MICROECONOMIC THEORY II

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考虑商品怎么来的
引入生产

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Production set 生产部门/消费部门

生产可能性集合(单个商品)

- Production possibility set: the all possible
- Example 1: labor as input; two outputs, beef and corn.
  Production function

  生产函数(生产出max)

$$b = 5l^{1/2}.$$

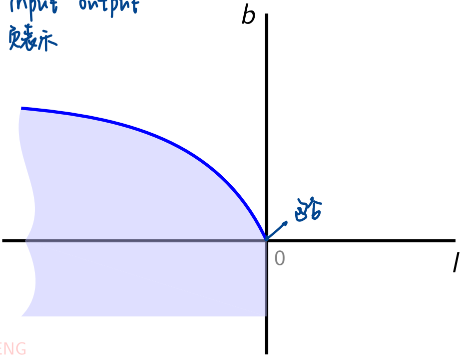
- The production possibility set 生产可能性集

凸集

convex set

$$Y_b = \{(\underbrace{-l}_{\text{input}}, \underbrace{b, c}_{\text{output}}) \in \mathbb{R}_- \times \mathbb{R} \times \mathbb{R} | b \leq 5l^{1/2}, c = 0\}$$

负表示



可能为负(消费资源)↓

Assumptions on PPS

- $0 \in Y_j$: can do nothing;
- $Y_j \cap \mathbb{R}^L_+ = \{0\}$: no free lunch;
- Y_j is convex: no increasing return; 不存在规模效应递增
- $\mathbb{R}^L_- \subseteq Y_j$.

AGGREGATE PRODUCTION 社会总生产

- y is possible if there exists $(y_1, \ldots, y_J)$,

$$\underset{总}{y} = \sum_j \underset{单个}{y_j} \quad \text{and} \quad (\forall\, j)\; y_j \in Y_j.$$

在相应生产可能性集合中.

Production Transformation Function

- Y is production efficient (有效) if Y is possible and there exists no possible alternative Y' such that $Y' > Y$. Y' 中每一项都≥Y中每一项, 至少有一项>

生产转换函数

- Definition: A transformation function (PTF) $T : A \subseteq \mathbb{R}^L \to \mathbb{R}$ represents Y over $A \subseteq \mathbb{R}^L$ if $(\forall y \in A)$, 生产转换函数

$$T(y) \leq 0 \Leftrightarrow y \in Y.$$

- $T(y) = 0 \Leftrightarrow y$ is efficient.
- Rule to get $T(y)$: pick one good l (use an output)

$$T(y) = y_l - \max\{y_l' | (y_l', y_{-l}) \in Y\}.$$

AGGREGATE PRODUCTION Example

- An example. The economy has 3 types of production:
  steel mill to produce iron (y_1);
  car parts factory (y_2);
  car assembly plant (y_3).

Steel mill

	y_1
labor	-5
iron	100
parts	0
car	0

parts factory

	y_2
labor	-5
iron	-100
parts	50
car	0

car factory

	y_3
labor	-4
iron	0
parts	-50
car	5

	y_1	y_2	y_3	y
labor	-5	-5	-4	-14
iron	100	-100	0	0
parts	0	50	-50	0
car	0	0	5	5

PRODUCTION TRANSFORMATION FUNCTION

- Example 1 continued: labor to produce corn and beef

$$Y_b = \{(-l_b, b, 0) | b \leq B(l_b)\}, \qquad Y_c = \{(-l_c, 0, c) | c \leq C(l_c)\}.$$

- Transformation function $T(y_1, y_2, y_3)$

$$\begin{aligned} T(-l, b, c) =& y_2 - \max\{y_2' | (y_1, y_2', y_3) \in Y_b + Y_c\} \\ =& b - B(l - C^{-1}(c)). \end{aligned}$$

Derivatives of Production Transformation Function

- Marginal product of labor:

生产肉的边际产品

$$TRS^{12} = \frac{T_1}{T_2} = MP_l^b,$$

$$TRS^{13} = \frac{T_1}{T_3} = MP_l^c.$$

- *Marginal rate of transformation* of beef for corn MRT^{bc} tells us the marginal opportunity cost of beef in terms of forgone units of corn

社会中多生产一单位牛肉
少生产一单位玉米

是生产两种产品
的边际成本之比

$$TRS^{23} = \frac{T_2}{T_3} = MRT^{bc}$$

边际转化率

- MRT implies that the economy can get one additional unit of beef by sacrificing MRT^{bc} unit of corn.

Derive transformation function (1)

- One input, two outputs:

$$b = 5l^{1/2}, \quad c = 10l^{1/2},$$

- The production transformation

可拿来生产牛肉的劳动力

$$T(-l, b, c) = b - 5\left(l - \frac{c^2}{100}\right)^{1/2}.$$

- Derivatives:

$$DT = \left(\frac{5}{2}\left(l - \frac{c^2}{100}\right)^{-1/2}, 1, \frac{c}{20}\left(l - \frac{c^2}{100}\right)^{-1/2}\right)$$

EXAMPLE 2 劳动力,资本品 → 牛肉,玉米

- Production possibility set

$$Y_b = \{b = (-k_b, -l_b, b, 0) | b \leq 4(k_b l_b)^{1/2}\},$$
$$Y_c = \{b = (-k_c, -l_c, 0, c) | c \leq 20(k_c l_c)^{1/2}\}.$$

- By definition 生产转换函数(选一种函数,找max)

$k_b = k - k_c \quad l_b = l - l_c$

保证 $T \leq 0$

$$T(-k, -l, b, c) = c - \max\{20(k_c l_c)^{1/2}) | 4[(k - k_c)(l - l_c)]^{1/2} \geq b\}$$

- Lagrangian for maximization problem and solve for FOC

$$\mathcal{L} = 20 k_c^{\frac{1}{2}} l_c^{\frac{1}{2}} + \lambda[4(k - k_c)^{\frac{1}{2}}(l - l_c)^{\frac{1}{2}} - b]$$

- From FOC:

少生产一单位牛肉,多生产玉米数量

$$\frac{\text{生产玉米 劳动力边际产量}}{\text{生产玉米 资本的边际产量}}$$

$$10 k_c^{\frac{1}{2}} l_c^{-\frac{1}{2}} = \lambda 2(k - k_c)^{\frac{1}{2}}(l - l_c)^{-\frac{1}{2}} = \lambda 2 k_b^{\frac{1}{2}} l_b^{-\frac{1}{2}};$$
$$10 k_c^{-\frac{1}{2}} l_c^{\frac{1}{2}} = \lambda 2(k - k_c)^{-\frac{1}{2}}(l - l_c)^{\frac{1}{2}} = \lambda 2 k_b^{-\frac{1}{2}} l_b^{\frac{1}{2}};$$
$$4(k - k_c)^{\frac{1}{2}}(l - l_c)^{\frac{1}{2}} - b = 0.$$

$$\frac{\text{生产牛肉 劳动力边际产量}}{\text{生产牛肉 资本的边际产量}}$$

边际技术替代率

边际技术替代率

Example 2 continued

- From previous conditions:

$$\lambda = 5, \quad , k_c^* = k\left(1 - \frac{b}{4(kl)^{1/2}}\right), \quad l_c^* = l\left(1 - \frac{b}{4(kl)^{1/2}}\right).$$

- Plugging into the solution into the PTF:

$$T(-k,-l,b,c) = c - 20(k_c^* l_c^*)^{\frac{1}{2}} = c - 20(kl)^{\frac{1}{2}}\left(1 - \frac{b}{4(kl)^{1/2}}\right).$$

- So

$$T(-k,-l,b,c) = c - 20(kl)^{\frac{1}{2}} + 5b.$$

最多生产的数量

  - If $b = 0 \quad \Rightarrow c = 20(kl)^{1/2}$;
  - If $c = 0 \quad \Rightarrow b = 4(kl)^{1/2}$.
  - Note that

边际转化率(由社会技术决定)

$$DT(-k,-l,b,c) = (10k^{\frac{-1}{2}} l^{\frac{1}{2}}, 10k^{\frac{1}{2}} l^{\frac{-1}{2}}, 5, 1),$$

多生产一单位肉少生产corn的数量

respectively, MP_k^c, MP_l^c, $MRT^{b,c}$

Production possibility frontier

社会生产可能性边界

- PPF: The equation for the production possibility frontier is given by setting the transformation function $T(y) = 0$
- For example 2:

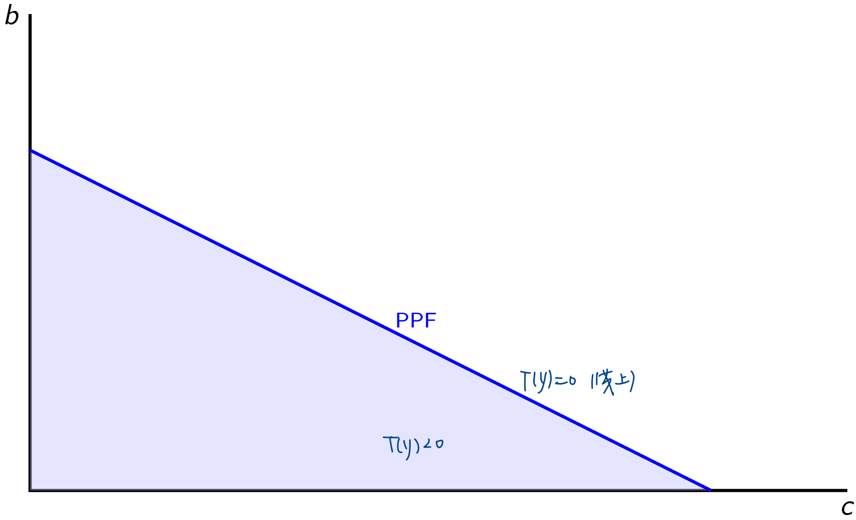
$$T(-k, -l, b, c) = 0 \Leftrightarrow c - 20(kl)^{\frac{1}{2}} + 5b = 0.$$

- If the economy is endowed with 5 units of k and l,

$$c + 5b - 100 = 0,$$

constant MRT along the PPF.

Graphical illustration of PPF



EFFICIENT ALLOCATIONS 有效分配

- Production
  - ➢ y is possible if $\forall j,\ y_j \in Y_j$;
  - ➢ y is possible if $y \in \sum_j Y_j$, or iff $T(y) \leq 0$;
  - ➢ y is efficient iff $T(y) = 0$.
- Consumption 消费可行
  - ➢ X is feasible iff

总产 总初始禀赋

$$(\exists\ y), \sum_i X_i \leq \sum_j y_j + \sum_i \omega_i \Leftrightarrow \sum_i X_i - \sum_i \omega_i \in \sum_j Y_j \Leftrightarrow$$

总消费 需要生产的

$$T(\sum_i X_i - \sum_i \omega_i) \leq 0.$$

分配有效

  - ➢ X is efficient if it is feasible and there does not exist a feasible X' such that

$$\forall i, X'_i \succeq X_i \quad \text{and} \quad \exists i,\ X'_i \succ_i X_i.$$

Pareto efficiency

- Theorem: Suppose $X \gg 0$ and $(\forall i)$, $\succeq_i$ is represented by a concave u_i which is twice continuously differentiable and strongly monotonic around X_i, and $\sum_j Y_j$ is represented by a convex function T, which is twice continuously differentiable around $(\sum_i X_i - \sum_i \omega_i)$. Then the following are equivalent
  (3) 边际替代率(消费者主观偏好决定) = 边际转化率(客观技术决定)

  条件

  - ➢ X is (Pareto) efficient;
  - ➢ $(\exists\ s_1, \ldots, s_I) \in \mathbb{R}^K_{++}$,

    生产转换函数对相同消费品求一阶偏导

    $i = (1, 2, \ldots, I)$ 向量

    福利权重

    (1) 对于all消费者对任意两种消费品的边际替代率相等.(交换有效)

$$(\forall i)\ s_i Du_i(X_i) = DT\left(\sum_i X_i - \sum_i \omega_i\right)$$

边际效用

(4) 社会总消费 − 总初禀 = 总生产

(2) 所有厂商的边际技术替代率相等(生产有效)

$$(\forall i)\ T\left(\sum_i X_i - \sum_i \omega_i\right) = 0.$$

↓PE ✓

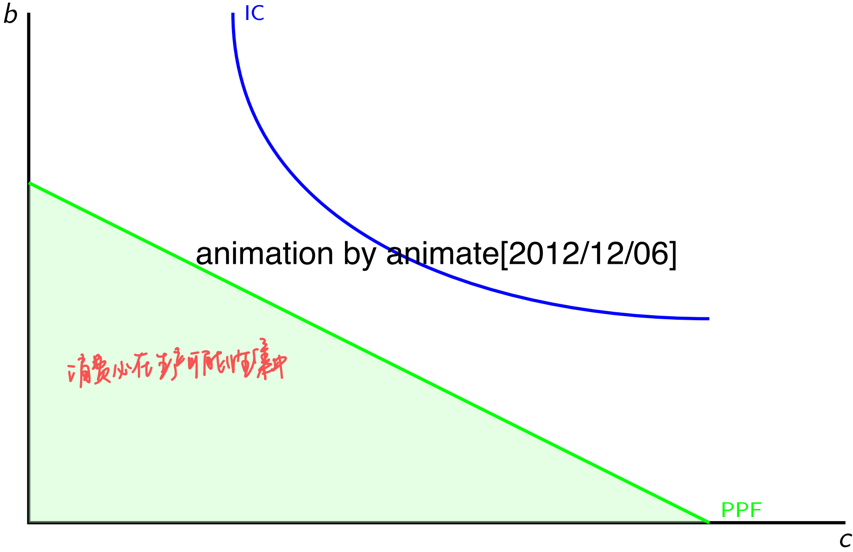
  - ➢ Remark: marginal rate of substitution (equal across consumers) equals marginal rate of transformation at PE allocations.

Graphical illustration

边际替代率 = 边际转换率

Pareto E.

无差异曲线 代表性消费者



消费必在生产可能性集中

animation by animate[2012/12/06]

EXAMPLE 1

- Robinson Crusoe economy proudction

$$y = \{y \subset (-l, c) | c \leq 6\sqrt{l}\}$$

$$T(-l, c) = c - 6\sqrt{l} = y_2 - 6(-y_1)^{1/2}$$

社会的边际转换率

$$DT(-l, c) = (3l^{-1/2}, 1) = (3(-y_1)^{-1/2}, 1).$$

- Preference

$$u = 3l + 2c.$$

- Solving the problem yields two equations in two unknowns

$$MRS^{1,2} = \frac{3}{2} = TRS^{1,2} = 3(-y_1)^{-1/2}$$

$$T(y) = y_2 - 6(-y_1)^{-1/2} = 0 \Longrightarrow \ y_1 = -4, y_2 = 12.$$

- Efficient allocations:

$$X = \begin{bmatrix} -4 \\ 12 \end{bmatrix} + \begin{bmatrix} 9 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$$

Example 2

- Two-input, two-output production:

$$Y_b = \{b = (-k_b, -l_b, b, 0) | b \leq 4(k_b l_b)^{1/2}\}$$

$$Y_c = \{b = (-k_c, -l_c, 0, c) | c \leq 20(k_c l_c)^{1/2}\}$$

- Representative agent preference

$$u = 2b^{1/2}c^{1/2}.$$

- Total endowment: $k = 4, l = 4$
- Efficient allocation

交换有效

生产有效

⋮

(1)-(4)

$$X = \begin{bmatrix} -4 \\ -4 \\ 8 \\ 40 \end{bmatrix} + \begin{bmatrix} 4 \\ 4 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 8 \\ 40 \end{bmatrix}$$

EQUILIBRIUM

利润max → ①成本min ②生产产品数量 P=marginal cost (all厂商)

边际技术替代率=投入品价格之比

- Ownership shares θ_{ji}

消费者i在厂商j的股份比例

$$(\forall i,\ j),\ \theta_{ji} \in [0,1],$$

$$(\forall\ j)\ \sum_i \theta_{ji} = 1.$$

分配社会资源 价格

- Equilibrium:
  Given $\{Y_j\}_j$ and $\{\omega_i, \theta_i, \succeq_i\}_i$, (X^*, y^*, P^*) is an equilibrium if
  - $(\forall\ i)$, X_i^* is the maximal for $\succeq_i$ in $\{X_i | P^* X_i \leq P^* \omega_i + \sum_j \theta_{ji} P^* y_j^*\}$; 从初始禀赋中得到的收入
  - $(\forall\ j)$, $y_j^* \in arg\max\{P^* y_j | y_j \in Y_j\}$; 每个厂商利润max
  - $\sum_i X_i^* = \sum_i \omega_i + \sum_j y_j^*$. 社会总需求=总供给

First Welfare Theorem

局部不饱和消费者偏好

- First Welfare Theorem: Suppose each $\succeq_i$ is locally non-satiated. Then any equilibrium is efficient. 市场分配结果有效
- Implication: market is good!
- Proof:
  - Suppose (X^*, y^*, P^*) is an equilibrium and X^* is not efficient,
    - Parent improvement: $\exists\ X',\ y',$

$$(\forall\ i)\ X_i' \succeq_i X_i^*;$$

    - Feasibility condition: $(\exists\ i)\ X_i' \succ_i X_i^*$,

$$\sum_i X_i' \leq \sum_j y_j' + \sum_i \omega_i.$$

Proof continued

- The first two equations imply

$$(\exists\ i)\ P^* X_i' > P^*\omega_i + \sum_j \theta_{ji} P^* y_j^*$$

$$(\forall\ i)\ P^* X_i' \geq P^*\omega_i + \sum_i \theta_{ji} P^* y_j^* \Longrightarrow$$

$$\sum_i P^* X_i' > \sum_i P^*\omega_i + \sum_i \sum_j \theta_{ji} P^* y_j^*, \qquad \sum_i \theta_{ji} = 1$$

- From the feasibility condition,

$$\sum_i P^* X_i^* \leq \sum_j P^* y_j' + \sum_i P^*\omega_i$$

$$\sum_i P^* X_i' > \sum_j P^* y_j^* + \sum_i P^*\omega_i \Longrightarrow$$

$$P^* \sum_j y_j' > \sum_j P^* y_j^*.$$

- Contradiction, as y^* maximizes profits given P^*

Second Welfare Theorem

- Second welfare Theorem: Suppose that $(\forall j)$, Y_j is convex, $(\forall i)$, $\succeq_i$ is locally non-satiated and convex. Then for every Pareto efficient (X^*, Y^*) such that $X^* \gg 0$, there exists $P^* > 0$ so that (X^*, y^*, P^*) is an equilibrium.
- Implications:
  - ➢ Any efficient allocations can be achieved using market mechanism.
  - ➢ The problems of distribution and efficiency can be separated
  - ➢ We can redistribute endowments to obtain an ideal distribution
  - ➢ However, price should be used to allocation final consumption, as it reflects the relative scarcity of different resources in the economy.

EXAMPLE 1

①消费者效用最大化 → ②厂商利润max {成本min 边际技术替代率 =投入的价格之比

生产每种产品的成本函数
↓ 对商品求偏导
边际成本(即商品的价格)

- Blue collar worker $L_b = 150, K_b = 0$;
- White collar worker $L_w = 50, K_w = 50$;
- Production function: food (x), energy (y).

③→市场出清:总需求=总供给
↓
求出相对价格

$$x = L_x^{1/2} K_x^{1/2}, \qquad y = L_y^{1/2} K_y^{1/2}.$$

- Production transformation

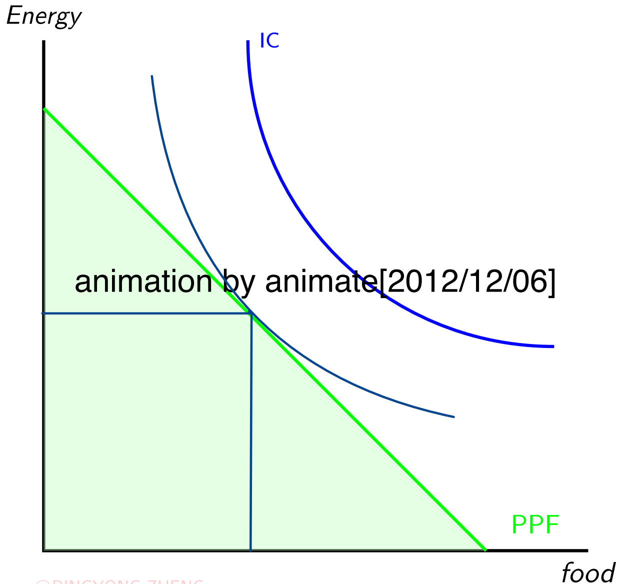
$$T(-K, -L, x, y) = x - (LK)^{1/2} + y \Longrightarrow$$

$$DT = \left(\frac{1}{2} K^{-1/2} L^{1/2}, \frac{1}{2} K^{1/2} L^{-1/2}, 1, 1\right)$$

- Preference

$$U_b(x_b, y_b) = (x_b y_b)^{1/2} \qquad U_w(x_w, y_w) = (x_w y_w)^{1/2}.$$

Production possibility frontier



Solve for equilibrium (1)

- From utility-maximization,

$$x_b = \frac{I_b}{2p_x},\ y_b = \frac{I_b}{2p_y},$$
$$x_w = \frac{I_w}{2p_x},\ y_w = \frac{I_w}{2p_y},$$

- Note

$$I_b = 150w \qquad I_w = 50w + 50r + \pi_x + \pi_y.$$

- From profit-maximization,

$$MRTS^x_{L,K} = MRTS^y_{L,K} = \frac{w}{r},$$

- So

$$wL_x = rK_x \qquad wL_y = rK_y \Longrightarrow$$
$$\underbrace{\frac{L_x}{K_x}}_{MRTS_x} = \underbrace{\frac{L_y}{K_y}}_{MRTS_y} = \frac{r}{w}.$$

Equilibrium (2)

- In equilibrium, labor market and capital market clear

$$200 = L_x + L_y \qquad 50 = K_x + K_y \Longrightarrow$$

$$\frac{L_x}{K_x} = \frac{L_y}{K_y} = \frac{L_x + L_y}{K_x + K_y} = 4 = \frac{r}{w}.$$

- Substitution efficiency:

$$MRT_{x,y} = \frac{MC_x}{MC_y} = \frac{p_x}{p_y} \Longrightarrow MRT_{x,y} = MRS^b_{x,y} = MRS^w_{x,y}.$$

- We know that in competitive equilibrium,

$$MRS^b_{x,y} = \frac{y_b}{x_b}.$$

- Let $w = 1$ and thus, $r = 4$.
- In equilibrium, $L_x = 4K_x$, $L_y = 4K_y$

$$MC_x = 4, \qquad MC_y = 4 \Longrightarrow \quad p_x = p_y = 4.$$

Equilibrium (3)

- Therefore, in equilibrium

$$x_b = y_b, x_w = y_w.$$

- Income for a typical white collar worker and blue collar worker

$$I_w = 50 + 4 \times 50 = 250, \qquad I_b = 150.$$

- Given the income we can find the equilibrium allocation

$$x_b = \frac{75}{4}, y_b = \frac{75}{4}; \qquad x_w = \frac{125}{4}, y_w = \frac{125}{4}.$$

- Summary of the competitive equilibrium
  - Price:

$$w = 1, r = 4, p_x = 4, p_y = 4;$$

  - Production:

$$L_x = 100, L_y = 100, K_x = 25, K_y = 25.$$

  - Allocations:

$$x_b = \frac{75}{4}, y_b = \frac{75}{4}; \qquad x_w = \frac{125}{4}, y_w = \frac{125}{4}.$$

Example 2 (1)

- The economy has 100 blue and white collar households;
- Each blue collar household endowed with 60 units of labor (L) and has preference

$$U^B = x^{\frac{3}{4}} y^{\frac{1}{4}}.$$

- Each white collar household endowed with 10 units of labor and 50 units of capital (K) and has preference

$$U^W = x^{\frac{1}{2}} y^{\frac{1}{2}}.$$

- The production function for the economy is

$$x = 1.89 L^{\frac{1}{3}} K^{\frac{2}{3}}, \qquad y = 2 L^{\frac{1}{2}} K^{\frac{1}{2}}.$$

➢ In this case, the economy's total inputs are

$$L = 100(10 + 60) = 7000, \qquad K = 100(50 + 0) = 5000.$$

Example 2 (2)

- Utility-maximization

$$x_B = \frac{3I_B}{4P_x}, \qquad y_B = \frac{I_B}{4P_y};$$

$$x_W = \frac{I_W}{2P_x}, \qquad y_W = \frac{I_W}{2P_y};$$

with $I_B = 60w$, $I_W = 10w + 50r$.

- Production efficiency
  - ➢ cost-minimization

$$\frac{K_x}{2L_x} = \frac{w}{r} \Longrightarrow L_x = \frac{rK_x}{2w}.$$

$$\frac{K_y}{L_y} = \frac{w}{r} \Longrightarrow L_y = \frac{rK_y}{w}.$$

  - ➢ Plugging L_x into production function, we can get condition input demand

$$x = 1.89\left(\frac{rK_x}{2w}\right)^{\frac{1}{3}} K_x^{\frac{2}{3}}, \qquad y = 2\left(\frac{rK_y}{w}\right)^{\frac{1}{2}} K_y^{\frac{1}{2}}$$

Example 2 (3)

- Production
  - Conditional input demand:

$$K_x = \frac{2x}{3}\left(\frac{w}{r}\right)^{\frac{1}{3}}, \quad L_x = \frac{x}{3}\left(\frac{r}{w}\right)^{\frac{2}{3}};$$

$$K_y = \frac{y}{2}\left(\frac{w}{r}\right)^{\frac{1}{2}}, \quad L_y = \frac{y}{2}\left(\frac{r}{w}\right)^{\frac{1}{2}}.$$

  - Given the demand curves, total cost

$$TC_x = wL_x + rK_x = \frac{x}{3}w^{\frac{1}{3}}r^{\frac{2}{3}} + \frac{2x}{3}w^{\frac{1}{3}}r^{\frac{2}{3}} = xw^{\frac{1}{3}}r^{\frac{2}{3}}.$$

$$TC_y = wL_y + rK_y = \frac{y}{2}w^{\frac{1}{2}}r^{\frac{1}{2}} + \frac{y}{2}w^{\frac{1}{2}}r^{\frac{1}{2}} = yw^{\frac{1}{2}}r^{\frac{1}{2}}.$$

  - Marginal cost

$$MC_x = w^{\frac{1}{3}}r^{\frac{2}{3}}, \quad MC_y = w^{\frac{1}{2}}r^{\frac{1}{2}}.$$

- In equilibrium

$$P_x = MC_x = w^{\frac{1}{3}}r^{\frac{2}{3}}, \qquad P_y = MC_y = w^{\frac{1}{2}}r^{\frac{1}{2}}.$$

Example 2 (3)

- Markets for x, y clears

$$x = 100x_B + 100x_W = \frac{50I_W + 75I_B}{P_x} = \frac{5000w + 2500r}{w^{\frac{1}{3}} r^{\frac{2}{3}}},$$

$$y = 100y_B + 100y_W = \frac{50I_W + 25I_B}{P_y} = \frac{2000w + 2500r}{w^{\frac{1}{2}} r^{\frac{1}{2}}}.$$

- Market for labor clears

$$\begin{aligned} 7000 &= L_x + L_y = \frac{x}{3}\left(\frac{r}{w}\right)^{\frac{2}{3}} + \frac{y}{2}\left(\frac{r}{w}\right)^{\frac{1}{2}} \\ &= \frac{5000w + 2500r}{3w^{\frac{1}{3}} r^{\frac{2}{3}}} \cdot \left(\frac{r}{w}\right)^{\frac{2}{3}} + \frac{2000w + 2500r}{2w^{\frac{1}{2}} r^{\frac{1}{2}}} \cdot \left(\frac{r}{w}\right)^{\frac{1}{2}} \\ &= \frac{5000w + 2500r}{3w} + \frac{2000w + 2500r}{2w}. \end{aligned}$$

- This gives

$$\frac{r}{w} = 2.08.$$

Example 2 (4)

- Market for capital clears

$$5000 = K_x + K_y = \frac{10000w + 5000r}{3r} + \frac{2000w + 2500r}{2r},$$

- This also gives

$$\frac{r}{w} = 2.08.$$

- If we let $w = 1$, we get $r = 2.08$.
- Plugging $r/w = 2.08$ into the equation for $x,\ y$,

$$x = 6300, \qquad y = 5000.$$

- We also get $P_x = 1.628$, $P_y = 1.4422$.

Example 2 (4)

- Market for capital clears

$$5000 = K_x + K_y = \frac{10000w + 5000r}{3r} + \frac{2000w + 2500r}{2r},$$

- This also gives

$$\frac{r}{w} = 2.08.$$

- If we let $w = 1$, we get $r = 2.08$.
- Plugging $r/w = 2.08$ into the equation for $x,\ y$,

$$x = 6300, \qquad y = 5000.$$

- We also get $P_x = 1.628$, $P_y = 1.4422$.