MICROECONOMIC THEORY II

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  - a utility function $\tilde{u}_i : S_i \times \Theta \to \mathbb{R}$:

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- Definition: A (pure strategy) Bayesian Nash equilibrium for the Bayesian game $[I, S, u, \mathit{\Theta}, F(\cdot)]$ is a profile of decision rules $(s_1(\cdot), \ldots, s_I(\cdot))$ such that, for all i,

$$\tilde{u}_i(s_i(\cdot), s_{-i}(\cdot)) \geq \tilde{u}_i(s_i'(\cdot), s_{-i}(\cdot))$$

AN EXAMPLE

- Boss and Tim play the game. Tim does not know the payoff of Boss.

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boss	M	3, 2	1, 1
	N	4, 3	2, 4

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- Hence Bayesian Nash equilibrium is:

$$(NM, W \ if \ \mu < 1/2 \ and \ S \ if \ \mu \geq 1/2).$$

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  - $M > w > s$ and $w > M/2 > s$


T-7

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ANALYSIS

- The strategic form

player 2

	AA	AN	NA	NN
AA	$\frac{M}{4} - \frac{s+w}{2}, \frac{M}{4} - \frac{s+w}{2}$	$\frac{M}{2} - \frac{s+w}{4}, \frac{M}{4} - \frac{s}{2}$	$\frac{3M}{4} - \frac{s+w}{4}, -\frac{w}{2}$	$M, 0$
AN	$\frac{M}{4} - \frac{s}{2}, \frac{M}{2} - \frac{s+w}{4}$	$\frac{M-s}{4}, \frac{M-s}{4}$	$\frac{M}{2} - \frac{s}{4}, \frac{M-w}{4}$	$\frac{M}{2}, 0$
NA	$-\frac{w}{2}, \frac{3M}{4} - \frac{s+w}{4}$	$\frac{M-w}{4}, \frac{M}{2} - \frac{s}{4}$	$\frac{M-w}{4}, \frac{M-w}{4}$	$\frac{M}{2}, 0$
NN	$0, M$	$0, \frac{M}{2}$	$0, \frac{M}{2}$	$0, 0$

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NN	$0,\ M$	$0,\ \frac{M}{2}$	$0,\ \frac{M}{2}$	$0,\ 0$

- Two pure strategy Bayesian NE:

$$(AA, AN), \qquad (AN, AA).$$

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- Two firms privately observe their own type, and then simultaneously decide to develop/not.

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- Thus, $\hat{\theta}_i$ is determined by the following condition:

$$\hat{\theta}_i^2 - c = \hat{\theta}_i^2 \left(1 - \hat{\theta}_j\right).$$

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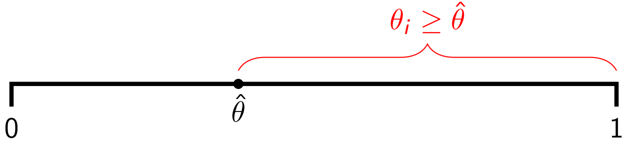

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- The Bayesian NE: For $i = A, B$,

$$s_i(\theta_i) = \begin{cases} 1 & \text{if} \quad \theta_i \geq \hat{\theta} = (c)^{1/3}; \\ 0 & \textit{otherwise} \end{cases}$$

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  - ➢ Europe's frenzied 2000 and 2001 auctions reaped nearly \$100 billion

EXAMPLE: US TREASURY BOND AUCTION

"As an auction neared, the primary dealers would work the phones, polling customers to gauge their appetite for bonds. A few seconds before the clock struck one p.m. on the appointed day, the dealers phoned "runners," who stood by a wall of phones at the Federal Reserve Building downtown, waiting to scribble down orders by hand and dash to the Fed clerk's wooden box, where they jammed them inside. At the stroke of one p.m., the clerk placed his hand over the slot. That ended the auction. The government had used this antiquated system for decades."

———-The Snowball

TWO IMPORTANT CHARACTERISTICS

- Uncertainty about the valuations of the bidders.
  - ➢ This uncertainty leads to the private values assumption about such valuations;
  - ➢ It is modeled as independent random variables from a common distribution.
- "Winner's curse": Because of the uncertainty of the value of the object for sale, a winner of an auction might wonder why all the other bidders' valuation were smaller than hers, and in particular whether this might have happened because of the others' more accurate information about the item's true value.

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    - ➢ Independent private valuation;
    - ➢ Common value;

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4. Risk preferences. Risk neutrality versus risk aversion.

DUTCH AUCTION

Maximum price

ENGLISH AUCTION

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- Since $b(0) = 0$, C should be zero,

  $$b(v) = \frac{1}{F^{N-1}(v)}\int_0^v xdF^{N-1}(x)$$

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$$\max_{b_i} EU(b_i, v) = Prob(b_i > \max_{j \neq i} b_j) \left(v - \max_{j \neq i} b_j \right) .$$

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- In the case of independent private value auctions, the English auction and the Vickrey auction are strategically equivalent.

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- In both auction, the bidder with the highest valuation wins and pays the second highest bidder's value.
- Without independent private values, English and second-price auction would not be strategically equivalent.

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REVENUE EQUIVALENCE BETWEEN FIRST AND SECOND-PRICE AUCTIONS

- k-order statistic of N draws from F_x

$$f_{x_k} = \frac{N!}{(k-1)!(N-k)!}[F(x)]^{N-k}[1-F(x)]^k f(x).$$

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PROOF OF EQUIVALENCE

- Revenue of first-price auction:

$$
\begin{aligned}
EV_F &= \int_0^1 N(N-1)\left[\int_0^v xF^{N-2}(x)f(x)dx\right] f(v)dv \\
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- Thus, we conclude

$$EV_F = EV_S = \int_0^1 N(N-1)vF^{N-2}(v)[1-F(v)]f(v)dv.$$

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    - ➢ Assign the item to the bidder with the highest valuation for it;
    - ➢ Giving incentive to bidders so that it is in their best interest to bid according to their true valuation

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- The reason for the limit on size: To prevent “short squeeze” in Treasury market.

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- Efficient rule

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- Violates budget-balancedness.