Problem Set 2

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Problem 1. Consider a more general choice-based approach to demand: assume that there exists a choice correspondence $x(p,w)$ defined on $\{B_{p,w} : p \gg 0, w > 0\}$. Assume that $x(p,w)$ satisfies the weak axiom of revealed preference and Walras' law. Show the following generalized compensated law of demand: for any $p \gg 0, w > 0$ and $p' \gg 0$, if $x \in x(p,w)$ and $w' = p' \cdot x$, then $[p' - p] \cdot [x' - x] \leq 0$ for any $x' \in x(p', w')$.

Answer to Problem 1.
Since $x(p,w)$ satisfies Walras' law, we have $\forall x \in x(p,w)$ s.t. $p \cdot x = w$. So we have $x \in x(p,w)$ and $w' = p' \cdot x$. We want to show that $px' \geq w$ for any $x' \in x(p', w')$. Assume that $px' < w$, thus $x' \in B_{p,w}$. By $x \in x(p,w)$, we have $x \succeq x'$. Now we have $w' = p' \cdot x$, so $x \in B_{p',w'}$. Since $x \succeq x'$, $x \in x(p', w')$. However, $x' \in x(p', w')$, contradicting to the WARP. So we have $px' \geq w$. $[p' - p] \cdot [x' - x] = p'x' - px' - p'x + px$. Since $p'x' = p'x = w'$, $[p' - p] \cdot [x' - x] = px - px') = w - px' \leq 0$. Hence $[p' - p] \cdot [x' - x] \leqslant 0$ for any $x' \in x(p', w')$.

Problem 2. Show that the lexicographic preference relation (as defined in the lecture notes, on $\mathbb{R}^2_+$) is complete, transitive, strongly monotone and strictly convex.

Answer to Problem 2.
Assume $X = \mathbb{R}^2_+$ for any $x, y \in X$, let $x \succeq y$ if $x_1 > y_1$ or $x_1 = y_1$ and $x_2 \geq y_2$.

- complete
  There are six relationships between x_1 and y_1, x_2 and y_2:
  $x_1 > y_1, x_2 \geqslant y_2$; $x_1 > y_1, x_2 < y_2$; $x_1 = y_1, x_2 \geqslant y_2$; $x_1 = y_1, x_2 < y_2$; $x_1 < y_1, x_2 \geqslant y_2$; $x_1 < y_1, x_2 < y_2$. Obviously, the three top relations suggest that $x \succeq y$, and the last three relations suggest that $y \succeq x$. Thus the lexicographic preference relation is complete.

- transitive
  Consider $x \succeq y$ and $y \succeq z$, w.t.s. $x \succeq z$. $x \succeq y$ suggests that $x_1 > y_1$ or $x_1 = y_1$ and $x_2 \geq y_2$, while $y \succeq z$ suggests that $y_1 > z_1$ or $y_1 = z_1$ and $y_2 \geq z_2$. If $x_1 > y_1$ and $y_1 > z_1$, we have $x_1 > z_1$, thus $x \succeq z$; if $x_1 > y_1$ and $y_1 = z_1$, we also have $x_1 > z_1$, thus $x \succeq z$; if $x_1 = y_1$, $x_2 \geq y_2$ and $y_1 > z_1$, we have $x_1 > z_1$, thus $x \succeq z$; if $x_1 = y_1$, $x_2 \geq y_2$ and $y_1 = z_1$, $y_2 \neq z_2$, we have $x_1 = z_1$, $x_2 \geq z_2$, thus $x \succeq z$. So the lexicographic preference relation is transitive.

- strongly monotone
  For any $x, y \in X$, $x \geq y$ and $x \neq y$, w.t.s. $x \succ y$. Assume to the contrary, $y \succeq x$. By the definition of the lexicographic preference relation, we have $y_1 > x_1$ or $y_1 = x_1$ and $y_2 \geq x_2$. If $y_1 > x_1$, we have $y \geq x$; if $y_1 = x_1$ and $y_2 \geq x_2$, we also have $y \geq x$, contradicting to $x \geq y$ and $x \neq y$. Thus the lexicographic preference relation is strongly monotone.

- strictly convex
  Consider $y \succeq x$, $z \succeq x$ and $y \neq z$, w.t.s $\alpha y + (1-\alpha)z \succ x$. Let $w = \alpha y + (1-\alpha)z$, we have $w_1 = \alpha y_1 + (1-\alpha)z_1$ and $w_2 = \alpha y_2 + (1-\alpha)z_2$. If $y_1 > x_1$ and $z_1 > x_1$, we have $\alpha y_1 > \alpha x$ and $(1-\alpha)z_1 > (1-\alpha)x_1$. Thus we have $w_1 = \alpha y_1 + (1-\alpha)z_1 > \alpha x_1 + (1-\alpha)x_1 = x_1$, $w \succeq x$. Since $x \not\succeq w$ we have $w \succ x$. If $y_1 > x_1$ and $z_1 = x_1$, $z_2 \geq x_2$, we also have $w_1 > x_1$, thus $w \succeq x$. If $y_1 = x_1$, $y_2 \geq x + 2$ and $z_1 > x_1$, we also have $w_1 > x_1$, thus $w \succeq x$. If $y_1 = x_1$, $y_2 \geq x + 2$ and $z_1 = x_1$, we have $w_1 = x_1$ and $w_2 \geq x_2$, thus $w \succeq x$.

Problem 3. Let u be a utility function representing a preference relation $\succeq$. Show that u is strictly quasiconcave if and only if $\succeq$ is strictly convex.

Answer to Problem 3.
u is strictly quasiconcave suggests that $u(\alpha y + (1-\alpha)z) > \min\{u(y), u(z)\}$; and $\succeq$ is strictly convex suggests that $\alpha y + (1-\alpha)z > x$ for any $x, y, z \in X$ with $y \succeq x$, $z \succeq x$ and $y \neq z$.

- Only if part
  Consider $u(\alpha y+(1-\alpha)z)>\min\{u(y),u(z)\}$ and $u(y)<u(z)$, we have $u(\alpha y+(1-\alpha)z)>u(y)$ and $\alpha y+(1-\alpha)z \succ y$. by $y \succeq x$ and transitivity, we have $\alpha y+(1-\alpha)z>x$. Also, when $u(z)<u(y)$, we have $\alpha y+(1-\alpha)z>y$.

- If part
  For any $x,y,z \in X$ with $y \succeq x$, $z \succeq x$ and $y \neq z$, we have $\alpha y+(1-\alpha)z>x$, w.t.s $u(\alpha y+(1-\alpha)z)>\min\{u(y),u(z)\}$. Assume to the contrary, $u(\alpha y+(1-\alpha)z) \leq \min\{u(y),u(z)\}$. If $u(y)<u(z)$, we have $u(\alpha y+(1-\alpha)z)<u(y)$, thus $y \geq \alpha y+(1-\alpha)z$. Now we have $(1-\alpha)y>(1-\alpha)z$ and $y>z$, contradicting to $u(y)<u(z)$. If $u(y)>u(z)$, we also have $u(\alpha y+(1-\alpha)z)>\min\{u(y),u(z)\}$.

Problem 4. Let u be a continuous utility function and $x(p,w)$ be the corresponding Walrasian demand correspondence derived from utility maximization. Then $x(p,w)$ can be considered as a choice correspondence defined on $\left\{B_{p,w} : p \gg 0, w>0\right\}$.
(a) Show that $x(p,w)$ satisfies WARP.
(b) Can $x(p,w)$ be rationalized? Explain your answer.

Answer to Problem 4.
(a) Assume to the contrary, $x \in x(p,w)$, $y \notin x(p,w)$ and $y \in x(p',w')$. Since $y \in x(p',w')$, we have $u(y) \geqslant u(y')$ for any $y' \in B_{p',w'}$, thus we have $u(y) \geqslant u(x)$. By $x \in x(p,w)$ and $y \notin x(p,w)$, we have $u(x)>u(y)$, contradicting to $u(y) \geqslant u(x)$. Thus $x(p,w)$ satisfies WARP.
(b) By the definition of $x(p,w)$, we have $x \in x(p,w)$ that there doesn't exist $y \in B_{p,w}$ such that $u(y)>u(x)$.

- complete
  For any $x_1,x_2 \in B_{p,w}$, if $x_1 \succeq x_2$, we have $u(x_1) \geqslant u(x_2)$. If $x_2 \succeq x_1$, we have $u(x_2) \geq u(x_1)$.

- transitive
  Assume to the contrary, if $u(x_1) \geq u(x_2)$, $u(x_2) \geq u(x_3)$ and $u(x_1)<u(x_3)$. By

$u(x_1) \geq u(x_2)$ and $u(x_2) \geq u(x_3)$, we have $x_1 \succeq x_2$, $x_3 \succeq x_3$, so $x_1 \succeq x_3$. Thus, we have $u(x_1) \geq u(x_3)$, contradicting to $u(x_1) < u(x_3)$.

Problem 5. Let $u : \mathbb{R}^2_+ \to \mathbb{R}$ be a continuous utility function, and let $v(p, w)$ be the corresponding indirect utility function.
(a) Prove that for any price vector $p \gg 0$ and consumption bundle $x \in \mathbb{R}^2_+$ with $x \neq 0$, $v(p, p \cdot x) \geq u(x)$.
(b) Given a consumption bundle $x \in \mathbb{R}^2_+$, $x \neq 0$, does there always exist a price vector $p \gg 0$ such that $v(p, p \cdot x) = u(x)$? If so, prove it. Otherwise provide a counterexample.

Answer to Problem 5.
(a) Consider $w' = p \cdot x$, we have $v(p, px) = v(p, w')$. By $w' = px$, we have $x \in B_{p,w'}$. According to the definition of indirect utility function, we know for any $x \in B_{p,w'}$, $v(p, w') \geq u(x)$, so $v(p, px) \geq u(x)$.
(b) In some cases, there doesn't exist a price vector $p \gg 0$ s.t. $v(p, px) = u(x)$.
Counterexample: When the utility function $u(x_1, x_2) = x_1$, which is continuous, to let $v(p, px) = u(x)$, we have $\frac{P_1}{P_2} = \frac{\partial u(x^*)/\partial x_2}{\partial u(x^*)/\partial x_1} = \frac{0}{1} = 0$. In this case, we can not find a price vector that $p \gg 0$ satisfies $v(p, px) = u(x)$.

Problem 6. For each of the following utility functions, derive the Hicksian demand and expenditure function, at prices $(p_1, p_2) \gg 0$ and utility $u > 0$.
(a) $u(x_1, x_2) = \min\{2x_1, 3x_2\}$
(b) $u(x_1, x_2) = 3x_1 + 2x_2$
(c) $u(x_1, x_2) = x_1^\alpha x_2^\beta$, $\alpha > 0, \beta > 0$

Answer to Problem 6.
(a) $u(x_1, x_2) = \min\{2x_1, 3x_2\}$
If $2x_1 \leq 3x_2$, $u(x_1, x_2) = 2x_1$, suppose that u is differentiable, the Lagrangian $\mathscr{L}(x, \lambda) =$

$p_1x_1+p_2x_2+\lambda(u-2x_1)$. We have

$$\begin{cases} \frac{\partial \mathscr{L}}{\partial x_1} = p_1 - 2\lambda = 0 \\ \frac{\partial \mathscr{L}}{\partial x_2} = p_2 = 0 \\ \frac{\partial \mathscr{L}}{\partial \lambda} = u - 2x_1 = 0 \end{cases} \Rightarrow \begin{cases} \lambda = \frac{p_1}{2} \\ p_2 = 0 \\ x_1 = \frac{u}{2} \end{cases}$$

So the Hicksian demand is $h_1(p,u) = \frac{1}{2}u$, $h_2(p,u) \geqslant \frac{2}{3}h_1(p,u)$, the expenditure function is $p_1h_1(p,u) + p_2h_2(p,u) = \frac{1}{2}p_1u$. If $2x_1 \geq 3_x2$, we have $u(x_1,x_2) = 3x_2$, and the Lagrangian $\mathscr{L}(x,\lambda) = p_1x_1 + p_2x_2 + \lambda(u-3x_2)$, we have

$$\begin{cases} \frac{\partial \mathscr{L}}{\partial x_1} = p_1 = 0 \\ \frac{\partial \mathscr{L}}{\partial x_2} = p_2 - 3\lambda = 0 \\ \frac{\partial \mathscr{L}}{\partial \lambda} = u - 3x_2 = 0 \end{cases} \Rightarrow \begin{cases} p_1 = 0 \\ \lambda = \frac{1}{3}p_2 \\ x_2 = \frac{1}{3}u \end{cases}$$

the Hicksian demand is $h_1(p,u) \geqslant \frac{3}{2}h_2(p,u)$, $h_2(p,u) = \frac{u}{3}$, and the expenditure function is $\frac{1}{3}p_2u$. So the Hicksian demand is:

$$\begin{cases} h_1(p,u) = \frac{1}{2}u, \quad h_2(p,u) \geqslant \frac{2}{3}h_1(p,u) \\ h_2(p,u) = \frac{1}{3}u, \quad h_1(p,u) \geqslant \frac{3}{2}h_2(p,u) \end{cases}$$

the expenditure function is:

$$\begin{cases} \frac{1}{2}p_1u & \left(x_2 \geqslant \frac{2}{3}x_1\right) \\ \frac{1}{3}p_2u & \left(x_1 \geqslant \frac{3}{2}x_2\right) \end{cases}$$

(b) $u(x_1,x_2) = 3x_1 + 2x_2$
The Lagrangian $\mathscr{L}(x,\lambda) = p_1x_1 + p_2x_2 + \lambda(u - 3x_1 - 2x_2)$, we have

$$\begin{cases} \frac{\partial \mathscr{L}}{\partial x_1} = p_1 - 3\lambda = 0 \\ \frac{\partial \mathscr{L}}{\partial x_2} = p_2 - 2\lambda = 0 \\ \frac{\partial \mathscr{L}}{\partial \lambda} = u - 3x_1 - 2x_2 = 0 \end{cases} \Rightarrow \begin{cases} p_1 = 3\lambda \\ p_2 = 2\lambda \\ u = 3x_1 + 2x_2 \end{cases}$$

$$p_1x_1 + p_2x_2 = 3\lambda x_1 + 2\lambda x_2 = \lambda(3x_1 + 2x_2) = \lambda u = \frac{1}{3}p_1u = \frac{1}{2}p_2u$$

So the Hicksian demand function is $3h_1(p,u) + 2h_2(p,u) = u$ and the expenditure function is $\frac{1}{3}p_1 u$ or $\frac{1}{2}p_2 u$.

(c) $u(x_1,x_2) = x_1^\alpha x_2^\beta$, $\alpha > 0, \beta > 0$

The Lagrangian $\mathscr{L}(x,\lambda) = p_1x_1 + p_2x_2 + \lambda\left(u - x_1^\alpha x_2^\beta\right)$, we have

$$\begin{cases} \frac{\partial \mathscr{L}}{\partial x_1} = p_1 - \alpha \cdot x_2^\beta \cdot \lambda x_1^{\alpha-1} = 0 \\ \frac{\partial \mathscr{L}}{\partial x_2} = p_2 - \beta \cdot x_1^\alpha \cdot \lambda {x_2}^{\alpha-1} = 0 \\ \frac{\partial \mathscr{L}}{\partial \lambda} = u - x_1^\alpha x_2^\beta = 0 \end{cases} \Rightarrow \begin{cases} p_1 = \alpha \cdot x_2^\beta \cdot \lambda x_1^{\alpha-1} \\ p_2 = \beta \cdot x_1^\alpha \cdot \lambda x_2^{\beta-1} \\ u = x_1^\alpha x_2^\beta \end{cases}$$

$$\frac{p_1}{p_2} = \frac{\alpha}{\beta} \cdot \frac{x_2^\beta}{x_1^\alpha} \cdot \frac{x_1^{\alpha-1}}{x_2^{\beta-1}} \Rightarrow x_2 = \frac{\beta p_1}{\alpha p_2} \cdot x_1$$

Thus we have $u = x_1^{\alpha+\beta}\beta^\beta p_1^\beta \cdot \alpha^{-\beta} \cdot p_2^{-\beta}$, so $x_1 = \left(\frac{\alpha p_2}{\beta p_1}\right)^{\frac{\beta}{\alpha+\beta}} \cdot u^{\frac{1}{\alpha+\beta}}$, $x_2 = \left(\frac{\alpha p_2}{\beta p_1}\right)^{-1} \cdot \left(\frac{\alpha p_2}{\beta p_1}\right)^{\frac{\alpha}{\alpha+\beta}} \cdot u^{\frac{\alpha+\beta}{\alpha+\beta}} = \left(\frac{\alpha p_2}{\beta p_1}\right)^{\frac{-\alpha}{\alpha+\beta}} \cdot u^{\frac{1}{\alpha+\beta}}$. The Hicksian demand function is:

$$\begin{cases} h_1(p,u) = \left(\frac{\alpha P_2}{\beta P_1}\right)^{\frac{\beta}{\alpha+\beta}} \cdot u^{\frac{1}{\alpha+\beta}} \\ h_2 \cdot (p,u) = \left(\frac{\alpha P_2}{\beta p_1}\right)^{\frac{-\alpha}{\alpha+\beta}} \cdot u^{\frac{\alpha}{\alpha+\beta}} \end{cases}$$

and the expenditure function is:

$$h_1(p,u) \cdot p_1 + h_2(p,u) \cdot p_2 = p_1 \cdot \left(\frac{\alpha p_2}{\beta p_1}\right)^{\frac{\beta}{\alpha+\beta}} \cdot u^{\frac{1}{\alpha+\beta}} + p_2 \cdot \left(\frac{\alpha p_2}{\beta p_1}\right)^{\frac{-\alpha}{\alpha+\beta}} \cdot u^{\frac{1}{\alpha+1}}$$

Problem 7. Suppose that the utility function $u(x)$ is homogeneous of degree one. Show that for any $p \gg 0$, $w > 0$ and $\alpha > 0$,

$$x(p, \alpha w) = \left\{x \in \mathbb{R}_+^L : x = \alpha y, y \in x(p,w)\right\}$$

and

$$v(p, \alpha w) = \alpha v(p, w)$$

(Hint: in the first part you have to show the two sets are the same. That is, if $x \in x(p, w)$, then $\alpha x \in x(p, \alpha w)$, and if $x \in x(p, \alpha w)$, then $\frac{1}{\alpha} x \in x(p, w)$.)

Answer to Problem 7.

- $x(p, \alpha w) = \left\{x \in \mathbb{R}_+^L : x = \alpha y, y \in x(p, w)\right\}$
  Consider $x \in x(p, w)$, we have $u(x) \geq u(y)$ for any $y \in B_{p,w}$. Consider $z \neq \alpha x$ and $z \in x(p, \alpha w)$, which means $u(z) \geq u(\alpha x)$. Since $u(x)$ is homogeneous of degree one, $u(\alpha x) = \alpha u(x)$ and $u\left(\frac{1}{\alpha} z\right) = \frac{1}{\alpha} u(z)$, we have $\frac{1}{\alpha} z \cdot p \leq w$, $\frac{1}{\alpha} z \in B_{p,w}$, contradicting to $u(x) \geq u(y)$ for any $y \in B_{p,w}$. Thus if $x \in x(p, w)$, then $\alpha x \in x(p, \alpha w)$. Similarly, we can conclude that if $x \in x(p, \alpha w)$, then $\frac{1}{\alpha} x \in x(p, w)$.

- $v(p, \alpha w) = \alpha v(p, w)$
  Let $v(p, aw) = u(x)$, which means $x \in x(p, \alpha w)$, thus we have $\frac{1}{\alpha} x \in x(p, w)$, which means $v(p, w) = u\left(\frac{1}{\alpha} x\right) = \frac{1}{\alpha} u(x)$. Thus we can conclude that $v(p, \alpha w) = u(x) = \alpha \cdot \frac{1}{\alpha} u(x) = \alpha v(p, w)$.