Open Economy Macro: Problem Set 2 Solution 2023 Fall

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Question 1. Consider a small open economy inhabited by identical consumers with preferences described by the utility function

$$\sum_{t=0}^{\infty} \beta^t \left[\ln c_t - \gamma h_t\right]$$

where c_t denotes consumption, h_t denotes hours worked, and $\beta \in (0,1)$ and $\gamma > 0$ are parameters. The consumption good is a composite made of tradable and nontradable goods via a Leontief aggregator. Formally,

$$c_t = \min\{c_t^{\mathrm{T}}, c_t^{\mathrm{N}}\}$$

where c_t^{T} and c_t^{N} denote, respectively, domestic absorption of tradables and nontradables in period t. To produce his nontraded consumption, each consumer operates a linear technology that uses labor as the sole input:

$$c_t^{\mathrm{N}} = Ah_t$$

where $A > 0$ is a parameter. In addition, households can borrow or lend in the international financial market at the rate $r > 0$. Their sequential budget constraint is given by

$$c_t^{\mathrm{T}} + (1+r)d_{t-1} = y^{\mathrm{T}} + d_t$$

where d_t denotes the level of net external debt assumed in period t and maturing in period $t+1$, and $y^{\mathrm{T}} > 0$ denotes a constant endowment of tradable goods. In period 0, households start with outstanding debt equal to $d_{-1} > 0$. Finally, households are subject to a no-Ponzi game constraint of the form

$$\lim_{t\to\infty} \frac{d_t}{(1+r)^t} \leq 0$$

1. Characterize the equilibrium levels of consumption, consumption of nontradables, and hours worked.

   Solution: Given the Leontief aggregator, the optimal choice of consumption should deliver

$$c_t = c_t^{\mathrm{T}} = c_t^{\mathrm{N}}$$

Therefore, the maximization problem is given by

$$
\begin{aligned}
\max_{\{d_t,h_t\}} \quad & \sum_{t=0}^{\infty} \beta^t \left[\ln Ah_t - \gamma h_t\right] \\
\text{s.t.} \quad & Ah_t + (1+r)d_{t-1} = y^{\mathrm{T}} + d_t \\
& \lim_{t\to\infty} \frac{d_t}{(1+r)^t} \leq 0
\end{aligned}
$$

Note that we replace c_t and c_t^{T} with $c_t^{\mathrm{N}} = Ah_t$. Denote λ_t as the Lagrangian multiplier of the budget constraint. The first order conditions with respect to d_t and h_t are respectively

$$
\begin{aligned}
[d_t] \quad & \lambda_t - \beta\lambda_{t+1}(1+r) = 0 \\
[h_t] \quad & \frac{1}{Ah_t}A - \gamma - \lambda_t A = 0
\end{aligned}
$$

Combining these two conditions to eliminate λ_t, we have

$$
\frac{1}{h_t} - \gamma = \beta(1+r)\left[\frac{1}{h_{t+1}} - \gamma\right]
$$

which further implies that

$$
h_t = \frac{1}{\gamma}
$$

and

$$
c_t = c_t^{\mathrm{T}} = c_t^{\mathrm{N}} = \frac{A}{\gamma}
$$

Note that from the sequential budget constraint, we can solve for d_t

$$
\begin{aligned}
d_t &= \left[(1+r)^t - 1\right]\frac{\frac{A}{\gamma} - y^{\mathrm{T}}}{r} + (1+r)^t d_{-1} \\
\Rightarrow \quad \frac{d_t}{(1+r)^t} &= d_{-1} + \left[1 - \frac{1}{(1+r)^t}\right]\frac{\frac{A}{\gamma} - y^{\mathrm{T}}}{r}
\end{aligned}
$$

and no-Ponzi game constraint implies that

$$
d_{-1} \leq \frac{\frac{A}{\gamma} - y^{\mathrm{T}}}{r}
$$

Since $d_{-1} > 0$, we need to assume that

$$
y^{\mathrm{T}} > \frac{A}{\gamma}
$$

$\square$

2. Suppose that in period 0, foreign lenders unexpectedly decide to forgive an amount $\Delta^d > 0$ of the debt. Assuming that Δ^d is relatively small, characterize the effect of this debt forgiveness shock on consumption, consumption of nontradables, and hours worked.

   Solution: Given a small $\Delta^d > 0$, all the allocations are not affected while only the level of debt will decrease. □

3. Now suppose $\Delta^d = 0$. Instead, assume that in period 0, the nontraded sector experience a permanent increase in productivity. Specifically, the productivity factor A increases by $\Delta^A > 0$. Characterize the effect of this positive productivity shock on consumption, consumption of nontradables, and hours worked.

   Solution: Suppose ΔA only increases by a relatively small amount such that $y^{\mathrm{T}} > \frac{A+\Delta^A}{\gamma}$ hold. Then we still have

$$h_t = \frac{1}{\gamma}$$

   and the consumption turn to

$$c_t = c_t^{\mathrm{T}} = c_t^{\mathrm{N}} = \frac{A + \Delta^A}{\gamma}$$

□

Question 2. We generally assume that the consumption aggregator function $A(c^{\mathrm{T}}, c^{\mathrm{N}})$ is increasing, concave, and linearly homogeneous. Thus, the household's demand for nontradables can be given by

$$p = \frac{A_2(c^{\mathrm{T}}, c^{\mathrm{N}})}{A_1(c^{\mathrm{T}}, c^{\mathrm{N}})}$$

where p is the relative price of nontradables in terms of tradables.

1. Show that the assumptions are sufficient to ensure that the demand schedule of nontradables is downward sloping in the space (c^{N}, p), holding c^{T} constant.

   Solution: Suppose the function is twice differentiable. We need to show that $\frac{\partial p}{\partial c^{\mathrm{N}}} \leq 0$. Given the demand for nontradables, we have

   $$\frac{\partial p}{\partial c^{\mathrm{N}}} = \frac{A_{22}(c^{\mathrm{T}}, c^{\mathrm{N}})A_1(c^{\mathrm{T}}, c^{\mathrm{N}}) - A_2(c^{\mathrm{T}}, c^{\mathrm{N}})A_{12}(c^{\mathrm{T}}, c^{\mathrm{N}})}{A_1(c^{\mathrm{T}}, c^{\mathrm{N}})^2}$$

   Since function A is increasing and concave, we have

   $$\begin{aligned} A_1(c^{\mathrm{T}}, c^{\mathrm{N}}) &\geq 0 \\ A_2(c^{\mathrm{T}}, c^{\mathrm{N}}) &\geq 0 \\ A_{11}(c^{\mathrm{T}}, c^{\mathrm{N}}) &\leq 0 \\ A_{22}(c^{\mathrm{T}}, c^{\mathrm{N}}) &\leq 0 \end{aligned}$$

   In addition, function A is linearly homogeneous which implies that $A_1(c^{\mathrm{T}}, c^{\mathrm{N}})$ is homogeneous of degree zero, i.e.

   $$A_1(\alpha c^{\mathrm{T}}, \alpha c^{\mathrm{N}}) = A_1(c^{\mathrm{T}}, c^{\mathrm{N}})$$

   Taking derivative with respect to α on both sides, we have

   $$A_{11}(\alpha c^{\mathrm{T}}, \alpha c^{\mathrm{N}})c^{\mathrm{T}} + A_{12}(\alpha c^{\mathrm{T}}, \alpha c^{\mathrm{N}})c^{\mathrm{N}} = 0$$

   Since $A_{11}(c^{\mathrm{T}}, c^{\mathrm{N}}) \leq 0$, we must have $A_{12}(c^{\mathrm{T}}, c^{\mathrm{N}}) \geq 0$. In sum, we must have

   $$\frac{\partial p}{\partial c^{\mathrm{N}}} \leq 0$$

   □

2. Show that the aforementioned assumptions about the aggregator $A(c^{\mathrm{T}}, c^{\mathrm{N}})$ are sufficient to guarantee that increases (decreases) in c^{T} shift the demand schedule up and to the right (down and to the left).

   Solution: We need to prove $\frac{\partial p}{\partial c^{\mathrm{T}}} \geq 0$. The expression is

   $$\frac{\partial p}{\partial c^{\mathrm{T}}} = \frac{A_{21}(c^{\mathrm{T}}, c^{\mathrm{N}})A_1(c^{\mathrm{T}}, c^{\mathrm{N}}) - A_2(c^{\mathrm{T}}, c^{\mathrm{N}})A_{11}(c^{\mathrm{T}}, c^{\mathrm{N}})}{A_1(c^{\mathrm{T}}, c^{\mathrm{N}})^2}$$

Following the same steps as the last question, we can easily show that

$$
\begin{aligned}
A_1(c^{\mathrm{T}}, c^{\mathrm{N}}) &\geq 0 \\
A_2(c^{\mathrm{T}}, c^{\mathrm{N}}) &\geq 0 \\
A_{11}(c^{\mathrm{T}}, c^{\mathrm{N}}) &\leq 0 \\
A_{22}(c^{\mathrm{T}}, c^{\mathrm{N}}) &\leq 0 \\
A_{21}(c^{\mathrm{T}}, c^{\mathrm{N}}) &\geq 0
\end{aligned}
$$

which implies that

$$\frac{\partial p}{\partial c^{\mathrm{T}}} \geq 0$$

□

3. Assume that the aggregator function takes the Cobb-Douglas form

$$A(c^{\mathrm{T}}, c^{\mathrm{N}}) = \sqrt{c^{\mathrm{T}} c^{\mathrm{N}}}$$

Find the demand function of nontradables.

Solution: The demand function is

$$p = \frac{c^{\mathrm{T}}}{c^{\mathrm{N}}}$$

□

4. Now assume the CES form

$$A(c^{\mathrm{T}}, c^{\mathrm{N}}) = \left[a(c^{\mathrm{T}})^{1-\frac{1}{\xi}} + (1-a)(c^{\mathrm{N}})^{1-\frac{1}{\xi}}\right]^{\frac{1}{1-\frac{1}{\xi}}}$$

Derive the demand function of nontradables. Interpret the parameter ξ.

Solution: The demand function is

$$p = \frac{1-a}{a}\left[\frac{c^{\mathrm{T}}}{c^{\mathrm{N}}}\right]^{\frac{1}{\xi}}$$

The parameter ξ is the elasticity of substitution between c^{N} and c^{T}. □

Question 3. Consider an open economy that lasts for only two periods 1 and 2. Households are endowed with 10 units of tradables in period 1 and 13.2 unites in period 2, i.e. $y_1^{\rm T} = 10$ and $y_2^{\rm T} = 13.2$. The country interest rate is 10 percent, or $r = 0.1$. The nominal exchange rate is fixed and equal to 1 in both periods ($\varepsilon_1 = \varepsilon_2 = 1$). Suppose that the foreign currency price of tradable goods is constant and equal to one in both periods, and that the law of one price holds for tradable goods in both periods. Nominal wages are downwardly rigid. Specifically, assume that the nominal wage W, measured in terms of domestic currency, is subject to the constraint

$$W_t \geq W_{t-1}$$

for $t = 1, 2$, with $W_0 = 8.25$. Suppose the economy starts period 1 with no assets or debts carried over from the past ($d_1 = 0$). Households are subject to the no-Ponzi game constraint $d_3 \leq 0$.

Suppose that the household's preferences are defined over tradables and nontradables in period 1 and 2, described by the following utility function:

$$\ln C_1^{\rm T} + \ln C_1^{\rm N} + \ln C_2^{\rm T} + \ln C_2^{\rm N}$$

where $C_i^{\rm T}$ and $C_i^{\rm N}$ denote consumption of tradables and nontradables in period $i = 1, 2$, respectively. Let p_1 and p_2 denote the relative price of nontradables in terms of tradables in period 1 and 2. Households supply inelastically $\bar{h} = 1$ unit of labor to the market each period. Finally, firms produce nontradable goods using labor as the sole input. The production technology is given by

$$y_t^{\rm N} = h_t^{\alpha}$$

for $t = 1, 2$, where $y_t^{\rm N}$ and h_t denote, respectively, nontradable output and hours employed in period $t = 1, 2$. The parameter α is equal to 0.75.

1. Compute the equilibrium levels of consumption of tradables, employment, nontradable output, the relative price of nontradables, and the trade balance in period 1 and 2.

   Solution: The household's maximization problem is given by

$$\max_{\{C_1^{\rm T}, C_1^{\rm N}, d_2, C_2^{\rm T}, C_2^{\rm N}\}} \quad \ln C_1^{\rm T} + \ln C_1^{\rm N} + \ln C_2^{\rm T} + \ln C_2^{\rm N}$$

$$\text{s.t.} \quad C_1^{\rm T} + p_1 C_1^{\rm N} = W_1 h_1 + y_1^{\rm T} + d_2 + \Phi_1 \tag{1}$$

$$C_2^{\rm T} + p_2 C_2^{\rm N} + (1+r) d_2 = W_2 h_2 + y_2^{\rm T} + \Phi_2 \tag{2}$$

   where Φ_1 and Φ_2 are firms' profits. Note that we use $\varepsilon_1 = \varepsilon_2 = 1$ to derive the two budget constraints and it is obvious that $d_3 = 0$ in equilibrium. The first-order conditions are given by

$$
\begin{aligned}
[C_1^{\mathrm{T}}] \quad & \frac{1}{C_1^{\mathrm{T}}} - \lambda_1 = 0 \\
[C_1^{\mathrm{N}}] \quad & \frac{1}{C_1^{\mathrm{N}}} - \lambda_1 p_1 = 0 \\
[d_2] \quad & \lambda_1 - \lambda_2(1+r) = 0 \\
[C_2^{\mathrm{T}}] \quad & \frac{1}{C_2^{\mathrm{T}}} - \lambda_2 = 0 \\
[C_2^{\mathrm{N}}] \quad & \frac{1}{C_2^{\mathrm{N}}} - \lambda_2 p_2 = 0
\end{aligned}
$$

where λ_1 and λ_2 are the Lagrangian multiplier of the two budget constraints, respectively. From the conditions, we can get the following equations:

$$\frac{C_1^{\mathrm{T}}}{C_1^{\mathrm{N}}} = p_1 \tag{3}$$

$$\frac{C_2^{\mathrm{T}}}{C_2^{\mathrm{N}}} = p_2 \tag{4}$$

$$C_2^{\mathrm{T}} = (1+r)C_1^{\mathrm{T}} \tag{5}$$

Nontradable goods producers' maximization problem in period $t = 1, 2$ is

$$\max_{\{h_t\}} \quad \Phi_t = p_t h_t^{\alpha} - W_t h_t$$

and the first-order conditions are

$$p_t \alpha h_t^{\alpha-1} = W_t \tag{6}$$

Nominal wage downward rigidity delivers the following conditions

$$
\begin{aligned}
& (W_t - W_{t-1})(\bar{h} - h_t) = 0 \\
& W_t \geq W_{t-1} \\
& h_t \leq \bar{h}
\end{aligned}
\tag{7}
$$

where $t = 1, 2$. In addition, nontradable goods market clears in period $t = 1, 2$

$$C_t^{\mathrm{N}} = y_t^{\mathrm{N}} = h_t^{\alpha} \tag{8}$$

Finally, combining the two budget constraints to cancel out d_2, we get the intertemporal budget constraint

$$C_1^{\mathrm{T}} + \frac{C_2^{\mathrm{T}}}{1+r} = y_1^{\mathrm{T}} + \frac{y_2^{\mathrm{T}}}{1+r} \tag{9}$$

The equilibrium consists of $\{C_t^{\mathrm{T}}, C_t^{\mathrm{N}}, h_t, p_t, W_t\}_{t=1,2}$ that solve (3) - (9).

Define $Y^{\mathrm{T}} = y_1^{\mathrm{T}} + \frac{y_2^{\mathrm{T}}}{1+r}$ which represents the present value of the endowment stream. From (5) and (9) we have

$$C_1^{\mathrm{T}} = \frac{Y^{\mathrm{T}}}{2} = 11$$

and therefore

$$C_2^{\mathrm{T}} = (1+r)C_1^{\mathrm{T}} = 12.1$$

From (3), (4), (6) and (8) we have

$$\begin{aligned} C_1^{\mathrm{T}} &= \frac{W_1 h_1}{\alpha} \\ C_2^{\mathrm{T}} &= \frac{W_2 h_2}{\alpha} \end{aligned}$$

Suppose the economy achieves full employment in both periods, i.e. $h_1 = h_2 = \bar{h} = 1$, then from the last two equations we have

$$\begin{aligned} W_1 &= \alpha C_1^{\mathrm{T}} = 0.75 \times 11 = 8.25 \\ W_2 &= \alpha C_2^{\mathrm{T}} = 0.75 \times 12.1 = 9.075 \end{aligned}$$

It is trivial to check that the downward nominal rigidity constraint is satisfied in both periods. So this is the solution. The rest of the equilibrium levels are

$$\begin{aligned} C_1^{\mathrm{N}} &= C_2^{\mathrm{N}} = 1 \\ p_1 &= \frac{C_1^{\mathrm{T}}}{C_1^{\mathrm{N}}} = 11 \\ p_2 &= \frac{C_2^{\mathrm{T}}}{C_2^{\mathrm{N}}} = 12.1 \\ tb_1 &= y_1^{\mathrm{T}} - C_1^{\mathrm{T}} = 10 - 11 = -1 \\ tb_2 &= y_2^{\mathrm{T}} - C_2^{\mathrm{T}} = 13.2 - 12.1 = 1.1 \end{aligned}$$

where tb_t represents the trade balance in period t.

□

2. Suppose now that the country interest rate increases to 32 percent. Calculate the equilibrium levels of consumption of tradables, the trade balance, consumption of nontradables, the level of unemployment, and the relative price of nontradables in period 1 and 2. Provide intuition.

   Solution: Given the increase in the country interest rate, the present value of endowment decreases to $Y^{\mathrm{T}} = 20$ and the consumption of tradables turn into

$$\begin{aligned} C_1^{\mathrm{T}} &= \frac{Y^{\mathrm{T}}}{2} = 10 \\ C_2^{\mathrm{T}} &= (1+r)C_1^{\mathrm{T}} = 1.32 \times 10 = 13.2 \end{aligned}$$

Compared with the previous part, the increase in borrowing cost makes the relative price of tradables goes up and the real wage goes up as well in period 1, while in period 2 the opposite happens. As a result, the economy cannot reach full employment in period 1, which implies

$$\begin{aligned} W_1 &= W_0 = 8.25 \\ h_1 &= \frac{\alpha C_1^{\mathrm{T}}}{W_1} = \frac{0.75 \times 10}{8.25} = \frac{10}{11} \approx 0.91 < 1 \end{aligned}$$

However, it still has full emplyment in period 2

$$\begin{aligned} h_2 &= 1 \\ W_2 &= \alpha C_2^{\mathrm{T}} = 0.75 \times 13.2 = 9.9 > W_1 = 8.25 \end{aligned}$$

The other equilibrium levels are

$$\begin{aligned} C_1^{\mathrm{N}} &= 0.91^{0.95} = 0.93 \\ C_2^{\mathrm{N}} &= 1 \\ p_1 &= \frac{C_1^{\mathrm{T}}}{C_1^{\mathrm{N}}} = \frac{10}{0.93} = 10.7 \\ p_2 &= \frac{C_2^{\mathrm{T}}}{C_2^{\mathrm{N}}} = 13.2 \\ tb_1 &= y_1^{\mathrm{T}} - C_1^{\mathrm{T}} = 10 - 10 = 0 \\ tb_2 &= y_2^{\mathrm{T}} - C_2^{\mathrm{T}} = 13.2 - 13.2 = 0 \end{aligned}$$

Note that at this level of country interest rate, $d_2 = 0$ so there is no borrowing. □

3. Given the situation in the previous question, calculate the minimum devaluation rates in period 1 and 2 consistent with full employment in both periods. To answer this question, assume that the nominal exchange rate in period 0 was also fixed at unity. Explain.

   Solution: When the nominal exchange rate can vary, i.e. $\varepsilon_1 = \varepsilon_2 = 1$ do not hold, the budget constraints turn into

$$\begin{aligned} C_1^{\mathrm{T}} + p_1 C_1^{\mathrm{N}} &= \frac{W_1}{\varepsilon_1} h_1 + y_1^{\mathrm{T}} + d_2 + \Phi_1 \\ C_2^{\mathrm{T}} + p_2 C_2^{\mathrm{N}} + (1+r) d_2 &= \frac{W_2}{\varepsilon_2} h_2 + y_2^{\mathrm{T}} + \Phi_2 \end{aligned}$$

   Note that households' first-order conditions do not alter given the new budget constraints. In addition, the intertemporal budget constraint (9) holds as well. Therefore, we still have

$$\begin{aligned} C_1^{\mathrm{T}} &= \frac{Y^{\mathrm{T}}}{2} = 10 \\ C_2^{\mathrm{T}} &= (1+r) C_1^{\mathrm{T}} = 1.32 \times 10 = 13.2 \end{aligned}$$

However, the firms' optimization problem delivers new first-order conditions

$$p_1 \alpha h_1^{\alpha-1} = \frac{W_1}{\varepsilon_1}$$
$$p_2 \alpha h_2^{\alpha-1} = \frac{W_2}{\varepsilon_2}$$

which implies that

$$C_1^{\mathrm{T}} = p_1 C_1^{\mathrm{N}} = \frac{W_1 h_1}{\alpha \varepsilon_1}$$
$$C_2^{\mathrm{T}} = p_2 C_2^{\mathrm{N}} = \frac{W_2 h_2}{\alpha \varepsilon_2}$$

To achieve full employment in both periods, we can set

$$\varepsilon_1 = \frac{11}{10}$$
$$\varepsilon_2 = \frac{5}{6}$$

which requires a 10% devaluation in period 1 as $\varepsilon_0 = 1$, and a 24% appreciate in period 2. Given the equilibrium conditions, it is easy to check that given these values of exchange rate, we have

$$\begin{aligned}
&h_1 = h_2 = 1 \\
&C_1^{\mathrm{N}} = C_2^{\mathrm{N}} = 1 \\
&C_1^{\mathrm{T}} = 10 \\
&C_2^{\mathrm{T}} = 13.2 \\
&p_1 = 10 \\
&p_2 = 13.2 \\
&W_1 = 8.25 = W_0 \\
&W_2 = 8.25 = W_1 \\
&tb_1 = tb_2 = 0 \\
&d_2 = 0
\end{aligned}$$

Intuitively, depreciation in period 1 can lower real wage and facilitate employment to maximum.

□

4. Continue to assume that $W_0 = 8.25$ and r is 32 percent. Assume also that the government is not willing to devalue the domestic currency, so that $\varepsilon_1 = \varepsilon_2 = 1$. Instead, the government chooses to apply capital controls in period 1. Specifically, the government imposes a proportional tax τ_1 on borrowed funds. If $\tau_1 < 0$, it serves as a subsidy. Suppose that this tax (subsidy) is rebated (financed) in a lump-sum fashion. Calculate the Ramsey optimal level of τ_1.

Solution: Given the capital control tax, the budget constraint in period 1 turns to

$$C_1^{\mathrm{T}} + p_1 C_1^{\mathrm{N}} = W_1 h_1 + y_1^{\mathrm{T}} + (1-\tau_1) d_2 + \Phi_1 + T_1$$

where $T_1 = \tau_1 d_2$. Note that the intertemporal budget constraint is still (9). The first-order condition associated with d_2 will change to

$$C_2^{\mathrm{T}} = \frac{1+r}{1-\tau_1} C_1^{\mathrm{T}}$$

while the other equilibrium condition are the same. The Ramsey problem is therefore given by

$$\begin{aligned}
\max_{\left\{\tau_1, \{C_t^{\mathrm{T}}, C_t^{\mathrm{N}}, h_t, p_t, W_t\}_{t=1,2}\right\}} \quad & \ln C_1^{\mathrm{T}} + \ln C_1^{\mathrm{N}} + \ln C_2^{\mathrm{T}} + \ln C_2^{\mathrm{N}} \\
\text{s.t.} \quad & C_1^{\mathrm{T}} = p_1 C_1^{\mathrm{N}} \\
& C_2^{\mathrm{T}} = p_2 C_2^{\mathrm{N}} \\
& C_2^{\mathrm{T}} = \frac{1+r}{1-\tau_1} C_1^{\mathrm{T}} \\
& p_1 \alpha h_1^{\alpha-1} = W_1 \\
& p_2 \alpha h_2^{\alpha-1} = W_2 \\
& C_1^{\mathrm{N}} = h_1^{\alpha} \\
& C_2^{\mathrm{N}} = h_2^{\alpha} \\
& C_1^{\mathrm{T}} + \frac{C_2^{\mathrm{T}}}{1+r} = y_1^{\mathrm{T}} + \frac{y_2^{\mathrm{T}}}{1+r} \\
& (W_2 - W_1)(\bar{h} - h_2) = 0,\ W_2 \geq W_1,\ h_2 \leq \bar{h} \\
& (W_1 - W_0)(\bar{h} - h_1) = 0,\ W_1 \geq W_0,\ h_1 \leq \bar{h}
\end{aligned}$$

This problem can be reduced to

$$\begin{aligned}
\max_{\{\tau_1, \{h_t, W_t\}_{t=1,2}\}} \quad & \ln W_1 h_1 + \alpha \ln h_1 + \ln W_2 h_2 + \alpha \ln h_2 - 2 \ln \alpha \\
\text{s.t.} \quad & W_2 h_2 = \frac{1+r}{1-\tau_1} W_1 h_1 \\
& W_1 h_1 + \frac{W_2 h_2}{1+r} = \alpha Y^{\mathrm{T}} \\
& (W_2 - W_1)(\bar{h} - h_2) = 0,\ W_2 \geq W_1,\ h_2 \leq \bar{h} \\
& (W_1 - W_0)(\bar{h} - h_1) = 0,\ W_1 \geq W_0,\ h_1 \leq \bar{h}
\end{aligned}$$

Next, we solve for the capital control tax τ_1 that can support full employment. Suppose $h_1 = h_2 = \bar{h} = 1$. Plugging $h_1 = h_2 = \bar{h} = 1$ and replacing W_1 and W_2 with τ_1, the Ramsey problem is equivalent to

$$\begin{aligned}
\max_{\{\tau_1\}} \quad & U = \ln(1-\tau_1) - 2\ln(2-\tau_1) + 2\ln Y^{\mathrm{T}} + \ln(1+r) \\
\text{s.t.} \quad & -r \leq \tau_1 \leq 1 - \frac{1}{\frac{\alpha Y^{\mathrm{T}}}{W_0} - 1}
\end{aligned}$$

Note that the constraints for τ_1 are derive from the nominal wage rigidity constraint, namely $W_2 \geq W_1$ and $W_1 \geq W_0$. Given the values of the parameters, the constraints for τ_1 are indeed

$$-0.32 \leq \tau_1 \leq -\frac{2}{9}$$

Taking first-order derivative of the objective function with respect to τ_1, we have

$$\frac{\partial U}{\partial \tau_1} = \frac{-\tau_1}{(1-\tau_1)(2-\tau_1)} > 0$$

given the range of τ_1. Therefore, the optimal is achieved when $\tau_1 = -\frac{2}{9} \approx -0.22$ which is a 22% subsidy. Given this level of capital control subsidy, it is easy to check the equilibrium is

$$\begin{aligned}
&h_1 = h_2 = 1 \\
&C_1^{\mathrm{N}} = C_2^{\mathrm{N}} = 1 \\
&C_1^{\mathrm{T}} = 11 \\
&C_2^{\mathrm{T}} = 11.88 \\
&p_1 = 11 \\
&p_2 = 11.88 \\
&W_1 = 8.25 = W_0 \\
&W_2 = 8.91 > W_1 \\
&tb_1 = 10 - 11 = -1 \\
&tb_2 = 13.2 - 11.88 = 1.32 \\
&d_2 = 1
\end{aligned}$$

Intuitively, by providing a subsidy, households will borrow tradables from abroad such that the relative price of nontradables increases while the real wage decreases. As a result, firms will employ more workers and the economy achieves full employment.

□

Question 4. Consider a small open perfect-foresight economy populated by a large number of identical and infinitely-lived consumers with preferences described by the utility function

$$\sum_{t=0}^{\infty} \beta^t \ln c_t$$

where consumption c_t is a composite good made of tradable and nontradables goods, denoted c_t^{T} and c_t^{N}, respectively, via the aggregator function

$$c_t = \sqrt{c_t^{\mathrm{T}} c_t^{\mathrm{N}}}$$

The sequential budget constraint is given by

$$(1+\tau_t)(c_t^{\mathrm{T}} + p_t c_t^{\mathrm{N}}) + (1+r)d_{t-1} = y^{\mathrm{T}} + p_t y^{\mathrm{N}} + d_t + s_t$$

where d_t denote debt acquired in period t and maturing in period $t+1$, τ_t is proportional consumption tax, p_t denotes the relative price of nontradables in terms of tradables, $y^{\mathrm{T}} = 1$ is an endowment of tradable goods, $y^{\mathrm{N}} = 1$ is an endowment of nontradable goods, and s_t denotes a lump-sum transfer received from the government. The interest rate r satisfies $1 + r = \beta^{-1} = 1.04$. Debt is denominated in terms of tradables. Consumers are subject to the no-Ponzi game constraint

$$\lim_{j\to\infty} \frac{d_{t+j}}{(1+r)^j} \leq 0$$

Assume that the household's initial debt position is nil, i.e. $d_{-1} = 0$.

The government runs a balanced budget period by period, that is,

$$s_t = \tau_t(c_t^{\mathrm{T}} + p_t c_t^{\mathrm{N}})$$

Suppose that before period 0 the economy was in a steady state with constant consumption of tradables and nontradables and no external debt.

1. Compute the equilibrium paths of c_t^{T}, p_t, the trade balanc, and the current account under two alternative tax policies:

$$\text{policy 1}: \ \tau_t = 0, \ t \geq 0$$

$$\text{policy 2}: \ \tau_t = \begin{cases} 0 & 0 \leq t \leq 11 \\ 0.3 & t \geq 12 \end{cases}$$

Solution: Given the sequential budget constraint, we can write down the intertemporal budget constraint in period 0

$$(1+r)d_{-1} = \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t \left[y^{\mathrm{T}} + p_t y^{\mathrm{N}} + s_t - (1+\tau_t)(c_t^{\mathrm{T}} + p_t c_t^{\mathrm{N}})\right] \tag{10}$$

To derive this constraint, we use the no-Ponzi game constraint. Denote λ_0 as the Lagrangian multiplier of this constraint. The first-order conditions are given by

$$\frac{1}{2}\frac{1}{c_t^{\mathrm{T}}} = \lambda_0(1+\tau_t) \tag{11}$$

$$p_t = \frac{c_t^{\mathrm{T}}}{c_t^{\mathrm{N}}} \tag{12}$$

Note that we use $\beta(1+r)=1$ to derive (11). Nontradable goods market clears implies that

$$c_t^{\mathrm{N}} = y^{\mathrm{N}}$$

As government runs a balanced budget, (10) reduces to

$$(1+r)d_{-1} = \sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t\left[y^{\mathrm{T}} - c_t^{\mathrm{T}}\right] \tag{13}$$

Under policy 1, we get from (11) that c_t^{T} are constant across time, denoted by c. Since $d_{-1}=0$, from (13) we have

$$c_t^{\mathrm{T}} = c = y^{\mathrm{T}} = 1$$

Since $c_t^{\mathrm{N}} = y^{\mathrm{N}} = 1$, from (12) we have $p_t = 1$. Finally, from the sequential budget constraint, it is easy to get $d_t = 0$. The trade balance $tb_t = y^{\mathrm{T}} - c_t^{\mathrm{T}} = 0$ and the current account is therefore constant at zero.

Under policy 2, from (11), we know c_t^{T} are constant are constant before $t=12$ and after $t=12$, respectively. We can therefore write

$$c_t^{\mathrm{T}} = \begin{cases} c^1 & 0 \le t \le 11 \\ c^2 & t \ge 12 \end{cases}$$

On one hand, from (11), we have

$$\frac{1}{2c^1} = \lambda_0(1+0)$$
$$\frac{1}{2c^2} = \lambda_0(1+0.3)$$

which implies that

$$c^1 = 1.3c^2$$

On the other hand, from (13) we have

$$y^{\mathrm{T}} = (1-\beta^{12})c^1 + \beta^{12}c^2$$

From the last two equations, we get

$$c^1 = 1.17$$
$$c^2 = 0.9$$

Then it is easy to get

$$p_t = \begin{cases} 1.17 & 0 \le t \le 11 \\ 0.9 & t \ge 12 \end{cases}$$

$$tb_t = y^{\mathrm{T}} - c_t^{\mathrm{T}} = \begin{cases} -0.17 & 0 \le t \le 11 \\ 0.1 & t \ge 12 \end{cases}$$

For $0 \le t \le 11$, since $d_{-1} = 0$, we can derive that

$$\begin{aligned} d_t &= (c^1 - y^{\mathrm{T}}) \sum_{j=0}^{t} (1+r)^j \\ &= 0.16 \cdot (1.04^t - 1) \end{aligned}$$

and current account

$$\begin{aligned} ca_t &= -(d_t - d_{t-1}) \\ &= (y^{\mathrm{T}} - c^1)(1+r)^t \\ &= -0.17 \cdot 1.04^t \end{aligned}$$

Using the sequential constraint after $t = 12$, we can easily check that

$$d_t = d_{11} = 0.09, \quad t \ge 12$$

and therefore

$$ca_t = 0, \quad t \ge 12$$

$\square$

2. Compute the welfare cost of policy 2 relative to policy 1, defined as the percentage increase in the consumption stream of a consumer living under policy 2 required to make him as well off as living under policy 1. Formally, the welfare cost of policy 2 relative to policy 1 is given by $\lambda \times 100$, where λ is implicitly given by

$$\sum_{t=0}^{\infty} \beta^t \ln\left[c_t^{p2}(1+\lambda)\right] = \sum_{t=0}^{\infty} \beta^t \ln c_t^{p1}$$

where c_t^{p1} and c_t^{p2} denote consumption in period t under policy 1 and 2, respectively.

Solution: From the last question, we can get

$$c_t^{p1} = 1$$

$$c_t^{p2} = \begin{cases} 1.17 & 0 \le t \le 11 \\ 0.9 & t \ge 12 \end{cases}$$

which implies that

$$\begin{aligned}
&\ln(1+\lambda) + \left[(1-\beta^{12})\ln c^1 + \beta^{12}\ln c^2\right] = \ln c \\
\Rightarrow\ &\ln(1+\lambda) + \left[(1-(\frac{1}{1.04})^{12})\ln 1.17 + (\frac{1}{1.04})^{12}\ln 0.9\right] = 0 \\
\Rightarrow\ &\lambda = 0.57\%
\end{aligned}$$

Therefore, the welfare cost of policy 2 is about 0.57% in terms of consumption stream.

□