Real Business Cycle Models

Ei Yang

SUFE

April 24, 2024

Introduction

- New Keynesian model builds on real business cycle (RBC) model
- RBC model, key features
  - intertemporal utility maximisation
  - rational expectations
  - complete asset markets / representative agent
  - perfect competition in goods and factor markets
- RBC model, key implications
  - business cycles are Pareto efficient
  - business cycles driven by exogenous productivity shocks (and other exogenous real shocks: terms-of-trade, government spending, etc)
  - money is neutral
- Established use of *dynamic stochastic general equilibrium* (DSGE) models and *quantitative theory*

Settings

A benchmark classical monetary model

- Representative household and firms
  - price taking
- general equilibrium

Households

- Household preferences over consumption and labor supply

$$U(C_t, N_t)$$

- Intertemporal preferences

$$\mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t U(C_t, N_t) \right\}, \qquad 0 < \beta < 1$$

- Flow budget constraint at every date and state

$$P_t C_t + Q_t B_t \leq B_{t-1} + W_t N_t - T_t$$

- We rule out *Ponzi games*
  (e.g., impose arbitrarily large bounds on real debt issuance)

- This is a *cashless* economy

Household intertemporal optimisation

- Lagrangian with nonnegative, stochastic, multipliers $\{\lambda_t\}$

$$L = \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \left[\beta^t U(C_t, N_t) + \lambda_t \left(B_{t-1} + W_t N_t - T_t - P_t C_t - Q_t B_t \right) \right] \right\}$$

- First order conditions

$$\begin{aligned}
C_t : \quad & \beta^t U_c(C_t, N_t) = \lambda_t P_t \\
N_t : \quad & -\beta^t U_n(C_t, N_t) = \lambda_t W_t \\
\\
B_t : \quad & \lambda_t Q_t = \mathbb{E}_t \left\{ \lambda_{t+1} \right\}
\end{aligned}$$

These hold at every date and state

Household first order conditions

- Let

$$U_{c,t} \equiv U_c(C_t, N_t), \qquad U_{n,t} \equiv U_n(C_t, N_t)$$

- Labor supply

$$-\frac{U_{n,t}}{U_{c,t}} = \frac{W_t}{P_t}$$

- Intertemporal consumption Euler equation

$$Q_t = \mathbb{E}_t \left\{ \beta \frac{U_{c,t+1}}{U_{c,t}} \frac{P_t}{P_{t+1}} \right\}$$

Firms

- Competition in goods and factor markets
- Production function

$$Y_t = A_t F(N_t)$$

- Profits

$$P_t Y_t - W_t N_t$$

- Labor demand

$$A_t F'(N_t) = \frac{W_t}{P_t}$$

Equilibrium

- A competitive equilibrium involves
  - households optimising taking prices as given
  - firms optimising taking prices as given
  - prices such that markets clear
- Optimality conditions for labor supply and demand give

$$-\frac{U_n(C_t, N_t)}{U_c(C_t, N_t)} = \frac{W_t}{P_t} = A_t F'(N_t)$$

- Goods market clearing

$$Y_t = C_t$$

- Bond market clears if goods market clears

Standard functional forms

- Utility function

$$U(C,N) = \frac{C^{1-\sigma}}{1-\sigma} - \frac{N^{1+\varphi}}{1+\varphi}$$

σ is (constant) coefficient of relative risk aversion (CRRA)

$$\sigma = -\frac{U_{cc}(\cdot)\,C}{U_c(\cdot)}$$

$1/\sigma$ is (constant) intertemporal elasticity of substitution (IES)

$$\frac{C^{1-\sigma}-1}{1-\sigma} \to \log C \qquad \text{as } \sigma \to 1$$

$1/\varphi$ is (λ-constant) Frisch elasticity of labor supply

- Typical numbers in macro, $\sigma = 1$ or 2 and $\varphi = 0.4$

Standard functional forms

- Production function

$$Y = A\,F(N) = A\,N^{1-\alpha}$$

$1-\alpha$ is labor's (constant) share in final output

$$1-\alpha = \frac{W\,N}{P\,Y}$$

- Implicitly α share is paid to capital and other fixed factors

- Typical number in macro, $\alpha = 1/3$ so that labor's share is $2/3$

Solving the model

- With these functional forms

$$N^{\varphi}\, C^{\sigma} = \frac{W}{P} = (1-\alpha) A\, N^{1-\alpha}$$

- Goods market clearing

$$C = Y = A\, N^{1-\alpha}$$

- Two equations in two unknowns C, N given exogenous A
- Solve for real variables independently of all nominal variables

Notation

- Little letters denote logs (or other proportional variables)

$$c \equiv \log C, \qquad n \equiv \log N, \qquad a \equiv \log A$$

- Bars denote non-stochastic steady state

$$\bar{c} \equiv \log \bar{C}, \qquad \bar{n} \equiv \log \bar{N}, \qquad \bar{a} \equiv \log \bar{A}$$

where

$$\bar{A} \equiv \mathbb{E}\{A\}$$

- Hats denote log deviations

$$\hat{c} \equiv c - \bar{c}, \qquad \hat{n} \equiv n - \bar{n}, \qquad \hat{a} \equiv a - \bar{a}$$

(approximate percentage deviation from steady state)

Solution

- Equilibrium labor

$$\hat{n} = \psi_{na}\,\hat{a}, \qquad \psi_{na} \equiv \frac{1-\sigma}{(1-\alpha)\sigma + \alpha + \varphi}$$

- Equilibrium output and consumption

$$\hat{y} = \psi_{ya}\,\hat{a}, \qquad \psi_{ya} \equiv \frac{1+\varphi}{(1-\alpha)\sigma + \alpha + \varphi}, \qquad \hat{c} = \hat{y}$$

Interpreting the solution

- Coefficients are *elasticities*

$$\frac{dn}{da} = \psi_{na} = \frac{1-\sigma}{(1-\alpha)\sigma + \alpha + \varphi}$$

one percent change in exogenous productivity a gives ψ_{na} percent change in equilibrium labor n

- Easy to calculate volatilities

$$\text{std}\{n\} = |\psi_{na}|\,\text{std}\{a\}$$

and

$$\text{std}\{y\} = |\psi_{ya}|\,\text{std}\{a\}$$

Interpreting the solution

- Relative volatilities

$$\frac{\text{std}\{n\}}{\text{std}\{y\}} = \frac{|\psi_{na}|}{|\psi_{ya}|} = \frac{|1-\sigma|}{1+\varphi}$$

- Correlations

$$\text{corr}\{n, y\} \equiv \frac{\text{cov}\{n, y\}}{\text{std}\{n\}\,\text{std}\{y\}} = \frac{\psi_{na}\,\psi_{ya}}{|\psi_{na}|\,|\psi_{ya}|}$$

- Endogenous real variables $n, c, y, w-p$ inherit time series properties of exogenous productivity a (autocorrelation, impulse responses, etc)

Euler equation

- Recall nominal bond prices Q_t given by

$$Q_t = \mathbb{E}_t \left\{ \beta \frac{U_{c,t+1}}{U_{c,t}} \frac{P_t}{P_{t+1}} \right\}$$

- With our standard separable preferences

$$Q_t = \mathbb{E}_t \left\{ \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{P_t}{P_{t+1}} \right\}$$

- Write in terms of nominal interest rate i_t, inflation π_t etc

$$i_t \equiv -\log Q_t, \quad \rho \equiv -\log \beta, \quad p_t \equiv \log P_t, \quad \pi_{t+1} \equiv p_{t+1} - p_t$$

- So Euler equation can be written

$$1 = \mathbb{E}_t \left\{ \exp(-\rho - \sigma \Delta c_{t+1} - \pi_{t+1} + i_t) \right\}$$

Log-linearised Euler equation

- Euler equation

$$1 = \mathbb{E}_t \{\exp(z_{t+1})\}$$

where

$$z_{t+1} \equiv -\rho - \sigma \Delta c_{t+1} - \pi_{t+1} + i_t$$

- First order approximation of $\exp(z)$ around $z \approx 0$ is

$$\exp(z) \approx \exp(0) + \exp(0)(z - 0) = 1 + z$$

- Treat approximation as exact and simplify

$$i_t = \rho + \sigma \mathbb{E}_t\{\Delta c_{t+1}\} + \mathbb{E}_t\{\pi_{t+1}\}$$

Euler equation and Fisher equation

- Define ex ante real interest rate r_t by Fisher equation

$$r_t \equiv i_t - \mathbb{E}_t\{\pi_{t+1}\}$$

- And since log linear Euler equation is

$$i_t = \rho + \sigma \mathbb{E}_t\{\Delta c_{t+1}\} + \mathbb{E}_t\{\pi_{t+1}\}$$

- Can eliminate nominal interest rate i_t to get

$$r_t = \rho + \sigma \mathbb{E}_t\{\Delta c_{t+1}\}$$

- Familiar version of Euler equation

$$\mathbb{E}_t\{\Delta c_{t+1}\} = \frac{r_t - \rho}{\sigma}$$

(recall, $1/\sigma$ is intertemporal elasticity of substitution)

Equilibrium real interest rates

- Up to log linear approximation, real interest rates r_t given by

$$r_t = \rho + \sigma \mathbb{E}_t\{\Delta c_{t+1}\}$$

- Aside: in new Keynesian literature, this often re-written as

$$y_t = -\frac{r_t - \rho}{\sigma} + \mathbb{E}_t\{y_{t+1}\}$$

and called the *dynamic IS curve* (why?)

- Equilibrium real interest rates therefore

$$r_t = \rho + \sigma \psi_{ya} \mathbb{E}_t\{\Delta a_{t+1}\}$$

Equilibrium real interest rates

- Conditional expectation determined by productivity process
- AR(1) example

$$a_{t+1} = \phi_a \, a_t + \varepsilon_{a,t+1} \qquad \varepsilon_{a,t+1} \sim \text{IID and } N(0, \sigma_\varepsilon^2)$$

with persistence $0 < \phi_a < 1$

Then

$$r_t = \rho - \sigma\psi_{ya}(1 - \phi_a)a_t$$

moves in opposite direction to productivity, since $\phi_a < 1$

- All interesting real variables now determined, all independent of nominal variables

Real variables

- Real variables all independent of nominal variables
- Equilibrium labor (in log deviations)

$$\hat{n}_t = \psi_{na}\,\hat{a}_t$$

- Equilibrium output and consumption

$$\hat{c}_t = \hat{y}_t = \psi_{ya}\,\hat{a}_t$$

- Equilibrium real wage

$$\hat{w}_t - \hat{p}_t = \psi_{wa}\hat{a}_t$$

- Equilibrium real interest rate

$$r_t = \rho + \sigma\psi_{ya}\mathbb{E}_t\{\Delta a_{t+1}\}$$

Nominal variables

- What about nominal variables p_t, π_t, i_t etc?
- To solve for them, need *money demand* and *money supply*

Money-in-the-utility function

- Household preferences now include *real money* M/P

$$U\left(C_t,\, \frac{M_t}{P_t},\, N_t\right)$$

- Intertemporal preferences

$$\mathbb{E}_0\left\{\sum_{t=0}^{\infty} \beta^t U\left(C_t,\, \frac{M_t}{P_t},\, N_t\right)\right\}, \qquad 0 < \beta < 1$$

- Flow budget constraint at every date and state

$$P_t C_t + Q_t B_t + M_t \leq B_{t-1} + M_{t-1} + W_t N_t - T_t$$

Household intertemporal optimisation

- Lagrangian with nonnegative, stochastic, multipliers $\{\lambda_t\}$

$$L = \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \left[\beta^t U\left(C_t, \frac{M_t}{P_t}, N_t\right) \right. \right.$$
$$\left. \left. + \quad \lambda_t \left(B_{t-1} + M_{t-1} + W_t N_t - T_t - P_t C_t - Q_t B_t - M_t\right) \right] \right\}$$

- First order conditions

$$C_t : \qquad \beta^t U_{c,t} = \lambda_t P_t$$
$$N_t : \qquad -\beta^t U_{n,t} = \lambda_t W_t$$

$$B_t : \qquad \lambda_t Q_t = \mathbb{E}_t \{\lambda_{t+1}\}$$
$$M_t : \qquad \lambda_t = \mathbb{E}_t \{\lambda_{t+1}\} + \beta^t U_{m,t} \frac{1}{P_t}$$

As usual, these hold at every date and state

Household first order conditions

- Money demand

$$\frac{U_{m,t}}{U_{c,t}} = 1 - Q_t = 1 - \exp(-i_t)$$

- Labor supply

$$-\frac{U_{n,t}}{U_{c,t}} = \frac{W_t}{P_t}$$

- Intertemporal consumption Euler equation

$$Q_t = \mathbb{E}_t \left\{ \beta \frac{U_{c,t+1}}{U_{c,t}} \frac{P_t}{P_{t+1}} \right\}$$

Standard functional forms

- Separable utility function

$$U\left(C, \frac{M}{P}, N\right) = \frac{C^{1-\sigma}}{1-\sigma} + \frac{(M/P)^{1-\nu}}{1-\nu} - \frac{N^{1+\varphi}}{1+\varphi}$$

- Money demand

$$\frac{U_{m,t}}{U_{c,t}} = \frac{(M_t/P_t)^{-\nu}}{C_t^{-\sigma}} = 1 - \exp(-i_t)$$

or

$$\frac{M_t}{P_t} = C_t^{\sigma/\nu}\left[1 - \exp(-i_t)\right]^{-1/\nu}$$

- Take logs

$$m_t - p_t = \frac{\sigma}{\nu} c_t - \frac{1}{\nu} \log\left[1 - \exp(-i_t)\right]$$

Log-linear money demand

- Approximate last term in i_t around some steady-state $\bar{\imath}$ to get

$$m_t - p_t \approx \frac{\sigma}{\nu} c_t - \eta i_t + \text{constant}$$

  where $\eta > 0$ is the *interest semi-elasticity* of money demand (a coefficient from the linearisation, depends on $\nu, \bar{\imath}$)

- Treating this as exact and writing it in log deviations

$$\hat{m}_t - \hat{p}_t = \frac{\sigma}{\nu} \hat{c}_t - \eta i_t$$

- Money supply: central bank sets $\hat{m}_t$, then price level $\hat{p}_t$ and interest rate i_t determined endogenously ($\hat{c}_t$ already determined)

Price level determination

- Fisher equation

$$i_t = r_t + \mathbb{E}_t\{\pi_{t+1}\}$$

- Money demand

$$\hat{m}_t - \hat{p}_t = \frac{\sigma}{\nu}\hat{c}_t - \eta i_t$$

with $\{\hat{m}_t\}$ process given exogenously and real variables independent of nominal

- For simplicity, set real variables to steady state

- *Stochastic difference equation* in endogenous process $\{\hat{p}_t\}$

$$\hat{p}_t = \frac{\eta}{1+\eta}\mathbb{E}_t\{\hat{p}_{t+1}\} + \frac{1}{1+\eta}\hat{m}_t$$

Equilibrium price level

- Solve the stochastic difference equation by iterating forward

$$\hat{p}_t = \frac{1}{1+\eta}\mathbb{E}_t\left\{\sum_{k=0}^{\infty}\left(\frac{\eta}{1+\eta}\right)^k \hat{m}_{t+k}\right\}$$

- We typically work with a process for *money growth*

$$\Delta\hat{m}_t \equiv \hat{m}_t - \hat{m}_{t-1}$$

- So rewrite price level in terms of money growth. Key steps

$$(1+\eta)\hat{p}_t = \hat{m}_t + \mathbb{E}_t\left\{\sum_{k=1}^{\infty}\left(\frac{\eta}{1+\eta}\right)^k \left[\Delta\hat{m}_{t+k} + \hat{m}_{t+k-1}\right]\right\}$$

$$\Rightarrow$$

$$\hat{p}_t = \hat{m}_t + \mathbb{E}_t\left\{\sum_{k=1}^{\infty}\left(\frac{\eta}{1+\eta}\right)^k \Delta\hat{m}_{t+k}\right\}$$

Equilibrium price level

- AR(1) money growth example

$$\Delta \hat{m}_{t+1} = \phi_m \, \Delta \hat{m}_t + \varepsilon_{m,t+1} \quad \varepsilon_{m,t+1} \sim \text{IID and } N(0, \sigma_\varepsilon^2)$$

  with persistence $0 < \phi_m < 1$

- Impulse response function with AR(1) money growth

$$\mathbb{E}_t\{\Delta \hat{m}_{t+k}\} = \phi_m^k \Delta \hat{m}_t, \qquad k \geq 1$$

- So price level evaluates to

$$\hat{p}_t = \hat{m}_t + \eta \frac{\phi_m}{1 + \eta(1 - \phi_m)} \Delta \hat{m}_t$$

  responds more than one-for-one to money shocks on impact

- With solution for $\hat{p}_t$, can now determine other nominal variables

Inflation

- Implies inflation

$$\pi_t = \Delta \hat{p}_t = \frac{1+\eta}{1+\eta(1-\phi_m)} \Delta \hat{m}_t - \frac{\eta \phi_m}{1+\eta(1-\phi_m)} \Delta \hat{m}_{t-1}$$

- Note that in 'steady-state' $\Delta \hat{m}_t = \Delta \hat{m}_{t-1}$ so

$$\pi_t = \Delta \hat{m}_t$$

- Long-run inflation rate determined by long-run money growth rate

Nominal interest rates

- Can calculate from either Fisher equation or money demand

- Easier from money demand

$$i_t = \frac{\hat{p}_t - \hat{m}_t}{\eta} = \frac{\phi_m}{1 + \eta(1 - \phi_m)} \Delta \hat{m}_t$$

- Nominal interest rates and money growth positively correlated (i.e., no *liquidity effect*)

- Long-run nominal interest rate likewise determined by long-run money growth

Velocity

- Classical *exchange equation*, implicitly defines *velocity*

$$MV = PC$$

- For this example, log velocity is just

$$\hat{v}_t = \hat{p}_t - \hat{m}_t = \eta i_t = \frac{\eta\phi_m}{1+\eta(1-\phi_m)}\Delta\hat{m}_t$$

- Moves with interest rates, positively correlated with money growth

- Long-run velocity again determined by long-run money growth