Advanced Microeconomics I
Note 8: Choice under uncertainty

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Lotteries

- Individual decision-making under uncertainty
- Let C denote the set of all possible outcomes (alternatives, or, *consequences*).
- An economic agent's problem is not to choose an alternative, but a *risky alternative*.
- A risky alternative is represented by a **lottery**.
- We first assume that C is finite and $|C| = N$.
- A **simple lottery** is a list $L = (p_1, ..., p_N)$ with $p_n \geq 0$ for all $n = 1, ..., N$ and $\sum_{n=1}^{N} p_n = 1$.
- $\mathcal{L} = \left\{ (p_1, ..., p_N) \in \mathbb{R}_+^N : \sum_{n=1}^{N} p_n = 1 \right\}$ is the set of all the possible simple lotteries.

- A **compound lottery** is a list $(L_1, ..., L_K; \alpha_1, ..., \alpha_K)$ where each $L_k = (p_1^k, ..., p_N^k)$ is a simple lottery, $\alpha_k \geq 0$ for all k, and $\sum_{k=1}^K \alpha_k = 1$.
  - That is, a compound lottery is a probability distribution over some simple lotteries.
- Given a compound lottery $(L_1, ..., L_K; \alpha_1, ..., \alpha_K)$, the ultimate probability of outcome n is
$$\alpha_1 p_n^1 + \alpha_2 p_n^2 + ... + \alpha_K p_n^K$$
- So, in terms of the final distribution over outcomes, the compound lottery can be reduced to the simple lottery $\sum_{k=1}^K \alpha_k L_k \in \mathcal{L}$.
- Different compound lotteries may be reduced to the same simple lottery.

- Example: $N = 3$. Consider the following two compound lotteries

$$\left\{(1,0,0),(\frac{1}{4},\frac{3}{8},\frac{3}{8}),(\frac{1}{4},\frac{3}{8},\frac{3}{8});\;\; (\frac{1}{3},\frac{1}{3},\frac{1}{3})\right\}$$

$$\left\{(1,0,0),(0,\frac{1}{2},\frac{1}{2});\;\; (\frac{1}{2},\frac{1}{2})\right\}$$

They are both reduced to the simple lottery

$$(\frac{1}{2},\frac{1}{4},\frac{1}{4})$$

- We make a consequentialist assumption: the agent only cares about the final distribution over the sure outcomes C.

- We will adopt the preference-based approach and assume that the agent has some preference relation $\succeq$ defined on $\mathcal{L}$.
- In the rest of the note, we focus on two topics
  - Expected utility theory
  - Risk preferences

Expected utility theory

- Assume that $\succeq$ on $\mathcal{L}$ is *rational*.
- $\succeq$ on $\mathcal{L}$ is *continuous* if for any sequence $\{L^k\} \subseteq \mathcal{L}$ with $L^k \to L$, and $L' \in \mathcal{L}$ we have (1) $L^k \succeq L'$ for all k implies $L \succeq L'$, and (2) $L' \succeq L^k$ for all k implies $L' \succeq L$.
- If $\succeq$ is rational and continuous, then there exists a utility function $U : \mathcal{L} \to \mathbb{R}$ that represents $\succeq$: for any L and L', $L \succeq L'$ if and only if $U(L) \geq U(L')$.
- However, we want to impose more structure on U.
- We call a utility function $u : C \to \mathbb{R}$ a **Bernoulli utility function**.
  - Since we have assumed C is finite, let $u_1, \ldots, u_N$ denote the utilities of the sure outcomes.

The utility function $U : \mathcal{L} \to \mathbb{R}$ has an **expected utility form** if there exists a Bernoulli utility function $u : C \to \mathbb{R}$ such that for every $L = (p_1, ..., p_N) \in \mathcal{L}$,

$$U(L) = p_1 u_1 + ... + p_N u_N$$

A utility function U with the expected utility form is called a **Von Neumann-Morgenstern expected utility function**.

Proposition. *A utility function $U : \mathcal{L} \to \mathbb{R}$ has an expected utility form if and only if it satisfies*

$$U(\sum_{k=1}^{K} \alpha_k L_k) = \sum_{k=1}^{K} \alpha_k U(L_k)$$

for any compound lottery $(L_1, ..., L_K; \alpha_1, ..., \alpha_K)$.

There may not exist an expected utility function that represents a rational and continuous $\succeq$ on $\mathcal{L}$.

- $\succeq$ on $\mathcal{L}$ satisfies the **independence axiom** if for all $L_1, L_2, L_3 \in \mathcal{L}$ and $\alpha \in (0,1)$, we have

$$L_1 \succeq L_2 \text{ if and only if } \alpha L_1 + (1-\alpha)L_3 \succeq \alpha L_2 + (1-\alpha)L_3$$

- The independence axiom is at the heart of the theory of choice under uncertainty, but also controversial.
- If a preference relation $\succeq$ on $\mathcal{L}$ can be represented by an expected utility function, then $\succeq$ must satisfy rationality, continuity and the independence axiom.

Expected Utility Theorem. *Suppose that the preference relation $\succeq$ on the set of lotteries $\mathcal{L}$ satisfies rationality, continuity and the independence axiom, then there exists a Von Neumann-Morgenstern expected utility function that represents $\succeq$.*

Money lotteries

- In the rest of this note, we focus on *money lotteries*.
- The set of possible outcomes is $C = \mathbb{R}_+$.
- A lottery is a *cumulative distribution function* (CDF) $F : \mathbb{R}_+ \to [0, 1]$.
  - For any $x \geq 0$, $F(x)$ is the probability that the realized monetary payoff is less than or equal to x.
  - If f is the probability density function (PDF) associated with the lottery F, then $F(x) = \int_{-\infty}^{x} f(t)dt$.
- Now, $\mathcal{L}$ is the set of all the CDFs on $\mathbb{R}_+$.
- The agent has a preference relation $\succeq$ on $\mathcal{L}$.
- By a generalized version of the expected utility theorem, under similar conditions there exists an expected utility function $U : \mathcal{L} \to \mathbb{R}$ that represents $\succeq$. That is, there also exists a Bernoulli utility function $u : \mathbb{R}_+ \to \mathbb{R}$ such that for all $F \in \mathcal{L}$

$$U(F) = \int u(x)dF(x)$$

- Assume that u is continuous and strictly increasing.

Risk preferences

- Given any lottery F, let F^d denote the degenerate lottery that yields $\int x dF(x)$ with certainty.
- We first define risk preferences without utility functions.

- An agent is **risk averse** if for any $F \in \mathcal{L}$, $F^d \succeq F$.
- The agent is **strictly risk averse** if for any $F \in \mathcal{L}$ with $F \neq F^d$, $F^d \succ F$.
- The agent is **risk neutral** if for any $F \in \mathcal{L}$, $F^d \sim F$.

- Sometimes, we also say that the agent is **risk loving** if for any $F \in \mathcal{L}$, $F \succeq F^d$.

- The agent is risk averse if and only if

$$\int u(x)dF(x) \leq u\left(\int x dF(x)\right) \quad \text{for all } \; F \in \mathcal{L}$$

  - The expected utility is less than or equal to the utility of expected value.
- This inequality is actually *Jensen's inequality*, and it is a defining property of concave functions.
- Hence, (strict) risk aversion is equivalent to (strict) concavity of u.
- Being risk loving is equivalent to convexity of u.
- Being risk neutral is equivalent to linearity of u.

- Two useful concepts for analyzing risk preferences: *certainty equivalent* and *probability premium*.

- The **certainty equivalent** of a lottery F, denoted $c(F,u)$, is defined in the following equation

$$u(c(F,u)) = \int u(x)dF(x)$$

- $c(F,u)$ is the amount of money which makes the agent indifferent between the lottery F and the certain amount $c(F,u)$.
- If the agent is risk averse, then $c(F,u) \leq \int xdF(x)$.
  - $\int xdF(x) - c(F,u)$ can be interpreted as the amount of expected return the agent wants to pay to get rid of the risk.
- In fact, risk aversion is equivalent to $c(F,u) \leq \int xdF(x)$ for all $F \in \mathcal{L}$:

$$c(F,u) \leq \int xdF(x)$$

$$\Leftrightarrow u\left(c(F,u)\right) \leq u\left(\int xdF(x)\right)$$

$$\Leftrightarrow \int u(x)dF(x) \leq u\left(\int xdF(x)\right)$$

- Fix an amount of money x and a number $\epsilon \in (0, x)$. The **probability premium**, denoted by $\pi(x, \epsilon, u)$, is defined in the following equation

$$u(x) = \left(\frac{1}{2} + \pi(x, \epsilon, u)\right) u(x + \epsilon) + \left(\frac{1}{2} - \pi(x, \epsilon, u)\right) u(x - \epsilon)$$

- $\pi(x, \epsilon, u)$ is the excess in winning probability over fair odds that makes the agent indifferent between the certain amount x, and a gamble between $x + \epsilon$ and $x - \epsilon$.
- If the agent is risk averse, then $\pi(x, \epsilon, u) \geq 0$:

$$u(x) = \frac{1}{2} u(x + \epsilon) + \frac{1}{2} u(x - \epsilon) + \pi(x, \epsilon, u) \left(u(x + \epsilon) - u(x - \epsilon)\right)$$

$\pi(x, \epsilon, u) \geq 0$, since $u(x) \geq \frac{1}{2} u(x + \epsilon) + \frac{1}{2} u(x - \epsilon)$ and $u(x + \epsilon) > u(x - \epsilon)$.

- It can also be shown that $\pi(x, \epsilon, u) \geq 0$ for all x and ϵ implies risk aversion.

In sum, risk aversion can be characterized in multiple ways:

Proposition. *The following statements are equivalent*:

(i) The agent is risk averse.

(ii) u is concave.

(iii) $c(F, u) \leq \int x dF(x)$ *for all* $F \in \mathcal{L}$.

(iv) $\pi(x, \epsilon, u) \geq 0$ *for all* x *and* ϵ.

Applications

St.Petersburg paradox

- Now, I will flip a fair coin.
- I will keep flipping until the first time it comes up heads.
- If heads on flip 1, I will pay you \$2.
- If heads on flip 2, I will pay you \$4.
- Generally, if heads on flip n, I will pay you 2^n.
- How much would you be willing to pay to play this game?

Expected value of this game:

$$\sum_{n=1}^{+\infty} (0.5)^n 2^n = 1 + 1 + 1 + \ldots = +\infty$$

Will you pay infinite amounts of money to play this game?

Assume risk aversion, with (Bernoulli) utility function $u(x) = \sqrt{x}$.

Expected utility from playing the game:

$$\sum_{n=1}^{+\infty} (0.5)^n \sqrt{2^n}$$

We can find the certainty equivalent of this game:

$$\sqrt{c} = \sum_{n=1}^{+\infty} (0.5)^n \sqrt{2^n}$$

Solving this equation, we get

$$c = 5.828429$$

A simple lesson here is that an agent's decision making under uncertainty depends on his preferences, or expected utilities, not on expected values.

Two-Envelope Paradox

You are given a choice between two envelopes. You are told, reliably, that each envelope has some money in it and that one envelope contains twice as much money as the other. You don't know which has the higher amount and which has the lower. You choose one, but are given the opportunity to switch to the other. Would you switch?

Insurance I - a problem of an insurance company

- A consumer has wealth w. With probability $p \in (0,1)$ the consumer will loose L.
- Suppose the cost of full insurance is R and the consumer cannot purchase partial insurance.
- Do not buy the insurance: $(1-p)u(w) + pu(w-L)$
- Buy the insurance: $(1-p)u(w-R) + pu(w-L-R+L) = u(w-R)$
- Optimal premium R^* for the insurance company:

$$(1-p)u(w) + pu(w-L) = u(w-R^*)$$

- If the consumer is strictly risk averse

$$(1-p)w + p(w-L) > w - R^* \Rightarrow R^* > pL$$

  So the consumer is willing to pay more than the expected loss.
- The consumer sells his lottery (or risk) to the insurance company at a negative price.
- Why would the insurance company want to buy the risk from the consumer?
  - difference in risk preferences: the insurance company is usually risk-neutral or much less risk averse than each individual consumer
  - for instance, when the insurance company is risk neutral, its expected profit (or utility) $= (1-p)R^* + p(R^*-L) = R^* - pL > 0$

Insurance II - a consumer's problem

- A strictly risk averse consumer has wealth w. With probability $p \in (0, 1)$ the consumer will loose L.
- The consumer can choose to buy coverage x, with a cost of px.
  - *Actuarially fair* insurance: price per dollar of coverage is equal to the probability of loss
- How much insurance coverage does the consumer want to buy?

$$\underset{x \geq 0}{\text{Max}} \;\; (1-p)u(w-px) + pu(w-px-L+x)$$

$$\text{FOC:} \quad -p(1-p)u'(w-px^*) + p(1-p)u'(w-px^*-L+x^*) \leq 0$$
$$\text{with equality if} \quad x^* > 0$$

If $x^* = 0$, then FOC implies

$$p(1-p)\left[u'(w-L) - u'(w)\right] \leq 0$$

which is not possible since u is strictly concave and u' is strictly decreasing. Hence

$$p(1-p)u'(w-px^*-L+x^*) = p(1-p)u'(w-px^*)$$
$$u'(w-px^*-L+x^*) = u'(w-px^*)$$
$$w-px^*-L+x^* = w-px^*$$
$$x^* = L$$

- So a strictly risk averse consumer will always purchase full insurance if the price is actuarially fair.
- In fact, the first-order method is redundant.
- Alternatively, think about the consumer's decision in the following way.
- If the consumer chooses x, he faces the following lottery, which depends on x

$$\text{with probability } 1-p\text{, he receives } w-px$$
$$\text{with probability } p\text{, he receives } w-px-L+x$$

- The expected value of the consumer's lottery is $w-pL$, which is independent of x.
- By choosing $x=L$, the consumer receives $w-pL$ for sure.

Investment

- A risk averse agent has wealth w and wants to invest his wealth into two assets.
- For each dollar invested in the safe asset, the return is 1 dollar.
- For each dollar invested in the risky asset, the return is a random variable z with the distribution $F(z)$ and $\int z dF(z) > 1$.
- The agent's problem is to choose the optimal amount of wealth, α, to invest in the risky asset:

$$\underset{\alpha \in [0,w]}{\text{Max}} \quad \int u(\alpha z + w - \alpha) dF(z)$$

If at the optimal, $\alpha^* = 0$, then

$$\int u'(\alpha^* z + w - \alpha^*)(z - 1) dF(z) > 0$$

So we must have $\alpha^* > 0$.

- If the risk is *actuarially favorable*, then a risk averse agent will accept at least some of it.