Advanced Microeconomics I
Note 5: Connection of UMP and EMP

Xiang Han (SUFE)

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Proposition. *Given a continuous utility function $u(x)$ representing a locally nonsatiated preference relation $\succeq$, and prices $p \gg 0$, we have the following:*

(i) Let $w > 0$. If $x^* \in x(p, w)$, then $x^* \in h(p, u(x^*))$.

(ii) Let $u > u(0)$. If $x^* \in h(p, u)$, then $x^* \in x(p, p \cdot x^*)$.

It follows from this proposition that, for all $p \gg 0$, $w > 0$ and $u > u(0)$

$$e(p, v(p, w)) = w$$
$$v(p, e(p, u)) = u$$

Then we have

$$h(p, u) = x(p, e(p, u))$$
$$x(p, w) = h(p, v(p, w))$$

Proof of the proposition. (i). Assume to the contrary, $x^* \in x(p, w)$ but $x^* \notin h(p, u(x^*))$. Then there exists $y \geq 0$ such that $u(y) \geq u(x^*)$ and $p \cdot y < p \cdot x^*$. Since $x^* \in x(p, w)$, $p \cdot x^* \leq w$. So $p \cdot y < w$ and $y \in B_{p,w}$. Then $u(y) \geq u(x^*)$ implies $y \in x(p, w)$. But $p \cdot y < w$, so Walras' law is violated, contradiction.

(ii) Assume to the contrary, $x^* \in h(p, u)$ but $x^* \notin x(p, p \cdot x^*)$. Then there exists $y \geq 0$ such that $p \cdot y \leq p \cdot x^*$ and $u(y) > u(x^*)$. Since $x^* \in h(p, u)$, $u(x^*) \geq u$. So $u(y) > u$. Then $p \cdot y \leq p \cdot x^*$ implies $y \in h(p, u)$. But $u(y) > u$, so "no excess utility" is violated, contradiction. $\square$

Slutsky v.s. Hicksian compensation

Establishing the *law of demand* is important in consumer theory. And it requires us to eliminate income effects (which are usually small for many goods in practice) by a certain type of income compensation.

Recall the choice-based approach to demand. It is assumed that for each $B_{p,w}$, the consumer chooses $x(p,w)$. WARP is imposed on $x(p,w)$.

Given p and w, if $p \to p'$, then let $w \to w' = p' \cdot x(p,w)$: **Slutsky compensation**.

Compensated law of demand is established under Slutsky compensation: $(p'-p) \cdot [x(p',w') - x(p,w)] \leq 0$, with strict inequality if $x(p',w') \neq x(p,w)$.

Consider a price increase, is Slutsky compensation too much?

Now consider the preference-based approach.

Let the consumer's initial demand be $x(p,w)$, with a utility of $u(x(p,w)) = v(p,w) = u$ (assume $x(p,w)$ is a singleton).

Suppose there is a price change: $p \to p'$. Income has to be adjusted to eliminate income effects.

Slutsky compensation requires $w' = p' \cdot x(p,w)$. But alternatively, we can change income from w to w'' such that the consumer enjoys the same utility, u.

What is w''?

w'' should satisfy $v(p', w'') = u$. According to $v(p', e(p',u)) = u$, $w'' = e(p',u)$.

After $p \to p'$, the income adjustment $w \to w''$ is called **Hicksian compensation.**

(Notice that, the adjusted income from Hicksian compensation is always lower compared to Slutsky compensation: $w'' = e(p',u) \leq p' \cdot x(p,w) = w'$.)

Since $w = e(p,u)$ and $w'' = e(p',u)$, the following equation

$$h(p,u) = x(p, e(p,u))$$

implies that the Hicksian demand can be understood as the *Walrasian demand under Hicksian compensation*.

Another compensated law of demand

Proposition. *Suppose that u is a continuous utility function representing a strictly convex preference relation. The Hicksian demand function $h(p, u)$ satisfies the compensated law of demand. That is, for any p and p',*

$$(p' - p) \cdot [h(p', u) - h(p, u)] \leq 0$$

Proof. The inequality can be written as

$$p' \cdot h(p', u) - p' \cdot h(p, u) + p \cdot h(p, u) - p \cdot h(p', u) \leq 0$$

which follows from the fact that

$$p' \cdot h(p', u) \leq p' \cdot h(p, u)$$

$$p \cdot h(p, u) \leq p \cdot h(p', u)$$

□

A few more results from UMP and EMP

From now on, we always assume that u is a continuous and differentiable utility function representing a locally nonsatiated and strictly convex preference relation.

First, the relationship between $e(p, u)$ and $h(p, u)$:

Proposition. *For all p and u, we have*

$$h_l(p, u) = \frac{\partial e(p, u)}{\partial p_l}, \;\; \forall l$$

That is, the Hicksian demand function can simply be found by differentiating the expenditure function with respect to prices.

Proof. We only provide the proof for the simple case where $h(p, u) \gg 0$, and assume $h(p, u)$ is differentiable. Consider any good l.

$$\frac{\partial e(p, u)}{\partial p_l} = \frac{\partial (p \cdot h(p, u))}{\partial p_l} = \sum_{k=1}^{L} p_k \frac{\partial h_k(p, u)}{\partial p_l} + h_l(p, u)$$

Recall that, in the first order conditions of EMP, for any k

$$p_k = \lambda \frac{\partial u(h(p, u))}{\partial x_k}$$

Then

$$\frac{\partial e(p, u)}{\partial p_l} = \lambda \sum_{k=1}^{L} \frac{\partial u(h(p, u))}{\partial x_k} \frac{\partial h_k(p, u)}{\partial p_l} + h_l(p, u)$$

The first term on the right hand side is equal to zero, which can be seen by differentiating both sides of $u(h(p, u)) = u$ with respect to p_l. □

Proposition. *Suppose that $h(p, u)$ is continuously differentiable, then its derivatives matrix $D_p h(p, u)$ has the following properties.*

(i) $D_p h(p, u) = D_p^2 e(p, u)$.

(ii) $D_p h(p, u)$ *is negative semidefinite.*

(iii) $D_p h(p, u)$ *is symmetric.*

Second, the relationship between derivatives of $h(p, u)$ and $x(p, w)$:

Proposition: Slutsky Equation. *Given any (p,w), let $u = v(p, w)$, then we have*

$$\frac{\partial h_l(p, u)}{\partial p_k} = \frac{\partial x_l(p, w)}{\partial p_k} + \frac{\partial x_l(p, w)}{\partial w} x_k(p, w) \quad \forall l, k$$

or equivalently, in matrix notations:

$$D_p h(p, u) = D_p x(p, w) + D_w x(p, w) x(p, w)^T$$

Slutsky equation decomposes the effect of p_k on h_l (which is the Walrasian demand for good l under Hicksian compensation) into two parts: the total effect of p_k on x_l and the effect of Hicksian compensation on x_l.

Proof of Slutsky equation. Consider p, w and $u = v(p, w)$. We know that $h_l(p, u) = x_l(p, e(p, u))$. Differentiating both sides with respect to p_k

$$\frac{\partial h_l(p, u)}{\partial p_k} = \frac{\partial x_l(p, e(p, u))}{\partial p_k} + \frac{\partial x_l(p, e(p, u))}{\partial w} \frac{\partial e(p, u)}{\partial p_k}$$

Since $\frac{\partial e(p,u)}{\partial p_k} = h_k(p, u)$

$$\frac{\partial h_l(p, u)}{\partial p_k} = \frac{\partial x_l(p, e(p, u))}{\partial p_k} + \frac{\partial x_l(p, e(p, u))}{\partial w} h_k(p, u)$$

Since $e(p, u) = w$ and $h(p, u) = x(p, w)$

$$\frac{\partial h_l(p, u)}{\partial p_k} = \frac{\partial x_l(p, w)}{\partial p_k} + \frac{\partial x_l(p, w)}{\partial w} x_k(p, w)$$

$\square$

For a differential change dp_k, $x_k(p, w)dp_k$ represents the amount of Hicksian compensation. Why?

Let $p_k \to p'_k$ be a small change, denote $p_{-k} = (p_1, ..., p_{k-1}, p_{k+1}, ..., p_L)$, then $\Delta w_H = e(p'_k, p_{-k}, u) - e(p_k, p_{-k}, u) \approx h_k(p, u)(p'_k - p_k) = x_k(p, w)(p'_k - p_k)$

$D_p h(p, u)$ is actually the *Slutsky matrix* $S(p, w)$.

A key implication of the Slutsky equation is that, *the derivatives of the Walrasian demand under Slutsky compensation are the same as the derivatives of the Walrasian demand under Hicksian compensation.*

This is because for a differential price change, Slutsky compensation and Hicksian compensation are the same: given a price change $p_k \to p'_k$, $\Delta w_S = x_k(p, w)(p'_k - p_k)$.

Finally, the Slutsky matrix is symmetric and negative semidefinite.

Third, the relationship between $v(p, w)$ and $x(p, w)$:

Proposition: Roy's identity. *For any p and w, we have*

$$x_l(p, w) = -\frac{\frac{\partial v(p,w)}{\partial p_l}}{\frac{\partial v(p,w)}{\partial w}}, \quad \forall l$$

Proof of Roy's identity. We only provide the proof for the case of $x(p, w) \gg 0$.

$$\frac{\partial v(p, w)}{\partial p_l} = \frac{\partial u(x(p, w))}{\partial p_l} = \sum_{k=1}^{L} \frac{\partial u(x(p, w))}{\partial x_k} \frac{\partial x_k(p, w)}{\partial p_l}$$

Recall that, in the first-order conditions of UMP, for any k

$$\frac{\partial u(x(p, w))}{\partial x_k} = \lambda p_k$$

Then

$$\frac{\partial v(p, w)}{\partial p_l} = \sum_{k=1}^{L} \lambda p_k \frac{\partial x_k(p, w)}{\partial p_l} = \lambda \sum_{k=1}^{L} p_k \frac{\partial x_k(p, w)}{\partial p_l}$$

Recall the *Cournot aggregation*

$$\sum_{k=1}^{L} p_k \frac{\partial x_k(p, w)}{\partial p_l} + x_l(p, w) = 0$$

So

$$\frac{\partial v(p, w)}{\partial p_l} = -\lambda x_l(p, w)$$

Then Roy's identity follows from the fact that $\frac{\partial v(p,w)}{\partial w} = \lambda$.

Recall that $e(p, v(p, w)) = w$. If we let $u = v(p, w)$, then $w = e(p, u)$. So v and e are inverse functions of each other.

Therefore, for the four analytic tools, $x(p, w), v(p, w), h(p, u)$ and $e(p, u)$, if we know $v(p, w)$, then the other three can be calculated from $v(p, w)$. If we know $e(p, u)$, then the other three can be calculated from $e(p, u)$.