# Problem Set 2 Solutions

**Problem 1.** Consider a more general choice-based approach to demand: assume that there exists a choice correspondence $x(p,w)$ defined on $\{B_{p,w} : p \gg 0, w > 0\}$. Assume that $x(p,w)$ satisfies the weak axiom of revealed preference and Walras' law. Show the following generalized compensated law of demand: for any $p \gg 0, w > 0$ and $p' \gg 0$, if $x \in x(p,w)$ and $w' = p' \cdot x$, then $[p' - p] \cdot [x' - x] \leq 0$ for any $x' \in x(p', w')$.

**Answer.** Assume to the contrary, there exist $p \gg 0, w > 0$ and $p' \gg 0$ such that for some $x \in x(p,w)$ and $x' \in x(p', w' = p' \cdot x)$, we have $[p' - p] \cdot [x' - x] > 0$. Then $p' \cdot x' - p' \cdot x - p \cdot x' + p \cdot x > 0$. By Walras' law, $p' \cdot x' = w'$ and $p \cdot x = w$. It follows that $p \cdot x' < w$. By Walras' law again, $x' \notin x(p,w)$. Therefore, we have $x, x' \in B_{p,w}$, $x \in x(p,w), x' \notin x(p,w)$, $x, x' \in B_{p',w'}$ and $x' \in x(p', w')$, contradicting to WARP.

**Problem 2.** Show that the lexicographic preference relation (as defined in the lecture notes, on $\mathbb{R}^2_+$) is complete, transitive, strongly monotone and strictly convex.

**Answer.** Recall that the lexicographic preference relation is defined as follows, on $X = \mathbb{R}^2_+$: for all $x, y \in X$, $x \succeq y$ if $x_1 > y_1$, or, $x_1 = y_1$ and $x_2 \geq y_2$.
*Completeness.* Given any $x, y \in X$, there are four possible cases. Case 1: $x_1 > y_1 \Rightarrow x \succeq y$. Case 2: $x_1 < y_1 \Rightarrow y \succeq x$. Case 3: $x_1 = y_1$ and $x_2 \geq y_2 \Rightarrow x \succeq y$. Case 4: $x_1 = y_1$ and $x_2 < y_2 \Rightarrow y \succeq x$. In sum, either $x \succeq y$ or $y \succeq x$.
*Transitivity.* Suppose that $x \succeq y$ and $y \succeq z$. Then we have $x_1 \geq y_1 \geq z_1$. There are two possible cases. Case 1: $x_1 > y_1$ or $y_1 > z_1$ (or both). Then $x_1 > z_1$ and $x \succeq z$. Case 2: $x_1 = y_1 = z_1$. Then we must have $x_2 \geq y_2 \geq z_2$, hence $x \succeq z$.
*Strong monotonicity.* Suppose that $x \geq y$ and $x \neq y$. Then $x_1 > y_1$, or, $x_1 = y_1$ and $x_2 > y_2$. In either case, we have $x \succeq y$ and $y \not\succeq x$, so $x \succ y$.
*Strict convexity.* Consider the upper contour set of any $x \in X$. Let $y \succeq x, z \succeq x$, $y \neq z$ and $\alpha \in (0,1)$. Then $y_1 \geq x_1$ and $z_1 \geq x_1$. We want to show that $\alpha y + (1-\alpha) z \succ x$. There are two possible cases. Case 1: $y_1 > x_1$ or $z_1 > x_1$. Then $\alpha y_1 + (1-\alpha) z_1 > x_1$, so $\alpha y + (1-\alpha) z \succ x$. Case 2: $y_1 = z_1 = x_1$. Then $y_2 \geq x_2$ and $z_2 \geq x_2$. Since $y \neq z$, we have $y_2 > x_2$ or $z_2 > x_2$. Hence $\alpha y_2 + (1-\alpha) z_2 > x_2$. Given that $\alpha y_1 + (1-\alpha) z_1 = x_1$ in this case, it follows that $\alpha y + (1-\alpha) z \succ x$.

**Problem 3.** Let $u$ be a utility function representing a preference relation $\succeq$. Show that $u$ is strictly quasiconcave if and only if $\succeq$ is strictly convex.

**Answer.** "*If*" *part.* Let $x, y \in X$, $x \neq y$ and $\alpha \in (0,1)$. We want to show that

$$u(\alpha x + (1-\alpha)y) > \min\{u(x), u(y)\} \tag{1}$$

As $u$ represents $\succeq$, $\succeq$ is complete. So $x \succeq y$ or $y \succeq x$.

If $x \succeq y$, then by the strict convexity of $\succeq$ we have $\alpha x + (1-\alpha)y \succ y$. So $u(\alpha x + (1-\alpha)y) > u(y) = \min\{u(x), u(y)\}$, and (1) is proved.

If $y \succeq x$, then by the strict convexity of $\succeq$ we have $\alpha x + (1-\alpha)y \succ x$. So $u(\alpha x + (1-\alpha)y) > u(x) = \min\{u(x), u(y)\}$, and (1) is proved.

"*Only if*" *part.* Consider the upper contour set of any $x \in X$. Let $y \succeq x$, $z \succeq x$, $y \neq z$ and $\alpha \in (0,1)$. We want to show that

$$\alpha y + (1-\alpha)z \succ x \tag{2}$$

By the strict quasiconcavity of $u$, and the fact that $u(y) \geq u(x)$ and $u(z) \geq u(x)$, we have $u(\alpha y + (1-\alpha)z) > \min\{u(y), u(z)\} \geq u(x)$. It follows that (2) is true.

**Problem 4.** Let $u$ be a continuous utility function and $x(p,w)$ be the corresponding Walrasian demand correspondence derived from utility maximization. Then $x(p,w)$ can be considered as a choice correspondence defined on $\left\{B_{p,w} : p \gg 0, w > 0\right\}$.

(a) Show that $x(p,w)$ satisfies WARP.

(b) Can $x(p,w)$ be rationalized? Explain your answer.

**Answer.** (a). Suppose that for some $B_{p,w}$ with $x, y \in B_{p,w}$ we have $x \in x(p,w)$ and $y \notin x(p,w)$. Then $u(x) > u(y)$ as $x(p,w)$ is derived from utility maximization. It follows that for any $B_{p',w'}$ with $x, y \in B_{p',w'}$, $y$ is not a solution to the utility maximization problem with respect to $p'$ and $w'$, i.e., $y \notin x(p',w')$. This shows that WARP is satisfied.

(b) Yes. Define $\succeq$ as follows: for any $x, y$, let $x \succeq y$ if $u(x) \geq u(y)$. As $u$ obviously represents $\succeq$, $\succeq$ is rational. Recall that, for any $B_{p,w}$,

$$C_{\succeq}(B_{p,w}) = \arg\max_{x \in B_{p,w}} u(x) = x(p,w)$$

Hence, $x(p,w)$ can be rationalized by $\succeq$.

**Problem 5.** Let $u : \mathbb{R}^2_+ \to \mathbb{R}$ be a continuous utility function, and let $v(p,w)$ be the corresponding indirect utility function.
(a) Prove that for any price vector $p \gg 0$ and consumption bundle $x \in \mathbb{R}^2_+$ with $x \neq 0$, $v(p, p \cdot x) \geq u(x)$.
(b) Given a consumption bundle $x \in \mathbb{R}^2_+$, $x \neq 0$, does there always exist a price vector $p \gg 0$ such that $v(p, p \cdot x) = u(x)$? If so, prove it. Otherwise provide a counterexample.

**Answer.** (a). As $p \gg 0$, $x \geq 0$ and $x \neq 0$, $p \cdot x > 0$. Then $x \in B_{p,p\cdot x}$ implies $v(p, p \cdot x) \geq u(x)$.

(b). There may not exist such a price vector. An example is as follows. $u(x) = \min\{x_1, x_2\}$ and $x = (1,2)^T$. For any $p \gg 0$, consider the UMP with respect to $p$ and income $p \cdot x = p_1 + 2p_2$. The Walrasian demand is $(\frac{p_1+2p_2}{p_1+p_2}, \frac{p_1+2p_2}{p_1+p_2})^T$. Hence $v(p, p \cdot x) = \frac{p_1+2p_2}{p_1+p_2} > 1 = u(x)$.

**Problem 6.** For each of the following utility functions, derive the Hicksian demand and expenditure function, at prices $(p_1, p_2) \gg 0$ and utility $u > 0$.
(a) $u(x_1, x_2) = \min\{2x_1, 3x_2\}$
(b) $u(x_1, x_2) = 3x_1 + 2x_2$
(c) $u(x_1, x_2) = x_1^\alpha x_2^\beta$, $\alpha > 0, \beta > 0$

**Answer.** All of the three utility functions are continuous and unbounded, so EMP always has a solution, and "no excess utility" is satisfied.

(a). If $(x_1^*, x_2^*)^T \in h(p,u)$, then we must have

$$2x_1^* = 3x_2^*$$

By no excess utility,

$$2x_1^* = 3x_2^* = u$$

Then there exists a unique solution

$$x_1^* = \frac{u}{2}, \;\; x_2^* = \frac{u}{3}$$

and

$$e(p,u) = \frac{p_1 u}{2} + \frac{p_2 u}{3}$$

(b). We solve this problem by exploring the linear nature of the utility function. By no excess utility,

$$h(p,u) \subseteq \{z : z_1 \geq 0, z_2 \geq 0, 3z_1 + 2z_2 = u\} = S$$

That is, any solution to EMP must be in the set $S$. Consider the following two bundles

$$x = (\frac{u}{3}, 0)^T \in S$$

$$y = (0, \frac{u}{2})^T \in S$$

For any $z \in S$, by the linearity of the utility function, there exists $\alpha \in [0,1]$ such that $z = \alpha x + (1-\alpha)y$. (How to find such $\alpha$? Let $z = \alpha x + (1-\alpha)y = (\frac{\alpha u}{3}, \frac{(1-\alpha)u}{2})^T = (z_1, z_2)^T$. Then $\alpha = \frac{3z_1}{u} = \frac{u-2z_2}{u} \in [0,1]$ ).

It follows that for any $z \in S$, there exists $\alpha \in [0,1]$ such that $p \cdot z = \alpha(p \cdot x) + (1-\alpha)(p \cdot y)$. That is, the expenditure of any possible solution to EMP must be a weighted average of the expenditures of $x$ and $y$. Then the solutions can be easily identified by inspecting the expenditures of $x$ and $y$:

If $\frac{p_1 u}{3} < \frac{p_2 u}{2}$, then $h(p,u) = \{x\}$. $e(p,u) = \frac{p_1 u}{3}$.
If $\frac{p_1 u}{3} > \frac{p_2 u}{2}$, then $h(p,u) = \{y\}$. $e(p,u) = \frac{p_2 u}{2}$.
If $\frac{p_1 u}{3} = \frac{p_2 u}{2}$, then $h(p,u) = S$. $e(p,u) = \frac{p_1 u}{3}$.

(c). Lagrangian:

$$\mathscr{L} = p_1 x_1 + p_2 x_2 + \lambda(u - x_1^\alpha x_2^\beta)$$

As $u > 0$, any solution must be interior and hence satisfy the following conditions.

$$\frac{\partial \mathcal{L}}{\partial x_1} = p_1 - \lambda \alpha x_1^{\alpha-1} x_2^{\beta} = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = p_2 - \lambda \beta x_1^{\alpha} x_2^{\beta-1} = 0$$

$$u = x_1^{\alpha} x_2^{\beta}$$

Solving these equations, there exists a unique solution

$$x_1^* = u^{\frac{1}{\alpha+\beta}} (\frac{p_2}{p_1}\frac{\alpha}{\beta})^{\frac{\beta}{\alpha+\beta}}$$

$$x_2^* = u^{\frac{1}{\alpha+\beta}} (\frac{p_1}{p_2}\frac{\beta}{\alpha})^{\frac{\alpha}{\alpha+\beta}}$$

Then

$$e(p,u) = p_1 u^{\frac{1}{\alpha+\beta}} (\frac{p_2}{p_1}\frac{\alpha}{\beta})^{\frac{\beta}{\alpha+\beta}} + p_2 u^{\frac{1}{\alpha+\beta}} (\frac{p_1}{p_2}\frac{\beta}{\alpha})^{\frac{\alpha}{\alpha+\beta}}$$

**Problem 7.** Suppose that the utility function $u(x)$ is homogeneous of degree one. Show that for any $p \gg 0$, $w > 0$ and $\alpha > 0$,

$$x(p, \alpha w) = \left\{x \in \mathbb{R}_+^L : x = \alpha y, y \in x(p,w)\right\}$$

and

$$v(p, \alpha w) = \alpha v(p, w)$$

(Hint: in the first part you have to show the two sets are the same. That is, if $x \in x(p,w)$, then $\alpha x \in x(p, \alpha w)$, and if $x \in x(p, \alpha w)$, then $\frac{1}{\alpha} x \in x(p,w)$.)

**Answer.** Consider any $p \gg 0$, $w > 0$ and $\alpha > 0$. We first show that if $x \in x(p,w)$, then $\alpha x \in x(p, \alpha w)$. Assume to the contrary, there exists some $x \in x(p,w)$ with $\alpha x \notin x(p, \alpha w)$. Then there exists $y \geq 0$ such that

$$p \cdot y \leq \alpha w, \text{ and } u(y) > u(\alpha x)$$

Since $u$ is homogeneous of degree one, we have

$$p \cdot (\frac{1}{\alpha} y) \leq w, \text{ and } u(\frac{1}{\alpha} y) > u(x)$$

This contradicts to $x \in x(p, w)$, since $\frac{1}{\alpha} y \in B_{p,w}$ achieves a higher utility.

Now, we show that if $x \in x(p, \alpha w)$, then $\frac{1}{\alpha} x \in x(p, w)$. Assume to the contrary, there exists some $x \in x(p, \alpha w)$ with $\frac{1}{\alpha} x \notin x(p, w)$. Then there exists $y \geq 0$ such that

$$p \cdot y \leq w \text{ and } u(y) > u(\frac{1}{\alpha} x)$$

By the homogeneity of $u$, we have

$$p \cdot (\alpha y) \leq \alpha w \text{ and } u(\alpha y) > u(x)$$

This contradicts to $x \in x(p, \alpha w)$, since $\alpha y$ achieves a higher utility in $B_{p,\alpha w}$.

Pick any $x^* \in x(p, w)$. Since $\alpha x^* \in x(p, \alpha w)$, we have

$$v(p, \alpha w) = u(\alpha x^*) = \alpha u(x^*) = \alpha v(p, w)$$

That is, the indirectly utility function is homogeneous of degree one in $w$.

**Problem 8.** Consider the indirect utility function:

$$v(p_1, p_2, w) = \alpha \ln \frac{\alpha}{p_1} + (1 - \alpha) \ln \frac{(1 - \alpha)}{p_2} + \ln w, \quad \alpha \in (0, 1)$$

(a) Derive the Walrasian demand function.
(b) Derive the expenditure function.
(c) Derive the Hicksian demand function.

**Answer.**

We can first simplify the indirect utility function:

$$v(p_1, p_2, w) = \alpha \ln \alpha - \alpha \ln p_1 + (1 - \alpha) \ln (1 - \alpha) - (1 - \alpha) \ln p_2 + \ln w$$

(a). The Walrasian demand function can be found by using Roy's identity:

$$x_1(p_1,p_2,w) = -\frac{\frac{\partial v(p_1,p_2,w)}{\partial p_1}}{\frac{\partial v(p_1,p_2,w)}{\partial w}} = -\frac{\frac{-\alpha}{p_1}}{\frac{1}{w}} = \frac{\alpha w}{p_1}$$

$$x_2(p_1,p_2,w) = -\frac{\frac{\partial v(p_1,p_2,w)}{\partial p_2}}{\frac{\partial v(p_1,p_2,w)}{\partial w}} = -\frac{\frac{-(1-\alpha)}{p_2}}{\frac{1}{w}} = \frac{(1-\alpha)w}{p_2}$$

(b). The inverse function of $v(p_1,p_2,w)$ is the expenditure function. So let

$$v(p_1,p_2,w) = \alpha \ln \alpha - \alpha \ln p_1 + (1-\alpha)\ln(1-\alpha) - (1-\alpha)\ln p_2 + \ln w = u$$

Solving $w$:

$$\ln w = u - \alpha \ln \alpha + \alpha \ln p_1 - (1-\alpha)\ln(1-\alpha) + (1-\alpha)\ln p_2$$

$$w = e^u \alpha^{-\alpha} p_1^{\alpha}(1-\alpha)^{\alpha-1} p_2^{1-\alpha}$$

Therefore

$$e(p_1,p_2,u) = e^u \alpha^{-\alpha} p_1^{\alpha}(1-\alpha)^{\alpha-1} p_2^{1-\alpha}$$

(c). The Hicksian demand can be found by differentiating the expenditure function with respect to prices:

$$h_1(p,u) = \frac{\partial e(p_1,p_2,u)}{\partial p_1} = e^u \alpha^{1-\alpha} p_1^{\alpha-1}(1-\alpha)^{\alpha-1} p_2^{1-\alpha} = e^u\left(\frac{\alpha}{1-\alpha}\frac{p_2}{p_1}\right)^{1-\alpha}$$

$$h_2(p,u) = \frac{\partial e(p_1,p_2,u)}{\partial p_2} = e^u \alpha^{-\alpha} p_1^{\alpha}(1-\alpha)^{\alpha} p_2^{-\alpha} = e^u\left(\frac{1-\alpha}{\alpha}\frac{p_1}{p_2}\right)^{\alpha}$$