Advanced Microeconomics

Problem set 3

1. Consider the N-player version of the (Rubinstein's) bargaining game presented in class. At dates 1, $N+1$, $2N+1$, ..., player 1 offers a division $(x_1, ..., x_N)$ of the pie with $x_i \geq 0$ for all i, and $\sum_{i=1}^{N} x_i \leq 1$. At dates 2, $N+2$, $2N+2$, ..., player 2 offers a division, and so on. When player i offers a division, the other players simultaneously accept or veto the division. If all accept, the pie is divided; if at least one vetoes, player $i+1$ $(mod\ N)$ offers a division in the following period. Assuming that the players have common discount factor δ, show that, for all i, player i offering a division

$$\left(\frac{v}{1+\cdots+\delta^{N-1}}, \frac{v\delta}{1+\cdots+\delta^{N-1}}, ..., \frac{v\delta^{N-1}}{1+\cdots+\delta^{N-1}}\right)$$

for players i, $i+1$, ...,$i+N-1$ $(mod\ N)$ at each date $(kN+i)$ and the other players' accepting any offer equal or higher than those amounts is a subgame-perfect Nash equilibrium.

Answer: For $i \in \{1, \dots, N\}$, let $(i_1, \dots, i_N)$ be defined as follows: i_1 be player i who offers the division, and i_2 be the next player to make an offer if player i's offer is rejected. $i_3, \dots, i_N$ are similarly defined. Assuming that players always make the same offer whenever it is their turn in equilibrium (stationary strategies). Assuming that for all i, v_{i1} is the division made to herself when player i offers. Without loss of generality we assume that in equilibrium, $v_{i1} \in [\underline{v}_i, \bar{v}_i]$, that is, player i offers herself some value in the range between $\underline{v}_i$ and $\bar{v}_i$. $\underline{v}_i$, $\bar{v}_i$ exist and $[\underline{v}_i, \bar{v}_i] \subset [0, v]$.

Player i can not offer a player i_j less than $\delta^{j-1}\underline{v}_j$, as player j can be better-off by rejecting the offer and get at least $\underline{v}_j$ after $(j-1)$ periods. Thus the maximum player i expects to get is

$$\bar{v}_i \leq v - \delta\underline{v}_{i2} - \ldots - \delta^{N-1}\underline{v}_{iN}. \tag{1}$$

Player i will not offer the other players more than their respective $\bar{v}_j$ $(j \neq i)$ discounted to current peirod, so $\underline{v}_i$ is bounded below

$$\underline{v}_i \geq v - \delta\bar{v}_{i2} - \ldots - \delta^{N-1}\bar{v}_{iN}. \tag{2}$$

For same reasoning, for $j \neq i$, the maximum and the minimum she expects to get can be expressed as

$$\bar{v}_j \leq v - \delta\underline{v}_{j2} - \ldots - \delta^{N-1}\underline{v}_{jN}. \tag{3}$$

$$\underline{v}_j \geq v - \delta \bar{v}_{j2} - \ldots - \delta^{N-1} \bar{v}_{jN}. \tag{4}$$

By symmetry, for all $i = 1, \ldots, N$, $\bar{v}_i = \bar{v}_j$, $\underline{v}_i = \underline{v}_j$ in equilibrium. Simplifying the equations gives

$$\bar{v}_i \leq v - (\delta + \ldots + \delta^{N-1}) \underline{v}_j, \tag{5}$$

$$\underline{v}_i \geq v - (\delta + \ldots + \delta^{N-1}) \bar{v}_j. \tag{6}$$

As $\bar{v}_j = \bar{v}_i$, $\underline{v}_i = \underline{v}_j$, (6) is equivalent to

$$v \leq \underline{v}_i + (\delta + \ldots + \delta^{N-1}) \bar{v}_i.$$

Thus (5) implies

$$\bar{v}_i \leq \underline{v}_i + (\delta + \ldots + \delta^{N-1}) \bar{v}_i - (\delta + \ldots + \delta^{N-1}) \underline{v}_i \Longrightarrow$$

$$\bar{v}_i \leq \underline{v}_i.$$

However, by assumption, $\underline{v}_i \leq \bar{v}_i$, so in equilibrium, $\bar{v}_i = \underline{v}_i = v_{i1}$. Given this condition, solving either (5) or (6) to get

$$\bar{v}_i = \underline{v}_i = \frac{v}{1 + \ldots + \delta^{N-1}}.$$

In equilibrium, when player i starts to make an offer, she can get

$$v_i = \frac{v}{1 + \ldots + \delta^{N-1}}.$$

Thus in equilibrium, a player i accepts any offer giving her some value equal or greater than $\frac{v}{1+\delta \ldots + \delta^{N-1}}$ discounted to the period she is offered, that is, at period 1, player 2 will accept any offer equal or greater than $\frac{\delta v}{1+\delta \ldots + \delta^{N-1}}$, player 3 accepts any offer equal or more than $\frac{\delta^2 v}{1+\delta \ldots + \delta^{N-1}}$, etc. In equilibrium, each player i makes the offer

$$\left(\frac{v}{1 + \cdots + \delta^{N-1}}, \frac{\delta v}{1 + \cdots + \delta^{N-1}}, \ldots, \frac{\delta^{N-1} v}{1 + \ldots + \delta^{N-1}} \right) , \tag{7}$$

and rejects any offer made by player j giving her less than $\frac{\delta^k v}{1+\cdots+\delta^{N-1}}$, where k is number of periods she has to wait to offer assuming no offers has been accepted by then.

The strategy profile specified above constitutes NE in any subgame when a player i is to make an offer, and by definition, it is a SPNE.

2. Two persons A and B can play one of the following games: G1:

G1 (A = row player, B = column player)

	L	R
U	4, 4	0, 0
D	0, 0	1, 1

G2 (A = row player, B = column player)

	L	R
U	-1, -1	0, 0
D	0, 0	4, 4

   (a) Suppose both A and B know they play game G1. Find all NE of the game.

   (b) Now suppose they play G1 and G2 with equal probabilities, which is common knowledge among them. In addition, A knows which game they are playing but B does not know if they are as in G1 or as in G2. Model the game as a Bayesian game and find all pure strategy Bayesian NE of the game.

Answer:

   (a) Three NE: (U, L), (D, R), ((1/5, 4/5), (1/5, 4/5).

   (b) Given G1 and G2 occur with equal probability, the two player's payoff are

A = row player, B = column player

	L	R
UU	1.5, 1.5	0, 0
UD	2, 2	2, 2
DU	-0.5, -0.5	0.5, 0.5
DD	0, 0	2.5, 2.5

   Two pure BNE: (UD, L), (DD, R).

3. Suppose that Michael and John are playing the following game of incomplete information. Michael(the row player) is perfectly aware of the payoffs but John (the column player) does not know if they are as in G1 or as in G2.

G1 (Michael = row player, John = column player)

	L	R
U	1, 1	0, 0
D	0, 0	0, 0

G2 (Michael = row player, John = column player)

	L	R
U	0, 0	0, 0
D	0, 0	2, 2

a) Model this situation as a Bayesian game.

b) Assuming that it is common knowledge that payoffs as in G1 or as in G2 with equal probabilities, find *all* Bayesian-Nash equilibria of the game.

Answer: (a) Bayesian game

- Players: $I = \{M, J\}$;
- Types: $\{G1 : \mu; G2 : 1 - \mu\}$;
- Strategies: M: $\{UU, UD, DU, DD\}$

  J: $\{L, R\}$
- Payoffs

$$\tilde{u}_i(s_M(), s_J()).$$

(b) BNE.

Given $mu = 1/2$, the two player's payoff are

John

		L	R
Michael	UU	1/2, 1/2	0, 0
	UD	1/2, 1/2	1, 1
	DU	0, 0	0, 0
	DD	0, 0	1, 1

Three pure strategy BNE: (UU, L), (UD, R), (DD, R)

Mixed strategy BNE

$$\{(UD, DD; \lambda, 1 - \lambda), R | \lambda \in [0, 1]\}$$

$$\{(UU, UD; \tau, 1 - \tau), L | \tau \in [1/2, 1]\}$$

4. There are two individuals $i = 1, 2$ who need to decide simultaneously whether to contribute to a public good or not ("C" or "NC"). Each player derives a benefit of 1 if at least one contributes and 0 if none does. A player's cost of contributing is c_i. While the benefit is common knowledge, each agent's cost c_i is known only to himself. However, it is common knowledge that for $i = 1, 2$, c_i is independently drawn from a uniform distribution on [0,2]. Identify the Bayesian Nash equilibria of this game.

Answer: A Bayesian NE of the game $(s_1(), s_i())$ should satisfy that for every $c_i \in [0, 1]$,

$$E[u_i(C, s_j(c_j), c_i)] \geq E[u_i(NC, s_j(c_j), c_i)]$$

if and only if $s_i(c_i) = C$.

Take any c_i and suppose that $(s_1(), s_i())$ is a BNE, then $s_i(c_i) = C$ if and only if

$$c_i \leq 1 - Prob(s_j(c_j) = C).$$

Hence player i's strategy $s_i(c_i)$ should be a cutoff strategy and we denote the cutoff by $\bar{c}_i$. $s_i(c_i) = C$ if and only if $c_i \leq \bar{c}_i$. Similarly, $s_j(c_j) = C$ if an only if $c_j \leq \bar{c}_j$.

As c_i is uniformly distributed on $[0, 1]$, we have

$$Prob(s_j(c_j) = C) = Prob(c_j \leq \bar{c}_j) = \int_0^{\bar{c}_j} \frac{1}{2} dc = \frac{1}{2}\bar{c}_j$$

and

$$Prob(s_i(c_i) = C) = Prob(c_i \leq \bar{c}_i) = \int_0^{\bar{c}_i} \frac{1}{2} dc = \frac{1}{2}\bar{c}_i.$$

This implies

$$\bar{c}_i = 1 - \frac{1}{2}\bar{c}_j, \qquad \bar{c}_j = 1 - \frac{1}{2}\bar{c}_i.$$

Hence we have

$$\bar{c}_i = \frac{2}{3}, \qquad \bar{c}_j = \frac{2}{3}.$$

The unique BNE:

$$s_1(c_1) = C \text{ iff } c_1 \leq \frac{2}{3}, \quad s_2(c_2) = C \text{ iff } c_2 \leq \frac{2}{3}.$$

5. In a first-price, all-pay auction, the bidders simultaneously submit sealed bids. The highest bid wins the objects and every bidder pays the seller the amount of his bid. The all-pay auction is a useful model of lobbying activity. In such models, different interest groups spend money (their bids) in order to influence government policy and the one spending the most (the highest bidder) is able to tilt policy in its favored direction (winning the auction). Since money spent on lobbying is a sunk cost borne by all groups regardless of which group is suceessful in obtaining its preferred policy, such situations have a natural all-pay aspects.

   Consider the independent private values model with symmetric bidders whose values are each distributed according to the distribution function F, with density f.

   (a) Find the unique symmetric equilibrium bidding function. Interpret.

(b) Do bidders bid higher or lower than in a first-price auction?

(c) Find an expression for the seller's expected revenue.

(d) Show that the seller's expected revenue is the same as in a first-price auction.

Answer:

(a) Without loss of generality, assume that player i uses a bidding function $b_i(r_i)$, and that r_i is the value she reports. By symmetry, $b_i(\cdot) = b_j(\cdot) = b(\cdot)$.

Given (r_i, r_{-i}), the expected utility for player i is

$$u_i = [v_i - b(r_i)]F^{N-1}(r_i) - b(r_i)(1 - F^{N-1}(r_i)).$$

FOC, let $r_i^* = v_i$,

$$-b'(v_i)(1 - F^{N-1}(v_i)) + b(v_i)(N-1)F^{N-2}(v_i)f(v_i) = 0 \Longrightarrow$$
$$b(v_i) = \int_0^{v_i} x dF^{N-1}(x).$$

(b) Recall that in the first-price auction, each bidder uses the bidding function

$$\widehat{b}(v_i) = \frac{1}{F^{N-1}(v_i)} \int_0^{v_i} x dF^{N-1}(x).$$

Obviously, they are bidding less aggressively in the all-pay auction than they do in the first-price auction, since $\frac{1}{F^{N-1}(v_i)} > 1$.

(c) The expected revenue for the seller in the all-pay auction is R^{AP}. Without loss of generality, let $v_i \in [0, \bar{v}]$.

$$\begin{aligned} R^{AP} =& N \int_0^{\bar{v}} b(v_i) f(v_i) dv_i \\ =& N \int_i^{\bar{v}} \left[\int_0^{v_i} x dF^{N-1}(x) \right] dF(v_i). \end{aligned}$$

(d) The expected revenue for the seller in first price auction is

$$\begin{aligned} R^{FP} =& \int_0^{\bar{v}} \widehat{b}(v_i) N F^{N-1}(v_i) f(v_i) dv_i \\ =& N \int_0^{\bar{v}} \left[\frac{1}{F^{N-1}(v_i)} \int_0^{v_i} x dF^{N-1}(x) \right] F^{N-1}(v_i) dF(v_i) \\ =& N \int_0^{\bar{v}} \left[\int_0^{v_i} x dF^{N-1}(x) \right] dF(v_i). \end{aligned}$$

Clearly, $R^{AP} = R^{FP}$, the two auction brings the same expected revenue for the seller.