MICROECONOMIC THEORY II

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  - ➢ Preferences

$$\{\succeq\}_{i=1}^{I} = \{\succeq_1, \succeq_2, \ldots, \succeq_I\}$$

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- Consumption

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- Consumption allocation, X, consumption bundle for agent i, X_i
- Prices

Implicit assumptions

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- Complete information or symmetric information

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  - $\exists q = (q_1, \ldots, q_L) \in \mathbb{R}^L_{++}$, shadow prices; $\exists (s_1, \ldots, S_I) \in \mathbb{R}^I_{++}$; $\forall i$,
$$s_i Du_i(X_i^*) = q, \qquad \sum_i X_i = \sum_i \omega_i.$$

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  $$s_i Du_i(X_i^*) = q, \qquad \sum_i X_i = \sum_i \omega_i.$$
  - ➢ $\forall i \in \{1, 2, \ldots, I-1\}$ and $\forall l \in \{1, 2, \ldots, L-1\}$,
  $$MRS_i^{l,l+1} = MRS_{i+1}^{l,l+1}$$
  $$\sum_i X_i = \sum_i \omega_i.$$

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- Contract curve gives all efficient allocation in the Edgeworth box.

One efficient allocation

Contract curve: all P.E.

Another example

Linear preferences: contract curve

- Preferences

$$U_A = x_{1A} + 2x_{2A}, \qquad U_B = 2x_{1B} + x_{2B}$$

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$$\omega^A = (7, 3), \qquad \omega^B = (3, 7).$$

Linear preference

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Draw the contract curve

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- This is equivalent to the optimization problem: $(\forall i)$

$$\max_X u_i(X_i) \quad s.t.$$

$$\sum_i X_i = \sum_i \omega_i$$

$$\forall h \neq i \quad u_h(X_h) \geq u_h(X_h^*)$$

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- The Lagrangian

$$\mathcal{L} = u_2(X_2) + \sum_{l=1}^{L} q_l \left[\sum_{i=1}^{I} \omega_{li} - \sum_{i=1}^{I} x_{li} \right] + \sum_{i \neq 2} s_i \left[u_i(X_i) - u_i(X_i^*) \right]$$

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- First-order condition gives

$$\forall i, \; s_i Du_i(X_i^*) = q$$

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- FOC yields

$$D_{x_{11}}\mathcal{L} = -q_1 + s_1 \frac{\partial u_1}{\partial x_{11}} = 0$$

$$D_{x_{21}}\mathcal{L} = -q_2 + s_1 \frac{\partial u_1}{\partial x_{21}} = 0$$

$$D_{x_{12}}\mathcal{L} = -q_1 + \frac{\partial u_2}{\partial x_{12}} = 0$$

$$D_{x_{22}}\mathcal{L} = -q_2 + \frac{\partial u_2}{\partial x_{22}} = 0$$

$$D_{x_{13}}\mathcal{L} = -q_1 + s_3 \frac{\partial u_3}{\partial x_{13}} = 0$$

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- Set $s_2 = 1$, we have

$$s_i D u_i = q.$$

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- UPF gives all possible combinations of utilities at P.E. allocations.
- How to find the UPF: identify all PE allocations.

Draw the UPF

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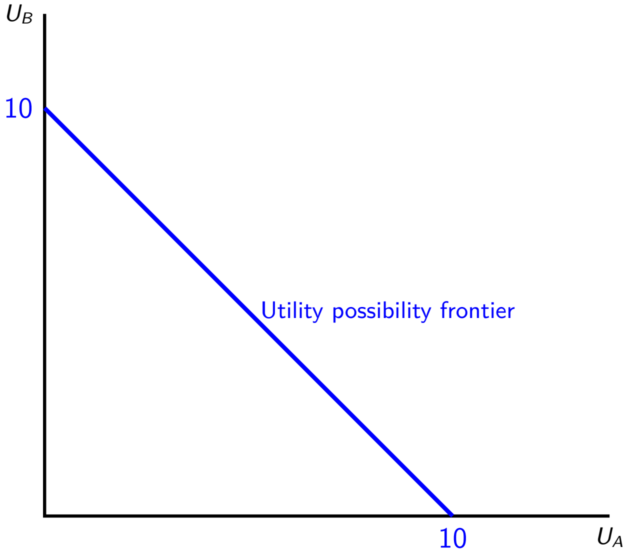

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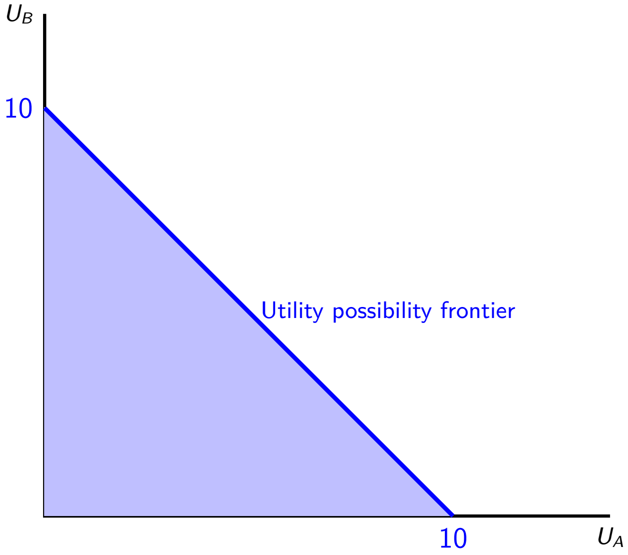

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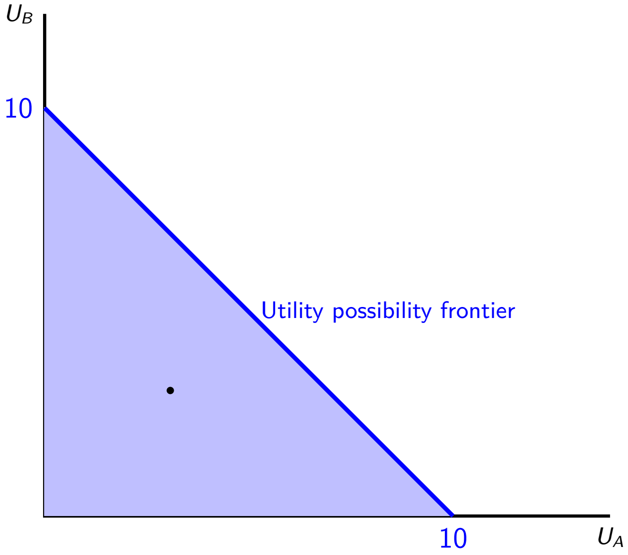

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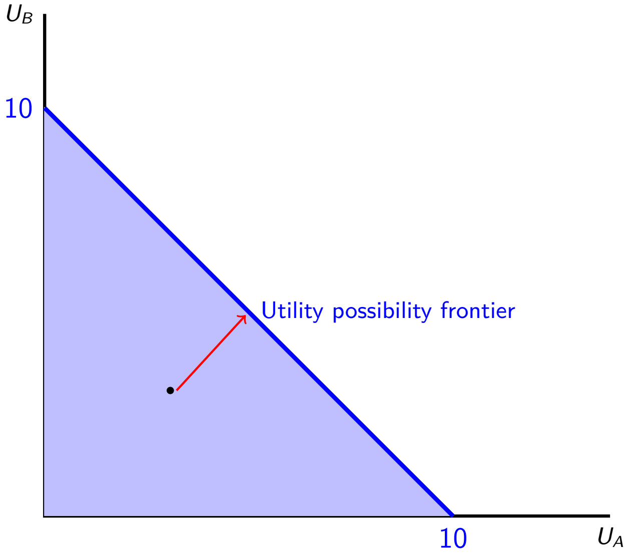

Competitive equilibrium example

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- Utility maximizing for consumers A and B,

$$x_{1A} = \frac{m_A}{2P_1},\; x_{2A} = \frac{m_A}{2P_2} \text{ where } \; m_A = 7P_1 + 3P_2$$
$$x_{1B} = \frac{m_B}{2P_1},\; x_{2B} = \frac{m_B}{2P_2} \text{ where } \; m_B = 3P_2 + 7P_2.$$

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- Plugging m_A, m_b into the allocations yields

$$x_{1A} = \frac{7P_1 + 3P_2}{2P_1},\ x_{2A} = \frac{7P_1 + 3P_2}{2P_2},$$

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C.E. example continued

- Market clears

$$5 + \frac{5P_2}{P_1} = 10, \quad 5 + \frac{5P_1}{P_2} = 10.$$

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- The competitive equilibrium (let $P_1 = 1$)

$$P_1 = 1,\ P_2 = 1,\ ,x_{1A} = x_{2A} = 5,\ x_{1B} = x_{2B} = 5.$$

Equilibrium and P.E. allocations

P.E. allocations

C.E. allocations

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C.E. allocations

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   (a) $MRS^i_{1,2} = \dfrac{P_1}{P_2} \Longrightarrow$

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   (b) $P_1x_{1A} + P_2x_{2A} = m_A$

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   (b) $P_1 x_{1A} + P_2 x_{2A} = m_A$

   $P_1 x_{1B} + P_2 x_{2B} = m_B$

2. Market clears, $j = 1, 2$

$$x_{1A} + x_{1B} = \omega_{1A} + \omega_{1B}$$

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    $$\sum_i X_i^* = \sum_i \omega_i;$$
    - For each i
    $$PX_i^* = P\omega_i.$$

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- Market dominates other mechanism to allocation resources in an economy.

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- However, Pareto improvement implies:

$$\sum_i PX'_i > \sum_i P\omega_i.$$

Contradicts the feasibility constraint.

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  - ➢ In competitive market, this is achieved through consumers' marginal decision to consume more or less given the price, which measures the relative scarcity of the goods.
- To achieve distribution goal, all that is needed is to transfer the purchasing power of the endowment.

Graphical illustration

Non-convex preferences

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  - Equilibrium must be in the core.

Some examples

- Three individual exchange economy

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- Endowment
$\omega = \begin{bmatrix} 5 & 9 & 1 \\ 5 & 1 & 9 \end{bmatrix}$

Determine core allocations

- Three allocations:
$$X = \begin{bmatrix} 7 & 6 & 2 \\ 4 & 3 & 8 \end{bmatrix} \qquad X' = \begin{bmatrix} 7 & 4 & 4 \\ 7 & 4 & 4 \end{bmatrix}$$
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- If not, find a blocking coalition that will block it.

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- This implies $\sum_{i \in S} PX'_i > \sum_{i \in S} P\omega_i$.
- Contradiction as:

$$\sum_{i \in S} X'_i \leq \sum_{i \in S} \omega_i \Longrightarrow \sum_{i \in S} PX'_i \leq \sum_{i \in S} P\omega_i.$$

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- Equilibrium is achieved through market exchange.

Core of the example

Excess demand function

- Excess demand for i:

$$Z_i(P) = X_i(P, \omega_i) - \omega_i,$$

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- Consumers A and B:

$$U_A = x_{1A}x_{2A} \qquad \omega_A = (4,1)$$
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$$Z(P) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- Walras' Law

$$PZ(P) = 0.$$

- What are the equilibrium prices?

Example 4

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$$u_A = x_{1A}^{1/2} + x_{2A}^{1/2}, \qquad u_B = x_{1B}.$$

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- Does an equilibrium exist?

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- Proposition 17C.1 (MWG): A Walrasian equilibrium exists in any pure exchange economy in which $\sum_i \omega_i \gg 0$ and $\forall i$, $X_i \in \mathbb{R}_+^L$, $\succeq_i$ is continuous, strictly convex and strongly monotonic.