PROBLEM SET 3 ANSWERS

1. (50) The question is related to the GK model with financial intermediaries. Consider a bank that is endowed with net worth N_t at the beginning of period t and raise deposits D_t from the depositors at the interest rate R_t. The bank grant loans L_t to entrepreneurs using its net worth and deposits, which implies that,

$$N_t + D_t = L_t. \tag{1}$$

The bank obtain loan repayments and repay the deposits at the beginning of period $t+1$. Denote R^l_{t+1} as the bank's realized return on loans L_t, where R^l_{t+1} realizes in period $t+1$. It is known that $E_t R^l_{t+1} > R_t$.

After the bank obtain funds from the depositors in period t, the banker managing the bank may transfer a fraction θ of the total assets L_t to his or her family. If a bank diverts assets for personal gain, it defaults on its deposits and is shut down. The depositors re-claim the remaining fraction $1-\theta$ of funds.

(a).(5) Please write down the bank's net worth N_{t+1} at the beginning of period $t+1$ after it collects the loan repayments and makes interest payments on deposits.

Answer:

$$N_{t+1} = R^l_{t+1} L_t - R_t D_t.$$

(b).(5) Suppose, with i.i.d probability $1-\sigma$, a bank exits in each period. The bank pays out its net worth as dividends when it exits. Suppose, in period t, the bank maximizes the expected present value of future dividends, where the discount factor for period $t+j$ dividend is $\beta^j \frac{\Lambda_{t+j}}{\Lambda_t}$. Please write down the bank's objective function.

Answer:

$$\sum_{j=1}^{+\infty} \mathrm{E}_t (1-\sigma)\sigma^{j-1}\beta^j \frac{\Lambda_{t+j}}{\Lambda_t} N_{t+j}.$$

(c).(10) As the depositors recognize the bank's incentive to divert funds, they will restrict the amount of deposits they saves at banks. Please write down the incentive constraint to ensure the bank does not divert funds.

Answer:

$$V_t \geq \theta L_t.$$

where V_t is the bank's objective function under optimal decisions and is given by

$$V_t = \max_{L_t, D_t} \sum_{j=1}^{+\infty} \mathrm{E}_t(1-\sigma)\sigma^{j-1}\beta^j \frac{\Lambda_{t+j}}{\Lambda_t} N_{t+j}.$$

Note: it is okay to include a denominator Λ_t in the bank's objective function in question (b) and (c).

(d).(20) Denote V_t as the bank's expected present value of future dividends in period t under optimal decisions of L_t and D_t subject to the budget constraint and the incentive constraint. Suppose $V_t = \gamma_t N_t$. Please solve for γ_t as an expression of R^l_{t+1}, R_t, γ_{t+1}, Λ_t and Λ_{t+1}.

Answer: First, solve for D_t and L_t using the budget constraint and the incentive constraint:

$$L_t = \frac{\gamma_t N_t}{\theta},$$

$$D_t = \frac{\gamma_t N_t}{\theta} - N_t,$$

Then rewrite V_t as follows,

$$V_t = \mathrm{E}_t(1-\sigma)\beta\frac{\Lambda_{t+1}}{\Lambda_t} + \sigma\beta\frac{\Lambda_{t+1}}{\Lambda_t}V_{t+1}.$$

Therefore,

$$\begin{aligned}
\gamma_t N_t = & \quad \mathrm{E}_t(1-\sigma)\beta\frac{\Lambda_{t+1}}{\Lambda_t}N_{t+1} + \sigma\beta\frac{\Lambda_{t+1}}{\Lambda_t}\gamma_{t+1}N_{t+1} \\
= & \quad \mathrm{E}_t(1-\sigma+\sigma\gamma_{t+1})\beta\frac{\Lambda_{t+1}}{\Lambda_t}N_{t+1} \\
= & \quad \mathrm{E}_t(1-\sigma+\sigma\gamma_{t+1})\beta\frac{\Lambda_{t+1}}{\Lambda_t}(R^l_{t+1}L_t - R_t D_t) \\
= & \quad \mathrm{E}_t(1-\sigma+\sigma\gamma_{t+1})\beta\frac{\Lambda_{t+1}}{\Lambda_t}[R^l_{t+1}\frac{\gamma_t N_t}{\theta} - R_t(\frac{\gamma_t N_t}{\theta} - N_t)] \\
= & \quad \mathrm{E}_t(1-\sigma+\sigma\gamma_{t+1})\beta\frac{\Lambda_{t+1}}{\Lambda_t}[R^l_{t+1}\frac{\gamma_t}{\theta} - R_t(\frac{\gamma_t}{\theta} - 1)]N_t.
\end{aligned}$$

Therefore,

$$\gamma_t = \mathrm{E}_t(1-\sigma+\sigma\gamma_{t+1})\beta\frac{\Lambda_{t+1}}{\Lambda_t}[R^l_{t+1}\frac{\gamma_t}{\theta} - R_t(\frac{\gamma_t}{\theta} - 1)].$$

γ_t is then given by,

$$\gamma_t = \frac{\mathrm{E}_t(1-\sigma+\sigma\gamma_{t+1})\beta\frac{\Lambda_{t+1}}{\Lambda_t}R_t}{1 - \mathrm{E}_t(1-\sigma+\sigma\gamma_{t+1})\beta\frac{\Lambda_{t+1}}{\Lambda_t}(R^l_{t+1} - R_t)\frac{1}{\theta}}.$$

(e).(10) Holding others equal, will an increase in the expected external financial premium $E_t R^l_{t+1} - R_t$ leads to an increase or a decrease in bank leverage $\frac{L_t}{N_t}$? Please explain the economic intuition behind.

Answer: The answer for (d) suggests that an increase in the expected external financial premium $E_t R^l_{t+1} - R_t$ leads to an increase in γ. The economic intuition is that, the higher the expected external financial premium, the higher the bank's return to net worth γ_t. The higher bank profitability raises the bank's value of continuing operation and increases the bank's borrowing capacity given by its incentive constraint. In particular, the bank leverage is given by $\frac{\gamma_t}{\theta}$. So higher γ_t leads to higher bank leverage.

2. (50) The question is related to the GK model with financial intermediaries. Consider a bank that is endowed with net worth N_t at the beginning of period t and raise deposits D_t from the depositors at the interest rate R_t. Assume that the bank also borrow discount window loans M_t from the central bank at the interest rate R_t^m. The bank grant loans L_t to entrepreneurs using its net worth, deposits and discount window loans, which implies that,

$$N_t + D_t + M_t = L_t. \tag{2}$$

The bank obtain loan repayments and make interest repayments to the depositors and the central bank at the beginning of period $t+1$. Denote R_{t+1}^l as the bank's realized return on loans L_t, where R_{t+1}^l realizes in period $t+1$. It is known that $E_t R_{t+1}^l > R_t$.

After the bank obtain funds from the depositors in period t, the banker managing the bank may transfer a fraction θ of the divertible assets $L_t - \omega M_t$ to his or her family, where $0 < \omega < 1$ denotes the fraction of discount window loans that cannot be diverted. If a bank diverts assets for personal gain, it default on its deposits and is shut down. The depositors may re-claim the remaining fraction $1 - \theta$ of funds.

Assume that the supply of discount window loans (M_t^s) is set by the government as a constant fraction of the beginning-of-period net worth of the bank:

$$M_t^s = \psi N_t, \tag{3}$$

where $\psi > 0$ is a constant.

(a).(5) Please write down the bank's net worth N_{t+1} at the beginning of period $t+1$ after it collects the loan repayments and makes interest payments on deposits.

Answer:

$$N_{t+1} = R_{t+1}^l L_t - R_t D_t - R_t^m M_t.$$

(b).(5) Suppose, with i.i.d probability $1-\sigma$, a bank exits in each period. The bank pays out its net worth as dividends when it exits. Suppose, in period t, the bank maximizes the expected present value of future dividends, where the discount factor for period $t+j$ dividend is $\beta^j \frac{\Lambda_{t+j}}{\Lambda_t}$. Please write down the bank's objective function.

Answer:

$$\sum_{j=1}^{+\infty} \mathrm{E}_t (1-\sigma)\sigma^{j-1}\beta^j \frac{\Lambda_{t+j}}{\Lambda_t} N_{t+j}.$$

(c).(10) As the depositors recognize the bank's incentive to divert funds, they will restrict the amount of deposits they saves at banks. Please write down the incentive constraint to ensure the bank does not divert funds.

Answer:

$$V_t \geq \theta(L_t - \omega M_t).$$

where V_t is the bank's objective function under optimal decisions and is given by

$$V_t = \max_{L_t, D_t, M_t} \sum_{j=1}^{+\infty} \mathrm{E}_t (1-\sigma)\sigma^{j-1}\beta^j \frac{\Lambda_{t+j}}{\Lambda_t} N_{t+j}.$$

Note: it is okay to include a denominator Λ_t in the bank's objective function in question (b) and (c).

(d).(20) Denote V_t as the bank's expected present value of future dividends in period t under optimal decisions of L_t, D_t and M_t subject to the budget constraint and the incentive constraint. Suppose $V_t = \gamma_t N_t$, and the discount window loan market clears in the equilibrium ($M_t = M_t^s$). Please solve for γ_t as an expression of R_{t+1}^l, R_t, R_t^m, γ_{t+1}, Λ_t and Λ_{t+1}.

Answer: First, the discount window loan market clearing condition implies that:

$$M_t = M_t^s = \psi N_t.$$

Then solve for D_t and L_t using the budget constraint and the incentive constraint

$$L_t = \frac{\gamma_t N_t}{\theta} + \omega M_t = (\frac{\gamma_t}{\theta} + \omega\psi)N_t,$$

$$D_t = \frac{\gamma_t + \omega\psi}{\theta} N_t, -N_t - M_t = (\frac{\gamma_t}{\theta} + \omega\psi - 1 - \psi)N_t,$$

Then rewrite V_t as follows,

$$V_t = \mathrm{E}_t(1-\sigma)\beta\frac{\Lambda_{t+1}}{\Lambda_t}N_{t+1} + \sigma\beta\frac{\Lambda_{t+1}}{\Lambda_t}V_{t+1}.$$

Therefore,

$$\begin{aligned}
\gamma_t N_t = & \mathrm{E}_t(1-\sigma)\beta\frac{\Lambda_{t+1}}{\Lambda_t}N_{t+1} + \sigma\beta\frac{\Lambda_{t+1}}{\Lambda_t}\gamma_{t+1}N_{t+1} \\
= & \mathrm{E}_t(1-\sigma+\sigma\gamma_{t+1})\beta\frac{\Lambda_{t+1}}{\Lambda_t}N_{t+1} \\
= & \mathrm{E}_t(1-\sigma+\sigma\gamma_{t+1})\beta\frac{\Lambda_{t+1}}{\Lambda_t}(R_{t+1}^l L_t - R_t D_t - R_t^m M_t) \\
= & \mathrm{E}_t(1-\sigma+\sigma\gamma_{t+1})\beta\frac{\Lambda_{t+1}}{\Lambda_t}[R_{t+1}^l(\frac{\gamma_t}{\theta}+\omega\psi)N_t - R_t(\frac{\gamma_t}{\theta}+\omega\psi-1-\psi)N_t - R_t^m\psi N_t] \\
= & \mathrm{E}_t(1-\sigma+\sigma\gamma_{t+1})\beta\frac{\Lambda_{t+1}}{\Lambda_t}[(R_{t+1}^l - R_t)(\frac{\gamma_t}{\theta}+\omega\psi) + R_t + (R_t - R_t^m)\psi]N_t.
\end{aligned}$$

Therefore,

$$\gamma_t = \mathrm{E}_t(1-\sigma+\sigma\gamma_{t+1})\beta\frac{\Lambda_{t+1}}{\Lambda_t}[(R_{t+1}^l - R_t)(\frac{\gamma_t}{\theta}+\omega\psi) + R_t + (R_t - R_t^m)\psi].$$

γ_t is then given by,

$$\gamma_t = \frac{\mathrm{E}_t(1-\sigma+\sigma\gamma_{t+1})\beta\frac{\Lambda_{t+1}}{\Lambda_t}[(R^l_{t+1}-R_t)\omega\psi + R_t + (R_t - R^m_t)\psi]}{1-\mathrm{E}_t(1-\sigma+\sigma\gamma_{t+1})\beta\frac{\Lambda_{t+1}}{\Lambda_t}(R^l_{t+1}-R_t)\frac{1}{\theta}}.$$

It is also okay if the student uses $R^m_t = \omega(E_t R^l_{t+1} - R_t) + R_t$ to substitute out R^m_t in the above equation.

(e).(10) Suppose M_t, D_t and L_t have positives values under the bank's optimal decisions. Please express R^m_t as a function of $E_t R^l_{t+1}$ and R_t based on the bank's optimal decisions. Do you expect the interest rate on discount window lending R^m_t to be higher or lower than the deposit interest rate R_t? Please explain the economic intuition behind your answer.

Answer:

$$R^m_t = \omega(E_t R^l_{t+1} - R_t) + R_t.$$

As long as $\omega > 0$, then $R^m_t > R_t$. The bank is willing to pay higher interest on discount window lending than on deposits because discount window loans cannot be fully diverted, which helps relax the bank's incentive constraint.