PROBLEM SET 1 ANSWERS

1. (40) Consider a standard New Keynesian model with price stickiness and working capital channel. Assume that the evolution of the economy is determined by the following three linearized equations.

$$\pi_t = \gamma x_t + \alpha R_t + \beta E_t \pi_{t+1} \quad (1)$$

$$x_t = E_t \left[x_{t+1} - (R_t - \pi_{t+1}) \right] \quad (2)$$

$$R_t = r_\pi \pi_t. \quad (3)$$

where π_t, x_t and R_t denote the inflation, the output gap and the nominal interest rate, respectively. Eq.(1) is the Phillips curve that links the inflation π_t to the output gap x_t and the nominal interest rate R_t. Eq.(2) is the intertemporal Euler equation. Eq.(3) characterizes the monetary policy.

(a).(10) Please substitute out x_t and R_t using Eq.(1), Eq.(2) and Eq.(3) and derive a second-order different equation in π_t.

Answer:

$$(1 - \alpha r_\pi + \gamma r_\pi)\pi_t + (\alpha r_\pi - \beta - 1 - \gamma)\pi_{t+1} + \beta \pi_{t+2} = 0.$$

Including the expectation operator is okay.

(b).(10) Under what condition, the equilibrium with zero inflation ($\pi_t = 0$) is stable? And under what condition, inflation expectations are self-fulfilling?

Answer: The general set of solutions to the equation in (a) is given by, $\pi_t = \alpha_0 \lambda_1^t + \alpha_1 \lambda_2^t$, where λ_1, λ_2 are the roots of the following equation:

$$(1 - \alpha r_\pi + \gamma r_\pi) + (\alpha r_\pi - \beta - 1 - \gamma)\lambda + \beta \lambda^2 = 0.$$

When $|\lambda_1| > 1$ and $|\lambda_2| > 1$, the equilibrium with zero inflation ($\pi_t = 0$) is stable.

When $|\lambda_1| <= 1$ or $|\lambda_2| <= 1$, there are many solutions to the equilibrium conditions, therefore, inflation expectations are self-fulfilling.

(c).(20) Assume $\alpha = 0$. Please derive the necessary and sufficient condition for r_π under which the equilibrium with zero inflation ($\pi_t = 0$) is stable. Please also explain the economic intuition behind your answer.(Hint: $0 < \beta < 1$ and $\gamma > 0$)

Answer: Let $\alpha = 0$. The roots of the following equation:

$$(1 + \gamma r_\pi) + (-\beta - 1 - \gamma)\lambda + \beta \lambda^2 = 0.$$

is given by,

$$\lambda_1 = \frac{\beta + 1 + \gamma + \sqrt{(\beta+1+\gamma)^2 - 4\beta(1+\gamma r_\pi)}}{2\beta}.$$

$$\lambda_2 = \frac{\beta + 1 + \gamma - \sqrt{(\beta+1+\gamma)^2 - 4\beta(1+\gamma r_\pi)}}{2\beta}.$$

When $|\lambda_1| > 1$ and $|\lambda_2| > 1$, the equilibrium with zero inflation ($\pi_t = 0$) is stable.

It is easy to see that, $|\lambda_1| > \frac{\beta+1}{2\beta} > 1$ and $\lambda_2 > 0$. Then, $\lambda_2 > 1$ requires that,

$$\frac{\beta + 1 + \gamma - \sqrt{(\beta+1+\gamma)^2 - 4\beta(1+\gamma r_\pi)}}{2\beta} > 1.$$

The above condition can be reduced to:

$$r_\pi > 1.$$

which is the necessary and sufficient condition under which the equilibrium with zero inflation ($\pi_t = 0$) is stable.

The economic intuition is that, if the inflation deviates above from zero, then the central bank will raise nominal interest rate more ($r_\pi > 1$), so that output gap falls based on Eq.(2). The fall in output gap leads to a decline in the inflation through the Phillips curve, and therefore stabilizes the inflation.

2. (30) Assume that there is a continuum of perfectly competitive firms that produce physical capital, hold it and obtain revenues by renting it out. Note that there are no trades of physical capital among firms. In particular, the representative firm is endowed with capital K_{t-1} at the beginning of period t and obtains capital rents $r_t^k K_{t-1}$ in period t where r_t^k is the capital rent rate. In period t, the representative firm uses part of the capital rents to makes investments I_t and distributes the rest to its shareholders as dividends D_t. The evolution of the physical capital follows,

$$K_t = (1-\delta)K_{t-1} + [1 - \frac{\Omega}{2}(\frac{I_t}{I_{t-1}} - 1)^2]I_t. \tag{4}$$

where δ denotes the capital depreciation rate. Ω denotes the size of the capital adjustment cost.

a. (5) Please write down the expression of the dividends D_t.

Answer:

$$D_t = r_t^k K_{t-1} - I_t.$$

b. (5) Suppose the representative firm maximizes the discounted sum of current and future dividend flows. The discount factor for period t dividend is $\beta^t \Lambda_t$. Please write down the representative firm's objective function in period t.

Answer:

$$\sum_{j=0}^{\infty} \beta^{t+j} \Lambda_{t+j} D_{t+j} = \sum_{j=0}^{\infty} \beta^t \Lambda_{t+j} (r_{t+j}^k K_{t+j-1} - I_{t+j}).$$

Note: It is enough to write down $\sum_{j=0}^{\infty} \beta^{t+j} \Lambda_{t+j} D_{t+j}$. It is also okay to say $\sum_{j=0}^{\infty} \beta^j \Lambda_{t+j} D_{t+j}$

c. (10) Please solve for the firm's investment problem that chooses K_t and I_t to maximize its objective function taking as given the constraint (4), and write down the first order conditions for K_t and I_t.

Answer: Denote μ_t as the Lagrange multiplier on the capital evolution process (4). Then the first order condition on K_t is given by,

$$-\mu_t + \beta E_t[\Lambda_{t+1} r_{t+1}^k + \mu_{t+1}(1-\delta)] = 0.$$

The first order condition on I_t is given by,

$$-\Lambda_t + \mu_t[1 - \frac{\Omega}{2}(\frac{I_t}{I_{t-1}} - 1)^2 - \Omega(\frac{I_t}{I_{t-1}} - 1)\frac{I_t}{I_{t-1}}] + \beta E_t \mu_{t+1} \Omega(\frac{I_{t+1}}{I_t} - 1)(\frac{I_{t+1}}{I_t})^2 = 0.$$

d. (10) Denote μ_t as the Lagrange multiplier on the capital evolution equation Eq.(4) for firm's investment problem. We usually interpret the ratio $q_t \equiv \frac{\mu_t}{\Lambda_t}$ as the market price of physical capital if there are trades of physical capital. Describe the relationship between $q_t \equiv \frac{\mu_t}{\Lambda_t}$ and the capital rents based on your answers in (c). Is current capital price q_t affected by current capital rent r_t^k or expected future capital rent $E_t r^k_{t+1}$? Is the effect positive or negative? Explain the economic intuition behind this relationship.

Answer: Substitute $q_t \equiv \frac{\mu_t}{\Lambda_t}$ into the first order condition for K_t, we have:

$$q_t = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t}[r^k_{t+1} + q_{t+1}(1-\delta)].$$

Taking else equal, the current capital price q_t is positively affected by the expected value of future capital rent $E_t r^k_{t+1}$, not current capital rent. This is because 1) The physical capital that is purchased in the current period can only generate capital rents from the next period. 2) Higher expected value of future capital rents makes physical capital more valuable, and raise the demand and thus the current price of physical capital.

PROBLEM SET 2 ANSWERS

1 (50) The question is related to the BGG model with financial frictions. Consider an entrepreneur that is endowed with net worth N initially and borrow B from the bank. The entrepreneur purchase capital K at the price Q^k and finance the capital acquisition using his net worth and bank loans, which implies that,

$$N + B = Q^k K. \tag{1}$$

The entrepreneur then experiences an idiosyncratic shock that convert capital K into efficient units ωK, where ω follows an i.i.d. process across firms and time, drawn from the distribution $F(\cdot)$ with a nonnegative support. The entrepreneur can obtain revenues of R^k for each effective unit of capital he holds. The idiosyncratic shock ω realizes after the capital acquisition.

a. (5) Assume that the bank charges a state-contingent interest rate Z on its loans. There then exists a cutoff ratio $\bar{\omega}$ such that the a. (5) Assume that the bank charges a state-contingent interest rate Z on its loans. There then exists a cutoff ratio $\bar{\omega}$ such that the entrepreneur will be not able to repay the bank loans and default if $\omega < \bar{\omega}$. Please write down the expression for $\bar{\omega}$.

Answer:

$$\bar{\omega} R^k K = ZB.$$

b. (10) If the entrepreneur defaults, the bank pays a liquidation cost to obtain the entrepreneur's generated revenues. The liquidation cost is a fraction μ of the entrepreneur's generated revenues. Please derive expressions for the entrepreneur's expected income and the bank's expected income before the idiosyncratic shock ω realizes.

Answer: The entrepreneur's expected income is given by,

$$\int_{\bar{\omega}}^{+\infty} (\omega R^k K - ZB) dF(\omega) = \int_{\bar{\omega}}^{+\infty} (\omega - \bar{\omega}) dF(\omega) R^k K.$$

The bank's expected income is given by,

$$\int_0^{\bar{\omega}} (1-\mu)\omega dF(\omega) R^k K + [1 - F(\bar{\omega})] ZB = \{\int_0^{\bar{\omega}} (1-\mu)\omega dF(\omega) + [1 - F(\bar{\omega})]\bar{\omega}\} R^k K.$$

c. (5) The bank is risk neutral and has a participation constraint that requires that its expected income to be able to cover its funding costs RB, where R is the risk-free interest rate. Please write down the expression for the bank's participation constraint.

Answer:

$$\{\int_0^{\bar{\omega}}(1-\mu)\omega dF(\omega)+[1-F(\bar{\omega})]\bar{\omega}\}R^kK \geq RB.$$

d. (10) Please solve for the optimal financial contract that maximizes the entrepreneur's expected income subject to the bank's participation constraint.

Answer: Rewrite the bank's participation constraint as:

$$\{\int_0^{\bar{\omega}}(1-\mu)\omega dF(\omega)+[1-F(\bar{\omega})]\bar{\omega}\}R^kK \geq R(Q^kK-N).$$

Denote λ as the Lagrange multiplier on the bank's participation constraint. Define:

$$g(\bar{\omega})=\int_{\bar{\omega}}^{+\infty}(\omega-\bar{\omega})dF(\omega).$$

$$h(\bar{\omega})=\{\int_0^{\bar{\omega}}(1-\mu)\omega dF(\omega)+[1-F(\bar{\omega})]\bar{\omega}\}.$$

The optimal financial contract problem is given by,

$$\max g(\bar{\omega})R^kK$$

subject to:

$$h(\bar{\omega})R^kK \geq R(Q^kK-N).$$

The first order condition on K is given by,

$$g(\bar{\omega})R^k+\lambda[h(\bar{\omega})R^k-RQ^k]=0.$$

The first order condition on $\bar{\omega}$ is given by,

$$g'(\bar{\omega})R^kK+\lambda h'(\bar{\omega})R^kK=0.$$

Using the above equations together provides the optimal financial contract:

$$\frac{g(\bar{\omega})R^k}{g'(\bar{\omega})}=\frac{h(\bar{\omega})R^k-RQ^k}{h'(\bar{\omega})}.$$

where

$$g'(\bar{\omega})=-1+F(\bar{\omega}).$$

$$h'(\bar{\omega})=-\mu f(\bar{\omega})\bar{\omega}+1-F(\bar{\omega}).$$

e. (10) Please describe the financial accelerator mechanism in the BGG framework. In other words, please explain why fluctuations in capital prices and capital investments are amplified in the BGG framework.

Answer: Consider a fall in capital price. The fall in capital price reduces entrepreneurs' realized return to capital and therefore leads to a decline in entrepreneur's net worth. In the BGG framework, entrepreneurs' borrowing capacity depends on their net worth. The lower net worth makes entrepreneurs borrow less and their demand for capital declines, which amplifies the fall in asset prices and capital investments.

f. (10) Define investment efficiency as the marginal efficiency of investment in producing capital. The higher the investment efficiency, the less investment used in producing capital. Both an unexpected increase in entrepreneur's risk (ω) and an unexpected decrease in investment efficiency have contractionary output effect but have opposite effects on entrepreneur's net worth and credits spread $Z - R$. Please explain why these two shocks have have opposite effects on entrepreneur's net worth and credits spread $Z - R$.

Answer: An unexpected increase in entrpereneur's risk (ω) increases the expected default ratio of entrepreneurs. The higher expected default ratio makes banks less willing to lend and leads to higher credit spread $Z - R$. In response to higher credit spread, entrepreneurs borrow less and their demand for capital declines. The capital price falls. So entrepreneur's net worth decline.

An unexpected decrease in investment efficiency makes it more costly to produce capital stock. The supply of capital stock falls and leads to an increase in the capital price. The higher capital price makes entrepreneur's return to capital higher and raises their net worth. One one hand, with higher net worth, entrepreneurs can take lower leverage ratio. On the other hand, the reduced supply of capital stock increases the expected value of future capital rent and hence entrepreneur expected return to capital. Both makes entrepreneur less likely to default and therefore their credit spread falls.

Note: The key is to mention these two risks have opposite effects on capital price.

2. (50) The question is related to the BGG model with financial frictions. Consider a firm that hires labor hours H at the wage rate W and produces goods Y with the following production function:

$$Y = A\omega H. \tag{2}$$

where A denotes the aggregate productivity shock that equals across firms. ω denotes the firm-specific idiosyncratic shock that realizes after the production occurs. ω follows an i.i.d. process across firms and time, drawn from the distribution $F(\cdot)$ with a nonnegative support.

The firm has to pay the wage payments before the production occurs, and finance the wage payments with its own net worth N and bank loans B. This implies that,

$$N + B = WH. \tag{3}$$

After the production occurs, the idiosyncratic shock ω realizes and the firm sells its output Y at the price p.

a. (10) Assume that the bank charges a state-contingent interest rate Z on its loans. There then exists a cutoff ratio $\bar{\omega}$ such that the firm will be not able to repay the bank loans and default if $\omega < \bar{\omega}$. Please write down the expression for $\bar{\omega}$.

$$PA\bar{\omega}H = ZB.$$

b. (10) If the firm defaults, the bank pays a liquidation cost to obtain the firm's generated revenues. The liquidation cost is a fraction μ of the firm's production revenues. Please derive expressions for the firm's expected income and the bank's expected income before the idiosyncratic shock ω realizes.

Answer: The firm's expected income is given by,

$$\int_{\bar{\omega}}^{+\infty} (PA\omega H - ZB)dF(\omega) = \int_{\bar{\omega}}^{+\infty} (\omega - \bar{\omega})dF(\omega)PAH.$$

The bank's expected income is given by,

$$\int_0^{\bar{\omega}} (1-\mu)\omega dF(\omega)PAH + [1 - F(\bar{\omega})]ZB = \{\int_0^{\bar{\omega}} (1-\mu)\omega dF(\omega) + [1 - F(\bar{\omega})]\bar{\omega}\}PAH.$$

c. (10) The bank is risk neutral and has a participation constraint that requires that its expected income to be able to cover its funding costs RB, where R is the risk-free interest rate. Please write down the expression for the bank's participation constraint.

Answer:

$$\{\int_0^{\bar{\omega}} (1-\mu)\omega dF(\omega) + [1 - F(\bar{\omega})]\bar{\omega}\}PAH \geq RB.$$

d. (10) Please solve for the optimal financial contract that maximizes the firm's expected income subject to the bank's participation constraint.

Answer: Rewrite the bank's participation constraint as:

$$\{\int_0^{\bar{\omega}} (1-\mu)\omega dF(\omega) + [1-F(\bar{\omega})]\bar{\omega}\}PAH \geq R(WH-N).$$

Denote λ as the Lagrange multiplier on the bank's participation constraint. Define:

$$g(\bar{\omega}) = \int_{\bar{\omega}}^{+\infty} (\omega-\bar{\omega})dF(\omega).$$

$$h(\bar{\omega}) = \{\int_0^{\bar{\omega}} (1-\mu)\omega dF(\omega) + [1-F(\bar{\omega})]\bar{\omega}\}.$$

The optimal financial contract problem is given by,

$$\max g(\bar{\omega})PAH$$

subject to:

$$h(\bar{\omega})PAH \geq R(WH-N).$$

The first order condition on H is given by,

$$g(\bar{\omega})PA + \lambda[h(\bar{\omega})PA - RW] = 0.$$

The first order condition on $\bar{\omega}$ is given by,

$$g'(\bar{\omega})PAH + \lambda h'(\bar{\omega})PAH = 0.$$

Using the above equations together provides the optimal financial contract:

$$\frac{g(\bar{\omega})PA}{h(\bar{\omega})PA - RW} = \frac{g'(\bar{\omega})}{h'(\bar{\omega})}.$$

where

$$g'(\bar{\omega}) = -1 + F(\bar{\omega}).$$

$$h'(\bar{\omega}) = -\mu f(\bar{\omega})\bar{\omega} + 1 - F(\bar{\omega}).$$

e. (10) Please consider an increase in aggregate productivity A and answer how an increase in A will affect the output. Please explain why this output effect can be amplified in the presence of financial frictions. (Hint: You should describe how the increase in A will affect the firm's net worth and how changes in the firm's net worth will in turn affect the output.)

Answer: An increase in the aggregate productivy will increase the output directly, because you can produce more for a certain amount of labor inputs.

This positive output effect can be amplified because the increase in A raises the firm's realized return to labor inputs and therefore leads to a rise in firm's net worth. In the

BGG framework,firms' borrowing capacity depends on their net worth. The higher net worth makes firms borrow more and hire more workers, which amplifies the rise in output.

PROBLEM SET 3 ANSWERS

1. (50) The question is related to the GK model with financial intermediaries. Consider a bank that is endowed with net worth N_t at the beginning of period t and raise deposits D_t from the depositors at the interest rate R_t. The bank grant loans L_t to entrepreneurs using its net worth and deposits, which implies that,

$$N_t + D_t = L_t. \tag{1}$$

The bank obtain loan repayments and repay the deposits at the beginning of period $t+1$. Denote R^l_{t+1} as the bank's realized return on loans L_t, where R^l_{t+1} realizes in period $t+1$. It is known that $E_t R^l_{t+1} > R_t$.

After the bank obtain funds from the depositors in period t, the banker managing the bank may transfer a fraction θ of the total assets L_t to his or her family. If a bank diverts assets for personal gain, it defaults on its deposits and is shut down. The depositors re-claim the remaining fraction $1-\theta$ of funds.

(a).(5) Please write down the bank's net worth N_{t+1} at the beginning of period $t+1$ after it collects the loan repayments and makes interest payments on deposits.

Answer:

$$N_{t+1} = R^l_{t+1} L_t - R_t D_t.$$

(b).(5) Suppose, with i.i.d probability $1-\sigma$, a bank exits in each period. The bank pays out its net worth as dividends when it exits. Suppose, in period t, the bank maximizes the expected present value of future dividends, where the discount factor for period $t+j$ dividend is $\beta^j \frac{\Lambda_{t+j}}{\Lambda_t}$. Please write down the bank's objective function.

Answer:

$$\sum_{j=1}^{+\infty} \mathrm{E}_t (1-\sigma)\sigma^{j-1} \beta^j \frac{\Lambda_{t+j}}{\Lambda_t} N_{t+j}.$$

(c).(10) As the depositors recognize the bank's incentive to divert funds, they will restrict the amount of deposits they saves at banks. Please write down the incentive constraint to ensure the bank does not divert funds.

Answer:

$$V_t \geq \theta L_t.$$

where V_t is the bank's objective function under optimal decisions and is given by

$$V_t = \max_{L_t, D_t} \sum_{j=1}^{+\infty} \mathrm{E}_t (1-\sigma)\sigma^{j-1}\beta^j \frac{\Lambda_{t+j}}{\Lambda_t} N_{t+j}.$$

Note: it is okay to include a denominator Λ_t in the bank's objective function in question (b) and (c).

(d).(20) Denote V_t as the bank's expected present value of future dividends in period t under optimal decisions of L_t and D_t subject to the budget constraint and the incentive constraint. Suppose $V_t = \gamma_t N_t$. Please solve for γ_t as an expression of R^l_{t+1}, R_t, γ_{t+1}, Λ_t and Λ_{t+1}.

Answer: First, solve for D_t and L_t using the budget constraint and the incentive constraint:

$$L_t = \frac{\gamma_t N_t}{\theta},$$
$$D_t = \frac{\gamma_t N_t}{\theta} - N_t,$$

Then rewrite V_t as follows,

$$V_t = \mathrm{E}_t(1-\sigma)\beta\frac{\Lambda_{t+1}}{\Lambda_t} + \sigma\beta\frac{\Lambda_{t+1}}{\Lambda_t}V_{t+1}.$$

Therefore,

$$\begin{aligned}
\gamma_t N_t = & \quad \mathrm{E}_t(1-\sigma)\beta\frac{\Lambda_{t+1}}{\Lambda_t}N_{t+1} + \sigma\beta\frac{\Lambda_{t+1}}{\Lambda_t}\gamma_{t+1}N_{t+1} \\
= & \quad \mathrm{E}_t(1-\sigma+\sigma\gamma_{t+1})\beta\frac{\Lambda_{t+1}}{\Lambda_t}N_{t+1} \\
= & \quad \mathrm{E}_t(1-\sigma+\sigma\gamma_{t+1})\beta\frac{\Lambda_{t+1}}{\Lambda_t}(R^l_{t+1}L_t - R_t D_t) \\
= & \quad \mathrm{E}_t(1-\sigma+\sigma\gamma_{t+1})\beta\frac{\Lambda_{t+1}}{\Lambda_t}[R^l_{t+1}\frac{\gamma_t N_t}{\theta} - R_t(\frac{\gamma_t N_t}{\theta} - N_t)] \\
= & \quad \mathrm{E}_t(1-\sigma+\sigma\gamma_{t+1})\beta\frac{\Lambda_{t+1}}{\Lambda_t}[R^l_{t+1}\frac{\gamma_t}{\theta} - R_t(\frac{\gamma_t}{\theta} - 1)]N_t.
\end{aligned}$$

Therefore,

$$\gamma_t = \mathrm{E}_t(1-\sigma+\sigma\gamma_{t+1})\beta\frac{\Lambda_{t+1}}{\Lambda_t}[R^l_{t+1}\frac{\gamma_t}{\theta} - R_t(\frac{\gamma_t}{\theta} - 1)].$$

γ_t is then given by,

$$\gamma_t = \frac{\mathrm{E}_t(1-\sigma+\sigma\gamma_{t+1})\beta\frac{\Lambda_{t+1}}{\Lambda_t}R_t}{1-\mathrm{E}_t(1-\sigma+\sigma\gamma_{t+1})\beta\frac{\Lambda_{t+1}}{\Lambda_t}(R^l_{t+1}-R_t)\frac{1}{\theta}}.$$

(e).(10) Holding others equal, will an increase in the expected external financial premium $E_t R^l_{t+1} - R_t$ leads to an increase or a decrease in bank leverage $\frac{L_t}{N_t}$? Please explain the economic intuition behind.

Answer: The answer for (d) suggests that an increase in the expected external financial premium $E_t R^l_{t+1} - R_t$ leads to an increase in γ. The economic intuition is that, the higher the expected external financial premium, the higher the bank's return to net worth γ_t. The higher bank profitability raises the bank's value of continuing operation and increases the bank's borrowing capacity given by its incentive constraint. In particular, the bank leverage is given by $\frac{\gamma_t}{\theta}$. So higher γ_t leads to higher bank leverage.

2. (50) The question is related to the GK model with financial intermediaries. Consider a bank that is endowed with net worth N_t at the beginning of period t and raise deposits D_t from the depositors at the interest rate R_t. Assume that the bank also borrow discount window loans M_t from the central bank at the interest rate R_t^m. The bank grant loans L_t to entrepreneurs using its net worth, deposits and discount window loans, which implies that,

$$N_t + D_t + M_t = L_t. \tag{2}$$

The bank obtain loan repayments and make interest repayments to the depositors and the central bank at the beginning of period $t+1$. Denote R_{t+1}^l as the bank's realized return on loans L_t, where R_{t+1}^l realizes in period $t+1$. It is known that $E_t R_{t+1}^l > R_t$.

After the bank obtain funds from the depositors in period t, the banker managing the bank may transfer a fraction θ of the divertible assets $L_t - \omega M_t$ to his or her family, where $0 < \omega < 1$ denotes the fraction of discount window loans that cannot be diverted. If a bank diverts assets for personal gain, it default on its deposits and is shut down. The depositors may re-claim the remaining fraction $1 - \theta$ of funds.

Assume that the supply of discount window loans (M_t^s) is set by the government as a constant fraction of the beginning-of-period net worth of the bank:

$$M_t^s = \psi N_t, \tag{3}$$

where $\psi > 0$ is a constant.

(a).(5) Please write down the bank's net worth N_{t+1} at the beginning of period $t+1$ after it collects the loan repayments and makes interest payments on deposits.

Answer:

$$N_{t+1} = R_{t+1}^l L_t - R_t D_t - R_t^m M_t.$$

(b).(5) Suppose, with i.i.d probability $1-\sigma$, a bank exits in each period. The bank pays out its net worth as dividends when it exits. Suppose, in period t, the bank maximizes the expected present value of future dividends, where the discount factor for period $t+j$ dividend is $\beta^j \frac{\Lambda_{t+j}}{\Lambda_t}$. Please write down the bank's objective function.

Answer:

$$\sum_{j=1}^{+\infty} \mathrm{E}_t (1-\sigma)\sigma^{j-1} \beta^j \frac{\Lambda_{t+j}}{\Lambda_t} N_{t+j}.$$

(c).(10) As the depositors recognize the bank's incentive to divert funds, they will restrict the amount of deposits they saves at banks. Please write down the incentive constraint to ensure the bank does not divert funds.

Answer:

$$V_t \geq \theta(L_t - \omega M_t).$$

where V_t is the bank's objective function under optimal decisions and is given by

$$V_t = \max_{L_t, D_t, M_t} \sum_{j=1}^{+\infty} \mathrm{E}_t(1-\sigma)\sigma^{j-1}\beta^j \frac{\Lambda_{t+j}}{\Lambda_t} N_{t+j}.$$

Note: it is okay to include a denominator Λ_t in the bank's objective function in question (b) and (c).

(d).(20) Denote V_t as the bank's expected present value of future dividends in period t under optimal decisions of L_t, D_t and M_t subject to the budget constraint and the incentive constraint. Suppose $V_t = \gamma_t N_t$, and the discount window loan market clears in the equilibrium ($M_t = M_t^s$). Please solve for γ_t as an expression of R_{t+1}^l, R_t, R_t^m, γ_{t+1}, Λ_t and Λ_{t+1}.

Answer: First, the discount window loan market clearing condition implies that:

$$M_t = M_t^s = \psi N_t.$$

Then solve for D_t and L_t using the budget constraint and the incentive constraint

$$L_t = \frac{\gamma_t N_t}{\theta} + \omega M_t = (\frac{\gamma_t}{\theta} + \omega\psi)N_t,$$

$$D_t = \frac{\gamma_t + \omega\psi}{\theta} N_t, -N_t - M_t = (\frac{\gamma_t}{\theta} + \omega\psi - 1 - \psi)N_t,$$

Then rewrite V_t as follows,

$$V_t = \mathrm{E}_t(1-\sigma)\beta\frac{\Lambda_{t+1}}{\Lambda_t}N_{t+1} + \sigma\beta\frac{\Lambda_{t+1}}{\Lambda_t}V_{t+1}.$$

Therefore,

$$\begin{aligned}
\gamma_t N_t = & \quad \mathrm{E}_t(1-\sigma)\beta\frac{\Lambda_{t+1}}{\Lambda_t}N_{t+1} + \sigma\beta\frac{\Lambda_{t+1}}{\Lambda_t}\gamma_{t+1}N_{t+1} \\
= & \quad \mathrm{E}_t(1-\sigma+\sigma\gamma_{t+1})\beta\frac{\Lambda_{t+1}}{\Lambda_t}N_{t+1} \\
= & \quad \mathrm{E}_t(1-\sigma+\sigma\gamma_{t+1})\beta\frac{\Lambda_{t+1}}{\Lambda_t}(R_{t+1}^l L_t - R_t D_t - R_t^m M_t) \\
= & \quad \mathrm{E}_t(1-\sigma+\sigma\gamma_{t+1})\beta\frac{\Lambda_{t+1}}{\Lambda_t}[R_{t+1}^l(\frac{\gamma_t}{\theta}+\omega\psi)N_t - R_t(\frac{\gamma_t}{\theta}+\omega\psi-1-\psi)N_t - R_t^m\psi N_t] \\
= & \quad \mathrm{E}_t(1-\sigma+\sigma\gamma_{t+1})\beta\frac{\Lambda_{t+1}}{\Lambda_t}[(R_{t+1}^l - R_t)(\frac{\gamma_t}{\theta}+\omega\psi) + R_t + (R_t - R_t^m)\psi]N_t.
\end{aligned}$$

Therefore,

$$\gamma_t = \mathrm{E}_t(1-\sigma+\sigma\gamma_{t+1})\beta\frac{\Lambda_{t+1}}{\Lambda_t}[(R_{t+1}^l - R_t)(\frac{\gamma_t}{\theta}+\omega\psi) + R_t + (R_t - R_t^m)\psi].$$

γ_t is then given by,

$$\gamma_t = \frac{\mathrm{E}_t(1-\sigma+\sigma\gamma_{t+1})\beta\frac{\Lambda_{t+1}}{\Lambda_t}[(R^l_{t+1}-R_t)\omega\psi + R_t + (R_t - R^m_t)\psi]}{1-\mathrm{E}_t(1-\sigma+\sigma\gamma_{t+1})\beta\frac{\Lambda_{t+1}}{\Lambda_t}(R^l_{t+1}-R_t)\frac{1}{\theta}}.$$

It is also okay if the student uses $R^m_t = \omega(E_t R^l_{t+1} - R_t) + R_t$ to substitute out R^m_t in the above equation.

(e).(10) Suppose M_t, D_t and L_t have positives values under the bank's optimal decisions. Please express R^m_t as a function of $E_t R^l_{t+1}$ and R_t based on the bank's optimal decisions. Do you expect the interest rate on discount window lending R^m_t to be higher or lower than the deposit interest rate R_t? Please explain the economic intuition behind your answer.

Answer:

$$R^m_t = \omega(E_t R^l_{t+1} - R_t) + R_t.$$

As long as $\omega > 0$, then $R^m_t > R_t$. The bank is willing to pay higher interest on discount window lending than on deposits because discount window loans cannot be fully diverted, which helps relax the bank's incentive constraint.

PROBLEM SET 4

1. (25) The question is related to the open economy setup. Consider a representative household that is endowed with funds W_0. The household allocates funds between domestic assets B and foreign assets B^*. R and R^* denote the risk-free interest rate on domestic assets and foreign assets, respectively. Assume that investing in foreign assets B^* incurs an adjustment cost $\Omega_b(B^*)^2/2$, where $\Omega_b > 0$.

a. (10) Suppose the household's objective is to maximize the expected income from investing in domestic and foreign assets. Please solve for the optimal choices of domestic assets B and foreign assets B^*.

Answer: The household's problem is given by,

$$\max RB + R^*B^*$$

subject to

$$B + B^* + \Omega_b(B^*)^2/2 = W_0$$

The household's problem can be rewriten as,

$$\max R[W_0 - B^* - \Omega_b(B^*)^2/2] + R^*B^*$$

The optimal condition for B^* is given by,

$$R^* = R(1 + \Omega_b B^*)$$

The optimal choices are given by,

$$B^* = (R^*/R - 1)/\Omega_b, \quad B = W_0 - (R^*/R - 1)/\Omega_b.$$

b. (10) Suppose $\Omega_b = 0$ and the government taxes the return on foreign assets so that the after-tax interest rate on foreign assets becomes $(1-\tau)R^*$, where $\tau > 0$. In this case, what are the optimal choices of domestic assets B and foreign assets B^*?

Answer: The household's problem is given by,

$$\max RB + R^*(1-\tau)B^*$$

subject to

$$B + B^* = W_0$$

The household's problem can be rewriten as,

$$\max R(W_0 - B^*) + R^*(1-\tau)B^*$$

The first order condition for B^* is given by,

$$R^*(1-\tau) - R$$

If $R^*(1-\tau) - R > 0$, the optimal choices are given by,

$$B^* = W_0, \quad B = 0.$$

If $R^*(1-\tau) - R < 0$, the optimal choices are given by,

$$B^* = 0, \quad B = W_0.$$

If $R^*(1-\tau) - R = 0$, any combination of (B, B^*) that satisfies $B + B^* = W_0$ is optimal.

c. (5) Given $B^* \in (0, W_0)$, under what condition, will the tax on foreign asset τ and the adjustment cost Ω_b generates the same interest rate spread R^*/R?

Answer:

$$1 + \Omega_b B^* = 1/(1-\tau).$$

2. (45) The question is related to the open economy setup. Consider an economy where the private sector holds B^*_{pt} units of foreign assets and the government holds B^*_{gt} units of foreign reserves by the end of period, both earning the risk-free foreign interest rate R^*_t. The foreign assets are denominated with foreign currency. The foreign exchange rate is given by e_t, so that one unit of foreign currency could be exchanged to e_t units of domestic currency. The net export is given by NX_t.

a. (5) Plese write down the balance of payment condition of the economy.

Answer: The balance of payment condition requires that the amount of foreign capital outflows equals the current account surplus,

$$e_t(B^*_{pt} + B^*_{gt}) - e_t(B^*_{p,t-1} + B^*_{g,t-1}) = NX_t + e_t(R^*_{t-1} - 1)(B^*_{p,t-1} + B^*_{g,t-1})$$

b. (5) Under a floating exchange rate regime, how the foreign reserves B^*_{gt} and the foreign exchange rate e_t are determined? Does the government achieve monetary policy independence?

Answer: The foreign reserves B^*_{gt} is zero or constant. The foreign exchange rate e_t is determined by the balance of payment condition. Yes, the government achieves monetary policy independence.

c. (5) Under a fixed exchange rate regime, how the foreign reserves B^*_{gt} and the foreign exchange rate e_t are determined?

Answer: The foreign exchange rate e_t is determined by the policy target. The foreign reserves B^*_{gt} is determined by the balance of payment condition.

d. (15) Consider an unexpected increase in foreign interest rate. Will the foreign asset held by the private sector B^*_{pt} increase or decrease? In the floating exchange rate regime, will the foreign exchange rate e_t increase or decrease? In the fixed exchange rate regime, will the foreign reserves B^*_{gt} increase or decrease? Please explain your answer.

Answer: An unexpected increase in foreign interest rate will raise the return on foreign assets and thus encourage the private sector to hold more foreign assets. So B^*_{pt} will increase.

In the floating exchange rate regime, the increase in private capital outflows will raise the demand for foreign currency. As a result, the foreign exchange rate e_t will increase, which means domestic currency depreciates and foreign currency appreciates.

In the fixed exchange rate regime, based on the balance of payment condition, the government has to reduce foreign reserves B^*_{gt} to offset the increase in private capital outflows to maintain the stability of exchange rate.

e.(15) If the foreign reserves are financed only with money supply, how will the unexpected increase in foreign interest rate affect domestic interest rate and inflation? Under what condition, will the government be able to achieve monetary policy independence and exchange rate stability at the same time? Please explain your answer.

Answer: If the foreign reserves are financed only with money supply, the unexpected increase in foreign interest rate will raise domestic interest rate and reduce inflation. Because the consequent decrease in foreign reserves will lead to a decrease in money supply. The contractionary monetary policy will raise domestic interest rate and reduce inflation by depressing aggregate demand.

The government will be able to achieve monetary policy independence and exchange rate stability at the same time if the private sector cannot freely trade foreign assets (i.e. capital controls are present). If the private sector can freely trade foreign assets, the no-arbitrage condition requires equal returns on foreign assets and on domestic assets. Given fixed exchange rates, domestic interest rates would be pinned down by foreign interest rates.