Open Economy Macro: Problem Set 1 Solution
2023 Fall

Question 1. Compute the relevant business cycle statistics for South Korea and the United States using 1) log-linear detrending; 2) log-quadratic detrending; 3) HP filtering with $\lambda = 100$. The data should be downloaded from the World Bank's WDI database. The sample period use 1971-2019 annually. Specifically, use the following time series

Variable	Series Name	Series Code
y	GDP per capita (constant LCU)	NY.GDP.PCAP.KN
c	Households and NPISHs final consumption expenditure (% of GDP)	NE.CON.PRVT.ZS
i	Gross capital formation (% of GDP)	NE.GDI.TOTL.ZS
g	General government final consumption expenditure (% of GDP)	NE.CON.GOVT.ZS
im	Imports of goods and services (% of GDP)	NE.IMP.GNFS.ZS
ex	Exports of goods and services (% of GDP)	NE.EXP.GNFS.ZS

Make a graph showing the natural logarithm of real GDP per capital and the trend for the two countries. Then identify the recessions in the two countries. Are the results the same across the three detrending methods? Also make a table showing the statistics. Attach your code for cleaning data and making tables and plots.

Solution: The results are summarized in the table below

	Log-linear		**Log-quadratic**		**HP filter**	
	U.S.	Korea	U.S.	Korea	U.S.	Korea
std (%)						
y	3.95	15.0	2.72	4.81	1.94	2.83
c/y	1.18	6.79	0.97	5.23	0.68	2.65
i/y	6.46	12.2	6.38	9.22	4.99	6.97
g/y	4.82	8.52	4.61	6.14	2.78	4.60
im/y	10.7	16.5	8.76	15.4	5.50	8.93
ex/y	11.0	16.6	10.9	16.6	7.94	10.5
corr(x,y)						
c/y	0.47	-0.56	0.11	0.16	-0.44	-0.43
i/y	0.5	0.66	0.56	0.15	0.79	0.65
g/y	-0.4	-0.81	-0.28	-0.64	-0.63	-0.58
im/y	0.51	-0.43	0.17	-0.32	0.43	-0.11
ex/y	-0.13	-0.15	-0.33	-0.51	-0.14	-0.22

Similar as the comparison between rich and emerging economies, the output in Korea is more volatile than that in the US, and consumption is less volatile than output in the U.S., while the opposite is true in Korea. In terms of business cycles, one distinct feature is that the import share is procyclical in the U.S., while it is countercyclical in Korea.

The following two figure show the result for detrending. The US experienced recessions during mid 1970s, early 1980s, and after the 2007 Great Recession. This result is robust among the three detrending methods. The log-linear detrending implies that Korea experienced recessions in almost the same periods, while the other two methods only see them in very small magnitude.

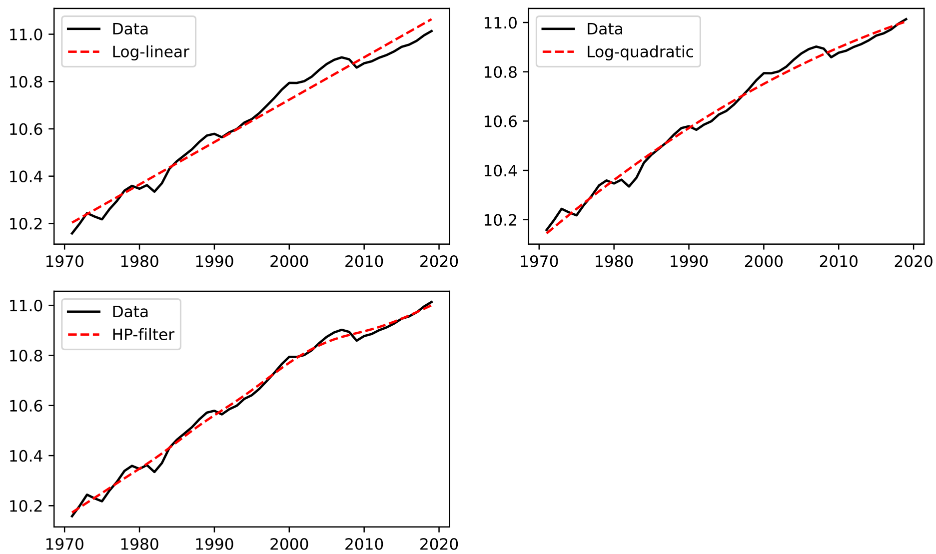

Figure 1: USA

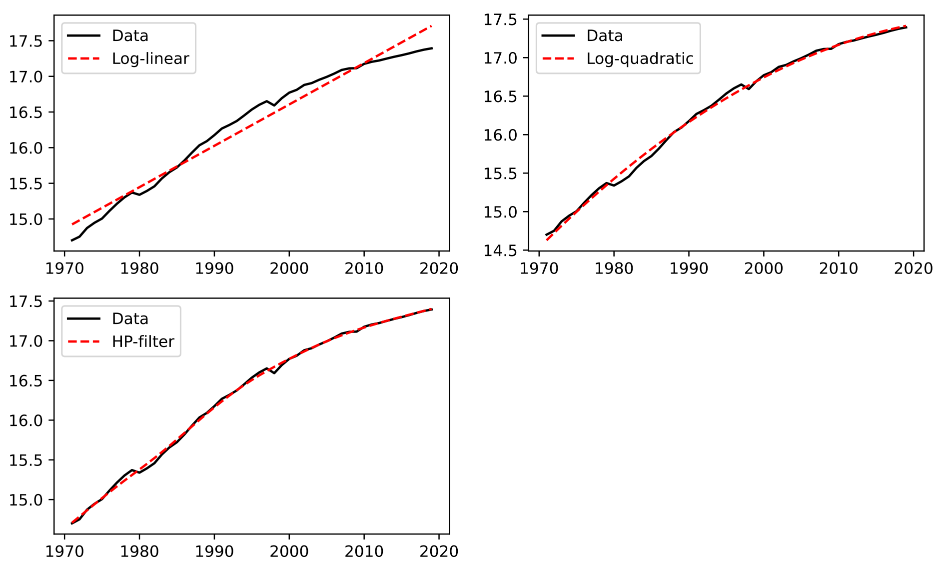

Figure 2: Korea

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Question 2. Consider a SOE model with nondurable and durable consumption goods. Let $c_{N,t}$ denote consumption of nondurable goods in period t, and $c_{D,t}$ denote purchases of durables in period t. The stock of durable goods s_t is assumed to evolve over time as

$$s_t = (1-\delta)s_{t-1} + c_{D,t}$$

where $\delta \in (0,1]$ denotes the depreciation rate of durable goods. Households have preferences of the form

$$\sum_{t=0}^{\infty} \beta^t U(c_t)$$

where U is increasing in consumption and concave. Consumption c_t is a composite of nondurable consumption and the service flow provided by the stock of consumer durables. Specifically,

$$c_t = \left[(1-\alpha)^{\frac{1}{\eta}} c_{N,t}^{1-\frac{1}{\eta}} + \alpha^{\frac{1}{\eta}} s_t^{1-\frac{1}{\eta}}\right]^{\frac{1}{1-\frac{1}{\eta}}}$$

where $\eta > 0$ and $\alpha \in (0,1)$. Households have access to an internationally traded risk-free one-period bond, which pays the interest rate r_t when held between periods t and $t+1$. The relative price of durables in terms of nondurables is one. The household is subject to a borrowing limit that prevents it from engaging in Ponzi schemes. Output y_t is produced with physical capital k_t according to a production function of the form

$$y_t = F(k_t)$$

The capital stock evolves over time as

$$k_{t+1} = (1-\delta_k)k_t + i_t$$

where i_t denotes investment in period t, and δ_k is the depreciation rate on physical capital.

1. Describe the household's budget constraint
2. State the optimization problem of the household
3. Present the complete set of equilibrium conditions
4. Suppose the interest rate is constant over time and equal to $r_t = r = \beta^{-1} - 1$. Assume that up to period -1, the economy was in a steady state equilibrium in which all variables were constant and $d = \bar{d}$, where d denotes net external debt in the steady state. Find the share of expenditures on durables in consumption expenditures in the steady state in terms of the parameters δ, r, α, and η
5. Assume that in period 0, the economy unexpectedly receives a positive income shock as a consequence of the rest of the world forgiving part of the country's net foreign debt. Assume that the positive income shock results in a 1 percent increase in the consumption of nondurables in period 0. Find the percentage increase in purchases of durables and in total consumption expenditures in period 0. Compare your answer to the one you would have obtained if all consumption goods were nondurables

6. Continuing to assume that consumption of nondurables has increased by 1 percent, find the change in trade balance in period 0 expressed as a share of steady-state consumption expenditure. Is the response of the trade balance countercyclical? Compare your findings to those you would have obtained if all consumption goods were nondurables. How much amplification is there due to the presence of durables?

Solution: 1. The budget constraint is

$$c_{N,t} + c_{D,t} + i_t + (1 + r_{t-1})d_{t-1} = y_t + d_t \tag{1}$$

2. The households optimal problem is

$$\max_{\{c_{N,t},s_t,k_{t+1},d_t\}} \quad \sum_{t=0}^{\infty} \beta^t U(c_t)$$

$$\text{s.t.} \quad c_{N,t} + s_t + k_{t+1} + (1 + r_{t-1})d_{t-1} = F(k_t) + (1 - \delta)s_{t-1} + (1 - \delta_k)k_t + d_t \tag{2}$$

$$c_t = A(c_{N,t}, s_t) = \left[(1 - \alpha)^{\frac{1}{\eta}} c_{N,t}^{1-\frac{1}{\eta}} + \alpha^{\frac{1}{\eta}} s_t^{1-\frac{1}{\eta}}\right]^{\frac{1}{1-\frac{1}{\eta}}} \tag{3}$$

$$\lim_{j\to\infty} \mathbb{E}_t \frac{d_{t+j}}{(1 + r)^j} \le 0 \tag{4}$$

3. Given $\{s_{-1}, k_0, d_{-1}\}$ and $\{r_{t-1}\}_{t=0}^{\infty}$, an equilibrium consists of $\{c_t, c_{N,t}, s_t, k_{t+1}, d_t\}_{t=0}^{\infty}$ that satisfy (2)-(4) and the following conditions (5)-(7) which are derived from FOCs

$$r_t = F'(k_{t+1}) - \delta_k \tag{5}$$

$$1 = \beta\frac{U'(c_{t+1})}{U'(c_t)} \cdot \frac{A_1(c_{N,t+1}, s_{t+1})}{A_1(c_{N,t}, s_t)} \cdot (1 + r_t) \tag{6}$$

$$\frac{s_t}{c_{N,t}} = \frac{\alpha}{1 - \alpha} \cdot \left(\frac{1 + r_t}{r_t + \delta}\right)^{\eta} \tag{7}$$

4. In steady state, from condition (7), we can get

$$\frac{c_D}{e} = \frac{\delta}{\delta + \frac{1-\alpha}{\alpha} \cdot \left(\frac{r+\delta}{1+r}\right)^{\eta}} \tag{8}$$

where $e = c_N + c_D$ is the consumption expenditure.

5. From condition (7) we can get

$$\eta\log\left(\frac{r + \delta}{1 + r}\right) = \log\left(\frac{1 - \alpha}{\alpha}\right) - \left[\log s_t - \log c_{N,t}\right]$$

which implies

$$\%\Delta s_t = \%\Delta c_{N,t} \tag{9}$$

In other words, the percentage of durable goods stock equals to the percentage change of non-durable goods.

In period 0, we have

$$s_0 = (1-\delta)s_{-1} + c_{D,0}$$

In period -1, as it is in steady state, we have

$$c_{D,-1} = \delta s_{-1}$$

From the above two equations, it's easy to get at $t = 0$

$$\%\Delta s_t = \delta \cdot \%\Delta c_{D,t} \tag{10}$$

Combining (9) and (10), we know when non-durable goods consumption increases by 1%, durable goods consumption increases by $\frac{1}{\delta}\%$. Plus condition (8) in period -1, percentage change in consumption expenditure e is

$$\frac{e_0 - e_{-1}}{e_{-1}} = \frac{1 + \frac{\alpha}{1-\alpha} \cdot \left(\frac{1+r}{r+\delta}\right)^{\eta}}{\delta + \frac{\alpha}{1-\alpha} \cdot \left(\frac{1+r}{r+\delta}\right)^{\eta}} \cdot 1\% > 1\% \tag{11}$$

If all consumption are non-durable, the increase would simply be 1%.

6. By definition of trade balance and condition (5), which implies k, i and y do not change, the change in trade balance tb in period 0 scaled by e_{-1} satisfies

$$\frac{tb_0 - tb_{-1}}{e_{-1}} = -\frac{e_0 - e_{-1}}{e_{-1}} \tag{12}$$

By condition (11), this change is greater than 1% in absolute value and is counter-cyclical. If there are only non-durable goods, the change is still counter-cyclical but is simply 1%.

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Question 3. Modify the GHH period utility function of the SOE-RBC-EDEIR model as follows:

$$U(c,h) = \frac{\left[c^{1-\omega}(1-h)^{\omega}\right]^{1-\sigma} - 1}{1-\sigma}$$

All other features of the model are unchanged.

1. Derive analytically the steady state of the model.
2. Keep the value of all parameters the same as the example in class, except for ω. Set ω such that $h = 1/3$ in steady state, i.e. households spend one-third of their time working.
3. Solve the model and produce a table of predicted second moments. You might find it convenient to use the Python code posted on Blackboard, or write your own code.

4. Compare the predictions of the present Cobb-Douglas preference model with those of its GHH preference counterpart.

Solution: 1. We have derived the first-order conditions for the RBC model. Given the current preference, we have

$$U_c(c,h) = \left[c^{1-\omega}(1-h)^{\omega}\right]^{1-\sigma} \cdot \frac{1-\omega}{c}$$
$$U_h(c,h) = \left[c^{1-\omega}(1-h)^{\omega}\right]^{1-\sigma} \cdot \left(-\frac{\omega}{1-h}\right)$$

In steady states, we still have $A=1$, $d=\bar{d}$, $r=r^*$, and

$$\frac{k}{h} = \left(\frac{\alpha}{\frac{1}{\beta}-1+\delta}\right)^{\frac{1}{1-\alpha}} \equiv \kappa$$

The condition for labor supply decision is

$$-\frac{U_h(c,h)}{U_c(c,h)} = (1-\alpha)\left(\frac{k}{h}\right)^{\alpha}$$

which implies

$$c = \frac{(1-\alpha)(1-\omega)}{\omega}\kappa^{\alpha}$$

Using the budget constraint in steady state

$$r^*\bar{d} + c + \delta k = k^{\alpha}h^{1-\alpha}$$

and plugging in the expressions for k and c in terms of h, we can get the analytical solution for h

$$h = \frac{r^*\bar{d} + \frac{(1-\alpha)(1-\omega)}{\omega}\kappa^{\alpha}}{\frac{1-\alpha(1-\omega)}{\omega}\kappa^{\alpha} - \delta\kappa} \tag{13}$$

and therefore the values for k and c.

2. Plugging $h = \frac{1}{3}$ into (13), and given the values for all the other parameters, we have

$$\omega = 0.6567$$

The solution can be found using some root finding routines in Python. See the attached code for detail.

3. Given the current preference, we have

$$U_{cc}(c,h) = \left[c^{1-\omega}(1-h)^{\omega}\right]^{1-\sigma} \cdot \frac{(1-\sigma)(1-\omega)^2-(1-\omega)}{c^2}$$
$$U_{ch}(c,h) = \left[c^{1-\omega}(1-h)^{\omega}\right]^{1-\sigma} \cdot \left(-\frac{(1-\sigma)\omega(1-\omega)}{c(1-h)}\right)$$
$$U_{hh}(c,h) = \left[c^{1-\omega}(1-h)^{\omega}\right]^{1-\sigma} \cdot \frac{(1-\sigma)\omega^2-\omega}{(1-h)^2}$$

We can further define

$$
\begin{aligned}
\epsilon_{cc} &= \frac{U_{cc}c}{U_c} = (1-\sigma)(1-\omega) - 1 \\
\epsilon_{ch} &= \frac{U_{ch}h}{U_c} = (1-\sigma)\omega\Big(-\frac{h}{1-h}\Big) \\
\epsilon_{hc} &= \frac{U_{hc}c}{U_h} = (1-\sigma)(1-\omega) \\
\epsilon_{hh} &= \frac{U_{hh}h}{U_h} = \big[(1-\sigma)\omega - 1\big]\Big(-\frac{h}{1-h}\Big)
\end{aligned}
$$

for convenience to construct the coefficient matrix of the linear system. Finally use the Python code to solve this problem.

4. The comparison is summarized in the following table:

	GHH	Cobb-Douglas
std.(%)		
y	3.08	2.79
c	2.71	0.85
i	9.04	11.3
h	2.12	1.80
tb/y	1.78	2.59
ca/y	1.45	2.35
corr(x,y)		
c	0.84	0.25
i	0.67	0.51
h	1	0.95
tb/y	-0.04	0.48
ca/y	0.05	0.36

The Cobb-Douglas preference implies wealth effect on labor supply compared with GHH preference. As a result, we can see employment is less volatile with the former preference. In addition, consumption is much less volatile, and investment, trade balance and current account are more volatile. These results can also be confirmed with the impulse response functions below.

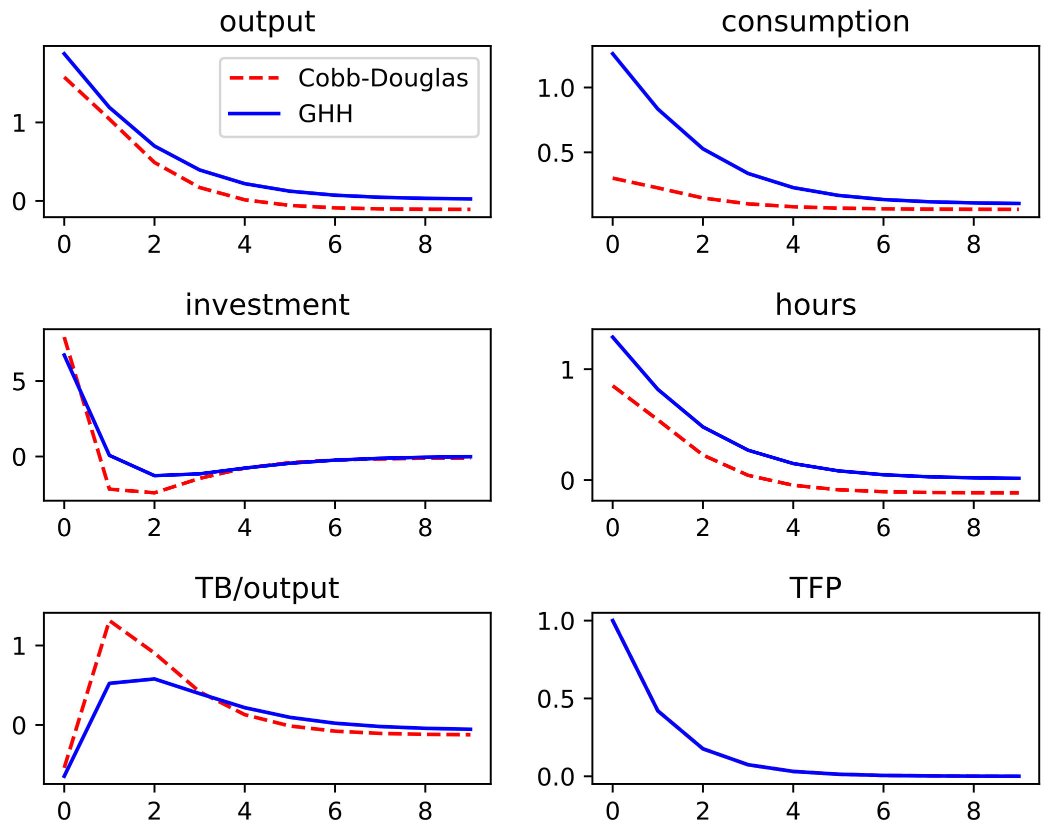

Figure 3: Impulse Response

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Question 4. Consider a two-period small open economy populated by identical households with preferences given by

$$\ln c_1 + \ln c_2$$

where c_1 and c_2 denote consumption in period 1 and 2, respectively. The household's budget constraints in the two periods are

$$c_1 + d_1 + q(k_1 - k) = F(k) + d_2$$
$$c_2 + d_2 = F(k_1)$$

where d_1 denotes debt due in period 1 and d_2 denote debt assumed in period 1 and due in period 2, k denotes an exogenous initial stock of capital, k_1 denotes capital purchased in period 1, q denotes the relative price of capital in terms of consumption, and $F(\cdot)$ is an increasing and concave production function. Debt accumulation in period 1 is subject to the following collateral constraint:

$$d_2 \leq \kappa q k_1$$

where $\kappa < 1$ is a parameter.

1. Derive the first-order conditions associated with the household's optimization problem.

2. Assume that the aggregate stock of capital is fixed. Derive the complete set of equilibrium conditions.

3. Characterize the range of the initial debt positions, d_1, for which the collateral constraint does not bind in equilibrium.

4. Find sufficient conditions on the initial level of debt, d_1, for which the economy possesses multiple equilibria (in particular, at least one equilibrium in which the collateral constraint binds and one in which the constraint does not).

Solution: 1. The Lagrangian associated with the household's optimization problem is given by

$$\begin{aligned}\mathcal{L} = \ln c_1 + \ln c_1 + &\lambda_1\Big[F(k) + d_2 - d_1 - q(k_2 - k_1) - c_1\Big] \\ &+ \lambda_2\Big[F(k_1) - d_2 - c_2\Big] \\ &+ \mu\Big(\kappa q k_1 - d_2\Big)\end{aligned}$$

and the first-order conditions are

$$\begin{aligned}&[c_1] \quad \frac{1}{c_1} - \lambda_1 = 0 \\ &[c_2] \quad \frac{1}{c_2} - \lambda_2 = 0 \\ &[k_1] \quad -\lambda_1 q + \lambda_2 F'(k_1) + \mu\kappa q = 0 \\ &[d_2] \quad \lambda_1 - \lambda_2 - \mu = 0\end{aligned}$$

2. The equilibrium conditions are as follows

$$\frac{q}{c_1} = \frac{F'(k)}{c_2} + \mu\kappa q \tag{14}$$

$$\mu = \frac{1}{c_1} - \frac{1}{c_2} \tag{15}$$

$$c_1 = F(k) + d_2 - d_2 \tag{16}$$

$$c_2 = F(k) - d_2 \tag{17}$$

$$\mu(\kappa q k - d_2) = 0 \tag{18}$$

$$d_2 \leq \kappa q k \tag{19}$$

$$\mu \geq 0, \; c_1, c_2 > 0 \tag{20}$$

3. When the constraint does not bind, we have $\mu = 0$. From condition (15), (16) and (17), we should have

$$\begin{aligned}&c_1 = c_2 = F(k) - \frac{d_1}{2} \\ &d_2 = \frac{d_1}{2}\end{aligned}$$

Plug these equations and $\mu = 0$ to (14) delivers

$$q = F'(k)$$

Therefore, by (18), we should have

$$d_1 \leq 2\kappa F'(k)k \tag{21}$$

4. First, we derive the conditions on d_1 for which an equilibrium in which the constraint binds exists. From conditions (14) to (17), we can get the following equation:

$$\kappa qk = \kappa F'(k)k \frac{F(k) + d_2 - d_1}{(1-\kappa)\Big[F(k) - d_2\Big] + \kappa\Big[F(k) + d_2 - d_1\Big]} \tag{22}$$

Note that the RHS of this equation is increasing in d_2. When the constraint binds, we also have

$$d_2 = \kappa qk \tag{23}$$

Equation (22) and (23) determine the binding equilibrium. Note that $d_2 = \frac{d_1}{2}$, $q = F'(k)$ is a solution to (22). Therefore a sufficient condition to ensure existence of a binding solution is when $q = 0$, the d_2 that satisfying (22) should be positive. This sufficient condition implies

$$d_1 > F(k) \tag{24}$$

Finally, if the economy possesses multiple equilibia, both (21) and (24) should hold at the same time, i.e.

$$F(k) < d_1 \leq 2\kappa F'(k)k \tag{25}$$

or in another form

$$1 < \frac{d_1}{y} \leq \frac{2\kappa F'(k)k}{F(k)}$$

Two points need to pay attention to. First, since the production $F(\cdot)$ is concave, we have $F(k) > F'(k)k$. The set for d_1 in not empty only if $\kappa > \frac{1}{2}$. If $\kappa \leq \frac{1}{2}$, there will be only one equilibrium. Second, given a value of κ, the smaller the k is, the larger the set for d_1. In sum, mutiple equilibria is easier to take place when output level is low and the stock collateral requirement is not very tight.

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Question 5. Modify the Eaton-Gersovitz model of sovereign default to allow for a proportional output loss function of the form $L(y) = a_1 y$. Calibrate a_1 to match an average output loss due to default of about 5 percent of output per period conditional on the country being in bad financial standing. Use my Matlab code, or write your own code, to produce quantitative predictions of the model including default probability, average debt-output ratio, average and standard deviation of spreads, and correlations of spreads with output and trade balance. Discuss your findings, paying particular attention to how the present output loss specification affects the model's ability to predict country default in bad times.

Solution: By setting $a_1 = 0.05$, we can match the target. The statistics are summarized in the table below.

	Linear	Asymmetrically quadratic
default prob.(%)	0.11	2.68
mean(debt/output)(%)	139	59
mean(spreads)(%)	0.1	3.53
std(spreads)(%)	0.08	4.2
corr(spreads,output)	-0.04	-0.42
corr(spreads,tb)	0.06	0.82

Using linear penalty function, we can see that the default probability is close to zero and is much lower than using the asymmetric and quadratic penalty function. The reason is that the output loss is higher in bad times with linear penalty. A smaller default probability results in much lower spreads. Knowing that the country is less likely to default, the lenders are willing to lend more. As a consequence, the country can accumulate a much higher level of debt. $\square$