Advanced Microeconomics I
Note 9: Normal form games

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Introduction

What is game theory?

- **Robert J. Aumann, 1985:** Briefly put, game and economic theory are concerned with the interactive behavior of rational man. . . [An] important function of game theory is the classification of interactive decision situations.
- **Roger B. Myerson, 1991:** Game theory can be defined as the study of mathematical models of conflict and cooperation between intelligent rational decision-makers. Game theory provides general mathematical techniques for analyzing situations in which two or more individuals make decisions that will influence one another's welfare.
- **Nobel Prize Citation, 1994:** Game theory is a mathematical method for analyzing strategic interaction.

- Economic applications of game theory
  - Trading processes (auctions, bargaining)
  - Labor and financial markets
  - Decision problems in organizations: performance wages, competing for promotions, competing for resources
  - International economics: countries choose tariffs and trade policies
  - Macroeconomics: effects of various government policies
- Game theory is also used in political science, psychology, philosophy, logic, computer science, biology, etc.

- **Noncooperative games**: *self-interested* and *rational* decision-makers act independently.
  - The main focus in our course.
- **Cooperative games**: situations where groups of agents can make binding agreements.
  - The primitives are groups or subgroups of agents.

- A simple classification of noncooperative games:
  - Simultaneous move games v.s. dynamic games
  - Games with complete (perfect) information v.s. games with incomplete (imperfect) information

- To describe a game, we need the following four elements:
  - Players: agents who are involved in the strategic situation
  - Rules of the game: timeline, information, possible actions
  - Outcomes: what will happen as a result of each possible combination of actions
  - Payoffs: preferences over the possible outcomes (utilities, profits, etc.)

- Example: *Matching Pennies* game.
  - Players: player 1 and player 2
  - Rules: each player simultaneously puts down a penny.
  - Outcomes: If the pennies match (both heads up or both tails up), player 1 pays \$1 to player 2; otherwise player 2 pays \$1 to player 1.
  - Payoffs
- Example: *Meeting in New York* game.
  - Players: player 1 and player 2
  - Rules: they are separated and cannot communicate. They are supposed to meet in NYC for lunch but have forgotten to specify where. They can either go to Empire State Building or Grand Central Station.
  - Outcomes: If they meet each other, they get to enjoy each other's company at lunch; otherwise they have to eat alone.
  - Payoffs: they each attach a value of \$100 to the other's company.

- The matching pennies game is a *zero-sum game* with pure conflicts of interests, while the meeting in New York game is a coordination game.
  - Noncooperative games are not limited to pure or even partial conflicts. They can also be used to study cooperation.

Norm form representation

- Two ways to model a game formally and mathematically: **Normal Form** and **Extensive Form**.
- In this note, we focus on **simultaneous move games**, and normal form representation is sufficient.

- A normal form game: $\Gamma = [N, \{S_i\}_{i\in N}, \{u_i\}_{i\in N}]$

- $N = \{1, 2, ..., n\}$: the set of players
- S_i: the set of all possible **strategies** of player $i \in N$
- A **strategy profile**: $s = (s_1, s_2, ..., s_n)$, where $s_i \in S_i$ for each i
- $S = \prod_{i\in N} S_i$ (Cartesian product) is the set of all strategy profiles.
  - $s_{-i} = (s_1, ..., s_{i-1}, s_{i+1}, ..., s_n) \in S_{-i}$ denotes a combination of strategies of all players except i.
- Then $u_i : S \to \mathbb{R}$ is player i's utility/payoff function.
  - Player i's utility depends not only on s_i but also on s_{-i}: strategic interaction.

The Matching Pennies game can be represented as $\Gamma = [N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N}]$, where

- $N = \{\text{player 1, player 2}\}$
- $S_1 = \{H, T\}$, $S_2 = \{H, T\}$
  - four possible strategy profiles: $S_1 \times S_2 = \{(H, H), (H, T), (T, H), (T, T)\}$
- $u_1(H, H) = -1, u_1(H, T) = 1, u_1(T, H) = 1, u_1(T, T) = -1$
  $u_2(H, H) = 1, u_2(H, T) = -1, u_2(T, H) = -1, u_2(T, T) = 1$

Simple normal form games can be represented using matrices (game tables):

	H	T
H	-1,1	1, -1
T	1,-1	-1, 1

Example: *Prisoner's Dilemma* game.

	S	C
S	-2,-2	-10, -1
C	-1,-10	-5, -5

- $N = \{$row player (prisoner 1), column player (prisoner 2)$\}$
- $S_1 = \{$S (stay silent), C (confess)$\}$, $S_2 = \{$S (stay silent), C (confess)$\}$
  - Four possible strategy profiles: $S_1 \times S_2 = \{(S, S), (S, C), (C, S), (C, C)\}$
- $u_1(S, S) = -2, u_1(S, C) = -10, u_1(C, S) = -1, u_1(C, C) = -5$
  $u_2(S, S) = -2, u_2(S, C) = -1, u_2(C, S) = -10, u_2(C, C) = -5$

- Assume that the game structure $\Gamma = [N, \{S_i\}_{i\in N}, \{u_i\}_{i\in N}]$ is *common knowledge* among all players.
  - Of course, this is much stronger than requiring complete information.
- Usually the term "common knowledge" is used loosely (what exactly common knowledge means is by no means common knowledge).
- An event E is **common knowledge** if (1) everyone knows E, (2) everyone knows that everyone knows E, and so on ad infinitum.
  - Why bother? Game theory typically involves situations where players engage in strategic thinking. It is important and fundamental to account for players' knowledge/beliefs about uncertainty (physical environment as well as opponents' knowledge/beliefs).

- Given the game $\Gamma = [N, \{S_i\}_{i\in N}, \{u_i\}_{i\in N}]$, strictly speaking each $s_i \in S_i$ is a **pure strategy**. A player can also randomize over his pure strategies, giving rise to *mixed strategies*.
- Given player i's (finite) set of pure strategies S_i, a **mixed strategy** for player i, $\sigma_i : S_i \to [0,1]$, assigns each of his pure strategy $s_i \in S_i$ a probability $\sigma_i(s_i)$, where $\sum_{s_i\in S_i} \sigma_i(s_i) = 1$.
- Let $\Delta(S_i)$ denote player i's set of all possible mixed strategies.
- When each player chooses a mixed strategy, we have a mixed strategy profile $\sigma = (\sigma_1, ..., \sigma_n) \in \prod_{i\in N} \Delta(S_i)$, which induces a probability distribution over pure strategy profiles: for each $s = (s_1, ..., s_n) \in S$, the probability of s is

$$\prod_{i\in N} \sigma_i(s_i) \quad \text{(assume each player randomizes on his own)}$$

- For player i, u_i can be considered as his Bernoulli utility function. Hence, given a mixed strategy profile σ, his Von Neumann-Morgenstern expected utility is

$$u_i(\sigma) = \sum_{s\in S} \left\{ \left[\prod_{j\in N} \sigma_j(s_j) \right] u_i(s) \right\}$$

$$u_i(\sigma) = \sum_{s \in S} \left\{ \left[\prod_{j \in N} \sigma_j(s_j) \right] u_i(s) \right\}$$

A player's (expected utility) from a mixed strategy profile has two other useful representations:

$$u_i(\sigma) = u_i(\sigma_i, \sigma_{-i}) = \sum_{s_i \in S_i} \{\sigma_i(s_i)\ u_i(s_i, \sigma_{-i})\}$$

$$u_i(\sigma) = u_i(\sigma_i, \sigma_{-i}) = \sum_{s_{-i} \in S_{-i}} \left\{ \prod_{j \neq i} \sigma_j(s_j)\ u_i(\sigma_i, s_{-i}) \right\}$$

If mixed strategies are allowed, we use $\Gamma = [N, \{\Delta(S_i)\}_{i \in N}, \{u_i\}_{i \in N}]$ to represent a normal form game.

Dominance

- Given a game, how would the players play the game, or what is the most likely outcome?
- We need **solution concepts**.
- We start from a non-equilibrium solution concept: **dominance**.
- If a rational player has a unique strategy that is obviously the best one, he should play this strategy.

Prisoner's Dilemma game.

	S	C
S	-2,-2	-10, -1
C	-1,-10	-5, -5

For each player, C is his best strategy, regardless what the other player chooses.

We focus on pure strategies first.

In the game $\Gamma = [N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N}]$, a strategy $s_i \in S_i$ is a **strictly dominant strategy** for player i if

$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$$

for all $s_i' \in S_i \setminus \{s_i\}$ and $s_{-i} \in S_{-i}$.

Hence in the Prisoner's Dilemma game, we predict that rational players will play (C, C): self-interested and rational behavior leads to suboptimal or Pareto inefficient outcome.

Strictly dominant strategies often do not exist. But the idea of dominance can be used to eliminate some "bad" strategies.

In the game $\Gamma = [N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N}]$, a strategy $s_i \in S_i$ is **strictly dominated** for player i if there exists another strategy $s_i' \in S_i$ such that for all $s_{-i} \in S_{-i}$

$$u_i(s_i', s_{-i}) > u_i(s_i, s_{-i})$$

In this case, we also say that s_i' **strictly dominates** s_i.

Obviously, a strategy $s_i \in S_i$ is a strictly dominant strategy for player i in the game $\Gamma = [N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N}]$ if and only if s_i strictly dominates every other strategy in S_i.

	L	R
U	1,-1	-1, 1
M	-1,-1	1, -1
D	-2,5	-3, 2

D is strictly dominated for player 1.

	L	R
U	1,-1	-1, 1
M	-1,-1	1, -1
D	0,5	0, 2

Is *D* strictly dominated for player 1?

There is also a weaker notion of dominance.

In the game $\Gamma = [N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N}]$, a strategy $s_i' \in S_i$ **weakly dominates** $s_i \in S_i$ for player i if

$$u_i(s_i', s_{-i}) \geq u_i(s_i, s_{-i})$$

for all $s_{-i} \in S_{-i}$, and for some $s_{-i} \in S_{-i}$

$$u_i(s_i', s_{-i}) > u_i(s_i, s_{-i})$$

Then in the game $\Gamma = [N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N}]$, a strategy $s_i \in S_i$ is **weakly dominated** if there exists $s_i' \in S_i$ that weakly dominates s_i. A strategy is **weakly dominant** for player i if it weakly dominates every other strategy of player i.

	L	R
U	5,1	4, 0
M	6,0	3, 1
D	6,4	4, 4

Both U and M are weakly dominated for player 1.

However, we cannot justify eliminating weakly dominated strategies.

Moreover, elimination of weakly dominated strategies might be quite problematic, which will be discussed in more details later.

Now, consider the following example.

	L	R
U	10,0	0,0
M	0,0	0,0
D	0,0	10,0

Although M is not strictly dominated by any pure strategy, it is "strictly dominated" by the mixed strategy of $\frac{1}{2}U + \frac{1}{2}D$.

From now on, we allow mixed strategies.

In the game $\Gamma = [N, \{\Delta(S_i)\}_{i \in N}, \{u_i\}_{i \in N}]$, a strategy $\sigma_i' \in \Delta(S_i)$ **strictly dominates** $\sigma_i \in \Delta(S_i)$ for player i if for all $\sigma_{-i} \in \prod_{j \neq i} \Delta(S_j)$

$$u_i(\sigma_i', \sigma_{-i}) > u_i(\sigma_i, \sigma_{-i})$$

In the game $\Gamma = [N, \{\Delta(S_i)\}_{i \in N}, \{u_i\}_{i \in N}]$, a strategy $\sigma_i \in \Delta(S_i)$ is **strictly dominated** if there exists $\sigma_i' \in \Delta(S_i)$ that strictly dominates σ_i. A strategy is **strictly dominant** for player i if it strictly dominates every other strategy of player i.

A strictly dominant strategy, if it exists, must be a pure strategy.

Allowing mixed strategies helps us eliminate more (pure) strategies. When we want to check whether a strategy is strictly dominated for player i, it is sufficient to check against pure strategies of i's opponents:

$$u_i(\sigma_i', \sigma_{-i}) > u_i(\sigma_i, \sigma_{-i}) \text{ for all } \sigma_{-i} \in \prod_{j \neq i} \Delta(S_j)$$

if and only if

$$u_i(\sigma_i', s_{-i}) > u_i(\sigma_i, s_{-i}) \text{ for all } s_{-i} \in S_{-i}$$

When a pure strategy s_i is strictly dominated, so is any mixed strategy that assigns s_i a positive probability. But a mixed strategy can also be strictly dominated even if it does not assign any positive probability to any strictly dominated pure strategy (homework question).

- A rational player will never use a strictly dominated pure strategy. So we can eliminate this strategy from the original game.
- After eliminating all the strictly dominated pure strategies for the players, some pure strategies, which are not strictly dominated before, may become strictly dominated in the reduced game. We can eliminate these strategies and continue this process until no more strategies can be eliminated: **iterative elimination of strictly dominated strategies** (ISD).
- Example:

	L	R
T	3,1	2,2
M	0,0	3,1
B	1,1	1,0

Eliminate $B \Rightarrow$ eliminate $L \Rightarrow$ eliminate $T \Rightarrow$ unique outcome (M, R)

- If ISD leads to a unique outcome, then this game is **dominance solvable** (Moulin, 1979).
- Notice that, ISD requires more than just rationality, it requires **common knowledge of rationality**.

- A very nice feature of ISD is that this procedure is *order-independent* for finite games.
  - Given a normal form game $\Gamma = [N, \{\Delta(S_i)\}_{i \in N}, \{u_i\}_{i \in N}]$, where each S_i is finite, the set of pure strategies that remains after iteratively removing strictly dominated pure strategies does not depend on the order in which the strictly dominated pure strategies are removed.
- In contrast, iterative elimination of weakly dominated strategies is not order independent.
- Example:

	L	R
U	5,1	4, 0
M	6,0	3, 1
D	6,4	4, 4

$$U \Rightarrow L \Rightarrow M \Rightarrow (D, R)$$
$$M \Rightarrow R \Rightarrow U \Rightarrow (D, L)$$

Beauty contest (John Maynard Keynes, 1936, Moulin, 1986)

Everyone simultaneously picks a number between 0 and 100. The winner of the contest is the person(s) whose number is closest to $\frac{2}{3}$ times the average of all numbers submitted.

Beauty Contest is often presented as one (simple) model of stock markets. When making an investment, some investors are not necessarily trying to identify desirable companies, but rather are attempting to identify those assets that others will consider to be desirable (and purchase these assets before their prices are driven up by the masses).

Nash Equilibrium

- Most games are not dominance solvable.
- We now turn to *Nash Equilibrium* (Nash, 1950), which is the most widely used solution concept in game theory.

  In the game $\Gamma = [N, \{\Delta(S_i)\}_{i \in N}, \{u_i\}_{i \in N}]$, a strategy profile $\sigma = (\sigma_1, ..., \sigma_n)$ is a **Nash Equilibrium** if for every player $i = 1, ..., n$

  $$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i})$$

  for all $\sigma_i' \in \Delta(S_i)$.

- Therefore, a strategy profile is a Nash equilibrium if and only if given others' strategies, no one has an incentive to deviate.

- An alternative way to describe a Nash equilibrium is to use *best responses*.
- For each player i, his **best response correspondence** is $b_i : \prod_{j \neq i} \Delta(S_j) \to \Delta(S_i)$, where for each $\sigma_{-i} \in \prod_{j \neq i} \Delta(S_j)$

$$b_i(\sigma_{-i}) = \{\sigma_i \in \Delta(S_i) : u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i}), \forall \sigma_i' \in \Delta(S_i)\}$$

- Then, a strategy profile σ is a Nash equilibrium if and only if

$$\sigma_i \in b_i(\sigma_{-i}) \text{ for all } i \in N$$

- Hence, in a Nash equilibrium, every player is best-responding to other players.

Example:

	L	C	R
U	10,0	7,9	15,8
D	10,15	5,11	12,12

Pure strategy NE can be found using cell-by-cell inspection: $(U, C), (D, L)$

Matching Pennies game.

	H	T
H	-1,1	1, -1
T	1,-1	-1, 1

There does not exist any pure strategy NE.

But there is a mixed strategy NE: $(\frac{1}{2}H + \frac{1}{2}T, \frac{1}{2}H + \frac{1}{2}T)$.

It is not surprising to see a unique NE in mixed strategies in this game. We have the same conclusion in games like rock–paper–scissors, or penalty-shooting in soccer. Mixed strategies are indeed often observed in these games (one possible exception would be the 2008 UEFA Champions League Final).

Notice that each player is indifferent between choosing H and T given that the other player chooses $\frac{1}{2}H + \frac{1}{2}T$. That is, each player randomizes to make the other player indifferent. This is a general principle behind any mixed strategy NE.

Proposition. *σ is a Nash equilibrium if and only if for any player i*

(i) $u_i(s_i, \sigma_{-i}) = u_i(s_i', \sigma_{-i})$ for all s_i and s_i' with $\sigma_i(s_i) > 0$ and $\sigma_i(s_i') > 0$.

(ii) $u_i(s_i, \sigma_{-i}) \geq u_i(s_i', \sigma_{-i})$ for all s_i and s_i' with $\sigma_i(s_i) > 0$ and $\sigma_i(s_i') = 0$.

While this result can be easily proved, it has two important implications.

First, the "only if" part provides a guideline for finding mixed strategy NE: in a player's NE strategy, he must be indifferent among all the pure strategies that are assigned a positive probability.

Second, the "if" part provides a useful test for NE. In particular, it implies that, to find a pure strategy NE, it is sufficient to consider the case where every player is only allowed to play pure strategies.

Proof. "only if" part. Assume to the contrary, for some player i, either (i) or (ii) is not true. Then there exist some $s_i^1, s_i^2 \in S_i$ such that $\sigma_i(s_i^1) > 0$ and $u_i(s_i^2, \sigma_{-i}) > u_i(s_i^1, \sigma_{-i})$. Then we can construct the following mixed strategy σ_i^* for player i:

$$\sigma_i^*(s_i^2) = \sigma_i(s_i^1) + \sigma_i(s_i^2)$$

$$\sigma_i^*(s_i^1) = 0$$

$$\sigma_i^*(s_i) = \sigma_i(s_i) \text{ if } s_i \in S_i \setminus \left\{s_i^1, s_i^2\right\}$$

Then we have

$$\begin{aligned} u_i(\sigma_i^*, \sigma_{-i}) &= [\sigma_i(s_i^1) + \sigma_i(s_i^2)]u_i(s_i^2, \sigma_{-i}) + \sum_{s_i \in S_i \setminus \left\{s_i^1, s_i^2\right\}} \sigma_i(s_i)u_i(s_i, \sigma_{-i}) \\ &> \sigma_i(s_i^1)u_i(s_i^1, \sigma_{-i}) + \sigma_i(s_i^2)u_i(s_i^2, \sigma_{-i}) + \sum_{s_i \in S_i \setminus \left\{s_i^1, s_i^2\right\}} \sigma_i(s_i)u_i(s_i, \sigma_{-i}) \\ &= u_i(\sigma_i, \sigma_{-i}) \end{aligned}$$

This contradicts to $\sigma_i \in b_i(\sigma_{-i})$.

"if part". Consider any player i. First, it is obvious that

$$\forall \sigma_i' \in \Delta(S_i), u_i(\sigma_i', \sigma_{-i}) = \sum_{s_i \in S_i} \sigma_i'(s_i) u_i(s_i, \sigma_{-i}) \leq \text{Max}\{u_i(s_i, \sigma_{-i}) : s_i \in S_i\} \quad (1)$$

If (i) and (ii) are true, then

$$u_i(\sigma_i, \sigma_{-i}) = \text{Max}\{u_i(s_i, \sigma_{-i}) : s_i \in S_i\}$$

Hence, according to (1), $\sigma_i \in b_i(\sigma_{-i})$. That is, σ is a NE. □

Example: *Battle of sexes* game

	Soccer	Opera
Soccer	2,1	0,0
Opera	0,0	1,2

Pure strategy NE: $(S, S), (O, O)$.

What about mixed strategy NE?

If there exists a mixed strategy NE (which is not a pure strategy NE), then it must be the case that both players mix.

Suppose $(\alpha S + (1-\alpha)O, \beta S + (1-\beta)O)$ is a NE, with $\alpha \in (0,1)$ and $\beta \in (0,1)$, then by the previous proposition, player 1 is indifferent between S and O:

$$2\beta + 0(1-\beta) = 0\beta + 1(1-\beta) \Rightarrow \beta = \frac{1}{3}$$

Similarly, player 2 is also indifferent between S and O:

$$1\alpha + 0(1-\alpha) = 0\alpha + 2(1-\alpha) \Rightarrow \alpha = \frac{2}{3}$$

Hence the following is a mixed strategy NE:

$$(\frac{2}{3}S + \frac{1}{3}O, \frac{1}{3}S + \frac{2}{3}O)$$

Relationship between NE and dominance

- We briefly investigate the relationship between the two solution concepts so far. In this part, we always assume the game is finite.
- If a game is dominance solvable, the resulting pure strategy profile is a NE.
- Moreover, iterative elimination of strictly dominated strategies will not eliminate any NE.
  - If a game is dominance solvable, then the outcome is the unique NE.
  - To find NE, we can always apply ISD first.
  - Eliminating weakly dominated strategies might eliminate some NE. Example:

	L	R
T	1,1	0,0
B	0,0	0,0

  - However, the beauty contest game is weak dominance solvable and the corresponding outcome is also the unique pure strategy NE of this game.

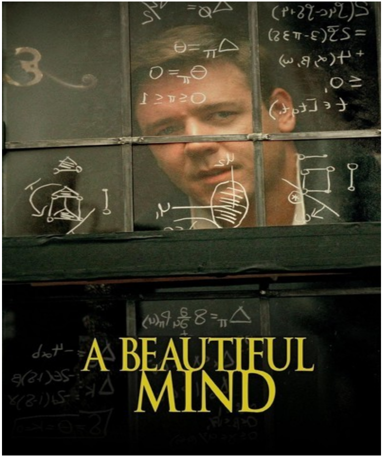
Existence of Nash equilibrium

Nash equilibrium requires that the players have correct conjectures about each other's play. This is a quite strong requirement, but a Nash equilibrium exists in a large class of games.

Theorem. *(Nash, 1950) Every finite normal form game has a Nash equilibrium.*

John Forbes Nash Jr. (June 13, 1928 – May 23, 2015) was an American mathematician who made fundamental contributions to game theory, differential geometry, and the study of partial differential equations.

He shared the 1994 Nobel Memorial Prize in Economic Sciences with game theorists Reinhard Selten and John Harsanyi.



CARNEGIE INSTITUTE OF TECHNOLOGY
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DEPARTMENT OF MATHEMATICS
COLLEGE OF ENGINEERING AND SCIENCE

February 11, 1948

Professor S. Lefschetz
Department of Mathematics
Princeton University
Princeton, N. J.

Dear Professor Lefschetz:

This is to recommend Mr. John F. Nash, Jr. who has applied for entrance to the graduate college at Princeton.

Mr. Nash is nineteen years old and is graduating from Carnegie Tech in June. He is a mathematical genius.

Yours sincerely,

Richard J. Duffin

Richard J. Duffin

RJD:hl

Chess and go are finite games.

To prove this result, we need the following mathematical results.

Given $A \subseteq \mathbb{R}^M$ and a compact set $Y \subseteq \mathbb{R}^K$, the correspondence $f : A \to Y$ is **upper hemicontinuous** if for any two convergent sequences $x^n \to x \in A$ and $y^n \to y$ with $x^n \in A$ and $y^n \in f(x^n)$ for all n, we have $y \in f(x)$.

Kakutani's Fixed Point Theorem. *Suppose that $A \subseteq \mathbb{R}^M$ is a nonempty, convex and compact set, and $f : A \to A$ is an upper hemicontinuous correspondence such that $f(x)$ is a nonempty and convex set for every $x \in A$. Then f has a fixed point. That is, there exists $x \in A$ such that $x \in f(x)$.*

Lemma. *For each player i, $b_i(\sigma_{-i})$ is a nonempty and convex set for all $\sigma_{-i} \in \prod_{j \neq i} \Delta(S_j)$; moreover, b_i is upper hemicontinous.*

Proof. For any σ_{-i}, $b_i(\sigma_{-i})$ is the set of solutions to the problem

$$\text{Max } \; u_i(\sigma_i, \sigma_{-i})$$

$$s.t. \; \sigma_i \in \Delta(S_i)$$

Since u_i is continuous and $\Delta(S_i)$ is compact, $b_i(\sigma_{-i})$ is nonempty. It is also convex since u_i is quasiconcave in σ_i. Finally, to see upper hemicontinuity, consider any two sequences $\sigma^n_{-i} \to \sigma_{-i} \in \prod_{j \neq i} \Delta(S_j)$ and $\sigma^n_i \to \sigma_i$ with $\sigma^n_{-i} \in \prod_{j \neq i} \Delta(S_j)$ and $\sigma^n_i \in b_i(\sigma^n_{-i})$ for all n. For any $\sigma'_i \in \Delta(S_i)$, we have

$$u_i(\sigma^n_i, \sigma^n_{-i}) \geq u_i(\sigma'_i, \sigma^n_{-i})$$

for all n. Then the continuity of u implies

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma'_i, \sigma_{-i})$$

That is, $\sigma_i \in b_i(\sigma_{-i})$. Hence b_i is upper hemicontinuous. $\square$

Proof of the existence of NE.

Define the correspondence $b : \prod_i \Delta(S_i) \to \prod_i \Delta(S_i)$ such that

$$b(\sigma) = b_1(\sigma_{-1}) \times ... \times b_n(\sigma_{-n})$$

$\prod_i \Delta(S_i)$ is nonempty, convex and compact. Using the previous lemma, it is not difficult to see that $b(\sigma)$ is nonempty and convex for any σ, and b is upper hemicontinuous. Therefore, by the Kakutani's fixed point theorem, there exists a fixed point. That is, there exists $\sigma \in \prod_i \Delta(S_i)$ such that $\sigma \in b(\sigma)$. Clearly, σ is a Nash equilibrium. $\square$

Applications - Models of competition

- We briefly discuss some issues regarding market structures.
  - An excellent reference book in this area would be *The Theory of Industrial Organization* by Jean Tirole.
- Besides the perfectly competitive market, another market structure that does not involve strategic interaction is one with a single firm in it: the monopoly.
- The monopoly produces a single output with a cost function $c(\cdot)$, and faces its (inverse) demand curve $p(q)$.
- The monopoly's profit-maximizing problem is

$$\underset{q \geq 0}{\text{Max}} \ p(q)q - c(q)$$

  At an interior solution

$$p'(q^M)q^M + p(q^M) = c'(q^M)$$

- With a downward sloping demand curve, $p'(\cdot) < 0$, hence $p(q^M) > c'(q^M)$. That is, the monopoly's optimal price is above its marginal cost, and the quantity produced is below the Pareto efficient level.

- Consider a concrete case with linear demand $p(q) = a - bq$ and constant marginal cost c. Assume that $a > c > 0, b > 0$.
- Solving the monopoly's problem, we get

$$q^M = \frac{a-c}{2b}, \;\; p^M = \frac{a+c}{2}$$

- In contrast, the socially optimal (or, perfectly competitive) level of output and price are

$$q^* = \frac{a-c}{b}, \;\; p^* = c$$

- We now turn to those intermediate cases between a monopoly and a perfectly competitive market: situations of *oligopoly*. We mainly focus on the case of a *duopoly*.
- Two firms can compete in prices (Bertrand) or in quantities (Cournot).
- Quantity competition: the two firms produce the same product and simultaneously decide on their quantities q_1, q_2.
- They face the (inverse) demand curve $p(q)$ together, where $q = q_1 + q_2$.
- Cost functions: $c_1(q_1)$ and $c_2(q_2)$.
- To solve for the Nash equilibrium (q_1, q_2), we need to find each firm's best response. For firm i, given q_j, its best response q_i is the solution to the following problem

$$\underset{q_i \geq 0}{\text{Max}} \;\; p(q_i + q_j)q_i - c_i(q_i)$$

- For the sake of concreteness, consider the case of linear demand $p(q) = a - bq$ and constant marginal cost c, with $a > c > 0, b > 0$.
- Solving for the best response functions, we get

$$q_1 = \frac{a - c - bq_2}{2b}, \quad q_2 = \frac{a - c - bq_1}{2b}$$

At the intersection of these two best response functions we find the NE:

$$q_1^C = q_2^C = \frac{a - c}{3b}$$

The total quantity produced is therefore $q^C = \frac{2(a-c)}{3b}$, and the resulting market price is $p^C = \frac{a+2c}{3}$.

- A simple comparison shows that

$$p^* < p^C < p^M$$

and

$$q^* > q^C > q^M$$

- In the Cournot NE, the two firms' profits are

$$\pi_1^C = \pi_2^C = \frac{(a-c)^2}{9b}$$

- If these two firms can collude (form a cartel), they can jointly earn

$$\pi^M = \frac{(a-c)^2}{4b}$$

- Therefore, the Cournot NE is not Pareto efficient for these two firms.
  - Each firm is producing "too much" compared to the cartel case.
  - This is because each firm is maximizing its own profit, without considering the effect of its action on the other firm (**externality**).

- Consider price competition: the Bertrand model.
- Two firms produce the same product.
- Each firm has constant marginal cost c.
- Market demand is $x(p)$: continuous and strictly decreasing
- Firms are competing in prices (p_1, p_2).
  - If one firm's price is lower, then it gets the whole market.
  - If $p_1 = p_2$, then the two firms split the market.
- Unique Nash Equilibrium: $p_1 = p_2 = c$.
  - This is a NE in weakly dominated strategies.

Applications - Auctions

- *Dutch auction (descending clock auction)*: price starts high and descends until one bidder declares he will buy, object is assigned to him, he pays his bid.
- *First price sealed bid auction*: all bidders submit bids in sealed envelopes so that each bidder does not know others' bids at the time of submission, winner is the one with the highest bid and he pays his bid.
  - Dutch auction and first price sealed bid auction are **strategically equivalent**.
- *English auction (ascending clock auction)*: bids are submitted sequentially in an ascending manner until bidding stops, the last person to submit a bid wins, he pays his bid.
- *Second price sealed bid auction (Vickrey auction)*: bidders submit sealed bids, the object is allocated to the one who submitted the highest bid, the winner pays the price of the second highest bid.
  - English auction and second price sealed bid auction are strategically equivalent.

- Consider the games under the first price (sealed bid) auction and the second price (sealed bid) auction.
  - There are two players, their valuations of the object are v_1, v_2, respectively.
  - Each player submits a bid b_i, which is a nonnegative integer.
  - If $b_1 = b_2$, each player wins the auction with probability 0.5.
  - payoff $=$ probability of winning * (payment - valuation)
  - Assume complete information: players know each other's valuation.
- Nash equilibria are not difficult to find, and there are many.
- Example: first price auction, $v_1 = 500, v_2 = 700$
  - NE: (499,500),(500,501),(501,502),...,(697,698),(698,699)
- Example: second price auction, $v_1 = 500, v_2 = 700$
  - Some NE: (b_1, b_2) with $b_1 < b_2$ and $500 \leq b_2 \leq 700$
- *Under second price auction, for each player, bidding his true valuation is his (unique) weakly dominant strategy.*