Advanced Microeconomics I
Note 6: Production

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Production sets

- The firm, or producer, is viewed as a "black box": transforming inputs into outputs.
- A **production plan** is a vector $y = (y_1, y_2, ..., y_L)^T \in \mathbb{R}^L$ that describes the *net outputs* of the L goods from a production process.
  - Suppose $L = 4$. Then $y = (-1, 2, 4, -3)$ means 1 unit of good 1 and 3 units of good 4 are used to produce 2 units of good 2 and 4 units of good 3.
- The primitive of our producer's model is the **production set** $Y \subseteq \mathbb{R}^L$: the set of all the feasible production plans.
- The production set Y fully describes the firm's technology.
- We are often interested in a *single-output technology*, which can be described by a *production function* $f(z)$, where $f : \mathbb{R}_+^{L-1} \to \mathbb{R}_+$. f corresponds to the production set:

$$Y = \{(-z, q) : q \leq f(z) \ \text{ and } \ z \geq 0\}$$

where $z = (z_1, z_2, ..., z_{L-1})^T$, good L is the output, and we have "free disposal".

Properties of production sets

- Y is **nonempty**.
- Y is **closed**.
- **No free lunch**: If $y \in Y$ and $y \geq 0$, then $y = 0$.
- **Possibility of inaction**: $0 \in Y$.
  - This may not be satisfied for a production set with *sunk cost*.
- **Free disposal**: If $y \in Y$ and $y' \leq y$, then $y' \in Y$.
- **Nonincreasing returns to scale**: For any $y \in Y$ and $\alpha \in [0, 1]$, $\alpha y \in Y$.
  - Any production plan can be scaled down.
- **Nondecreasing returns to scale**: For any $y \in Y$ and $\alpha \geq 1$, $\alpha y \in Y$.
  - Any production plan can be scaled up.
- **Constant returns to scale**: For any $y \in Y$ and $\alpha \geq 0$, $\alpha y \in Y$.
- **Convexity**.
  - If Y is convex and $0 \in Y$, then Y exhibits nonincreasing returns to scale.

Similar to the relationship between preferences and utility functions, for a single-output technology, properties of the production set translate into properties of the production function.

Proposition. *Let f be a production function and Y be the corresponding production set. Then we have*

(i) Y exhibits nonincreasing returns to scale if and only if for any $\lambda > 1$ and $z \geq 0$, $f(\lambda z) \leq \lambda f(z)$.

(ii) Y exhibits nondecreasing returns to scale if and only if for any $\lambda > 1$ and $z \geq 0$, $f(\lambda z) \geq \lambda f(z)$.

(iii) Y exhibits constant returns to scale if and only if f is homogeneous of degree one.

Proof of (i). "only if" part. Consider any $\lambda > 1$ and $z \geq 0$. We want to show

$$\frac{1}{\lambda} f(\lambda z) \leq f(z)$$

which is equivalent to

$$(-z, \frac{1}{\lambda} f(\lambda z)) \in Y$$

This follows from the fact that $(-\lambda z, f(\lambda z)) \in Y$ and Y exhibits nonincreasing returns to scale.

"if" part. Consider any $(-z, q) \in Y$ and $\alpha \in (0, 1)$ (if $\alpha = 0$ or $\alpha = 1$, we already have $(-\alpha z, \alpha q) \in Y$). Clearly $q \leq f(z)$. We want to show

$$(-\alpha z, \alpha q) \in Y$$

which is equivalent to

$$\alpha q \leq f(\alpha z)$$
$$q \leq \frac{1}{\alpha} f(\alpha z)$$

Since $\frac{1}{\alpha} > 1$, this follows from

$$\frac{1}{\alpha} f(\alpha z) \geq f(\frac{1}{\alpha} \alpha z) = f(z) \geq q$$

Proof of (ii). "only if" part. Consider any $\lambda > 1$ and $z \geq 0$. We want to show

$$f(\lambda z) \geq \lambda f(z)$$

which is equivalent to

$$(-\lambda z, \lambda f(z)) \in Y$$

This follows from the fact that $(-z, f(z)) \in Y$ and Y exhibits nondecreasing returns to scale.

"if" part. Consider any $(-z, q) \in Y$ and $\alpha > 1$. Clearly $q \leq f(z)$. We want to show

$$(-\alpha z, \alpha q) \in Y$$

which is equivalent to

$$\alpha q \leq f(\alpha z)$$

Since $\alpha > 1$, this follows from

$$f(\alpha z) \geq \alpha f(z) \geq \alpha q$$

Proof of (iii): It follows from (i) and (ii). $\square$

Proposition. *Let f be a production function and Y be the corresponding production set. Then f is concave if and only if Y is convex.*

So, if a production function is concave, then it exhibits nonincreasing returns to scale. Is the converse true?

No!

If a production function is increasing, homogeneous, quasiconcave and exhibits nonincreasing to returns to scale, then it is concave. (Friedman, Econometrica 1976, Prada-Sarmiento, Economics Bulletin 2011).

Cobb-Douglas production function $f(z_1, z_2) = z_1^{\alpha} z_2^{\beta}$, $\alpha > 0$, $\beta > 0$.

It is always quasiconcave.

It is concave if $\alpha + \beta \leq 1$.

Proof of the proposition. "only if" part. Consider any $(-z, q) \in Y$, $(-z', q') \in Y$ and $\alpha \in [0, 1]$. We want to show

$$(-\alpha z - (1-\alpha)z', \alpha q + (1-\alpha)q') \in Y$$

which is equivalent to

$$f(\alpha z + (1-\alpha)z') \geq \alpha q + (1-\alpha)q'$$

This is true because the concavity of f implies

$$f(\alpha z + (1-\alpha)z') \geq \alpha f(z) + (1-\alpha)f(z')$$

and

$$f(z) \geq q, \;\; f(z') \geq q'$$

"if" part. Consider any z, z' and $\alpha \in [0, 1]$. We want to show

$$f(\alpha z + (1-\alpha)z') \geq \alpha f(z) + (1-\alpha)f(z')$$

which is equivalent to

$$(-\alpha z - (1-\alpha)z', \alpha f(z) + (1-\alpha)f(z')) \in Y$$

and this follows from the fact that $(-z, f(z)) \in Y$, $(-z', f(z')) \in Y$, and Y is convex.

Profit maximization

- In this course, we always assume that a firm's goal is to maximize profits.
  - Is this really true?
- Prices: $p = (p_1, p_2, ..., p_L) \gg 0$
  - *Price-taking assumption*: competitive market
- The firm's *profit maximization problem* (PMP):

$$\text{Max} \quad p \cdot y$$
$$s.t. \quad y \in Y$$

The solution set of PMP is denoted as $y(p)$: the firm's *supply correspondence*.

The maximized profit is $\pi(p)$: the firm's *profit function*.

PMP often has no solution, i.e., it is possible that $y(p) = \phi$ for some p.

Proposition. *If the production set Y exhibits nondecreasing returns to scale, then either $\pi(p) \leq 0$ or $\pi(p) = +\infty$.*

Nevertheless, the **law of supply** can be easily established:

Proposition. *For any $p \gg 0, p' \gg 0$, $y \in y(p)$ and $y' \in y(p')$, we have*

$$(p' - p) \cdot (y' - y) \geq 0$$

Proof of the first proposition. Case 1: there exists some $y \in Y$ such that $p \cdot y > 0$. Since Y exhibits nondecreasing returns, we can always scale up y by a large positive number α such that $\alpha y \in Y$ and $\pi(p) \geq \alpha(p \cdot y)$. Hence we must have $\pi(p) = +\infty$.

Case 2: there does not exist any $y \in Y$ such that $p \cdot y > 0$, then obviously $\pi(p) \leq 0$.

Proof of the second proposition. This inequality is equivalent to

$$(p' \cdot y' - p' \cdot y) + (p \cdot y - p \cdot y') \geq 0$$

This is true, because $y' \in y(p')$ implies $p' \cdot y' = \pi(p') \geq p' \cdot y$, and $y \in y(p)$ implies $p \cdot y = \pi(p) \geq p \cdot y'$.

$\square$

Cost minimization

- From now on, we focus on a single-output technology represented by a production function $f(z)$, where $f : \mathbb{R}_+^{L-1} \to \mathbb{R}_+$.
  - Assume that $f(0) = 0$, and f is unbounded.
- Given input prices $w \gg 0$ and a quantity of output $q \geq 0$, the firm's *cost minimization problem* (CMP):

$$\text{Min} \;\; w \cdot z$$
$$s.t. \;\; z \geq 0 \text{ and } f(z) \geq q$$

- Cost minimization is a necessary condition for profit maximization. Why do we need an independent study of CMP?
  - The solution to CMP is usually better behaved than PMP. In fact, CMP and EMP are almost mathematically identical (yeh!).
  - In the future analysis of market structures, the firm usually has some market power in its output market, but is still assumed to be a price taker in its input market. In that case, the cost function is very useful (while the profit function is useless).

The solution set of CMP is denoted $z(w, q)$: the *conditional factor demand correspondence*.

The minimized cost is denoted $c(w, q)$: the *cost function*.

Proposition. *Suppose that f is a continuous production function.*

(i) $z(w, q)$ is homogeneous of degree zero in w.

(ii) For any $z \in z(w, q)$, $f(z) = q$.

(ii) If f is quasiconcave, then $z(w, q)$ is a convex set; if f is strictly quasiconcave, then $z(w, q)$ is a singleton.

(iii) $c(w, q)$ is homogeneous of degree one in w.

(iv) $c(w, q)$ is concave in w.

(v) (Shepard's lemma) Suppose $z(w, q)$ is a singleton. $\frac{\partial c(w,q)}{\partial w_l} = z_l(w, q), \forall l.$

(vi) *$c(w, q)$ is strictly increasing in q.*

(vii) If f is homogeneous of degree one (constant returns to scale), then $z(w, q)$ and $c(w, q)$ are homogeneous of degree one in q.

(viii) If f is concave, then $c(w, q)$ is convex in q.

Proof of (viii). Consider any q, q' and $\alpha \in [0,1]$. We want to show that

$$c(w, \alpha q + (1-\alpha)q') \leq \alpha c(w,q) + (1-\alpha)c(w,q')$$

Let $z \in z(w,q)$ and $z' \in z(w,q')$, then the above inequality is equivalent to

$$c(w, \alpha q + (1-\alpha)q') \leq \alpha w \cdot z + (1-\alpha) w \cdot z'$$

or

$$c(w, \alpha q + (1-\alpha)q') \leq w \cdot (\alpha z + (1-\alpha)z')$$

To show the above inequality, it is sufficient to show

$$f(\alpha z + (1-\alpha)z') \geq \alpha q + (1-\alpha)q'$$

This is true, because the concavity of f implies

$$f(\alpha z + (1-\alpha)z') \geq \alpha f(z) + (1-\alpha)f(z')$$

and $f(z) \geq q$, $f(z') \geq q'$. $\square$

Revisiting PMP

Under the single-output technology, the firm's profit maximization problem takes the following form:

$$\underset{z \geq 0}{\text{Max}} \;\; pf(z) - w \cdot z$$

Alternatively, we can decompose PMP into two stages: determine cost function first, then determine optimal output. The second stage is the following simple optimization problem regarding a one-variable function:

$$\underset{q \geq 0}{\text{Max}} \;\; pq - c(w, q)$$

$$\text{FOC:} \;\; p - \frac{\partial c(w, q^*)}{\partial q} \leq 0, \;\; \text{with equality if} \;\; q^* > 0$$

The (existence of) profit-maximizing output level can be easily inferred from the cost function and its relation with price.

When f is concave so $c(w, q)$ is convex in q, FOC is also sufficient.

At an interior solution, *marginal revenue equals marginal cost*. In future analysis of non-competitive markets, marginal revenue may take a different form (i.e., not equal to price), but this principle remains the same.