MICROECONOMIC THEORY II

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WHY EXTENSIVE FORM GAME? 扩展式博弈

策略式博弈　①连续删被占优 ② mutual best response

发展 ↔ proper equilibrium

完美均衡 $< \epsilon$ $\vdash \epsilon$

纳什均衡 < 0

- Strategic form games describe a game by its strategies—complete contingent plans of how to react in each possible scenario—and play down the temporal aspect of the situation—who moves first, who moves second, etc. It is like a computer chess program. Once each player submit the programs, the computer will take over and decide which side will win. You don't get to see the actual step-by-step plays.

Why extensive form game?

NE - 信念限制有时不那么合理

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- Extensive-form games explicitly describe how the game is played through time, including details about who moves first, who moves second, and so on.
- In this sense, extensive form game provides more information than the strategic form.

静态博弈 - 动态博弈
选择 (同时) (有先后)
策略式不适用动态博弈

common knowledge rationality
↓
在任何条件下选择均为最优
静态 → NE保证
动态 → 每点都保证
SPNE

Game between a young kid with his parents

Analyzing the example

not all reasonable

(nb, GG)
(nb, SG)
(b, GS)

- IF we just look at NE, then we may have some problem:

不合理(有不可信威胁存在)

3个 纯策略的纳什均衡

		kid			
		GG	GS	SG	SS
Parent	buy	-2, 1	-2, 1	-5, -10	-5,-10
	not buy	0, -2	-5, -10	0, -2	-5, -10

backward induction (nb, GG)

- How the game gets played

GS、SG为弱被占优策略

2player 博弈 完美均衡

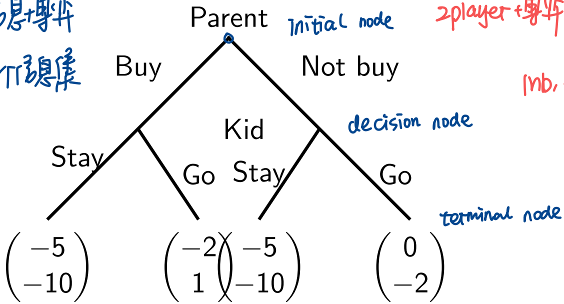
↓

(nb, GG) 唯一

完美信息博弈

↓

每个点都对应一个信息集



Extensive form

扩展式博弈

Definition: An extensive form game Γ_E contains the following information:

1. Set of nodes $\mathscr{X}$, set of actions $\mathscr{A}$ and set of players $\{1, \ldots, I\}$.
2. The order of moves—i.e., who moves when
   - Predecessor function: $p : \mathscr{X} \to \mathscr{X} \cup \varnothing$; $p(x_0) = \varnothing$
   - Successor function: $s(x) = p^{-1}(x)$; $T = \{x \in \mathscr{X} : s(x) = \varnothing\}$
   - $\alpha : \mathscr{X} \backslash \{x_0\} \to \mathscr{A}$ giving the action that leads to x from $p(x)$

   ↓ decision node

$$c(x) = \{\alpha \in \mathscr{A} : a = \alpha(x'), x' \in s(x)\}$$

3. Information sets: $H : \mathscr{X} \to \mathscr{H}$,

信息集 → 决策点的集合

(对应策略式中的 "一种情况")

$$c(x) = c(x') \quad \text{if} \quad H(x) = H(x')$$

Extensive form continued

一个外生事件的概率分布

4. The probability distribution over any exogenous events: (信息经济学)

$$\rho : \mathscr{H}_0 \times \mathscr{A} \to [0,1], \quad \rho(H,a) = 0 \quad \text{if } a \notin C(H) \text{ and}$$

$$\sum_{a \in C(H)} \rho(H,a) = 1$$

5. The players' payoffs as a function of the moves that were made.

Perfect recall

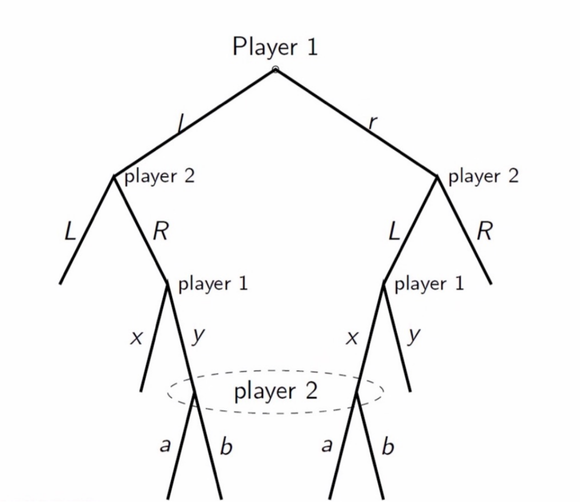
完美记忆

策略式博弈中的假设都需要

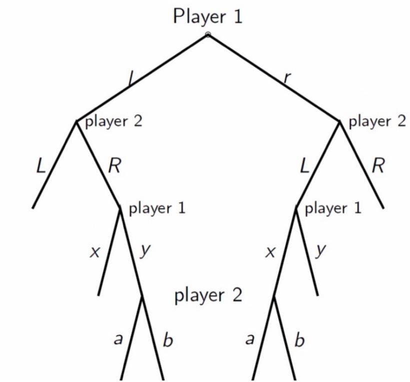
自身/他人行为

- We say a game is of *perfect recall* if "no player ever forgets any information he once knew, and all players know the actions they have chosen previously.
- More formally, if x and x' belongs to the same information set of player i, then it must be that 1) the sequence of moves that leads to x and the sequence of moves that leads to x' must pass through the same sequence of information sets for player i, and 2) in each of the information set of players i that leads to x and x', the same action must be chosen by player i.

EXAMPLE 1: IMPERFECT RECALL

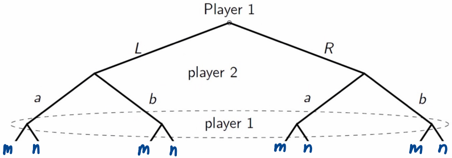


Another example



player 1 有2个信息集，4个纯策略 Lm Ln Rm Rn

player 2 有2个信息集，4个纯策略 aa ba ab bb



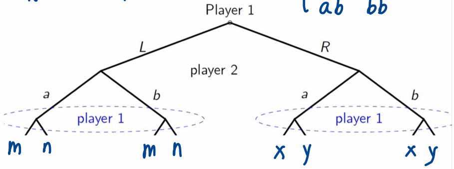
每个信息集有2个选项

player 1 有3个信息集，有8个纯策略

Rmx Rmy
Rnx Rny
Lmx Lmy
Lnx Lny

player 2 有2个信息集，有4个纯策略

aa ba
ab bb



More on extensive form

- One problem with this way of writing down a game is that there is no natural way to express a simultaneous move.
- Example: Battle of Sexes. When we write down this game in extensive form, we write it as if someone moves first, and the second player does not observe the move of the first mover. This maintain the same information structure as the simultaneous game but change the sequence of moves.
- The point is: although the extensive form tells us more about the sequence of moves, it is not a completely accurate description (when the game involves simultaneous moves).

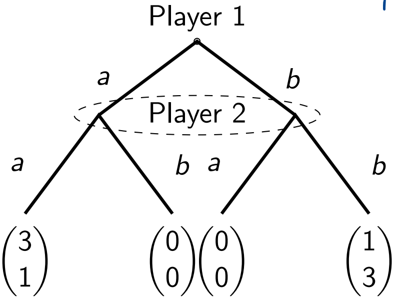
Battle of sexes game

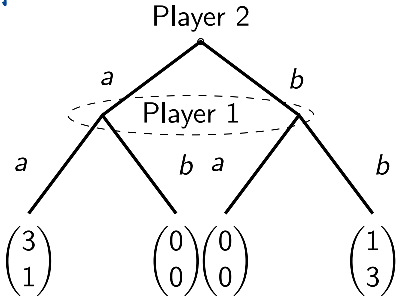
G

	a	b
a	3,1	0,0
b	0,0	1,3

B 球赛 电影

↑均表示





Pure strategy 纯策略

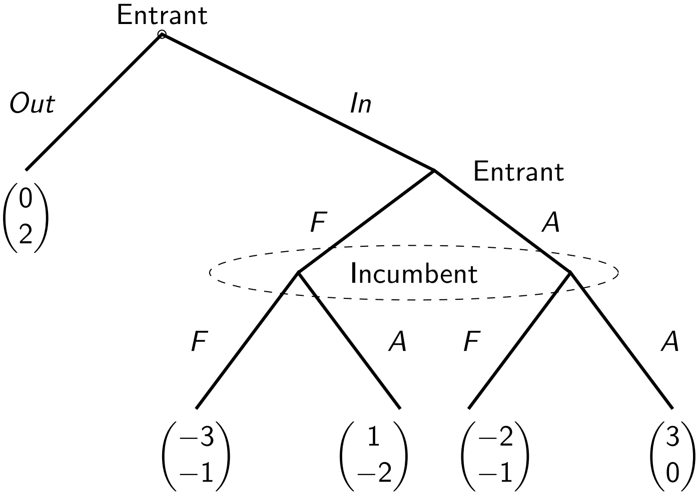
每个信息集下对应player i的选项

- A pure strategy of an extensive form game for a player i prescribes an action at each information set of player i.
- Example: In the Entrant-incumbent game given below:
  - ➢ Firm I has two pure strategies: fight, accommodate
  - ➢ Firm E has four pure strategies:
    Out and Fight if In (OF), Out and Accommodate if In (OA),
    In and Fight if In (IF), In and Accommodate if In (IA).

Entrant-incumbent game

Entrant firm 4个纯策略

Incumbent 2



Interpretation on strategy

策略 → belief 对应
(自身) (对方)

1. A strategy is not really a plan of actions, as it requires a player to specify his actions at information sets that are impossible to be reached if he carries out his plan.
2. So, if a player wants to let an agent to play the game for him, there is no need to tell the agent what to do in those information sets.
3. In fact, two strategies that are different only in information sets ruled out by own strategy are *strategically equivalent* in the sense they always lead to the same payoffs.
4. One way to interpret this is that a strategy for player i actually includes two parts:
   - a rational plan for player i at information set that he may be called upon to play;
   - and a prediction about i's future behavior should she deviates from her plan.

Mixed strategy

纯策略上的概率分布 不同纯策略取不同概率 (或者定义为在不同结点上概率不同)

不同信息集上选项的概率分布

策略式 扩展式

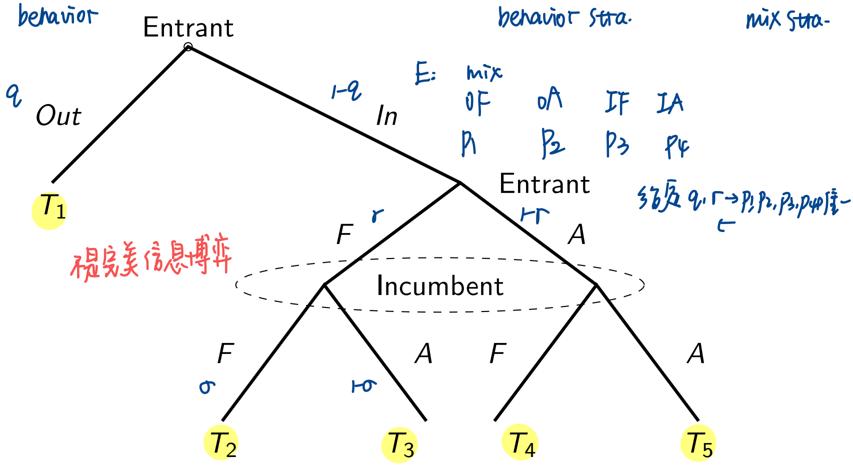
- There are two ways to define mixed strategies.
- 策略式 A *mixed strategy* σ_i for player i assigns to each pure strategy $s_i \in S_i$ a probability $\sigma_i(s_i) \geq 0$ that it will be played, where $\sum_{s_i \in S_i} \sigma_i(s_i) = 1$
- 扩展式 A *behavioral strategy* for player i prescribes, for every information set H and action a_i a probability $\lambda_i(a_i, H)$, with $\sum_{a_i} \lambda_i(a_i, H) = 1$ for all H. 同一信息集上选项的概率分布之和为1
- Mixed strategies and behavior strategies are strategically equivalent in games of perfect recall. (在完美记忆中等价)
- This implies that we can use either one of the two at our convenience.
- We typically use *behavior* strategies for *extensive* form game, and *mixed* strategies for *strategical* form game.

(序贯均衡) → behavior strategy

Equivalence of two mixed strategies

给定一个mix stra. 有唯一对应 behavior stra.

behavior stra. mix stra.



不完美信息博弈

Sketch of proof

1. For any mixed strategy of Firm E (OF,OA,IF,IA; p_1, p_2, p_3, p_4), there exists a unique behavior strategy such that the probability of reaching terminal nodes $T_1, \ldots, T_5$ is the same.
2. To see this, let the mixed strategy for Firm I be (F, A; σ, $1-\sigma$), we show there is a unique behavior strategy for Firm E that assigns q and $1-q$ to "Out" and "In" at the first information set, and assigns r and $1-r$ to "F" and "A" at the second information set.
3. Given the mixed strategy of Firm E and Firm I:

$$Pr(T_1) = p_1 + p_2, \quad Pr(T_2) = p_3\sigma, \quad Pr(T_3) = p_3(1-\sigma)$$
$$Pr(T_4) = p_4\sigma, \quad Pr(T_5) = p_4(1-\sigma).$$

4. Hence we have the unique behavior strategy

$$q = p_1 + p_2, \quad r = \frac{p_3}{1-(p_1+p_2)}.$$

5. On the other hand, given a behavior strategy $(q, 1-q)$ at the first information set and $(r, 1-r)$ at the second information set, we can get the mixed strategy.
6. The unique mixed strategy that is equivalent to the behavior strategy is:

$$p_1 = qr, \quad p_2 = q(1-r), \quad p_3 = (1-q)r, \quad p_4 = (1-q)(1-r).$$

NE

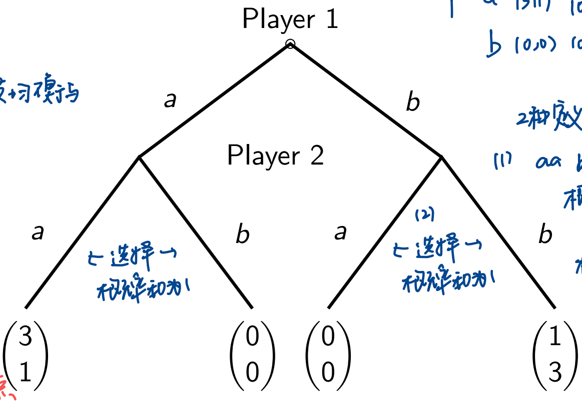
扩展式→策略式→找NE

- Nash Equilibrium is defined in the usual way.
- The sequential battle of sexes game:

策略式 完美均衡

↓

应为扩展式中的序贯均衡与子博弈完美均衡



完美信息博弈

every information set is a singleton

每个信息集都只含一个决策点

1 \ 2	aa	ba	ab	bb
a	(3,1)	(0,0)	(3,1)	(0,0)
b	(0,0)	(0,0)	(1,3)	(1,3)

2种定义混合策略的方式

(1) aa ba ab bb 概率和为1

a b 概率和为1

Sequential battle of sexes game

- Strategical form of the game

proper Equilibrium (a,ab)

	aa	ab	ba	bb
a	3,1	3,1	0,0	0,0
b	0,0	1,3	0,0	1,3

- Obviously every Nash equilibrium of the extensive form game is a Nash equilibrium in the strategic form game, and vice versa.
- The strategic form however does not capture all the information, namely, the order of moves, contained in an extensive form game.
- Two extensive form games may have the same strategic form. For, example, the game above may also be a 2x4 simultaneous-move game.

Principle of sequential rationality

序贯理性 每个player在每个信息集时选项是最优的

- NE may involves incredible threat:
  believe opponent will make a choice that is not optimal off the equilibrium path
- The principle of sequential rationality: a player's strategy should specify optimal actions *at every point in the game tree*.
- 方法 Backward induction ensures that a player's strategies specify optimal behavior at every decision node of the game.

Problem with NE

- There are three Nash equilibria in SBoS:

子博弈完美均衡(逆向归纳)

$$(a, aa), \quad (a, ab), \quad (b, bb).$$ ← 均衡策略

不满足每个player在每个信息集选择最优 满足 序贯理性

- In the last equilibrium,

1选a时，2威胁要选b，尽管对1,2均harm

  - ➢ Player 2 threatens to choose b when player 1 chooses a, even when doing so harms both players.
  - ➢ Believing the threat, player 1 chooses b, leading to the outcome (b,b).
  - ➢ This is a Nash equilibrium because Player 2 will not be called upon to carry out the threat. NE中有不可信威胁路

如果1接受2的挑战，2说"1选a时它选b"的威胁不可信不会被真的行使.

- Most people will think this type of threat is not credible. If player 1 calls player 2's bluff by choosing a, it is not in the interest of player 2 to actually carry the threat.
- The concept of Nash equilibrium does not distinguish whether a threat is credible because as long as a threat is effective, it has no payoff consequences.

Backward induction

- In games of perfect information, we can formalize the idea that each player should act rationally in decisions nodes off the equilibrium path by the procedure of backward induction.

完美信息博弈

- Definition: An extensive form game is of perfect information if every information set is a singleton (which means there is no exogenous uncertainty and each player also all the moves up to that point).

逆向归纳法

- Backward Induction: 最后一个决策点(信息集)
  1. Start with the decision nodes in the final stage (those whose successors are all terminal nodes).
  2. At each of these nodes, selects one of the best alternatives for the player who is making the decision and eliminates the rest.
  3. Repeat the same procedure until the initial node is reached. The resulting payoff profile is called a backward induction solution.
- Backward induction solutions are all Nash equilibrium, but the converse is false. The solution is unique if no player is ever indifferent between two actions.

Subgame perfect NE

博弈的一部分　　　　① 单决策点的信息集开始

- Subgame: The portion of the game tree that follows a decision node x is a *subgame* if it constitutes a well-defined extensive form game. That is, if
  1. the information set that contains x is a singleton; and
  2. if x belongs to the subgame, then every x' in the same information set as x must also belong to the subgame.

② 和其在同一信息集的决策点也必在同一子博弈中

得子博弈中的 NE(也是SPNE)
与解到每个子博弈中

子博弈完美均衡

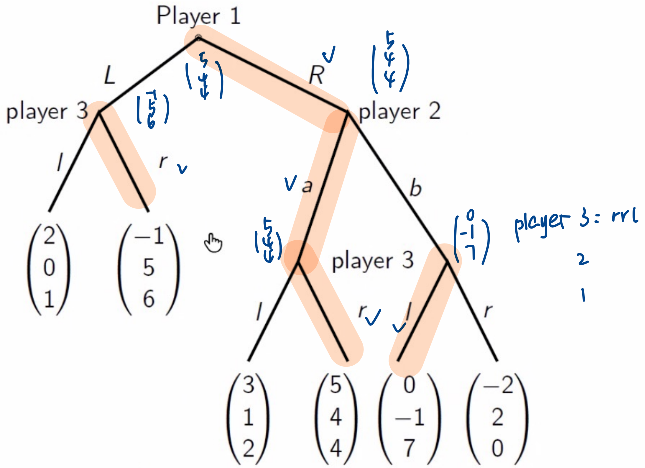
用backward induction 找到的NE

- Subgame Perfection: A Nash equilibrium is subgame perfect if it prescribes a Nash equilibrium in every subgame.
- Subgame perfection generalizes the idea of backward induction to games of imperfect information. The backward induction solution is always subgame perfect.
- The way to find subgame perfect equilibrium is similar to backward induction: starting from the subgame near the end and work backward.

An application

完美信息博弈

5个博弈



Strategical form of the game

Player 3

	lll	llr	lrl	lrr	rrl	rlr	rll	rrr
L	2, 0, 1	2, 0, 1	2, 0, 1	2, 0, 1	-1, 5, 6	-1, 5, 6	-1, 5, 6	-1, 5, 6
R	3, 1, 2	3, 1, 2	5, 4, 4	5, 4, 4	5, 4, 4	3, 1, 2	3, 1, 2	5, 4, 4

Player 2 plays a

backward induction

子博弈完美均衡.

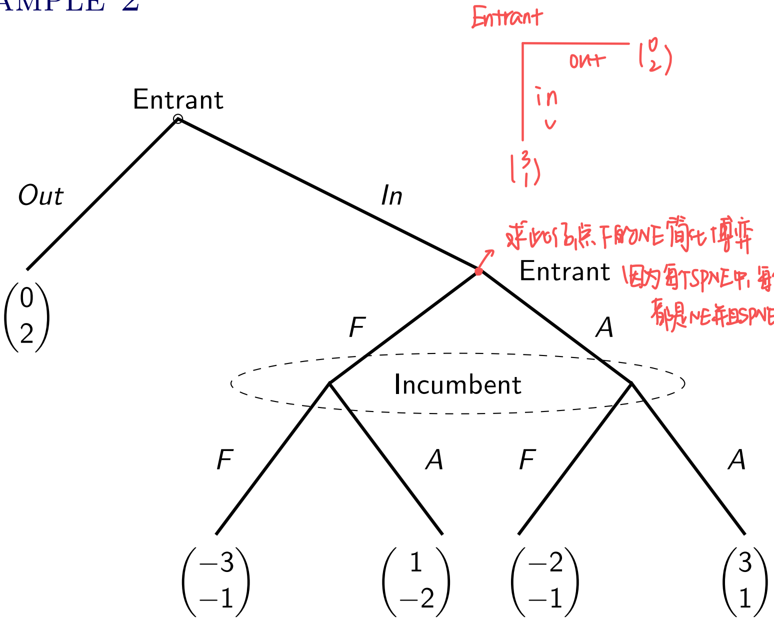
其它5个包含不可信威胁

Player 3

	lll	llr	lrl	lrr	rrl	rlr	rll	rrr
L	2, 0, 1	2, 0, 1	2, 0, 1	2, 0, 1	-1, 5, 6	-1, 5, 6	-1, 5, 6	-1, 5, 6
R	0, -1, 7	-2, 2, 0	0, -1, 7	-2, 2, 0	0, -1, 7	-2, 2, 0	0, -1, 7	-2, 2, 0

Player 2 plays b

Example 2



求此节点下的NE简化博弈
(因为每个SPNE中，每个子博弈都是NE并且SPNE的)

Example 2: NE

- Strategic form

player 2

player 1		F	A
	OF	0, 2	0, 2
	OA	0, 2	0, 2
	IF	-3, -1	1, -2
	IA	-2, -1	3, 1

Example 2: NE

- Strategic form

		player 2	
		F	A
player 1	OF	0, 2	0, 2
	OA	0, 2	0, 2
	IF	-3, -1	1, -2
	IA	-2, -1	3, 1

- Three pure NE:

用逆向归纳

$$(OF, F), \quad (OA, F) \quad (IA, A)$$

唯一的SPNE

Example 2: NE

- Strategic form

player 2

player 1

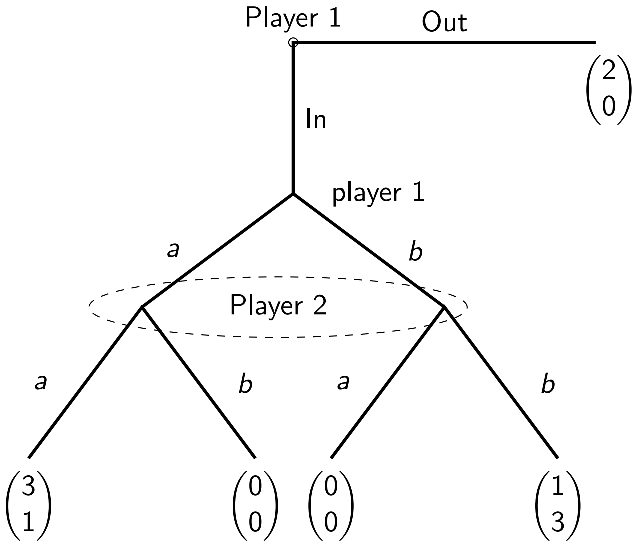
	F	A
OF	0, 2	0, 2
OA	0, 2	0, 2
IF	-3, -1	1, -2
IA	-2, -1	3, 1

- Three pure NE:

$$(OF, F), \quad (OA, F) \quad (IA, A)$$

- Whichof them involves incredible threat?

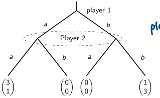
Example 3



Player 1
Out
$\binom{2}{0}$
In
player 1
a
b
Player 2
a
b
a
b
$\binom{3}{1}$
$\binom{0}{0}$
$\binom{0}{0}$
$\binom{1}{3}$

Example 3: find SPNE

① 先找子博弈的NE



		player 2	
		q a	$1-q$ b
player 1	p a	(3,1)	(0,0)
	$1-p$ b	(0,0)	(1,3)

(a,a) (b,b) 均为子博弈的NE (pure)

求混合NE：设 p. q 等

$$\begin{cases} 3q+0(1-q)=0q+1(1-q) \Rightarrow 3q=1-q \quad q=\frac{1}{4} \\ 1p+0(1-p)=0p+3(1-p) \Rightarrow p=3-3p \quad p=\frac{3}{4} \end{cases}$$

$(\frac{3}{4},\frac{1}{4};\frac{1}{4},\frac{3}{4})$ 为子博弈的混合NE

② 逆向归纳简化博弈

player 1 —out— $\binom{2}{0}$
in
√ $\binom{3}{1}$ [(a,a)情形]

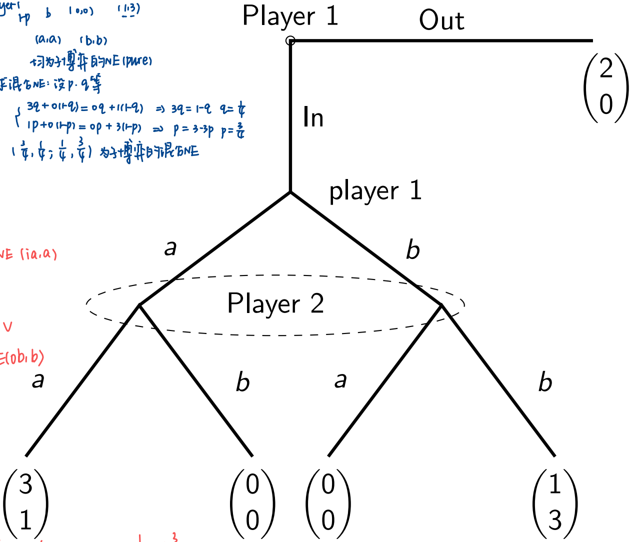
SPNE (ia,a)

player 1 —out— $\binom{2}{0}$ √
in
$\binom{1}{3}$ [(b,b)情形]

SPNE(ob,b)

player 1 —out— $\binom{2}{0}$ √
in
$\binom{3/4}{3/4}$ $(\frac{3}{4},\frac{1}{4};\frac{1}{4},\frac{3}{4})$ 情形

SPNE $(oa\frac{3}{4}, ob\frac{1}{4}, ia0, ib0; a\frac{1}{4}, b\frac{3}{4})$



Example 3: find SPNE

player 2

player 1		a	b
	Oa	2, 0	2, 0
	Ob	2, 0	2, 0
	Ia	3, 1	0, 0
	Ib	0, 0	1, 3

Example 3: find SPNE

player 2

player 1

	a	b
Oa	2, 0	2, 0*
Ob	2, 0	2, 0*
Ia	3, 1*	0, 0
Ib	0, 0	1, 3

Example 3: find SPNE

- Subgame after IN

	a	b
a	3, 1	0, 0
b	2, 0	1, 3

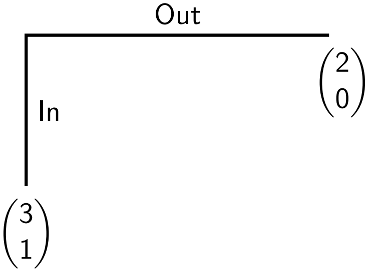
- Three NE in the subgame:

混合策略NE

$$(a, a); \qquad (b, b); \qquad \left(\frac{3}{4}, \frac{1}{4}; \frac{1}{4}, \frac{3}{4}\right).$$

分别用payoff替代

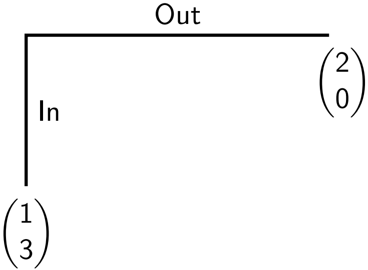
Example 3: find SPNE



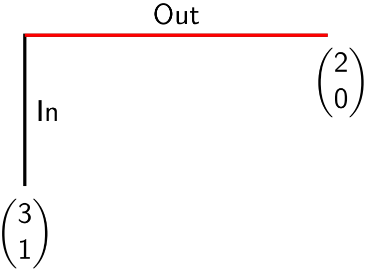
Example 3: find SPNE

SPNE 1: (Ia, a)

Example 3: find SPNE

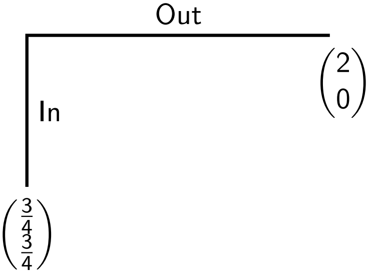


Example 3: find SPNE

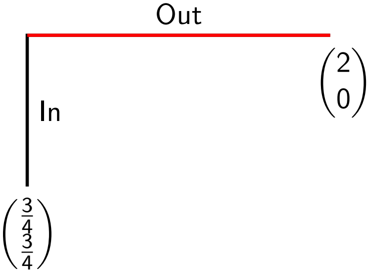


SPNE 2: (Ob, b)

Example 3: find SPNE



Example 3: find SPNE



SPNE 3: $\left(Oa\frac{3}{4}, Ob\frac{1}{4}, Ia0, Ib, 0; a\frac{1}{4}, b\frac{3}{4}\right)$

More on SPNE

对不完美信息
NE存在SPNE存在

(完美信息)

SPNE存在性

- Existence of SPNE: Every finite extensive form game of perfect information has a pure strategy SPNE. (纯策略有SPNE
- For finite extensive form game of perfect information, SPNE is unique if there is no tie in payoffs for any player.
- Intuitively, backward induction rules out incredible threats. The backward induction solution of SBoS is (a,ab).
- This example illustrates the value of commitment in strategic situations.
- Note that here the second mover is harmed by his own rationality—he will be better off if he can convince the first mover that he is irrational.
- That's one reason why young children often get what they want from parents.

Backward induction and common knowledge

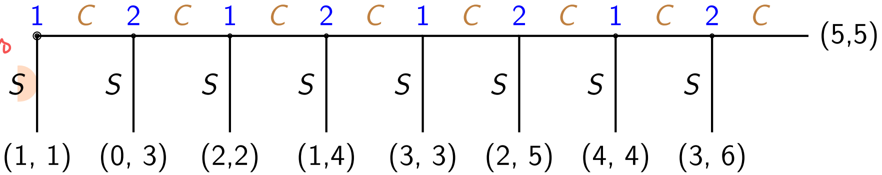
*前提

- Backward induction in some sense relies on the common knowledge of rationality at every decision node.
- But it is problematic to maintain the assumption of rationality off the equilibrium path.
- According to backward induction logic, a rational player should not deviate in the first place.
- There is no completely satisfactory solution to this problem.

Centipede game

Common knowledge of Rationality (一旦不是, player 1/2 continue 都正常)

唯一的 SPNE



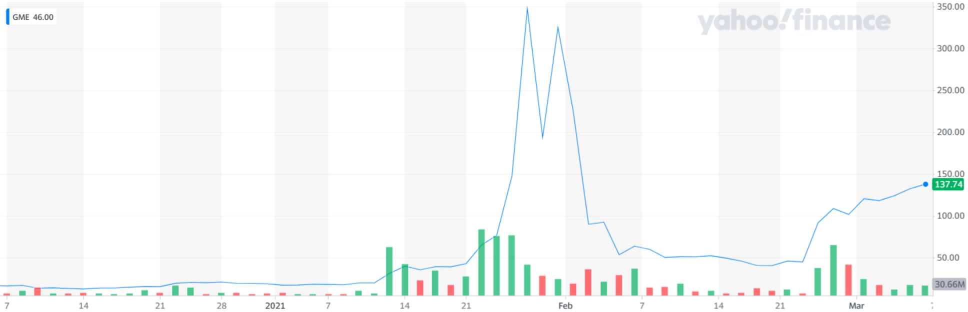
- The unique SPNE is for 1 & 2 to choose "S" , which follows from Iterated deletion of weakly dominated strategies.
- But this SPNE is rather doubtful.

Wallstreetbets day traders vs. Hedge fund

散户讨论组

美国股市：机构投资者占比大

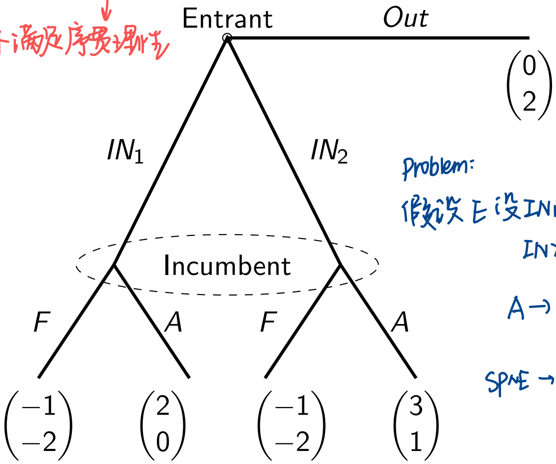
GME SHORT SQUEEZE



SPNE MAY HAVE NO POWER

逆向归纳不能保证对all 不完美信息博弈 序贯理性(每个player在每个信息集的选项是最优的)

或者说其SPNE不满足序贯理性



Problem:

假设E设IN1 → 选A

IN2 → 选A

A → 占优选项

SPNE → 序贯均衡

FIGURE : Entrant incumbent example 1

Find SPNE of the entry game

不可信威胁

Incumbent → 没有理由选F(只要当其需要做选择时)

Entrant		F	A
	Out	0, 2 NE	0, 2
	IN_1	-1, -2	2, 0
	IN_2	-1, -2	3, 1 NE

NE / 也为SPNE

(out,F)

不满足序贯理性

The problem with SPNE

- The SPNE are identical to NE for the entrant-incumbent game
- Definition: An extensive form game is of imperfect information if not all information sets are singletons.
- Backward induction may not work in games of imperfect information.
- In view of this problem, a natural solution is to require each player to make optimal choices at every information set. This solves the problem in the above example.
- But is this enough?

Example 2: Entrant-incumbent game 2

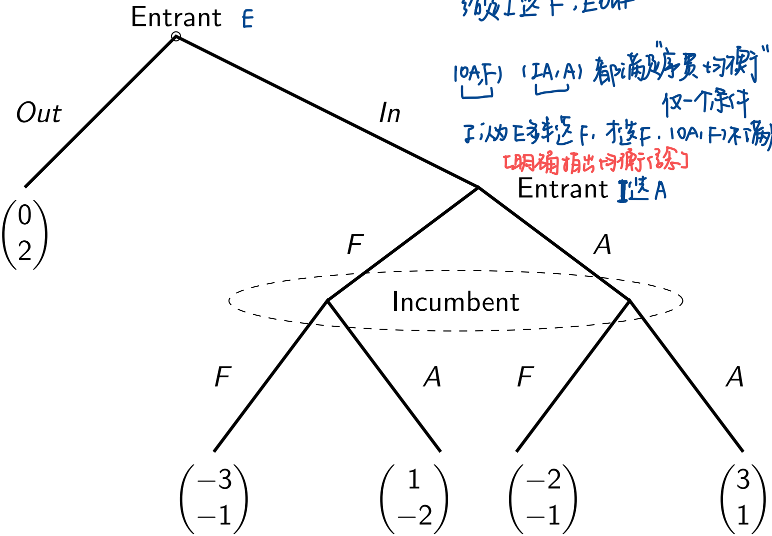
预设I选F, E out

(OA, F) (IA, A) 都满足"序贯均衡"

仅一个条件

I认为E多半选F, 才选F. (OA, F)不满足

[明确指出均衡路径]



Entrant-incumbent game 2 continued

- The strategic form

	F	A
OF	0, 2	0, 2
OA	0, 2	0, 2
IF	-3, -1	1, -2
IF	-2, -1	3, 1

- NE

$$(OF, F), \quad (OA, F), \quad (IA, A).$$

A ×

不满足序贯理性

Entrant-incumbent game 2 continued

- The strategic form

	F	A
OF	0, 2	0, 2
OA	0, 2	0, 2
IF	-3, -1	1, -2
IF	-2, -1	3, 1

- NE
$$(OF, F), \quad (OA, F), \quad (IA, A).$$
- If we require every player to make optimal choice at every information set:
$$(OA, F), \quad (IA, A).$$

Entrant-incumbent game 2 continued

- The strategic form

	F	A
OF	0, 2	0, 2
OA	0, 2	0, 2
IF	-3, -1	1, -2
IF	-2, -1	3, 1

- NE

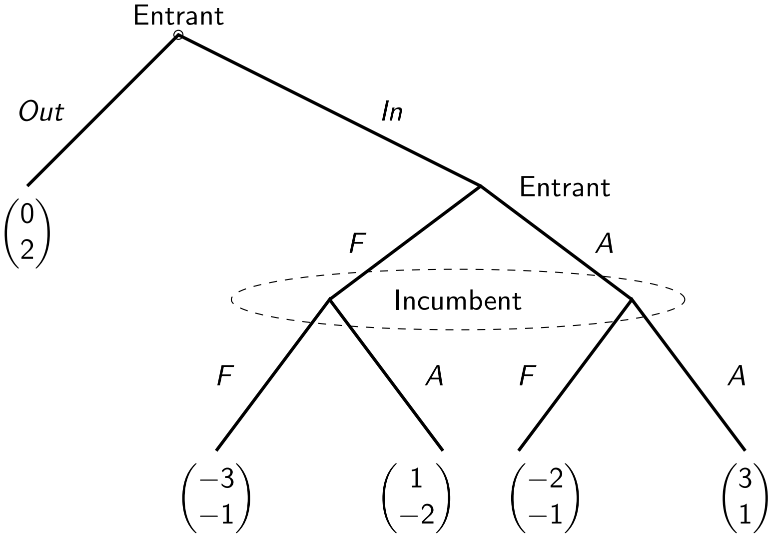
$$(OF, F), \quad (OA, F), \quad (IA, A).$$

- If we require every player to make optimal choice at every information set:

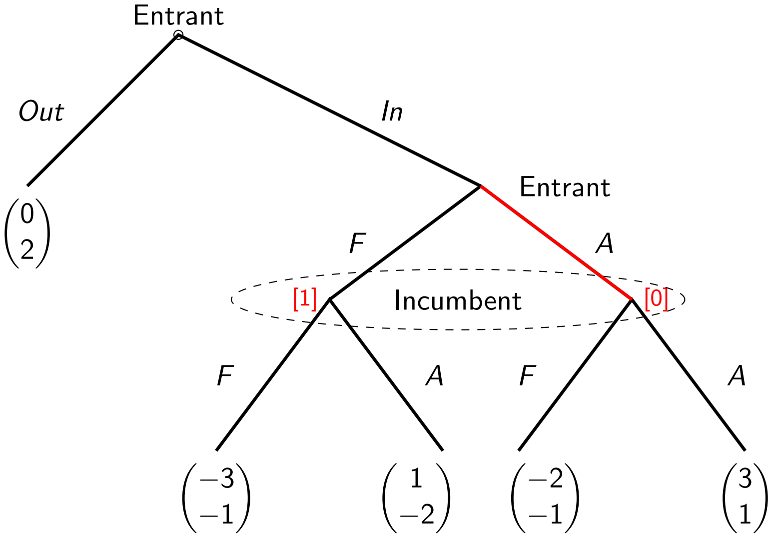
$$(OA, F), \quad (IA, A).$$

- (OA, F) meets the requirement!

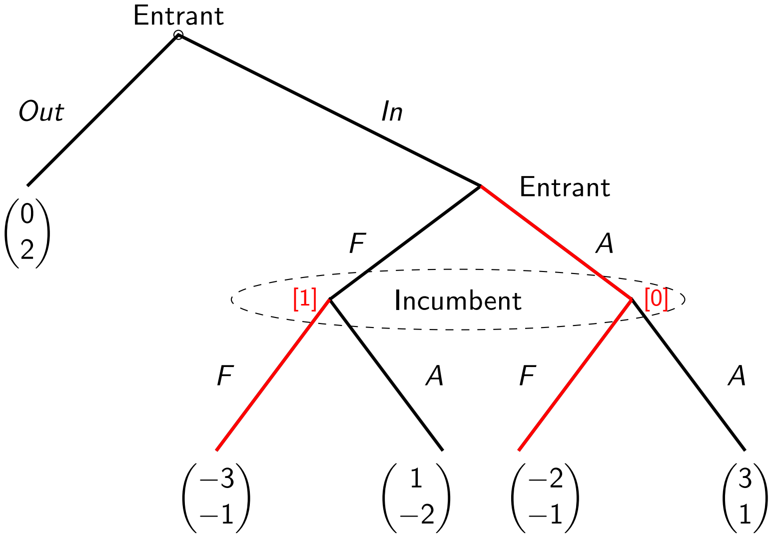
Entrant-incumbent game 2 continued



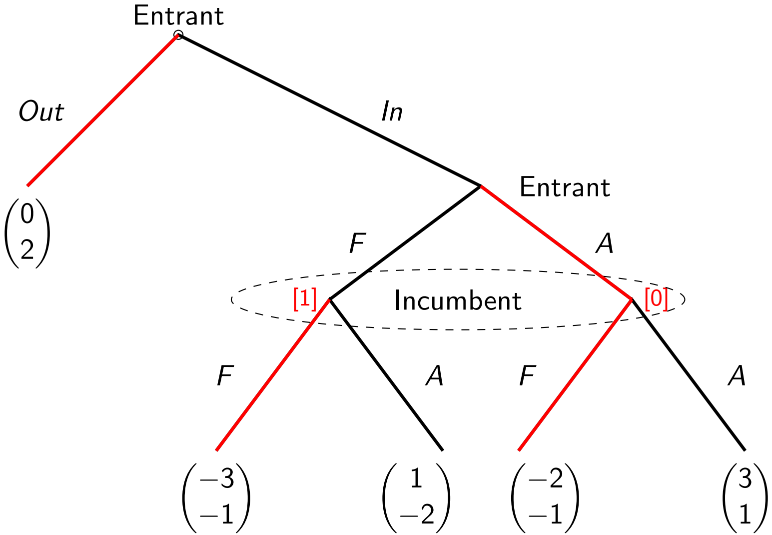
Entrant-incumbent game 2 continued



Entrant-incumbent game 2 continued



Entrant-incumbent game 2 continued



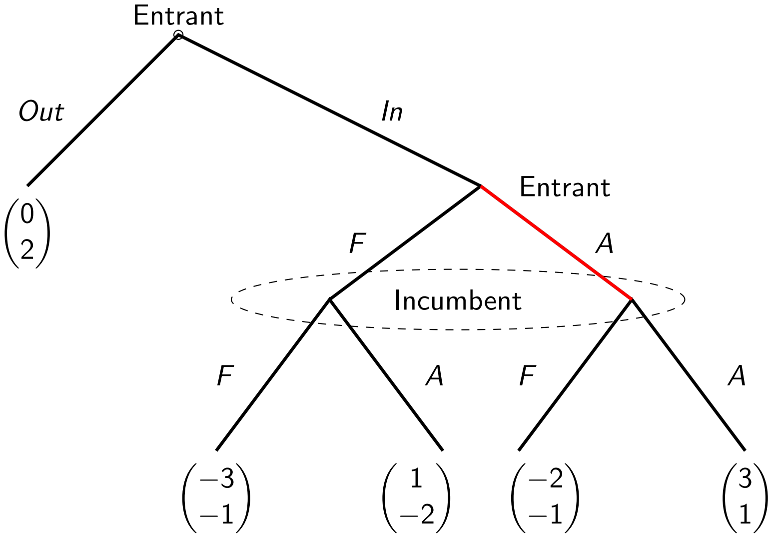
Entrant-incumbent game 2 continued

- Subgame after entry

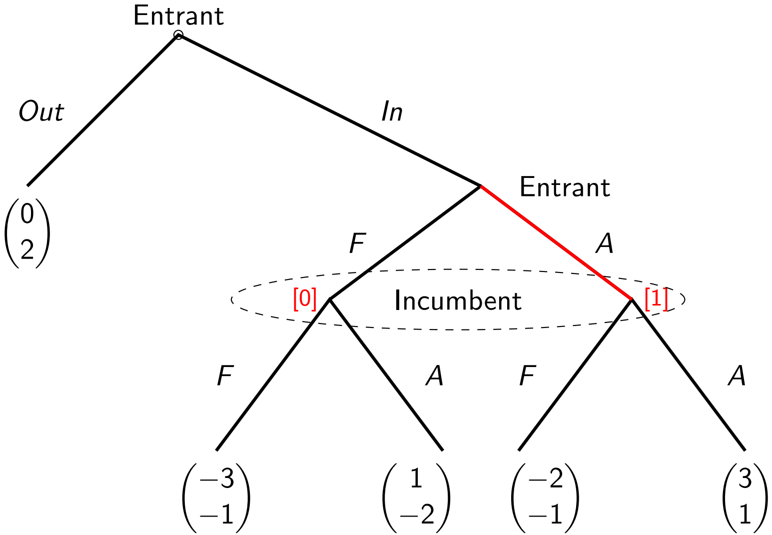
	F	*A*
F	-3, -1	1, -2
A	-2, -1	3, 1

- So (OA, F) is not SPNE!
- In addition to restriction on choices, there needs to be restrictions on off-equilibrium path beliefs!

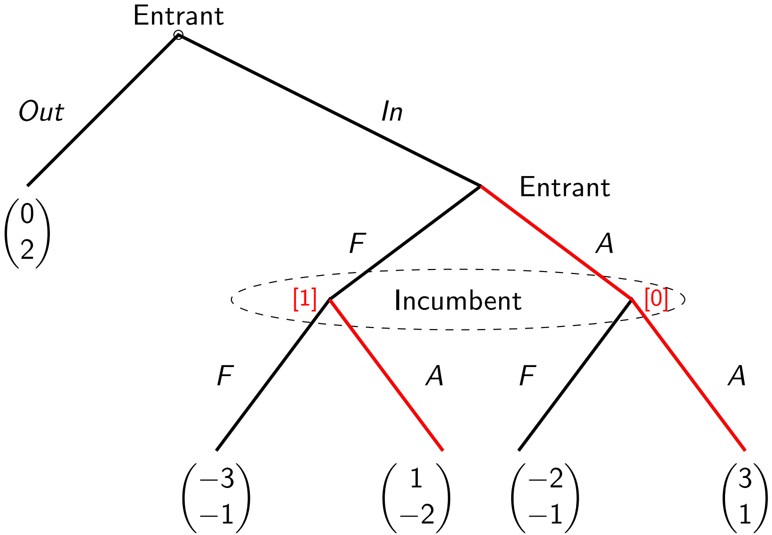
Belief consistent with choices



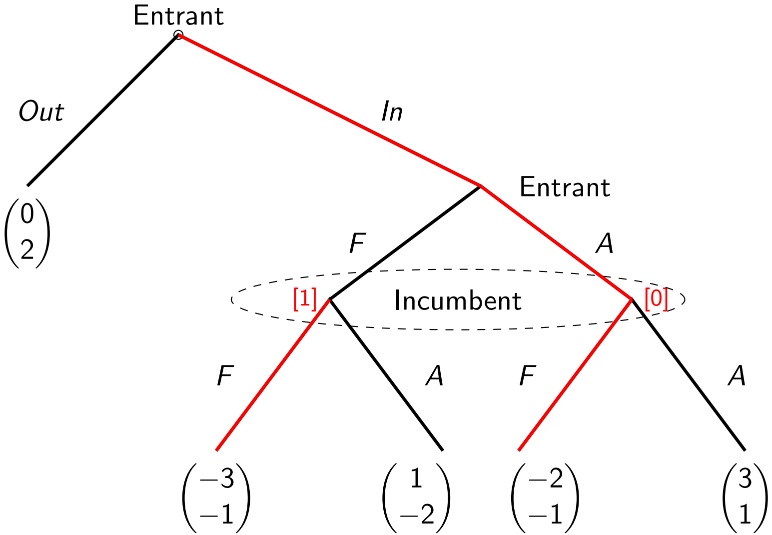
Belief consistent with choices



Belief consistent with choices



Belief consistent with choices



Some definitions

- Definition: A system of beliefs μ in an extensive form game Γ_E is a specification of probability $\mu(x) \in [0,1]$ for each decision node x in Γ_E such that for all information set $h \in \mathbf{H}$,

$$\sum_{x \in h} \mu(x) = 1.$$

Sequential equilibrium 序贯均衡

← NE存在
一定存在

NE→策略式 中定义
SE→扩展式
序贯均衡

给定信念 → 序贯理性

- A strategy profile and system of beliefs (σ, μ) is a sequential equilibrium of Γ_E if

  策略组合 信念

  I. The strategy profile σ is sequentially rational given μ.

  完全混合策略组合
  ↓极限
  均衡策略

  II. There exists completely mixed strategies $\{\sigma^k\}_{k=1}^{\infty}$ with $\lim_{k\to\infty}\sigma^k = \sigma$, such that $\mu = \lim_{k\to\infty}\mu^k$, where μ^k is derived from σ^k using Bayes rule. 贝叶斯法则

  均衡策略

  信念
  均衡信念应与对手的均衡策略完全一致

如果对手在前信息集选择为A 我们应认为其选A 而非B

- The concept of sequential equilibrium is a strengthening of the concept of subgame perfection.
- Any sequential equilibrium is necessarily subgame perfect, but the converse is not true. The difference of the two, of course, only lies in imperfect information game.
- Consistency requirement: There are sequential equilibrium in which consistency may impose restrictions on the possible sequences of totally mixed strategy, and in turn also on the possible belief players may have off-the-equilibrium path.

Interpretation of the definition

- The concept of sequential equilibrium captures the intuition of backward induction - - each player believes the other players are rational and thus will play optimally in any continuation of the game - - by defining an equilibrium to be a pair consisting of a behavioral strategy and a system of beliefs.
- The behavior strategy is *sequentially rational* with the system of beliefs, namely that at every information set at which a player moves, the player's behavioral strategy maximizes his conditional payoff, given his belief at that information set and the strategies of the other players.
- The system of belief is *consistent* with the behavioral strategy, that is, it is the limit of a sequence of beliefs each being the actual conditional distribution on nodes of the various information sets induced by a sequence of totally mixed behavioral strategies converging to the given behavioral strategy.

Apply the definition of SE

- Consider NE (OF, F)
  - ➢ σ not sequentially rational for any μ;
  - ➢ F is not optimal for Entrant at the information set after IN.

序贯均衡实操

- Consider NE (OA, F)
  - ➢ σ OK if $\mu = (1, 0)$;
  - ➢ Given σ, totally mixed strategy

$$\sigma_E^k = (Out\ 1 - \epsilon^k, IN\ \epsilon^k; F\ \eta^k, A\ 1 - \eta^k).$$

  - ➢ Use Bayes' rule, x_L is left decision node,

$$\mu^k(x_L) = Pr(x_L | h, \sigma^k) = \frac{Pr(x_L, h | \sigma^k)}{Pr(h | \sigma^k)} = \frac{\epsilon^k \eta^k}{\epsilon^k} = \eta^k;$$

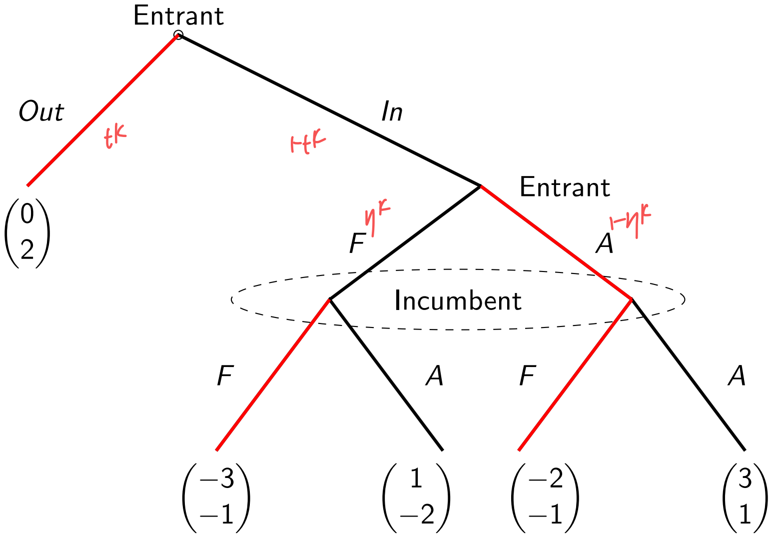
  - ➢ Thus,

$$\mu(x_L) = \lim_{k \to \infty} \mu^k(x_L) = 0;$$

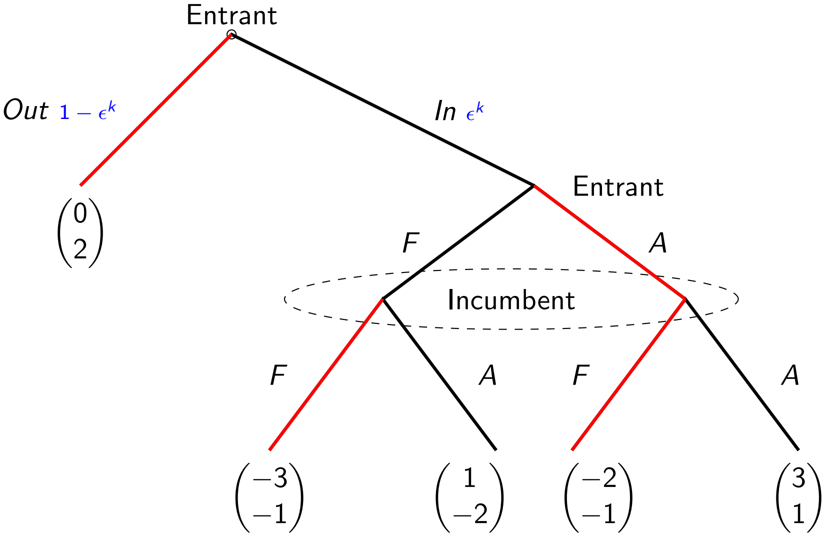
  - ➢ But given $\mu(x_L) = 0$, optimal choice for Incumbent is A, not F!
- (OA, F) not S.E. either.

Construct σ^k: (OA, F)

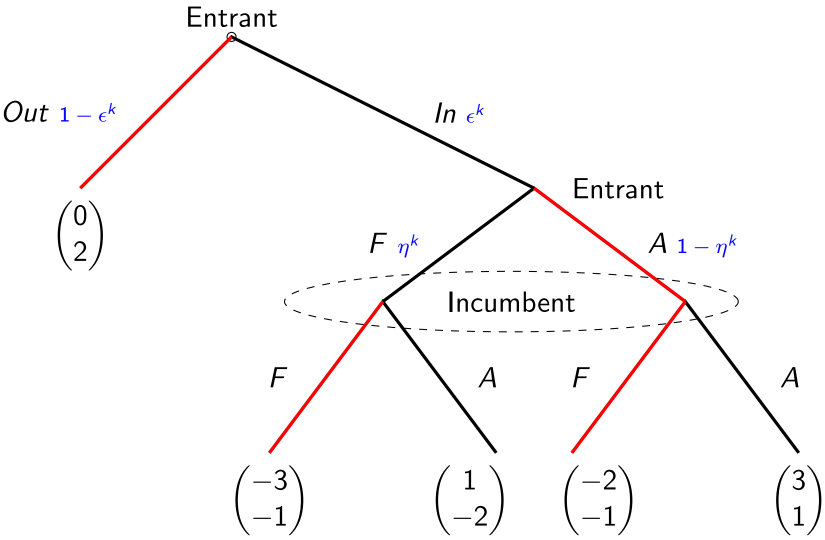
(IA , A)



Construct σ^k: (OA, F)



Construct σ^k: (OA, F)



S.E. of Entrant-incumbent example 2

- Strategy

$$(IA, A);$$

- Belief:

$$\mu = (0, 1).$$

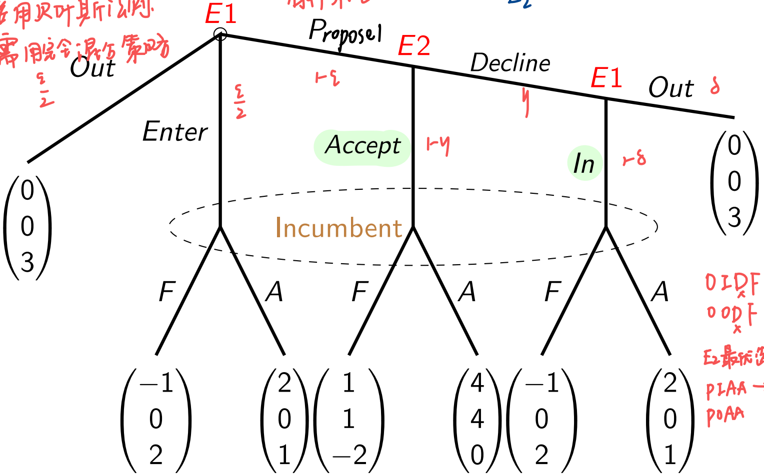
- That is, Incumbent assigns probability 0 to the decision node after F, and probability 1 to the decision after A.

Entrant incumbent game 3

信息集在均衡路径上
可直接用贝叶斯法则
而不需用完全混合策略

只有一个子博弈
↓
有1个SPNE

E1 2个信息集 6个pure stratigies
E2 1 2



OIDF
OODF

E2最优选择为A
PIAA → 序贯均衡
POAA 唯一

Firm E2

Firm E1

	Accept	Decline
OI	0, 0, 3	0, 0, 3
OO	0, 0, 3	0, 0, 3
EI	-1, 0, 2	-1, 0, 2
EO	-1, 0, 2	-1, 0, 2
PI	1, 1, -2	-1, 0, 2
PO	1, 1, -2	0, 0, 3

I fight

Firm E1

	Accept	Decline
OI	0, 0, 3	0, 0, 3
OO	0, 0, 3	0, 0, 3
EI	2, 0, 1	2, 0, 1
EO	2, 0, 1	2, 0, 1
PI	4, 4, 0	2, 0, 1
PO	4, 4, 0	0, 0, 3

I accomodate

- While this game has several pure strategy NE, there is only one SE.

Selten's Horse

都只有1个信息集，2个选择，无先后顺序

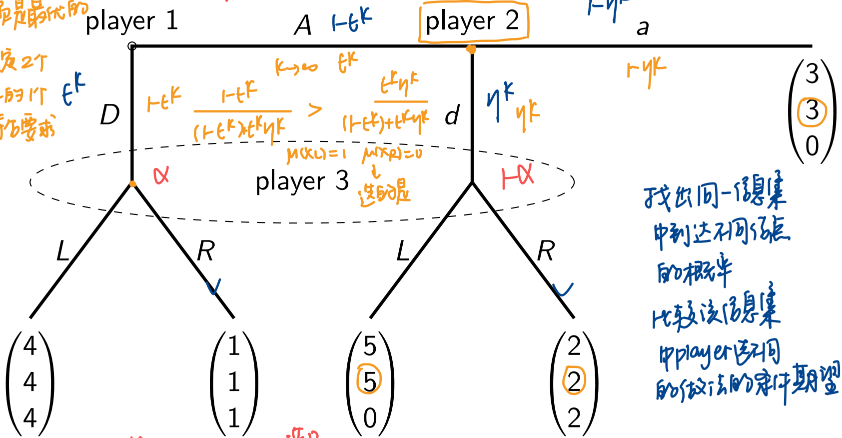
D,a,L　A,a,R

↓

非序贯理性　仍应该选 d 非a

序贯理性：
每个玩家在每个信息集的选项是最优的

依次固定2个
看剩下的1个 ϵ^k
是否符合要求



$1-\epsilon^k$　$1-\eta^k$　η^k

α　$1-\alpha$

$k\to\infty$　ϵ^k

$1-\epsilon^k$　$\frac{1-\epsilon^k}{(1-\epsilon^k)+\epsilon^k\eta^k} > \frac{\epsilon^k\eta^k}{(1-\epsilon^k)+\epsilon^k\eta^k}$　η^k

$\mu(x_L)=1$　$\mu(x_R)=0$

↓

选R混

选L　选R

$4\alpha \le 1\cdot\alpha + 2\cdot(1-\alpha)$

$\boxed{\text{SE}}$实操　Dal

① 算条件期望，比大小

② 比不出就设α, $1-\alpha$
算期望payoff，得结果

打Dal

找出同一信息集
中到达不同结点
的概率
比较该信息集
中player选不同
的做法的条件期望

- The strategical form

Player 2

Player 1

	a	d
A	(3, 3, 0)	(5, 5, 0)
D	(4, 4, 4)	(4, 4, 4)

3 plays L

Player 1

	a	d
A	(3, 3, 0)	(2, 2, 2)
D	(1, 1, 1)	(1, 1, 1)

R

SELTEN'S HORSE CONTINUED

- The strategical form

Player 2

Player 1		a	d
	A	(3, 3, 0)	(5, 5, 0)
	D	(4, 4, 4)	(4, 4, 4)

3 plays L

Player 1		a	d
	A	(3, 3, 0)	(2, 2, 2)
	D	(1, 1, 1)	(1, 1, 1)

R

- Two pure NE:

SPNE → 2 → $(D, a, L), \quad (A, a, R).$

由拟对博弈

Another Example

子博弈 1 个

SPNE/NE: 2 个

RCU ✓

RSV ← X

序贯均衡(实为 SPNE)

① 找 NE

② 找序贯理性

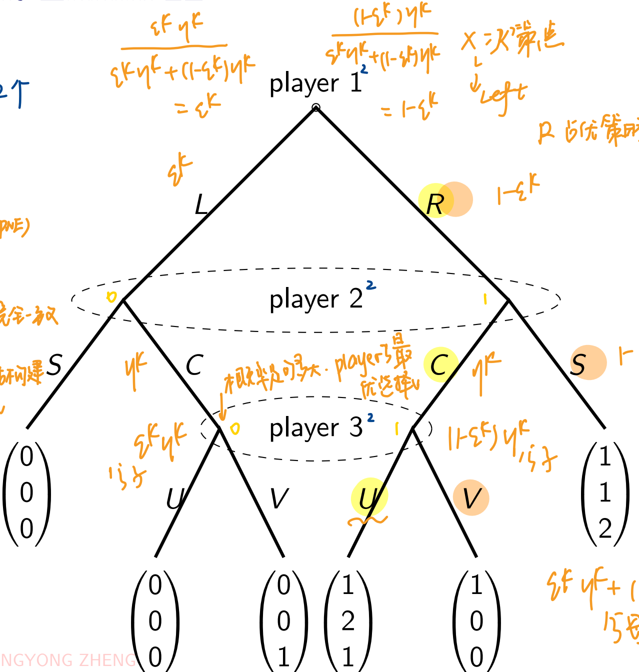
③ 信念与均衡完全一致

(0,1) 看出来 ✓

完全混合策略构建 看不出来 ✓

X: 次序靠 Left

R 占优策略



$\varepsilon^k y^k + (1-\varepsilon^k) y^k = y^k$ 分母

Find S.E.

- Strategic form

Player 2

	S	C
L	(0, 0, 0)	(0, 0, 0)
R	(1, 1, 2)	(1, 2, 1)

3 plays U

	S	C
L	(0, 0, 0)	(0, 0, 1)
R	(1, 1, 2)	(1, 0, 0)

V

FIND S.E.

- Strategic form

Player 2

	S	C
L	(0, 0, 0)	(0, 0, 0)
R	(1, 1, 2)	(1, 2, 1)

3 plays U

	S	C
L	(0, 0, 0)	(0, 0, 1)
R	(1, 1, 2)	(1, 0, 0)

V

- Two pure NE: (R, C, U), (R, S, V).

Implications of conditions imposed by SE

- On behavior: In NO circumstances should a player makes a choices that is dominated by other choices. Therefore, the strategy should specify optimal choice at every information set given the beliefs about what has happened previously, thus the probability distribution over different decision nodes at the information set, as well as what the other players are playing.
- On belief (off-the equilibrium-path behavior by the other players):
  - ➢ Simply put, belief about what has happened thus which decision note one faces should be consistent with sequential rationality on the part of opponents;
  - ➢ At EVERY information sets when an opponent played the game, one should think that she has played her best response.
  - ➢ As a direct consequence, at every information set, if the player has a dominant choice, one that is better than the rest of choices regardless of what the choices of other players, then her opponent's belief should put probability one to the dominant choice, and zero to the rest of choices.

More on SE beliefs

完全混合策略的组合→贝叶斯法则

- SE belief is consistent, derived from equilibrium strategies using totally mixed strategies.
- But it may not be stracturally consistent.
- *Structural consistency*: A belief system μ is structurally consistent if for each information set h, there exists some strategy profile σ such that for all $x \in h$,

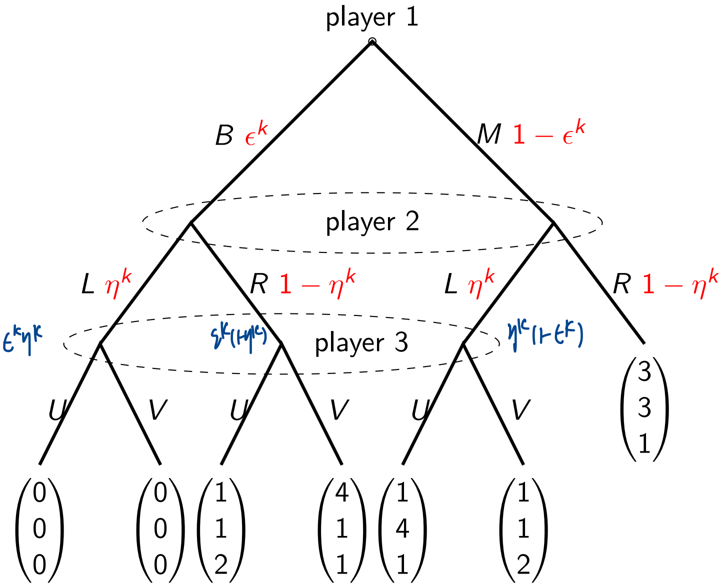
混合策略的σ

x节点

$$\mu(x) = \frac{prob(x|\sigma)}{prob(h|\sigma)}.$$

h信息集

Example 228.2 of Osborne and Rubinstein



Solve for SE

- The NE of this game is 无NE，仅一混合NE

$$(M, R, (\alpha, 1-\alpha) | \alpha \in [1/3, 2/3]).$$

- S.E of this game:
  - Strategy σ

$$\left\{ M; R; (\alpha, 1-\alpha) | \alpha \in \left[\frac{1}{3}, \frac{2}{3}\right] \right\};$$

  - Belief μ
    - Player 2's belief: $(0, 1)$
    - Player 3's belief: $(0, 0.5, 0.5)$. 用序贯均衡定义算出

Derive SE belief: player 2

- Let the totally mixed strategy profit σ^k be
$$(\epsilon^k, 1-\epsilon^k; \eta^k, 1-\eta^k; \alpha, 1-\alpha)$$
such that
$$\lim_{k\to\infty} \epsilon^k = 0, \qquad \lim_{k\to\infty} \eta^k = 0.$$
- Denote the 2 decision nodes, in the order of left, right, as z_L, z_R.
$$\mu^k(z_L) = \frac{\epsilon^k}{1} = \epsilon^k$$
$$\mu^k(z_R) = \frac{1-\epsilon^k}{1} = 1-\epsilon^k$$
- Taking limit we have
$$\mu(z_L) = \lim_{k\to\infty} \mu^k(z_L) = 0$$
$$\mu(x_M) = \lim_{k\to\infty} \mu^k(z_R) = 1.$$

Player 3's belief

- Recall the totally mixed strategy profit σ^k:
$$(\epsilon^k, 1-\epsilon^k; \eta^k, 1-\eta^k; \alpha, 1-\alpha).$$
- Denote the 3 decision nodes, in the order of left, middle and right, as x_L, x_M, x_R.
$$\mu^k(x_L) = \frac{\eta^k\epsilon^k}{\eta^k\epsilon^k + \epsilon^k(1-\eta^k) + (1-\epsilon^k)\eta^k} = \frac{\eta^k\epsilon^k}{\epsilon^k + (1-\epsilon^k)\eta^k}$$
$$\mu^k(x_M) = \frac{(1-\eta^k)\epsilon^k}{\eta^k\epsilon^k + \epsilon^k(1-\eta^k) + (1-\epsilon^k)\eta^k} = \frac{(1-\eta^k)\epsilon^k}{\epsilon^k + (1-\epsilon^k)\eta^k}$$
$$\mu^k(x_R) = \frac{(1-\epsilon^k)\eta^k}{\eta^k\epsilon^k + \epsilon^k(1-\eta^k) + (1-\epsilon^k)\eta^k} = \frac{\eta^k(1-\epsilon^k)}{\epsilon^k + (1-\epsilon^k)\eta^k}$$
- Taking limit we have
$$\mu(x_L) = \lim_{k\to\infty} \mu^k(x_L) = 0$$
$$\mu(x_M) = \lim_{k\to\infty} \mu^k(x_M) = \frac{1}{2}.$$

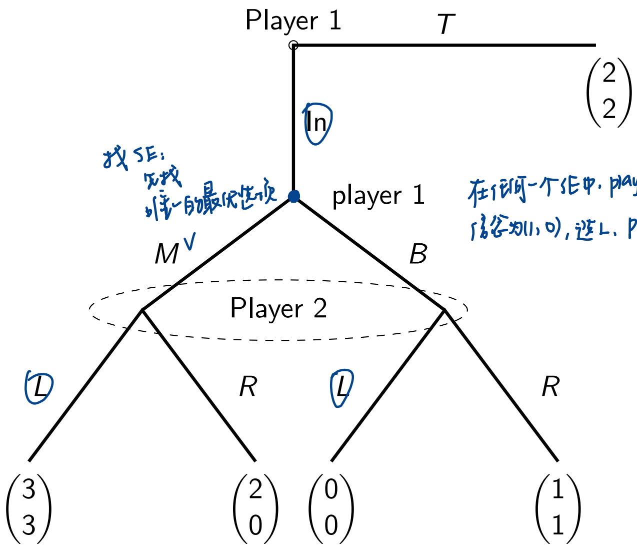
Example continued

- While in general, the totally mixed strategy for player 1 could be $(1-\epsilon, \epsilon)$ and for player 2 could be $(1-\eta, \eta)$, for consistency, it nevertheless must be true that $\eta(1-\epsilon) = \epsilon(1-\eta)$, so that player 3 assigns equal probability to the upper and the middle decision nodes.
- Only in this case will player 3 be indifferent between the two pure strategies U and V, which makes the mixed strategy best response motivated.
- Hence, it must be true that $\epsilon = \eta$.
- The belief for player 3, while consistent with the totally mixed strategy, is not structurally consistent. $(0, \frac{1}{2}, \frac{1}{2})$

Flaw in SE

- Conformity with backward induction, while being necessary, is not sufficient for strategic stability.
- A basic flaw in the concept of "sequential equilibrium": it depends on all the arbitrary details with which the game tree is drawn.
- Sequential equilibrium may involve players playing dominated strategies.
- The main problems: the requirement of consistency on belief allows unreasonable beliefs.
  - ➢ Requirement on strategies: OK, players should make optimal choice at any point in the game tree given the belief.
  - ➢ Requirement on beliefs: only requirement the belief to come from a sequence of totally mixed strategies. But some sequence of totally mixed strategies may not make sense at all, thereby leading to unreasonable belief in sequential equilibrium.

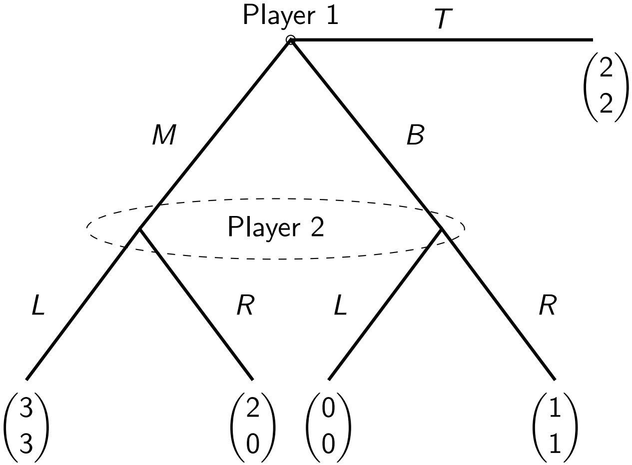
Example 1



找SE:
先找
小[illegible]的最优选项

在任何一个SE中, player 2的
信念为(1,0), 选L. Player1选In

Example 2



Example 2

Strategic form of game 2

Player 2 {ML 为SE, TR}

纯策略NE

Player 1		L	R
	M	(3, 3)	(2, 0)
	B	(0, 0)	(1, 1)
	T	(2, 2)	(2, 2)

but * T为弱被占优策略

Compare the two examples

- There is minimal difference between the two games;
- There is only one S.E. in Example 1 with 1 playing IM and 2 playing L;
- However, there are two S.E. in Example 2.
- One S.E.
- σ

$$\sigma: \ (T, R); \qquad \mu: \ (0, 1).$$

- The belief (0,1) comes from the totally mixed strategy

$$\sigma_1^k = (M \ \epsilon^2, B \ \epsilon, T \ 1 - \epsilon - \epsilon^2).$$

- Clearly,

$$\mu^k(x_L) = \frac{\epsilon^2}{\epsilon + \epsilon^2}, \qquad \mu(x_L) = \lim_{k \to \infty} \mu^k(x_L) = 0.$$

Agent norm form perfect equilibrium

- The *agent normal form* of an extensive form game is the normal form of the game between agents, obtained by letting each information set be manned by a different agent, and by giving any agent of the same player that player's payoff.

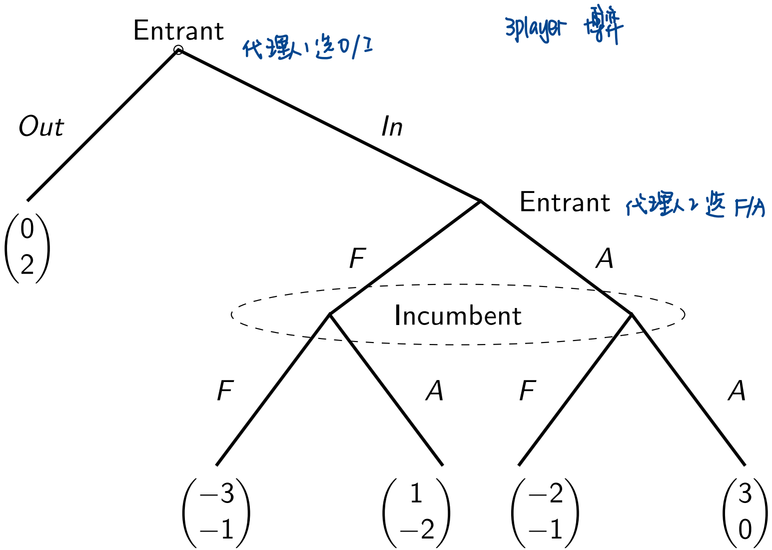
Agent norm form perfect equilibrium 贝叶均衡

/extensive

是SPNE

- The *agent normal form* of an extensive form game is the normal form of the game between agents, obtained by letting each information set be manned by a different agent, and by giving any agent of the same player that player's payoff.
- Also called extensive form perfect equilibrium.

Example: Entrant-incumbent game



Find the equilibrium

- Agent norm form

E1

	F	A
Out	0, 2	0, 2
In	-3, -1	1, -2

E_2 plays F

E1

	F	A
Out	0, 2	0, 2
In	-2, -1	3, 0

E_2 plays A

- Perfect equilibrium:

$$(IA, A).$$

IA, A ✓

PBE

完美贝叶斯均衡

是SPNE

完美信息博弈：3个均衡概念同 SPNE, Agent, PBE

不完美. SPNE < Agent/PBE

- A profile of strategies and system of beliefs (σ, μ) is Perfect Bayesian equilibrium if
    - I. σ is sequentially rational given μ; 俗称说，策略的序贯理性
    - II. The system of belief μ is obtained using Bayes rule **whenever possible**.

      只要有可能
- Perfect Bayesian equilibrium is also weak PBE as it requires Bayesian updating for all information set that is reached with positive probability under equilibrium strategy σ; but it also requires:
    - III. If an information set I is reached with zero probability under σ (off the equilibrium path), the belief at I is derived from σ using Bayes' rule, if possible.
- The restriction (iii) is evidently vague. One can interpret it as follows:

  If an information set I is reached with zero probability under σ (off the equilibrium path), the belief at I is derived, using Bayes' rule, from the beliefs at the information sets that precede I and players' continuation strategies as specified by σ, if possible.

PBE vs SE

- Both SE and PBE are subgame perfect.
- Sequential equilibrium is equivalent to a PBE in a general class of games.
- However, for some games, sequential equilibrium imposes more restrictions on off-the-equilibrium beliefs.
- Sequential equilibrium requires the beliefs of players at information sets not reached in the equilibrium to be derived from the SAME sequence of mixed strategies. PBE imposes no such restrictions on off-the-equilibrium beliefs.

Forward induction

- While SPNE and sequential equilibrium concept can help to rule out noncredible threat in extensive form game, a large range of off-equilibrium behavior can be justified by picking off-equilibrium-path behavior appropriately.
- Furthermore, PBE as well as sequential equilibrium can be sensitive to what may seem like irrelevant changes in the extensive form game.
- The key underlying forward induction is that players maintain the assumption that their opponents have maximized their utility in the past as long as the assumption is tenable, even if unexpected is observed.
- That is, while finding himself off the equilibrium path, he should not interpret it as a result of unintentional mistake by his opponents as long as the deviations by his opponents are *rationalizable*.

Forward induction vs backward induction

对于 Rationality 假设，对手会作出最优选择。 认为有犯错可能

不会将博弈割裂开考虑一个一个的子博弈

- A crucial consequence of forward induction is: a subgame can not be treated as a game on its own.
- In other words, a forward induction of a subgame need not be part of the solution of the whole game.
- This is *different* from backward induction.
- REMEMBER: A backward induction solution of a subgame is part of the subgame perfect NE of the whole game.
- In forward induction, how a subgame is reached conveys information about intended play in the subgame.
- One can use reduced strategic form game for forward induction, rather than the extensive form game used in backward induction.

向前归纳与重复删除弱劣策略会得同样结果

- For a large class of generic games, forward induction and iterated deletion of weakly dominated strategies yield the same set of solutions.

Outside option game

μ^k_{IRF} ε^k Ia,a σ^k μ^k

$\sigma=(1,0)$ $\mu^k(L)=\frac{(1-\eta^k)(1-\varepsilon^k)}{1-\eta^k}$ $=1-\varepsilon^k$

判定SE. Ia,a

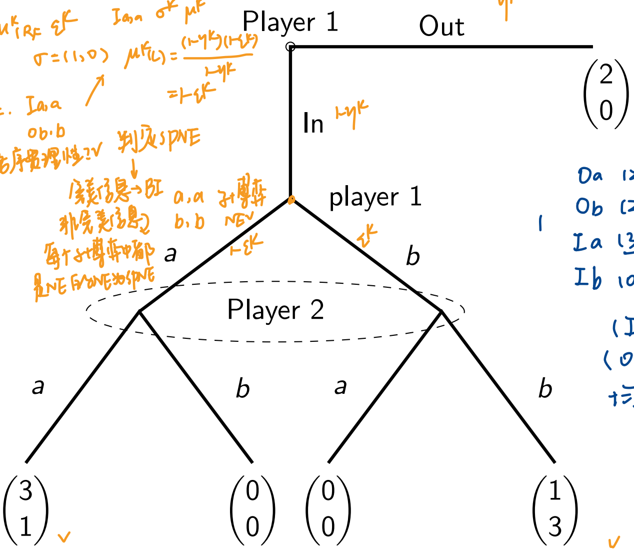
Ob,b

是否序贯理性?√ 判定SPNE

完美信息→BI a,a 子博弈

非完美信息? b,b NE√

每个子博弈中都是NE的NE为SPNE



1 \ 2	a	b
Oa	(2,0)	(2,0)
Ob	(2,0)	(2,0)
Ia	(3,1)	(0,0)
Ib	(0,0)	(1,3)

(Ia, a)
(Ob, b)
均为NE

Analyze the game

- Two pure NE. Both are also SPNE.

$$(Ia, a), \quad (Ob, b).$$

- Two pure SE
  - First one:
    - Strategy profile: (Ia, a);
    - Belief (1,0), that is, at her information set, player 2 believes 1 has played a.
  - Second one: (不满足Forward Induction)
    - Strategy profile: (Ob, b);
    - Belief (0, 1), that is, at her information set, player 2 believes 1 has played b.
- The second one does not pass the forward induction test: deviation by opponent should be rationalized first.

China-US trade talk

讨价还价　　完美信息动态博弈

Finite Bargaining game

确定outcome → 都知双方的payoff

Common knowledge

2player 假设 (BARGAINING 通常)

双方有确定的最后一期 + 完美信息 → 逆向归纳 → SPNE

- Split the dollar game: Two players divide a dollar between themselves.
- Let x_i denote the share of player i, $i = 1, 2$. The set of agreements is

最终达成的协议

$$X = \{(x_1, x_2) : x_i \geq 0, i = 1, 2 \text{ and } x_1 + x_2 = 1\}.$$

- The game last for T periods. (轮流)
  - ➢ In period 1, player 1 makes an offer to player 2.
  - ➢ If player 2 accepts, then they split the dollar according to the offer.
  - ➢ If player 2 rejects, then they move to period 2.
  - ➢ In period 2, they exchange roles with player 2 making an offer and player 1 decides whether to accept.
  - ➢ In general, player 1 makes offer in odd periods; player 2 makes offers in even periods.
  - ➢ The game continues until an agreement is reached or after the end of period T.

Bargaining continued

- If an agreement (x_1, x_2) is reached in period t, then player i receives payoff
$$u_i(x_i, t) = \delta_i^t x_i.$$

Bargaining continued

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$$(s, 1 - s)$$
is enforced in period T.

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- For finite T, we can solve the game by backward induction.

Bargaining continued

大家都不愿往后拖

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$$u_i(x_i, t) = \delta_i^t x_i.$$
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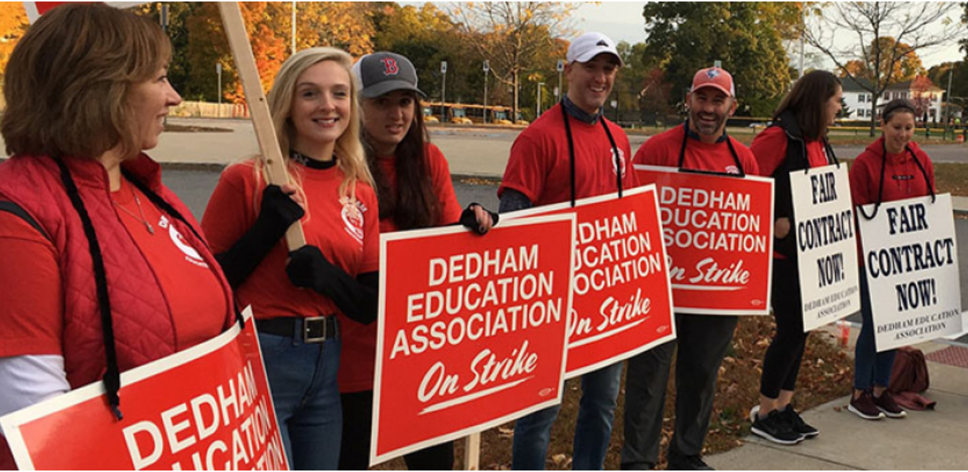
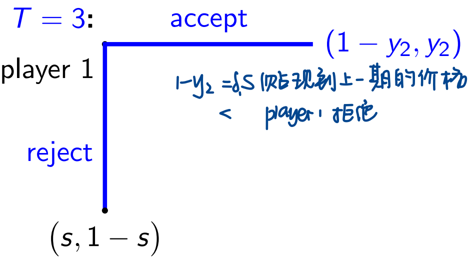
- If no agreement is reached after $T - 1$ periods, then a settlement
$$(s, 1-s)$$
分1块钱

  is enforced in period T.
- For finite T, we can solve the game by backward induction.
- This is commonly known as the Rubinstein bargaining game.

The consequence of no deal



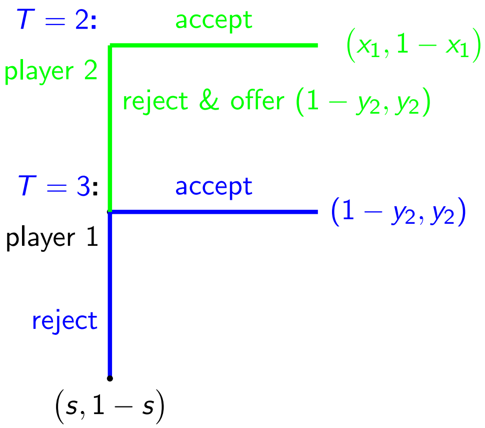
THE CASE OF $T = 3$



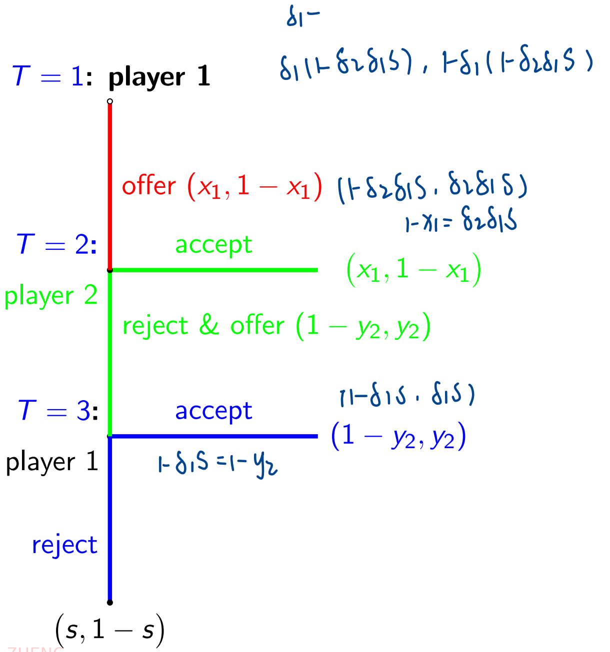
$1 - y_2 = \delta_1 s$ (贴现到上一期的价格)

$<$ player1 拒绝

THE CASE OF $T = 3$



THE CASE OF $T = 3$



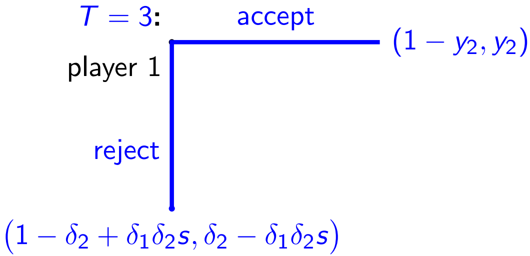
Solve the game: $T = 3$

- In period $t = 2$, Player 1 can obtain s in the next period by rejecting player 2's present offer.
- Thus, player 1 will reject any offer if and only if it is strictly worse than $(\delta_1 s, 1 - \delta_1 s)$.
- We will assume that a player accepts whenever he is indifferent between accepting and rejecting. All payoffs are evaluated from the current period.
- In period 1, player 2 knows that he can obtain $1 - \delta_1 s$ in the next period. Hence, by the same reasoning, he will accept the present offer iff

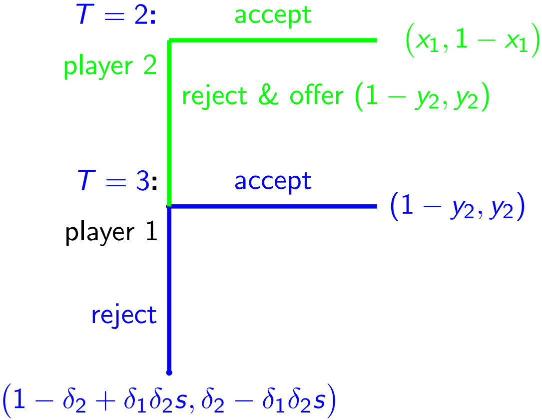
$$x_2 \geq \delta_2 (1 - \delta_1 s) = \delta_2 - \delta_1 \delta_2 s.$$

- Hence, in equilibrium player 1 proposes $(\underbrace{1 - \delta_2 + \delta_1 \delta_2 s}_{s'}, \underbrace{\delta_2 - \delta_1 \delta_2 s}_{1-s'})$ in $T = 1$ and player 2 accepts.

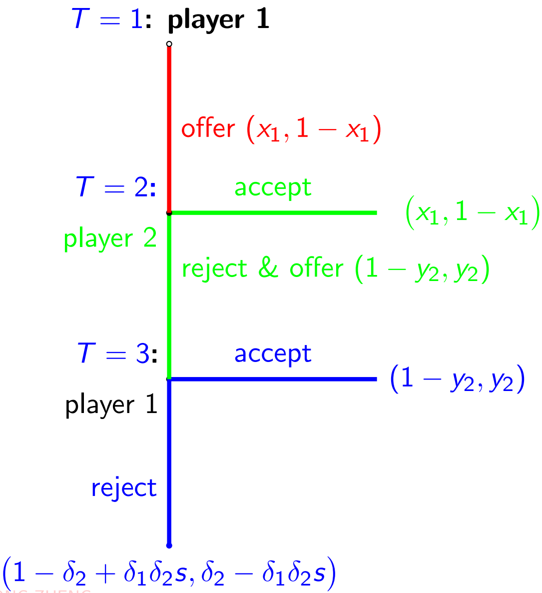
The case of $T = 5$



THE CASE OF $T = 5$



The case of $T = 5$



The case of $T = 5$

- The case of $T = 5$ is equivalent to $T = 3$ with the breakdown's payoff equal to

$$(1 - \delta_2 + \delta_1 \delta_2 s, \delta_2 - \delta_1 \delta_2 s) \, .$$

The case of $T = 5$

- The case of $T = 5$ is equivalent to $T = 3$ with the breakdown's payoff equal to

$$\left(1 - \delta_2 + \delta_1\delta_2 s, \delta_2 - \delta_1\delta_2 s\right).$$

- Substituting the new breakdown payoff into the equilibrium for $T = 3$ gives the first period offer:

$$\begin{aligned} x_1 &= 1 - \delta_2 + \delta_1\delta_2\left(1 - \delta_2 - \delta_1\delta_2 s\right) \\ &= \left(1 - \delta_2\right)\left(1 + \delta_1\delta_2\right) + \left(\delta_1\delta_2\right)^2 s, \end{aligned}$$

$$x_2 = 1 - \left(1 - \delta_2\right)\left(1 + \delta_1\delta_2\right) - \left(\delta_1\delta_2\right)^2 s.$$

The general case

- In general, when $T = 2n + 1$, we have player 1's equilibrium share

$$x_1^* (2n+1) = (1-\delta_2) \sum_{i=1}^{n} (\delta_1 \delta_2)^{i-1} + (\delta_1 \delta_2)^n s.$$

The general case 奇数期

- In general, when $T = 2n+1$, we have player 1's equilibrium share

数学归纳

$$x_1^*(2n+1) = (1-\delta_2)\sum_{i=1}^{n}(\delta_1\delta_2)^{i-1} + (\delta_1\delta_2)^n s.$$

- When $T = 2n+2$, we know that if the game proceeds to period 2, player 2 will obtain

$$(1-\delta_1)\sum_{i=1}^{n}(\delta_1\delta_2)^{i-1} + (\delta_1\delta_2)^n s.$$

The general case

- In general, when $T = 2n + 1$, we have player 1's equilibrium share

$$x_1^*\left(2n+1\right) = \left(1-\delta_2\right)\sum_{i=1}^{n}\left(\delta_1\delta_2\right)^{i-1} + \left(\delta_1\delta_2\right)^n s.$$

偶数期

- When $T = 2n + 2$, we know that if the game proceeds to period 2, player 2 will obtain

↓数学归纳

$$\left(1-\delta_1\right)\sum_{i=1}^{n}\left(\delta_1\delta_2\right)^{i-1} + \left(\delta_1\delta_2\right)^n s.$$

- So, in this case, player 1 offers in period 1

$$x_1^*\left(2n+2\right) = 1 - \delta_2\left(1-\delta_1\right)\sum_{i=1}^{n}\left(\delta_1\delta_2\right)^{i-1} - \delta_2\left(\delta_1\delta_2\right)^n s.$$

The limit case

- We can take limit to see how increasing T affects the result.
- As T goes to infinity,

有耐心/等得起 δ比较小
未来很重要 δ为1
→ player 2 非常有耐心
固定 δ_1, δ_2↑, player 1
得到的越小

$$\lim_T x_1^*(T) \equiv x_1^* = \frac{1-\delta_2}{1-\delta_1\delta_2};$$

$$\lim_T x_2^*(T) \equiv x_2^* = \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}.$$

- Note that the limit is the same whether T is odd or even.

On bargaining result

- There are several things that are remarkable about this result.
  - ➢ First, there is no delay. Agreement is reached immediately.
  - ➢ Second, the breakdown share is irrelevant; the division is entirely driven by the discounts factor.
  - ➢ Third, there is a first-mover advantage even though there are many periods of negotiation.
- Let $(y_1^*(T), y_2^*(T))$ denote the equilibrium division if player 2 proposes in the first period.

$$\lim_T y_1^*(T) \equiv y_1^* = \frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2};$$
$$\lim_T y_2^*(T) \equiv y_2^* = \frac{1-\delta_1}{1-\delta_1\delta_2}.$$

- Note that

$$x_2^* = \delta_2 y_2^* \text{ and } y_1^* = \delta_1 x_1^*.$$

Infinite bargaining

- When $T = \infty$, we can no longer solve the game by backward induction (since there is no final period).
- **The one-step deviation proof principle:** In any perfect information extensive-form game with either finite horizon or discounting, a strategy profile is a subgame perfect equilibrium if and only if no player can be better off in any subgame (including those not reached by the original equilibrium strategies) by deviating in only one information set in the subgame.
- Note that the principle only works for subgame perfect equilibrium in perfect information games. It is not true for Nash equilibrium, and it is not true for SPNE in games of imperfect information.

Main result on infinite bargaining

- Theorem: In the Rubinstein bargaining game with infinite horizon, there is a unique subgame perfect equilibrium where in every odd period, player 1 proposes (x_1^*, x_2^*) and player 2 accepts any $x_2 \geq x_2^*$, and in every even period player 2 proposes (y_1^*, y_2^*) and player 1 accepts any $y_1 \geq y_1^*$.
- Proof: To show that the strategy profile is subgame perfect, we need to show that no player can gain by deviating once immediately and follow the equilibrium strategy in the future.
- In all odd periods, player 1 obviously would not gain by proposing proposing $x_2 > x_2^*$.
- If player 1 proposes $x_2 < x_2^*$, then player 2 will rejects the offer and player 1 will obtain

$$y_1^* = \delta_1 x_1^* < x_1^*$$

in the next period, making him worse off.

Proof continued

- On the other hand, player 1 can get at most $\overline{x}_1$ in the next period by rejecting player 2's offer. So

$$\underline{y}_2 \geq 1 - \delta_1 \overline{x}_1.$$

- Combining the two equations, we have

$$\overline{x}_1 \leq 1 - \delta_2 + \delta_1 \delta_2 \overline{x}_1,$$

- So

$$\overline{x}_1 \leq \frac{1 - \delta_2}{1 - \delta_1 \delta_2}.$$

- Interchanging the roles of the players, the same argument implies that

$$\underline{x}_1 \geq 1 - \delta_2 \overline{y}_2.$$
$$\overline{y}_2 \leq 1 - \delta_1 \underline{x}_1.$$

- Combining the equations mean that

$$\underline{x}_1 \geq \frac{1 - \delta_2}{1 - \delta_1 \delta_2}.$$

Proof continued

- We end up with

$$\bar{x}_1 = \frac{1-\delta_2}{1-\delta_1\delta_2} \geq x_1^* \geq \underline{x}_1 = \frac{1-\delta_2}{1-\delta_1\delta_2}.$$

- Similarly we can get

$$\bar{y}_2 = \frac{1-\delta_1}{1-\delta_1\delta_2} \geq y_2^* \geq \underline{y}_2 = \frac{1-\delta_1}{1-\delta_1\delta_2}.$$

- The equilibrium is unique.