Advanced Microeconomics I
Note 2: Choice-based approach to demand theory

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The consumer's decison-making problem

- There are L commodities, or *goods*.
- A vector $x = (x_1, x_2, ..., x_L)^T \in \mathbb{R}^L$ is a commodity *bundle*.
- The *consumption set*:

$$X = \mathbb{R}^L_+ = \left\{x \in \mathbb{R}^L : x_l \geq 0, \forall l = 1, 2, ..., L\right\}$$

- Price vector: $p = (p_1, p_2, ..., p_L)^T \gg 0$
- Wealth/income: $w > 0$
- *Walrasian, or competitive budget set*:

$$B_{p,w} = \left\{x \in \mathbb{R}^L_+ : p \cdot x \leq w\right\} \subseteq X$$

- The consumer's problem: given any $B_{p,w}$, choose a bundle x from $B_{p,w}$.

Walrasian demand function

- Choice-based approach: there is a Walrasian demand function $x(p, w)$.
- It is homogeneous of degree zero: $x(\alpha p, \alpha w) = x(p, w)$, for any $p \gg 0$, $w > 0$ and $\alpha > 0$.
- Walrasian demand function $x(p, w)$ satisfies **Walras'law** if for any $p \gg 0$ and $w > 0$, $p \cdot x(p, w) = w$.

Comparative statics

- Assume that $x(p, w)$ is differentiable.
- Income effects: $D_w x(p, w) = (\frac{\partial x_1(p,w)}{\partial w}, ..., \frac{\partial x_L(p,w)}{\partial w})^T$
- Good l is *normal* at (p, w) if $\frac{\partial x_l(p,w)}{\partial w} \geq 0$.
- Good l is *inferior* at (p, w) if $\frac{\partial x_l(p,w)}{\partial w} < 0$.
- Price effects: $D_p x(p, w)$
- Good l is a *Giffen good* at (p, w) if $\frac{\partial x_l(p,w)}{\partial p_l} > 0$.

Proposition. *For any* $p \gg 0$ *and* $w > 0$*, we have*

$$\sum_{k=1}^{L} \frac{\partial x_l(p,w)}{\partial p_k} p_k + \frac{\partial x_l(p,w)}{\partial w} w = 0, \forall l \in \{1, ..., L\}$$

Notice that this can be written in terms of elasticities:

$$\sum_{k=1}^{L} e_{lk}(p,w) + e_{lw}(p,w) = 0, \forall l \in \{1, ..., L\}$$

Proposition. *Suppose that $x(p, w)$ satisfies Walras' law. Then for any $p \gg 0$ and $w > 0$, we have*

(Cournot aggregation)

$$\sum_{k=1}^{L} p_k \frac{\partial x_k(p,w)}{\partial p_l} + x_l(p,w) = 0, \forall l$$

$$\textit{equivalently,} \;\; p \cdot D_p x(p,w) + x(p,w)^T = 0^T$$

(Engel aggregation)

$$\sum_{k=1}^{L} p_k \frac{\partial x_k(p,w)}{\partial w} = 1$$

$$\textit{equivalently,} \;\; p \cdot D_w x(p,w) = 1$$

WARP and the law of demand

We want to establish the *law of demand*.

Now, impose WARP on the Walrasian demand function.

Proposition. *The Walrasian demand function $x(p, w)$ satisfies the weak axiom of revealed preference if and only if for any (p, w) and (p', w'):*

$$\textit{If } \; p' \cdot x(p, w) \leq w' \; \textit{ and } \; x(p', w') \neq x(p, w), \; \textit{ then } \; p \cdot x(p', w') > w.$$

$(p, w) \rightarrow (p', w')$ is a **compensated price change** if $p' \cdot x(p, w) = w'$.

The Walrasian demand function $x(p, w)$ satisfies the **compensated weak axiom of revealed preference** (CWARP) if for any (p, w) and (p', w') with $p' \cdot x(p, w) = w'$,

$$\textit{If } \; x(p', w') \neq x(p, w), \quad \textit{then } \; p \cdot x(p', w') > w$$

Lemma. *Suppose that $x(p, w)$ satisfies Walras' law. Then $x(p, w)$ satisfies WARP if and only if it satisfies CWARP.*

Proof. The "only if" part is obvious. We show the "if" part. Assume to the contrary, CWARP is satisfied, but WARP is not. Then there exist (p, w) and (p', w') such that

$$p' \cdot x(p, w) \leq w', x(p', w') \neq x(p, w) \;\; and \;\; p \cdot x(p', w') \leq w$$

If $p' \cdot x(p, w) = w'$ or $p \cdot x(p', w') = w$, then CWARP is violated. So we consider the case where

$$p' \cdot x(p, w) < w' \;\; and \;\; p \cdot x(p', w') < w \tag{1}$$

Consider the following function

$$f(\alpha) = [\alpha p + (1 - \alpha)p'] \cdot [x(p, w) - x(p', w')]$$

By (1) and Walras' law, we have $f(0) < 0$ and $f(1) > 0$. Since f is continuous on $[0, 1]$, there exists some $\alpha^* \in (0, 1)$ such that $f(\alpha^*) = 0$. Let $p^* = \alpha^* p + (1 - \alpha^*)p'$, and $p^* \cdot x(p, w) = p^* \cdot x(p', w') = w^*$. Clearly $p^* \gg 0$ and $w^* > 0$.

Since $x(p, w) \neq x(p', w')$, without loss of generality, let $x(p^*, w^*) \neq x(p, w)$. Since $(p, w) \rightarrow (p^*, w^*)$ is a compensated price change, by CWARP we have

$$p \cdot x(p^*, w^*) > w \tag{2}$$

Then $p \cdot x(p', w') < w$ implies $x(p^*, w^*) \neq x(p', w')$. Since $(p', w') \to (p^*, w^*)$ is also a compensated price change, by CWARP again we have

$$p' \cdot x(p^*, w^*) > w' \tag{3}$$

A contradiction can be reached by using (1), (2) and (3), and considering the relation between w^* and $\alpha^* w + (1 - \alpha^*) w'$.

First, from (2) and (3) we have

$$(\alpha^* p) \cdot x(p^*, w^*) + [(1 - \alpha^*) p'] \cdot x(p^*, w^*) = p^* \cdot x(p^*, w^*) > \alpha^* w + (1 - \alpha^*) w'$$

By Walras' law,

$$w^* > \alpha^* w + (1 - \alpha^*) w' \tag{4}$$

Second, from (1) we have $p' \cdot x(p, w) < w'$; by Walras' law we have $p \cdot x(p, w) = w$. So

$$\alpha^* p \cdot x(p, w) + (1 - \alpha^*) p' \cdot x(p, w) = p^* \cdot x(p, w) < \alpha^* w + (1 - \alpha^*) w'$$

Since $(p, w) \to (p^*, w^*)$ is a compensated price change, $p^* \cdot x(p, w) = w^*$. It follows that $w^* < \alpha^* w + (1 - \alpha^*) w'$, contradicting to (4). $\square$

Theorem. *Suppose that the Walrasian demand function $x(p, w)$ satisfies Walras' law. Then $x(p, w)$ satisfies the weak axiom of revealed preference if and only if the following is true:*

For any (p, w) and (p', w') with $w' = p' \cdot x(p, w)$, we have

$$(p' - p) \cdot [x(p', w') - x(p, w)] \leq 0$$

with strict inequality if $x(p, w) \neq x(p', w')$.

That is, the Walrasian demand function satisfies WARP if and only if it satisfies the **compensated law of demand**.

Assume that $x(p, w)$ is differentiable. Consider a differential price change dp, and let $dw = x(p, w) \cdot dp$. Then by the compensated law of demand, $dp \cdot dx \leq 0$.

Take the total differential of $x(p, w)$:

$$dx = D_p x(p, w) dp + D_w x(p, w) dw$$

Then we have

$$dx = D_p x(p, w) dp + D_w x(p, w)[x(p, w) \cdot dp]$$
$$dx = [D_p x(p, w) + D_w x(p, w) x(p, w)^T] dp$$
$$dx = S(p, w) dp$$

where $S(p, w) = D_p x(p, w) + D_w x(p, w) x(p, w)^T$ is the **Slutsky matrix**.

The *lk-th* entry of $S(p, w)$ is the substitution effect of p_k on x_l:

$$S_{lk}(p, w) = \frac{\partial x_l(p, w)}{\partial p_k} + \frac{\partial x_l(p, w)}{\partial w} x_k(p, w)$$

Since $dp \cdot S(p, w) dp \leq 0$, it follows that $S(p, w)$ is *negative semidefinite*, which implies that $S_{ll}(p, w) \leq 0$ for all $l \in \{1, ..., L\}$.

This also shows that a good is a Giffen good at (p, w) only if it is an inferior good at (p, w).

However, generally, $S(p, w)$ is not symmetric.