1.

(a).

$\sqrt{n}(\hat{\beta}-\beta)=(\dfrac{X'X}{n})^{-1}\dfrac{1}{\sqrt{n}}X'u$

by LLN ,$(\dfrac{X'X}{n})\xrightarrow{P} S_{X'X}$ (finite)

$\dfrac{1}{\sqrt{n}}X'u=\dfrac{1}{\sqrt{n}}\sum x_i u_i \qquad Ex_iu_i=0 \qquad Var(x_iu_i)=E(x_iu_iu_ix_i')=\sigma^2E(x_ix_i')$(finite)

$\therefore \hat{\beta}-\beta=O_p(\dfrac{1}{\sqrt{n}})$

(b).

$\hat{u}_i=y_i-x_i'\hat{\beta}=x_i'(\beta-\hat{\beta})+u_i$

$\dfrac{1}{n}\sum \hat{u}_i{}^2=\dfrac{1}{n}\sum[(\beta-\hat{\beta})'x_ix_i'(\beta-\hat{\beta})+2u_ix_i'(\beta-\hat{\beta})+u_i{}^2]=(\beta-\hat{\beta})'[\dfrac{1}{n}\sum x_ix_i'](\beta-\hat{\beta})+\dfrac{2}{n}[\sum u_ix_i'](\beta-\hat{\beta})+\dfrac{1}{n}\sum u_i^2=O_p(\dfrac{1}{\sqrt{n}})O_p(1)O_p(\dfrac{1}{\sqrt{n}})+O_p(\dfrac{1}{\sqrt{n}})O_p(\dfrac{1}{\sqrt{n}})+\dfrac{1}{n}\sum u_i^2\rightarrow\dfrac{1}{n}\sum u_i^2$

$\therefore \dfrac{1}{n}\sum u_i^2\sim N(\sigma^2,\dfrac{1}{n}Var(u_i^2))$

$\therefore \dfrac{1}{n}\sum(u_i^2-\sigma^2)\sim N(0,\dfrac{1}{n}B), Var(u_i^2)=B$

$\therefore \dfrac{1}{n}\sum \hat{u}_i{}^2=O_p(\dfrac{1}{n})+O_p(\dfrac{1}{\sqrt{n}})+\sigma^2$

$\therefore \dfrac{1}{n}\sum \hat{u}_i{}^2-\sigma^2=O_p(\dfrac{1}{\sqrt{n}})$

Finally, $\dfrac{1}{n-k}=\dfrac{1}{n}(\dfrac{n}{n-k})=\dfrac{1}{n}(\dfrac{n-k}{n-k}+\dfrac{k}{n-k})=\dfrac{1}{n}+O(\dfrac{1}{n^2})$

$\therefore \hat{\sigma^2}-\sigma^2\rightarrow\dfrac{1}{n}\sum \hat{u}_i{}^2-\sigma^2=O_p(\dfrac{1}{\sqrt{n}})$

2.

By Yule-Walker method

$X_t = Z_t + \theta_1 Z_{t-1}$

$X_t X_t = X_t Z_t + \theta_1 X_t Z_{t-1}$

$X_t X_t = (Z_t + \theta_1 Z_{t-1})Z_t + \theta_1(Z_t + \theta_1 Z_{t-1})Z_{t-1}$

By taking E $\Rightarrow \gamma(0) = \sigma^2 + \theta_1^2\sigma^2$

$X_{t-1}X_t = (Z_{t-1} + \theta_1 Z_{t-2})Z_t + \theta_1(Z_{t-1} + \theta_1 Z_{t-2})Z_{t-1} \Rightarrow \gamma(1) = \theta_1\sigma^2$

$X_{t-2}X_t = (Z_{t-2} + \theta_1 Z_{t-3})Z_t + \theta_1(Z_{t-2} + \theta_1 Z_{t-3})Z_{t-1} \Rightarrow \gamma(2) = 0$

For AR(1) process, $X_t = \phi X_{t-1} + Y_t$

$Y_t = X_t - \phi X_{t-1}$

$Y_t Y_t = Y_t X_t - \phi Y_t X_{t-1} = (X_t - \phi X_{t-1})X_t - \phi(X_t - \phi X_{t-1})X_{t-1}$

$\because E(Y_t) = 0 \qquad \therefore \tilde{\gamma}(0) = E(Y_t Y_t) = \gamma(0) - \phi\gamma(1) - \phi\gamma(1) + \phi^2\gamma(0)$

$Y_{t-1}Y_t = (X_{t-1} - \phi X_{t-2})X_t - \phi(X_{t-1} - \phi X_{t-2})X_{t-1}$

$\therefore \tilde{\gamma}(1) = \gamma(1) - \phi\gamma(0) + \phi^2\gamma(1)$

$Y_{t-2}Y_t = (X_{t-2} - \phi X_{t-3})X_t - \phi(X_{t-2} - \phi X_{t-3})X_{t-1}$

$\therefore \tilde{\gamma}(2) = -\phi\gamma(1)$

$Y_{t-3}Y_t = (X_{t-3} - \phi X_{t-4})X_t - \phi(X_{t-3} - \phi X_{t-4})X_{t-1}$

$\therefore \tilde{\gamma}(3) = 0$

$$\because \phi = \frac{\gamma(1)}{\gamma(0)} = \frac{\theta_1\sigma^2}{\sigma^2 + \theta_1^2\sigma^2} = \frac{\theta_1}{1+\theta_1^2}$$

$$\therefore \tilde{\gamma}(0) = \gamma(0) - 2\frac{\gamma(1)}{\gamma(0)}\cdot\gamma(1) + \frac{\gamma(1)^2}{\gamma(0)^2}\cdot\gamma(0) = \frac{\sigma^2 + \theta_1^2\sigma^2 + \theta_1^4\sigma^2}{1+\theta_1^2}$$

$$\tilde{\gamma}(1) = \gamma(1) - \frac{\gamma(1)}{\gamma(0)}\cdot\gamma(0) + \frac{\gamma(1)^2}{\gamma(0)^2}\cdot\gamma(1) = \frac{\theta_1^3\sigma^2}{(1+\theta_1^2)^2}$$

$$\tilde{\rho}(1) = \frac{\tilde{\gamma}(1)}{\tilde{\gamma}(0)} = \frac{\theta_1^3}{(1+\theta_1^2)(1+\theta_1^2+\theta_1^4)}$$

$$\tilde{\gamma}(2) = -\frac{\gamma(1)}{\gamma(0)}\gamma(1) = -\frac{\theta_1^2\sigma^2}{1+\theta_1^2}$$
$$\tilde{\rho}(2) = \frac{\tilde{\gamma}(2)}{\tilde{\gamma}(0)} = -\frac{\theta_1^2}{1+\theta_1^2+\theta_1^4}$$
$$\tilde{\rho}(3) = 0$$

3.

$$E(\frac{1}{n}\sum y_{t-1}^2) = \frac{1}{n}\cdot nE(y_{t-1}^2) = \sigma_y^2$$
$$Var(\frac{1}{n}\sum y_{t-1}^2) = \frac{1}{n^2}(\sum Var(y_{t-1}^2)) + 2\sum_{t=1}^n\sum_{s>t}^n cov(y_{t-1}^2, y_{s-1}^2)$$
$$\because y_s^2 = \rho^2 y_{s-1}^2 + u_s^2 + 2\rho y_{s-1}u_s$$
$$\therefore cov(y_s^2, y_{s-1}^2) = \rho^2 Var(y_{s-1}^2) = C\cdot\rho^2$$

Similarly, $cov(y_s^2, y_{s-T}^2) = C\cdot\rho^{2T}$

$$Var(\frac{1}{n}\sum y_{t-1}^2) = \frac{1}{n^2}(n\cdot C + 2\sum_{t=1}^n\sum_{s>t}^n C\cdot\rho^{2(s-t)})$$
$$\sum_{t=1}^n\sum_{s>t}^n \rho^{2(s-t)} = \frac{\rho^2(1-\rho^{2(n-1)})}{1-\rho^2} + \frac{\rho^2(1-\rho^{2(n-2)})}{1-\rho^2} + \cdots + \frac{\rho^2(1-\rho^2)}{1-\rho^2} = \frac{\rho^2}{1-\rho^2}[(n-1) - \frac{\rho^2(1-\rho^{2(n-1)})}{1-\rho^2}] = O(n)$$
$$\therefore Var(\frac{1}{n}\sum y_{t-1}^2) = O(\frac{1}{n}) \to 0$$
$$\therefore \frac{1}{n}\sum y_{t-1}^2 \overset{MSE}{\to} \sigma_y^2 \qquad \frac{1}{n}\sum y_{t-1}^2 \overset{P}{\to} \sigma_y^2$$

4.

(a).

$$y_t = \begin{pmatrix} 0.5 & 0.7 \\ -0.2 & 0.4 \end{pmatrix} y_{t-1} + \begin{pmatrix} 0.3 & 0 \\ 0 & 0 \end{pmatrix} y_{t-2} + \epsilon_t$$

(b).

$$\Gamma_1 = \begin{pmatrix} 0.5 & 0.7 \\ -0.2 & 0.4 \end{pmatrix} \qquad \Gamma_2 = \begin{pmatrix} 0.3 & 0 \\ 0 & 0 \end{pmatrix}$$

$\Gamma_1^2 = \begin{pmatrix} 0.11 & 0.63 \\ -0.18 & 0.02 \end{pmatrix}$

The effect on a_{t+2} of a one-unit shock in b_t is 0.63

5.

$\begin{pmatrix} \sqrt{n}(\hat{\beta}_1 - \beta_1) \\ n(\hat{\beta}_2 - \beta_2) \end{pmatrix} = D_n(X'X)^{-1}D_n \cdot (D_n)^{-1}X'u$

$(D_n)^{-1}X'u = \begin{pmatrix} \frac{1}{\sqrt{n}} & 0 \\ 0 & \frac{1}{n} \end{pmatrix} \sum \begin{pmatrix} x_{1t}u_t \\ x_{2t}u_t \end{pmatrix}$

$\frac{1}{\sqrt{n}} \sum x_{1t}u_t \xrightarrow{d} N(0, \sigma^2 E(x_{1t}^2)) = W(\lambda) \qquad \sum \frac{x_{2t}}{\sqrt{n}} \cdot \frac{u_t}{\sqrt{n}} \xrightarrow{d} \int_0^1 \sigma_v W_v(r) \sigma_u dW_u(r)$

$(D_n(X'X)^{-1}D_n)^{-1} = (D_n)^{-1}(X'X)(D_n)^{-1} = \begin{pmatrix} \frac{1}{n}\sum x_{1t}^2 & \frac{1}{n^{\frac{3}{2}}}\sum x_{1t}x_{2t} \\ \frac{1}{n^{\frac{3}{2}}}\sum x_{1t}x_{2t} & \frac{1}{n^2}\sum x_{2t}^2 \end{pmatrix}$

$\frac{1}{n^{\frac{3}{2}}}\sum x_{1t}x_{2t} = \frac{1}{n}\mu_1 \sum \frac{x_{2t}}{\sqrt{n}} + \frac{1}{\sqrt{n}} \sum \frac{x_{2t}}{\sqrt{n}} \frac{\epsilon_{1t}}{\sqrt{n}} = \mu_1 \sigma_v \int_0^1 W_v(r)dr + \frac{1}{\sqrt{n}} \int_0^1 \sigma_v W_v(r) \sigma_\epsilon dW_\epsilon(r) \rightarrow$
$\mu_1 \sigma_v \int_0^1 W_v(r)dr$

$\frac{1}{n}\sum x_{1t}^2 \rightarrow E(X_{1t}^2) \qquad \frac{1}{n^2}\sum x_{2t}^2 \rightarrow \sigma_v^2 \int_0^1 W_v(r)^2 dr$

$\therefore D_n(X'X)^{-1}D_n(D_n)^{-1}X'u \rightarrow \frac{\begin{pmatrix} \sigma_v^2 \int_0^1 W_v(r)^2 dr & -\mu_1\sigma_v \int_0^1 W_v(r)dr \\ -\mu_1\sigma_v \int_0^1 W_v(r)dr & E(x_{1t}^2) \end{pmatrix}}{E(x_{1t}^2) \cdot \sigma_v^2 \int_0^1 W_v(r)^2 dr - \mu_1^2 \sigma_v^2 [\int_0^1 W_v(r)dr]^2} \begin{pmatrix} W(\lambda) \\ \int_0^1 \sigma_v \sigma_u W_v(r) dW_u(r) \end{pmatrix}$

8.

$y_{it} - \bar{y_i} = (x_{it} - \bar{x_i})'\beta + (\epsilon_{it} - \bar{\epsilon_i})$

$\ddot{y_{it}} = \ddot{x_{it}}'\beta + \ddot{\epsilon_{it}}$

$\ddot{y_{it}} = \ddot{x_{it}}'b + \ddot{e_{it}}$

$\ddot{x_{it}}'\beta + \ddot{\epsilon_{it}} = \ddot{x_{it}}'b + \ddot{e_{it}}$

$\ddot{e_{it}} = \ddot{\epsilon_{it}} - \ddot{x_{it}}'(b-\beta)$

$\because$ We want to prove $\hat{\sigma_\epsilon^2} \xrightarrow{P} \sigma_\epsilon^2$ as $n \to \infty$

$\therefore$ k can be ignored

$\therefore \frac{1}{n(T-1)} \sum_{i=1}^n \sum_{t=1}^T \ddot{e_{it}}^2 = \frac{1}{n(T-1)} \sum_{i=1}^n \sum_{t=1}^T (\ddot{\epsilon_{it}} - \ddot{x_{it}}'(b-\beta))^2 = \frac{1}{n(T-1)} \sum_{i=1}^n \sum_{t=1}^T (\ddot{\epsilon_{it}}^2 + (b-\beta)'\ddot{x_{it}}\ddot{x_{it}}'(b-\beta) - 2(b-\beta)'\ddot{x_{it}}\ddot{\epsilon_{it}}) = \frac{T}{T-1}\frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T \ddot{\epsilon_{it}}^2 + \frac{T}{T-1}(b-\beta)'\frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T \ddot{x_{it}}\ddot{x_{it}}'(b-\beta) - \frac{2T}{T-1}(b-\beta)'\frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T \ddot{x_{it}}\ddot{\epsilon_{it}} \xrightarrow{P} \frac{T}{T-1}E\ddot{\epsilon_{it}}^2 + \frac{T}{T-1}(b-\beta)'E\ddot{x_{it}}\ddot{x_{it}}'(b-\beta) - \frac{2T}{T-1}(b-\beta)'E\ddot{x_{it}}\ddot{\epsilon_{it}}$

$E\ddot{\epsilon_{it}}^2 = E(\epsilon_{it}^2 - 2\epsilon_{it}\bar{\epsilon_i} + \bar{\epsilon_i}^2) = E(\epsilon_{it}^2 - 2\epsilon_{it}\frac{1}{T}\sum_{t=1}^T \epsilon_{it} + (\frac{1}{T}\sum_{t=1}^T \epsilon_{it})^2) = \sigma_\epsilon^2 - \frac{2}{T}\sigma_\epsilon^2 + \frac{1}{T}\sigma_\epsilon^2 = \frac{T-1}{T}\sigma_\epsilon^2$

$(b-\beta) = O_p(\frac{1}{\sqrt{n}})$ $\quad E\ddot{x_{it}}\ddot{x_{it}}'$ is finite $\quad E\ddot{x_{it}}\ddot{\epsilon_{it}} = 0$

$\therefore \hat{\sigma_\epsilon^2} = \frac{1}{n(T-1)-k} \sum_{i=1}^n \sum_{t=1}^T \ddot{e_{it}}^2 \to \frac{1}{n(T-1)} \sum_{i=1}^n \sum_{t=1}^T \ddot{e_{it}}^2 \xrightarrow{P} \sigma_\epsilon^2$ as $n \to \infty$