PROBLEM SET 2 ANSWERS

1 (50) The question is related to the BGG model with financial frictions. Consider an entrepreneur that is endowed with net worth N initially and borrow B from the bank. The entrepreneur purchase capital K at the price Q^k and finance the capital acquisition using his net worth and bank loans, which implies that,

$$N + B = Q^k K. \tag{1}$$

The entrepreneur then experiences an idiosyncratic shock that convert capital K into efficient units ωK, where ω follows an i.i.d. process across firms and time, drawn from the distribution $F(\cdot)$ with a nonnegative support. The entrepreneur can obtain revenues of R^k for each effective unit of capital he holds. The idiosyncratic shock ω realizes after the capital acquisition.

a. (5) Assume that the bank charges a state-contingent interest rate Z on its loans. There then exists a cutoff ratio $\bar{\omega}$ such that the a. (5) Assume that the bank charges a state-contingent interest rate Z on its loans. There then exists a cutoff ratio $\bar{\omega}$ such that the entrepreneur will be not able to repay the bank loans and default if $\omega < \bar{\omega}$. Please write down the expression for $\bar{\omega}$.

Answer:

$$\bar{\omega} R^k K = ZB.$$

b. (10) If the entrepreneur defaults, the bank pays a liquidation cost to obtain the entrepreneur's generated revenues. The liquidation cost is a fraction μ of the entrepreneur's generated revenues. Please derive expressions for the entrepreneur's expected income and the bank's expected income before the idiosyncratic shock ω realizes.

Answer: The entrepreneur's expected income is given by,

$$\int_{\bar{\omega}}^{+\infty} (\omega R^k K - ZB) dF(\omega) = \int_{\bar{\omega}}^{+\infty} (\omega - \bar{\omega}) dF(\omega) R^k K.$$

The bank's expected income is given by,

$$\int_0^{\bar{\omega}} (1-\mu)\omega dF(\omega) R^k K + [1 - F(\bar{\omega})] ZB = \{\int_0^{\bar{\omega}} (1-\mu)\omega dF(\omega) + [1 - F(\bar{\omega})]\bar{\omega}\} R^k K.$$

c. (5) The bank is risk neutral and has a participation constraint that requires that its expected income to be able to cover its funding costs RB, where R is the risk-free interest rate. Please write down the expression for the bank's participation constraint.

Answer:

$$\{\int_0^{\bar{\omega}}(1-\mu)\omega dF(\omega)+[1-F(\bar{\omega})]\bar{\omega}\}R^kK \geq RB.$$

d. (10) Please solve for the optimal financial contract that maximizes the entrepreneur's expected income subject to the bank's participation constraint.

Answer: Rewrite the bank's participation constraint as:

$$\{\int_0^{\bar{\omega}}(1-\mu)\omega dF(\omega)+[1-F(\bar{\omega})]\bar{\omega}\}R^kK \geq R(Q^kK-N).$$

Denote λ as the Lagrange multiplier on the bank's participation constraint. Define:

$$g(\bar{\omega})=\int_{\bar{\omega}}^{+\infty}(\omega-\bar{\omega})dF(\omega).$$

$$h(\bar{\omega})=\{\int_0^{\bar{\omega}}(1-\mu)\omega dF(\omega)+[1-F(\bar{\omega})]\bar{\omega}\}.$$

The optimal financial contract problem is given by,

$$\max g(\bar{\omega})R^kK$$

subject to:

$$h(\bar{\omega})R^kK \geq R(Q^kK-N).$$

The first order condition on K is given by,

$$g(\bar{\omega})R^k+\lambda[h(\bar{\omega})R^k-RQ^k]=0.$$

The first order condition on $\bar{\omega}$ is given by,

$$g'(\bar{\omega})R^kK+\lambda h'(\bar{\omega})R^kK=0.$$

Using the above equations together provides the optimal financial contract:

$$\frac{g(\bar{\omega})R^k}{g'(\bar{\omega})}=\frac{h(\bar{\omega})R^k-RQ^k}{h'(\bar{\omega})}.$$

where

$$g'(\bar{\omega})=-1+F(\bar{\omega}).$$

$$h'(\bar{\omega})=-\mu f(\bar{\omega})\bar{\omega}+1-F(\bar{\omega}).$$

e. (10) Please describe the financial accelerator mechanism in the BGG framework. In other words, please explain why fluctuations in capital prices and capital investments are amplified in the BGG framework.

Answer: Consider a fall in capital price. The fall in capital price reduces entrepreneurs' realized return to capital and therefore leads to a decline in entrepreneur's net worth. In the BGG framework, entrepreneurs' borrowing capacity depends on their net worth. The lower net worth makes entrepreneurs borrow less and their demand for capital declines, which amplifies the fall in asset prices and capital investments.

f. (10) Define investment efficiency as the marginal efficiency of investment in producing capital. The higher the investment efficiency, the less investment used in producing capital. Both an unexpected increase in entrepreneur's risk (ω) and an unexpected decrease in investment efficiency have contractionary output effect but have opposite effects on entrepreneur's net worth and credits spread $Z - R$. Please explain why these two shocks have have opposite effects on entrepreneur's net worth and credits spread $Z - R$.

Answer: An unexpected increase in entrpereneur's risk (ω) increases the expected default ratio of entrepreneurs. The higher expected default ratio makes banks less willing to lend and leads to higher credit spread $Z - R$. In response to higher credit spread, entrepreneurs borrow less and their demand for capital declines. The capital price falls. So entrepreneur's net worth decline.

An unexpected decrease in investment efficiency makes it more costly to produce capital stock. The supply of capital stock falls and leads to an increase in the capital price. The higher capital price makes entrepreneur's return to capital higher and raises their net worth. One one hand, with higher net worth, entrepreneurs can take lower leverage ratio. On the other hand, the reduced supply of capital stock increases the expected value of future capital rent and hence entrepreneur expected return to capital. Both makes entrepreneur less likely to default and therefore their credit spread falls.

Note: The key is to mention these two risks have opposite effects on capital price.

2. (50) The question is related to the BGG model with financial frictions. Consider a firm that hires labor hours H at the wage rate W and produces goods Y with the following production function:

$$Y = A\omega H. \tag{2}$$

where A denotes the aggregate productivity shock that equals across firms. ω denotes the firm-specific idiosyncratic shock that realizes after the production occurs. ω follows an i.i.d. process across firms and time, drawn from the distribution $F(\cdot)$ with a nonnegative support.

The firm has to pay the wage payments before the production occurs, and finance the wage payments with its own net worth N and bank loans B. This implies that,

$$N + B = WH. \tag{3}$$

After the production occurs, the idiosyncratic shock ω realizes and the firm sells its output Y at the price p.

a. (10) Assume that the bank charges a state-contingent interest rate Z on its loans. There then exists a cutoff ratio $\bar{\omega}$ such that the firm will be not able to repay the bank loans and default if $\omega < \bar{\omega}$. Please write down the expression for $\bar{\omega}$.

$$PA\bar{\omega}H = ZB.$$

b. (10) If the firm defaults, the bank pays a liquidation cost to obtain the firm's generated revenues. The liquidation cost is a fraction μ of the firm's production revenues. Please derive expressions for the firm's expected income and the bank's expected income before the idiosyncratic shock ω realizes.

Answer: The firm's expected income is given by,

$$\int_{\bar{\omega}}^{+\infty} (PA\omega H - ZB)dF(\omega) = \int_{\bar{\omega}}^{+\infty} (\omega - \bar{\omega})dF(\omega)PAH.$$

The bank's expected income is given by,

$$\int_0^{\bar{\omega}} (1-\mu)\omega dF(\omega)PAH + [1 - F(\bar{\omega})]ZB = \{\int_0^{\bar{\omega}} (1-\mu)\omega dF(\omega) + [1 - F(\bar{\omega})]\bar{\omega}\}PAH.$$

c. (10) The bank is risk neutral and has a participation constraint that requires that its expected income to be able to cover its funding costs RB, where R is the risk-free interest rate. Please write down the expression for the bank's participation constraint.

Answer:

$$\{\int_0^{\bar{\omega}} (1-\mu)\omega dF(\omega) + [1 - F(\bar{\omega})]\bar{\omega}\}PAH \geq RB.$$

d. (10) Please solve for the optimal financial contract that maximizes the firm's expected income subject to the bank's participation constraint.

Answer: Rewrite the bank's participation constraint as:

$$\{\int_0^{\bar{\omega}} (1-\mu)\omega dF(\omega) + [1-F(\bar{\omega})]\bar{\omega}\}PAH \geq R(WH-N).$$

Denote λ as the Lagrange multiplier on the bank's participation constraint. Define:

$$g(\bar{\omega}) = \int_{\bar{\omega}}^{+\infty} (\omega - \bar{\omega}) dF(\omega).$$

$$h(\bar{\omega}) = \{\int_0^{\bar{\omega}} (1-\mu)\omega dF(\omega) + [1-F(\bar{\omega})]\bar{\omega}\}.$$

The optimal financial contract problem is given by,

$$\max g(\bar{\omega})PAH$$

subject to:

$$h(\bar{\omega})PAH \geq R(WH-N).$$

The first order condition on H is given by,

$$g(\bar{\omega})PA + \lambda[h(\bar{\omega})PA - RW] = 0.$$

The first order condition on $\bar{\omega}$ is given by,

$$g'(\bar{\omega})PAH + \lambda h'(\bar{\omega})PAH = 0.$$

Using the above equations together provides the optimal financial contract:

$$\frac{g(\bar{\omega})PA}{h(\bar{\omega})PA - RW} = \frac{g'(\bar{\omega})}{h'(\bar{\omega})}.$$

where

$$g'(\bar{\omega}) = -1 + F(\bar{\omega}).$$

$$h'(\bar{\omega}) = -\mu f(\bar{\omega})\bar{\omega} + 1 - F(\bar{\omega}).$$

e. (10) Please consider an increase in aggregate productivity A and answer how an increase in A will affect the output. Please explain why this output effect can be amplified in the presence of financial frictions. (Hint: You should describe how the increase in A will affect the firm's net worth and how changes in the firm's net worth will in turn affect the output.)

Answer: An increase in the aggregate productivy will increase the output directly, because you can produce more for a certain amount of labor inputs.

This positive output effect can be amplified because the increase in A raises the firm's realized return to labor inputs and therefore leads to a rise in firm's net worth. In the

BGG framework,firms' borrowing capacity depends on their net worth. The higher net worth makes firms borrow more and hire more workers, which amplifies the rise in output.