Math Notes

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1 静态（均衡）分析

1.1 对角矩阵

对角矩阵：非对角线元素均为0的对称矩阵。

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad A = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}$$

- 平方和：$x'x = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 + x_2^2$
- 加权平方和：$x'Ax = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = a_{11}x_1^2 + a_{22}x_2^2$

1.2 拉普拉斯展开

拉普拉斯展开（按行/列展开）：$|B| = \sum_{j=1}^{n} b_{i,j} |C_{ij}| = \sum_{i=1}^{n} b_{i,j} |C_{i,j}|$

- **子式**：删除行列式$|A|$的第i行第j列而得到的子行列式$|M_{ij}|$。
- **余子式**：$|C_{ij}| = (-1)^{i+j} |M_{ij}|$。

1.3 矩阵的秩

- 矩阵的**秩**：**线性无关**的**最大**行/列数。
- $r(A) \leqslant \min\{m, n\}$。
- **非奇异矩阵**$M_{n\times n}$的**秩**必为n：$r(A) = n$。
- $r(AB) \leqslant \min\{r(A), r(B)\}$。
- B为**非奇异矩阵**，则$r(A) = r(AB)$或$r(A) = r(BA)$。

1.4 克莱姆法则

按**异行余子式**进行**拉普拉斯展开**的行列式为$\mathbf{0}$：

$$\sum_{j=1}^{n} a_{i,j}\,|C_{i',j}| = \sum_{i=1}^{n} a_{i,j}\,|C_{i,j'}| = 0$$

伴随矩阵$\mathrm{adj}A$：给定**非奇异矩阵** $A_{n\times n} = [\ |a_{ij}|\ ]$，$|A| \neq 0$，以**余子式**$|C_{i,j}|$置换$A$中每一个元素$a_{ij}$而形成一个**余子式矩阵**$C_{n\times n} = [\ |C_{ij}|\ ]$，则**转置矩阵**$C'$为$A$的**伴随矩阵**，即$C'_{n\times n} = \mathrm{adj}A = [\ |C_{ji}|\ ]$。

$$\textbf{矩阵的逆}：A^{-1} = \frac{\mathrm{adj}A}{|A|}$$

方程组$A_{n\times n}x_{n\times 1} = d_{n\times 1}$（**A**为**非奇异矩阵**）的**解**：$x^* = A^{-1}d = \frac{\mathrm{adj}A}{|A|}\cdot d$

$$\textbf{克莱姆法则}：x_j^* = \frac{|A_j|}{|A|} = \frac{1}{|A|}\begin{bmatrix} a_{11} & a_{12} & \cdots & d_1 & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & d_2 & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & d_n & \cdots & a_{nn} \end{bmatrix}$$

2 比较静态分析

2.1 隐函数定理

雅可比行列式:

$$|J| \equiv \left|\frac{\partial\left(y_1, y_2, \cdots, y_n\right)}{\partial\left(x_1, x_2, \cdots, x_n\right)}\right| \equiv \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \cdots & \frac{\partial y_n}{\partial x_n} \end{vmatrix} \equiv \begin{vmatrix} f_1 & \cdots & f_n \\ \vdots & & \vdots \\ f_1^n & \cdots & f_n^n \end{vmatrix}$$

隐函数法则：给定方程$F(y, x)=0$，

$$\frac{dy}{dx}=-\frac{F_x}{F_y}$$

推广到联立方程组的情况:

$$\begin{aligned} F^1\left(y_1, \cdots, y_n ; x_1, \cdots, x_m\right)&=0, \\ F^2\left(y_1, \cdots, y_n ; x_1, \cdots, x_m\right)&=0, \\ &\cdots \\ F^n\left(y_1, \cdots, y_n ; x_1, \cdots, x_m\right)&=0. \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} y_1&=f^1\left(x_1, \cdots, x_m\right), \\ y_2&=f^2\left(x_1, \cdots, x_m\right), \\ &\cdots \\ y_n&=f^n\left(x_1, \cdots, x_m\right). \end{aligned}$$

- 对所有y变量和x变量，**隐函数**$F^1, \cdots F^n$均具有**连续偏导数**。
- 在点$(y_{10}, \cdots, y_{n0} ; x_{10}, \cdots, x_{m0})$满足上述联立方程组。
- **雅可比行列式** $|J| \equiv \left|\frac{\partial\left(F^1, \cdots, F^n\right)}{\partial\left(y_1, \cdots, y_n\right)}\right| \equiv \begin{vmatrix} \frac{\partial F^1}{\partial y_1} & \frac{\partial F^1}{\partial y_2} & \cdots & \frac{\partial F^1}{\partial y_n} \\ \frac{\partial F^2}{\partial y_1} & \frac{\partial F^2}{\partial y_2} & \cdots & \frac{\partial F^2}{\partial y_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial F^n}{\partial y_1} & \frac{\partial F^n}{\partial y_2} & \cdots & \frac{\partial F^n}{\partial y_n} \end{vmatrix} \neq 0$

$$\begin{bmatrix} \frac{\partial F^1}{\partial y_1} & \frac{\partial F^1}{\partial y_2} & \cdots & \frac{\partial F^1}{\partial y_n} \\ \frac{\partial F^2}{\partial y_1} & \frac{\partial F^2}{\partial y_2} & \cdots & \frac{\partial F^2}{\partial y_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial F^n}{\partial y_1} & \frac{\partial F^n}{\partial y_2} & \cdots & \frac{\partial F^n}{\partial y_n} \end{bmatrix} \begin{bmatrix} \left(\frac{\partial y_1}{\partial x_1}\right) \\ \left(\frac{\partial y_2}{\partial x_1}\right) \\ \vdots \\ \left(\frac{\partial y_n}{\partial x_1}\right) \end{bmatrix} = \begin{bmatrix} -\frac{\partial F^1}{\partial x_1} \\ -\frac{\partial F^2}{\partial x_1} \\ \vdots \\ -\frac{\partial F^n}{\partial x_1} \end{bmatrix} \Rightarrow \left(\frac{\partial y_j}{\partial x_1}\right)=\frac{\left|J_j\right|}{|J|}, \quad (j=1,2, \cdots, n)$$

3 最优化

3.1 麦克劳林级数与泰勒级数

n次多项式函数：$f(x)=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\cdots+a_nx^n$

麦克劳林级数：$f(x)=\dfrac{f(0)}{0!}+\dfrac{f'(0)}{1!}x+\dfrac{f''(0)}{2!}x^2+\dfrac{f''(0)}{3!}x^3+\cdots+\dfrac{f^{(n)}(0)}{n!}x^n$

泰勒级数：$f(x)=\dfrac{f(x_0)}{0!}+\dfrac{f'(x_0)}{1!}(x-x_0)+\dfrac{f''(x_0)}{2!}(x-x_0)^2+\cdots+\dfrac{f^{(n)}(x_0)}{n!}(x-x_0)^n$

任意函数展开：

$$\begin{aligned}\phi(x)=&\left[\frac{\phi(x_0)}{0!}+\frac{\phi'(x_0)}{1!}(x-x_0)+\frac{\phi''(x_0)}{2!}(x-x_0)^2+\cdots+\frac{\phi^{(n)}(x_0)}{n!}(x-x_0)^n\right]+R_n\\ \equiv&P_n+R_n\end{aligned}$$

$$R_n=\frac{\phi^{(n+1)}(p)}{(n+1)!}(x-x_0)^{n+1}$$

$$n=0\quad\Rightarrow\quad\phi(x)=P_0+R_0=\phi(x_0)+\phi'(p)(x-x_0)$$

$$\Downarrow$$

$$\phi(x)-\phi(x_0)=\phi'(p)(x-x_0)$$

$$\Downarrow$$

$$\begin{aligned}f(x)-f(x_0)=&f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2\\&+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+\frac{f^{(n+1)}(p)}{(n+1)!}(x-x_0)^{n+1}\end{aligned}$$

3.2 增长率

变量$y=f(t)$的**瞬时增长率**为

$$r_y\equiv\frac{\mathrm{d}y/\mathrm{d}t}{y}=\frac{f'(t)}{f(t)}=\frac{\text{边际函数}}{\text{总函数}}$$

变量$V=Ae^{rt}$的**瞬时增长率**为

$$r_V=\frac{V'(t)}{V(t)}=\frac{d\ln V}{dt}=\frac{d(\ln A+rt)}{dt}=r\qquad(\frac{\ln V}{dt}=\frac{V'(t)}{V(t)})$$

3.3 多变量函数极值

一阶条件是极值存在的**必要条件**，但不是**充分条件**。

- **鞍点**：某一方向上的函数**极大值点**，另一方向上是函数**极小值点**。
- **拐点**：函数**凹凸性**发生变化的点。
- **驻点（稳定点）**：函数**一阶导数为0**的点。

杨氏定理：$f_{xy} = f_{yx}$ （两个交叉偏导数是连续的）

二阶条件

- 极值的二阶**充分条件**：对于任意不同时为0的$\mathrm{d}x$和$\mathrm{d}y$，$\mathrm{d}^2 z \gtrless 0$。
- 极值的二阶**必要条件**：对于任意不同时为0的$\mathrm{d}x$和$\mathrm{d}y$，$\mathrm{d}^2 z \leq\geq 0$。

3.4 二次型

$$
\begin{aligned}
&\mathrm{d}^2 z = f_{xx}\,\mathrm{d}x^2 + 2f_{xy}\,\mathrm{d}x\,\mathrm{d}y + f_{yy}\,\mathrm{d}y^2 \\
&\qquad\Downarrow \\
&\quad q = au^2 + 2huv + bv^2 = \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} a & h \\ h & b \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}
\end{aligned}
$$

$$
\text{若 } q \text{ 恒为} \begin{cases} \text{正} & (>0) \\ \text{非负} & (\geqslant 0) \\ \text{非正} & (\leqslant 0) \\ \text{负} & (<0) \end{cases}, \quad \text{则 } q \text{ 为} \begin{cases} \text{正定} \\ \text{半正定} \\ \text{半负定} \\ \text{负定} \end{cases}
$$

海塞行列式：对称行列式（杨氏定理$f_{xy} = f_{yx}$）

$$
|H| = \begin{vmatrix} a & h \\ h & b \end{vmatrix} = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}
$$

二次型的判别：

$$\begin{cases} f_{xx} \gtrless 0 \\ |H| > 0 \end{cases} \quad \Longleftrightarrow \quad \mathrm{d}^2 z\text{为}\begin{cases} \text{正定} \\ \text{负定} \end{cases}$$

n-**变量**二次型

$$\begin{aligned} q(u_1, u_2, \cdots, u_n) &= \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij} u_i u_j \quad [\text{其中} d_{ij} = d_{ji}] \\ &= \underset{(1\times n)}{u'} \underset{(n\times n)}{D} \underset{(n\times 1)}{u} \end{aligned}$$

- **正定**的充要条件为：**主子式**均为正。

$$|D_1| \equiv d_{11} > 0, \quad |D_2| \equiv \begin{vmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{vmatrix} > 0, \quad \cdots, \quad |D_n| \equiv \begin{vmatrix} d_{11} & d_{12} & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & d_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ d_{n1} & d_{n2} & \cdots & d_{nn} \end{vmatrix} > 0$$

- **负定**的充要条件为：**主子式**交替改变符号。

$$(-1)^n |D_n| > 0: \quad |D_1| < 0, \quad |D_2| > 0, \quad |D_3| < 0, \quad \cdots$$

二次型有定符号的**特征根检验**

$$\underset{(n\times n)}{D} \underset{(n\times 1)}{x} = r \underset{(n)}{x} \quad \Longleftrightarrow \quad (D - rI)x = 0$$

$$\underset{\text{特征向量}}{x}\ \textbf{存在非零解} \Rightarrow \underset{\text{特征矩阵}}{(D - rI)}\text{为}\textbf{奇异矩阵} \Rightarrow \underset{\text{特征方程}}{|D - rI| = 0}$$

3.5 对称矩阵对角化

1. 将解出的特征根r_i带入矩阵方程$Dx = r_i x$。

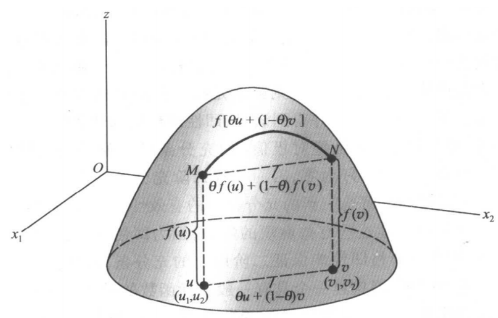
2. 施加限制$x'x = x_1^2 + x_2^2 + \cdots + x_n^2 = 1$以使解**正规化**。

3. **标准正交**特征向量v_i。（正规化：$v_i'v_i = 1$；正交/垂直：$v_i'v_j = 0$）

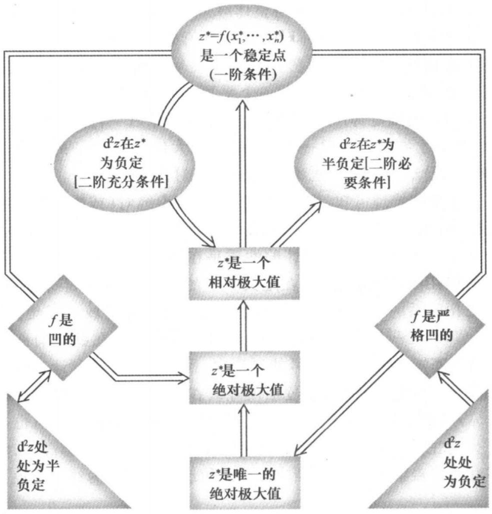
4. **对角矩阵**$R \equiv T'DT \leftarrow T_{n\times n} = [v_1, v_2, \cdots, v_n]$。

$$\underset{(n\times 1)}{u} = \underset{(n\times n)}{T} \times \underset{(n\times 1)}{y} \quad \Rightarrow \quad u'Du = (Ty)'D(Ty) = y'(T'DT)y = y'Ry$$

3.6 函数的凸性和凹性（免除检验二阶条件）



(a) 严格凸函数



(b) 二次连续可微函数的极值与凹凸性

$$\underbrace{\theta f(u)+(1-\theta)f(v)}_{\text{线段的高度}}\left\{\begin{matrix}\leqslant\\ \geqslant\end{matrix}\right\}\underbrace{f\left[\theta u+(1-\theta)v\right]}_{\text{弧的高度}}\Longleftrightarrow f(x)\text{为}\left\{\begin{matrix}\text{凹函数}\\ \text{凸函数}\end{matrix}\right.$$

- 如果函数**可微**:

$$f(v)\left\{\begin{matrix}\leqslant\\ \geqslant\end{matrix}\right\}f(u)+f'(u)(v-u)\Longleftrightarrow \text{可微函数}f(x)\text{为}\left\{\begin{matrix}\text{凹函数}\\ \text{凸函数}\end{matrix}\right.$$

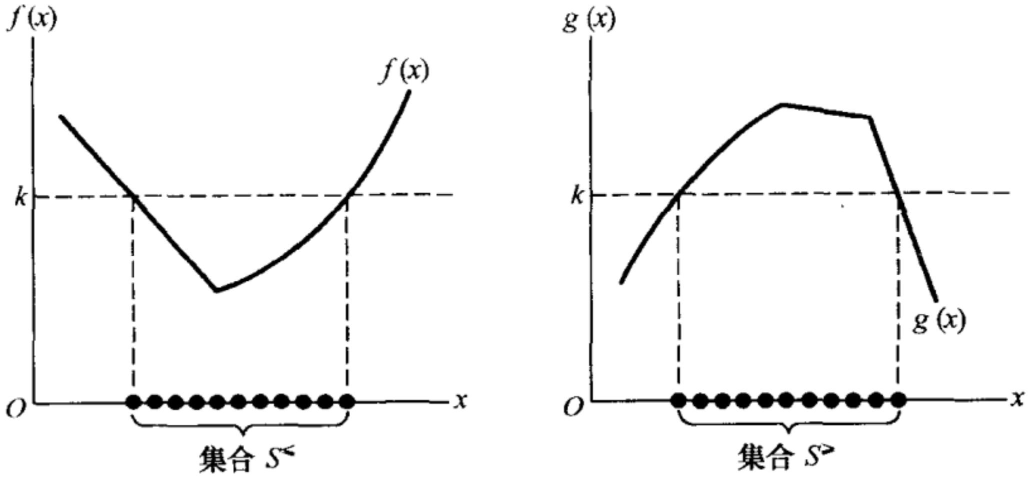
- 多个自变量:

$$f(v)\left\{\begin{matrix}\leqslant\\ \geqslant\end{matrix}\right\}f(u)+\sum_{j=1}^{n}f_j(u)\left(v_j-u_j\right)\Longleftrightarrow \text{可微函数}f(\boldsymbol{x})\text{为}\left\{\begin{matrix}\text{凹函数}\\ \text{凸函数}\end{matrix}\right.$$

- 二次连续可微函数:
  - **当且仅当** d^2z 处处为 $\left\{\begin{matrix}\text{负}\\ \text{正}\end{matrix}\right\}$ 半定，$z=f(x_1,x_2,\cdots,x_n)$ 为 $\left\{\begin{matrix}\text{凹函数}\\ \text{凸函数}\end{matrix}\right.$
  - 当（**但不是仅当**）d^2z 处处为 $\left\{\begin{matrix}\text{负}\\ \text{正}\end{matrix}\right\}$ 定，z 为**严格** $\left\{\begin{matrix}\text{凹函数}\\ \text{凸函数}\end{matrix}\right.$

3.7 凸函数与凸集

- 向量**凸组合**（两个向量的加权平均）：$\theta u+(1-\theta)v, \quad (0 \leqslant \theta \leqslant 1)$
- 任意两点$u, v \in S$，当且仅当$w=\theta u+(1-\theta)v \in S$时，$S$为**凸集**。



$$\begin{cases} S^{\leqslant} \equiv \{x \mid f(x) \leqslant k\}, & [f(x)\text{为凸函数}] \\ S^{\geqslant} \equiv \{x \mid f(x) \geqslant k\}, & [f(x)\text{为凹函数}] \end{cases}$$

3.8 具有约束方程的最优化

3.8.1 一阶条件：$\boxed{\text{拉格朗日乘数法}}$

目标函数z满足约束条件:

$$z=f\left(x_1, x_2, \cdots, x_n\right)$$
$$\text{s.t.} \quad g\left(x_1, x_2, \cdots, x_n\right)=c$$
$$\Uparrow$$
$$(\mathrm{d} g=) g_{x_1} \mathrm{d} x_1+g_{x_2} \mathrm{d} x_2+\cdots+g_{x_n} \mathrm{d} x_n=0(=\mathrm{d} c)$$
$$\text{即 } \mathrm{d} g=0$$

拉格朗日函数为:

$$Z=f\left(x_1, x_2, \cdots, x_n\right)+\lambda\left[c-g\left(x_1, x_2, \cdots, x_n\right)\right]$$

一阶条件为:

$$Z_\lambda = c - g\left(x_1, x_2, \cdots, x_n\right) = 0 \text{（满足约束条件）}$$

$$\left.\begin{array}{l} Z_1 = f_1 - \lambda g_1 = 0 \\ Z_2 = f_2 - \lambda g_2 = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ Z_n = f_n - \lambda g_n = 0 \end{array}\right\} \Rightarrow \lambda = \frac{f_1}{g_1} = \frac{f_2}{g_2} = \cdots = \frac{f_n}{g_n}$$

拉格朗日乘数λ^*的解释：度量**目标函数**Z^*对**约束条件**的**敏感性**。

$$\frac{\mathrm{d}Z^*}{\mathrm{d}c} = \lambda^*$$

3.8.2 二阶条件：$\boxed{\text{海塞加边行列式}}$

满足线性约束的两个变量的二次型:

$$q = au^2 + 2huv + bv^2$$
$$\text{s.t.} \quad \alpha u + \beta v = 0$$

将q写成仅有一个变量的函数:

$$q = au^2 - 2h\frac{\alpha}{\beta}u^2 + b\frac{\alpha^2}{\beta^2}u^2 = \left(\alpha\beta^2 - 2h\alpha\beta + b\alpha^2\right)\frac{u^2}{\beta^2}$$

当且仅当$|\bar{H}| = \begin{vmatrix} 0 & g_x & g_y \\ g_x & Z_{xx} & Z_{xy} \\ g_y & Z_{yx} & Z_{yy} \end{vmatrix} = \begin{vmatrix} 0 & \alpha & \beta \\ \alpha & a & h \\ \beta & h & b \end{vmatrix} \lesseqgtr 0$，$q(\mathrm{d}z^2)$为$\left\{\begin{array}{c} \text{正定} \\ \text{负定} \end{array}\right\}$，
且满足$\alpha u + \beta v = 0$（即$\mathrm{d}g = 0$）

二阶充分条件：给定$Z = f(x,y) + \lambda[c - g(x,y)]$，$|\bar{H}|$为**正/负**是稳定值为$z$的**极大/小值**的**充分条件**。

n个变量:

$$z = f\left(x_1, x_2, \cdots, x_n\right)$$
$$\text{s.t.} \quad g\left(x_1, x_2, \cdots, x_n\right) = c$$
$$\Uparrow$$
$$(\mathrm{d}g =) g_1\,\mathrm{d}x_1 + g_2\,\mathrm{d}x_2 + \cdots + g_n\,\mathrm{d}x_n = 0$$

海塞加边行列式:

$$|\bar{H}|=\begin{vmatrix} 0 & g_1 & g_2 & \cdots & g_n \\ g_1 & Z_{11} & Z_{12} & \cdots & Z_{1n} \\ g_2 & Z_{21} & Z_{22} & \cdots & Z_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ g_n & Z_{n1} & Z_{n2} & \cdots & Z_{nn} \end{vmatrix}$$

其**逐次加边主子式**为:

$$\left|\bar{H}_2\right| \equiv \begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & Z_{11} & Z_{12} \\ g_2 & Z_{21} & Z_{22} \end{vmatrix}, \quad \left|\bar{H}_3\right| \equiv \begin{vmatrix} 0 & g_1 & g_2 & g_3 \\ g_1 & Z_{11} & Z_{12} & Z_{13} \\ g_2 & Z_{21} & Z_{22} & Z_{23} \\ g_3 & Z_{31} & Z_{32} & Z_{33} \end{vmatrix}, \quad \cdots$$

当且仅当$\begin{cases} \left|\bar{H}_2\right|, \left|\bar{H}_3\right|, \cdots, \left|\bar{H}_n\right| < 0 \\ \left|\bar{H}_2\right| > 0, \left|\bar{H}_3\right| < 0, \left|\bar{H}_4\right| > 0, \cdots \end{cases}$，$\mathrm{d}^2 z$为$\begin{cases} \text{正定} \\ \text{负定} \end{cases}$满足$\mathrm{d}g=0$。

多重约束下

- 拉格朗日函数

$$Z=f\left(x_1, \cdots, x_n\right)+\sum_{i=1}^{m} \lambda_j\left[c_j-g^j\left(x_1, \cdots, x_n\right)\right]$$

- 海塞加边行列式

$$|\bar{H}| \equiv \begin{vmatrix} 0 & 0 & \cdots & 0 & g_1^1 & g_2^1 & \cdots & g_n^1 \\ 0 & 0 & \cdots & 0 & g_1^2 & g_2^2 & \cdots & g_n^2 \\ 0 & 0 & \cdots & 0 & g_1^m & g_2^m & \cdots & g_n^m \\ g_1^1 & g_1^2 & \cdots & g_1^m & Z_{11} & Z_{12} & \cdots & Z_{1n} \\ g_2^1 & g_2^2 & \cdots & g_2^m & Z_{21} & Z_{22} & \cdots & Z_{2n} \\ g_n^1 & g_n^2 & \cdots & g_n^m & Z_{n1} & Z_{n2} & \cdots & Z_{nn} \end{vmatrix}$$

- 二阶充分条件：$(n-m)$个加边主子式的代数符号**相同**（**交替变换**）

$$\left|\bar{H}_{m+1}\right|, \left|\bar{H}_{m+2}\right|, \cdots, \left|\bar{H}_n\right| (=|\bar{H}|)$$

3.9 拟凸性与拟凹性（免除检验二阶条件）

- 当且仅当函数 f 定义域（凸集）中的两点 u 和 v，且$0<\theta<1$，

$$f(v) \geqslant f(u) \Longrightarrow f[\theta u+(1-\theta)v]\left\{\begin{array}{l} >\geqslant f(u) \\ <\leqslant f(v) \end{array}\right., f\text{为}\left\{\begin{array}{l} \text{(严格)拟凹} \\ \text{(严格)拟凸} \end{array}\right.$$

- 对于任意常数k，当且仅当$\left\{\begin{array}{l} S^{\leqslant} \equiv \{x \mid f(x) \leqslant k\} \\ S^{\geqslant} \equiv \{x \mid f(x) \geqslant k\} \end{array}\right.$ 是凸集，$f(x)$是$\left\{\begin{array}{l} \text{是拟凸的} \\ \text{是拟凹的} \end{array}\right.$

- 对于**可微函数** $f\left(x_1, \cdots, x_n\right)$定义域中任意两点$u=\left(u_1, \cdots, u_n\right)$ 和$v=\left(v_1, \cdots, v_n\right)$，当且仅当

$$f(v) \geqslant f(u) \Rightarrow \left.\begin{array}{l} \sum_{j=1}^n f_j(u)\left(v_j-u_j\right) \\ \sum_{j=1}^n f_j(v)\left(v_j-u_j\right) \end{array}\right\} \geqslant 0, f(x)\text{为}\left\{\begin{array}{l} \text{拟凹函数} \\ \text{拟凸函数} \end{array}\right.$$

若$z=f\left(x_1, \cdots, x_n\right)$ 为**二阶连续可微函数**，拟凹性和拟凸性可以用函数的一阶导数和二阶导数（整理成加边行列式）的方法检验：

$$|B|=\left|\begin{array}{ccccc} 0 & f_1 & f_2 & \cdots & f_n \\ f_1 & f_{11} & f_{12} & \cdots & f_{1n} \\ f_2 & f_{21} & f_{22} & \cdots & f_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ f_n & f_{n1} & f_{n2} & \cdots & f_{nn} \end{array}\right|$$

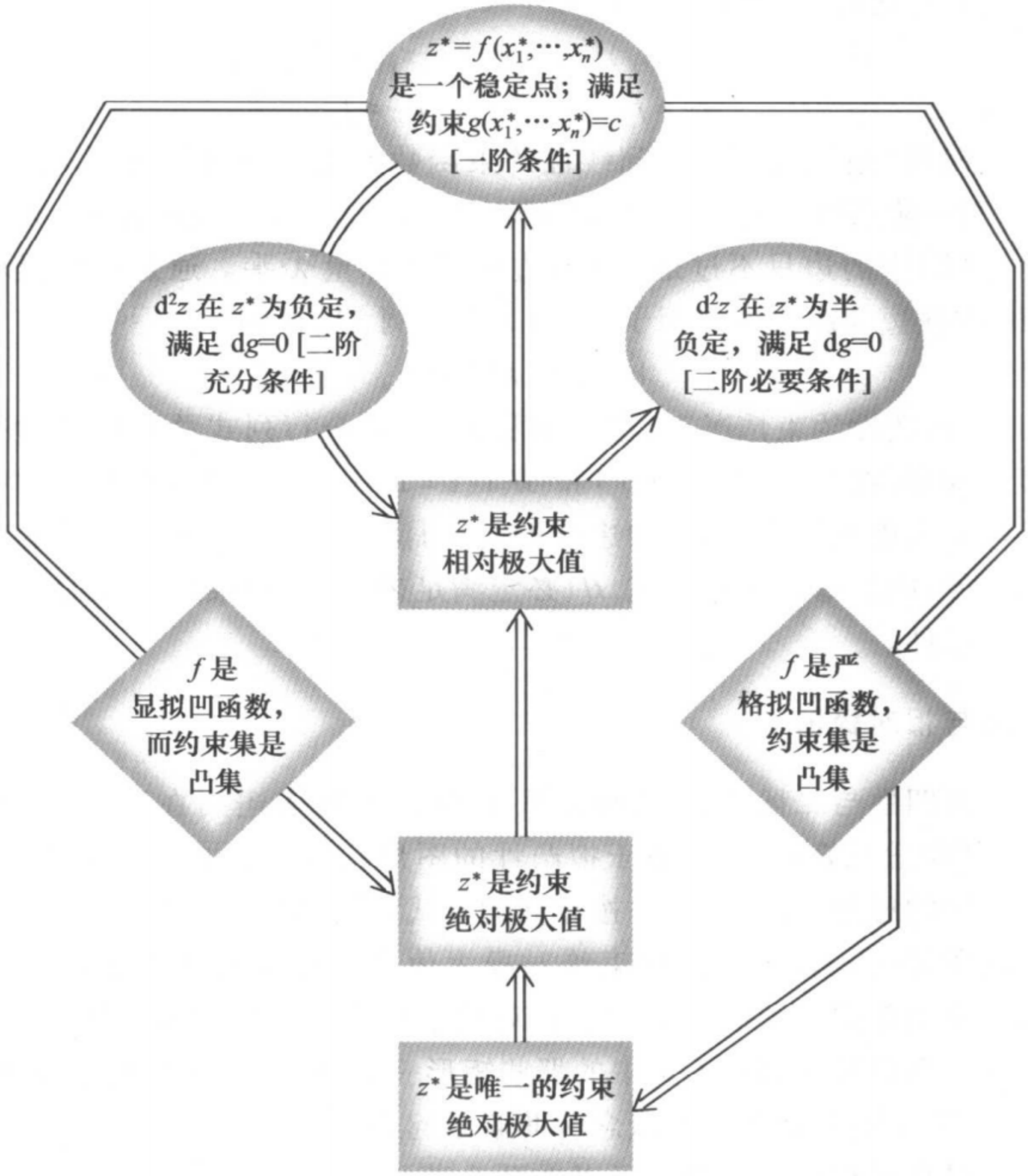
拟凹函数的必要条件：

$$\left|B_1\right|=\left|\begin{array}{cc} 0 & f_1 \\ f_1 & f_{11} \end{array}\right| \leqslant 0,\left|B_2\right|=\left|\begin{array}{ccc} 0 & f_1 & f_2 \\ f_1 & f_{11} & f_{12} \\ f_2 & f_{21} & f_{22} \end{array}\right| \geqslant 0, \quad \cdots, \quad \left|B_n\right|=|B|\left\{\begin{array}{l} \leqslant 0 \\ \geqslant 0 \end{array}\right.$$

$$Z_j=f_j-\lambda a_j=0 \quad \Rightarrow \quad f_j=\lambda a_j=\lambda g_j \quad \Rightarrow \quad \bigstar \quad \boxed{|B|=\lambda^2|\bar{H}|}$$

- **显拟凹函数**

当且仅当 $\underbrace{f(v)>f(u)}_{\text{区别(严格)拟凸拟凹}} \Longrightarrow f[\theta u+(1-\theta)v]>f(u)$，$f$为**显拟凹函数**


$z^* = f(x_1^*, \cdots, x_n^*)$
是一个稳定点；满足
约束$g(x_1^*, \cdots, x_n^*) = c$
[一阶条件]
d^2z 在 z^* 为负定，
满足 dg=0 [二阶
充分条件]
d^2z 在 z^* 为半
负定，满足 dg=0
[二阶必要条件]
z^* 是约束
相对极大值
f 是
显拟凹函数，
而约束集是
凸集
f 是严
格拟凹函数，
约束集是
凸集
z^* 是约束
绝对极大值
z^* 是唯一的约束
绝对极大值

3.10 齐次函数 $f\left(jx_1,\cdots,jx_n\right)=j^r f\left(x_1,\cdots,x_n\right)$

生产函数为线性齐次: $Q=f(L,K) \quad [=L\phi(k)]$

- $\mathrm{APP}_L \equiv \frac{Q}{L}=f(\frac{K}{L},1)=f(k,1)=\phi(k), \quad \mathrm{APP}_K \equiv \frac{Q}{K}=\frac{Q}{L}\frac{L}{K}=\frac{\phi(k)}{k}$
- $\frac{\partial k}{\partial K}=\frac{\partial}{\partial K}\left(\frac{K}{L}\right)=\frac{1}{L}, \quad \frac{\partial k}{\partial L}=\frac{\partial}{\partial L}\left(\frac{K}{L}\right)=\frac{-K}{L^2}$

$$\begin{aligned}\mathrm{MPP}_K \equiv \frac{\partial Q}{\partial K}=\frac{\partial}{\partial K}[L\phi(k)]&=L\frac{\partial\phi(k)}{\partial K}\\ &=L\frac{\mathrm{d}\phi(k)}{\mathrm{d}k}\frac{\partial k}{\partial K}=L\phi'(k)\left(\frac{1}{L}\right)=\phi'(k)\end{aligned}$$

$$\begin{aligned}\mathrm{MPP}_L \equiv \frac{\partial Q}{\partial L}=&\frac{\partial}{\partial L}[L\phi(k)]\\ &=\phi(k)+L\frac{\partial\phi(k)}{\partial L}=\phi(k)+L\phi'(k)\frac{\partial k}{\partial L}\\ &=\phi(k)+L\phi'(k)\frac{-K}{L^2}=\phi(k)-k\phi'(k)\end{aligned}$$

- **欧拉定理**

$$K\frac{\partial Q}{\partial K}+L\frac{\partial Q}{\partial L}\equiv Q$$

★ 推广：包含n个变量的**线性齐次函数** $y=f\left(x_1,x_2,\cdots,x_n\right)$

$$y=x_1\phi\left(\frac{x_2}{x_1},\frac{x_3}{x_1},\cdots,\frac{x_n}{x_1}\right) \quad [\text{一次齐次性}]$$

$$\sum_{i=1}^{n}x_i f_i \equiv y \quad [\text{欧拉定理}]$$

$$y=x_1^r\phi\left(\frac{x_2}{x_1},\frac{x_3}{x_1},\cdots,\frac{x_n}{x_1}\right) \quad [r\text{次齐次性}]$$

$$\sum_{i=1}^{n}x_i f_i \equiv ry \quad [\text{修正的欧拉定理}]$$

3.11 非线形规划

3.11.1 一阶条件：库恩塔克条件

4 动态分析