Liquidity Models

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Roadmap of Lecture

Holmstrom and Tirole 1997

Holmstrom and Tirole 1998

Appendix

Model

- There is a continuum of firms with access to the same investment technology and different amounts of capital A.
- The distribution of assets across firms is described by the cumulative distribution function $G(A)$.
- The investment required is I, so a firm needs to raise $I - A$ in external resources. The return is either 0 or R, and the probability depends on the type of project that the firm chooses.
- The firm may choose a lower type to enjoy private benefits.

Projects

Project	Good	Bad (no monitoring)	Bad (monitoring)
Prob of success	p_H	$p_L < p_H$	p_L
Private benefit	0	$B > 0$	$b < B$

Model (cont.)

- The rate of return demanded by investors is denoted as γ, which can either be fixed or coming from a supply function $S(\gamma)$.

- The assumption is that only the good project is viable:

$$p_H R - \gamma I > 0 > p_L R - \gamma I + B.$$

- The incentive of the firm to choose the good project will depend on how much "skin in the game" it has.

- Hence, it would be easier to finance firms with large assets A, since they are more likely to internalize the monetary benefit and choose the good project.

Financial Intermediaries

- In addition to investors who demand a rate of return γ, there are financial intermediaries, who can monitor the firm.

- Monitoring is assumed to prevent the firm from taking a B project, hence reducing the opportunity cost of the firm from B to b.

- Monitoring yields a private cost of c to the financial intermediary.

- Intermediary capital K_m will be important to provide incentives to the intermediary to monitor the firm (the Diamond solution of diversification is not considered here).

Direct Finance I

- Consider a contract where the firm invests A, the investor invests $I - A$, no one gets anything if the project fails, and in case of success the firm gets R_f and the investor gets R_u :

$$R_f + R_u = R$$

- A necessary condition is that the firm has an incentive to choose the good project:

$$p_H R_f \geq p_L R_f + B$$

- Denoting $\Delta p = p_H - p_L$, we get the incentive compatibility constraint:

$$R_f \geq B/\Delta p$$

- This implies that the maximum amount that can be promised to the investors (the pledgeable expected income) is:

$$p_H(R - B/\Delta p)$$

Direct Finance II

- Due to the participation constraint:

$$\gamma(I - A) \leq p_H(R - B/\Delta p)$$

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Indirect Financing I

- An intermediary can help relax the financing constraint of the firm by monitoring it and reducing its temptation to take the bad project.
- Now, the intermediary will get a share R_m of the return of the successful project

$$R_f + R_u + R_m = R$$

- The incentive constraint of the firm is now:

$$R_f \geq b/\Delta p$$

- There is also an incentive constraint for the intermediary:

$$R_m \geq c/\Delta p$$

- Then, the pledgeable expected income becomes:

$$p_H(R - (b + c)/\Delta p)$$

Indirect Financing II

- Suppose that the intermediary is making a return of β (which has to exceed γ due to the monitoring cost), and invests $I_m : \beta = p_H R_m / I_m$, because of the incentive constraint it will contribute a least: $I_m(\beta) = p_H c/(\Delta p)\beta$.
- Now, we can look at the financing constraint imposed by the participation constraint of the investors:

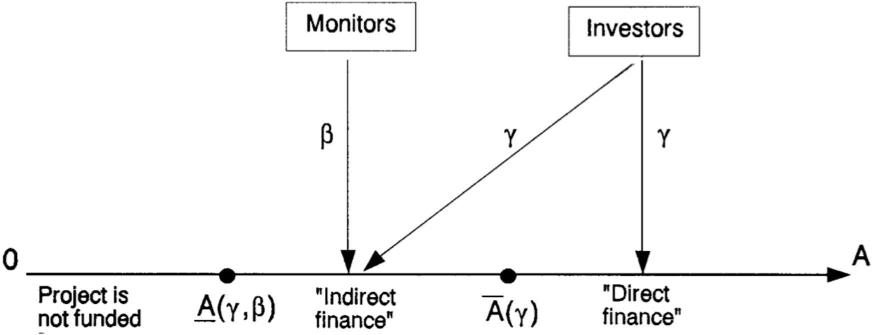
$$\gamma\left(I - A - I_m(\beta)\right) \leq p_H(R - (b+c)/\Delta p)$$

- This can be rewritten as:

$$A \geq \underline{A}(\gamma, \beta) = I - I_m(\beta) - p_H/\gamma(R - (b+c)/\Delta p)$$

- A firm with capital less than $\underline{A}(\gamma, \beta)$ cannot convince investors to supply it with capital even in the presence of intermediation. The firm will not increase reliance on intermediaries, as their capital is more expensive.

Main results



Main Results

- There are conditions in the paper guaranteeing that $\underline{A}(\gamma, \beta)$ is below $\bar{A}(\gamma)$
- The result is that small firms are not financed at all, intermeidate firms are financed by intermediaries and investors, and large firms are financed solely by investors.
- In equilibrium, the demand for capital equals the supply.
- The paper analyze the effect of decrease in the supply of capital.
- The main result is that the small firms are hurt most, as the squeeze leads to an increase $\underline{A}(\gamma, \beta)$.

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Appendix

Holmstrom-Tirole Setup

- Three dates $t = \{0, 1, 2\}$
- Firm has endowment A, chooses investment scale I at $t = 0$
- Liquidity shock $\rho \geq 0$ realised at $t = 1$
  - continuation scale $i(\rho) \leq I$
  - required reinvestment $\rho i(\rho)$, else project ceases
- Returns at $t = 2$
  - liquid (pledgeable) return $\rho_0 i(\rho)$
  - illiquid (private) return $(\rho_1 - \rho_0)\, i(\rho)$ to entrepreneur

Timing

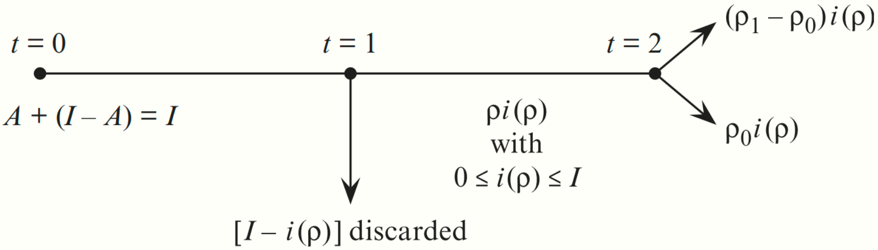

Figure 2.1
Timing

Binary liquidity shocks

- Two possible values
$$\rho \in \{\rho_L, \rho_H\}$$
with probabilities f_L, f_H respectively
- To focus on interesting cases, suppose
$$0 \leq \rho_L < \rho_0 < \rho_H < \rho_1$$
- Low shock ρ_L does not require pre-arranged financing, but high shock ρ_H does
- Also assumed that project is (i) socially desirable, and (ii) not self-financing

Second-best contract

- Specifies three terms

$$I, \quad i_L \equiv i(\rho_L), \quad i_H \equiv i(\rho_H)$$

  and payments to outside investors and entrepreneurs
- These maximise expected social return

$$\max_{I, i_L, i_H} \left[f_L (\rho_1 - \rho_L) i_L + f_H (\rho_1 - \rho_H) i_H - I \right]$$

  subject to the investor's budget constraint

$$f_L (\rho_0 - \rho_L) i_L + f_H (\rho_0 - \rho_H) i_H \geq I - A$$

  and feasibility

$$0 \leq i_L, i_H \leq I$$

- When low shock, firm pays investors $\rho_0 - \rho_L > 0$. When high shock, investors pay firm $\rho_H - \rho_0$
- Contract trades off ex ante scale vs. ex post liquidity

Entrepreneur rent

- Using budget constraint to eliminate I, we get an equivalent optimization problem that involves maximising net entrepreneurial rent

$$U = \max_{i_L, i_H} \left[f_L \left(\rho_1 - \rho_0 \right) i_L + f_H \left(\rho_1 - \rho_0 \right) i_H - A \right]$$

  subject to

$$0 \le i_L, i_H \le I$$

- Full social surplus goes to the entrepreneur (investors get their outside option)

Solving the contract

- If low liquidity shock, no tension. Since

$$\rho_1 - \rho_L > 0 \text{ and } \rho_0 - \rho_L > 0$$

  it is in everyone's interest to continue at full scale. Hence

$$i_L = I$$

  for some I to be determined
- Tension between I and i_H, both involve outlays by investors
- Fraction of project continued if high shock

$$x \equiv \frac{i_H}{I}$$

- Expected unit cost of continuing project

$$\bar{\rho}(x) \equiv f_L \rho_L + f_H \rho_H x$$

Solving the contract (cont.)

- Implies entrepreneurial rent (net social surplus)

$$U(x) = (\mu(x) - 1)A$$

where $\mu(x)$ is gross value of extra unit of entrepreneurial capital A

$$\mu(x) \equiv \frac{(\rho_1 - \rho_0)(f_L + f_H x)}{(1 + \bar{\rho}(x)) - \rho_0 (f_L + f_H x)}$$

- Original problem (SBC) is a linear program, hence solution is at one of the extreme points
- These correspond to $x = 0$ (continue project only if low shock) or $x = 1$ (always continue)

Summary of solution

- If $\rho = \rho_L$, project continues and $i_L = I$
- If $\rho = \rho_H$, project continues and $i_H = I$ if and only if

$$\rho_H < c \equiv \min\{1 + \bar{\rho}(1), \frac{1 + f_L \rho_L}{\rho_L}\}$$

  i.e., the unit cost of the liquidity shock is less than c, the unit cost of effective investment.
- Project is continued in both states if and only if

$$f_L(\rho_H - \rho_L) < 1$$

  Both a larger ρ_H and smaller ρ_L serve to increase ex-ante scale I at cost of reducing ex-post liquidity

Ex-ante scale

- From budget constraint

$$I = A + f_L\,(\rho_0 - \rho_L)\,i_L + f_H\,(\rho_0 - \rho_H)\,i_H$$

- Two cases:
  (i) $\rho_H < c$ so that $i_L = i_H = I$. Then

$$I = \frac{1}{1 + \bar{\rho}(1) - \rho_0} A$$

  (ii) $\rho_H > c$ so that $i_L = I$ but $i_H = 0$. Then

$$I = \frac{1}{1 + (\bar{\rho}(0) - \rho_0)\,f_L} A$$

Continuous liquidity shocks

- Continuous distribution of liquidity shocks $\rho \geq 0$
- Probability density function (PDF)

$$f(\rho) \geq 0, \quad \int_0^{\infty} f(\rho)\, d\rho = 1$$

- Cumulative distribution function (CDF)

$$F(\rho) = \int_0^{\rho} f(r)\, dr = \Pr[r \leq \rho]$$

Second best contract

- Maximises entrepreneur's expected rent

$$U = \max_{I, i(\rho)} \int (\rho_1 - \rho_0)\, i(\rho) f(\rho)\, d\rho$$

subject to the budget constraint

$$\int (\rho_0 - \rho)\, i(\rho) f(\rho)\, d\rho \geq I - A$$

and feasibility

$$0 \leq i(\rho) \leq I$$

Continuation policy

- Linearity of the optimisation problem implies continuation policy is a cutoff rule

$$i(\rho) = I \quad \text{for } \rho < \hat{\rho}$$

  and

$$i(\rho) = 0 \quad \text{for } \rho > \hat{\rho}$$

- Critical value $\hat{\rho}$ to be determined

Ex ante scale, continuous case

- Binding budget constraint implies

$$A = I - \int (\rho_0 - \rho)\, i(\rho) f(\rho)\, d\rho = I - \int_0^{\hat{\rho}} (\rho_0 - \rho)\, I f(\rho)\, d\rho$$
$$= \left(1 - \rho_0 F(\hat{\rho}) + \int_0^{\hat{\rho}} \rho f(\rho)\, d\rho\right) I$$

  or simply

$$I = k(\hat{\rho}) A$$

- Investment multiplier

$$k(\hat{\rho}) = \frac{1}{1 - \rho_0 F(\hat{\rho}) + \int_0^{\hat{\rho}} \rho f(\rho)\, d\rho}$$

- This is maximised at $\hat{\rho} = \rho_0$ with $k(\rho_0) > 1$ (continuing at full scale when $\rho_0 \geq \rho$), and is decreasing in $\hat{\rho}$ at ρ_1

Entrepreneurial rent

- Plugging back into objective

$$U(\hat{\rho}) = m(\hat{\rho})I = m(\hat{\rho})k(\hat{\rho})A$$

- Total expected return per unit investment (marginal return)

$$m(\hat{\rho}) = F(\hat{\rho})\rho_1 - 1 - \int_0^{\hat{\rho}} \rho f(\rho)\, d\rho$$

- This is maximised at $\hat{\rho} = \rho_1$ (continuing at full scale whenever $\rho_1 \geq \rho$), and is increasing in $\hat{\rho}$ at ρ_0

Fundamental Tradeoff

- Tension between investing in initial scale vs. saving funds to meet anticipated liquidity shocks
  - (i) lower $\hat{\rho}$ towards ρ_0 to increase size of investment $I = k(\hat{\rho})A$, or
  - (ii) increase $\hat{\rho}$ towards ρ_1 to increase ability to withstand liquidity shock ρ, this raises marginal return $m(\hat{\rho})$ on initial investment I

  (not both, binding IR constraint places limit on firm's investment)
- Solution is a ρ^* that balances $k(\hat{\rho})$ and $m(\hat{\rho})$ effects

$$\rho_0 < \rho^* < \rho_1$$

- Compromise between credit rationing initial scale and credit rationing reinvestment to meet liquidity shock

Solving for optimal ρ^*

- Can write entrepreneurial rent

$$U(\hat{\rho}) = \frac{\rho_1 - c(\hat{\rho})}{c(\hat{\rho}) - \rho_0} A$$

- Expected unit cost of effective investment

$$c(\hat{\rho}) = \frac{1 + \int_0^{\hat{\rho}} \rho f(\rho)\, d\rho}{F(\hat{\rho})}$$

- Maximising $U(\hat{\rho})$ is achieved by minimising $c(\hat{\rho})$, first order condition for this can be written

$$1 = \int_0^{\rho^*} F(\rho)\, d\rho$$

- Interior solutions depend only on $F(\rho)$, not ρ_0, ρ_1, A etc

Overview of second best contract

- Firm with capital A invests $I = k(\rho^*)\, A$
- Project continued if and only if $\rho < \rho^*$ where $\rho^* \in (\rho_0, \rho_1)$
- If project continued, then
  - firm paid $(\rho_1 - \rho_0)\, I$ for all ρ
  - outside investors paid $\rho_0 I$

Implementing the optimal contract

1. Credit line
   - outside investors lend $I - A$ at $t = 0$
   - credit line $\rho^* I$, can be used by firms at $t = 1$
   - such funds cannot be consumed, firm prefers to continue if possible [twist: credit line of $(\rho^* - \rho)\, I$ but allow investors claims to be diluted to cover shock]
2. Liquidity ratio
   - outside investors lend $(1 + \rho^*)\, I - A$ at $t = 0$
   - covenant that minimum $\rho^* I$ be kept in liquid assets, liquidity ratio

$$\frac{\rho^*}{1 + \rho^*}$$

Endogenous liquidity, no aggregate risk

- No storage technology, only assets created by firms can be used to store value
- Ex ante identical firms. Idiosyncratic liquidity shocks $\rho \sim \mathrm{IID}\, f(\rho)$ make firms heterogeneous ex post
- Risk neutral firms and consumers. Consumers have endowments large enough to finance any taxes and to finance all required investments. Cannot issue their own assets

Endogenous liquidity, no aggregate risk (cont.)

- To implement the second-best, additional funds needed at $t = 1$ are

$$D = I \int_0^{\rho^*} \rho f(\rho)\, d\rho$$

  (since firms are identical ex ante, I is the same for all firms)
- Credit line and liquidity ratio implementations of second best relied on exogenous supply of the liquid asset
- Can financial market generate endogenously the needed supply of liquid assets? Possible instruments
  - additional claims issued at date $t = 1$
  - holding shares in other firms

Distribution of liquidity

- Can show that without aggregate risk, total liquidity needs can be met endogenously
- Main problem is possible inefficient distribution of liquidity
- firms with $\rho < \rho_0$ have liquid assets they do not need
- firms with $\rho > \rho^*$ will shut down, release liquid assets
- firms with $\rho \in (\rho_0, \rho^*]$ want liquidity
- Need a way to transfer from excess liquidity firms to shortfall firms

Liquidity supply from financial intermediary

- Financial intermediation can pool the idiosyncratic risk of all firms thereby cross-subsidising unlucky firms
- With no aggregate uncertainty, financial intermediary can pool risk and second best can be implemented
- No particular role for government intervention

Endogenous liquidity with aggregate risk

- All firms receive the same ρ shock, perfectly correlated
- Firms cannot generally be self sufficient. For $\rho_0 < \rho < \rho^*$, firms need ρI but can only raise $\rho_0 I$
- Intermediaries cannot pool aggregate risk
- Role for government supplied liquid assets
- issue $(\rho^* - \rho_0)\, I$ bonds at $t = 0$, provides "storage facility" for cash
- firms invest $(1 + \rho^*)\, I - A$ at $t = 0$, spend $(\rho^* - \rho_0)\, I$ of this amount on bonds
- Government bonds "crowd-out" initial investment I at $t = 0$ but increase reinvestment at $t = 1$

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Appendix

Appendix!

Holmstrom-Tirole: Key Ingredients

Assumption 1: Limited pledgeable income and asset shortages.

- Inside liquidity: Es' income partially pledgeable (e.g., stocks/bonds).
- Fs' income (e.g., wages) not pledgeable. But pledgeable to G.
- Outside liquidity: Durable goods (e.g. mortgages), treasuries, bubbles, etc.

Assumption 2: All promises, e.g., insurance, backed by pledgeable income.

- Collateral constraints with liquid assets serving as collateral.
- Complete markets, but only on pledgeable income.

Limited pledgeability

- Risk-neutral entrepreneur with an investment opportunity
- Opportunity worth Z_1 to entrepreneur but $Z_0 < Z_1$ to investors
- Initial investment I required to implement project

$$Z_0 < I < Z_1$$

- Positive net present value $I < Z_1$, but not self-financing, $Z_0 < I$
- Shortfall $I - Z_0$ must be covered by the entrepreneur
- Entrepreneurial rent $Z_1 - Z_0$ cannot be pledged to investors (e.g., because of private benefits, different beliefs, non-transferability)

Limited pledgeability

- Value of project to entrepreneur Z_1
- Value of project to investors Z_0
- Entrepreneurial rent $Z_1 - Z_0$
- Investment shortfall $I - Z_0$

Credit Rationing with fixed investment scale

- Let $A > 0$ be entrepreneurial capital committed to project
- Project can proceed if and only if pledgeable income Z_0 exceeds financing need $I - A$, i.e.,

$$A \geq \bar{A} \equiv I - Z_0$$

- If $A < \bar{A}$, entrepreneur is credit-rationed
  - entrepreneurial rent $Z_1 - Z_0 > 0$ is necessary for credit-rationing (else all positive NPV projects are self-financing)
  - entrepreneur must also be capital poor $A < Z_1 - Z_0$ (else firm can pay ex-ante for ex-post rents)

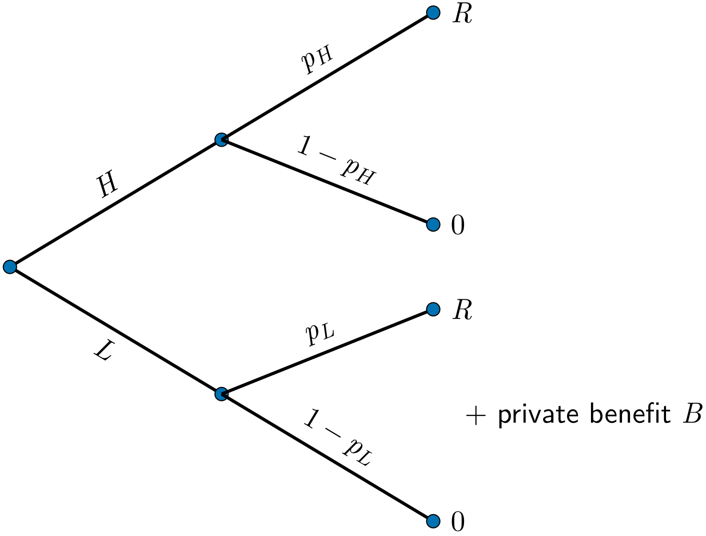
$$\mathrm{NPV} = Z_1 - I \geq Z_1 - Z_0 - A = \text{ net entrepreneurial rent}$$

- Positive NPV projects may go unfunded if capital poor

Moral Hazard and Limited Pledgeability

- Model of a wedge between project value and pledgeable income
- Two periods $t = \{0, 1\}$
- Project gross payoff R (success, s) or 0 (failure, f) at time $t = 1$
- Moral hazard problem: entrepreneur chooses probability of success
    - if diligent, probability of success is high p_H
    - if shirks, probability of success is low $p_L < p_H$, obtains private benefit B

Moral Hazard Timing



Moral hazard constraints

- Project returns shared between entrepreneur and investors
- Payments to entrepreneurs contingent on outcome, X_s or X_f
- **Individual rationality**: investors break even if

$$p_H (R - X_s) + (1 - p_H) \left(0 - X_f\right) \geq I - A$$

- **Incentive compatibility**: entrepreneur diligent if

$$p_H X_s + (1 - p_H) X_f \geq p_L X_s + (1 - p_L) X_f + B$$

or

$$X_s - X_f \geq \frac{B}{\Delta p}, \quad \Delta p \equiv p_H - p_L$$

- Limited liability: $X_f, X_s \geq 0$

Moral hazard and pledgeable income

- Limited liability and incentive compatibility together imply an entrepreneurial rent

- Entrepreneurial rent minimized by setting

$$X_s = \frac{B}{\Delta p}, \quad X_f = 0$$

- Pledgeable income is the maximum that can be promised to investors

$$Z_0 = p_H (R - X_s) = p_H \left(R - \frac{B}{\Delta p} \right)$$

Factors influencing pledgeable income

- Bias towards less risky projects (if the entrepreneur has a portfolio of projects to choose from)
- But diversification across projects increases pledgeable income from the portfolio (if projects are not perfectly correlated)
- Financial intermediation, loan covenants, costly monitoring etc

Variable investment scale

- Now I is the scale of investment, not the fixed amount
- Let ρ_1 denote expected return per unit investment, ρ_0 denote pledgeable return per unit investment

$$0 < \rho_0 < 1 < \rho_1$$

- Total project payoff $\rho_1 I$, with $\rho_0 I$ pledged to investors, entrepreneurial rent $(\rho_1 - \rho_0)\, I$
- Entrepreneur's endowed with capital A, $\rho_0 I$ raised from investors, remaining $(1 - \rho_0)\, I$ covered by own capital

$$(1 - \rho_0)\, I \leq A$$

Equity multiplier

- If this constraint is binding (maximum scale), I is a proportion of own funds

$$I = kA, \quad k \equiv \frac{1}{1-\rho_0} > 1$$

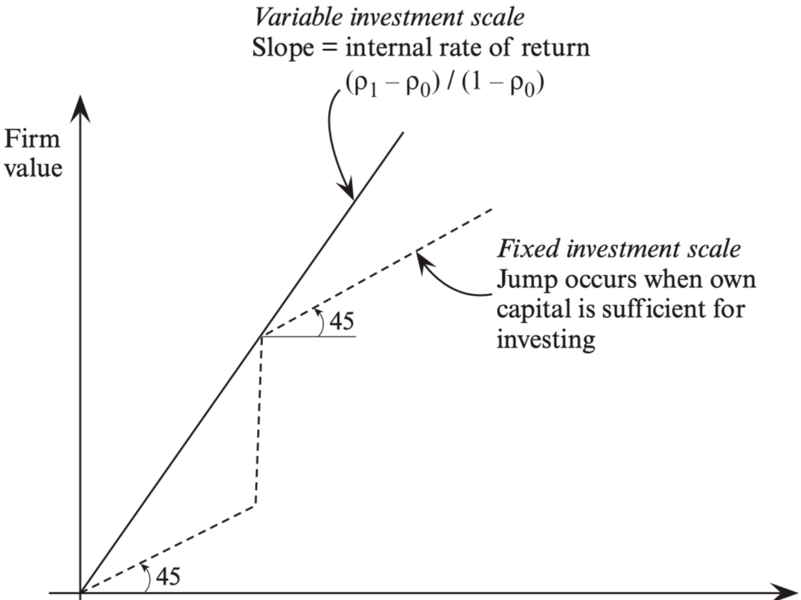
- A measure of leverage
- Gross payoff to entrepreneur

$$(\rho_1 - \rho_0)\, I = \frac{\rho_1 - \rho_0}{1-\rho_0} A \equiv \mu A, \quad \mu > 1$$

  where μ is the gross rate of return on own capital (internal rate of return), greater than market return $(= 1)$
- Net payoff to entrepreneur

$$U = (\mu - 1) A$$

Internal Rate of Return



NPV v.s. pledgeable income

- Consider the portfolio of projects distinguished by ρ_0, ρ_1
- Rate of return

$$\mu = \frac{\rho_1 - \rho_0}{1 - \rho_0}$$

- Holding μ fixed

$$\frac{d\rho_1}{d\rho_0} = 1 - \mu < 0$$

- Substitute NPV for more pledgeable income. Each ρ_0 is worth $\mu - 1$ units of ρ_1