Advanced Microeconomics I
Note 10: Extensive form games

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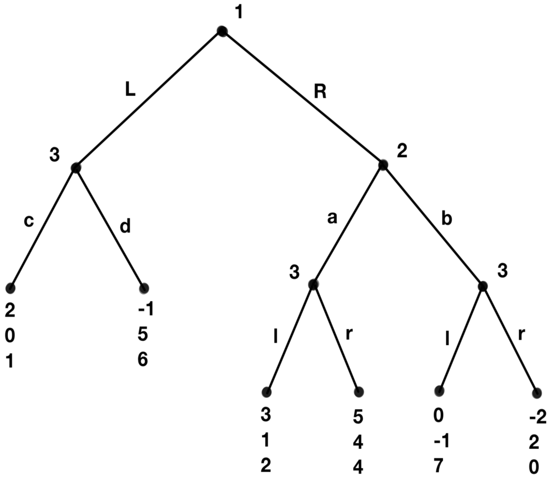
Extensive form representation

- A dynamic game generally has a richer set of rules compared to simultaneous move games.
  - Who moves when, what actions each player can take when it is his turn to move, what he knows when he moves
- To study dynamic games, we usually use the **extensive form representation**: a game tree


1
L
R
3
2
c
d
a
b
3
3
l
r
l
r
2 0 1
-1 5 6
3 1 2
5 4 4
0 -1 7
-2 2 0

- Extensive form: Γ_E
- A set of nodes $\mathcal{X}$, a set of possible actions $\mathcal{A}$, and a finite set of players $\{1, 2, ..., n\}$

- A function $p : \mathcal{X} \to \{\mathcal{X} \cup \phi\}$: $p(x)$ is the immediate predecessor of node x.
  - notice that $p(x)$ is unique
- There is a single node x_0 with $p(x_0) = \phi$: the initial node.
- A correspondence $s : \mathcal{X} \to \mathcal{X}$: $s(x)$ is the set of immediate successors of x.
- For any x, its set of predecessors and set of successors are disjoint.
  - No cycles are allowed.
- $T = \{x \in \mathcal{X} : s(x) = \phi\}$: terminal nodes
- $\mathcal{X} \setminus T$: decision nodes

- For each $x \in \mathcal{X}$ and $x' \in s(x)$, the branch connecting x and x' is labeled by an action $a \in \mathcal{A}$.
- For each decision node $x \in \mathcal{X} \setminus T$, $c(x)$ is the set of actions (choices) available at x.
    - For $a \in \mathcal{A}$, $a \in c(x)$ if and only if there is $x' \in s(x)$ such that x and x' are connected by a branch labeled by the action a.
    - If $x', x'' \in s(x)$ and $x' \neq x''$, then the two actions leading to x' and x'' from x cannot be the same.

- An **information set** $H \subseteq \mathcal{X} \setminus T$ is a collection of decision nodes. Each information set is labeled by one player and it represents a circumstance in which this player might be called upon to move.
- Each decision node belongs to one and only one information set. That is, all the information sets form a partition of $\mathcal{X} \setminus T$.
- If $x, x' \in H$, then $c(x) = c(x')$.
- Then, given an information set H, pick some $x \in H$ and denote $C(H) = c(x)$.
- Let $\mathcal{H}_i$ denote the collection of player i's information sets. Then $\mathcal{H} = \cup_i \mathcal{H}_i$ is the collection of all the information sets.
- Finally, each player i has a (Bernoulli) utility function $u_i : T \to \mathbb{R}$. This finishes the definition of Γ_E.

- The extensive form game Γ_E is finite if $\mathcal{X}$ is finite.
  - The game can be infinite due to infinite player set, infinite action set, or infinite time horizon.

Γ_E is an extensive form game with **perfect information** if every information set contains only one decision node. Otherwise it is an extensive form game with **imperfect information**.

However, we maintain the assumption of complete information in this note: the game structure Γ_E is common knowledge.

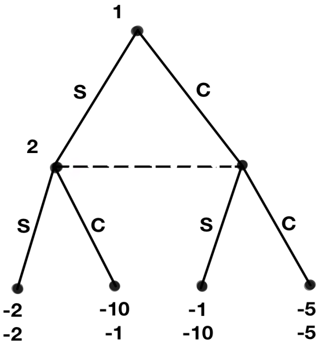
A simultaneous move game can be represented as an extensive form game with imperfect information.

Example: Prisoner's dilemma game.

Normal form representation:

	S	C
S	-2,-2	-10, -1
C	-1,-10	-5, -5

Extensive form representation:

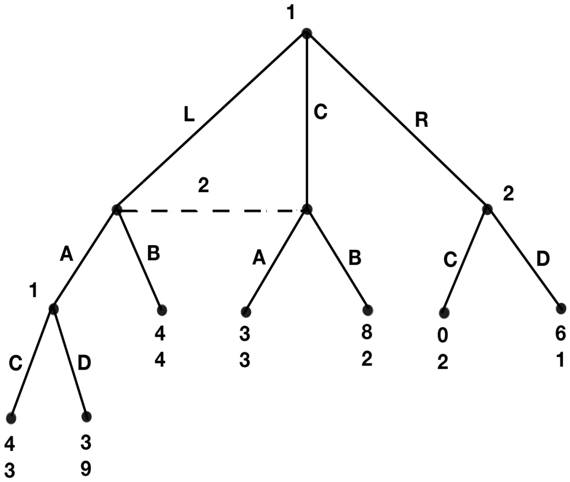


This cannot be overemphasized: *In an extensive form game, a strategy is a complete contingent plan that specifies how each player will act in every possible circumstance in which he might be called upon to move.*

Formally, in the extensive form game Γ_E, a **(pure) strategy** for player i is a function $s_i : \mathcal{H}_i \to \mathcal{A}$ such that $s_i(H) \in C(H)$ for all $H \in \mathcal{H}_i$.

Given any extensive form game, there is a unique normal form representation.

Example: find the normal form of the following game



In an extensive form game, a player can also play mixed strategies, as in normal form games. Additionally, a player has another way to randomize: he can randomize separately over the possible actions at each of his information sets. This is called a **behavior strategy**.

These two types of randomizations are equivalent in finite games with **perfect recall** (Kuhn, 1953).

A player has perfect recall if he never forgets what he once knew.

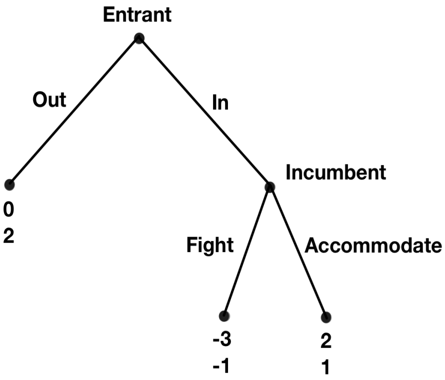
In analyzing dynamic games, we usually consider behavior strategies instead of mixed strategies.

Subgame perfection

- A natural starting point of solving a dynamic game is to find its Nash equilibria.
  - Given a dynamic game, we can (completely) represent this game using an extensive form.
  - Once the players' strategies are specified, we can derive a normal form representation of this game and apply the solution concept of Nash equilibrium.
- However, there is an important issue in this approach: the *credibility* of strategies in the NE of a dynamic game.

Example: *Entry deterrence game*

Extensive form:



Normal form:

	F	A
Out	0,2	0,2
In	-3,-1	2,1

There are two pure strategy NE: (Out, F) and (In, A), and the first one involves a *non-credible threat*.

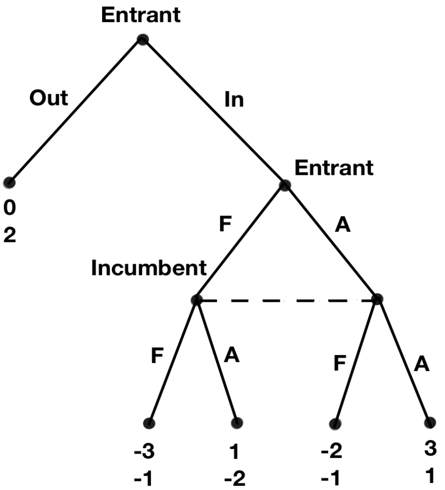
- To rule out such unreasonable equilibrium, generally in the dynamic setting we want players' equilibrium strategies to be **sequentially rational**: *the equilibrium strategies should specify optimal choice from any point in the game onward.*
- The principle of sequential rationality is first captured by a stronger solution concept called *subgame perfect Nash equilibrium*.
- Given an extensive form game Γ_E, a **subgame** is a subset of the game with the following two properties.
  - It begins with an information set containing a single decision node, contains all the decision nodes that are successors of this node, and contains only these nodes.
  - If a decision node x is in the subgame and $x, x' \in H$ for some information set H, then x' is also in the subgame.

Given an extensive form game Γ_E, a strategy profile σ is a **subgame perfect Nash equilibrium** (SPE) if it induces a Nash equilibrium in every subgame of Γ_E.

Since a game is a subgame of itself, SPE is a refinement on NE.

In the entry deterrence game, (Out, F) is not a SPE because it does not induce a NE in the subgame that starts from the incumbent's decision node (information set).

Example: Entry deterrence II



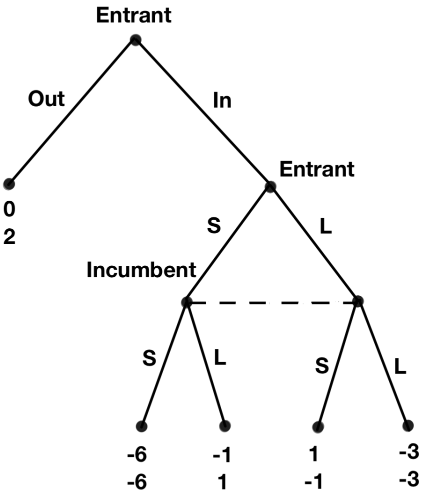
Three pure strategy NE: ((Out, A), F), ((Out, F), F), ((In, A), A)

Only the last one is SPE.

- Generally, for extensive form games, SPE can be found using the method of **backward induction**.
  - Start at the end of the game tree Γ_E, and identify the NE for all the final subgames.
  - Select one NE in each of the final subgames, and replace these final subgames using the payoffs in the selected NE.
  - Repeat this procedure for the reduced game tree, until every move in Γ_E is specified. The resulting collection of moves at each information set constitutes a SPE.

Example: find the SPE in the game on page 3 using backward induction.

Example: Entry deterrence III



Three SPE: ((In, L), S), ((Out, S), L), ((Out, $\frac{2}{9}$S+$\frac{7}{9}$L), $\frac{2}{9}$S+$\frac{7}{9}$L).

Example: the **Centipede game**

- Basic settings:
  - Two players each start with \$1 in front of them.
  - They alternate saying "stop" or "continue", starting with player 1.
  - When a player says "continue", \$1 is taken from his pile and \$2 are added to his opponent's pile.
  - When a player says "stop", the game ends and each player receives the money in his current pile.
  - The game also ends if both players' piles reach \$100.
- The unique SPE can be easily found using backward induction, and the SPE makes a sharp prediction.

Example: **sequential bargaining**

- Basic settings:
  - Player 1 and 2 are bargaining over one dollar.
  - At the beginning of $t = 1$, player 1 proposes $(s_1, 1 - s_1)$.
  - Player 2 either accepts, or rejects the offer (in which case play continues to period 2).
  - At the beginning of $t = 2$, player 2 proposes $(s_2, 1 - s_2)$.
  - Player 1 either accepts, or rejects the offer (in which case play continues to period 3).
  - At the beginning of $t = 3$, player 1 and 2 receive $(s, 1 - s)$: the *disagreement values*, with $s \in (0, 1)$.
  - Players are impatient: discount factor $\delta \in (0, 1)$.
- Agreement is immediate in the SPE.