Advanced Microeconomics I
Note 7: Partial equilibrium analysis

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Introduction

- The fundamental issue of Economics: **resource allocation**.
  - This issue should be addressed from two perspectives: one *normative* and the other *positive*.
- Normative: what is a good allocation?
  - Efficiency: an allocation (or, generally, an outcome) is efficient if it is not possible to make someone better-off without hurting anyone.
    - It serves as a minimal test for the desirability of an allocation.
  - Fairness: a more controversial issue.
- Positive: investigate the *mechanisms* that allocate resources.
  - Market
    - Private ownership
    - Price plays an important role.
    - Competitive and non-competitive markets
  - Mechanisms can also be specifically designed: pricing strategies, auctions, contracts...
    - Incentives
  - There are also allocation problems without monetary transfers
    - Example: school choice, organ transplant (kidney exchange), hospital-intern matching, refugee resettlement...and more recently, COVID-19 vaccines and convalescent plasma allocation
    - Matching theory and market design

This note looks at a general allocation problem: the organization of production and the allocation of the resulting commodities among consumers.

And we focus on the (mechanism of) competitive market.

The resource allocation problem

- Consider the following **economy**.

- N consumers: $i = 1, ..., N$.
- J firms: $j = 1, ..., J$.
- L goods: $l = 1, ..., L$.

- $\omega_l \geq 0$: the initial endowment of good l in the economy.

- Each consumer i's preferences over bundles $x_i = (x_{1i}, x_{2i}, ..., x_{Li})$ are represented by u_i.
- Each firm j's production technology is summarized by its production set Y_j. $y_j = (y_{1j}, ..., y_{Lj}) \in Y_j$ represents the net output of each good from y_j.

- An **allocation** $(x_1, ..., x_N, y_1, ..., y_J)$ consists of a consumption vector x_i for each consumer i and a production vector $y_j \in Y_j$ for each firm j, such that

$$\sum_{i=1}^{N} x_{li} \leq \omega_l + \sum_{j=1}^{J} y_{lj} \ \ \forall \ l = 1, ..., L$$

Efficiency

An allocation $(x'_1, ..., x'_N, y'_1, ..., y'_J)$ **Pareto dominates** another allocation $(x_1, ..., x_N, y_1, ..., y_J)$ if

$$u_i(x'_i) \geq u_i(x_i) \text{ for all } i = 1, ..., N$$

and

$$u_i(x'_i) > u_i(x_i) \text{ for some } i$$

An allocation is **Pareto efficient**, or simply **efficient**, if it cannot be Pareto dominated by any other allocation.

Maximization of total utility implies efficiency, but the converse is not necessarily true.

Competitive markets

- *Private ownership*: each consumer i owns ω_{li} of good l and $\omega_l = \sum_{i=1}^{N} \omega_{li}$; consumer i owns a share $\theta_{ij} \in [0, 1]$ of firm j and $\sum_{i=1}^{N} \theta_{ij} = 1$ for each j.
  - Different ownership structures correspond to different market mechanisms.
- *Complete markets*: a market exists for each of the L goods.
  - There is a price p_l for each good l.
- *Competitive*: all consumers and firms act as price takers.

- The resulting allocation of resources is given in the *competitive equilibrium*.

The allocation $(x_1^*, ..., x_N^*, y_1^*, ..., y_J^*)$ and price vector $p^* = (p_1^*, ..., p_L^*)$ constitute a **competitive equilibrium** if the following conditions are satisfied:

(i) *Profit maximization*: for each firm j, y_j^* solves

$$\text{Max} \;\; p^* \cdot y_j$$

$$s.t. \;\; y_j \in Y_j$$

(ii) *Utility maximization*: for each consumer i, x_i^* solves

$$\text{Max} \;\; u_i(x_i)$$

$$s.t. \;\; p^* \cdot x_i \leq p^* \cdot \omega_i + \sum_{j=1}^{J} \theta_{ij}(p^* \cdot y_j^*)$$

(iii) *Market clearing*: for each good $l = 1, ..., L$,

$$\sum_{i=1}^{N} x_{li}^* = \omega_l + \sum_{j=1}^{J} y_{lj}^*$$

Lemma. *If* $(x_1^*, ..., x_N^*, y_1^*, ..., y_J^*)$ *and* $p^* = (p_1^*, ..., p_L^*)$ *constitute a competitive equilibrium, then for any* $\alpha > 0$, $(x_1^*, ..., x_N^*, y_1^*, ..., y_J^*)$ *and* $\alpha p^* = (\alpha p_1^*, ..., \alpha p_L^*)$ *also constitute a competitive equilibrium*.

Lemma. *Suppose that every consumer's budget constraint is satisfied with equality. Given an allocation* $(x_1, ..., x_N, y_1, ..., y_J)$ *and a price vector* $p = (p_1, ..., p_L) \gg 0$, *if the market clearing condition is satisfied for* $L - 1$ *goods, then it is satisfied for all goods.*

Does a competitive equilibrium always exist?

If it exists, is the allocation in the equilibrium efficient?

Partial equilibrium analysis: two-good quasilinear model

- We now focus on the market for one particular good: good l.
- Two-good quasilinear model: good l and the *numeraire*
  - The numeraire represents the composite of all other goods, or the expenditure on all other goods.
- Each consumer i has a quasilinear utility function:

$$u_i(m_i, x_i) = m_i + \phi_i(x_i)$$

where m_i is the consumption of the numeraire.

- Assume that each consumer's consumption set is $\mathbb{R} \times \mathbb{R}_+$, i.e., the consumption of the numeraire is allowed to be negative.
- Assume that $\phi_i'(x_i) > 0, \phi_i''(x_i) < 0$ for all $x_i \geq 0$, $\lim_{x_i \to +\infty} \phi_i'(x_i) = 0$, and $\phi_i(0) = 0$.

- Each firm j produces good l from the numeraire, with the cost function $c_j(q_j)$: producing q_j units of good l requires $c_j(q_j)$ units of the numeraire.
- Assume that $c_j'(q_j) > 0$, $c_j''(q_j) > 0$ for all $q_j \geq 0$, $\lim_{q_j \to +\infty} c_j'(q_j) = +\infty$, and $c_j(0) = 0$.
- Assume $\omega_l = 0$, $\omega_m > 0$

- For simplicity, in this specialized two-good quasilinear model, an allocation is represented by the vector $(x_1, ..., x_N, q_1, ..., q_J, m_1, ..., m_N)$ with

$$\sum_{i=1}^{N} x_i = \sum_{j=1}^{J} q_j$$

$$\sum_{i=1}^{N} m_i + \sum_{j=1}^{J} c_j(q_j) = \omega_m$$

  - Each firm j's demand for the numeraire is given by $c_j(q_j)$.
  - We only focus on *non-wasteful* allocations.

Efficient allocations in the two-good quasilinear model

Given an allocation $(x_1, ..., x_N, q_1, ..., q_J, m_1, ..., m_N)$, the total utility is

$$\sum_{i=1}^{N} u_i(m_i, x_i) = \sum_{i=1}^{N} \phi_i(x_i) + \omega_m - \sum_{j=1}^{J} c_j(q_j)$$

Consider the following problem

$$\underset{x_i, q_j}{\text{Max}} \quad \sum_{i=1}^{N} \phi_i(x_i) + \omega_m - \sum_{j=1}^{J} c_j(q_j)$$

$$s.t. \;\; x_i \geq 0, \;\; \forall i$$

$$q_j \geq 0, \;\; \forall j$$

$$\sum_{i=1}^{N} x_i = \sum_{j=1}^{J} q_j$$

This problem has a unique solution $(x_1^*, ..., x_N^*, q_1^*, ..., q_J^*)$ that satisfies the following conditions:

There exists $\lambda \in \mathbb{R}$ such that

$$\text{For each } j, \quad \lambda \leq c_j'(q_j^*), \text{ with equality if } q_j^* > 0$$

$$\text{For each } i, \quad \phi_i'(x_i^*) \leq \lambda, \text{ with equality if } x_i^* > 0$$

$$\sum_{i=1}^{N} x_i^* = \sum_{j=1}^{J} q_j^*$$

Hence, all the allocations $(x_1^*, ..., x_N^*, q_1^*, ..., q_J^*, m_1, ..., m_N)$ are efficient.

Because utilities are perfectly transferable in this two-good quasilinear model, these are also the only efficient allocations.

In sum, all the efficient allocations involve $(x_1^*, ..., x_N^*, q_1^*, ..., q_J^*)$ and they only differ in the distribution of the numeraire among consumers. They all maximize

$$\sum_{i=1}^{N} \phi_i(x_i) + \omega_m - \sum_{j=1}^{J} c_j(q_j)$$

or

$$\sum_{i=1}^{N} \phi_i(x_i) - \sum_{j=1}^{J} c_j(q_j)$$

which is the **Marshallian aggregate surplus**. It can be considered as the total utility generated from production and consumption.

Competitive equilibrium in the two-good quasilinear model

- Private ownership: ω_{mi}, θ_{ij}
- Normalize the price of the numeraire to 1.
- Suppose that the allocation $(x_1^*, ..., x_N^*, q_1^*, ..., q_J^*, m_1^*, ..., m_N^*)$ and the price of good l, p^*, constitute a competitive equilibrium.
- For each firm j, given p^*, q_j^* solves

$$\underset{q_j \geq 0}{\text{Max}} \;\; p^* q_j - c_j(q_j)$$

which has the necessary and sufficient first-order condition:

$$p^* \leq c_j'(q_j^*), \;\; \text{with equality if } \; q_j^* > 0$$

- For each consumer i, m_i^* and x_i^* must solve

$$\underset{m_i, x_i}{\text{Max}} \;\; m_i + \phi_i(x_i)$$

$$s.t. \;\; m_i + p^* x_i \leq \omega_{mi} + \sum_{j=1}^{J} \theta_{ij}(p^* q_j^* - c_j(q_j^*)) \;\text{ and }\; x_i \geq 0$$

which is equivalent to

$$\underset{x_i \geq 0}{\text{Max}} \;\; \phi_i(x_i) - p^* x_i + \left\{ \omega_{mi} + \sum_{j=1}^{J} \theta_{ij}(p^* q_j^* - c_j(q_j^*)) \right\}$$

Necessary and sufficient first-order condition:

$$\phi_i'(x_i^*) \leq p^* \;\text{ with equality if }\; x_i^* > 0$$

- Moreover,

$$m_i^* = \left\{ \omega_{mi} + \sum_{j=1}^{J} \theta_{ij}(p^* q_j^* - c_j(q_j^*)) \right\} - p^* x_i^*$$

Therefore, the allocation $(x_1^*, ..., x_N^*, q_1^*, ..., q_J^*, m_1^*, ..., m_N^*)$ and the price of good l, p^*, constitute a competitive equilibrium if and only if the following conditions are satisfied

$$\text{For each } j, \quad p^* \leq c_j'(q_j^*), \;\; \text{with equality if } \; q_j^* > 0$$

$$\text{For each } i, \quad \phi_i'(x_i^*) \leq p^*, \;\; \text{with equality if } \; x_i^* > 0$$

$$\sum_{i=1}^{N} x_i^* = \sum_{j=1}^{J} q_j^*$$

Notice that we only need to make sure that the market for good l clears.

The existence of the competitive equilibrium can be studied using the traditional Marshallian graphical analysis.

- Let $x_i(p)$ be consumer i's demand function for good l. Then $x_i(p) = [\phi_i']^{-1}(p)$ if $p \leq \phi_i'(0)$; $x_i(p) = 0$ if $p > \phi_i'(0)$.
- The aggregate demand function for good l is then $x(p) = \sum_{i=1}^{N} x_i(p)$.
- $x(p) = 0$ if $p > \text{Max}_i\ \phi_i'(0)$, and $x(p)$ is strictly decreasing if $p \leq \text{Max}_i\ \phi_i'(0)$. Moreover, it is continuous, and $\lim_{p\to 0} x(p) = +\infty$.

- Let $q_j(p)$ be firm j's supply function for good l. Then $q_j(p) = [c_j']^{-1}(p)$ if $p \geq c_j'(0)$; $q_j(p) = 0$ if $p < c_j'(0)$;
- The aggregate supply function for good l is then $q(p) = \sum_{j=1}^{J} q_j(p)$.
- $q(p) = 0$ if $p < \text{Min}_j\ c_j'(0)$, and $q(p)$ is strictly increasing if $p \geq \text{Min}_j\ c_j'(0)$. Moreover, it is continuous, and $\lim_{p\to +\infty} q(p) = +\infty$.

- Finally, assume that $\text{Max}_i\ \phi_i'(0) > \text{Min}_j\ c_j'(0)$, then clearly there exists a unique $p^* > 0$ such that $x(p^*) = q(p^*) > 0$. Therefore a competitive equilibrium exists, and it is unique.

Fundamental welfare theorems

By comparing the conditions on p.13 and p.17, it follows that the allocation in competitive equilibrium is efficient.

Moreover, any efficient allocation can be achieved in the competitive equilibrium by appropriately transferring the initial endowments among consumers (see the last equation on p.16).

We have established the fundamental welfare theorems in a partial equilibrium context.

The First Fundamental Theorem of Welfare Economics: *If the allocation $(x_1^*, ..., x_N^*, q_1^*, ..., q_J^*, m_1^*, ..., m_N^*)$ and the price of good l, p^*, constitute a competitive equilibrium, then the allocation is efficient.*

The Second Fundamental Theorem of Welfare Economics: *Given an efficient allocation $(x_1^*, ..., x_N^*, q_1^*, ..., q_J^*, m_1^*, ..., m_N^*)$, there exist transfers of the numeraire $(T_1, ..., T_N)$ with $\sum_{i=1}^N T_i = 0$, and a price of good l, $p^* > 0$, such that $(x_1^*, ..., x_N^*, q_1^*, ..., q_J^*, m_1^*, ..., m_N^*)$ and p^* constitute a competitive equilibrium reached from initial endowments $(\omega_{m1} + T_1, ..., \omega_{mN} + T_N)$.*

The key implication of the welfare theorems is that, under some mild assumptions, an allocation is efficient if and only if it can be achieved by the competitive markets.

We now proceed with some further analysis that help better understand the welfare theorems.

Let $D(\cdot) = x^{-1}(\cdot)$ be the *inverse demand function*. Given a quantity x, how to interpret $D(x)$?

If the total consumption of good l is x, then the optimal distribution of x to the consumers should (try to) equate everyone's marginal utility:

$$\text{For each consumer } i, \quad \phi_i'(x_i) = D(x) \quad \text{if} \quad \phi_i'(0) \geq D(x)$$

Hence the inverse demand function $D(\cdot)$ represents the *marginal social utility* of good l.

Similarly, let $S(\cdot) = q^{-1}(\cdot)$ be the *inverse supply function*, and it can be interpreted as the *marginal social cost* of good l. Why? If the total production is q, then the optimal distribution of q among firms should (try to) equate every firm's marginal cost:

$$\text{For each firm } j, \quad c_j'(q_j) = S(q) \quad \text{if} \quad c_j'(0) \leq S(q)$$

Therefore, the total consumption (or production) of good l in the competitive equilibrium equates the marginal social utility and marginal social cost. The distribution of consumption and production is also optimal in the competitive equilibrium.

Finally, we consider the traditional graphical analysis.

For each consumer i, given a consumption x_i of good l

$$\int_0^{x_i} \phi_i'(t)dt = \phi_i(x_i) - \phi_i(0) = \phi_i(x_i)$$

Since the inverse demand function $D(\cdot)$ is the horizontal sum of all consumers' marginal utility functions, we have

$$\int_0^{x} D(t)dt = \sum_{i=1}^{N} \phi_i(x_i)$$

when a total consumption level x is optimally distributed. Similarly, when a total production level q is optimal distributed,

$$\int_0^{q} S(t)dt = \sum_{j=1}^{J} c_j(q_j)$$

Therefore, given any allocation $(x_1, ..., x_N, q_1, ..., q_J, m_1, ..., m_N)$, if the consumption and production of good l have been optimally distributed, then the aggregate Marshallian surplus can be identified using the inverse demand and inverse supply:

$$\sum_{i=1}^{N} \phi_i(x_i) - \sum_{j=1}^{J} c_j(q_j) = \int_0^x [D(t) - S(t)]dt$$

where $x = \sum_{i=1}^{N} x_i = \sum_{j=1}^{J} q_j$.

From the graph, clearly it is maximized at the intersection of the inverse demand and inverse supply.

Conclusion

We have established the welfare theorems in the special partial equilibrium context. More generally, these theorems can be stated as follows.

The First Fundamental Theorem of Welfare Economics: If every relevant good is traded in a market at publicly known prices (there is a complete set of markets), and if consumers and firms act as price takers, then the market outcome is efficient.

The Second Fundamental Theorem of Welfare Economics: If consumers' preferences and firms' production sets are convex, every relevant good is traded in a market at publicly known prices, and if consumers and firms act as price takers, then every efficient outcome can be achieved in a competitive equilibrium with appropriate transfers of income.

- The first welfare theorem is a formal expression of Adam Smith's claim about *invisible hand*.
  - Invisible hand was introduced by Adam Smith in his book 'The Wealth of Nations': an economy can work well in a free market scenario where everyone will work for his/her own interest.
- Any inefficiencies that arise in a market must be traceable to a violation of some assumption of the first welfare theorem.
- Market equilibrium fails to be efficient - **Market failure**
- Some common sources of marker failure:
  - Market power: monopoly, Cournot
  - Externality: non-marketed "goods" or "bads"
  - Asymmetric information
    - ⋆ postcontractual asymmetric information: moral hazard
    - ⋆ asymmetric information at the time of contracting: adverse selection