Asset Pricing: Mean Variance Analysis

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Portfolio Selection

- Portfolio selection is one of the most important problem in both academy and practice of financial economics.
- Markowits (1952) is the first academic paper to seriously discuss the trade-off in portfolio selection.
- The latter developed inter-temporal Capital Asset Pricing Model of Merton (1973) takes care of the multiperiod nature of investment-consumption trade-off.
- In this vein of literature, we usually refer portfolio selection problem to both of them.
- In this class, we mainly focus on the Markowits (1952).

Is Mean Variance Investors a special case?

- It seems that Markowits (1952) assume the utility function from intuition.
- Does the utility function just employs first and second order moments a reasonable application?
- Is it a strong assumption to employ the mean-variance utility function?

Is Mean Variance Investors a special case?

- The answer is NO. Actually, Mean-variance analysis is quite general.
- Let's assume a utility function $U(X)$, there is an equivalent utility function $e^{U(X)}$. It will generate same rank for same X.
- Let's apply Taylor Expansion:

$$e^{U(X)} \simeq e^{U'(X)X - \frac{1}{2}U''(X)X^2},$$

Especially for ranking order.

$$U(X) \sim U'(X)X + \frac{1}{2}U''(X)X^2 \sim X - \frac{1}{2}\frac{U''(X)}{U'(X)}X^2$$

Is Mean Variance Investors a special case?

- In your micro class, you should have learned that this is the absolute risk aversion:

$$\gamma(X) = -\frac{U''(X)}{U'(X)}$$

- Similarly, we could exponentially translate any utility function to mean-variance utility function!

Mean-Variance Analysis

- In Markowits (1952), he brillantly designs an investor to capture the psychology in investment decision of human-being. We just reframe everything in morden expression.

- Support there is one investor with initial wealth W_0, he does not consume at all at time t, but will consume everything at time $t+1$.

- There are a risk-free asset and N risky asset with returns R_f and R.

Mean-Variance Analysis

- The investor choose fraction of x_j to risky asset j for $j = 1, ..., N$. The final wealth is

$$W = W_0 x' R + W_0(1 - x' 1_N) R_f = W_0 R_f + W_0 x(R - R_f).$$

$$R_P = x' R + (1 - x' 1_N) R_f = R_f + x(R - R_f).$$

- The investor's problem is to find the optimal x to maximize $E[U(W)]$, i.e.,

$$\max_{x} E[U(R_f + x(R - R_f))]$$

Mean-Variance Analysis

- When the utility function follows a set of relative loose assumptions, we set

$$E[U(R_p)] = E(R_p) - \frac{A}{2} Var(R_p) = x'\mu - \frac{A}{2} x'\Sigma x$$

$$s.t. \quad x' 1_N = 1$$

- F.O.C.

$$\mu - A\Sigma x - \lambda I = 0$$

$$x' 1_N = 1$$

which yields

$$x^* = \frac{1}{A}\Sigma^{-1}(\mu - \lambda^* 1_N),$$

where

$$\lambda^* = \frac{\mu'\Sigma^{-1}1_N - A}{1_N'\Sigma^{-1}1_N}, \text{special case,} \quad x^* = \frac{\mu - r_f}{A\sigma^2}$$

Mean-Variance Portfolio Performance

- In most of times, mean-variance portfolio is not best, even not the relative good.
- By checking the historical data, we could observe that mean-variance portfolio could be easily beaten by equal-weight or minimum-variance portfolio.
- What's wrong with Markowits?

Parameter uncertainty

- When we run the MVA, we assume the parameter is a constant. What if the parameter is a random variable as well?

- There are two main challenges:
  1. Parameter uncertainty.
  2. Estimation risk.

- One potential reason of failure of MVA is that Markowits may be right, but how to get the right answer?

MVP: a portfolio that is easy to estimate

- Estimation risk is mainly from estimation of mean less than variance.

$$\hat{\mu} \sim N(\mu, \Sigma/T)$$

$$\hat{\sigma} \sim W_N(T-1, \Sigma)/T$$

- One could estimate variance more precisely.
- How about the weighting skeme relies less on mean but more on variance?
- In Zhou and Kan (2007), They set up a three fund separation and show that the solution could help to mitigate the estimation risk.

1/N: a portfolio without estimation

- If there is estimation risk, how about not estimate at all?
- $1/N$ is one easy way to get rid of estimation, but also diversify.
- In practice, DeMiguel, Garlappi and Uppal (2009), Tu and Zhou (2011) show that MVA could frequently be beaten by $1/N$.

Constraint: give up opportunity set for less estimation risk

- Sometimes, economic constraint could help.
- Like the estimated equity premium should be above 0.
- DeMiguel, Garlappi, Uppal (2009): More choices mean more opportunity, but could also mean more estimation risk.

Transaction cost

- Markowits assume a world without transaction cost.
- DeMiguel, Martin-Utrera, Nogales and Uppal (2020), if there is transaction cost, the cost could hinder some wrong trade.

Do we really want diversification?

- Diversification is because you don’t know what to do. –Warren Buffett.
- Many of the household, they don’t diversify their portfoilo at all.

Conclusion

- MVA is the starting point of modern finance in both academy and industry.
- Markowits opens a magic box. Somehow, it seems that most of us believe he is right.
- If the mean-variance portfolio does not make money for you, the world is wrong.