MICROECONOMIC THEORY II

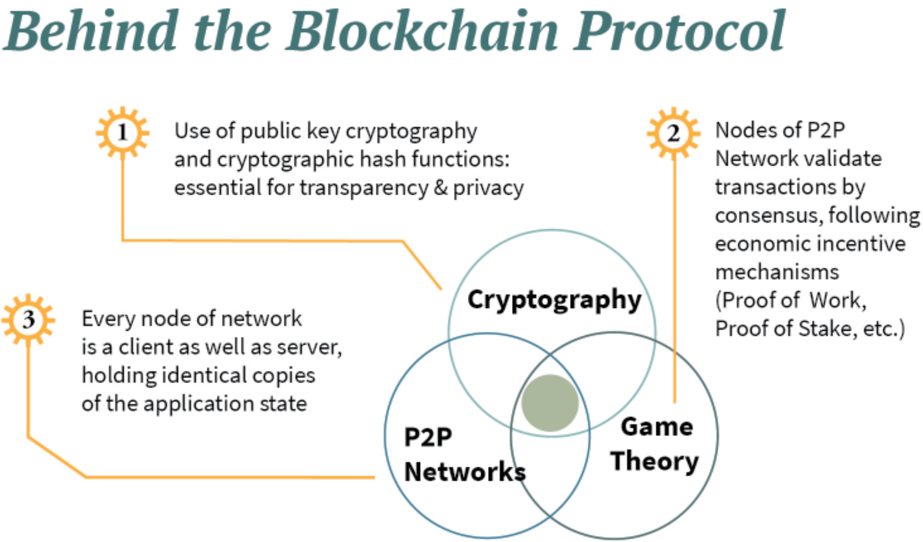
Bingyong Zheng

Email: bingyongzheng@gmail.com

What is game theory?

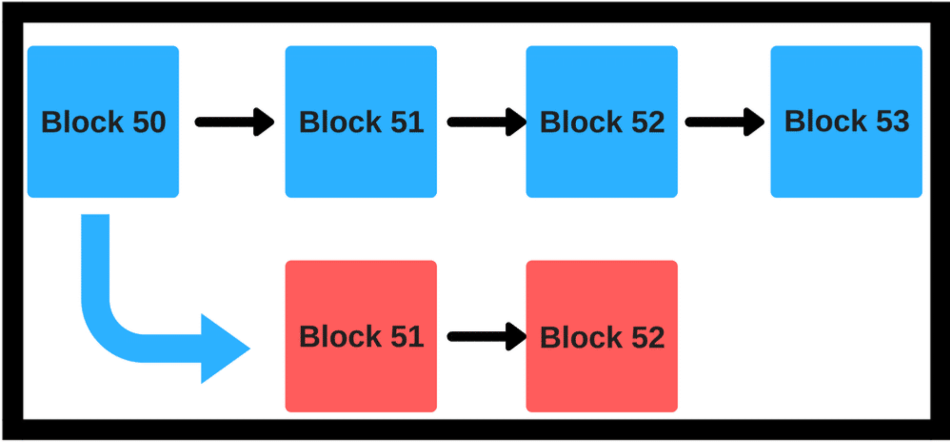
- Game theory is the branch of microeconomics concerned with the analysis of optimal decision making in competitive situations in which the actions of each decision maker have significant impact on the fortune of the others.
  Examples: Poker games; Go; Chess
- It helps us answer many questions traditional economic theory can not.

Motivation: Block chain and game theory



From the Book "**Token Economy**" by Shermin Voshmgir, 2019

Block chain and game theory (2)



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- In a stricter sense, it implies that every player always maximizes his utility, thus being able to perfectly calculate the probabilistic result of every action.

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- A fact is common knowledge among players if each player knows the fact, and each player knows everyone else knows, and each knows everyone else knows everyone else knows, and so on.
- For example, a handshake is common knowledge between the two persons involved. When I shake hand with you, I know you know I know you know,....., that we shake hand. Neither person can convince the other that she does not know that they shake hand. So, perhaps it is not entirely random that we sometimes use a handshake to signal an agreement or a deal.

Common Knowledge Example 1: Muddy children puzzle

n children playing together. Each child wants to keep clean, but each would love to see the others get dirty. Now it happens during their play that some of the children, say k ($k > 1$) of them, get mud on their foreheads. Each can see the mud on others but no on his own forehead. No one says a thing. Along comes the father, who says, "At least one of you has mud on your forehead."
The father then asks the following question, over and over: "Does any of you know whether you have mud on your own forehead?"
Assuming that all the children are perceptive, intelligent, truthful, and that they answer simultaneously, what will happen?

Ann

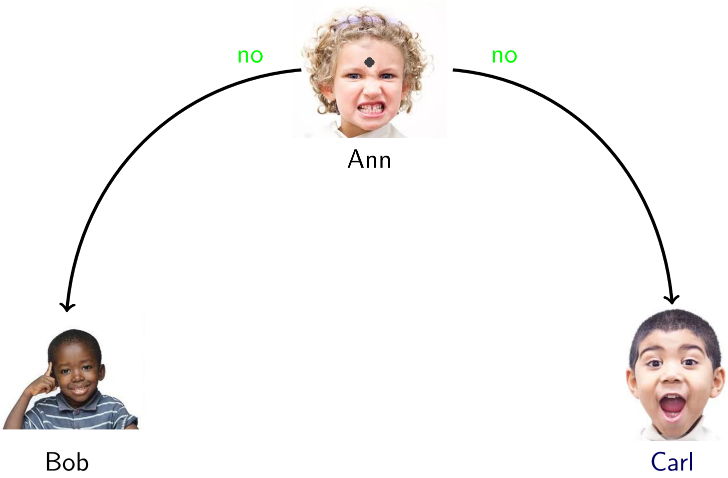
Bob

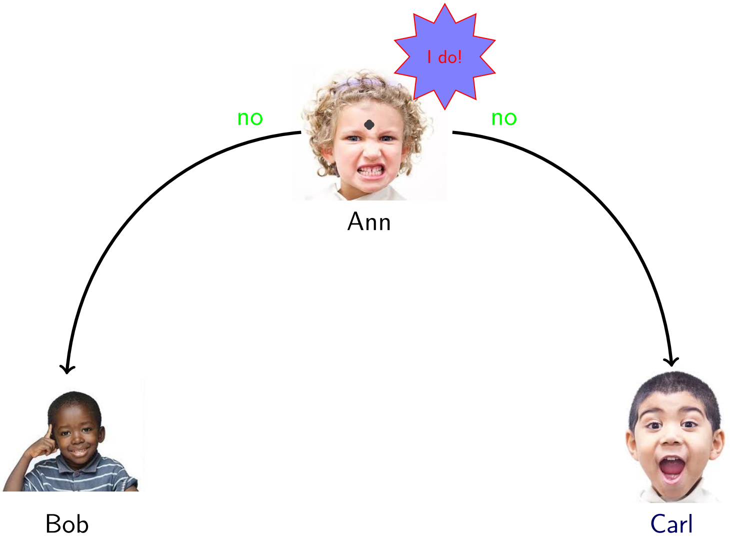
Carl

Ann

Bob

Carl


no
no
Ann
Bob
Carl


I do!
no
no
Ann
Bob
Carl

Ann

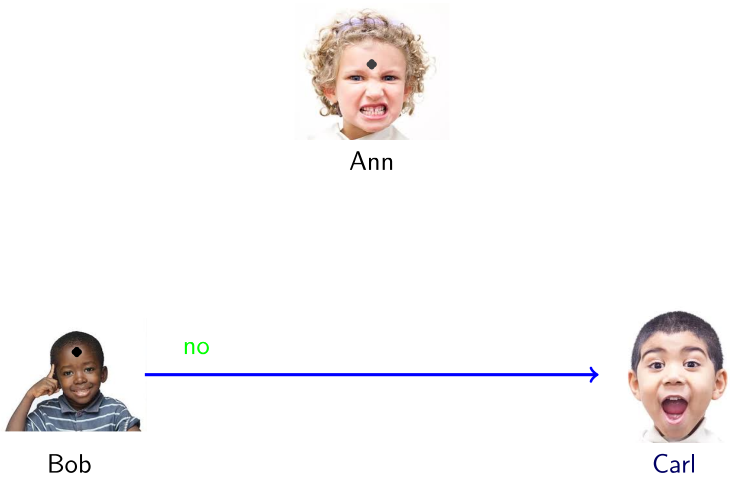
Bob

Carl

Ann

Bob

Carl


Ann
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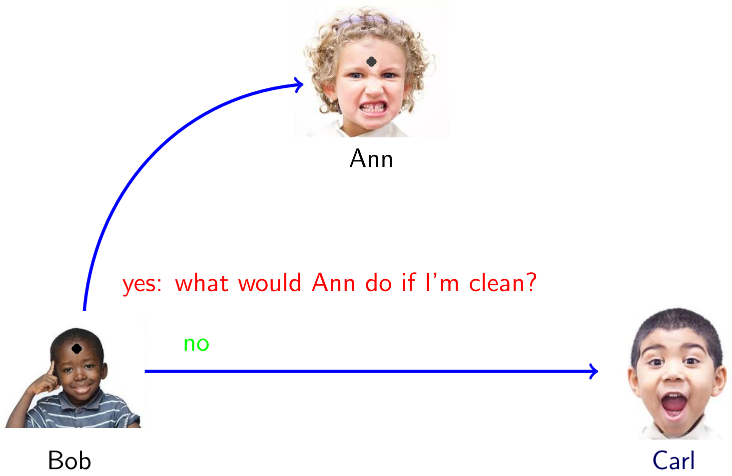
Ann
yes: what would Ann do if I'm clean?
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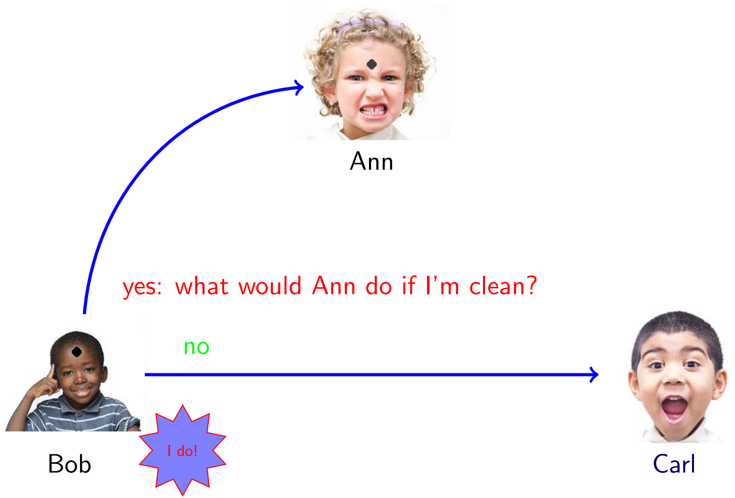
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Bob
I do!
Carl

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Bob

Carl

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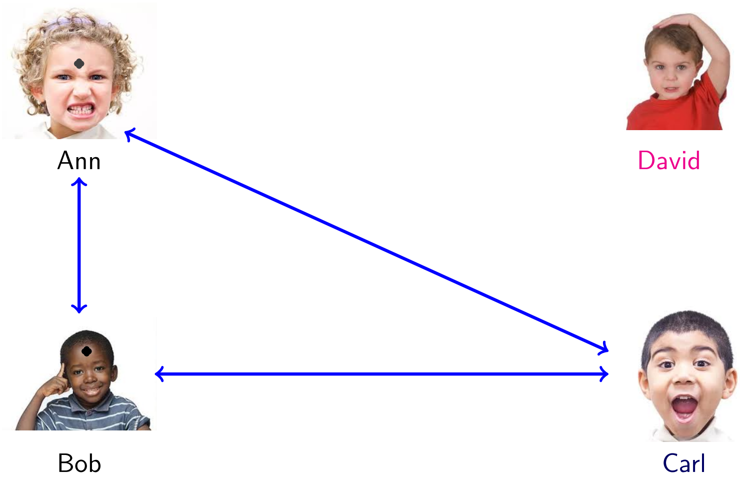
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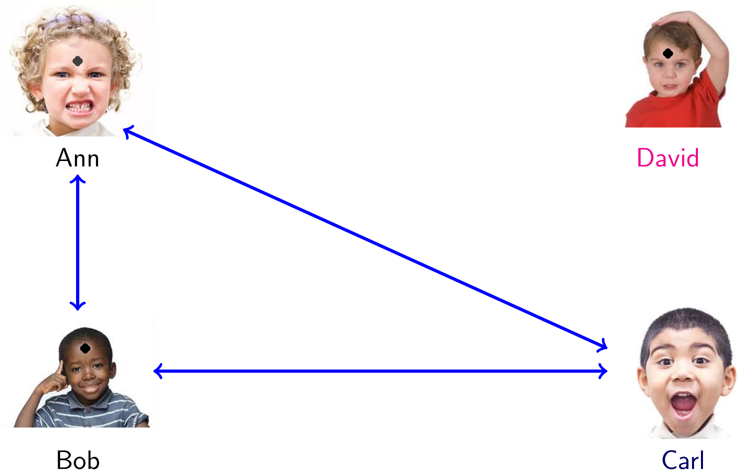
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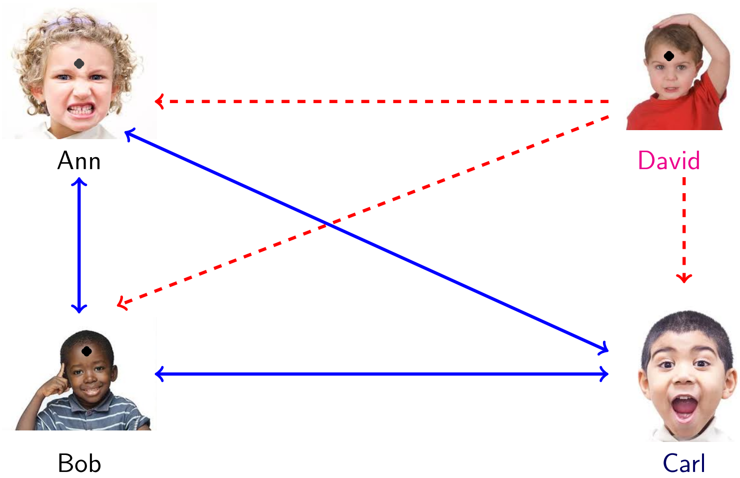
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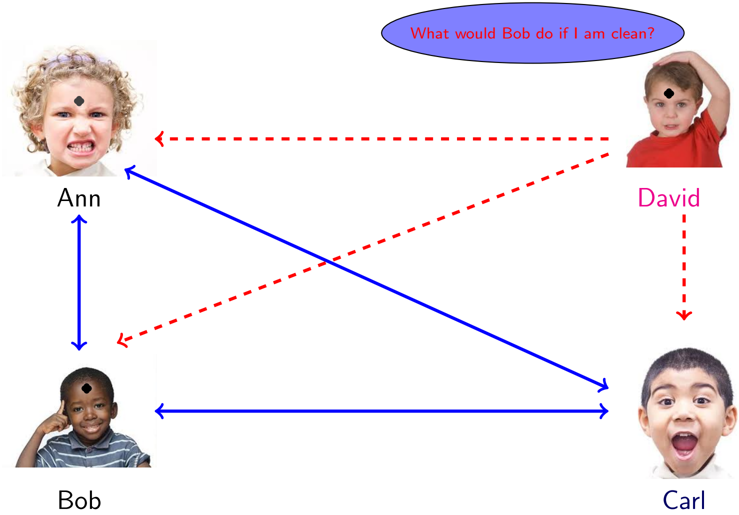
What would Bob do if I am clean?
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Carl

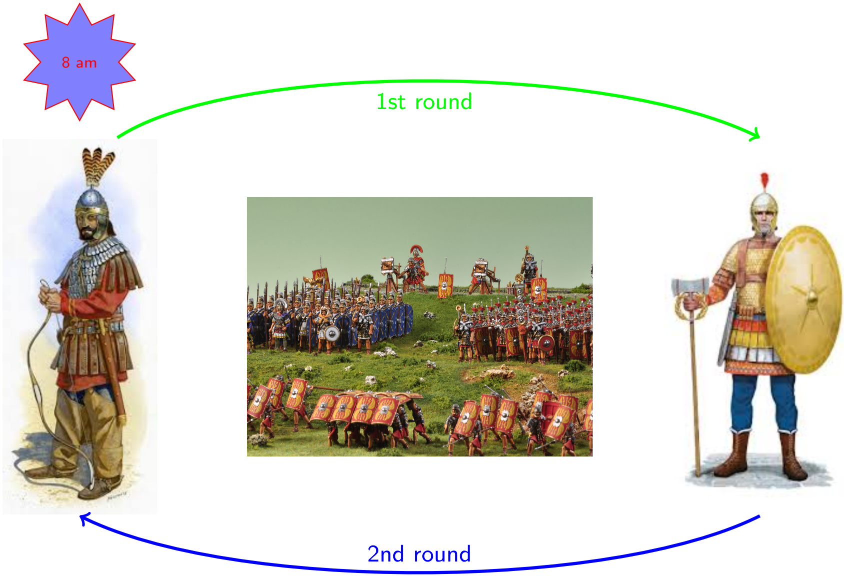
Example 2: The general's problem

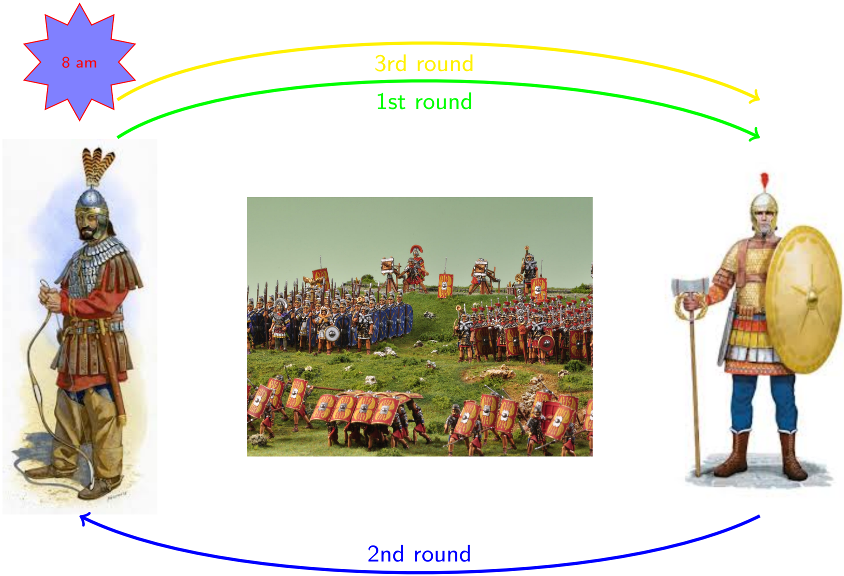
Two divisions of an army, each commanded by a general, camped on two hilltops overlooking a valley the enemy stays. If both divisions attack the enemy simultaneously they will win the battle, while if only one division attacks it will be defeated. Neither general will attack unless he is absolutely sure that the other will attack with him: a general will not attack if he receives no messages. The general of the first division wishes to coordinate a simultaneously attack (at some time the next day). They can communicate only by means of messengers. Normally, it takes a messenger one hour to get from on encampment to the other and on this particular night, everything goes smoothly.
How long will it takes them to coordinate an attack?

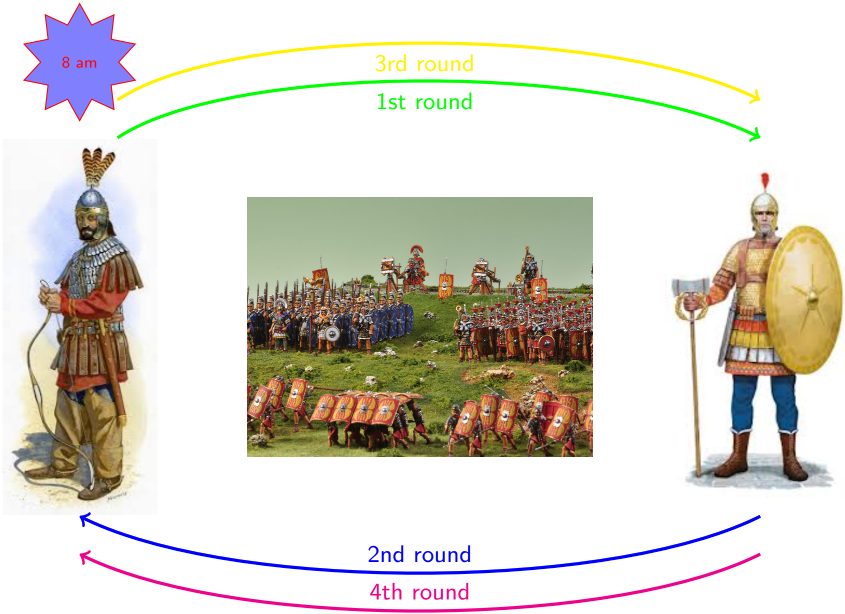
8 am

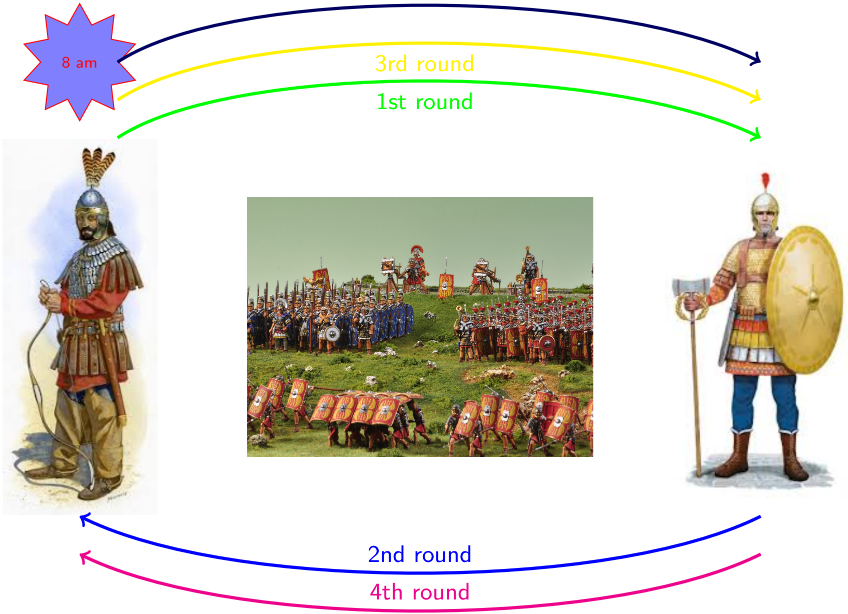
8 am

1st round


8 am
1st round
2nd round


8 am
3rd round
1st round
2nd round


8 am
3rd round
1st round
2nd round
4th round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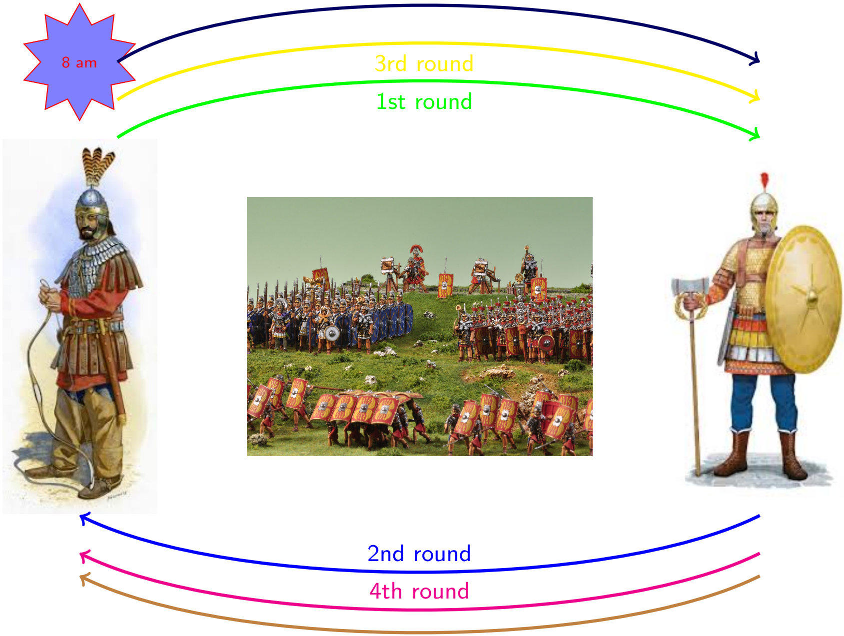
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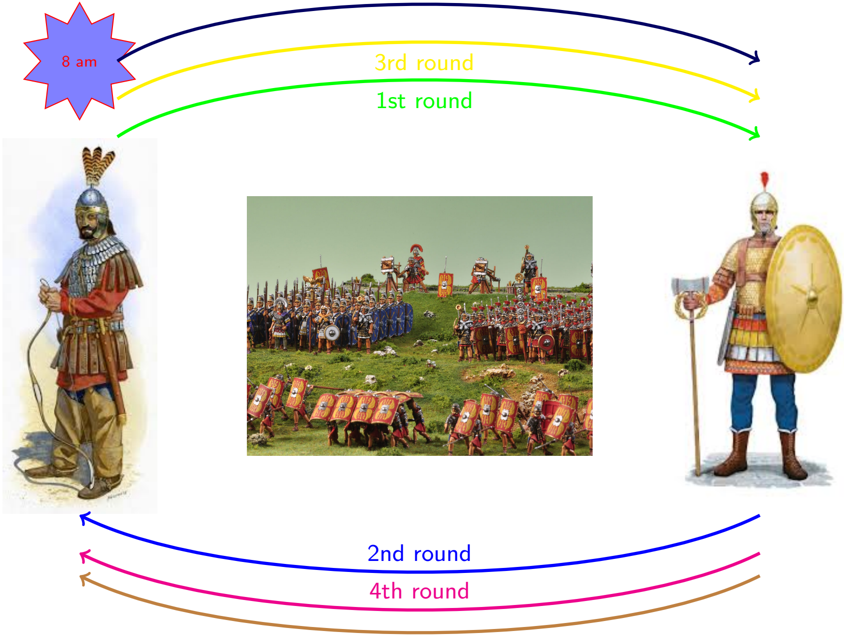
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  - ➢ Also known as an action profile or strategy profile

The capacity expansion game

- Strategic-form

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	Build	16, 16	20, 15
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- Nash equilibrium of this game: (Build, Build).

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- The concept of Nash equilibrium imposes a further restriction, player's belief is consistent with the actual play of her opponents.
- In view of this interpretation, each player in a game holds the belief $p^i(s_{-i})$, and choose s_i such that:

$$s_i^* \in \arg\max_{s_i \in S_i} \sum_{s_{-i}} p^i(s_{-i}) u_i(s_i, s_{-i}).$$

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- An important feature of the subjective expected utility approach is that it does not require randomization on the part of the players.
- Recall that the traditional interpretation of mixed strategies that assumes players explicitly randomize. The probabilistic nature of strategies now reflects the uncertainty of other players about a player's choice.
  Thinking about the traditional Chinese "Scissor-rock-cloth" game.

Pure strategies

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- A strategy is a complete contingency plan that says what a player will do at each of her information sets if she is called on to play there
- A player's strategy may include plans for actions that her own strategy makes irrelevant.

Interpretation of strategies

- According to Rubinstein (1991) and Reny (1992), a player's strategy can be partitioned into two parts, a *plan* that describes a rational play for i, and a *prediction* about i's future behavior should i deviates from his plan.
  - ➢ A *plan* for player i specifies a choice for player i only when he is called upon to move, and does not specify what he would do at an information set of his that can not be reached according to this plan.
  - ➢ In order that others are able to specify what they would do were i not to follow through his plan (something i must know in order to evaluate the soundness of this plan in the first place), it must provide others with a *prediction* about i's future behavior should i deviate.

Interpretation of strategies continued

- Given the SEUM approach discussed before, one natural interpretation for the specification of choices at information sets that won't be reached given a player's strategy is that they are *beliefs* of his opponents about what he would do in case he does not follow his strategy, i.e., the information sets were reached.
- The belief of his opponents is important as their choices at those information sets are based on this belief.
- Furthermore, what the player's opponents would do at those information sets rationalize his choice at at an upstream information set.
- Hence, this definition of strategy is not so odd when you interpret it as the way a player determines his strategy.

Mixed strategies in soccer

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- For example, the traditional Chinese game, rock, scissor and cloth.

player 2

Player 1

	scissor	rock	cloth
scissor	0,0	-1, 1	1,-1
rock	1, -1	0,0	-1, 1
cloth	-1,1	1,-1	0,0

Find mixed strategy NE of the game

- Let the mixed strategy equilibrium be

$$(p_1, p_2, 1 - p_1 - p_2; q_1, q_2, 1 - q_1 - q_2).$$

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- 2's payoff from Rock

$$u_2(\sigma_1, R) = p_1 + 0 \cdot p_2 - (1 - p_1 - p_2).$$

Mixed NE (2)

- 2's payoff from Cloth

$$u_2(\sigma_1, C) = -p_1 + p_2 + 0 \cdot (1 - p_1 - p_2).$$

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$$p_1 = p_2 = \frac{1}{3}.$$
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- Similarly, we solve player 1's problem to get

$$q_1 = q_2 = \frac{1}{3}.$$

Dominant strategy

- Dominant strategy: A strategy s_i is dominant for player i if for all $s_{-i} \in S_{-i}$ and $s_i' \in S_i / s_i$

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- In **any Nash equilibrium**, players who have a dominant strategy play the dominant strategy. Thus, it is easy to find the Nash equilibrium of games in which some of the players have a dominant strategy.
- Dominant strategies are rarity rather than norm. There is no dominant strategies in most interesting games.

Dominant strategy example

- Example: Prisoner's dilemma game

Prisoner 2

Prisoner 1		Confess	Not confess
	Confess	-5, -5	0, -10
	Not confess	-10, 0	-1, -1

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Prisoner 2

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Prisoner 1	Confess	-5, -5	0,-10
	Not confess	-10, 0	-1, -1

- "confess" is the dominant strategy for both prisoners.

Dominated strategy

- A pure strategy $s_i \in S_i$ is weakly dominated if there is another strategy $s_i' \in S_i$ such that for all $s_{-i} \in S_{-i}$,

$$u_i(s_i', s_{-i}) \geq u_i(s_i, s_{-i}),$$

with strict inequality for some s_{-i}.

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- A strategy is strictly **dominated** when the player has another strategy that gives her a higher payoff no matter what the other player plays.
- A player will not play a strictly dominated strategy in Nash equilibrium.

Allowing mixed strategy

- A strategy σ_i is strictly dominated for player i if there is another strategy $\sigma'_i \in \Delta(S_i)/\sigma_i$ such that for all $\sigma_{-i} \in \Delta(S_{-i})$

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- A pure strategy s_i is strictly dominated for player i if and only if there exists $\sigma_i \in \Delta(S_i)$ such that

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$$u_i\left(s_i, s_{-i}\right) < u_i\left(\sigma_i, s_{-i}\right)$$
for all s_{-i}.
- A mixed strategy that assigns positive probability to a pure strategy that is strictly dominated is also strictly dominated.

Dominated strategy example 1

- The game

Player 2

Player 1

	l	m	r
a	7, 5	2, 4	1, 2
b	2, 2	4, 5	0, 3
c	0, 4	7, 0	2, 1

Dominated strategy example 1

- The game

	Player 2			
		l	m	r
Player 1	a	7, 5	2, 4	1, 2
	b	2, 2	4, 5	0, 3
	c	0, 4	7, 0	2, 1

- While player 1's pure strategy *b* is not dominated by any pure strategy, it is dominated by a mixed strategy

$$\sigma_1 = (\frac{1}{2}a, 0b, \frac{1}{2}c).$$

Dominated strategy example 1

- The game

Player 2

Player 1		l	m	r
	a	7, 5	2, 4	1, 2
	b	2, 2	4, 5	0, 3
	c	0, 4	7, 0	2, 1

- While player 1's pure strategy b is not dominated by any pure strategy, it is dominated by a mixed strategy

$$\sigma_1 = \left(\frac{1}{2}a, 0b, \frac{1}{2}c\right).$$

- Player 2's r is also dominated by a mixed strategy.

Dominated strategy example 2

- Modifies capacity expansion game between Toyota and Honda

Toyota

Honda		large	small	not build
	large	0, 0	12, 8	18, 9
	small	8, 12	16, 16	20, 15
	not build	9, 18	15, 20	18, 18

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- Toyota knows that Honda will not play Large!

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Dominated strategy example 2 (cont.)

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- Note that Nash and IDSDS are based on different logic.
- IDSDS does not require that the players know that the equilibrium is going to be played, so it requires less coordination. However, common knowledge of rationality is itself a very strong assumption.

IDSDS application

- Consider the following game (Gibbons pp. 6) :

Player 2

Player 1

	L	M	R
U	1,0	1,2	0,1
D	0,3	0,1	2,0

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		L	M	R
Player 1	U	1,0	1,2	0,1
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- Note that R is strictly dominated by M for Player 2.
- Now, if Player 1 knows that Player 2 is rational, then Player 1 knows that Player 2 will never choose R. If R is eliminated, then D becomes dominated by U.
- Now, if Player 2 knows Player 1 knows that Player 2 is rational, then Player 2 knows that Player 1 will not choose D. In that case, Player 2 should choose M.

Another example

		Player 2			
		L	M	R	U
	A	(5, 6)	(2, 6)	(1, 5)	(0, 7)
Player 1	B	(2, 10)	(1, 2)	(0, 10)	(-1, 1)
	C	(4, 1)	(3, 4)	(1, 3)	(2, 0)
	D	(0, 2)	(1, 3)	(3, 2)	(1, 1)

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Another example

	M	U
A	(2, 6)	(0, 7)
C	(3, 4)	(2, 0)
D	(1, 3)	(1, 1)

Another example

	M	U
C	(3, 4)*	(2, 0)

Nash equilibrium

- Definition: A strategy profile $\sigma^* = (\sigma_1^*, ..., \sigma_n^*)$ is a **Nash Equilibrium** (NE) if for all $i \in N$ and for all $\sigma_i' \in \Delta(S_i)$

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  1. Subjective expected-utility maximization: Each player holds the belief $p^i(s_{-i})$, and choose s_i such that:

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  2. Beliefs coincides with opponents' equilibrium strategies:

  $$\forall\ i, \quad p^i(s_{-i}) = \sigma_{-i}^*.$$

How to understand NE?

- The central concept of noncooperative game theory is Nash equilibrium. A Nash equilibrium is a profile of strategies such that for each player in the game, given the strategy chosen by the other players, the strategy is a best response for the player, that is, the strategy gives the player the highest payoff.
- Early interpretation of the concept of Nash equilibrium.
  - ➢ In most of the early literature the idea of equilibrium was that it said something about how players would play the game or about how a game theorist might recommend that they play the game.
  - ➢ However, this interpretation runs into trouble in many cases. For example, how do we interpret mixed strategy Nash equilibrium? How to motivate the refinements of Nash equilibrium?

Recent interpretation of NE

- Recently, there has been a shift to thinking of equilibria as representing not recommendations to players of how to play the game but rather the expectations of the others as to how a player will play.

Recent interpretation of NE

- Recently, there has been a shift to thinking of equilibria as representing not recommendations to players of how to play the game but rather the expectations of the others as to how a player will play.
- Further, if the players all have the same expectations about the play of the other players we could as well think of an outside observer having the same information about the players as they have about each other.

More on interpretation of NE

- While the first interpretation can be problematic in case of mixed strategy equilibrium, the second interpretation can accommodate mixed strategies without any trouble. In this scenario, the mixed strategy of a player does not represent a conscious randomization on the part of that player, but rather the uncertainty in the minds of the others as to how that player will act.

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- Hence, the second interpretation of Nash equilibrium has become the preferred interpretation among game theorists.
- Thus the focus of the equilibrium analysis becomes, not the choices of the players, but the assessments of the players about the choices of the others.
- The basic consistency condition that we impose on the players' assessments is this: A player reasoning through the conclusions that others would draw from their assessments should not be led to revise his own assessment.

Find NE in two-player games

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- If players have dominated strategies, delete the dominated strategies from the game and work with a smaller game.
- NE must be a mutual best-response, that, given player 1 plays NE strategy, player 2 can not do better by playing some other strategies, similarly, given player 2 plays this strategy, player 1 can not do better by changing strategies. A NE is a strategy profile in which both players are playing the best-response given the other player's strategy.

Example: The game of chicken

- Strategic form

Jack

Tom		Swerve	Stay
	Swerve	0, 0	-10, 10
	Stay	10, -10	-100, -100

- Two pure strategy NE: (Swerve, Stay), (Stay, Swerve)
- One mixed strategy NE

$$((Swerve, Stay; 0.9, 0.1), (Swerve, Stay; 0.9, 0.1)).$$

Example 2

- The game

	L	M	R
T	3, 3	0, 2	3, 0
B	0, 0	3, 2	0, 3

Example 2

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	L	M	R
T	3, 3	0, 2	3, 0
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- How many NE are there, mixed included?

Example 2 NE

- One pure strategy NE
$$(T, L).$$

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Example 2 NE

- One pure strategy NE
$$(T, L).$$
- In addition, there exists two mixed NE.
- Let the mixed strategy NE be
$$(p, 1-p; q_1, q_2, 1-q_1-q_2).$$

Mixed NE of Ex. 2

- For player 1, given σ_2

$$u_1(T) = 3q_1 + 3(1 - q_1 - q_2),$$
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$$u_2(L) = 3p,$$
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- So one mixed NE:

$$\left(\frac{2}{3}, \frac{1}{3}; \frac{1}{2}, \frac{1}{2}, 0\right).$$

Mixed NE of Ex. 2 (cont.)

- If $p = \frac{1}{3}$,

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Mixed NE of Ex. 2 (cont.)

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- However, there is no NE in which both L and R are played with positive probabilities.
- If $u_2(L) = u_2(R)$, $p = \frac{1}{2}$, but
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- Given the belief 1 plays T and B with equal probabilities, player 2 should play M!

Example 3

- The game

Player 1 \ Player 2	A	B	C
A	5, 8	15, 10	10, 5
B	10, 15	20, 9	15, 0
C	20, 20	10, 10	10, 8

Example 3 continued

- Player 1's A is dominated by B; 2's C is also dominated;

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	A	B
B	10, 15	20, 9
C	20, 20	10, 10

- Unique NE: (C, A)

Example of 3-player game

Player 2

		U	D
1	A	(1, 1,0)	(2, -2, 5)
	B	(1,-2, -1)	(0, 3, 1)

Player 3 plays L

Player 2

	U	D
A	(1, 1,-2)	(2, -2, 5)
B	(2, 2, -1)	(2, 3, 7)

Player 3 plays R

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		U	D
1	A	(1, 1, 0)	(2, -2, 5)
	B	(1,-2, -1)	(0, 3, 1)

Player 3 plays L

Player 2

	U	D
A	(1, 1,-2)	(2, -2, 5)
B	(2, 2, -1)	(2, 3, 7)

Player 3 plays R

Example of 3-player game

Player 2

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Example of 3-player game

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Player 2

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A	(1, 1, -2)	(2, -2, 5)
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3-player game: example 2

1		U	V	W
	L	3, 0, 2	2, -1, 0	1, -2, 0
1	M	3, 2, 1	1, 4, -1	0, 0, -2
	R	1, 1, 10	0, 2, 1	-2, 0, 3

Player 3 plays A

	U	V	W
L	2, 1, 1	3, 0, 0	2, -2, -1
M	5, 4, 2	1, 3, 4	3, 0, -2
R	1, 1, 1	0, 2, 0	-2, 0, 2

Player 3 plays B

Example 2 (cont.)

	U	V	W
L	2, 1, -1	3, 0, -1	2, -2, -3
M	5, 4, -1	1, 3, -2	3, 0, -4
R	1, 1, -10	0, 2, -1	-2, 0, -2

Player 3 plays C

3-player game: example 2

		U	V	W
	L	3, 0, 2	2, -1, 0	1, -2, 0
1	M	3, 2, 1	1, 4, -1	0, 0, -2
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Player 3 plays A

		U	V	W
	L	2, 1, 1	3, 0, 0	2, -2, -1
	M	5, 4, 2	1, 3, 4	3, 0, -2
	R	1, 1, 1	0, 2, 0	-2, 0, 2

Player 3 plays B

3-player game: example 2

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	L	3, 0, 2	2, -1, 0	1, -2, 0
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Player 3 plays A

		U	V	W
	L	2, 1, 1	3, 0, 0	2, -2, -1
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Player 3 plays B

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Player 3 plays A

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Player 3 plays A

	U	V
L	2, 1, 1	3, 0, 0
M	5, 4, 2	1, 3, 4

Player 3 plays B

3-PLAYER GAME: EXAMPLE 2

1

	U	V
L	3, 0, 2	2, -1, 0
M	3, 2, 1	1, 4, -1

Player 3 plays A

	U	V
L	2, 1, 1	3, 0, 0
M	5, 4, 2	1, 3, 4

Player 3 plays B

3-player game: example 2

		U	V
	L	3, 0, 2	2, -1, 0
1	M	3, 2, 1	1, 4, -1

Player 3 plays A

	U	V
L	2, 1, 1	3, 0, 0
M	5, 4, 2	1, 3, 4

Player 3 plays B

3-player game: example 2

		U	V
	L	3, 0, 2*	2, -1, 0
1	M	3, 2, 1	1, 4, -1

Player 3 plays A

	U	V
L	2, 1, 1	3, 0, 0
M	5, 4, 2*	1, 3, 4

Player 3 plays B

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    - ➢ What are the means by which we are to identify "obvious way to play a game?"
    - ➢ What can one say about games that do not admit a "solution" When the game does not admit an "obvious way to play," looking at its NE can give precisely the wrong answer. The concept of NE is of no use when the game admits no "solution"

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- At other times, the requirement of NE is not strong enough to rule out multiple equilibria or implausible predictions. Two lines of research to address these difficulties:
  - ➢ Equilibrium selection: concerned with narrowing the prediction to a single prediction.
  - ➢ Refinement of NE: concerned with establishing necessary conditions for reasonable predictions.

Rationalizability

- A strategy σ_i is a best response for player i to her rivals' strategies σ_{-i} if

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i}).$$

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for all $\sigma_i' \in \Delta(S_i)$
- Strategy σ_i is never a best response if there is no σ_{-i} for which σ_i is a best response.
- The strategies in $\Delta(S_i)$ that survive iterated deletion removal of strategies that are never a best response are known as player i's rationalizable strategies.

Example

Player 2

Player 1

	b_1	b_2	b_3	b_4
a_1	0, 7	2, 5	7, 0	0, 1
a_2	5, 2	3, 3	5, 2	0, 1
a_3	7, 0	2, 5	0, 7	0, 1
a_4	0, 0	0, -2	0, 0	10, -1

Example

		Player 2			
		b_1	b_2	b_3	b_4
Player 1	a_1	0, 7	2, 5	7, 0	0, 1
	a_2	5, 2	3, 3	5, 2	0, 1
	a_3	7, 0	2, 5	0, 7	0, 1
	a_4	0, 0	0, -2	0, 0	10, -1

- b_4 is never a best response for player 2!

Example

		Player 2		
		b_1	b_2	b_3
	a_1	0, 7	2, 5	7, 0
Player 1	a_2	5, 2	3, 3	5, 2
	a_3	7, 0	2, 5	0, 7
	a_4	0, 0	0, -2	0, 0

- a_4 is never a best response for player 2!

Example

Player 2

Player 1		b_1	b_2	b_3
	a_1	0, 7	2, 5	7, 0
	a_2	5, 2	3, 3	5, 2
	a_3	7, 0	2, 5	0, 7

- The set of rationalizable strategies:

$$\{a_1, a_2, a_3; b_1, b_2, b_3\}.$$

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- For two-player games, rationalizable strategies are those remaining after the iterative deletion of strictly dominated strategies.
- For more than two player games, this is no longer true.
- A strictly dominated strategy is never a best response; but the reverse is not necessarily true for more than two-player game.

Example

- The game

	L	R
U	9	0
D	0	0

A

	L	R
U	0	9
D	9	0

B

	L	R
U	0	0
D	0	9

C

	L	R
U	6	0
D	0	6

D

Example

- The game

	L	R
U	9	0
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U	0	9
D	9	0

B

	L	R
U	0	0
D	0	9

C

	L	R
U	6	0
D	0	6

D

- In this example, D is not dominated, but never a best response for player 3.

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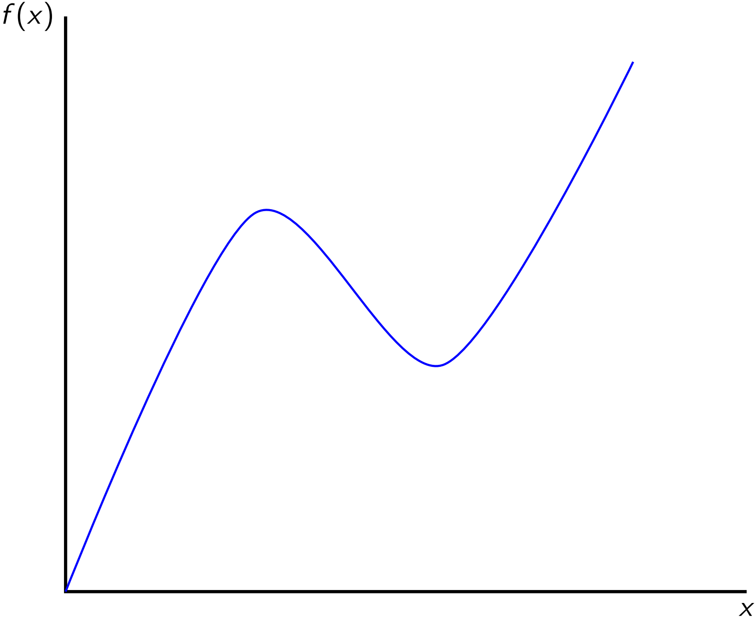
- Theorem 1 (Theorem 7.2, Jehle and Reny). Every finite strategic form game has at least one NE.
- Main idea of the proof:
  - Use fixed point theorem. A continuous function $f(x)$ mapping from a convex, compact set to itself has a fixed point $x = f(x)$.
  - Two steps: construct such a continuous function; show the fixed point is NE.

Fixed Point Theorem



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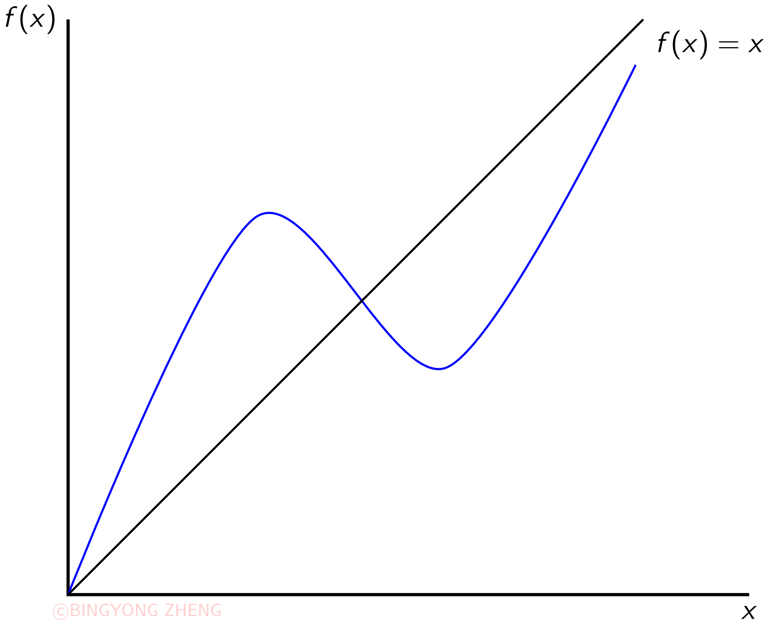

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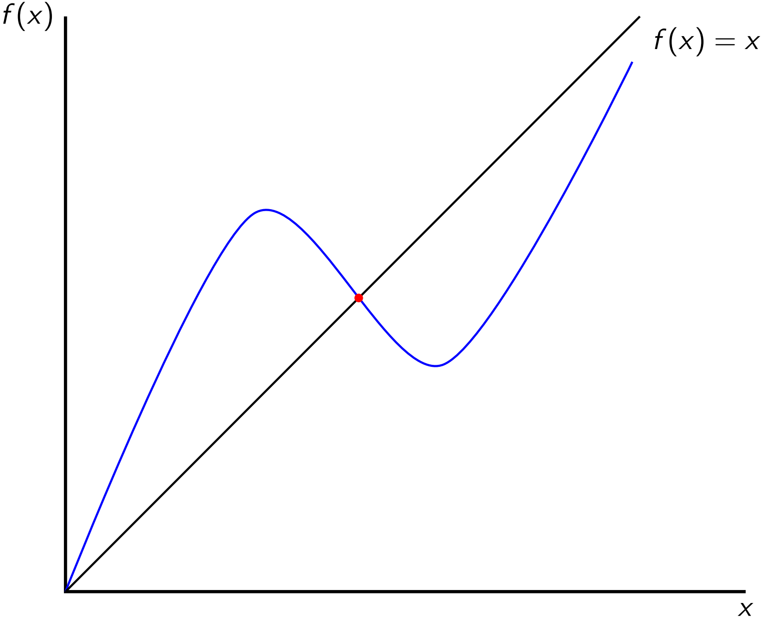

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- M_i is convex and compact, so is $M = \prod_i M_i$.
- Define $f : M \to M$ as follows

$$f_{ij}(m) = \frac{m_{ij} + \max\{0, u_i(j, m_{-i}) - u_i(m)\}}{1 + \sum_{j'=1}^{n} \max\{0, u_i(j, m_{-i}) - u_i(m)\}}. \qquad \text{(Func)}$$

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- For all i, j and for all m: $f_{ij} \in [0,1]$ and

$$\sum_{j=1}^{n} f_{ij}(m) = 1.$$

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- From (Func), we have

$$f_{ij}(\widehat{m}) + f_{ij}(\widehat{m}) \sum_{j'=1}^{n} \max\{0, u_i(j, \widehat{m}_{-i}) - u_i(\widehat{m})\} =$$
$$\widehat{m}_{ij} + \max\{0, u_i(j, \widehat{m}_{-i}) - u_i(\widehat{m})\}.$$

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$$\widehat{m}_{ij} + \max\{0, u_i(j, \widehat{m}_{-i}) - u_i(\widehat{m})\}.$$

- Using the fact $\widehat{m}_{ij} = f_{ij}(\widehat{m})$:

$$\widehat{m}_{ij} \sum_{j'=1}^{n} \max\{0, u_i(j, \widehat{m}_{-i}) - u_i(\widehat{m})\} = \max\{0, u_i(j, \widehat{m}_{-i}) - u_i(\widehat{m})\}$$

Proof of Theorem 1 (cont.)

- Multiplies both sides by $u_i(j, \widehat{m}_{-i}) - u_i(\widehat{m})$ and sum for all j:

$$\sum_{j'=1}^{n} \max\{0, u_i(j, \widehat{m}_{-i}) - u_i(\widehat{m})\} \sum_{j=1}^{n} \widehat{m}_{ij}[u_i(j, \widehat{m}_{-i}) - u_i(\widehat{m})]$$

$$= \sum_{j=1}^{n} [u_i(j, \widehat{m}_{-i}) - u_i(\widehat{m})] \max\{0, u_i(j, \widehat{m}_{-i}) - u_i(\widehat{m})\}.$$

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- So we end up with

$$\sum_{j=1}^{n} [u_i(j, \widehat{m}_{-i}) - u_i(\widehat{m})] \max\{0, u_i(j, \widehat{m}_{-i}) - u_i(\widehat{m})\} = 0.$$

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- Since $\max\{0, u_i(j, \widehat{m}_{-i}) - u_i(\widehat{m})\} \geq 0$ for all j, we have:

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- $\widehat{m}$ is a mixed strategy NE.

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  - ➢ Step 2: by Kukutani's fixed point theorem, a non-empty, convex-valued upper hemicontinuous correspondence $b_i(\sigma_{-i})$ mapping from $\Delta(S)$ to itself, there must exists a fixed point.

NE may involve unreasonable beliefs

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a_1	3, 3	0, 0
a_2	-5, -5	0, -5

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➢ But is (a_2, b_2) a good prediction of the game? Not likely.

NE BELIEF

- Player 1's belief

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- So b_2 a best response!
- Everything works according to the definition of NE:

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- So b_1 is the only best response!
- So the only reasonable belief should be

	b_1 ($1-\epsilon$)	b_2 (ϵ)
a_1 ($1-\epsilon$)	3, 3	0, 0
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Normal form perfect equilibrium

- An ϵ-*perfect equilibrium of the normal form game* is a totally mixed strategy $\sigma \equiv (\sigma_1, \ldots, \sigma_N)$, if for all i and for all $s_i, s_i' \in S_i$,

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- For two-player game, any NE in which no player plays dominated strategies is perfect.
- For more than two-player game, the above statement is not true. There are NE with no players playing dominated strategies that is not perfect.

NE with no weakly dominated strategy may not be perfect

- Consider the following example:

Player 2

Player 1

	a	d
A	(3, 3, 0)	(5, 5, 0)
D	(4, 4, 4)	(4, 4, 4)

3 plays L

	a	d
A	(3, 3, 0)	(2, 2, 2)
D	(1, 1, 1)	(1, 1, 1)

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- This game has two pure NE:

$$(D, a, L), \qquad (A, a, R).$$

- But (D, a, L) is not a perfect equilibrium, even if no weakly dominated strategy is played.

Example continued

- To see (D, a, L) is not perfect, note it is the limit of totally mixed strategy profile

$$(\epsilon, 1-\epsilon;\ 1-\eta, \eta;\ 1-\nu, \nu).$$

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$$\begin{aligned} u_2(a, \sigma^\epsilon) =& (1-\nu)[3\epsilon + 4(1-\epsilon)] + \nu[3\epsilon + 1 - \epsilon] \\ =& (1-\nu)(4-\epsilon) + \nu(1+2\epsilon), \end{aligned}$$

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- Since

$$u_2(d, \sigma^\epsilon) - u_2(a, \sigma^\epsilon) = 2\epsilon - 3\epsilon\nu,$$

which is greater than zero for small number ν. IN no ϵ-perfect equilibrium does a receive higher probability than d, indicating (D, a, L) is not a perfect equilibrium.

Another example

- The game

Player 2

Player 1

	L	R
T	(1, 1, 1)	(1, 0, 1)
B	(1,1, 1)	(0, 0, 1)

3 plays *l*

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- What are the pure strategy NE?

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- However, for any small probabilities player 2 assigns to R and player 3 to r, the expected payoff for player 1 from T is greater than that from B.
- Thus, there exists no ϵ-perfect equilibrium in which the totally mixed strategy profile assigns more than ϵ to B.

Example 2 (continued)

- If (B, L, l) if perfect, the ϵ-perfect equilibrium must be of the form
$$(\epsilon, 1-\epsilon; 1-\eta, \eta; 1-\nu, \nu).$$

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- But is this indeed an ϵ-perfect equilibrium?
- For player 1, given belief $(1-\eta, \eta; 1-\nu, \nu)$:

$$U_1(T) = (1-\nu) + \nu(1-\eta);$$

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- Clearly,
$$U_1(T) > U_1(B).$$
- So any ϵ-perfect equilibrium must have:
$$\sigma_1(B) < \epsilon.$$

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M_1	0, 0	0, 0

	L_2	M_2	R_2
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- Is this belief reasonable?

More on the belief

- Belief associated with (M_1, M_2)

	L_2 (ϵ^2)	M_2 $(1-\epsilon-\epsilon^2)$	R_2 (ϵ)
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- But then, 2's best response for 2 is L_2!

More on the belief (2)

- So player 1's "correct" belief should be

	L_2 ($1-\epsilon-\epsilon^2$)	M_2 (ϵ)	R_2 (ϵ^2)
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- But then, the best response for 1 should be L_1!
- And 2's belief should be consistent with the optimal choice of player 1 as well, so we end up with

	L_2 $(1-\epsilon^-\epsilon^2)$	M_2 (ϵ)	R_2 (ϵ^2)
L_1 $(1-\epsilon-\epsilon^2)$	1, 1	0, 0	-1, -2
M_1 (ϵ)	0, 0	0, 0	0, -2
R_1 (ϵ^2)	-2, -1	-2, 0	-2, -2

Other solution concept: proper equilibrium

- ϵ-proper equilibrium: totally mixed strategy profile σ is ϵ-proper if for all i and for all $s_i, s_i' \in S_i$,

$$u_i(s_i, \sigma) > u_i(s_i', \sigma) \Longrightarrow \frac{\sigma_i(s_i')}{\sigma_i(s_i)} < \epsilon.$$

- A proper equilibrium is the limit of a sequence of ϵ-proper equilibria.
- A proper equilibrium is also a perfect equilibrium.
- A proper equilibrium strategy is also sequential equilibrium strategy in the corresponding extensive form game.