1.

(a).

$$X' = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \end{pmatrix}$$

(b).

$$X'X = \begin{pmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{pmatrix} \qquad X'Y = \begin{pmatrix} \sum y_i \\ \sum x_i y_i \end{pmatrix}$$

$$\hat{\beta} = \frac{1}{n\sum x_i^2 - (\sum x_i)^2} \begin{pmatrix} \sum x_i^2 & -\sum x_i \\ -\sum x_i & n \end{pmatrix} \begin{pmatrix} \sum y_i \\ \sum x_i y_i \end{pmatrix}$$

$$\Rightarrow \hat{\beta}_2 = \frac{n\sum x_i y_i - \sum x_i \sum y_i}{n\sum x_i^2 - (\sum x_i)^2} = \frac{n(\sum x_i y_i - \bar{x}\sum y_i)}{n\sum x_i^2 - (\sum x_i)^2} = \frac{\frac{1}{n}(\sum (x_i - \bar{x}) y_i)}{\frac{1}{n}(\sum x_i^2 - n(\bar{x})^2)} = \frac{\frac{1}{n}\sum (x_i - \bar{x})(y_i - \bar{y})}{\frac{1}{n}\sum (x_i - \bar{x})^2}$$

(c).

$$FOC : X'e = 0 \Rightarrow \sum e_i = 0$$

$$\frac{1}{n}\sum y_i = \hat{\beta}_1 + \hat{\beta}_2 \cdot \frac{1}{n}\sum x_i + \frac{1}{n}\sum e_i$$

$$\bar{y} = \hat{\beta}_1 + \hat{\beta}_2 \bar{x} \Rightarrow \hat{\beta}_1 = \bar{y} - \bar{x}\hat{\beta}_2$$

2.

(a).

$$R = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad q = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

(b).

$u \sim N(0, \sigma^2 I_n) \Rightarrow \hat{\beta} \sim N(\beta, \sigma^2 (X'X)^{-1})$

Under H_0 $\quad d = R\hat{\beta} - q \sim N(0, \sigma^2 R(X'X)^{-1}R')$

Then we have $d'[Var(d)]^{-1}d = (R\hat{\beta} - q)'[\sigma^2 R(X'X)^{-1}R']^{-1}(R\hat{\beta} - q) \sim \chi^2(J)$

J is the number of restrictions

Because σ^2 is unknown $\quad$ Estimate it by $S^2 = \dfrac{e'e}{n-k}$

and $\dfrac{(n-k)S^2}{\sigma^2} = \dfrac{e'e}{\sigma^2} = \dfrac{u'Mu}{\sigma^2} = (\dfrac{u}{\sigma})'M(\dfrac{u}{\sigma}) \sim \chi^2(tr(M)) = \chi^2_{n-k}$

So we can construct $F(J, n{-}k)$ given the independent of the numerator and denominator as follows:

$$F(J, n{-}k) = \frac{\chi^2(J)/J}{\chi^2_{n-k}/n-k} = \frac{d'[Var(d)]^{-1}d/J}{[(n-k)S^2/\sigma^2]/n-k} = \frac{[(R\hat{\beta}-q)/\sigma]'[R(X'X)^{-1}R']^{-1}[(R\hat{\beta}-q)/\sigma]/J}{[(u/\sigma)'M(u/\sigma)]/n-k}$$

$[R\hat{\beta} - q]/\sigma \overset{H_0}{\Rightarrow} R(\hat{\beta} - \beta)/\sigma = R(X'X)^{-1}X'(u/\sigma) = A(u/\sigma)$

Since $AM = R(X'X)^{-1}X'M = 0$

$Cov[A(u/\sigma), M(u/\sigma)] = AE(uu')M'/\sigma^2 = A\sigma^2 I_n M/\sigma^2 = AM = 0$

Since $A(u/\sigma), M(u/\sigma)$ are joint normally distributed

Then $A(u/\sigma), M(u/\sigma)$ are independent with each other

$\Rightarrow Cov(A(u/\sigma), M(u/\sigma)) = 0$

As $n \rightarrow \infty$,the asymptotic distribution of the F test follows χ^2_J/J because $S^2 \xrightarrow{P} \sigma^2$

(c).

Under H_0,R&D spending does not affect firm's output and the production technology is constant return to scale

$$\Rightarrow y = L^{\alpha}K^{\beta} \qquad \ln y = \alpha \ln L + \beta \ln K \qquad \alpha + \beta = 1$$

3.

$$F(k, n-k) = \frac{R^2/k}{(1-R^2)/n-k} = \frac{\frac{b'x'xb}{y'y}/k}{\frac{\hat{u}'\hat{u}}{y'y}/n-k} = \frac{b'x'xb/k}{\hat{u}'\hat{u}/n-k} \qquad \hat{u}'\hat{u}/n-k \xrightarrow{P} \sigma^2$$

For numerator: $b = \beta + (x'x)^{-1}x'u = (x'x)^{-1}x'u$

$$\therefore b'x'xb = u'x(x'x)^{-1}x'xx(x'x)^{-1}x'u = u'x(x'x)^{-1}x'u$$

$$\therefore \frac{bx'xb}{\sigma^2} = (u/\sigma)'x(x'x)^{-1}x'(u/\sigma) \rightarrow \chi^2(tr(P_x)) = \chi^2_k$$

$$\therefore \frac{b'x'xb/k}{\hat{u}'\hat{u}/n-k} \rightarrow \frac{\sigma^2\chi^2_k/k}{\sigma^2}$$

Consider the numerator and the denominator together, $plimF(k, n-k) = \chi^2_k/k$

4.

$$corr^2(y^*, x) = \frac{cov^2(y^*, x)}{Var(y^*)Var(x)} \qquad cov(y^*, x) = cov(\beta x^* + \epsilon, x^* + u) = \beta Var(x^*)$$

$$Var(y^*) = Var(\beta x^* + \epsilon) = \beta^2 Var(x^*) + \sigma^2_{\epsilon}$$

$$Var(x) = Var(x^* + u) = Var(x^*) + \sigma^2_u$$

$$corr^2(y^*, x^*) = \frac{cov^2(y^*, x^*)}{Var(y^*)Var(x^*)} \qquad cov(y^*, x^*) = cov(\beta x^* + \epsilon, x^*) = \beta Var(x^*)$$

$$\because cov(y^*, x) = cov(y^*, x^*) \qquad Var(x) > Var(x^*)$$

$$\therefore (corr(y^*, x))^2 < (corr(y^*, x^*))^2$$

$$corr^2(y, x) = \frac{cov^2(y, x)}{Var(y)Var(x)} \qquad cov(y, x) = cov(\beta x^* + \epsilon + v, x^* + u) = \beta Var(x^*)$$

$$Var(y) = Var(y^*) + \sigma^2_v$$

$$\because cov(y, x) = cov(y^*, x^*) \qquad Var(y) > Var(y^*) \qquad Var(x) > Var(x^*)$$

$$\therefore (corr(y, x))^2 < (corr(y^*, x^*))^2$$

5.

(a).

For GMM $\quad \hat{Q}_n = \epsilon' w[Var(w'\epsilon)]^{-1} w'\epsilon = \frac{1}{\sigma^2}\epsilon' w(w'w)^{-1}w'\epsilon$

So we can minimize $\epsilon' w(w'w)^{-1}w'\epsilon$

$(y-\delta Z)'w(w'w)^{-1}w'(y-Z\delta) = (y-\delta Z)'P_w(y-Z\delta) = y'P_w y - 2\delta' Z' P_w y + \delta' Z' P_w Z\delta$

FOC: $-2Z'P_w y + 2Z'P_w Z\delta = 0 \Rightarrow \hat{\delta}_{GMM} = (Z'P_w Z)^{-1} Z' P_w y$

For 2SLS $\quad$ stage1. $\quad$ regress Z on w and predict Z as $\hat{Z} = P_w Z$

stage2. $\quad$ put $\hat{z}$ into original regression and use OLS to get $\hat{\delta}$ $\quad y = \hat{Z}\delta + \tilde{\epsilon} = P_w Z\delta + \tilde{\epsilon}$

$\Rightarrow \hat{\delta}_{2SLS} = [(P_w Z)'(P_w Z)]^{-1}(P_W Z)'y = (Z'P_w Z)^{-1}Z'P_w y$

(b).

The function that GMM minimizes is $f(\delta) = \frac{1}{\sigma^2}(y-Z\delta)'w(w'w)^{-1}w'(y-Z\delta)$

Evaluated at $\hat{\delta}$, $f(\hat{\delta}) = \frac{1}{\sigma^2}e'w(w'w)^{-1}w'e$,where $e = y - Z\hat{\delta}$

σ^2 can be estimated by $e'e/n$

$\therefore f(\hat{\sigma}) = n[e'w(w'w)^{-1}w'e/e'e]$

uncentered $R^2 = \frac{SSR}{SST}$

regress e on w $\quad SST = e'e \quad SSR = \hat{e}'\hat{e} = e'P_w e$

$\therefore nR^2 = n[e'w(w'w)^{-1}w'e/e'e]$

6.

GMM

$\hat{Q}_n = u'wA^{-1}w'u = (y-xb)'wA^{-1}w'(y-xb) = y'wA^{-1}w'y - 2b'x'wA^{-1}w'y + b'x'wA^{-1}w'xb$

FOC $\Rightarrow b_{GMM} = (x'wA^{-1}w'x)^{-1}x'wA^{-1}w'y = (w'x)^{-1}w'y$

$Var(\sqrt{n}b_{GMM}) = (G'\omega^{-1}G)^{-1} = [(-\frac{1}{n}Ew'x)'(\frac{1}{n}Ew'uu'w)^{-1}(-\frac{1}{n}Ew'x)]^{-1} = (\frac{1}{n}w'x)^{-1}(\frac{1}{n}w'\sum w)(\frac{1}{n}x'w)^{-1}$

$Var(b_{GMM}) = (w'x)^{-1}(w'\sum w)(x'w)^{-1}$

2SLS

$y = P_w x\beta + u$

$b_{2SLS} = (x'P_w x)^{-1}x'P_w y = (x'w(w'w)^{-1}w'x)^{-1}x'w(w'w)^{-1}w'y = (w'x)^{-1}w'y = \beta + (w'x)^{-1}w'u$

$\sqrt{n}(b_{2SLS} - \beta) = (\frac{w'x}{n})^{-1}(\frac{1}{\sqrt{n}}w'u) \to (\frac{w'x}{n})^{-1}N(0, \frac{1}{n}w'\sum w)$

$Var(\sqrt{n}b_{2SLS}) = (\frac{1}{n}w'x)^{-1}(\frac{1}{n}w'\sum w)(\frac{1}{n}x'w)^{-1}$

$Var(b_{2SLS}) = (w'x)^{-1}(w'\sum w)(x'w)^{-1}$

$Var(b_{GMM}) = Var(b_{2SLS})$ The same efficient

7.

(a).

$$\begin{pmatrix} \bar{y_A} - \mu \\ \bar{y_B} - (\mu+5) \end{pmatrix}' \begin{pmatrix} Var(\bar{y_A}) & 0 \\ 0 & Var(\bar{y_B}) \end{pmatrix}^{-1} \begin{pmatrix} \bar{y_A} - \mu \\ \bar{y_B} - (\mu+5) \end{pmatrix}$$

$$= \begin{pmatrix} 7-\mu \\ 4-\mu \end{pmatrix}' \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} 7-\mu \\ 4-\mu \end{pmatrix} = f(\mu)$$

$f(\mu) = (7-\mu)^2 + 5(4-\mu)^2$

$\frac{\partial f(\mu)}{\partial \mu} = -2(7-\mu) - 10(4-\mu) = 0 \Rightarrow \hat{\mu} = 4.5$

(b).

$H_0 : E(yingroupB) = E(yingroupA) + 5$

$f(\hat{\mu}) = (2.5)^2 + 5(0.5)^2 = 7.5$

The best asymptotically distributes as χ^2 with 1 degree of freedom

$(\mu_B - \mu_A - 5)[Var(\mu_B - \mu_A - 5)]^{-1}(\mu_B - \mu_A - 5) = 7.5$

8.

(a).

$\hat{\beta}_1 = (X_1' M_2 X_1)^{-1} X_1' M_2 y = \beta_1 + (X_1' M_2 X_1)^{-1} X_1' M_2 u$

$Var(\hat{\beta}_1) = \sigma^2 (X_1' M_2 X_1)^{-1}$

$$W = (\hat{\beta}_1 - 0)'[Var(\hat{\beta}_1)]^{-1}(\hat{\beta}_1 - 0) = \frac{1}{\sigma^2} y' M_2 X_1 (X_1' M_2 X_1)^{-1} (X_1' M_2 X_1)(X_1' M_2 X_1)^{-1} X_1' M_2 y$$
$$= \frac{1}{\sigma^2} y' M_2 X_1 (X_1' M_2 X_1)^{-1} X_1' M_2 y$$

(b).

$$g(\hat{\beta}) = \begin{pmatrix} X_1'(y - X_1\hat{\beta}_1 - X_2\hat{\beta}_2) \\ X_2'(y - X_1\hat{\beta}_1 - X_2\hat{\beta}_2) \end{pmatrix}$$
$$g(\tilde{\beta}) = \begin{pmatrix} X_1'(y - X_2\tilde{\beta}_2) \\ X_2'(y - X_2\tilde{\beta}_2) \end{pmatrix} = \begin{pmatrix} X_1'(y - X_2\tilde{\beta}_2) \\ 0 \end{pmatrix}$$
$$\therefore LM = ((y - X_2\tilde{\beta}_2)' X_1, 0)[Var \begin{pmatrix} X_1'(y - X_2\tilde{\beta}_2) \\ 0 \end{pmatrix}]^{-1} \begin{pmatrix} X_1'(y - X_2\tilde{\beta}_2) \\ 0 \end{pmatrix}$$
$$y - X_2\tilde{\beta}_2 = y - X_2(X_2'X_2)^{-1}X_2'y = M_2 y = M_2(X_1\beta_1 + X_2\beta_2 + u) = M_2X_1\beta_1 + M_2u = M_2u$$

$Var[X_1'(y - X_2\tilde{\beta}_2)] = Var[X_1'(M_2X_1\beta_1 + M_2u)] = \sigma^2(X_1'M_2X_1)$

$$\therefore LM = \frac{1}{\sigma^2} y' M_2 X_1 (X_1' M_2 X_1)^{-1} X_1' M_2 y$$