Advanced Microeconomics I
Note 4: Utility maximization and expenditure minimization

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The utility maximization problem

Suppose that the consumer has a preference relation $\succeq$ on $X = \mathbb{R}^L_+$. The consumer's problem is to choose the best bundles from the budget set $B_{p,w}$ (*preference maximization*).

$$C_{\succeq}(B_{p,w}) = \{x \in B_{p,w} : x \succeq y, \forall y \in B_{p,w}\}$$

Assume that $\succeq$ can be represented by a utility function $u : \mathbb{R}^L_+ \to \mathbb{R}$. Then the consumer's problem can be transformed to the following *utility maximization problem* (UMP).

$$\text{Max} \;\; u(x)$$

$$s.t. \;\; x \geq 0, \;\; p \cdot x \leq w$$

Notice that, the solution to the consumer's problem of choosing the best bundles based on $\succeq$ does not depend on the specific utility representation. That is, for any u that represents $\succeq$:

$$C_{\succeq}(B_{p,w}) = \arg\max_{x \in B_{p,w}} u(x), \;\; \text{for all} \;\; B_{p,w}$$

Proposition. *If u is continuous, then UMP has a solution.*

Proof. The existence follows from the fact that a continuous function has a maximum value on a compact set. It is easy to see that $B_{p,w}$ is bounded. To see that it is closed, consider any $\{x^n\} \subseteq B_{p,w}$ with $x^n \to x$. Then $p \cdot x^n \to p \cdot x$. Since $p \cdot x^n \leq w$ for all n, we have $p \cdot x \leq w$. Moreover, $x \geq 0$ since $x^n \geq 0$ for all n. Hence $x \in B_{p,w}$ and $B_{p,w}$ is closed. $\square$

Walrasian demand correspondence

Suppose that u is continuous. Given any $p \gg 0$ and $w > 0$, the solution set of UMP is denoted as $x(p, w)$: the *Walrasian demand correspondence*.

Proposition. *Suppose that u is a continuous utility function representing $\succeq$. Then $x(p, w)$ has the following properties:*

(i) $x(p, w)$ is homogeneous of degree zero in (p,w).

(ii) If $\succeq$ is locally nonsatiated, then $x(p, w)$ satisfies Walras' law.

(iii) If $\succeq$ is convex, then $x(p, w)$ is a convex set. If $\succeq$ is strictly convex, then $x(p, w)$ is a singleton.

Using calculus to solve UMP

$$\text{Max } \ u(x)$$
$$s.t. \ \ x \geq 0, \ \ p \cdot x \leq w$$

Suppose that u is differentiable. The Lagrangian

$$\mathcal{L}(x, \lambda) = u(x) + \lambda(w - p \cdot x)$$

Kuhn-Tucker conditions

$$\frac{\partial u(x^*)}{\partial x_l} - \lambda p_l \leq 0, \ \ \text{with equality if } \ x_l^* > 0, \forall l$$
$$\lambda(w - p \cdot x^*) = 0, \ \ \lambda \geq 0$$

If $x^* \gg 0$, then

$$MRS_{lk}(x^*) = \frac{\frac{\partial u(x^*)}{\partial x_l}}{\frac{\partial u(x^*)}{\partial x_k}} = \frac{p_l}{p_k}$$

where $MRS_{lk}(x^*)$ is the *marginal rate of substitution* of good l for good k at x^*.

How to interpret the Lagrangian multiplier λ? It measures the marginal effect of w on the maximized utility level:

assume that the underlying preference relation is locally nonsatiated, $x(p, w)$ is a differentiable **function** and $x(p, w) \gg 0$, then

$$\begin{aligned}
\frac{\partial u(x(p, w))}{\partial w} &= \sum_{l=1}^{L} \frac{\partial u(x(p, w))}{\partial x_l} \frac{\partial x_l(p, w)}{\partial w} \\
&= \sum_{l=1}^{L} \lambda p_l \frac{\partial x_l(p, w)}{\partial w} \\
&= \lambda
\end{aligned}$$

where the last equality follows from *Engel aggregation*.

Some concrete UMP

- *Cobb-Douglas* utility function: $u(x_1, x_2) = x_1^\alpha x_2^\beta$, where $\alpha > 0, \beta > 0$.
  - Notice that it is homogeneous of degree $\alpha + \beta$.
  - UMP has a unique interior solution: $x_1(p, w) = \frac{w}{p_1}\frac{\alpha}{\alpha+\beta}$, $x_2(p, w) = \frac{w}{p_2}\frac{\beta}{\alpha+\beta}$.
  - The **maximized utility** is $[\frac{w}{p_1}\frac{\alpha}{\alpha+\beta}]^\alpha[\frac{w}{p_2}\frac{\beta}{\alpha+\beta}]^\beta$
- *Leontief* utility function (perfect complements): $u(x_1, x_2) = \min\{x_1, x_2\}$
  - not differentiable
- Linear utility function (perfect substitutes): $u(x_1, x_2) = x_1 + x_2$
  - differentiable, but don't need to differentiate

Indirect utility function

Fix a continuous utility function u. Given any $p \gg 0$ and $w > 0$, the maximized utility level is denoted $v(p, w)$: the **indirect utility function**.

That is, if $x \in x(p, w)$, $v(p, w) = u(x)$. Moreover, for any $y \in B_{p,w}$, $v(p, w) \geq u(y)$.

Proposition. *Suppose that u is a continuous utility function representing $\succeq$. Then $v(p, w)$ is*

(i) Homogeneous of degree zero in (p,w).

(ii) Strictly increasing in w if $\succeq$ is locally nonsatiated.

(iii) Nonincreasing in p_l for any l.

(iv) Quasiconvex.

Proof of (ii). Let $w' > w$, and $x \in x(p, w)$. Since $p \cdot x < w'$, there exists some $\epsilon > 0$ such that $p \cdot y < w'$ for all $y \in X$ with $\|y - x\| < \epsilon$. By local nonsatiation, there exists some $y \in X$ such that $p \cdot y < w'$ and $y \succ x$. Since $y \in B_{p,w'}$, we have $v(p, w') \geq u(y)$. Hence $v(p, w') \geq u(y) > u(x) = v(p, w)$.

Proof of (iii). Let $p = (p_1, ..., p_l, ..., p_L)$, $p' = (p_1, ..., p_l', ..., p_L)$ and $p_l' > p_l$. Let $x \in x(p', w)$. Since $x \in B_{p,w}$, we have $v(p, w) \geq u(x) = v(p', w)$.

Proof of (iv). Consider any (p, w), (p', w') and $\alpha \in [0, 1]$. Denote $\bar{p} = \alpha p + (1 - \alpha)p'$ and $\bar{w} = \alpha w + (1 - \alpha)w'$. We want to show

$$v(\bar{p}, \bar{w}) \leq \max\{v(p, w), v(p', w')\} \tag{1}$$

Let $x \in x(\bar{p}, \bar{w})$. Then $\bar{p} \cdot x \leq \bar{w}$ implies

$$\alpha p \cdot x + (1 - \alpha)p' \cdot x \leq \alpha w + (1 - \alpha)w'$$

It follows that we have either $p \cdot x \leq w$ or $p' \cdot x \leq w'$. In the former case, $v(p, w) \geq u(x) = v(\bar{p}, \bar{w})$; in the latter case, $v(p', w') \geq u(x) = v(\bar{p}, \bar{w})$. Hence (1) is proved. $\square$

The expenditure minimization problem

Given a utility function $u(x)$, $p \gg 0$ and a utility level u, we consider the *expenditure minimization problem* (EMP):

$$\text{Min} \;\; p \cdot x$$

$$s.t. \;\; x \geq 0, \; u(x) \geq u$$

A solution to EMP exists under very general conditions. If u is continuous and there exists some $x' \geq 0$ such that $u(x') \geq u$, then a solution exists, since in this case EMP is equivalent to the following problem with a compact and nonempty constraint set:

$$\text{Min} \;\; p \cdot x$$

$$s.t. \;\; x \geq 0, \; u(x) \geq u, \;\; p \cdot x \leq p \cdot x'$$

(From now on, we only consider the case that $u > u(0)$, and assume that for any $u > u(0)$, there exists $x \geq 0$ with $u(x) \geq u$.)

Hicksian demand correspondence

Given any $p \gg 0$ and u, the solution set of EMP is denoted as $h(p, u)$: the **Hicksian demand correspondence**.

Proposition. *Suppose that u is a continuous utility function representing* $\succeq$*, then* $h(p, u)$ *has the following properties:*

(i) $h(p, u)$ *is homogeneous of degree zero in* p.

(ii) No excess utility: for any $x \in h(p, u)$, $u(x) = u$.

(iii) If $\succeq$ *is convex, then* $h(p, u)$ *is a convex set. If* $\succeq$ *is strictly convex, then* $h(p, u)$ *is a singleton.*

Using calculus to solve EMP

$$\text{Min } \ p \cdot x$$

$$s.t. \ \ x \geq 0, \ u(x) \geq u$$

Suppose that u is differentiable. The Lagrangian

$$\mathcal{L}(x, \lambda) = p \cdot x + \lambda(u - u(x))$$

Kuhn-Tucker conditions

$$\forall l : \quad p_l - \lambda \frac{\partial u(x^*)}{\partial x_l} \geq 0, \ \text{ with equality if } \ x_l^* > 0$$

$$\lambda \geq 0, \ \ \lambda(u - u(x^*)) = 0$$

Example: Cobb-Douglas utility function $u(x_1, x_2) = x_1^{\alpha} x_2^{\beta}$, where $\alpha > 0, \beta > 0$.

EMP has a unique interior solution if $u > 0$. For simplicity, let $\beta = 1 - \alpha$, then

$$h_1(p, u) = [\frac{\alpha}{1-\alpha} \cdot \frac{p_2}{p_1}]^{1-\alpha} u$$

$$h_2(p, u) = [\frac{1-\alpha}{\alpha} \cdot \frac{p_1}{p_2}]^{\alpha} u$$

Then the **minimized expenditure** is

$$p_1[\frac{\alpha}{1-\alpha} \cdot \frac{p_2}{p_1}]^{1-\alpha} u + p_2[\frac{1-\alpha}{\alpha} \cdot \frac{p_1}{p_2}]^{\alpha} u$$

Expenditure function

Fix a continuous utility function u. Give any $p \gg 0$ and u, the minimized expenditure level is denoted $e(p, u)$: the **expenditure function**.

That is, if $x \in h(p, u)$, then $e(p, u) = p \cdot x$. Moreover, if $y \geq 0$ and $u(y) \geq u$, then $p \cdot y \geq e(p, u)$.

Proposition. *Suppose that u is a continuous function. Then $e(p, u)$ is*

(i) Homogeneous of degree one in p.

(ii) Strictly increasing in u.

(iii) Nondecreasing in p_l for any l.

(iv) Concave in p.

Concavity of $e(p, u)$ is a very important property and has a nice graphical interpretation.

Proof of (ii). Let $u' > u$. Assume to the contrary, $e(p, u') \leq e(p, u)$. Consider any $x \in h(p, u')$. Since $u(x) \geq u' > u$ and $p \cdot x = e(p, u') \leq e(p, u)$, we have $x \in h(p, u)$, but this contradicts to "no excess utility", given that u is continuous.

$\square$