1. Consider the two-player game whose extensive form representation (excluding payoffs) is depicted in Figure 1.

   (a) What are player 1's possible strategies? Player 2's?

   (b) Show that for any behavior strategy that player 1 might play, there is a realization equivalent mixed strategy; that is, a mixed strategy that generates the same probability distribution over the terminal nodes for any mixed strategy choice by player 2.

   (c) Show that the converse is also true: For any mixed strategy that player 1 might play, there is a realization equivalent behavior strategy.

   (d) Suppose that we change the game by merging the information sets at player 1's second round of moves (so that all four nodes are now in a single information set). Which of the two results in (b) and (c) still holds?

Answer:

(a) Player 1's possible (pure) strategies are:

$$S_1 = \{Lxx, Lxy, Lyx, Lyy, Mxx, Mxy, Myx, Myy, Rxx, Rxy, Ryx, Ryy\}.$$

Player 2's possible (pure) strategies are: $S_2 = \{l, r\}$.

(b) Assume that player 2 plays l, r with probabilities of s_1, s_2 respectively. Suppose at the root, player 1 chooses L, M, R with probabilities of p_1, p_2, p_3 respectively, where $p_1 + p_2 + p_3 = 1$; at his second information set (where the outcomes are T_1 through T_4), player 1 chooses x, y with probabilities of q_1, q_2 resp.($q_1 + q_2 = 1$); at his third information set, player 1 chooses x, y with probabilities of r_1, r_2 resp.($r_1 + r_2 = 1$). Now for the above behavioral strategy of player 1, there is a realization equivalent mixed strategy σ_1, which is:

$$\sigma_1(Lxx) = p_1, \sigma_1(Mxx) = p_2q_1, \sigma_1(Myx) = p_2q_2, \sigma_1(Rxx) = p_3r_1, \sigma_1(Rxy) = p_3r_2.$$

Note that $p_1 + p_2q_1 + p_2q_2 + p_3r_1 + p_3r_2 = 1$. When player 1 is using this mixed strategy, the probabilities that we reach each terminal nodes will be the same compared with his behavioral strategy stated above. to show $\Sigma\sigma_1(\cdot) = 1$

(c) Assume that player 2 plays l, r with probabilities of s_1, s_2 respectively. Given player 1's any mixed strategy

$$\sigma_1 = (p_1, p_2, \ldots, p_{11}, p_{12}),$$

there is a realization equivalent behavioral strategy, which is:

At the root, player 1 chooses L, M, R with probabilities of $p_1 + p_2 + p_3 + p_4, p_5 + p_6 + p_7 + p_8, p_9 + p_{10} + p_{11} + p_{12}$ respectively; at his second information set, player 1 chooses x, y with probabilities of $(p_5 + p_6)/(p_5 + p_6 + p_7 + p_8), (p_7 + p_8)/(p_5 + p_6 + p_7 + p_8)$, resp.; at his third information set, player 1 chooses x, y with probabilities of $(p_9 + p_{11})/(p_9 + p_{10} + p_{11} + p_{12}), (p_{10} + p_{12})/(p_9 + p_{10} + p_{11} + p_{12})$ resp.

(d) In this case, the game does not satisfy perfect recall, since if player 1 reaches his (only) information set after player 2 moves, he will not remember whether he chose M or R. The result in (b) still holds, whereas the result in (c) does not always true, that is, there is a mixed strategy for player 1 that is realization equivalent to any behavioral strategy, however, there does not always exist a behavioral strategy that is realization equivalent to a mixed strategy. ① $S_1 = \{LX, Ly, MX, My, RX, Ry\}$ then ignore r_1, r_2, Given player 1's behavioral strategy, $\sigma_1(LX) = P_1$, $\sigma_1(MX) = P_2 q_1$ $\sigma_1(My) = P_2 q_2$ $\sigma_1(RX) = P_3 q_1$ $\sigma_1(Ry) = P_3 q_2$ and $P_1 + P_2 q_1 + P_2 q_2 + P_3 q_1 + P_3 q_2 = 1$, the first part is true
② Given $\sigma_1 = (P_1, \cdots, P_6)$, $\sigma_2 = (S_1, S_2)$ $P(L) = P_1 + P_2$, P

2. For the game given below:

Player 2 ② ④

		B1	B2	B3	B4
	A1	(0, 6)	(3, 1)	(2, 0)	(3, 7)
Player 1 ③	A2	~~(1, 0)~~	~~(9, 4)~~	~~(0, 12)~~	~~(1, 1)~~
⑤	A3	~~(0, 0)~~	~~(10, -1)~~	~~(2, -3)~~	~~(0, 1)~~
	A4	(7, 3)	(0, 0)	(5, 1)	(-1, 2)
①	A5	~~(2, 8)~~	~~(-2, 1)~~	~~(3, 1)~~	~~(1, 0)~~

(a) Does any player have any dominated strategies? If yes, what are they?

(b) Find all pure strategy NE of the game.

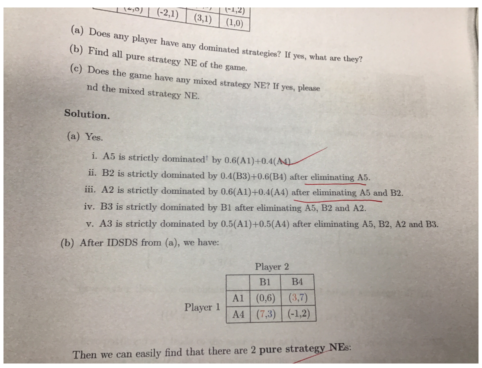
(c) Does the game have any mixed strategy NE? If yes, please find the mixed strategy NE.

Answer: $(A_1, A_4: a, 1-a)$ for $\frac{1}{2} < a < \frac{2}{3}$ or $(\frac{3}{5}A_1 + \frac{2}{5}A_4)$

(a) For player 1, A_5 is a strictly dominated strategy, dominated by $\sigma_1 = (\frac{4}{7}A_1, 0, 0, \frac{3}{7}A_4, 0)$.

For player 2, B_2 is a strictly dominated strategy, dominated by $\sigma_2 = (\frac{1}{3}B_1, 0, \frac{1}{3}B_3, \frac{1}{3}B_4)$.

(b) Iterative deletion of strictly dominated strategy gives:



(2,8)	(-2,1)	(3,1)	(1,0)

(a) Does any player have any dominated strategies? If yes, what are they?
(b) Find all pure strategy NE of the game.
(c) Does the game have any mixed strategy NE? If yes, please nd the mixed strategy NE.

Solution.

(a) Yes.

i. A5 is strictly dominated† by 0.6(A1)+0.4(A4).
ii. B2 is strictly dominated by 0.4(B3)+0.6(B4) after eliminating A5.
iii. A2 is strictly dominated by 0.6(A1)+0.4(A4) after eliminating A5 and B2.
iv. B3 is strictly dominated by B1 after eliminating A5, B2 and A2.
v. A3 is strictly dominated by 0.5(A1)+0.5(A4) after eliminating A5, B2, A2 and B3.

(b) After IDSDS from (a), we have:

Player 2

		B1	B4
Player 1	A1	(0,6)	(3,7)
	A4	(7,3)	(-1,2)

Then we can easily find that there are 2 pure strategy NEs:

Player 2

		β	$1-\beta$
		B1	B4
α	A1	(0, 6)	(3, 7)
$1-\alpha$	A4	(7, 3)	(-1, 2)

Thus, the two pure strategy NE are:

$$(A_4, B_1) \qquad (A_1, B_4).$$

Let the mixed strategy by $(\alpha, 1-\alpha;\ \beta, 1-\beta)$

for P1: $u_1(A_1) = 3(1-\beta) = 3-3\beta$

$u_1(A_4) = 7\beta - (1-\beta) = 8\beta - 1$

(c) Mixed strategy NE:

$$\left(\frac{1}{2}A_1, 0, 0, \frac{1}{2}A_4, 0; \frac{4}{11}B_1, 0, 0, \frac{7}{11}B_4\right).$$

$\Rightarrow \beta = \frac{4}{11}$

Similarly, $\alpha = \frac{1}{2}$

3. For the game depicted in Figure 2:

(a) Determine all SPNE of this game; NE: (U, al), (V, ar) (V, br)

Answer: SPNE

$$(U, al), \quad (V, ar).$$

(b) Determine all (pure strategy) sequential equilibrium of this game.

Answer: SPNE

- Strategy: (V, ar);
- Belief: player 2 puts probability one on his left decision node; i.e., he thinks player 1 plays V.

4. For the game depicted in Figure 3:

(a) Determine all normal-form perfect equilibrium of this game;

Answer: The set of normal-form perfect equilibria:

$$\left\{A, a, (\alpha_L, 1-\alpha_L) | \alpha_L \in \left[0, \frac{1}{3}\right]\right\}.$$

That is, player 1 plays A, 2 plays a, and player 3 plays a mixed strategy that puts probability of at least 2/3 on strategy R.

Note that:

- There are two sets of NE:

$$\{D, a, L)\}; \qquad \left\{A, a, (\alpha_L, 1-\alpha_L) | \alpha_L \in \left[0, \frac{1}{3}\right]\right\}.$$

- To see (D, a, L) is not perfect, note that given the totally mixed strategies of player 1 $(\epsilon, 1-\epsilon)$ and 2 $(1-\nu, \nu)$:

$$u_2(a, \sigma^\epsilon) = (1-\nu)[3\epsilon + 4(1-\epsilon)] + \nu[3\epsilon + 1 - \epsilon] = (1-\nu)(4-\epsilon) + \nu(1+2\epsilon),$$

$$u_2(d, \sigma^\epsilon) = (1-\nu)[5\epsilon + 4(1-\epsilon)] + \nu[2\epsilon + 1 - \epsilon] = (1-\nu)(4+\epsilon) + \nu(1+\epsilon).$$

  Since $u_2(d, \sigma^\epsilon) - u_2(a, \sigma^\epsilon) = 2\epsilon - 3\epsilon\nu$, which is greater than zero for any small number ν, there is no ϵ-perfect equilibrium in which a receives higher probability than d, indicating (D, a, L) is not a perfect equilibrium.

(b) Determine all sequential equilibrium of this game.

Answer: Two sets of S.E.:

- One

  Strategy: $\{A, a, R\}$;

  Belief: $\{(\mu_L, 1-\mu_L) | \mu_L \leq 2/5\}$; that is, player 3 puts a probability of at least 3/5 on his right decision node.

- Two

  Strategy:

$$\left\{ A, a, (\alpha_L, 1-\alpha_L) | \alpha_L \in \left[0, \frac{1}{3}\right] \right\};$$

  Belief:

$$\left(\frac{2}{5}, \frac{3}{5}\right);$$

  that is, player 3 puts exactly 2/5 on his left decision node.

(b) Determine all pure strategy sequential equilibrium of this game.

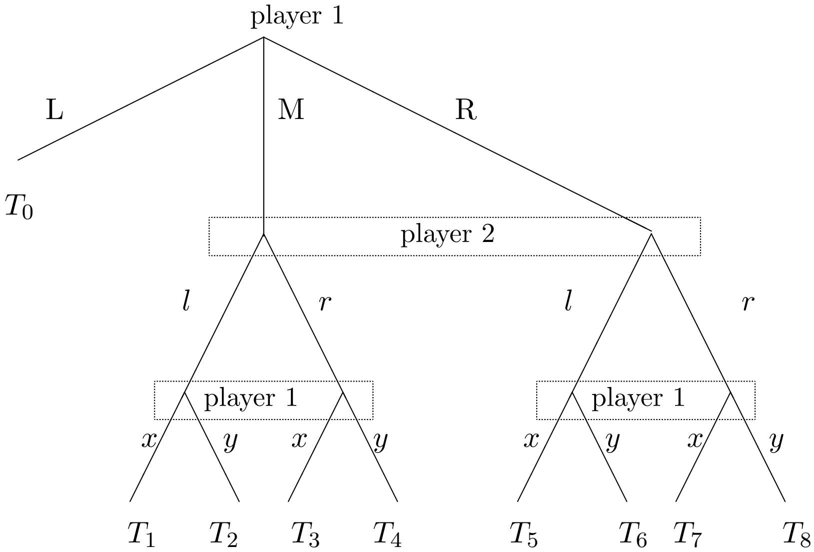


Figure 1

(u, al)
(V, ar)

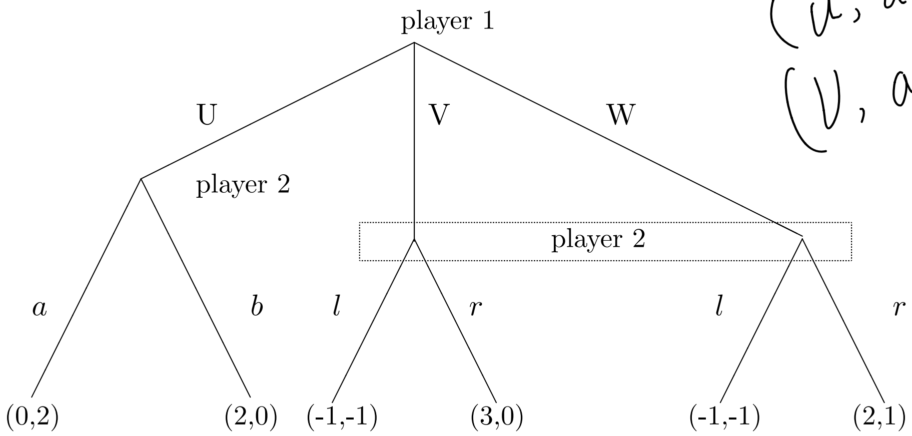


Figure 2

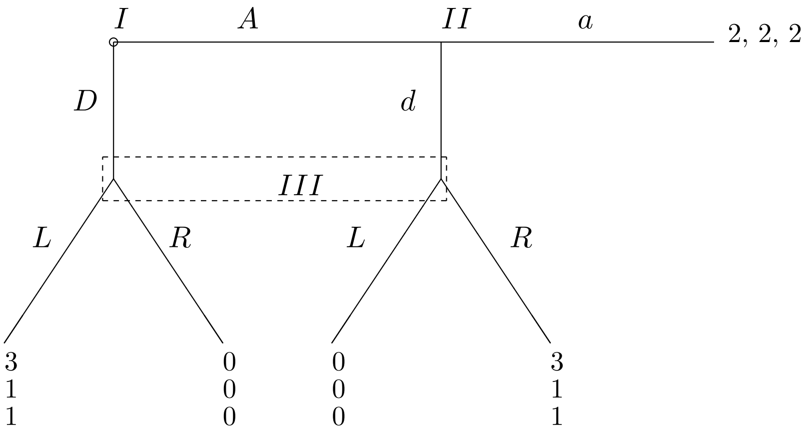


Figure 3

4. (a) the normal form

		a	d
I	A	(2,2,2)	(0,0,0)
	D	(3,1,1)	(3,1,1)

III play L

		a	d
I	A	(2,2,2)	(3,1,1)
	D	(0,0,0)	(0,0,0)

III play R

NE: (D, a, L) (D, d, L) (A, a, R)

It is easy to see (D, d, L) is not normal-form perfect eq (d is weakly dominated by a for P2)

To see that. let $\sigma^{\varepsilon} = (y, 1-y;\ \varepsilon, 1-\varepsilon;\ 1-v, v)$

for player 2 $\quad u_2(a, \sigma^{\varepsilon}) = (1-v)[2y + (1-y)] + 2yv$

$$u_2(d, \sigma^{\varepsilon}) = (1-v)(1-y) + yv$$

$$u_2(a, \sigma^{\varepsilon}) - u_2(d, \sigma^{3}) = (1-v)\cdot 2y + yv = 2y - yv = y(2-v) > 0$$

for any $v, y \in (0,1)$

So there is no ϵ-perfect equilibrium in which d receives higher probablity than d;

then (D, d, L) is not a normal-form perfect eq

It is easy to verify that (D, a, L) and (A, a, R) are normal form perfect equilibria. of this game.

(b) for (D, d, L), because of sequential rationality, II will not play d at his decision node, so it is not SE.

for (A, a, R), let the belief of III be (Left, Right) $= (\mu, 1-\mu)$

let $\sigma^k = (1-y^k, y^k;\ 1-v^k, v^k;\ \varepsilon^k, 1-\varepsilon^k)$ by Bayes' rule

$$\mu^k = \frac{y^k}{y^k + (1-y^k)v^k} = \frac{1}{1 + \frac{1-y^k}{y^k} v^k}, \text{ picking } v^k \geq \frac{y^k}{1-y^k}, \quad \mu = \lim_{k \to \infty} \mu^k \leq \frac{1}{2}$$

$\therefore$ (A, a, R) and belief $\mu \leq \frac{1}{2}$ is SE for (D, a, L) let $\sigma^k = (y^k, 1-y^k; 1-v^k, v^k; 1-\varepsilon^k, \varepsilon^k)$

$$\mu^k = \frac{1-y^k}{1-y^k+y^k v^k} \qquad \mu = \lim_{k\to\infty} \mu^k = 1$$

So (D, a, L) and $\mu = 1$ is SE

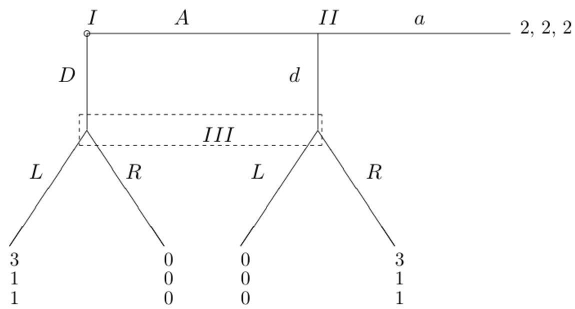


Figure 3

P1. (d)

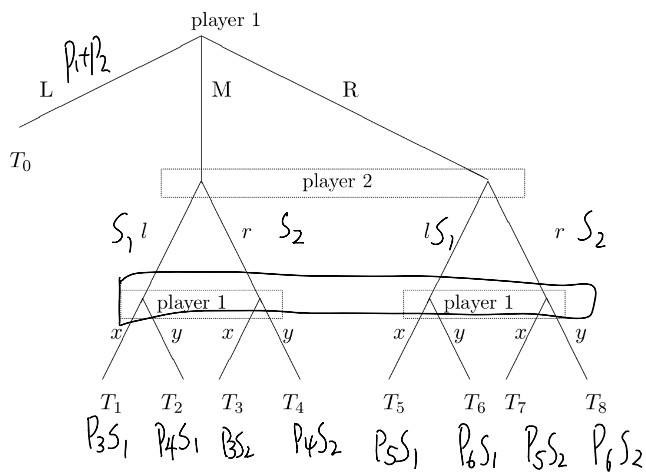


Figure 1

$$\begin{array}{cccccc} Lx & Ly & Mx & My & Rx & Ry \end{array}$$

$$\sigma = (p_1, p_2, p_3, p_4, p_5, p_6)$$

$P(L) = p_1 + p_2$ $\qquad$ $P(M+R) = p_3 + p_4 + p_5 + p_6$

$P(M) = p_3 + p_4$ $\qquad$ $P(R) = p_5 + p_6$

$P(x) = \dfrac{p_3}{p_3 + p_4}$ s.t. $Pr(T_1) = p_3 s_1 = P(M) \cdot P(x) \cdot s_1$

then $Pr(T_5) = p_5 s_1 \neq P(R) \cdot P(x) s_1 = (p_5 + p_6) \cdot \dfrac{p_3}{p_3 + p_4} \cdot s_1$

Since σ is arbitrary, contradiction. So the second part does not hold.