l(P))}{L\varepsilon + 1 + L \cdot 1} \\ &\geq \frac{\varepsilon}{L\varepsilon + 1 + L \cdot 1} \\ &\geq \frac{\varepsilon}{2L+1} \end{aligned}$$

as $\varepsilon < 1$. Therefore we can conclude

$$f : S_\varepsilon \to S_\varepsilon.$$

By Brouwer's fixed point theorem we know there is a fixed point $f(P^\varepsilon) = P^\varepsilon$,

$$P_l^\varepsilon = \frac{\varepsilon + P_l^\varepsilon + \max(0, \bar{Z}_l(P^\varepsilon))}{L\varepsilon + 1 + \sum_{k=1}^{L} \max(0, \bar{Z}_k(P^\varepsilon))} \Longrightarrow$$

$$P_l^\varepsilon \left[L\varepsilon + \sum_{k=1}^{L} \max(0, \bar{Z}_k(P^\varepsilon)) \right] = \varepsilon + \max(0, \bar{Z}_l(P^\varepsilon)). \tag{4.1}$$

As $\varepsilon \to 0$, $P^\varepsilon(P) \to P^*$ and $P^* \geq 0$, $P^* \neq 0$.

If $\exists\, l$ such that $P_l = 0$, then $\exists\, l'$ $Z_{l'}(P^*) \to \infty$.

$$P_{l'}^\varepsilon \left[L\varepsilon + \sum_{k=1}^{L} \max(0, \bar{Z}_k(P^\varepsilon)) \right] \to 0,$$

but

$$\varepsilon + \max(0, \bar{Z}_{l'}(P^{\varepsilon})) \to 1.$$

Contradiction. Hence $P^* \gg 0$.

At P^* we have from (**??**)

$$P_l^* \sum_{k=1}^{L} \max(0, \bar{Z}_k(P^*)) = \max(0, \bar{Z}_l(P^*))$$

Multiplying both sides by $\bar{Z}_l(P^*)$ and summing over l yields

$$P\bar{Z}(P^*) \left(\sum_{k=1}^{L} \max(0, \bar{Z}_k(P^*) \right) = \sum_{l=1}^{L} \bar{Z}_l(P^*) \max(0, \bar{Z}_l(P^*)).$$

Note that

$$P\bar{Z}(P) \leq PZ(P) = 0 \Longrightarrow P\bar{Z}(P^*) \left(\sum_{k=1}^{L} \max(0, \bar{Z}_k(P^*) \right) \leq 0.$$

This implies $\forall\ l$, $\bar{Z}_l(P) \leq 0$; otherwise,

$$\sum_{l=1}^{L} \bar{Z}_l(P^*) \max(0, \bar{Z}_l(P^*)) > 0.$$

Because $\bar{Z}_l(P^*) = \min(1, Z_l(P))$,

$$\bar{Z}_l(P) \leq 0 \Longrightarrow Z_l(P^*) \leq 0.$$

But as $P \gg 0$,

$$Z(P^*) = 0 \Longrightarrow Z(P^*) = 0.$$

Chapter 5

General equilibrium: production economy

5.1 Production

5.1.1 Production transformation

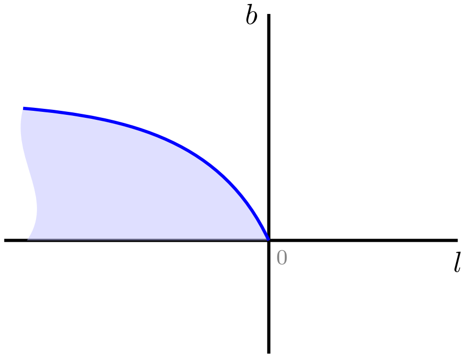
1. Production possibility set

   Example 1: labor as input; two outputs, beef and corn. Production function

   $$b = 5l^{1/2}.$$

   The production possibility set

   $$Y_b = \{(-l, b, c) \in \mathbb{R}_- \times \mathbb{R} \times \mathbb{R} | b \leq 5l^{1/2}, c = 0\}$$



2. Assumptions on production possibility set

- $0 \in Y_j$: can do nothing;
- $Y_j \cap \mathbb{R}^L_+ = \{0\}$: no free lunch;
- Y_j is convex: no increasing return;
- $\mathbb{R}^L_- \subseteq Y_j$.

3. Aggregate production

   y is possible if there exists $(y_1, \ldots, y_J)$,

$$y = \sum_j y_j \quad \text{and} \quad (\forall\ j)\ y_j \in Y_j.$$

4. Y is production efficient if Y is possible and there exists no possible alternative Y' such that $Y' > Y$.

5. Definition: A transformation function $T : A \subseteq \mathbb{R}^L \to \mathbb{R}$ represents Y over $A \subseteq \mathbb{R}^L$ if $(\forall y \in A)$,

$$T(y) \leq 0 \Leftrightarrow y \in Y$$

$$\text{and } T(y) = 0 \Leftrightarrow y \text{ is efficient.}$$

6. Rule to get $T(y)$: pick one good l (use an output)

$$T(y) = y_l - \max\{y'_l | (y'_l, y_{-l}) \in Y\}.$$

   Example 1 continued: labor, corn and beef

$$Y_b = \{(-l_b, b, 0) | b \leq B(l_b)\}, \qquad Y_c = \{(-l_c, 0, c) | c \leq C(l_c)\}.$$

$$\begin{aligned} T(y_1, y_2, y_3) &= T(-l, b, c) = y_2 - \max\{y'_2 | (y_1, y'_2, y_3) \in Y_b + Y_c\} \\ &= b - \max\{b' | (-l, b', c) \textit{ is possible}\} \\ &= b - \max\{b' | (\exists\ l_b, l_c) l_b + l_c \leq l,\ b' \leq B(l_b),\ c \leq C(l_c)\} \\ &= b - \max\{b' | (\exists\ l_c)\ b' \leq B(l - l_c),\ c \leq C(l_c)\} \\ &= b - B(l - C^{-1}(c)). \end{aligned}$$

7. Derivatives $DT(y)$

$$T_1(y_1, y_2, y_3) = \frac{\partial}{\partial y_1} T(y_1, y_2, y_3)$$

$$T_2(y_1, y_2, y_3) = \frac{\partial}{\partial y_2} T(y_1, y_2, y_3)$$
$$T_3(y_1, y_2, y_3) = \frac{\partial}{\partial y_3} T(y_1, y_2, y_3)$$

Note that

$$TRS^{12} = \frac{T_1}{T_2} = MP_l^b,$$
$$TRS^{13} = \frac{T_1}{T_3} = MP_l^c,$$

the marginal products of labor and

$$TRS^{23} = \frac{T_2}{T_3} = MRT^{bc}$$

MRT^{bc} is the *marginal rate of transformation* of beef for corn, which tells us the marginal opportunity cost of beef in terms of forgone units of corn, the economy can get one additional unit of beef by sacrificing MRT^{bc} unit of corn.

5.1.2 Examples

1. Example 1:

$$b = 5l^{1/2}, \quad c = 10l^{1/2}$$

The production transformation

$$T(-l, b, c) = b - 5\left(l - \frac{c^2}{100}\right)^{1/2} \Longrightarrow$$
$$DT = \left(\frac{5}{2}\left(l - \frac{c^2}{100}\right)^{-1/2}, 1, \frac{c}{20}\left(l - \frac{c^2}{100}\right)^{-1/2}\right)$$

2. Example 2: 2-inputs & 2-outputs

$$Y_b = \{b = (-k_b, -l_b, b, 0) | b \le 4(k_b l_b)^{1/2}\}$$
$$Y_c = \{b = (-k_c, -l_c, 0, c) | c \le 20(k_c l_c)^{1/2}\}$$

In this case

$$T(-k, -l, b, c) = \{c - \max\{20(k_c l_c)^{1/2}) | 4[(k - k_c)(l - l_c)]^{1/2} \ge b\}$$

To solve the problem, set Lagrangian for maximization problem and solve for FOC

$$\mathcal{L} = 20k_c^{\frac{1}{2}} l_c^{\frac{1}{2}} + \lambda[4(k - k_c)^{\frac{1}{2}}(l - l_c)^{\frac{1}{2}} - b]$$

FOC:

$$\lambda = 5,$$

$$k_c^* = k\left(1 - \frac{b}{4(kl)^{1/2}}\right)$$

$$l_c^* = l\left(1 - \frac{b}{4(kl)^{1/2}}\right) \Longrightarrow$$

$$T(-k,-l,b,c) = c - 20(k_c^* l_c^*)^{\frac{1}{2}} = c - 20(kl)^{\frac{1}{2}}\left(1 - \frac{b}{4(kl)^{1/2}}\right) \Longrightarrow$$

$$T(-k,-l,b,c) = c - 20(kl)^{\frac{1}{2}} + 5b.$$

If $b = 0 \quad \Rightarrow c = 20(kl)^{1/2}$;

If $c = 0 \quad \Rightarrow b = 4(kl)^{1/2}$.

- Note that

  $$DT(-k,-l,b,c) = (10k^{\frac{-1}{2}}l^{\frac{1}{2}}, 10k^{\frac{1}{2}}l^{\frac{-1}{2}}, 5, 1),$$

  respectively, MP_k^c, MP_l^c, $MRT^{b,c}$

3. Production possibility frontier: The equation for the production possibility frontier is given by setting the transformation function $T(y) = 0$

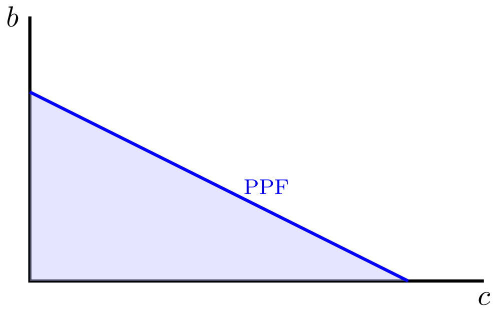
   Therefore, for example 2:

   $$T(-k,-l,b,c) = 0 \Leftrightarrow c - 20(kl)^{\frac{1}{2}} + 5b = 0.$$

   If the economy is endowed with 5 units of k and l,

   $$c + 5b - 100 = 0,$$

   constant MRT along the PPF.



5.1.3 Efficiency

1. Production

   - y is possible if $\forall j,\ y_j \in Y_j$;
   - y is possible if $y \in \sum_j Y_j$, or iff $T(y) \leq 0$;
   - y is efficient iff $T(y) = 0$.

2. Consumption

   - X is feasible iff

   $$(\exists\ y), \sum_i X_i \leq \sum_j y_j + \sum_i \omega_i \Leftrightarrow \sum_i X_i - \sum_i \omega_i \in \sum_j Y_j \Leftrightarrow$$

   $$T(\sum_i X_i - \sum_i \omega_i) \leq 0.$$

   - X is efficient if it is feasible and there does not exist a feasible X' such that

   $$\forall i, X_i' \succeq X_i \quad \text{and} \quad \exists i,\ X_i' \succ_i X_i.$$

3. Theorem: Suppose $X \gg 0$ and $(\forall i)$, $\succeq_i$ is represented by a concave u_i which is twice continuously differentiable and strongly monotonic around X_i, and $\sum_j Y_j$ is represented by a convex function T, which is twice continuously differentiable around $(\sum_i X_i - \sum_i \omega_i)$. Then the following are equivalent

   - X is (Pareto) efficient;
   - $(\exists\ s_1, \ldots, s_I) \in \mathbb{R}^K_{++}$,

   $$(\forall i)\ s_i Du_i(X_i) = DT\left(\sum_i X_i - \sum_i \omega_i\right)$$

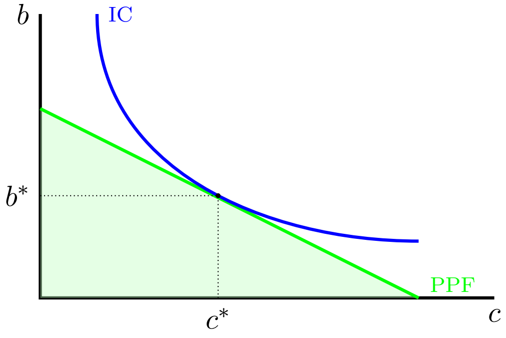
   $$(\forall i)\ T\left(\sum_i X_i - \sum_i \omega_i\right) = 0.$$

   Remark: marginal rate of substitution (equal across consumers) equals marginal rate of transformation at PE allocations.

4. Efficient allocation with representative consumer:

   - Production: beef-corn example continued

- There is a representative consumer in the economy
- graphical illustration



5. Example 1: Robinson Crusoe economy

$$y = \{y \subset (-l, c) | c \leq 6\sqrt{l}\}$$

$$T(-l, c) = c - 6\sqrt{l} = y_2 - 6(-y_1)^{1/2}$$

$$DT(-l, c) = (3l^{-1/2}, 1) = (3(-y_1)^{-1/2}, 1).$$

Preference $u = 3l + 2c$.

Solving the problem yields two equations in two unknows

$$MRS^{1,2} = \frac{3}{2} = TRS^{1,2} = 3(-y_1)^{-1/2}$$

$$T(y) = y_2 - 6(-y_1)^{-1/2} = 0 \Longrightarrow$$

$$y_1 = -4, y_2 = 12.$$

$$X = \begin{bmatrix} -4 \\ 12 \end{bmatrix} + \begin{bmatrix} 9 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$$

Remark: We could also use Leontief preference to get the same result,

$$u = \min \left\{ \frac{l}{5}, \frac{c}{12} \right\}.$$

6. Example 2:

$$Y_b = \{b = (-k_b, -l_b, b, 0) | b \leq 4(k_b l_b)^{1/2}\}$$

$$Y_c = \{b = (-k_c, -l_c, 0, c) | c \leq 20(k_c l_c)^{1/2}\}$$

Representative agent preference

$$u = 2b^{1/2}c^{1/2}.$$

Total endowment: $k = 4, l = 4$

$$X = \begin{bmatrix} -4 \\ -4 \\ 8 \\ 40 \end{bmatrix} + \begin{bmatrix} 4 \\ 4 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 8 \\ 40 \end{bmatrix}$$

5.1.4 Equilibrium

1. Ownership shares θ_{ji}

$$(\forall i,\ j),\ \theta_{ji} \in [0,1],$$

$$(\forall\ j)\ \sum_i \theta_{ji} = 1.$$

2. Equilibrium:

   Given $\{Y_j\}_j$ and $\{\omega_i, \theta_i, \succeq_i\}_i$, (X^*, y^*, P^*) is an equilibrium if

   - $(\forall\ i)$, X_i^* is the maximal for $\succeq_i$ in $\{X_i | P^* X_i \leq P^* \omega_i + \sum_j \theta_{ji} P^* y_j^*\}$;
   - $(\forall\ j)$, $y_j^* \in arg\max\{P^* y_j | y_j \in Y_j\}$;
   - $\sum_i X_i^* = \sum_i \omega_i + \sum_j y_j^*$.

3. First Welfare Theorem: Suppose each $\succeq_i$ is locally nonsatiated. Then any equilibrium is efficient.

4. Proof of the theorem: Suppose (X^*, y^*, P^*) is an equilibrium and X^* is not efficient, then $(\exists\ X',\ y')$,

$$(\forall\ i)\ X_i' \succeq_i X_i^*,$$

$$(\exists\ i)\ X_i' \succ_i X_i^*,$$

$$\sum_i X_i' \leq \sum_j y_j' + \sum_i \omega_i. \qquad (feasibility\ condition)$$

   The first two equations imply

$$(\exists\ i)\ P^* X_i' > P^* \omega_i + \sum_j \theta_{ji} P^* y_j^*$$

$$(\forall\ i)\ P^* X_i' \geq P^* \omega_i + \sum_i \theta_{ji} P^* y_j^* \Longrightarrow$$

$$\sum_i P^* X_i' > \sum_i P^* \omega_i + \sum_i \sum_j \theta_{ji} P^* y_j^*, \qquad \sum_i \theta_{ji} = 1$$

From the feasibility condition,

$$\sum_i P^* X_i^* \leq \sum_j P^* y_j' + \sum_i P^* \omega_i$$

$$\sum_i P^* X_i' > \sum_j P^* y_j^* + \sum_i P^* \omega_i \Longrightarrow$$

$$P^* \sum_j y_j' > \sum_j P^* y_j^*.$$

Contradiction, as y^* maximizes profits given P^*.

5. Second welfare Theorem: Suppose that $(\forall j)$, Y_j is convex, $(\forall i)$, $\succeq_i$ is locally nonsatiated and convex. Then for every Pareto efficient (X^*, Y^*) such that $X^* \gg 0$, there exists $P^* > 0$ so that (X^*, y^*, P^*) is an equilibrium.

6. Example 1: Two-inputs and two outputs example

   Blue collar worker $L_b = 150, K_b = 0$, white collar worker $L_w = 50, K_w = 50$, and profits go to white collar workers.

   - Production function: food (x), energy (y).

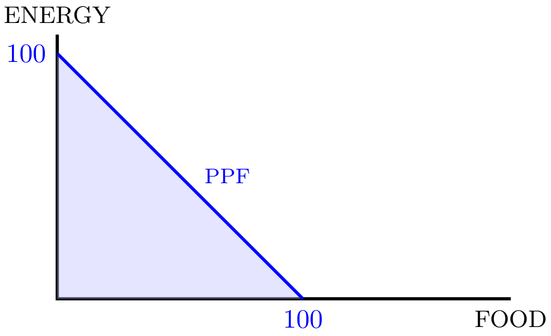
   $$x = L_x^{1/2} K_x^{1/2}, \qquad y = L_y^{1/2} K_y^{1/2}.$$

   - Production transformation

   $$T(-K, -L, x, y) = x - (LK)^{1/2} + y \Longrightarrow$$

   $$DT = (K^{-1/2} L^{1/2}, K^{1/2} L^{-1/2}, 1, 1)$$

   - Production possibility frontier



   - Preference

   $$U_b(x_b, y_b) = (x_b y_b)^{1/2} \qquad U_w(x_w, y_w) = (x_w y_w)^{1/2}.$$

- From utility-maximization,

$$x_b = \frac{I_b}{2p_x},\ y_b = \frac{I_b}{2p_y},$$

$$x_w = \frac{I_w}{2p_x},\ y_w = \frac{I_w}{2p_y},$$

  where

$$I_b = 150w \qquad I_w = 50w + 50r + \pi_x + \pi_y.$$

- From profit-maximization,

$$MRTS^x_{L,K} = MRTS^y_{L,K} = \frac{w}{r},$$

  which implies

$$wL_x = rK_x \qquad wL_y = rK_y \Longrightarrow$$

$$\frac{L_x}{K_x} = \frac{L_y}{K_y} = \frac{r}{w}.$$

  In equilibrium, no excess demand in labor market or capital market and

$$200 = L_x + L_y \qquad 50 = K_x + K_y \Longrightarrow$$

$$\frac{L_x}{K_x} = \frac{L_y}{K_y} = \frac{L_x + L_y}{K_x + K_y} = 4 = \frac{r}{w}.$$

- substitution efficiency:

$$MRT_{x,y} = \frac{MC_x}{MC_y} = \frac{p_x}{p_y} \Longrightarrow$$

$$MRT_{x,y} = MRS^b_{x,y} = MRS^w_{x,y}.$$

  We know that in competitive equilibrium,

$$MRS^b_{x,y} = \frac{y_b}{x_b}$$

- Solve for the competitive equilibrium (let $w = 1$ and thus, $r = 4$.)

  In equilibrium, $L_x = 4K_y$, $L_y = 4K_y$

$$MC_x = 4, \qquad MC_y = 4 \Longrightarrow$$

$$p_x = p_y = 4.$$

  Therefore, in equilibrium

$$x_b = y_b, x_w = y_w.$$

50 units of x and 50 units of y will be produced. Income for a typical white collar worker and blue collar worker

$$I_w = 50 + 4 \times 50 = 250, \qquad I_b = 150.$$

Given the income we can find the equilibrium allocation

$$x_b = \frac{75}{4}, y_b = \frac{75}{4}; \qquad x_w = \frac{125}{4}, y_w = \frac{125}{4}.$$

- We summarize the competitive equilibrium as follows

$$\text{Price: } w = 1, r = 4, p_x = 4, p_y = 4;$$

$$\text{Production: } L_x = 100, L_y = 100, K_x = 25, K_y = 25.$$

$$\text{Allocations: } x_b = \frac{75}{4}, y_b = \frac{75}{4}; \qquad x_w = \frac{125}{4}, y_w = \frac{125}{4}.$$

7. Example 2: The economy has 100 blue collar household; each household is endowed with 60 units of labor (L) and has preference

$$U^B = x^{\frac{3}{4}} y^{\frac{1}{4}}.$$

There is also 100 white collar household, each is endowed with 10 units of labor and 50 units of capital (K) and has preference

$$U^W = x^{\frac{1}{2}} y^{\frac{1}{2}}.$$

The production function for the economy is

$$x = 1.89 L^{\frac{1}{3}} K^{\frac{2}{3}}, \quad y = 2 L^{\frac{1}{2}} K^{\frac{1}{2}}.$$

- In this case, the economy's total inputs are

$$L = 100(10 + 60) = 7000, \qquad K = 100(50 + 0) = 5000.$$

- Exchange efficiency: utility-maximization Given $P_x,\ P_y,\ r,\ w$, we have

$$x_B = \frac{3I_B}{4P_x}, \qquad y_B = \frac{I_B}{4P_y};$$

$$x_W = \frac{I_W}{2P_x}, \qquad y_W = \frac{I_W}{2P_y};$$

and $I_B = 60w$, $I_W = 10w + 50r$.

- Production efficiency
  - ➢ cost-minimization

$$\frac{K_x}{2L_x} = \frac{w}{r} \Longrightarrow L_x = \frac{rK_x}{2w}.$$
$$\frac{K_y}{L_y} = \frac{w}{r} \Longrightarrow L_y = \frac{rK_y}{w}.$$

  - ➢ Plugging $L_{x,\,y}$ back into the production function, we can find the demand curves for labor and capital

$$x = 1.89\left(\frac{rK_x}{2w}\right)^{\frac{1}{3}} K_x^{\frac{2}{3}} \Longrightarrow K_x = \frac{2x}{3}\left(\frac{w}{r}\right)^{\frac{1}{3}}, \quad L_x = \frac{x}{3}\left(\frac{r}{w}\right)^{\frac{2}{3}}.$$
$$y = 2\left(\frac{rK_y}{w}\right)^{\frac{1}{2}} K_y^{\frac{1}{2}} \Longrightarrow K_y = \frac{y}{2}\left(\frac{w}{r}\right)^{\frac{1}{2}}, \quad L_y = \frac{y}{2}\left(\frac{r}{w}\right)^{\frac{1}{2}}.$$

  - ➢ Given the demand curves, we can find the total cost functions and marginal cost

$$TC_x = wL_x + rK_x = \frac{x}{3}w^{\frac{1}{3}}r^{\frac{2}{3}} + \frac{2x}{3}w^{\frac{1}{3}}r^{\frac{2}{3}} = xw^{\frac{1}{3}}r^{\frac{2}{3}}.$$
$$TC_y = wL_y + rK_y = \frac{y}{2}w^{\frac{1}{2}}r^{\frac{1}{2}} + \frac{y}{2}w^{\frac{1}{2}}r^{\frac{1}{2}} = yw^{\frac{1}{2}}r^{\frac{1}{2}}.$$

    And

$$MC_x = w^{\frac{1}{3}}r^{\frac{2}{3}}, \quad MC_y = w^{\frac{1}{2}}r^{\frac{1}{2}}.$$

  - ➢ In equilibrium

$$P_x = MC_x = w^{\frac{1}{3}}r^{\frac{2}{3}}, \qquad P_y = MC_y = w^{\frac{1}{2}}r^{\frac{1}{2}}.$$

- Market clearing conditions
  - ➢ Markets for x, y clears

$$x = 100x_B + 100x_W = \frac{50I_W + 75I_B}{P_x} = \frac{5000w + 2500r}{w^{\frac{1}{3}}r^{\frac{2}{3}}},$$
$$y = 100y_B + 100y_W = \frac{50I_W + 25I_B}{P_y} = \frac{2000w + 2500r}{w^{\frac{1}{2}}r^{\frac{1}{2}}}.$$

  - ➢ Market for labor clears

$$\begin{aligned} 7000 &= L_x + L_y = \frac{x}{3}\left(\frac{r}{w}\right)^{\frac{2}{3}} + \frac{y}{2}\left(\frac{r}{w}\right)^{\frac{1}{2}} \\ &= \frac{5000w + 2500r}{3w^{\frac{1}{3}}r^{\frac{2}{3}}} \cdot \left(\frac{r}{w}\right)^{\frac{2}{3}} + \frac{2000w + 2500r}{2w^{\frac{1}{2}}r^{\frac{1}{2}}} \cdot \left(\frac{r}{w}\right)^{\frac{1}{2}} \\ &= \frac{5000w + 2500r}{3w} + \frac{2000w + 2500r}{2w}. \end{aligned}$$

    This gives

$$\frac{r}{w} = 2.08.$$

- ➢ Market for capital clears

$$5000 = K_x + K_y = \frac{10000w + 5000r}{3r} + \frac{2000w + 2500r}{2r},$$

  which also gives

$$\frac{r}{w} = 2.08.$$

- If we let $w = 1$, we get $r = 2.08$.

  Plugging $r/w = 2.08$ into the equation for $x,\ y$, we have

$$x = 6300, \qquad y = 5000.$$

  We also get $P_x = 1.628,\ P_y = 1.4422$.

8. Existence of equilibrium

   An equilibrium exists in $\{\omega_i, \succeq_i, \theta_i\}_i$ and $\{Y_j\}_j$ if

   - No satiation bundle exists for any consumer (weaker than local nonsatiation);
   - $(\forall\ X_i)$, $\{X_i' | X_i' \succeq X_i\}$ and $\{X_i' | X_i' \preceq X_i\}$ are closed;
   - $(\forall X_i^1,\ X_i^2)$ $(\forall\ t \in (0,1))$,

$$X_i^2 \succ_i X_i^1,\ X_i^2 \neq X_i^1 \Longrightarrow tX_i^1 + (1-t)X_i^2 \succ_i X_i^1;$$

   - $(\forall\ i)$, $0 \ll \omega_i$;
   - $(\forall\ j)$, $0 \in Y_j$;
   - $(\forall\ j$, Y_j is closed and convex, decreasing return to scale;
   - $(\sum_j Y_j) \cap (-\sum_j Y_j) = \{0\}$;
   - $\mathbb{R}_-^L \subseteq \sum_j Y_j$.

Chapter 6

Information economics: adverse selection

6.1 Brief introduction of information economics

Orthodox economic theory has little to offer in terms of understanding how nonmarket organizations, like firms, form and function. This is so because traditional theory pays little or no attention to the role of information, which evidently lies at the heart of organizations. The development of information economics, which explicitly recognizes that agents have limited and different information, significantly improves the understanding of the intricacies of organizational design.

Adverse selection is commonly used in economics, insurance, and risk management that describes a situation where market participation is affected by asymmetric information. When buyers and sellers have different information, it is known as a state of asymmetric information. Traders with better private information about the quality of a product will selectively participate in trades which benefit them the most, at the expense of the other trader.

Adverse selection is not the only informational problem one can imagine. Agents to whom a task has been delegated by a principal may also choose actions which affect the value of trade or, more generally, the agent's performance. By the mere fact of delegation, the principal loses any ability to control those actions when those actions are no longer observable or verifiable. We will then say that there is moral hazard. Sometimes it is also referred to as *hidden action*, as contrast to *hidden information* that is used to refer to adverse selection.

6.2 Adverse selection

1. A situation in which different agents possess different information is said to be one of asymmetric information
2. The presence of asymmetric information typically lead to inefficient market outcomes.

6.2.1 Insurance market

1. Model
   - Consumer: initial wealth w, accident occurs with $\pi_i \in [0,1]$ in which L dollar loss
   - Insurance companies: identical and offer full insurance at price p
2. Symmetric information, Zero-profit condition

$$p_i = \pi_i L \quad \forall i$$

3. Asymmetric information and adverse selection

   (a)

$$\pi \in [\underline{\pi}, \bar{\pi}]$$

   (b) Consumer purchase policy iff

$$u(w-p) \geq \pi u(w-L) + (1-\pi)u(w)$$

   which implies consumer purchase policy iff accident probability

$$\pi \geq \frac{u(w)-u(w-p)}{u(w)-u(w-L)} \equiv h(p)$$

   (c) Competitive equilibrium price under asymmetric information

$$p^* = E(\pi|\pi \geq h(p^*))L,$$

$$E(\pi|\pi \geq h(p^*)) = \frac{\int_{h(p^*)}^{\bar{\pi}} \pi dF(\pi)}{1-F(h(p^*))}$$

4. Example: $\pi \sim U(0,1)$

   In this case

$$E(\pi|\pi \geq h(p)) = \frac{1+h(p)}{2}$$

Let $g(p) = E(\pi|\pi \geq h(p))L$, , there is a unique equilibrium p^*

$$p^* = \frac{1 + h(p^*)}{2}L,$$

which is $P^* = L$.

Only consumer that is certain to have an accident buy the insurance.

6.2.2 Used car market

1. Price of automobile p, quality $\mu(p)$
2. Two groups of traders
   (a) Group one: total income Y_1 and has N used cars

   $$u_1 = M + \sum_{i=1}^{n} x_i$$

   x_i quality of ith automobile

   (b) Group two: total income Y_2 and

   $$u_2 = M + \sum_{i=1}^{n} \frac{3x_i}{2}$$
3. Symmetric information: both groups only knows

$$x_i \sim U(0,2)$$

and expected quality $\mu = 1$.

- Supply

$$S(p) = \begin{cases} N, & p > 1 \\ 0 & p < 1 \end{cases}$$

- Demand

$$D(p) = \begin{cases} \frac{Y_2+Y_1}{p}, & p < 1 \\ \frac{Y_2}{p}, & 1 < p < \frac{3}{2} \\ 0 & p > \frac{3}{2} \end{cases}$$

- Equilibrium

$$p = \begin{cases} 1, & \text{if } Y_2 < N \\ \frac{Y_2}{N} & \text{if } \frac{2Y_2}{3} < N < Y_2 \\ \frac{3}{2} & \text{if } N < \frac{2Y_2}{3} \end{cases}$$

4. Asymmetric information: Group one knows quality of cars while group two does not.

   - Demand of group one

$$D_1(p) = \begin{cases} \frac{Y_1}{p}, & \mu > p \\ 0 & \mu < p \end{cases}$$

   - Demand of group two

$$D_2(p) = \begin{cases} \frac{Y_2}{p}, & \frac{3\mu}{2} > p \\ 0 & \frac{3\mu}{2} < p \end{cases}$$

   - Supply

$$S(p) = \frac{pN}{2}$$

     with average quality $p/2$.

   - Total demand

$$D(p,\mu) = \begin{cases} \frac{Y_1+Y_2}{p}, & \mu > p \\ \frac{Y_2}{p} & \mu < p < \frac{3\mu}{2} \\ 0 & \frac{3\mu}{2} < p \end{cases}$$

   - Note that with price p,

$$\mu = \frac{p}{2}$$

     There will be no trade in equilibrium, even if at *any given price* $p \in [0, 3]$, there are group one trader willing to sell at a price which group two are willing to pay.

6.2.3 Adverse selection in financial market

In stock market, a trade ultimately represents a difference of opinion. One side of the trade buys while the other side of the trade sells. Who is right and who is wrong? Do I know more about the situation than the contrary side, or am I being led like a lamb to slaughter? This is the question market participants ask themselves on each trade.

Adverse selection arises when a trader sells a security because he has private information that the security is overpriced.[1] An uninformed market maker or trader on the other side of the transaction

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[1] For example, in the wake of the space shuttle Challenger explosion at 11:39 a.m. EST on January 28, 1986, the stock market very quickly determined which of the four potential contracting manufacturers was at fault for the defective parts of the shuttle: within fifteen minutes, there was a sell-induced New-York Stock Exchange (NYSE) trading halt in the shares of only one company, Morton-Thiokol. By the end of the day its shares had fallen by 11.86 percent, while Lockheed, Martin-Marietta, and Rockwell fell by much less. By contrast, the general public did not learn of the cause of the crash until two weeks later, on February 11, when Nobel-winning physicist Richard

will try to protect himself by offering the seller a lower price. Similarly, a purchase may indicate positive information that buyers have about the security's price, and this will induce uninformed market makers and traders to ask for a higher selling price. The greater the extent of asymmetric information, the lower will be the selling price and the higher will be the buying price, that is, the greater the spread will be.

In the ecology of financial markets, liquidity providers offer to both purchase and sell an asset, and seek to earn the difference between the buy and sell price. Traders who implement this strategy in modern markets are still called "market-makers."

Market-makers typically aim to keep their net inventory as close to zero as possible, so as not to bear the risk of the asset's price going up or down. Their goal is to earn a profit by buying low (at their bid-price b) and selling high (at their ask-price a), and therefore earning the bid–ask spread

$$S = a - b$$

from each round-trip trade.

So the basic ecology of modern financial markets is: market-makers offer opportunities to buy and/or sell, with the aim of profiting from round-trip trades, while speculators buy or sell assets, with the aim of profiting from subsequent price changes. Based on this simple picture, it seems that market-makers have a much more favourable position than speculators. If speculators make incorrect predictions about future price moves, then they will experience losses, but they will still trade with market-makers, who will therefore still conduct round-trip trades. In this simplistic picture, speculators bear the risk of incorrect predictions, while market-makers seemingly always make a profit from the bid–ask spread. Is it really the case that market-makers can earn risk-free profits?

The simple answer to the last question is: no. Market-makers also experience several different types of risk. Perhaps the most important is adverse selection (also called the "winner's curse" effect). Adverse selection results from the fact that market-makers must post binding quotes, which can be "picked off" by more informed traders who see an opportunity to buy low or to sell high. This informational asymmetry is a fundamental concern for market-makers.

For a market-maker, the core question is how to choose the values of b and a. If the values of b and a remain constant at all times, then the market-maker always earns a profit of S for

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Feynman demonstrated that there were problems with Morton-Thiokol's booster rockets. This episode illustrates that insiders do have an information advantage over the public. *Market Liquidity: Theory, Evidence, and Policy* by Thierry Foucault, Marco Pagano, and Ailsa Roell each round-trip (buy and sell) trade. All else being equal, the larger the value of S, the larger the profit a market-maker earns per trade. However, the larger the value of S, the less attractive a market-maker's buy and sell prices are to liquidity takers. In situations where several different market-makers are competing, if one market-maker tries to charge too wide a spread, then another market-maker will simply undercut these prices by stepping in and offering better quotes. Most modern financial markets are indeed highly competitive, which prevents the spread from becoming too large. But why does this competition not simply drive S to zero?

Consider a market-maker trading stocks of a given company by offering a buy price of $b = \$54.50$ and a sell price of $a = \$55.50$. If the buy and sell order flow generated by liquidity takers was approximately balanced, then the market-maker would earn $s = \$1.00$ per round-trip trade. If, however, an insider knows that the given company is about to announce a drop in profits, they will revise their private valuations of the stock downwards, say to \$50.00. If the market-maker continues to offer the same quotes, then he or she would experience a huge influx of sell orders from insiders, who regard selling the stock at \$54.50 to be extremely attractive. The market-maker would therefore quickly accumulate a large net buy position by purchasing more and more stocks at the price \$54.50, which will likely be worth much less soon after, generating a huge loss. This is adverse selection.

To compensate for this potential loss, market-makers charge a non-zero spread– even in situations where market-making is fiercely competitive. To mitigate the risk of being adversely selected, market-makers must update their values of b and a to respond to their observations of order-flow imbalance. If a market-maker receives many more buy orders than sell orders, then he or she can attempt to reduce this order-flow imbalance by increasing the ask-price a (to dissuade future buyers), increasing the bid-price b (to encourage future sellers), or both. Similarly, if a market-maker receives many more sell orders than buy orders, then he or she can attempt to reduce this order-flow imbalance by decreasing b, decreasing a, or both. An important consequence of this fact is that trades have price impact: on average, the arrival of a buy trade causes prices to rise and the arrival of a sell trade causes prices to fall. This is precisely what the spread S compensates for.[2]

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[2] For detailed discussion on information asymmetry and adverse selection, see Chapter 1: The Ecology of Financial Markets, *Trades, Quotes and Prices: Financial Markets under the Microscope.*

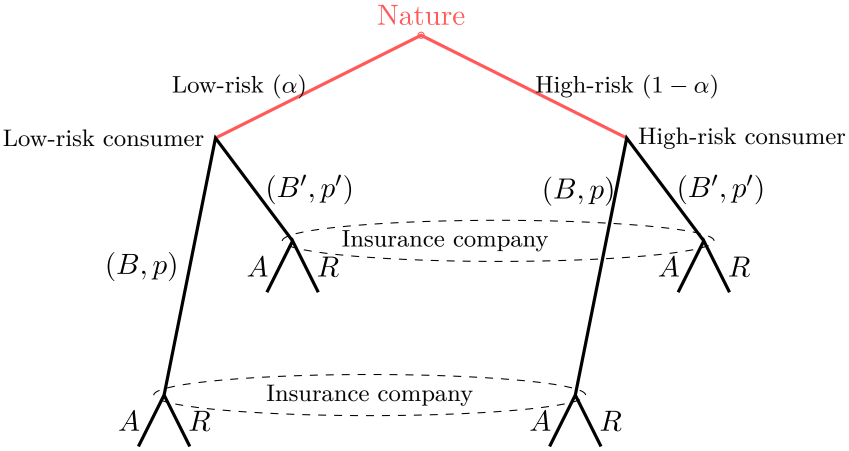
6.3 Signaling

Consumers can credibly communicate how risky they are to insurance companies, i.e., by purchasing different types of policies.

6.3.1 The game

1. Signaling game

   - Insurance signaling game



   - Nature choose risk type $t_i \in T \equiv \{h, l\}$
   - Prior belief: $Prob(\underline{\pi}) = \alpha,\ Prob(\bar{\pi}) = 1 - \alpha$.
   - Consumer (*Sender*) chooses message $m_i \in M \equiv \{(B, p)\}$.
   - Insurance company (*Receiver*) responds given belief $\beta(B, p)$: accept, reject.

2. A pure strategy for the low-risk consumer is a policy $\psi_l(B_l, p_l)$, and a pure strategy for the high-risk is $\psi_h(B_h, p_h)$.

3. Belief: $\beta(B, p)$–the consumer who proposes (B, p) is low-risk type

4. Signaling game pure strategy sequential equilibrium: $(\psi_l, \psi_h, \sigma(\cdot), \beta(\cdot))$ is a pure strategy sequential equilibrium of the insurance signaling game if

   - Given $\sigma(\cdot)$, ψ_l, ψ_h maximize low-risk, high-risk's expected utility respectively; $\sigma(\cdot)$ maximizes insurance company's expected profit given belief
   - Belief satisfy Bayes rule,

- ➢ $\beta(\psi) \in [0,1]$
- ➢ If $\psi_l \neq \psi_h$, then $\beta(\psi_l) = 1$, $\beta(\psi_h) = 0$
- ➢ If $\psi_l = \psi_h$, then $\beta(\psi_l) = \beta(\psi_h) = \alpha$

5. Example:

   "In mid-1985, Warren Buffett took out a nervy advertisement inviting big commercial customers who were hard pressed to find coverage to submit policies for any type of risk with premiums of \$1 million or more. There was a twist: respondents had to name their price. If Buffett deemed a proposal to be unreasonable, he would throw it out with the understanding that he would not grant a second chance. This poker ploy generated more than \$100 million in premiums."

   (Chaper 16, Buffett, The Making of American Capitalist)

6.3.2 Consumer's optimal choice

1. First note that expected utility of the two types:

$$u_l(B,p) = \underline{\pi} u(w - L + B - p) + (1 - \underline{\pi}) u(w - p)$$

$$u_h(B,p) = \bar{\pi} u(w - L + B - p) + (1 - \bar{\pi}) u(w - p)$$

2. Individual's optimal insurance problem:

   A strictly risk-averse individual with wealth w is facing a potential loss of L. The probability that the loss will happen is π. The individual must decide how much insurance he wants to purchase. One unit of insurance costs $q < 1$ dollars and pay one dollar in the event of the loss. Let B denote the amount of insurance. The individual's optimization problem is

$$\max_{B} \pi u\left(w - L + B\left(1 - q\right)\right) + \left(1 - \pi\right) u\left(w - Bq\right)$$
$$s.t.\ B \geq 0,\ B \leq w/q$$

3. To solve the problem, define the Lagrangian function

$$\mathcal{L} = \pi u\left(w - L + B\left(1 - q\right)\right) + \left(1 - \pi\right) u\left(w - Bq\right) + \lambda\left(w/q - B\right).$$

4. The first-order conditions are

$$\pi u'\left(w-L+B\left(1-q\right)\right)\left(1-q\right)-\left(1-\pi\right)u'\left(w-Bq\right)q-\lambda\leq 0, f$$
$$B\left[\pi u'\left(w-L+B\left(1-q\right)\right)\left(1-q\right)-\left(1-\pi\right)u'\left(w-Bq\right)q-\lambda\right]=0$$
$$\lambda\left(\alpha-w/q\right)=0, \lambda\geq 0, B\geq 0.$$

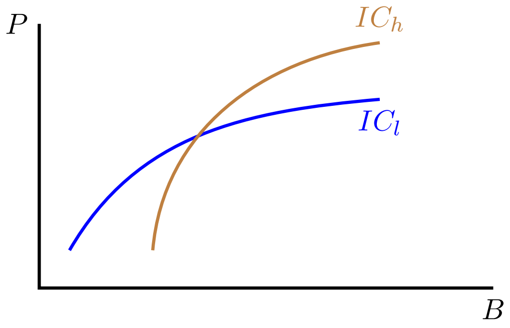
5. Thus, the optimal B satisfies

$$\frac{\pi u'\left(w-L+B\left(1-q\right)\right)}{\left(1-\pi\right)u'\left(w-Bq\right)}=\frac{q}{1-q}$$

6. Note that $P=Bq$ and $B(1-q)=B-P$, so we have:

$$MRS(B,P)=\frac{\pi u'\left(w-L+B-P\right)}{\pi u'(w-L+B-P)+\left(1-\pi\right)u'\left(w-P\right)}=\frac{P}{B}=q.$$

7. Single crossing property



8. Hence:

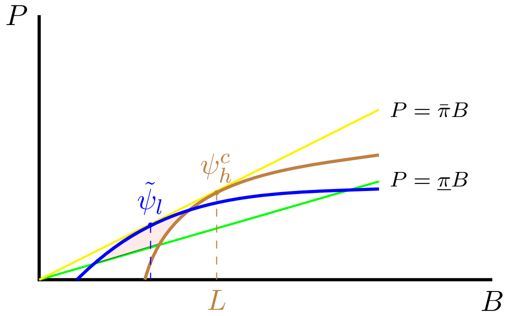
   - $u_l(B,p)$ and $u_h(B,p)$ are continuous, differentiable, strictly concave in (B,p)
   - $MRS_l(B,p)$ $(MRS_h(B,p))$ is greater than, equal to or less than $\underline{\pi}$ $(\bar{\pi})$ as B is less than, equal to or greater than L.
   - $MRS_l(B,p) < MRS_h(B,p)$

6.3.3 Analyzing the signaling game

1. Budget constraint

2. Consumers' preferences for risks

3. Consumer and insurance company's problem



4. Lemma 8.1. (Jehle & Reny) Let

$$\tilde{u}_l \equiv \max_{(B,p)} u_l(B,P) \quad s.t. p = \bar{\pi}B \leq w, \qquad u_h^c \equiv u_h(L, \bar{\pi}L).$$

   And let $(\psi_l, \psi_h, \sigma(\cdot), \beta(\cdot))$ be a sequential equilibrium in which the equilibrium utilities for low-risk and high-risk are, respectively, u_l^* and u_h^*. Then

$$u_l^* \geq \tilde{u}_l \qquad u_h^* \geq u_h^c.$$

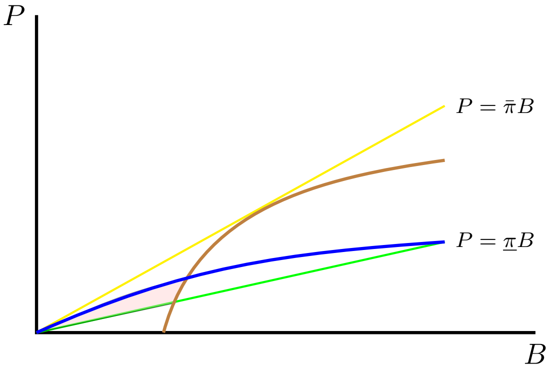
6.3.4 Separating equilibrium

1. An equilibrium is a separating equilibrium if the different types of consumers propose different policies.

2. Theorem 8.1. (Jehle & Reny) In separating equilibrium,

   - $\psi_l \neq \psi_h = (L, \bar{\pi}L)$
   - $p_l \geq \underline{\pi}B_l$
   - $u_l(\psi_l) \geq \tilde{u}_l \equiv \max_{(B,p)} u_l(B,p) \ s.t. \ \ p = \underline{\pi}B \leq w$
   - $u_h^c \equiv u_h(\psi_h) \geq u_h(\psi_l)$, where $u_h^c \equiv u_h(L, \bar{\pi}L)$ is high-risk's utility in competitive equilibrium with full information

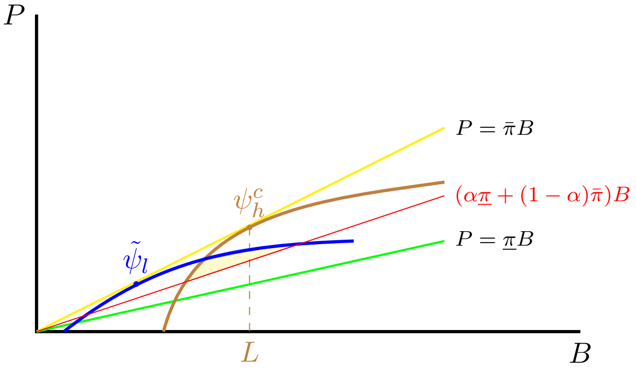
3. Existence of separating equilibrium



6.3.5 Pooling equilibria

1. An equilibrium is pooling equilibrium if both high-risk and low-risk propose the same policy.

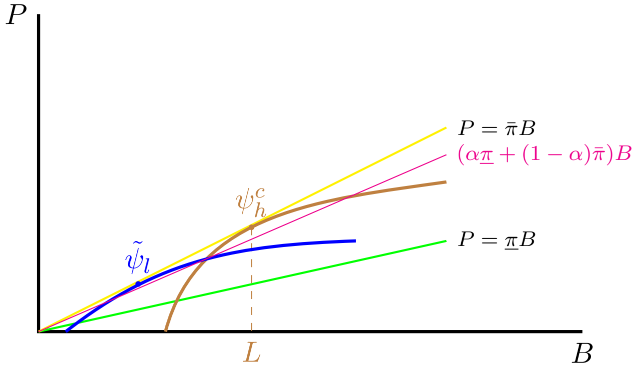
2. Identify the pooling equilibria



3. Theorem 8.2. (Jehle &Reny) $\psi = (B, p)$ is the outcome in some pooling equilibrium if and only if

    - $u_l(B, p) \geq \tilde{u}_l$, $u_h(B, p) \geq u_h^c$
    - $p \geq (\alpha\underline{\pi} + (1-\alpha)\bar{\pi})B$

4. Existence of pooling equilibra



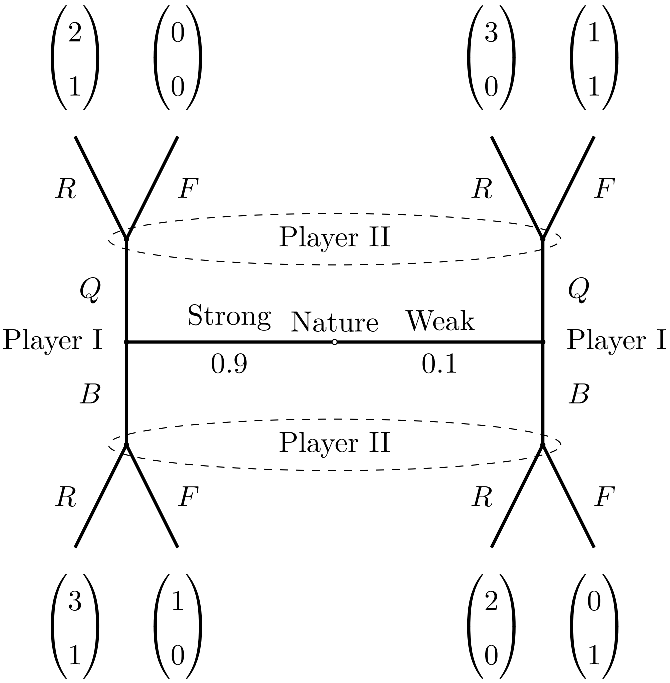
6.4 Intuitive criterion

6.4.1 Discrete message

1. Briefly, a sequential equilibrium satisfies Intuitive Criterion if no type of sender could obtain a payoff higher than his equilibrium payoff were he to choose a nonequilibrium message and the receiver responds with an optimal reply to the belief that imputes zero probability to Nature's choice of those types that can not gain from such a deviation regardless of the receiver's responses.

2. Application: Beer-Quiche game studied by Cho and Kreps (1987, Sec.II).

   The extensive-form game



3. The normal-form

II

I		FF	FR	RF	RR
	BB	0.9, 0.1	0.9, 0.1	2.9, 0.9	2.9, 0.9
	BQ	1, 0.1	1.2, 0	2.8, 1	3, 0.9
	QB	0, 0.1	1.8, 1	0.2, 0	2, 0.9
	QQ	0.1, 0.1	2.1, 0.9	0.1, 0.1	2.1, 0.9

- Player I's strategy:

  BB: plays B if strong, and plays B if weak; BQ: plays B if strong, and plays Q if weak

- Player 2's strategy:

  FF: plays F if Beer, and plays F if Quiche; FR: plays F if Beer, and plays R if Quiche

- To get the expected payoff from (BB, FF), note that the probability of weak and strong is (0.1, 0.9). Player I always sends the message B. Player II plays F no matter what I's signal is, so player I's expected payoff from this strategy profile is

  $$U_I(BB, FF) = 0.1 \times 0 + 0.9 \times 1 = 0.9.$$

  For player II, the expected payoff is

  $$U_{II}(BB, FF) = 0.1 \times 1 + 0.9 \times 0 = 0.1.$$

- If the strategy profile is (BB, RF), player I still sends B regardless of type. Player II plays R if signal is B and plays F if signal is Q. So player I's expected payoff is:

  $$U_I(BB, RF) = 0.1 \times 2 + 0.9 \times 3 = 2.9.$$

  For player II, the expected payoff is

  $$U_{II}(BB, RF) = 0.1 \times 0 + 0.9 \times 1 = 0.9.$$

- If the strategy profile is (QB, FF), player I sends signal Q if weak, and sends signal B if strong. Player 2 responds to both B and Q with F. Player I's expected payoff is:

  $$U_I(QB, FF) = 0.1 \times 0 + 0.9 \times 0 = 0.$$

  Player II's expected payoff is:

  $$U_{II}(QB, FF) = 0.1 \times 1 + 0.9 \times 0 = 0.1.$$

- The rest payoffs can be obtained similarly

4. Two pure strategy S.E.:

   - Equilibrium one
     - ➢ Equilibrium strategy σ^*

       $$(BB, RF)$$

     - ➢ Player II's belief:
       (a) player I is type S with probability 0.9 and type W with prob. 0.1 if Beer;
       (b) Player I is type W with probability greater than 0.5 if Quiche
   - Equilibrium two:
     - ➢ Equilibrium strategy:

       $$(QQ, FR)$$

     - ➢ Player II's belief:
       (a) Player I is type S with probability 0.9 and type W with prob. 0.1 if Quiche;
       (b) Player I is type W with probability greater than 0.5 if Beer

6.4.2 Continuous message

Consider the following sequential-move game between a worker and a firm first analyzed by Spence (1973). First, nature selects the type of a worker θ_k. The worker observes his own productivity level, but the firm does not. Observing his type, the worker chooses an education level, $e \geq 0$. Observing the education level of the worker, e, the firm offers wage $w(e)$.

The worker s utility function is

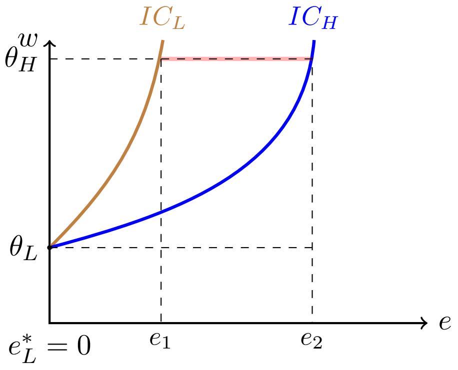
$$u_w(w, e_j, \theta) = w - \frac{e}{2\theta}$$

if he accepts a wage offer, and zero if he rejects. Note that θ only affects the worker 's cost of acquiring education.

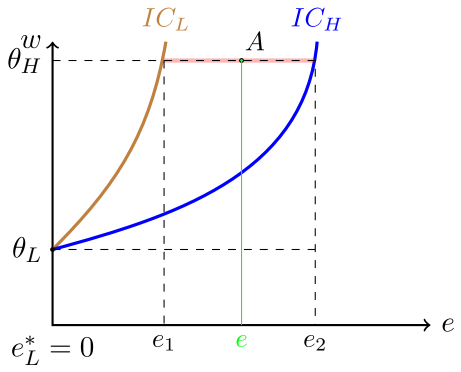
6.4.3 Two types

Suppose a worker can be of only two types, either θ_H (high productivity) or θ_L (low productivity), such that $\theta_H > \theta_L$. The set of separating equilibria is

$$e_L^* = 0, \quad e_H^* \in [e_1, e_2]; \qquad w(e_L^*) = \theta_L, \quad w(e_H^*) = \theta_H.$$



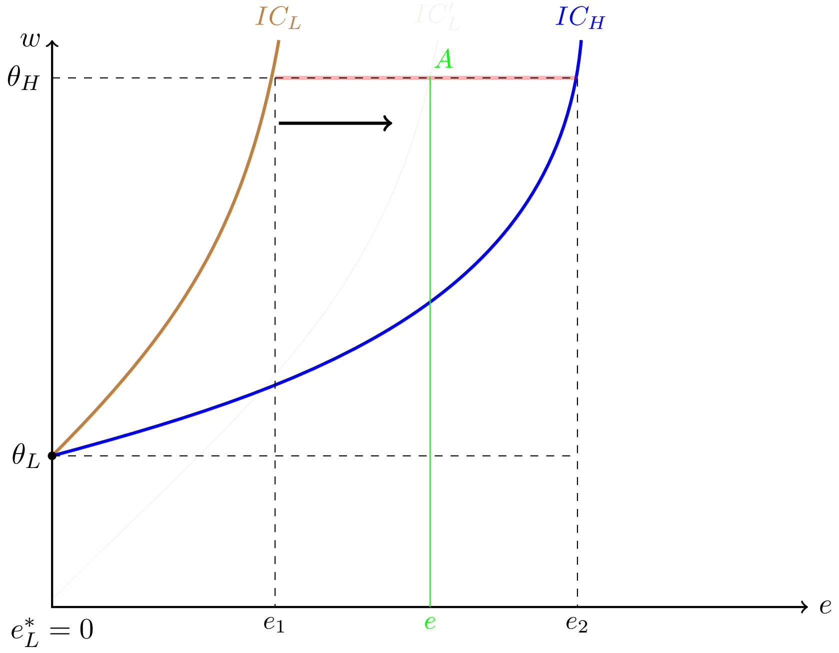
For a start, we check if the separating equilibrium $e_L^* = 0$, $e_H^* = e_2$ survives the Intuitive Criterion. Hence, let us consider any off-the-equilibrium message $e_2 \in (e_1, e_2)$. (the one in Green color).



Step one We check which type could potentially benefit from sending the off-the-equilibrium path message e. It is easy to check that θ_L type can never benefit from choosing this education level, because the best this type can get is strictly below that he can obtains in equilibrium:

$$\underbrace{u_L^*(\theta_L)}_{\text{Equilibrium payoff}} \quad > \quad \underbrace{\max_{w\in W^*(\Theta,m)} u_L(e,w,\theta_L)}_{\text{Max payoff from deviating to } e} \quad .$$

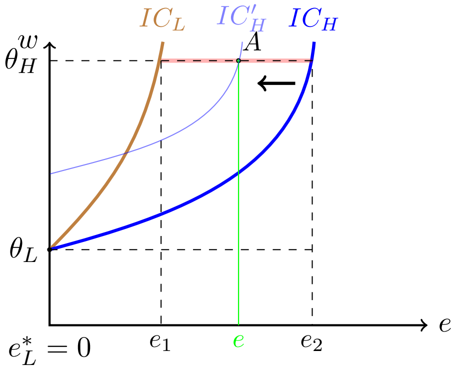
This can be easily seen from the figure below.



On the other hand, the θ_H type can benefit from such an off-equilibrium message if the firm would pay the best possible wage θ_H,

$$\underbrace{u_H^*(\theta_H)}_{\text{Equilibrium payoff}} \quad < \quad \underbrace{\max_{w\in W^*(\Theta,m)} u_H(e,w,\theta_H)}_{\text{Max payoff from deviating to } e} \quad .$$

This can also be seen from the figure below.



For the firm, it understand the worker's problem. Therefore, education levels in $e \in (e_1, e_2)$ can only come from the θ_H -worker. So , the firm's belief upon observing e concentrates on the type that could potentially benefit from such a deviation, θ_H:

$$\Theta^{**}(e) = \{\theta_H\}.$$

Now that the firm understands that the off-the-equilibrium message e can only be chosen by θ_H worker, the best response for the firm is to offer $w(e) = \theta_H$ for any worker with education level e. Knowing that the firm will offer $w(e) = \theta_H$, the best response for θ_H type worker is to choose education level e instead of e_2 because

$$\underbrace{\min_{w \in W^*(\Theta^{**}(e),e)} u_H(e, w, \theta_H)}_{\theta_H - c(e,\theta_H)} > \underbrace{u^*_H(\theta_H)}_{\theta_H - c(e_2,\theta_H)} .$$

The reason is simple, since $e_2 > e$, and the cost of eduction for e_2 is also higher, $C(\theta_H, e_2) > C(\theta_H, e)$. Hence

$$\theta_H - C(\theta_H, e_2) < \theta_H - C(\theta_H, e).$$

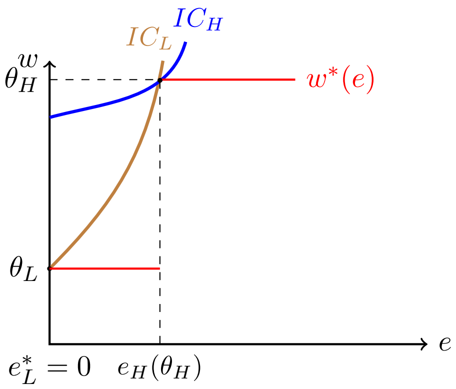
Intuitively, the lowest payoff θ_H type obtains from deviating to e is higher than the equilibrium payoff. Thus, we conclude that the separating equilibrium

$$\{e^*_L(\theta_L), e^*_H(\theta_H)\} = \{0, e_2)\}$$

violates IC.

In fact, similar procedure can be used to show that all separating equilibria in which the θ_H worker sends $e \in (e_1, e_2)$ violate the Intuitive Criterion. (Practice yourself). There is a unique separating equilibrium that survives the IC. The unique separating equilibrium surviving the Intuitive

Criterion is that in which the θ_H type worker sends $e = e_1$. This equilibria is usually referred as the efficient outcome (or Riley outcome).



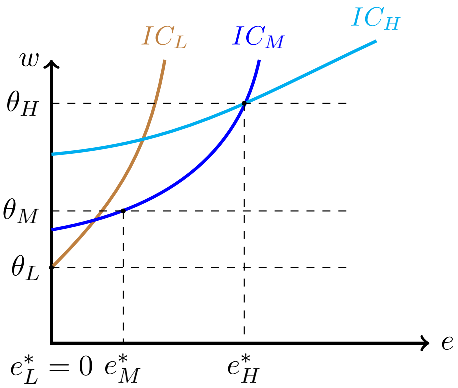
6.4.4 *D*1

In this subsection we consider the case of $n = 3$. We show that the Intuitive Criterion fails to eliminate unreasonable equilibria in this case. We introduce another refinement criterion that can eliminate unreasonable equilibria for any n types, not just the $n = 3$ case.

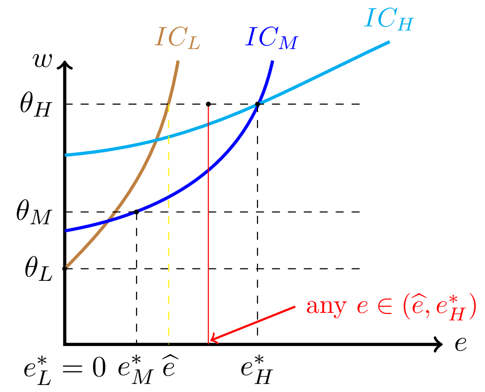
Suppose the worker has 3 types, θ_L, θ_M and θ_H with

$$\theta_L < \theta_M < \theta_H.$$

The figure below presents one separating equilibrium $e_L^* = 0,\ e_M^*,\ e_H^*$.



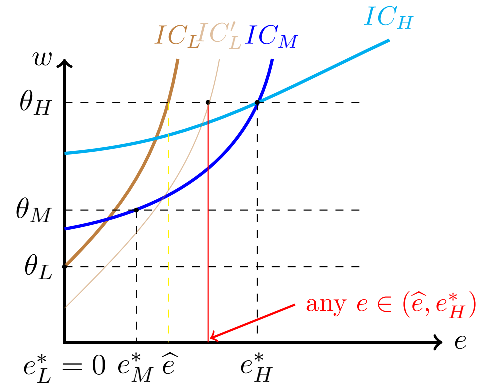
First, let us check whether the equilibrium survives the Intuitive Criterion, by choosing an off-the-equilibrium message $e \in (\widehat{e}, e_H^*)$



Step one: Let's consider workers' incentives to send the off-the-equilibrium message $e \in (\widehat{e}, e^*_H)$. First, the message e is equilibrium dominated for θ_L-type worker as

$$\underbrace{u^*_L(\theta_L)}_{\text{Equilibrium payoff}} > \underbrace{\max_{w \in W^*(\Theta,m)} u_L(e, w, \theta_L)}_{\text{Max payoff from deviating to } e}.$$

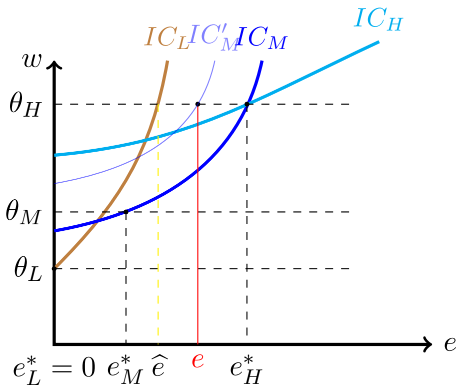
That is, even if the firm were to offer the highest wage to the worker, θ_L would still be worse off from choosing the eduction level e. See the figure below.



Second, the θ_M-type could send the message $e \in (\widehat{e}, e^*_H)$ because for the θ_M type,

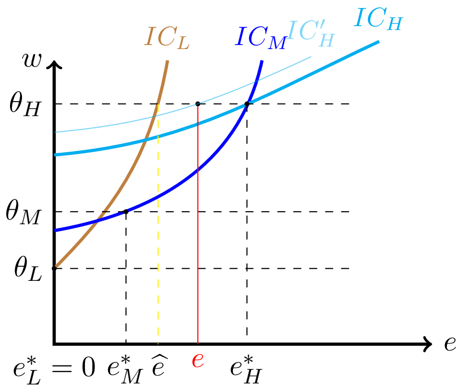
$$\underbrace{u^*_M(\theta_M)}_{\text{Equilibrium payoff}} < \underbrace{\max_{w \in W^*(\Theta,m)} u_M(e, w, \theta_M)}_{\text{Max payoff from deviating to } e}.$$

That is, if the firm were to offer the highest possible wage $w(e) = \theta_H$, type θ_M would be better off than from choosing the equilibrium education level e^*_M.



Finally, θ_H type could send the message $e \in (\widehat{e}, e^*_H)$ as well because

$$\underbrace{u^*_H(\theta_H)}_{\text{Equilibrium payoff}} \quad < \quad \underbrace{\max_{w \in W^*(\Theta, m)} u_H(e, w, \theta_H)}_{\text{Max payoff from deviating to } e} \quad .$$



Hence, after observing $e \in (\widehat{e}, e^*_H)$, the firm's beliefs will concentrate their beliefs on those types of workers for which these education levels are not equilibrium dominated: θ_M and θ_H:

$$\Theta^{**} = \{\theta_M, \theta_H\}.$$

Second step: Now that the firms' belief concentrates on only two types, the lowest wage the firm could offer upon observing any education level $e \in (\widehat{e}, e^*_H)$ is

$$\theta_M = \min\{w | w \in W^*(\Theta^{**}(e), e)\}.$$

First, given firm's offer θ_M, θ_M type worker has no incentives to deviate towards e as

$$\min_{w \in W^*(\Theta^{**}(e), e)} u_M(e, w, \theta_M) < u^*_M(\theta_M).$$

Next, given $w \in W^*(\Theta^{**}(e), e)$, θ_H type worker has no incentives to deviate towards e as well because

$$\min_{w \in W^*(\Theta^{**}(e), e)} u_H(e, w, \theta_H) < u_H^*(\theta_H).$$

Hence, there is no type of worker $\theta \in \Theta^{**}$ for whom deviation to $e \in (\widehat{e}, e_H^*)$ is profitable. So the equilibrium $e_L^* = 0$, e_M^*, e_H^* survives the Intuitive Criterion.

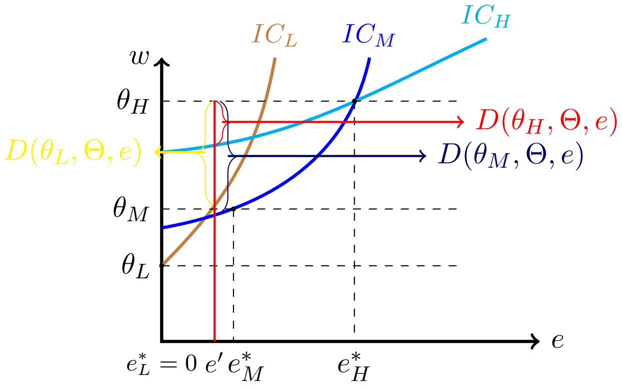
Next, let us now check if the previous separating equilibrium (e_L^*, e_M^*, e_H^*) survives the D1-Criterion. The main idea of $D1$ criterion is similar to IC in that for any fixed equilibrium it check whether any types would want to deviate and send an off-the-equilibrium message. Like IC, it also uses a two-step procedure. In the first step, it identifies the type that are most likely to send the off-the-equilibrium message. However, unlike the Intuitive criterion which asks who would never send such an message, it asks which type would be mostly likely send this message. That is, which types would benefit more often than the other types. This is the only difference between the two criterion. The second step of D1 is the same as the Intuitive criterion. Let's illustrate the criterion with our example.

Step one: Given the off-the-equilibrium message e', we identify the type(s) that will benefit more often from choosing the off-the-equilibrium message than sticking to the equilibrium message. That is, we need to construct sets $D(\theta_k, \widehat{\Theta}, e')$ for $k = L, M, H$, representing the set of wage offers for which a θ_k-worker is better-off when he deviates towards message e' than when he sends his equilibrium message:

$$D(\theta_k, \widehat{\Theta}, e') \equiv \{w \in [\theta_L, \theta_H] | u_k(e', w, \theta_k) > u_k^*(\theta_k)\}.$$

Also let be the set of wage offers for which θ_k is just indifferent between sending message e' and not,

$$D^o(\theta_k, \widehat{\Theta}, e') \equiv \{w \in [\theta_L, \theta_H] | u_k(e', w, \theta_k) = u_k^*(\theta_k)\}.$$



We see from the figure that

$$D(\theta_H, \widehat{\Theta}, e') \bigcup D^o(\theta_H, \widehat{\Theta}, e') \subset D(\theta_M, \widehat{\Theta}, e').$$

That is, θ_M type has more incentives to deviate to e' than θ_H type Similarly, we see that

$$D(\theta_L, \widehat{\Theta}, e') \bigcup D^o(\theta_L, \widehat{\Theta}, e') \subset D(\theta_M, \widehat{\Theta}, e').$$

So θ_M type has more incentives to deviate to e' than θ_L type. Applying the D1 criterion, the θ_M type is the most likely to deviate to e'

$$\Theta^{**}(e') = \{\theta_M\}.$$

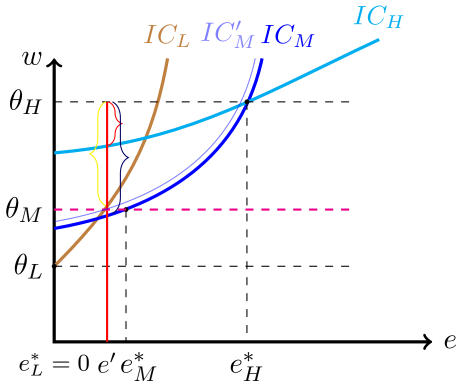
Second step: Given the firm's belief $\Theta^{**}(e')$, the firm should offer

$$w(e') = \theta_M.$$

Knowing that choosing the education level would receive the wage offer $w(e') = \theta_M$, the θ_M worker is better off choosing e' instead of the equilibrium education level e^*_M:

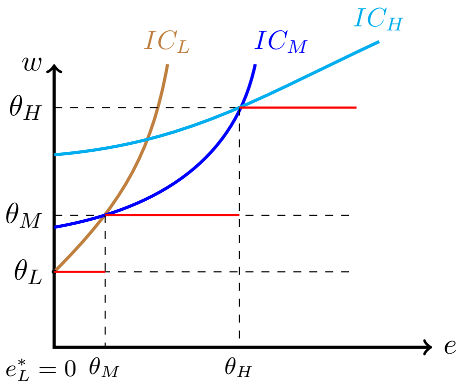
$$\min_{w \in W^*(\Theta^{**}(e'), e')} u_M(e', w, \theta_M) > u^*_M(\theta_M).$$

This can be seen from the figure below.



So the equilibrium (e^*_L, e^*_M, e^*_H) violates the D1 criterion

Applying the D1 criterion, we can eliminates most of the separating equilibria. There is a unique equilibrium that satisfies D1.



6.5 Application of Intuitive criterion to insurance model

1. Briefly, a sequential equilibrium satisfies Intuitive Criterion if no type of sender could obtain a payoff higher than his equilibrium payoff were he to choose a nonequilibrium message and the receiver responds with an optimal reply to the belief that imputes zero probability to Nature's choice of those types that can not gain from such a deviation regardless of the receiver's responses.

2. Intuitive criterion in current situation: Sequential equilibrium $(\psi_l, \psi_h, \sigma, \beta)$, in which utilities for low-risk and high-risk consumers are u_l^*, u_h^* respectively, satisfy the intuitive criterion if for all ψ ($\psi \neq \psi_l$ or $\psi \neq \psi_h$),

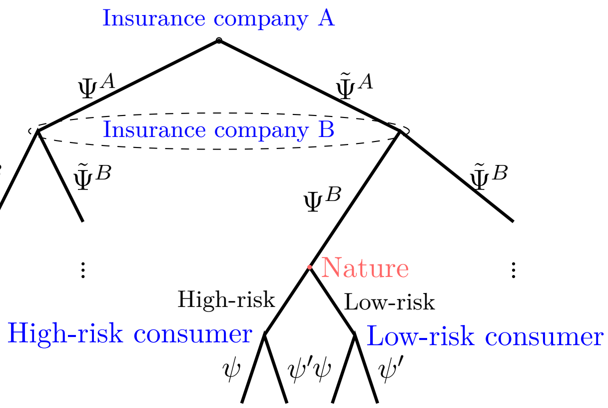
$$u_l(\psi) > u_l^* \text{ and } u_h(\psi) < u_h^* \Longrightarrow \beta(\psi) = 1$$

$$u_h(\psi) > u_h^* \text{ and } u_l(\psi) < u_l^* \Longrightarrow \beta(\psi) = 0$$

3. Theorem 8.3. (Jehle& Reny) There is a unique policy pair (ψ_l, ψ_h) that can be supported by a sequential equilibrium satisfying the intuitive criterion. And this equilibrium is the best separating equilibrium for the low-risk consumer.

6.6 Screening

1. Model: assume two insurance companies the engage in Bertrand competition.



2. Insurance companies offer consumers a menu of policies to screen consumers by tailing the contract such that high-risk type chooses one policy and low-risk choose another.

3. Cream skimming occurs when one insurance company takes strategic advantage of the set of policies offered by the other by offering a policy that would attract away only the low-risk consumers from the competing company.

4. Lemma 8.2. (Jehle & Reny) Insurance companies earn zero expected profits in equilibrium.

   Follows directly from Bertrand competition.

5. Theorem 8.4. (Jehle & Reny) Pooling equilibrium does not exist.

6. Theorem 8.5. (Jehle & Reny) Suppose ψ_l^* and ψ_h^* are the policies chosen by low- and high-risk consumers in a pure strategy separating equilibrium. Then $\psi_h^* = \psi_h^c$ and $\psi_l^* = \bar{\psi}_l$, where $\bar{\psi}_l$ is the best separating equilibrium for consumers in the insurance signaling game.

7. Theorem 8.6. (Jehle & Reny) No pure strategy equilibrium may exist if the proportion of high-risk is too low.

6.7 Screening by monopoly: nonlinear pricing

6.7.1 Two types

The firm delegates to an agent the production of q units of consumption goods. The value of q units for the principal is $S(q)$ with the the usual assumptions on S. The production cost of the agent is unknown to the principal, but it is common knowledge that it follows distribution F. For simplicity, suppose there are only two types of cost:

$$C(q,\theta) = \underline{\theta} q \qquad \text{with probability } \nu; \tag{6.1}$$

$$C(q,\theta) = \bar{\theta} q \qquad \text{with probability } 1-\nu. \tag{6.2}$$

We assume that $\Delta\theta = (\bar{\theta} - \underline{\theta}) > 0$.

The problem for the principal is to maximize profit subject to the agent's IC constraint and IR constraints:

$$\max_{\bar{q},\underline{q},\bar{t},\underline{t}} \Pi = \nu[S(\underline{q}) - \underline{t}] + (1-\nu)[S(\bar{q}) - \bar{t}]. \tag{6.3}$$

The agent's IC constraints are:

$$\underline{t} - \underline{\theta}\underline{q} \geq \bar{t} - \underline{\theta}\bar{q}; \tag{6.4}$$

$$\bar{t} - \bar{\theta}\bar{q} \geq \underline{t} - \bar{\theta}\underline{q}. \tag{6.5}$$

$$\tag{6.6}$$

The agent's participation constraints are:

$$\underline{t} - \underline{\theta}\underline{q} \geq 0; \tag{6.7}$$

$$\bar{t} - \bar{\theta}\bar{q} \geq 0. \tag{6.8}$$

$$\tag{6.9}$$

First, consider the complete information case. In this case, since the principal knows the agent's type, profit-maximization requires that for different types produces different quantity with:

$$S'(\underline{q}) = \underline{\theta}, \qquad S'(\bar{q}) = \bar{\theta}.$$

Next, we consider the case of pooling contract. Since the two types produce the same amount, the only constraint that needs to be satisfied is the inefficient type's participation constraint:

$$\underline{t} = \bar{t} = \bar{\theta} q,$$

so profit-maximization yields the quantity:

$$S'(q) = \bar{\theta}.$$

Third, let's analyze the situation in which the principal wants to shut down the least efficient type. So

$$\underline{t} = \bar{t} = \underline{\theta} q,$$

and profit-maximization yields the quantity:

$$S'(q) = \underline{\theta}.$$

In this case, the inefficient type does not produce because its participation constraint is violated:

$$\underline{\theta} q - \bar{\theta} q < 0,$$

and producing any positive amount results in a loss.

Finally, consider the separating contract case. In this case, the principal only needs to give information rent to the efficient type to induce separation. The inefficient type receives no rent, so its participation constraint will be satisfied with equality:

$$\bar{t} - \bar{\theta}\bar{q} = 0.$$

The information rent to the efficient type is just large enough for voluntary separation:

$$\underline{t} - \underline{\theta}\underline{q} = \bar{t} - \underline{\theta}\bar{q} \Longrightarrow \underline{t} = \bar{\theta}\bar{q} + \underline{\theta}(\underline{q} - \bar{q}).$$

So the problem for the principal becomes:

$$\max_{\underline{q},\bar{q}} \Pi = \nu[S(\underline{q}) - \bar{\theta}\bar{q} - \underline{\theta}(\underline{q} - \bar{q})] + (1-\nu)[S(\bar{q}) - \bar{\theta}\bar{q}].$$

Solving the problem gives:

$$S'(\underline{q}) = \underline{\theta} \tag{6.10}$$

$$S'(\bar{q}) = \frac{(1-\nu)\bar{\theta} + \nu(\bar{\theta} - \underline{\theta})}{1-\nu} = \bar{\theta} + \frac{\nu(\bar{\theta} - \underline{\theta})}{(1-\nu)} > \bar{\theta}. \tag{6.11}$$

This implies that $\bar{q}$ will be smaller than the first-best output as $S'(\cdot)$ is a decreasing function.

The information rent to the two types of agents are respectively:

$$U_h = \underline{t} - \underline{\theta}\underline{q} = (\bar{\theta} - \underline{\theta})\bar{q}, \qquad U_l = \bar{t} - \bar{\theta}\bar{q} = 0.$$

We can rewrite the objective function for the principal as:

$$\Pi = \{\nu[S(\underline{q}) - \underline{\theta}\underline{q}] + (1-\nu)[S(\bar{q}) - \bar{\theta}\bar{q}]\} - \{\nu U_h + (1-\nu)U_l\}.$$

The first part is the social welfare from trade, while the second part is the expected information rent. So the problem for the principal is to maximize the difference between total social welfare and information rent.

The inefficient type receives no information rent, and the efficient type's rent

$$U(\underline{\theta}) = \bar{q}(\bar{\theta} - \underline{\theta}),$$

the amount the efficient type can obtain from pretending to be an inefficient type and producing $\bar{q}$. Thus, the problem for the principal can be further simplified as

$$\Pi = \{\nu[S(\underline{q}) - \underline{\theta}\underline{q}] + (1-\nu)[S(\bar{q}) - \bar{\theta}\bar{q}]\} - \nu U_h.$$

The only difference between this problem and the maximization problem under complete information is the extra information rent term, which depends on the size of $\bar{q}$. This clearly implies that in general, $\bar{q}$ will be inefficient. In particular, it is determined by the condition:

$$(1-\nu)[S'(\bar{q}) - \bar{\theta}] = \nu(\bar{\theta} - \underline{\theta}).$$

Increasing in $\bar{q}$ increases allocation efficiency, but at the same time, also increases information rent to the efficient type.

6.7.2 A continuum of types

Now suppose there is a continuum of types. In particular, θ follows a continuous distribution F on the interval $[\underline{\theta}, \bar{\theta}]$. The principal now choose a menu of contracts $(t(\theta), q(\theta))$ to maximize profits subject to the agent's IC constraints and participation constraints. We consider only contracts that satisfy the Revelation Principle, that is, the agent reveals type truthfully in equilibrium.

First, note that given $(t(\theta), q(\theta))$, an agent of θ can choose any contract for type $\tilde{\theta}$ to maximize utility:

$$\max_{\tilde{\theta}} U(\theta, \tilde{\theta}) \equiv t(\tilde{\theta}) - \theta q(\tilde{\theta}).$$

The First-order condition is:

$$\frac{dU(\theta, \tilde{\theta})}{d\tilde{\theta}} = \frac{t(\tilde{\theta})}{d\tilde{\theta}} - \theta \frac{dq(\tilde{\theta})}{d\tilde{\theta}}.$$

For truth revealing, her utility is maximized at $\tilde{\theta} = \theta$,

$$\frac{t(\theta)}{d\theta} - \theta \frac{dq(\theta)}{d\theta} = 0 \Longrightarrow \frac{t(\theta)}{d\theta} = \theta \frac{dq(\theta)}{d\theta}.$$

Using Leibnitz Integral Rule and integrating the two sides of the equation, we have:[3]

$$t(\theta) = t(\tilde{\theta}) + \int_{\tilde{\theta}}^{\theta} x \frac{dq(x)}{dx} dx.$$

---

[3] **The Leibnitz integral rule**:

$$\frac{d}{dx}\int_{A(x)}^{B(x)} f(x,t)dt = f(x, B(x))B'(x) - f(x, A(x))A'(x) + \int_{A(x)}^{B(x)} \frac{\partial f(x,t)}{\partial x} dt.$$

To see that the constant term on the right-hand side equals $t(\tilde{\theta})$, simply note that second term on the right side of the equation would be zero when $\theta = \tilde{\theta}$. Furthermore, using the Product rule we have:[4]

$$\int_{\tilde{\theta}}^{\theta} x \frac{dq(x)}{dx} dx = \theta q(\theta) - \tilde{\theta} q(\tilde{\theta}) - \int_{\tilde{\theta}}^{\theta} q(x) dx.$$

Thus, we have

$$t(\theta) - t(\tilde{\theta}) = \theta q(\theta) - \tilde{\theta} q(\tilde{\theta}) - \int_{\tilde{\theta}}^{\theta} q(x) dx. \tag{6.12}$$

This is equivalent to

$$t(\theta) - \theta q(\theta) = t(\tilde{\theta}) - \theta q(\tilde{\theta}) + (\theta - \tilde{\theta}) q(\tilde{\theta}) - \int_{\tilde{\theta}}^{\theta} q(x) dx. \tag{6.13}$$

IC constraints immediately leads to the result that $q(\tilde{\theta})$ is nonincreasing. To see this, note that for any pair $\theta, \tilde{\theta}$, the IC constraints are

$$t(\theta) - \theta q(\theta) \geq t(\tilde{\theta}) - \theta q(\tilde{\theta}); \tag{6.14}$$

$$t(\tilde{\theta}) - \tilde{\theta} q(\tilde{\theta}) \geq t(\theta) - \tilde{\theta} q(\theta). \tag{6.15}$$

Combining the two conditions we have:

$$(\theta - \tilde{\theta})[q(\theta) - q(\tilde{\theta})] \leq 0.$$

Since $q(\tilde{\theta})$ is nonincreasing in $\tilde{\theta}$, we have

$$(\theta - \tilde{\theta}) q(\tilde{\theta}) - \int_{\tilde{\theta}}^{\theta} q(x) dx \geq 0.$$

Thus, it follows immediately from condition (**??**) that for all $\tilde{\theta}$:

$$t(\theta) - \theta q(\theta) \geq t(\tilde{\theta}) - \theta q(\tilde{\theta}).$$

From the previous discussion, we know that for a type θ agent,

$$U(\theta) = t(\theta) - \theta q(\theta) = t(\underline{\theta}) - \underline{\theta} q(\underline{\theta}) - \int_{\underline{\theta}}^{\theta} q(x) dx.$$

---

[4] From calculus, the **Product Rule** tells us that

$$\frac{d}{dv}[F^{N-1}(v)b(v)] = (N-1)F^{N-2}(v)f(v)b(v) + F^{N-1}(v)b'(v) \Longrightarrow$$

$$F^{N-1}(v)b(v) = \int [(N-1)F^{N-2}(v)f(v)b(v) + F^{N-1}(v)b'(v)]dv.$$

This gives the change rate of information rent as

$$\dot{U} = \frac{dU(\theta)}{d\theta} = -q(\theta).$$

Thus, the optimization problem facing the principal can be expressed as

$$\begin{aligned} \max_{\{(q(\cdot),U(\cdot))\}} & \int_{\underline{\theta}}^{\bar{\theta}} \{S(q(\theta)) - \theta q(\theta) - U(\theta)\} f(\theta) d\theta \qquad (6.16) \\ s.t. \quad & \dot{U} = -q(\theta); \\ & \dot{q}(\theta) \leq 0; \\ & U(\theta) \geq 0. \end{aligned}$$

For time being, we ignore the condition that q is nonincreasing first. Type $\bar{\theta}$ receives zero information rent, $U(\bar{\theta}) = 0$. Using the fact that

$$U(\bar{\theta}) - U(\theta) = -\int_{\theta}^{\bar{\theta}} q(x)dx,$$

we have

$$U(\theta) = \int_{\theta}^{\bar{\theta}} q(x)dx.$$

Using this equality, we can rewrite the principal's profit function as

$$\int_{\underline{\theta}}^{\bar{\theta}} \left\{ S(q(\theta)) - \theta q(\theta) - \int_{\theta}^{\bar{\theta}} q(x)dx \right\} f(\theta)d\theta.$$

We can integrate the profit function by parts:

$$\begin{aligned} & \int_{\underline{\theta}}^{\bar{\theta}} \left\{ S(q(\theta)) - \theta q(\theta) - \int_{\theta}^{\bar{\theta}} q(x)dx \right\} f(\theta)d\theta \\ = & \int_{\underline{\theta}}^{\bar{\theta}} \{S(q(\theta)) - \theta q(\theta)\} f(\theta)d\theta - \int_{\underline{\theta}}^{\bar{\theta}} \int_{\theta}^{\bar{\theta}} q(x)dx f(\theta)d\theta \\ = & \int_{\underline{\theta}}^{\bar{\theta}} \{S(q(\theta)) - \theta q(\theta)\} f(\theta)d\theta - \int_{\underline{\theta}}^{\bar{\theta}} \left[\int_{\underline{\theta}}^{x} f(\theta)d\theta \right] q(x)dx \\ = & \int_{\underline{\theta}}^{\bar{\theta}} \{S(q(\theta)) - \theta q(\theta)\} f(\theta)d\theta - \int_{\underline{\theta}}^{\bar{\theta}} q(x)F(x)dx \\ = & \int_{\underline{\theta}}^{\bar{\theta}} \{S(q(\theta)) - \theta q(\theta)\} f(\theta)d\theta - \int_{\underline{\theta}}^{\bar{\theta}} q(\theta)F(\theta)d\theta \\ = & \int_{\underline{\theta}}^{\bar{\theta}} \left\{ S(q(\theta)) - q(\theta) \left[\theta + \frac{F(\theta)}{f(\theta)} \right] \right\} f(\theta)d\theta \end{aligned}$$

Maximizing the profit function with respect to q we have the first-order condition:

$$S'(q(\theta)) = \theta + \frac{F(\theta)}{f(\theta}.$$

Provided that monotone hazard property rate condition holds, i.e.,

$$\frac{d}{d\theta}\frac{F(\theta)}{f(\theta)} \geq 0,$$

$q(\theta)$ is indeed decreasing in θ.

Even with a continuum of types, there is no quantity distortion for the most efficient type, as

$$\frac{F(\underline{\theta})}{f(\underline{\theta})} = 0$$

and so

$$S'(q(\underline{\theta})) = \underline{\theta}.$$

But there is distortion for all the other types. All types, except the least efficient types obtains an information rent as for all $\theta < \bar{\theta}$,

$$U(\theta) = \int_{\theta}^{\bar{\theta}} q(x)dx > 0.$$

Note that we can also use optimal control to solve for the optimal quantity. In this case, the state variable is

$$U(q(\theta)) = -\int_{\underline{\theta}}^{\theta} q(x)dx,$$

and the control variable is $q(\theta)$. The Hamiltonian can be written as[5]

$$H(q, U, \theta) = [S(q(\theta)) - \theta q(\theta) - U(q(\theta))]f(\theta) - \lambda q(\theta). \tag{6.17}$$

From the Pontryagin principle we have:

$$\lambda'(\theta) = -\frac{\partial H}{\partial U} = f(\theta).$$

---

[5] See page 127, Calculus and variation byKamien and Schwartz. The Hamiltonian is of the form:

$$H(t, x(t), u(t), \lambda(t) = f(t, x, u + \lambda g(t, x, u),$$

with $x'(t) = g(t, x, u)$. $x(t)$ is the state variable, and $u(t)$ is the control variable while $\lambda(t)$ is the costate variable. The conditions are:

$$\begin{aligned} \frac{\partial H}{\partial u} =& 0 \Longrightarrow f_u + \lambda g_u = 0; \\ -\frac{\partial H}{\partial x} =& \lambda' \Longrightarrow \lambda'(t) = -(f_x + \lambda g_x); \\ \frac{\partial H}{\partial \lambda} =& x'(t) \Longrightarrow x'(t) = \frac{\partial H}{\partial \lambda} = g(t, x, u); \\ x(t_0) =& x_0 \qquad \lambda(t_1) = 0. \end{aligned}$$

From this we immediately have:

$$\lambda(\theta)=\int_{\underline{\theta}}^{\theta} f(x)dx = F(\theta).$$

Also

$$\frac{\partial H}{\partial q}=[S'(q(\theta))-\theta]f(\theta)-\lambda=0\Longrightarrow$$

$$S'(q(\theta))=\theta+\frac{\lambda(\theta)}{f(\theta)}=\theta+\frac{F(\theta)}{f(\theta)}.$$

6.8 Bunching

Now suppose there are 3 types: $(\underline{\theta},\widehat{\theta},\bar{\theta})$ with probability $(\underline{\nu},\widehat{\nu},\bar{\nu})$ respectively. For simplicity, we assume that

$$\Delta\theta=\bar{\theta}-\widehat{\theta}=\widehat{\theta}-\underline{\theta}.$$

The principal's problem is to maximize profit

$$\max_{(\bar{q},\widehat{q},\underline{q})}\Pi=\underline{\nu}[S(\underline{q})-\underline{t}]+\widehat{\nu}[S(\widehat{q})-\widehat{t}]+\bar{\nu}[S(\bar{q})-\bar{t}], \tag{6.18}$$

subject to the agents' IC and participation constraints.

The IC constraints:

$$\begin{aligned}
\bar{U} =&\bar{t}-\bar{\theta}\bar{q}\geq\widehat{t}-\bar{\theta}\widehat{q};\\
\bar{U} \geq&\underline{t}-\bar{\theta}\underline{q};\\
\widehat{U} =&\widehat{t}-\widehat{\theta}\widehat{q}\geq\bar{t}-\widehat{\theta}\bar{q};\\
\widehat{U} \geq&\underline{t}-\widehat{\theta}\underline{q};\\
\underline{U} =&\underline{t}-\underline{\theta}\underline{q}\geq\widehat{t}-\underline{\theta}\widehat{q};\\
\underline{U} \geq&\bar{t}-\underline{\theta}\bar{q}.
\end{aligned}$$

Note that the IC constraints indicates that

$$(\widehat{\theta}-\bar{\theta})(\bar{q}-\widehat{q})\geq 0,$$

$$(\widehat{\theta}-\underline{\theta})(\underline{q}-\widehat{q})\geq 0;$$

hence $\underline{q}\geq\widehat{q}\geq\bar{q}$. The participation constraints are simply

$$\bar{U}\geq 0,\quad \widehat{U}\geq 0,\quad \underline{U}\geq 0.$$

In this example, we only need to consider local IC constraints, that is, IC constraint involves adjacent types. By contrast, global constraints involves IC constraints all pairs of types, not jut local constraints.

We can simplify the IC and participation constraints as:

$$\bar{U} = 0, \quad \widehat{U} = \bar{q}\Delta\theta, \quad \underline{U} = (\bar{q} + \widehat{q})\Delta\theta.$$

Thus,

$$\bar{t} = \bar{\theta}\bar{q}, \quad \widehat{t} = \widehat{\theta}\widehat{q} + \bar{q}\Delta\theta, \quad \underline{t} = \underline{\theta}\underline{q} + \bar{q}\Delta\theta + \widehat{q}\Delta\theta.$$

Hence we can rewrite the principal's problem as

$$\max_{(\bar{q},\widehat{q},\underline{q})} \underline{\nu}[S(\underline{q}) - \underline{\theta}\underline{q} - \bar{q}\Delta\theta - \widehat{q}\Delta\theta] + \widehat{\nu}[S(\widehat{q}) - \widehat{\theta}\widehat{q} - \bar{q}\Delta\theta] + \bar{\nu}[S(\bar{q}) - \bar{\theta}\bar{q}].$$

First-order condition gives:

$$S'(\underline{q}) = \underline{\theta}; \tag{6.19}$$

$$S'(\widehat{q}) = \widehat{\theta} + \frac{\underline{\nu}\Delta\theta}{\widehat{\nu}}; \tag{6.20}$$

$$S'(\bar{q}) = \bar{\theta} + \frac{(\underline{\nu} + \widehat{\nu})\Delta\theta}{\bar{\nu}}. \tag{6.21}$$

By assumption, $S'(\cdot)$ is a decreasing function. From previous discussion, we know that the IC constraints imply q is nonincreasing in θ. However, when $\bar{\nu}\underline{\nu} > \widehat{\nu}$,

$$\begin{aligned} \widehat{\theta} + \frac{\underline{\nu}\Delta\theta}{\widehat{\nu}} - \left[\bar{\theta} + \frac{(\underline{\nu} + \widehat{\nu})\Delta\theta}{\bar{\nu}}\right] &= \Delta\theta\left[\frac{\bar{\nu}\underline{\nu} - \widehat{\nu}(\underline{\nu} + \widehat{\nu} + \bar{\nu})}{\widehat{\nu}\bar{\nu}}\right] \\ &= \Delta\theta\left[\frac{\bar{\nu}\underline{\nu} - \widehat{\nu}}{\widehat{\nu}\bar{\nu}}\right] \end{aligned}$$

Hence, if $\widehat{\nu} > \bar{\nu}\underline{\nu}$, $S'(\widehat{q}) < S'(\bar{q})$ and $\widehat{q} > \bar{q}$. However, if $\widehat{\nu} \leq \bar{\nu}\underline{\nu}$, to satisfy the IC constraint, we have

$$\widehat{q} = \bar{q} = q^p.$$

To find q^p, replacing $\widehat{q} = \bar{q} = q^p$ in the objective function, we have

$$\max_{(\bar{q},\widehat{q},\underline{q})} \underline{\nu}[S(\underline{q}) - \underline{\theta}\underline{q} - 2q^p\Delta\theta] + \widehat{\nu}[S(q^p) - \widehat{\theta}q^p - q^p\Delta\theta] + \bar{\nu}[S(q^p) - \bar{\theta}q^p].$$

Taking first-order condition, we have

$$-2\underline{\nu}\Delta\theta + \widehat{\nu}[S'(q^p) - (\widehat{\theta} + \Delta\theta) + \bar{\nu}[S'(q^p) - \bar{\theta}] = 0 \Longrightarrow$$

$$S'(q^p) = \bar{\theta} + \frac{2\underline{\nu}\Delta\theta}{\widehat{\theta} + \bar{\theta}}.$$

- Inefficient type receives no rent:

$$\bar{t} - \bar{\theta}\bar{q} = 0.$$

- Information rent to the efficient type:

$$\underline{t} = \underline{\theta}\underline{q} + \Delta\theta\bar{q}.$$

- So the problem for the principal:

$$\max_{\underline{q},\bar{q}} \Pi = \nu[S(\underline{q}) - \underline{\theta}\underline{q} - \bar{q}(\bar{\theta} - \underline{\theta})] + (1-\nu)[S(\bar{q}) - \bar{\theta}\bar{q}].$$

- Solving the problem gives:

$$\begin{aligned} S'(\underline{q}) =& \underline{\theta} \\ S'(\bar{q}) =& \frac{(1-\nu)\bar{\theta} + \nu(\bar{\theta} - \underline{\theta})}{1-\nu} = \bar{\theta} + \frac{\nu(\bar{\theta} - \underline{\theta})}{(1-\nu)} > \bar{\theta}. \end{aligned}$$

Chapter 7

Moral hazard

7.1 Insurance market

1. Moral hazard is an economic phenomenon in which the agent can take an action that is unobservable to the principal but has a detrimental effect on the principal's payoff.

2. A car accident, or a house fire, may not be completely avoided because of precaution taken by individual insured, however, the probability is almost always affected by insured's actions. When individuals are fully insured against risks, they will not capture the full benefits of efforts to reduce risks and thus, may take less care than they would have done otherwise.

3. Example 1: Insurance symmetric information

    - One insurance company and one consumer.
    - Consumer initial wealth W. L losses, $l \in \{0, 1, \ldots, L\}$, each occurring with probability $\pi_l(e) > 0$.
    - Disutility of effort: $e \in \{0, 1\}$ and $d(1) > d(0)$.
    - Monotone likelihood ratio: Assume $\pi_l(0)/\pi_l(1)$ is strictly increasing in $l \in \{0, 1, \ldots, L\}$.
    - Insurance company chooses policy $(p, B_0, B_1, \ldots, B_L)$ to maximize profit.
    - With symmetric information:

$$\max_{e,p,B_l} p - \sum_{l=0}^{L} \pi_l(e) B_l, \qquad \textit{subject to}$$

$$\sum_{l=1}^{L} \pi_l(e) u(w-p-l+B_l) - d(e) \geq \bar{u}.$$

The optimal policy solves:

$$\mathcal{L} = p - \sum_{l=0}^{L} \pi_l(e) B_l + \lambda \left[\sum_{l=1}^{L} \pi_l(e) u(w-p-l+B_l) - d(e) - \bar{u} \right].$$

First order conditions:

$$\frac{\partial \mathcal{L}}{\partial p} = 1 - \lambda \left[\sum_{l=1}^{L} \pi_l(e) u'(w-p-l+B_l) \right] = 0, \tag{7.1}$$

$$\frac{\partial \mathcal{L}}{\partial B_l} = -\pi_l(e) + \lambda \pi_l(e) u'(w-p-l+B_l) = 0, \qquad \forall l \geq 0, \tag{7.2}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \sum_{l=1}^{L} \pi_l(e) u(w-p-l+B_l) - d(e) - \bar{u} \geq 0. \tag{7.3}$$

Thus it is optimal to have

$$B_l = l \qquad forl = 0, 1, \ldots, L.$$

4. Asymmetric information

$$\max_{e,p,B_l} p - \sum_{l=0}^{L} \pi_l(e) B_l, \qquad \textit{subject to}$$

$$\sum_{l=1}^{L} \pi_l(e) u(w-p-l+B_l) - d(e) \geq \bar{u};$$

$$\sum_{l=1}^{L} \pi_l(e) u(w-p-l+B_l) - d(e) \geq \sum_{l=1}^{L} \pi_l(e') u(w-p-l+B_l) - d(e').$$

Optimal policy $e = 0$:

Similar as the symmetric information case.

Optimal policy $e = 1$*:*

$$\begin{aligned}\mathcal{L} =& p - \sum_{l=0}^{L} \pi_l(1) B_l + \lambda \left[\sum_{l=1}^{L} \pi_l(e) u(w-p-l+B_l) - d(e) - \bar{u} \right] \\ & \beta \left[\sum_{l=1}^{L} \pi_l(1) u(w-p-l+B_l) - d(1) - \left(\sum_{l=1}^{L} \pi_l(0) u(w-p-l+B_l) - d(0) \right) \right].\end{aligned}$$

First order conditions:

$$\frac{\partial \mathcal{L}}{\partial p} = 1 - \lambda \left[\sum_{l=1}^{L} \pi_l(1) u'(w - p - l + B_l)\right] - \beta \left[\sum_{l=1}^{L} (\pi_l(1) - \pi_l(0)) u'(w - p - l + B_l)\right] = 0,$$

$$\frac{\partial \mathcal{L}}{\partial B_l} = -\pi_l(1) + [\lambda \pi_l(1) + \beta(\pi_l(1) - \pi_l(0))] u'(w - p - l + B_l) = 0, \qquad \forall l, \qquad (*)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \sum_{l=1}^{L} \pi_l(1) u(w - p - l + B_l) - d(1) - \bar{u} \geq 0,$$

$$\frac{\partial \mathcal{L}}{\partial \beta} = \sum_{l=1}^{L} (\pi_l(1) - \pi_l(0)) u(w - p - l + B_l) + d(0) - d(1) \geq 0.$$

Equation (*) implies

$$\frac{1}{u'(w - p + B_l - l)} = \lambda + \beta \left[1 - \frac{\pi_l(0)}{\pi_l(1)}\right] \qquad . \qquad \text{(CON-OP)}$$

Clearly, $\lambda > 0$, $\beta > 0$. Thus,

$$l - B_l \quad \text{is strictly increasing in} \quad l = 0, 1, \ldots, L.$$

5. NOTE:

- The incentive effect of deviating from optimal risk sharing is stronger the larger is $\pi_l(1) - \pi_l(0)$, and it is more costly (in terms of risk-sharing benefits) the greater is $\pi_l(1)$. Thus $\frac{\pi_l(1) - \pi_l(0)}{\pi_l(1)}$ may be interpreted as a benefit-cost ratio for deviation from optimal risk sharing, and condition (CON-OP) states that such deviations should be made in proportion to this ratio, with individual risk aversion taken into account.
- In contrast to perfect risk sharing, the second-best solution is crucially dependent on the distribution of l and its functional relation to effort e. This occurs because the outcome l can be used as a signal about the action which is not directly observable. In this sense,

  $$\frac{\pi_l(1) - \pi_l(0)}{\pi_l(1)}$$

  measures how strongly one is inclined to infer from l that the agent did not take the assumed action, and condition (CON-OP) says that penalties or bonuses (as expressed by deviations from first-best risk sharing) should be paid in proportion to this measure.
- The deviation from perfect risk sharing implies that the agent is forced to carry excess responsibility for the outcome and this points to the implicit costs involved in contracting under imperfect information.

7.2 Moral hazard with risk neutrality and limited liability (optional)

Assume the agent can chooses $e \in \{0, 1\}$. The cost of choosing $e = 1$ for the agent is C. And there are n levels of performance,

$$q_1 < q_2 < \cdots < q_n.$$

The principal's return from each performance is

$$S(q_i) = S_i.$$

Let the probability of each level of performance occurring be π_{ik}, with $k = 0, 1$. We assume that $\pi_{ik} > 0$ for all pairs (i, k) and $\sum_{i=1}^{n} \pi_{ik} = 1$.

We assume that the agent has limited liability, and so the payment $t_i \geq 0$ for all i. In this case, the problme for the principal is

$$\begin{aligned} \max_{\{(q_1,\cdots,q_n)\}} \Pi = &\sum_{i=1}^{n} \pi_{i1}(S_i - t_i) \\ s.t. &\sum_{i=1}^{n} \pi_{i1} t_i \geq C; && \text{(IR)} \\ &\sum_{i=1}^{n} (\pi_{i1} - \pi_{i0}) t_i \geq C; && \text{(IC)} \\ &t_i \geq 0 \qquad \forall i. \end{aligned}$$

Note that the IR constraint holds strictly as long as the IC constraint is satisfied, and so it is not a binding constraint. Set up the Lagrangian

$$\mathcal{L} = \sum_{i=1}^{n} \pi_{i1}(S_i - t_i) + \lambda \left[\sum_{i=1}^{n} (\pi_{i1} - \pi_{i0}) t_i - C \right] + \sum_{i=1}^{n} \mu_i t_i.$$

The first-order conditions are:

$$\begin{aligned} \frac{\partial L}{\partial t_i} =& \pi_{i1} + \lambda(\pi_{i1} - \pi_{i0}) + \mu_i = 0; \\ \frac{\partial L}{\partial \lambda} =& \sum_{i=1}^{n} (\pi_{i1} - \pi_{i0}) t_i - C = 0; \\ \mu_i t_i =& 0 \quad \forall i. \end{aligned}$$

Therefore, we conclude that for all i such that $t_i > 0$, which implies $\mu_i = 0$,

$$\lambda = \frac{\pi_{i1}}{\pi_{i1} - \pi_{i0}}.$$

Under the condition that for all $i \neq j$,

$$\frac{\pi_{i1}}{\pi_{i1} - \pi_{i0}} \neq \frac{\pi_{j1}}{\pi_{j1} - \pi_{j0}},$$

there is only one t_i that is strictly positive. That is, the optimal contract is:

$$\begin{aligned} t_{\hat{i}} =& \frac{C}{\pi_{\hat{i}1} - \pi_{\hat{i}0}}, \\ t_i =& 0 \qquad \forall i \neq \hat{i} \end{aligned} \tag{7.4}$$

where

$$\hat{i} = arg \max_j \left\{ \frac{\pi_{j1} - \pi_{j0}}{\pi_{j1}} \right\}.$$

Thus, the structure of the optimal payments is “bang-bang.”

OBSERVATION: The important point here is that the agent is rewarded in the state nature which is the most informative one about the fact that he has exerted a positive effort. Indeed, $\frac{\pi_{i1} - \pi_{i0}}{\pi_{i1}}$ can be interpreted as a *likelihood ratio*. The principal uses therefore a *maximum likelihood ratio criterion* to reward the agent. The agent is only rewarded when this likelihood ratio is maximum. Like an econometrician, the principal tries thus to infer from the distribution of observed outputs what has been the “parameter” (effort) underlying this distribution. But here the “parameter” is endogenous and affected by the incentive contract.

Definition: The probabilities of success satisfy the monotone likelihood ratio property (MLRP) if $\frac{\pi_{i1} - \pi_{i0}}{\pi_{i1}}$ is non-decreasing in i.

When this monotonicity property holds, the structure of the agent’s rewards is quite intuitive.[1]

PROPOSITION: If the probability of success satisfies MLRP, the second-best payment t_i^{SB} received by the agent increases with the level of production q_i.

7.2.1 An example when MLRP fails

Now consider an example when the MLRP is not satisfied. We show that the optimal payment will not be increasing in q. Suppose

$$\pi_{10} = \pi_{30} = \frac{1}{6}, \qquad \pi_{20} = \frac{2}{3}.$$

$$\pi_{11} = \pi_{21} = \pi_{31} = \frac{1}{3}.$$

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[1] Proposition 4.6, page 165, THE THEORY OF INCENTIVES I: THE PRINCIPAL-AGENT MODEL by Jean-Jacques Laffont and David Martimort.

In this case,

$$\frac{\pi_{11} - \pi_{10}}{\pi_{11}} = \frac{\pi_{31} - \pi_{30}}{\pi_{31}} = \frac{1}{2} > \frac{\pi_{21} - \pi_{20}}{\pi_{21}} = -1.$$

Suppose the principal benefits from positive effort. Since outputs q_1 and q_3 are equally informative on the fact the agent has exerted a positive effort, the agent must receive the same transfer in both states. Since output q_2 is also particularly informative on the fact that the agent has exerted no effort, the second-best payment should be zero in this state of nature. Hence, the non-monotonic schedule reduces the agent's incentives to shirk and reduces therefore the probability of state 2, which is bad from the principal's point of view.

On the other hand, the benefit of offering to the agent a schedule of rewards which is increasing in the level of production is that such a scheme does not create any incentive for the agent to sabotage or destroy production to increase his payment. Note that if the non-monotonic payment is offered, the agent has a strong incentive to destroy part of the production when the true output is q_2. It is reasonable to assume that in general, the principal does not observe the production q but that the agent can show hard evidence that he has produced some amount q. This evidence can always be hidden to the principal by destroying production. It is also reasonable to assume that "Lying upwards" and pretending having produced more than what has really been done is instead impossible.

7.3 Moral hazard with risk aversion (optional)

Suppose now that the agent is strictly risk averse, $u''(\cdot) < 0$. The problem for the principal

$$\max_{\{(t_1,\cdots,t_n)\}} \Pi = \sum_{i=1}^{n} \pi_{i1}(S_i - t_i)$$

$$s.t. \sum_{i=1}^{n} \pi_{i1} u(t_i) \geq C; \tag{IR}$$

$$\sum_{i=1}^{n} (\pi_{i1} - \pi_{i0}) u(t_i) \geq C. \tag{IC}$$

Set up the Lagrangian:

$$\mathcal{L} = \sum_{i=1}^{n} \pi_{i1}(S_i - t_i) + \lambda \left[\sum_{i=1}^{n} (\pi_{i1} - \pi_{i0}) u(t_i) - C \right] + \mu \left[\sum_{i=1}^{n} \pi_{i1} u(t_i) - C \right].$$

The first-order conditions are:

$$\frac{\partial L}{\partial t_i} = -\pi_{i1} + \lambda(\pi_{i1} - \pi_{i0})u'(t_i) + \mu\pi_{i1}u'(t_i) = 0 \qquad i = 1, \cdots, n; \tag{$IC-de$}$$

$$\frac{\partial L}{\partial \lambda} = \sum_{i=1}^{n}(\pi_{i1} - \pi_{i0})u(t_i) - C = 0; \tag{IC-de}$$

$$\frac{\partial L}{\partial \mu} = \sum_{i=1}^{n}\pi_{i1}u(t_i) - C = 0.$$

From $(IC - de)$ we have that for all $i = 1, \cdots, n$:

$$\frac{1}{u'(t_i)} = \mu + \lambda\left(\frac{\pi_{i1} - \pi_{i0}}{\pi_{i1}}\right). \tag{OP}$$

Multiplying both sides of (OP) by π_{i1} and summing over all i we have:

$$\mu = E\left[\frac{1}{u'(t_i)}\right] \equiv \sum_{i=1}^{n}\frac{\pi_{i1}}{u'(t_i)}.$$

Hence, we conclude that the multiplier $\mu > 0$.

Next, multiplying both sides of (OP) by $\pi_{i1}u(t_i)$ and summing over all i:

$$\sum_{i=1}^{n}\frac{\pi_{i1}u(t_i)}{u'(t_i)} = \mu\sum_{i=1}^{n}\pi_{i1}u(t_i) + \lambda\sum_{i=1}^{n}(\pi_{i1} - \pi_{i0})u(t_i)).$$

7.3.1 When MLRP satisfied

From previous discussion, we know that if $\frac{\pi_{i1}-\pi_{i0}}{\pi_{i1}}$ is increasing in i, then the monotonic likelihood ratio property is satisfied. In this case, we show that the optimal payment is increasing in q_i. Because $\sum_{i=1}^{n}\pi_{i0} = \sum_{i=1}^{n}\pi_{i1}$, MLRP indicates that the likelihood ratio $\frac{\pi_{i1}-\pi_{i0}}{\pi_{i1}}$ is negative for small i but gradually increases and becomes positive for large i.

We first show that the multiplier $\mu > 0$. Suppose instead that $\mu \leq 0$, then the right-hand side of (OP) could be negative for small i. A CONTRADICTION as marginal utility $u'(\cdot) > 0$ for all i!

Next we argue that $\lambda > 0$ as well. Suppose that $\lambda = 0$. Then $u'(t_i) = \frac{1}{\mu}$ for all i. This would indicates that $t_1 = \cdots = t_n = t$. If this were the case, however, $u(t_i) = u(t)$ for all i. But then (IC-de) would be violated as in this case

$$\sum_{i=1}^{n}(\pi_{i1} - \pi_{i0})u(t_i) = 0.$$

Also suppose that $\lambda < 0$. Since the likelihood ratio is increasing in i, this would indicates that the right-hand side of (OP) is decreasing in i, which indicates that $u'(t_i)$ is increasing and t_i is decreasing in i. By assumption $\pi_{i1} - \pi_{i0}$ is negative for small i and positive for large i and $\sum_{i=1}^{n} \pi_{i0} = \sum_{i=1}^{n} \pi_{i1}$, $u(t_i)$ is decreasing in i would indicates that

$$\sum_{i=1}^{n} (\pi_{i1} - \pi_{i0}) u(t_i) < 0.$$

Again, condition (IC-de) would be violated!

So we conclude that both multipliers $\mu > 0$ and $\lambda > 0$. In this case, t_i is increasing in i and in q_i.

7.4 Coninuous effort choice (optional)

1. The agent's preference

   $$V = -exp[-r(w - C(e))],$$

   $C(e) = ce^2/2$.

2. Signals

   $$y = e + \varepsilon,$$

   where

   $$\varepsilon \sim N(0, \sigma^2).$$

3. The Agent's expected utility: given wage contract $w = w_0 + \beta y$,

   $$\begin{aligned} EV &= \int -exp\{-r[w_0 + \beta(e + \varepsilon) - C(e)]\} f(\varepsilon) d\varepsilon \\ &= -\, exp\{-r[w_0 + \beta e - C(e^2)]\} exp\left\{\frac{r^2\beta^2\sigma^2}{2}\right\} \\ &= -\, exp\left\{-r\left[w_0 + \beta e - \frac{ce^2}{2} - \frac{r\beta^2\sigma^2}{2}\right]\right\}. \end{aligned}$$

   To see this, note that:

   $$\begin{aligned} &\int_{-\infty}^{\infty} exp\{-r\beta\varepsilon\} \frac{1}{\sqrt{2\pi\sigma^2}} exp\{-\frac{\varepsilon^2}{2\sigma^2}\} d\varepsilon \\ =&\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} exp\left\{-\frac{(\varepsilon + r\beta\sigma^2)^2}{2\sigma^2}\right\} exp\left\{\frac{r^2\beta_i^2\sigma^2}{2}\right\} d\varepsilon \\ =&exp\left\{\frac{r^2\beta^2\sigma^2}{2}\right\}. \end{aligned}$$

   Maximizing utility gives:

   $$ce^* = \beta \Longrightarrow e^* = \frac{\beta}{c}.$$

The Certainty Equivalent for the Agent from exerting effort e^* is

$$CE = C(e^*) + \frac{r\beta^2\sigma^2}{2}.$$

NOTE: The Certainty Equivalent is strictly greater than the cost of effort $C(e)$. The difference between CE and $C(e)$ is affected by two factors: risk-aversion represented by r and the variance of observation error σ^2.

4. For the Principal, maximize profit subject to Agent's IC constraint and Participation constraint. The Principal's problem is equivalent to the maximization problem:

$$S = e^* - C(e^*) - \frac{r\beta^2\sigma^2}{2}.$$

The Principal choose β to maximize S. The FOC gives:

$$\frac{\partial e^*}{\partial \beta} - ce^* \frac{\partial e^*}{\partial \beta} - r\beta\sigma^2 = 0.$$

As $\frac{\partial e^*}{\partial \beta} = 1/c$, we therefore have

$$\frac{1}{c} - \frac{\beta}{c} - r\beta\sigma^2 = 0.$$

Solving for β gives

$$\beta = \frac{1}{1 + rc\sigma^2}.$$