MICROECONOMIC THEORY II

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Why extensive form game?

- Strategic form games describe a game by its strategies—complete contingent plans of how to react in each possible scenario—and play down the temporal aspect of the situation—who moves first, who moves second, etc. It is like a computer chess program. Once each player submit the programs, the computer will take over and decide which side will win. You don't get to see the actual step-by-step plays.

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- Extensive-form games explicitly describe how the game is played through time, including details about who moves first, who moves second, and so on.
- In this sense, extensive form game provides more information than the strategic form.

Game between a young kid with his parents

Analyzing the example

- IF we just look at NE, then we may have some problem:

kid

		GG	GS	SG	SS
Parent	buy	-2, 1	-2, 1	-5, -10	-5,-10
	not buy	0, -2	-5, -10	0, -2	-5, -10

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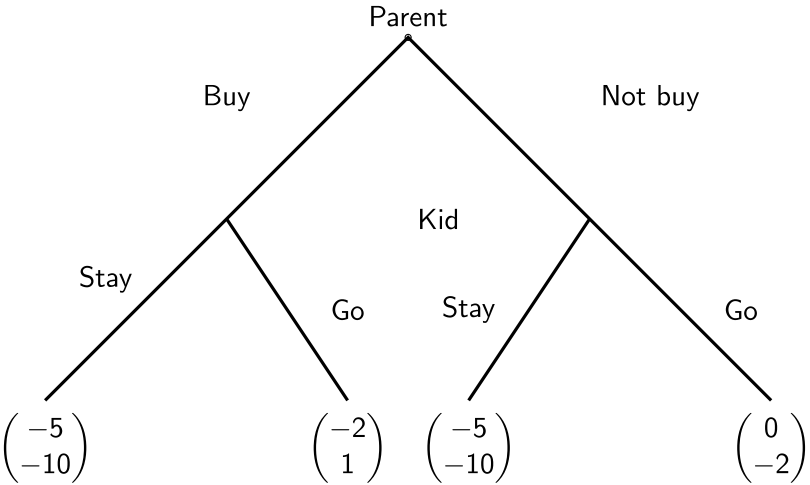
- Not all NE reasonable predictions!

How the game gets played



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   - $\alpha : \mathscr{X} \backslash \{x_0\} \to \mathscr{A}$ giving the action that leads to x from $p(x)$

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3. Information sets: $H : \mathscr{X} \to \mathscr{H}$,

$$c(x) = c(x') \quad \text{if} \quad H(x) = H(x')$$

Extensive form continued

4. The probability distribution over any exogenous events:

$$\rho : \mathscr{H}_0 \times \mathscr{A} \to [0,1], \quad \rho(H,a) = 0 \quad \text{if } a \notin C(H) \text{ and}$$
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5. The players' payoffs as a function of the moves that were made.

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- More formally, if x and x' belongs to the same information set of player i, then it must be that 1) the sequence of moves that leads to x and the sequence of moves that leads to x' must pass through the same sequence of information sets for player i, and 2) in each of the information set of players i that leads to x and x', the same action must be chosen by player i.

Example 1: imperfect recall

Another example

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- The point is: although the extensive form tells us more about the sequence of moves, it is not a completely accurate description (when the game involves simultaneous moves).

Battle of sexes game

	a	b
a	3,1	0,0
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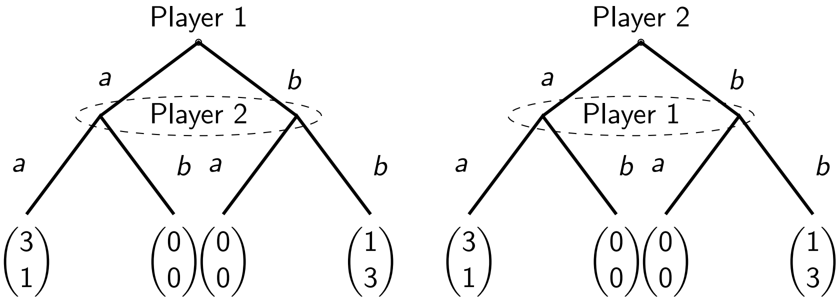

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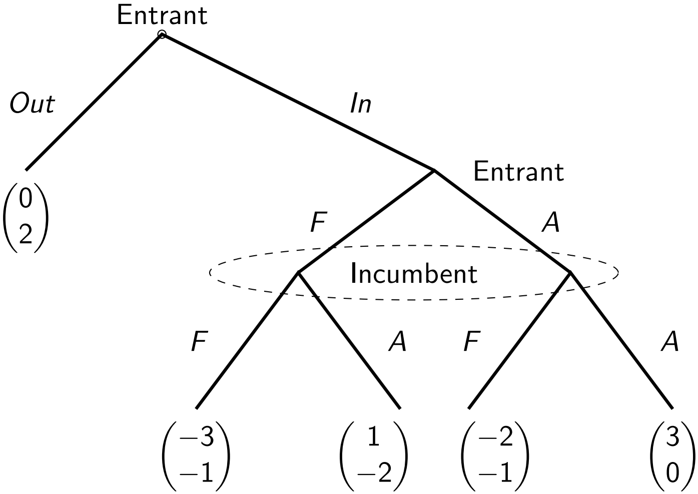
- Example: In the Entrant-incumbent game given below:
  - ➢ Firm I has two pure strategies: fight, accommodate
  - ➢ Firm E has four pure strategies:
    Out and Fight if In (OF), Out and Accommodate if In (OA), In and Fight if In (IF), In and Accommodate if In (IA).

Entrant-incumbent game



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   - and a prediction about i's future behavior should she deviates from her plan.

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7. In addition, to know the belief of other players about her play at those information set and how they would respond help rationalize player i's plan in the first place (her choices at information sets that could be reached given her PLAN).

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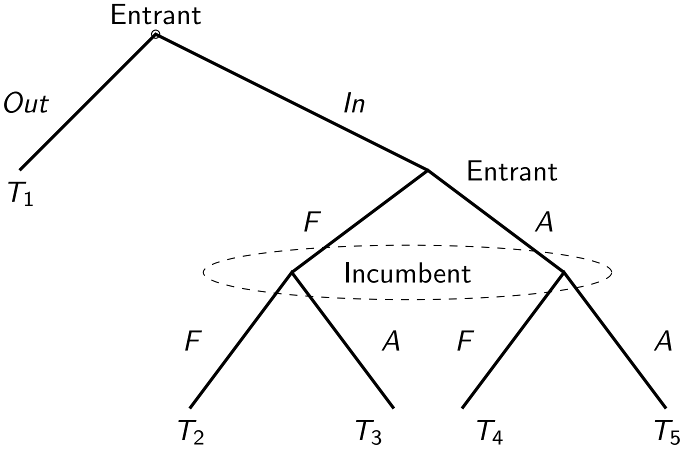
- This implies that we can use either one of the two at our convenience.
- We typically use *behavior* strategies for *extensive* form game, and *mixed* strategies for *strategical* form game.

Equivalence of two mixed strategies



Sketch of proof

1. For any mixed strategy of Firm E (OF,OA,IF,IA; p_1, p_2, p_3, p_4), there exists a unique behavior strategy such that the probability of reaching terminal nodes $T_1, \ldots, T_5$ is the same.

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2. To see this, let the mixed strategy for Firm I be (F, A; σ, $1 - \sigma$), we show there is a unique behavior strategy for Firm E that assigns q and $1 - q$ to "Out" and "In" at the first information set, and assigns r and $1 - r$ to "F" and "A" at the second information set.

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3. Given the mixed strategy of Firm E and Firm I:

$$Pr(T_1) = p_1 + p_2, \quad Pr(T_2) = p_3\sigma, \quad Pr(T_3) = p_3(1-\sigma)$$
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4. Hence we have the unique behavior strategy

$$q = p_1 + p_2, \quad r = \frac{p_3}{1-(p_1+p_2)}.$$

PROOF CONTINUED

5. On the other hand, given a behavior strategy $(q, 1-q)$ at the first information set and $(r, 1-r)$ at the second information set, we can get the mixed strategy.

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6. The unique mixed strategy that is equivalent to the behavior strategy is:

$$p_1 = qr, \quad p_2 = q(1-r), \quad p_3 = (1-q)r, \quad p_4 = (1-q)(1-r).$$

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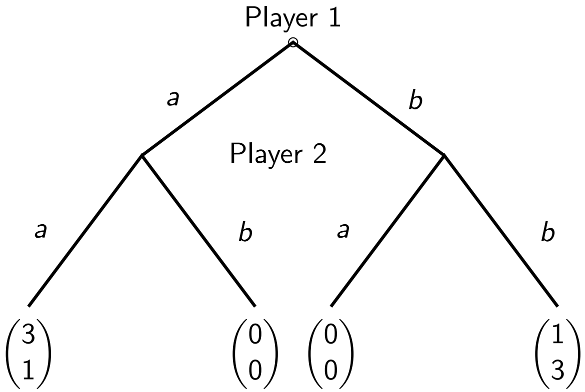
- The sequential battle of sexes game:



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- Two extensive form games may have the same strategic form. For, example, the game above may also be a 2x4 simultaneous-move game.

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- Backward induction ensures that a player's strategies specify optimal behavior at every decision node of the game.

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- The concept of Nash equilibrium does not distinguish whether a threat is credible because as long as a threat is effective, it has no payoff consequences.

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  3. Repeat the same procedure until the initial node is reached. The resulting payoff profile is called a backward induction solution.

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    3. Repeat the same procedure until the initial node is reached. The resulting payoff profile is called a backward induction solution.
- Backward induction solutions are all Nash equilibrium, but the converse is false. The solution is unique if no player is ever indifferent between two actions.

Subgame perfect NE

- Subgame: The portion of the game tree that follows a decision node x is a *subgame* if it constitutes a well-defined extensive form game. That is, if

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- Subgame perfection generalizes the idea of backward induction to games of imperfect information. The backward induction solution is always subgame perfect.
- The way to find subgame perfect equilibrium is similar to backward induction: starting from the subgame near the end and work backward.

An application

Strategical form of the game

Player 3

	lll	llr	lrl	lrr	rrl	rlr	rll	rrr
L	2, 0, 1	2, 0, 1	2, 0, 1	2, 0, 1	-1, 5, 6	-1, 5, 6	-1, 5, 6	-1, 5, 6
R	3, 1, 2	3, 1, 2	5, 4, 4	5, 4, 4	5, 4, 4	3, 1, 2	3, 1, 2	5, 4, 4

Player 2 plays a

Player 3

	lll	llr	lrl	lrr	rrl	rlr	rll	rrr
L	2, 0, 1	2, 0, 1	2, 0, 1	2, 0, 1	-1, 5, 6	-1, 5, 6	-1, 5, 6	-1, 5, 6
R	0, -1, 7	-2, 2, 0	0, -1, 7	-2, 2, 0	0, -1, 7	-2, 2, 0	0, -1, 7	-2, 2, 0

Player 2 plays b

Strategical form of the game

Player 3

	lll	llr	lrl	lrr	rrl	rlr	rll	rrr
L	2, 0, 1	2, 0, 1	2, 0, 1	2, 0, 1	-1, 5, 6	-1, 5, 6	-1, 5, 6	-1, 5, 6
R	3, 1, 2	3, 1, 2	5, 4, 4*	5, 4, 4*	5, 4, 4*	3, 1, 2	3, 1, 2	5, 4, 4*

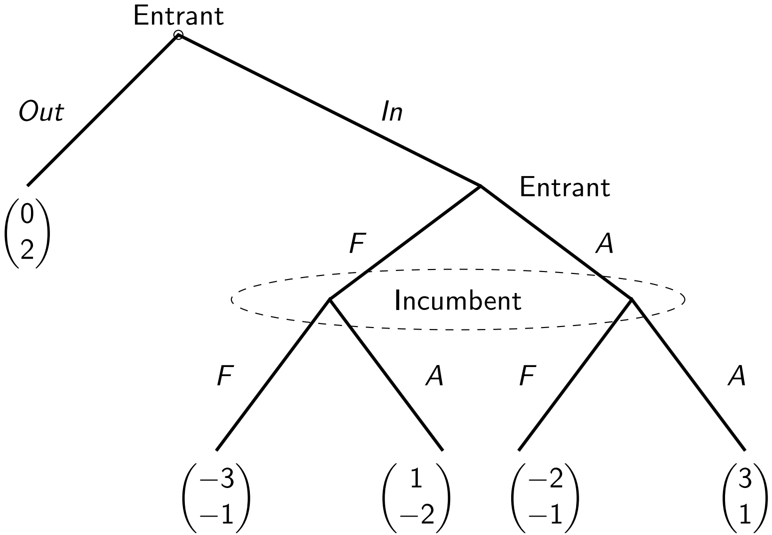
Player 2 plays a

Player 3

	lll	llr	lrl	lrr	rrl	rlr	rll	rrr
L	2, 0, 1	2, 0, 1	2, 0, 1	2, 0, 1	-1, 5, 6	-1, 5, 6*	-1, 5, 6	-1, 5, 6*
R	0, -1, 7	-2, 2, 0	0, -1, 7	-2, 2, 0	0, -1, 7	-2, 2, 0	0, -1, 7	-2, 2, 0

Player 2 plays b

Example 2



Entrant
Out
In
$\binom{0}{2}$
Entrant
F
A
Incumbent
F
A
F
A
$\binom{-3}{-1}$
$\binom{1}{-2}$
$\binom{-2}{-1}$
$\binom{3}{1}$

Example 2: NE

- Strategic form

player 2

player 1

	F	A
OF	0, 2	0, 2
OA	0, 2	0, 2
IF	-3, -1	1, -2
IA	-2, -1	3, 1

Example 2: NE

- Strategic form

player 2

player 1

	F	A
OF	0, 2	0, 2
OA	0, 2	0, 2
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IA	-2, -1	3, 1

- Three pure NE:

$$(OF, F), \quad (OA, F) \quad (IA, A)$$

Example 2: NE

- Strategic form

player 2

player 1

	F	A
OF	0, 2	0, 2
OA	0, 2	0, 2
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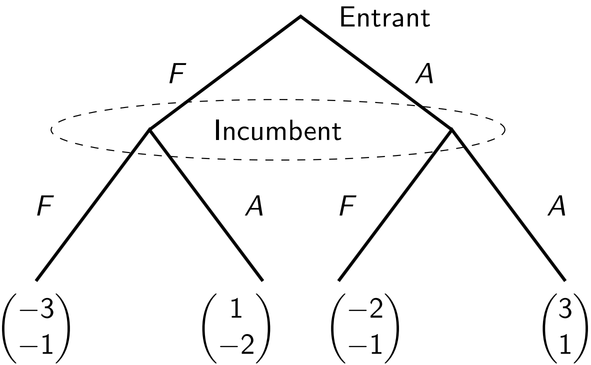
- Three pure NE:

$$(OF, F), \quad (OA, F) \quad (IA, A)$$

- Whic of them involves incredible threat?

Subgame after In



Find NE in subgame

- Strategic form of subgame

	F	A
F	-3, -1	1, -2
A	-2, -1	3, 1

Find NE in subgame

- Strategic form of subgame

	F	A
F	-3, -1	1, -2
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- In the subgame, A is a dominant strategy!

Find NE in subgame

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	F	A
F	-3, -1	1, -2
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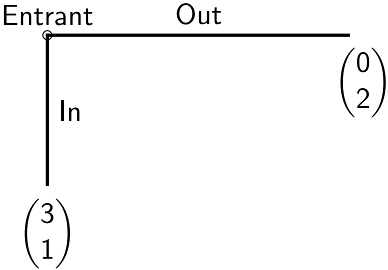
- In the subgame, A is a dominant strategy!
- Unique NE:

$$(A, A).$$

Example 2: Find SPNE



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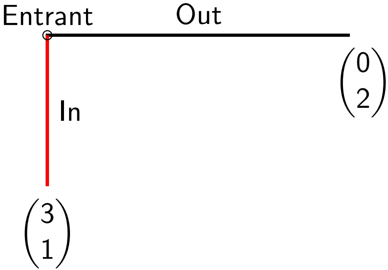

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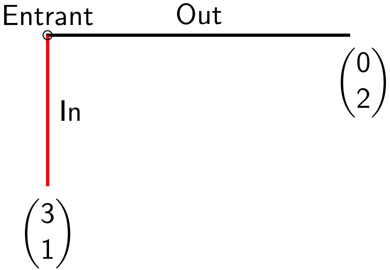
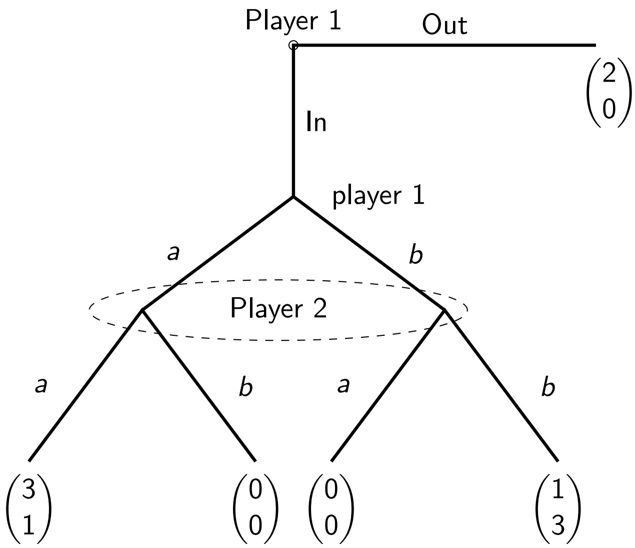


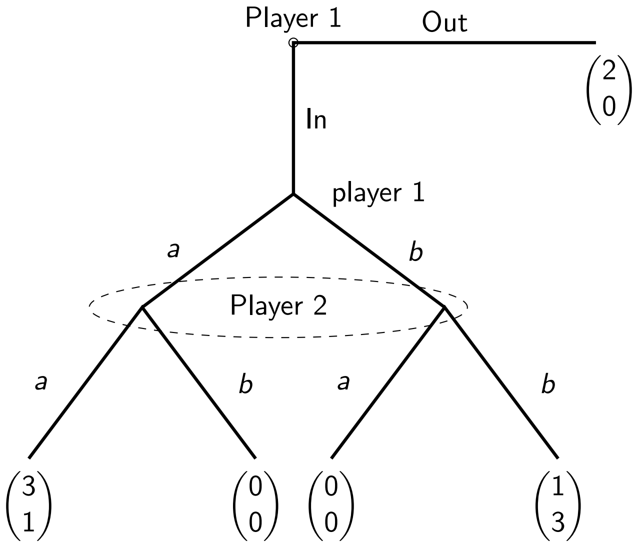
Entrant plays In; the unique SPNE: (IA, A)

EXAMPLE 3



Player 1
Out
$\binom{2}{0}$
In
player 1
a
b
Player 2
a
b
a
b
$\binom{3}{1}$
$\binom{0}{0}$
$\binom{0}{0}$
$\binom{1}{3}$

Example 3: find SPNE



Example 3: find SPNE

player 2

player 1

	a	b
Oa	2, 0	2, 0
Ob	2, 0	2, 0
Ia	3, 1	0, 0
Ib	0, 0	1, 3

Example 3: find SPNE

player 2

player 1

	a	*b*
Oa	2, 0	2, 0*
Ob	2, 0	2, 0*
Ia	3, 1*	0, 0
Ib	0, 0	1, 3

Example 3: find SPNE

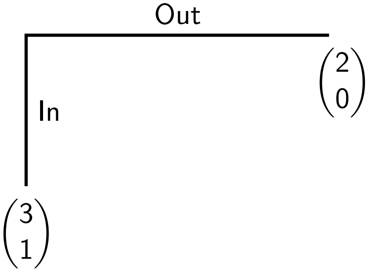
- Subgame after IN

	a	b
a	3, 1	0, 0
b	2, 0	1, 3

- Three NE in the subgame:

$$(a, a); \qquad (b, b); \qquad \left(\frac{3}{4}, \frac{1}{4}; \frac{1}{4}, \frac{3}{4}\right).$$

Example 3: find SPNE



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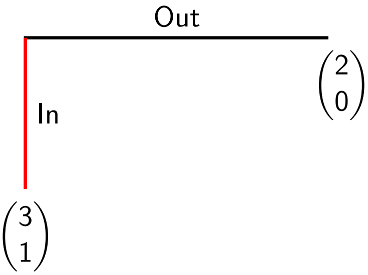
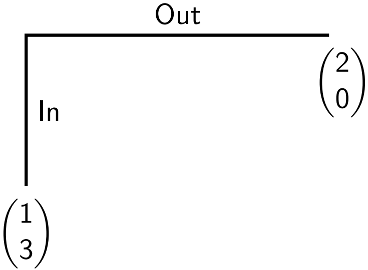


SPNE 1: (Ia, a)

Example 3: find SPNE



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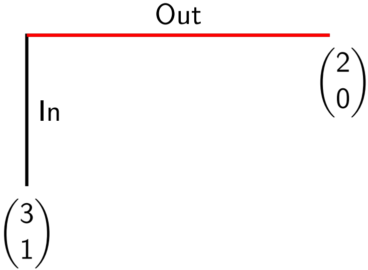
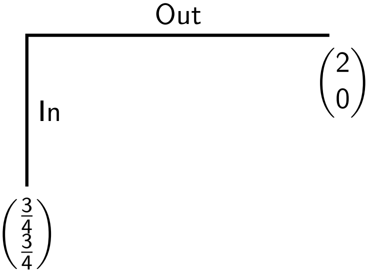
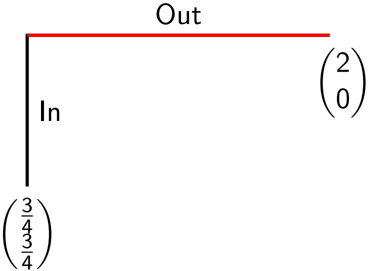


SPNE 2: (Ob, b)

Example 3: find SPNE



Example 3: find SPNE



SPNE 3: $\left(Oa\frac{3}{4}, Ob\frac{1}{4}, Ia0, Ib, 0; a\frac{1}{4}, b\frac{3}{4}\right)$

More on SPNE

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- Note that here the second mover is harmed by his own rationality—he will be better off if he can convince the first mover that he is irrational.
- That's one reason why young children often get what they want from parents.

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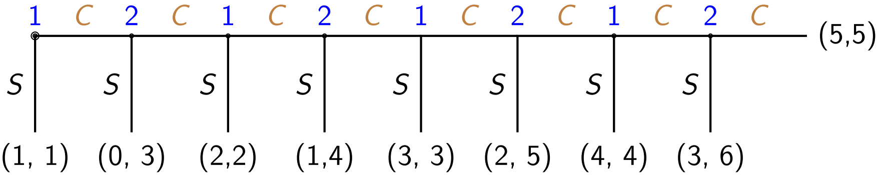
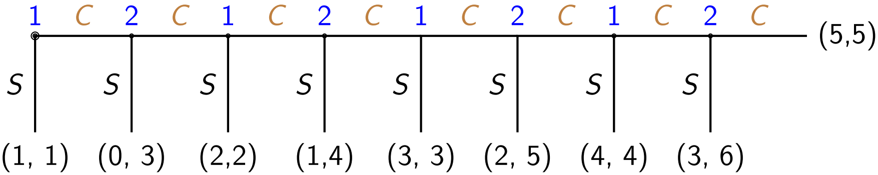
- But it is problematic to maintain the assumption of rationality off the equilibrium path.
- According to backward induction logic, a rational player should not deviate in the first place.
- There is no completely satisfactory solution to this problem.

Centipede game



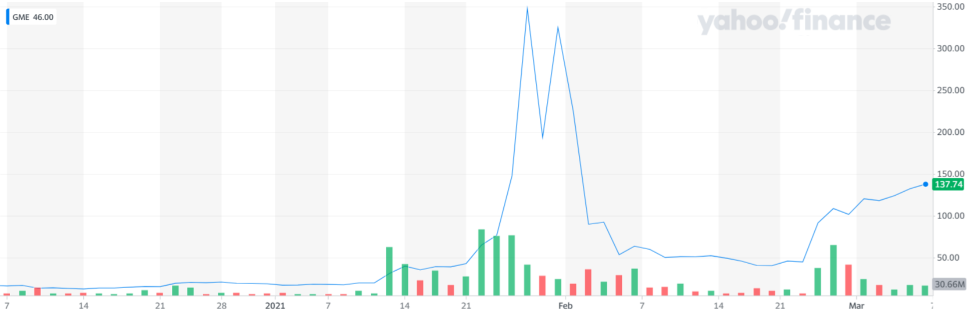
- The unique SPNE is for 1 & 2 to choose "S" , which follows from Iterated deletion of weakly dominated strategies.



- The unique SPNE is for 1 & 2 to choose "S" , which follows from Iterated deletion of weakly dominated strategies.
- But this SPNE is rather doubtful.

Wallstreetbets day traders vs. Hedge fund

GME short squeeze



SPNE may have no power

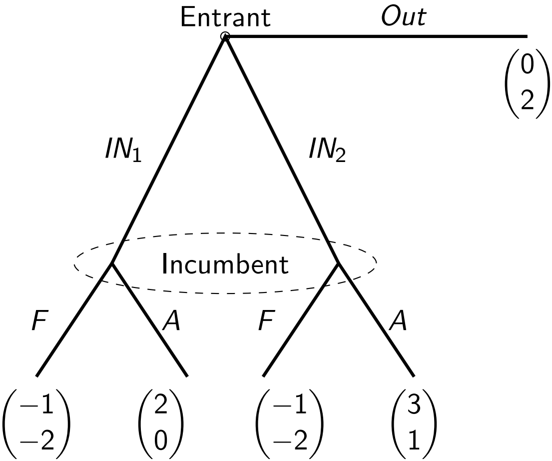


Figure : Entrant incumbent example 1

Find SPNE of the entry game

Incumbent

Entrant		F	A
	Out	0, 2	0, 2
	IN_1	-1, -2	2, 0
	IN_2	-1, -2	3, 1

The problem with SPNE

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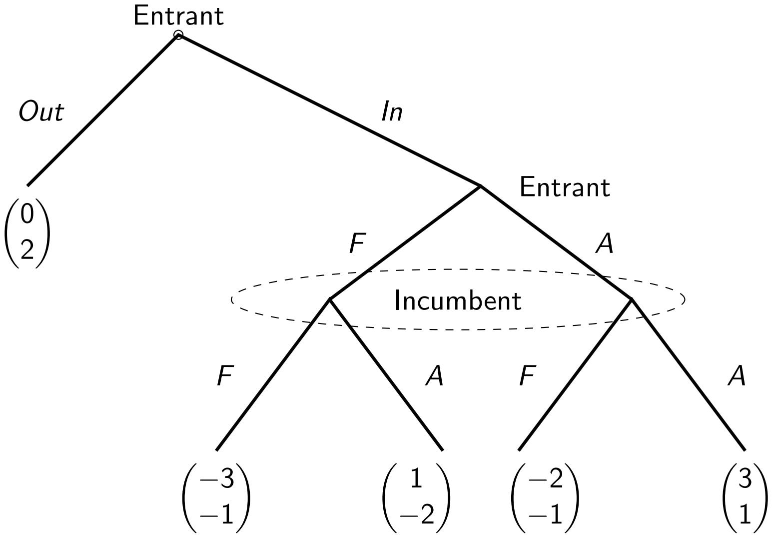
- But is this enough?

Example 2: Entrant-incumbent game 2



Entrant-incumbent game 2 continued

- The strategic form

	F	A
OF	0, 2	0, 2
OA	0, 2	0, 2
IF	-3, -1	1, -2
IA	-2, -1	3, 1

Entrant-incumbent game 2 continued

- The strategic form

	F	A
OF	0, 2	0, 2
OA	0, 2	0, 2
IF	-3, -1	1, -2
IA	-2, -1	3, 1

- NE

$$(OF, F), \quad (OA, F), \quad (IA, A).$$

Entrant-incumbent game 2 continued

- The strategic form

	F	A
OF	0, 2	0, 2
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IA	-2, -1	3, 1

- NE

$$(OF, F), \quad (OA, F), \quad (IA, A).$$

- If we require every player to make optimal choice at every information set:

$$(OA, F), \quad (IA, A).$$

Entrant-incumbent game 2 continued

- The strategic form

	F	A
OF	0, 2	0, 2
OA	0, 2	0, 2
IF	-3, -1	1, -2
IA	-2, -1	3, 1

- NE

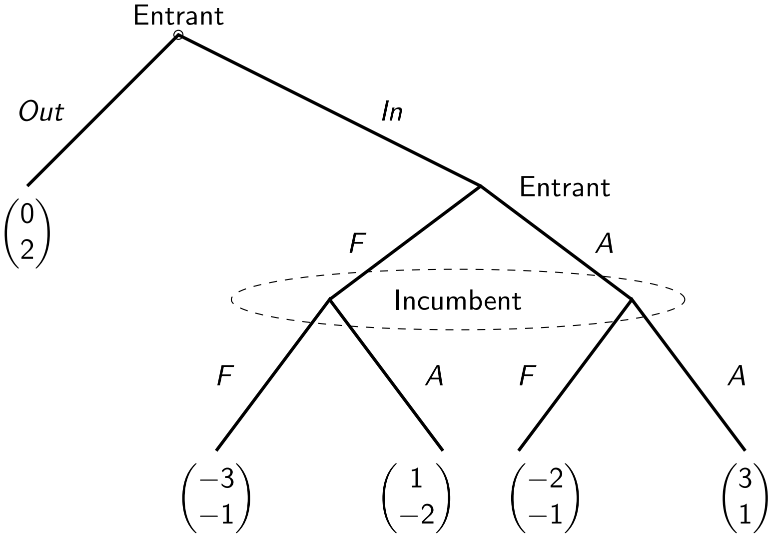
$$(OF, F), \quad (OA, F), \quad (IA, A).$$

- If we require every player to make optimal choice at every information set:

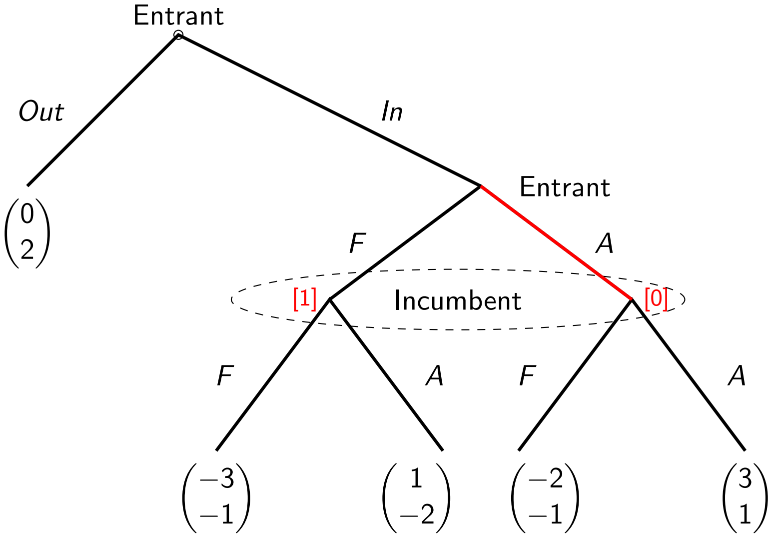
$$(OA, F), \quad (IA, A).$$

- (OA, F) meets the requirement!

Entrant-incumbent game 2 continued



Entrant-incumbent game 2 continued



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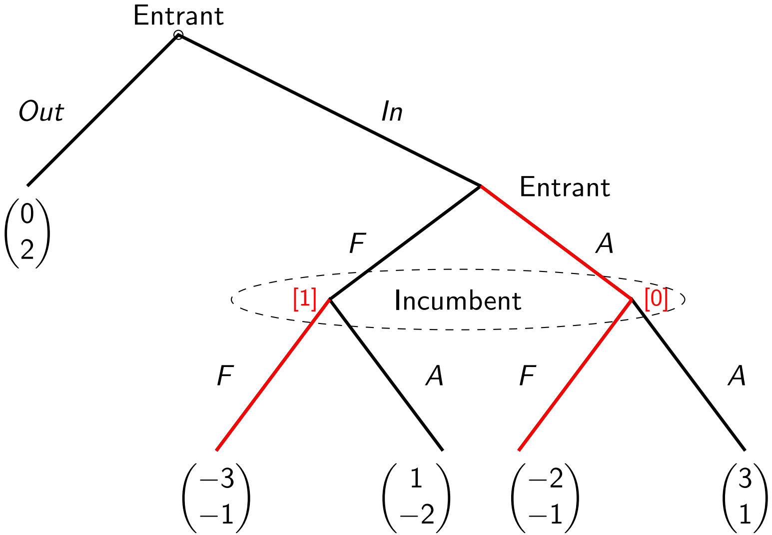
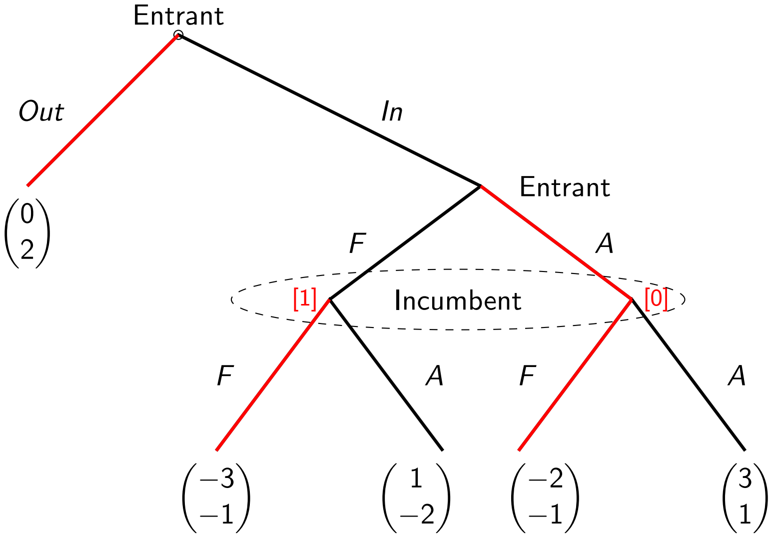


Entrant-incumbent game 2 continued



Entrant-incumbent game 2 continued

- Subgame after entry

	F	A
F	-3, -1	1, -2
A	-2, -1	3, 1

Entrant-incumbent game 2 continued

- Subgame after entry

	F	A
F	-3, -1	1, -2
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- So (OA, F) is not SPNE!

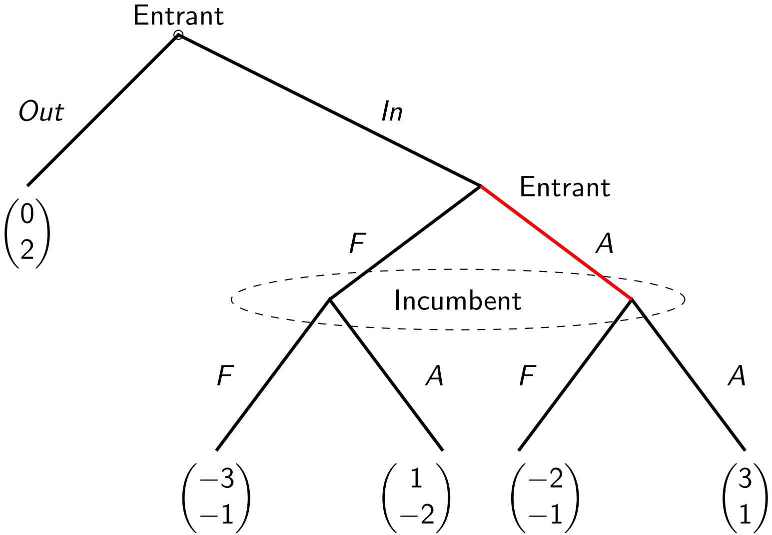
Entrant-incumbent game 2 continued

- Subgame after entry

	F	A
F	-3, -1	1, -2
A	-2, -1	3, 1

- So (OA, F) is not SPNE!
- In addition to restriction on choices, there needs to be restrictions on off-equilibrium path beliefs!

Belief consistent with choices



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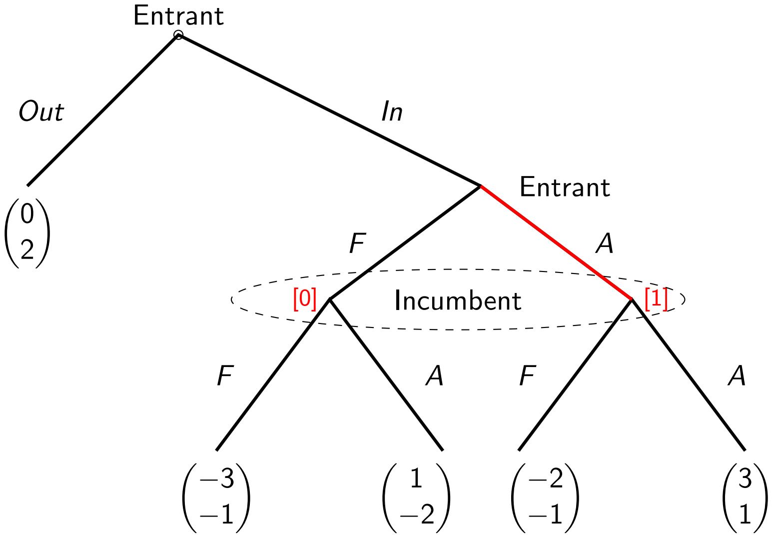

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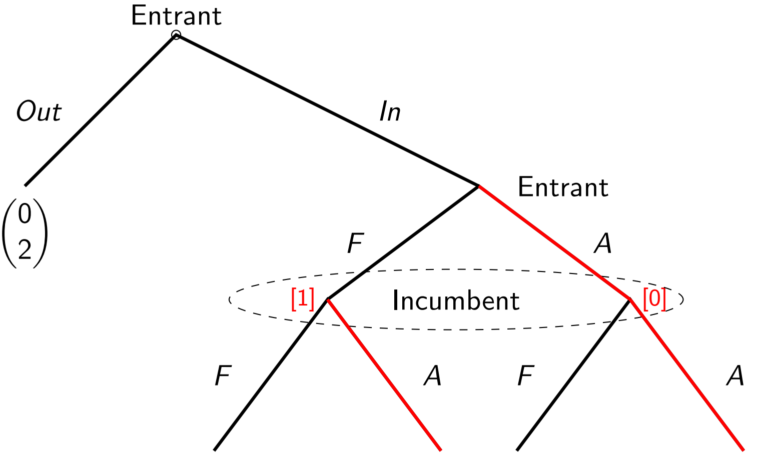
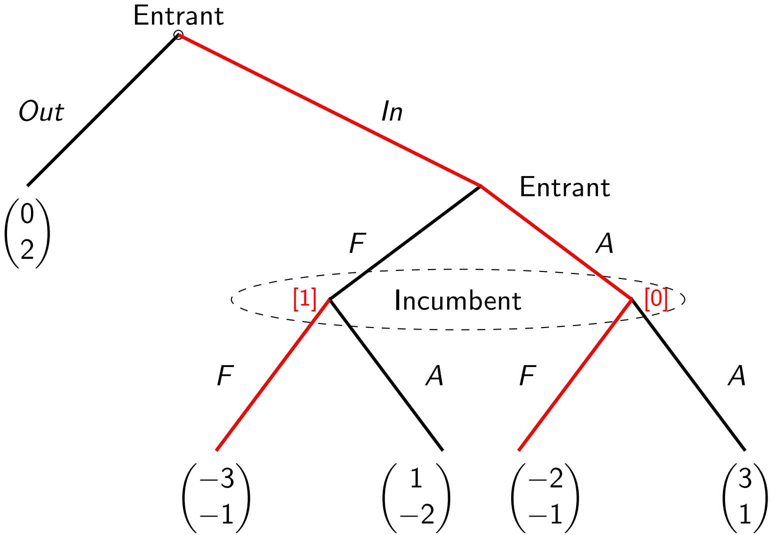


$$\begin{pmatrix}0\\2\end{pmatrix} \quad \begin{pmatrix}-3\\-1\end{pmatrix} \quad \begin{pmatrix}1\\-2\end{pmatrix} \quad \begin{pmatrix}-2\\-1\end{pmatrix} \quad \begin{pmatrix}3\\1\end{pmatrix}$$

Belief consistent with choices



- Definition: A system of beliefs μ in an extensive form game Γ_E is a specification of probability $\mu(x) \in [0,1]$ for each decision node x in Γ_E such that for all information set $h \in \mathbf{H}$,

$$\sum_{x \in h} \mu(x) = 1.$$

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$$\sum_{x \in h} \mu(x) = 1.$$

- Definition: A strategy profile $\sigma = (\sigma_1, \ldots, \sigma_I)$ is sequentially rational *at information set h* given belief μ if for player i who moves at information set h,

$$E[U_i|\mu, \sigma_i, \sigma_{-i}] \geq E[U_i|\mu, \sigma_i', \sigma_{-i}]$$

for all σ_i' of player i.
A strategy profile σ is sequentially rational given belief μ if this condition is satisfied for all information sets h.

Sequential equilibrium

- A strategy profile and system of beliefs (σ, μ) is a sequential equilibrium of Γ_E if

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- The concept of sequential equilibrium is a strengthening of the concept of subgame perfection.
- Any sequential equilibrium is necessarily subgame perfect, but the converse is not true. The difference of the two, of course, only lies in imperfect information game.

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  1. The strategy profile σ is sequentially rational given μ.
  2. There exists completely mixed strategies $\{\sigma^k\}_{k=1}^{\infty}$ with $\lim_{k\to\infty} \sigma^k = \sigma$, such that $\mu = \lim_{k\to\infty} \mu^k$, where μ^k is derived from σ^k using Bayes rule.
- The concept of sequential equilibrium is a strengthening of the concept of subgame perfection.
- Any sequential equilibrium is necessarily subgame perfect, but the converse is not true. The difference of the two, of course, only lies in imperfect information game.
- Consistency requirement: There are sequential equilibrium in which consistency may impose restrictions on the possible sequences of totally mixed strategy, and in turn also on the possible belief players may have off-the-equilibrium path.

Interpretation of the definition

- The concept of sequential equilibrium captures the intuition of backward induction - - each player believes the other players are rational and thus will play optimally in any continuation of the game - - by defining an equilibrium to be a pair consisting of a behavioral strategy and a system of beliefs.

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- The system of belief is *consistent* with the behavioral strategy, that is, it is the limit of a sequence of beliefs each being the actual conditional distribution on nodes of the various information sets induced by a sequence of totally mixed behavioral strategies converging to the given behavioral strategy.

Apply the definition of SE

- Consider NE (OF, F)

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  - ➢ Use Bayes' rule, x_L is left decision node,

$$\mu^k(x_L) = Pr(x_L | h, \sigma^k) = \frac{Pr(x_L, h|\sigma^k)}{Pr(h|\sigma^k)} = \frac{\epsilon^k \eta^k}{\epsilon^k} = \eta^k;$$

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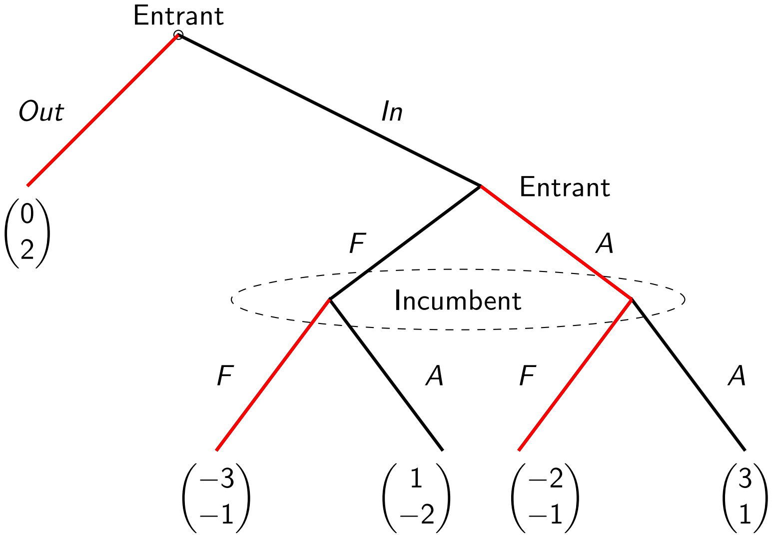
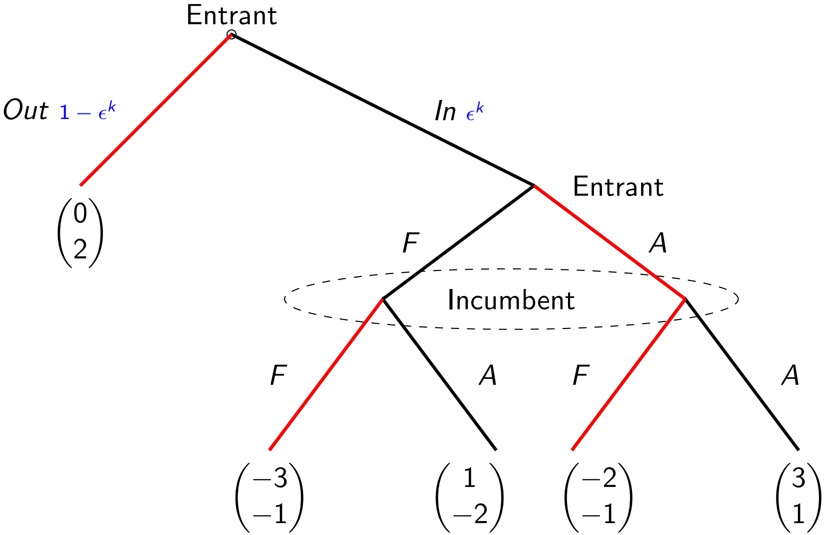
- (OA, F) not S.E. either.

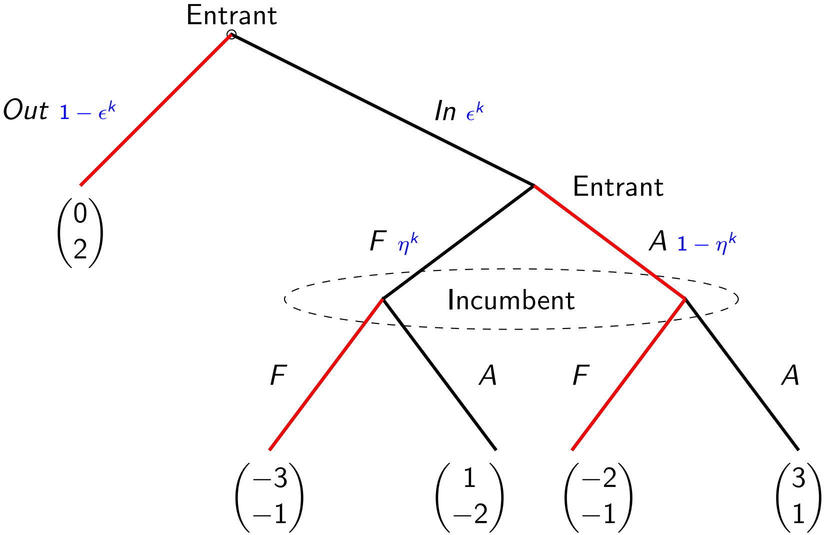
Construct σ^k: (OA, F)



Construct σ^k: (OA, F)



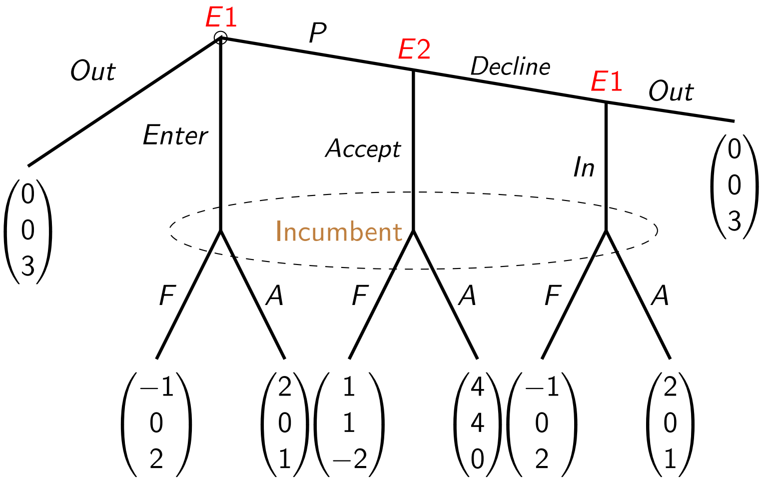
Construct σ^k: (OA, F)



S.E. of Entrant-incumbent example 2

- Strategy
$$(IA, A);$$
- Belief:
$$\mu = (0, 1).$$
- That is, Incumbent assigns probability 0 to the decision node after F, and probability 1 to the decision after A.

Entrant incumbent game 3



Firm E2

Firm E1

	Accept	Decline
OI	0, 0, 3	0, 0, 3*
OO	0, 0, 3	0, 0, 3*
EI	-1, 0, 2	-1, 0, 2
EO	-1, 0, 2	-1, 0, 2
PI	1, 1, -2	-1, 0, 2
PO	1, 1, -2	0, 0, 3

I fight

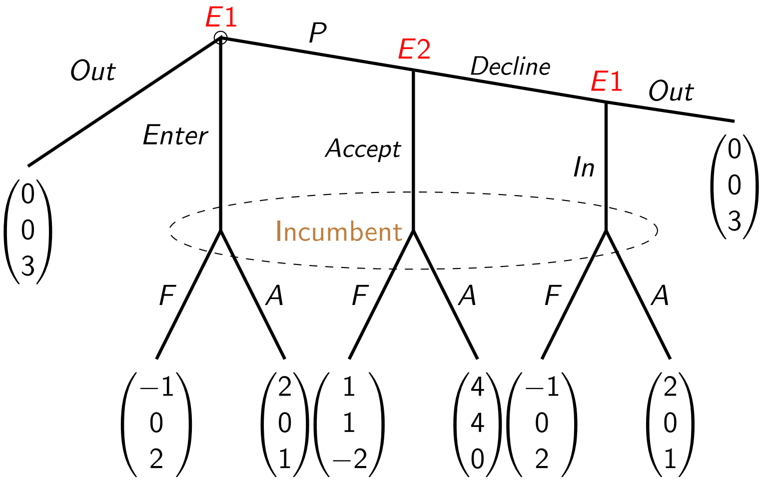
Firm E1

	Accept	Decline
OI	0, 0, 3	0, 0, 3
OO	0, 0, 3	0, 0, 3
EI	2, 0, 1	2, 0, 1
EO	2, 0, 1	2, 0, 1
PI	4, 4, 0*	2, 0, 1
PO	4, 4, 0*	0, 0, 3

I accommodate

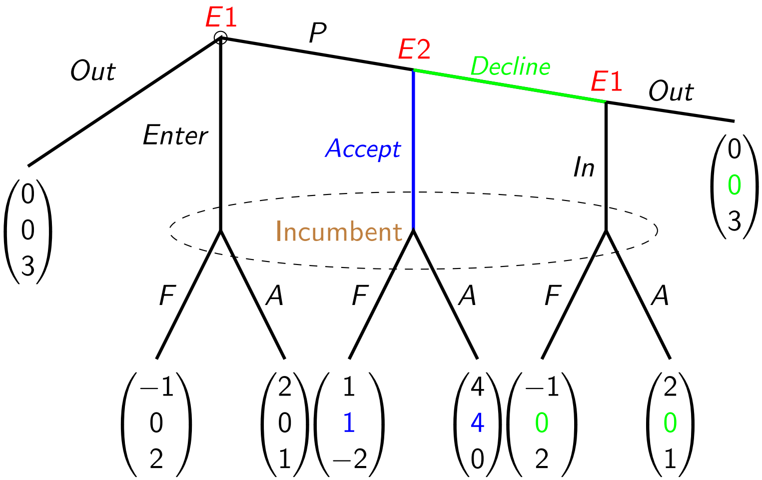
- While this game has several pure strategy NE, there is only one SE.

E2's problem



E1
P
E2
Decline
E1
Out
Out
Enter
Accept
In
Incumbent
F
A
F
A
F
A
$\begin{pmatrix}0\\0\\3\end{pmatrix}$
$\begin{pmatrix}0\\0\\3\end{pmatrix}$
$\begin{pmatrix}-1\\0\\2\end{pmatrix}$
$\begin{pmatrix}2\\0\\1\end{pmatrix}$
$\begin{pmatrix}1\\1\\-2\end{pmatrix}$
$\begin{pmatrix}4\\4\\0\end{pmatrix}$
$\begin{pmatrix}-1\\0\\2\end{pmatrix}$
$\begin{pmatrix}2\\0\\1\end{pmatrix}$

E2's problem



E1
P
E2
Decline
E1
Out
Out
Enter
Accept
In
Incumbent
F
A
F
A
F
A
$\begin{pmatrix}0\\0\\3\end{pmatrix}$
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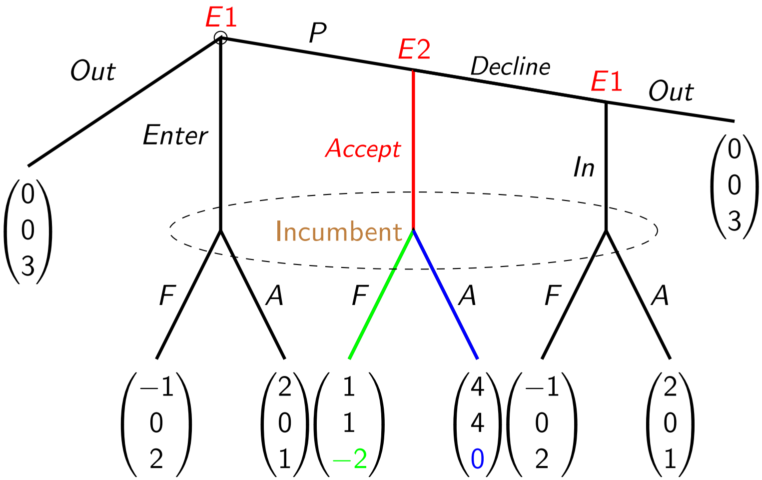
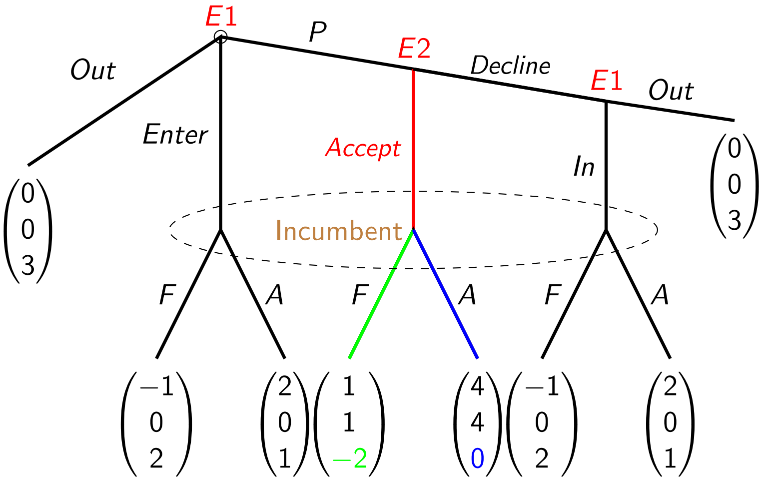
Incumbent's problem

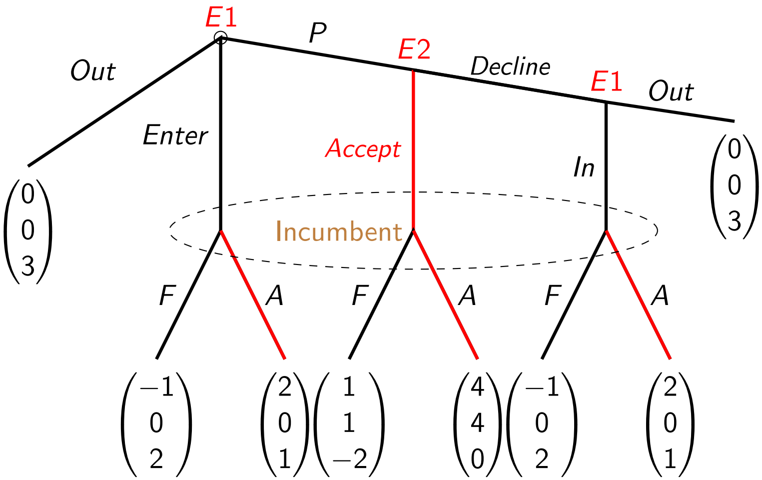


Incumbent's problem



Optimal for Incumbent to play A!

E1's problem


E1
P
E2
Decline
E1
Out
Out
Enter
Accept
In
Incumbent
F
A
F
A
F
A
$\begin{pmatrix}0\\0\\3\end{pmatrix}$
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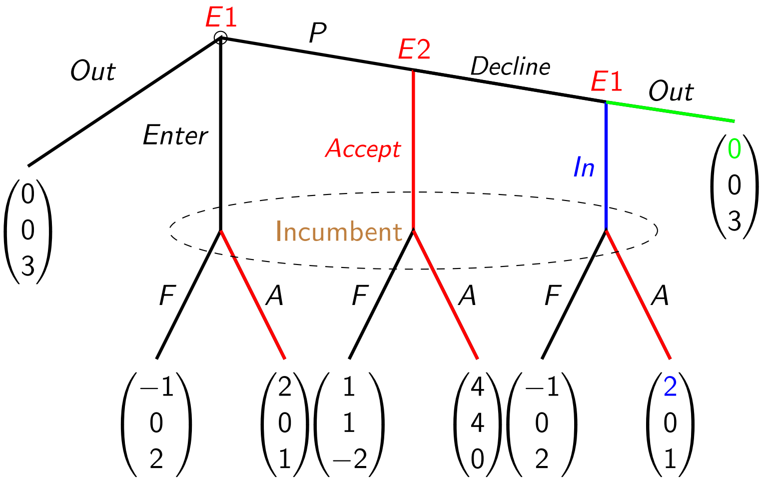
E1's problem



E1
P
E2
Decline
E1
Out
Out
Enter
Accept
In
Incumbent
F
A
F
A
F
A
(0, 0, 3)
(0, 0, 3)
(−1, 0, 2)
(2, 0, 1)
(1, 1, −2)
(4, 4, 0)
(−1, 0, 2)
(2, 0, 1)

S.E. of Entrant-incumbent game 3

- Optimal choice for E2 is Accept, so (OI, D, F) and (OO, DO, F) fails condition (i) of the definition of S.E.;

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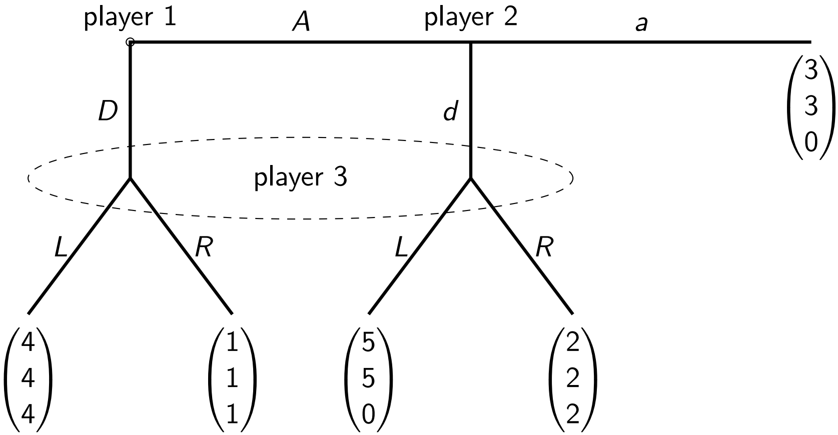
  - ➢ Strategy σ
    $$(PI, A, A);$$
  - ➢ Belief:
    $$\mu = (0, 1, 0).$$
    That is, Incumbent firm assigns probability 1 to the decision node after Accept.

Selten's Horse



Selten's horse continued

- The strategical form

Player 2

Player 1

	a	d
A	(3, 3, 0)	(5, 5, 0)
D	(4, 4, 4)	(4, 4, 4)

3 plays L

Player 1

	a	d
A	(3, 3, 0)	(2, 2, 2)
D	(1, 1, 1)	(1, 1, 1)

R

Selten's horse continued

- The strategical form

Player 2

Player 1		a	d
	A	(3, 3, 0)	(5, 5, 0)
	D	(4, 4, 4)	(4, 4, 4)

3 plays L

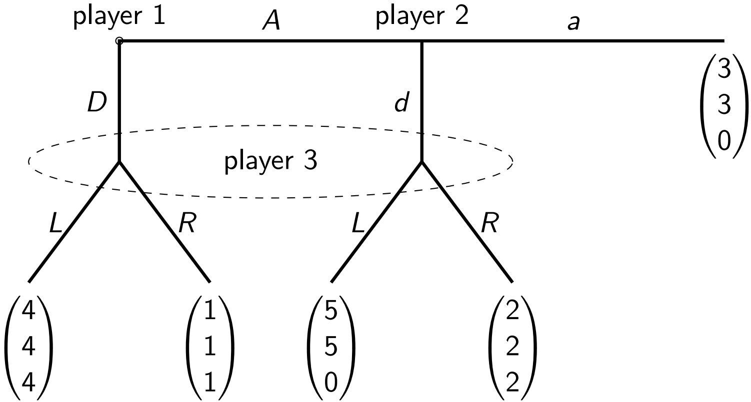
Player 1		a	d
	A	(3, 3, 0)	(2, 2, 2)
	D	(1, 1, 1)	(1, 1, 1)

R

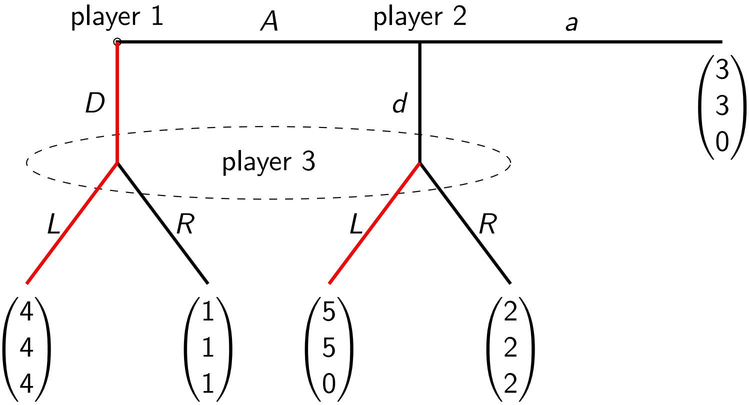
- Two pure NE:

$$(D, a, L), \quad (A, a, R).$$

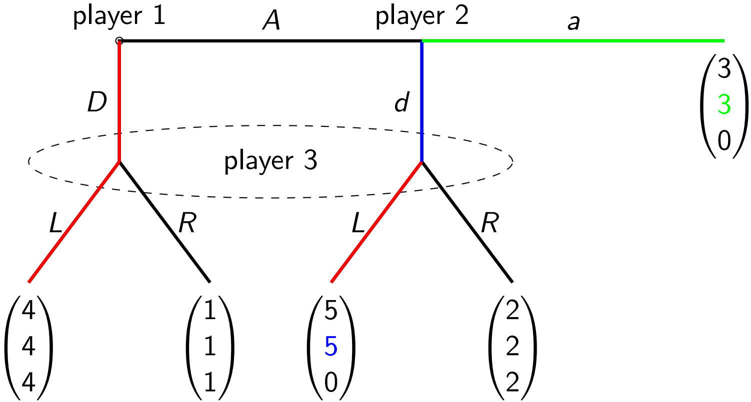
Selten's Hore: SE



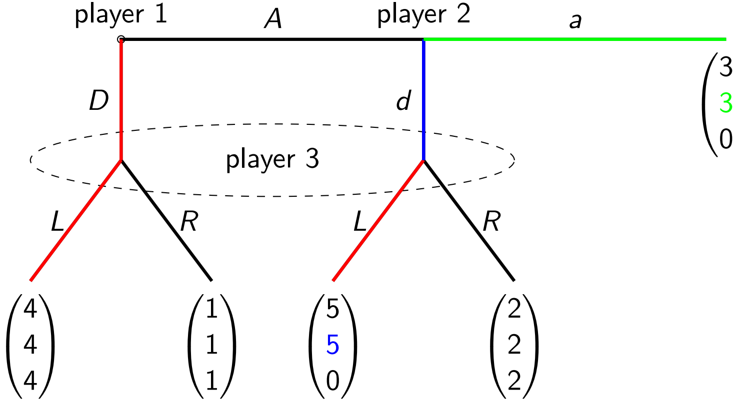
Consider (D, a, L): for player 2, given (D, L),



Selten's Hore: SE

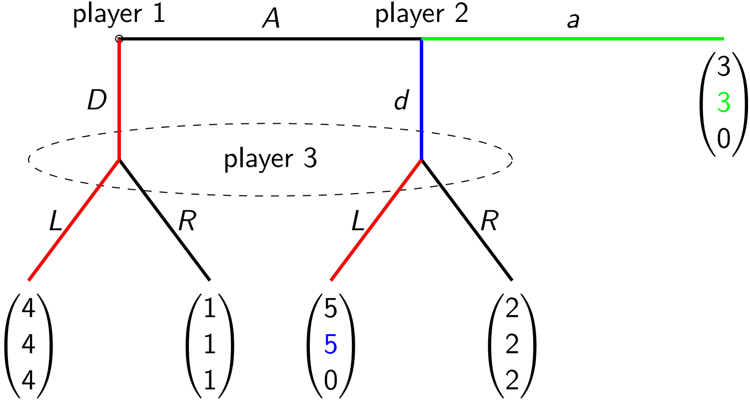


Selten's Hore: SE



Given (D, L), optimal choice should be *d*! (D,a,L) not S.E.

Selten's Hore: SE



(D,a,L) not S.E., and (A, a,R) is S.E.

S.E. of Selten's horse

- The only S.E.:

S.E. of Selten's horse

- The only S.E.:
  - ➢ σ

$$(A, a, R);$$

S.E. of Selten's horse

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  - σ
$$(A, a, R);$$
  - μ:
$$(\alpha, 1-\alpha), \qquad \alpha \leq \frac{2}{5}.$$

S.E. of Selten's horse

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- For player 3, given belief $(\alpha, 1-\alpha)$:
$$U_3(L, \mu) = 4\alpha, \qquad U_3(R, \mu) = \alpha + 2(1-\alpha).$$

S.E. of Selten's horse

- The only S.E.:
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$$(A, a, R);$$
  - ➢ μ:
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- For player 3, given belief $(\alpha, 1-\alpha)$:
$$U_3(L, \mu) = 4\alpha, \qquad U_3(R, \mu) = \alpha + 2(1-\alpha).$$
- R is optimal when $\alpha \leq \frac{2}{5}$, that is,
$$U_3(L, \mu) = 4\alpha \leq U_3(R, \mu) = \alpha + 2(1-\alpha).$$

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$$\mu^k(x_L) = \frac{\epsilon^k}{\epsilon^k + (1-\epsilon^k)\eta^k} = \alpha.$$

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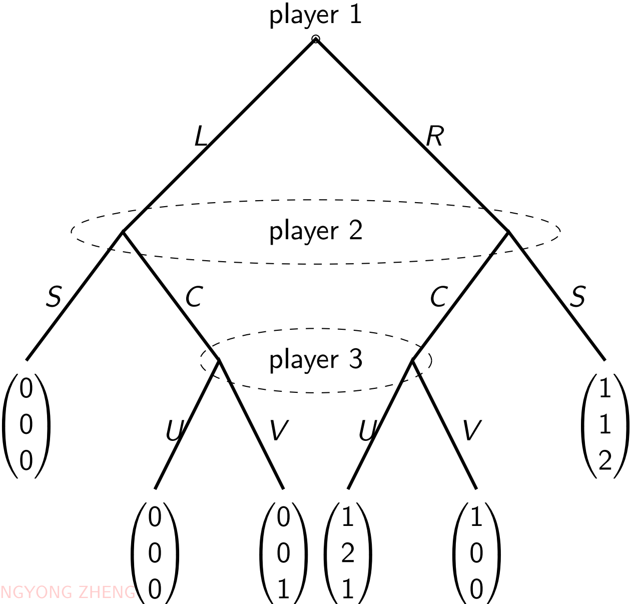
- Thus,

$$\mu(x_L) = \lim_{k\to\infty} = \alpha.$$

- The belief μ indeed comes from a totally mixed strategy profile σ^k!

Another Example



Find S.E.

- Strategic form

Player 2

	S	C
L	(0, 0, 0)	(0, 0, 0)
R	(1, 1, 2)	(1, 2, 1)

3 plays U

	S	C
L	(0, 0, 0)	(0, 0, 1)
R	(1, 1, 2)	(1, 0, 0)

V

Find S.E.

- Strategic form

Player 2

	S	C
L	(0, 0, 0)	(0, 0, 0)
R	(1, 1, 2)	(1, 2, 1)

3 plays U

	S	C
L	(0, 0, 0)	(0, 0, 1)
R	(1, 1, 2)	(1, 0, 0)

V

- Two pure NE: (R, C, U), (R, S, V).

S.E. continued

- But (R, S, V) is not S. E.

S.E. continued

- But (R, S, V) is not S. E.
  - Given (R, S, V), σ_1^k and σ_2^k are

$$\sigma_1^k = (L\ \epsilon^k, R\ (1-\epsilon^k)), \qquad \sigma_2^k = (S\ (1-\eta^k), C\ \eta^k)$$

S.E. continued

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$$\sigma_1^k = (L\ \epsilon^k, R\ (1-\epsilon^k)), \qquad \sigma_2^k = (S\ (1-\eta^k), C\ \eta^k)$$

  - ➢ Player 3' belief μ^3 should be (0, 1),

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S.E. CONTINUED

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  - ➢ U is optimal, not V.
- S.E. of the game

S.E. continued

- But (R, S, V) is not S. E.
  - Given (R, S, V), σ_1^k and σ_2^k are

$$\sigma_1^k = (L\ \epsilon^k, R\ (1-\epsilon^k)), \qquad \sigma_2^k = (S\ (1-\eta^k), C\ \eta^k)$$

  - Player 3' belief μ^3 should be (0, 1),

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Implications of conditions imposed by SE

- On behavior: In NO circumstances should a player makes a choices that is dominated by other choices. Therefore, the strategy should specify optimal choice at every information set given the beliefs about what has happened previously, thus the probability distribution over different decision nodes at the information set, as well as what the other players are playing.

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  - ➢ As a direct consequence, at every information set, if the player has a dominant choice, one that is better than the rest of choices regardless of what the choices of other players, then her opponent's belief should put probability one to the dominant choice, and zero to the rest of choices.

More on SE beliefs

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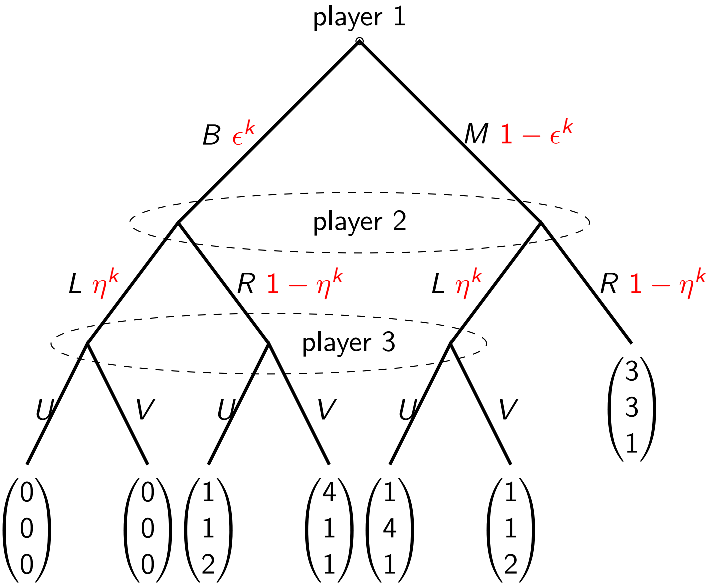
- *Structural consistency*: A belief system μ is structurally consistent if for each information set h, there exists some strategy profile σ such that for all $x \in h$,

$$\mu(x) = \frac{prob(x|\sigma)}{prob(h|\sigma)}.$$

Example 228.2 of Osborne and Rubinstein



Solve for SE

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    - ➠ Player 3's belief: $(0, 0.5, 0.5)$.

Derive SE belief: player 2

- Let the totally mixed strategy profit σ^k be

$$(\epsilon^k, 1-\epsilon^k; \eta^k, 1-\eta^k; \alpha, 1-\alpha)$$

such that

$$\lim_{k\to\infty} \epsilon^k = 0, \qquad \lim_{k\to\infty} \eta^k = 0.$$

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$$\mu^k(z_L) = \frac{\epsilon^k}{1} = \epsilon^k$$
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$$\mu(z_L) = \lim_{k\to\infty} \mu^k(z_L) = 0$$
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- Recall the totally mixed strategy profit σ^k:
$$(\epsilon^k, 1-\epsilon^k; \eta^k, 1-\eta^k; \alpha, 1-\alpha).$$
- Denote the 3 decision nodes, in the order of left, middle and right, as x_L, x_M, x_R.
$$\mu^k(x_L) = \frac{\eta^k\epsilon^k}{\eta^k\epsilon^k + \epsilon^k(1-\eta^k) + (1-\epsilon^k)\eta^k} = \frac{\eta^k\epsilon^k}{\epsilon^k + (1-\epsilon^k)\eta^k}$$
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Example continued

- While in general, the totally mixed strategy for player 1 could be $(1-\epsilon, \epsilon)$ and for player 2 could be $(1-\eta, \eta)$, for consistency, it nevertheless must be true that $\eta(1-\epsilon) = \epsilon(1-\eta)$, so that player 3 assigns equal probability to the upper and the middle decision nodes.

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- Only in this case will player 3 be indifferent between the two pure strategies U and V, which makes the mixed strategy best response motivated.
- Hence, it must be true that $\epsilon = \eta$.
- The belief for player 3, while consistent with the totally mixed strategy, is not structurally consistent.

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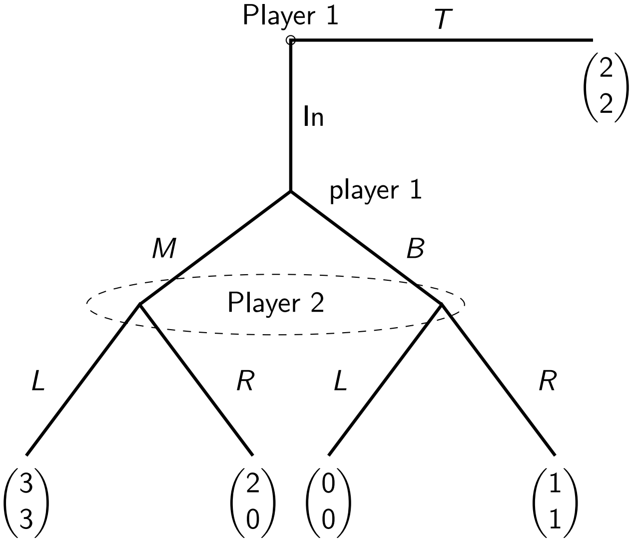
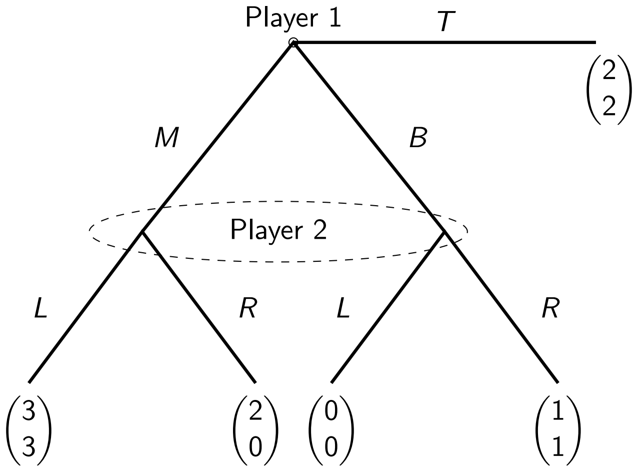
  - ➢ Requirement on strategies: OK, players should make optimal choice at any point in the game tree given the belief.
  - ➢ Requirement on beliefs: only requirement the belief to come from a sequence of totally mixed strategies. But some sequence of totally mixed strategies may not make sense at all, thereby leading to unreasonable belief in sequential equilibrium.

Example 1



Example 2



Player 1
T
M
B
Player 2
L
R
L
R
$\binom{2}{2}$
$\binom{3}{3}$
$\binom{2}{0}$
$\binom{0}{0}$
$\binom{1}{1}$

Example 2

Strategic form of game 2

Player 2

Player 1		L	R
	M	(3, 3)	(2, 0)
	B	(0, 0)	(1, 1)
	T	(2, 2)	(2, 2)

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- Clearly,

$$\mu^k(x_L) = \frac{\epsilon^2}{\epsilon + \epsilon^2}, \qquad \mu(x_L) = \lim_{k \to \infty} \mu^k(x_L) = 0.$$

Agent norm form perfect equilibrium

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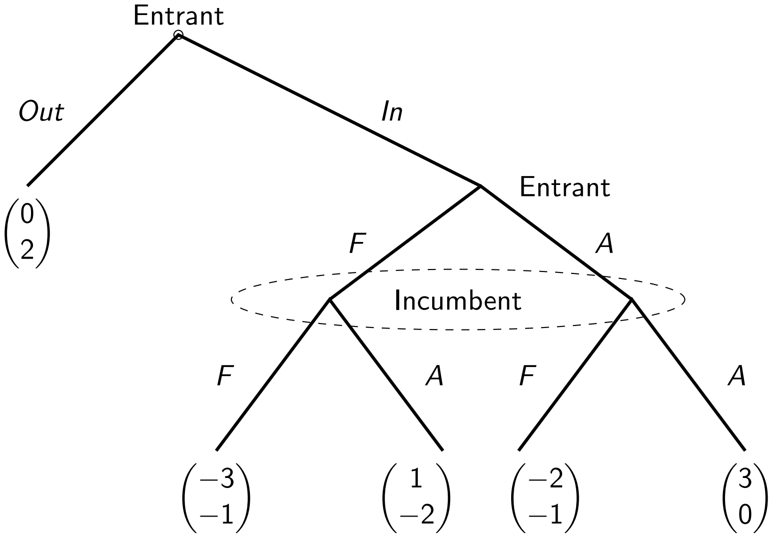
Entrant incumbent firm example



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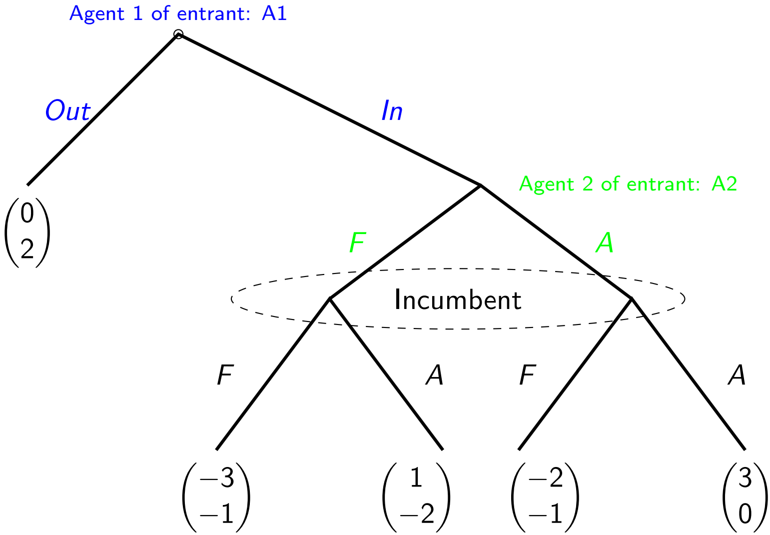

Find the equilibrium

- Agent norm form

A1		F	A
	Out	0, 2	0, 2
	In	-3, -1	1, -2

	F	A
Out	0, 2	0, 2
In	-2, -1	3, 0

A_2 plays F A

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- Perfect equilibrium:

$$(IA, A).$$

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- The restriction (iii) is evidently vague. One can interpret it as follows:

  If an information set I is reached with zero probability under σ (off the equilibrium path), the belief at I is derived, using Bayes' rule, from the beliefs at the information sets that precede I and players' continuation strategies as specified by σ, if possible.

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- Sequential equilibrium requires the beliefs of players at information sets not reached in the equilibrium to be derived from the SAME sequence of mixed strategies. PBE imposes no such restrictions on off-the-equilibrium beliefs.

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- The key underlying forward induction is that players maintain the assumption that their opponents have maximized their utility in the past as long as the assumption is tenable, even if unexpected is observed.
- That is, while finding himself off the equilibrium path, he should not interpret it as a result of unintentional mistake by his opponents as long as the deviations by his opponents are *rationalizable*.

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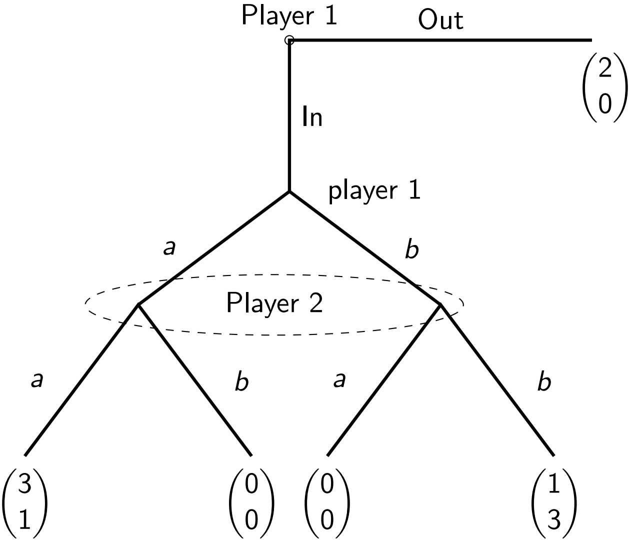
- For a large class of generic games, forward induction and iterated deletion of weakly dominated strategies yield the same set of solutions.

Outside option game



Analyze the game

- Two pure NE. Both are also SPNE.

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    - Belief (1,0), that is, at her information set, player 2 believes 1 has played a.

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China-US trade talk

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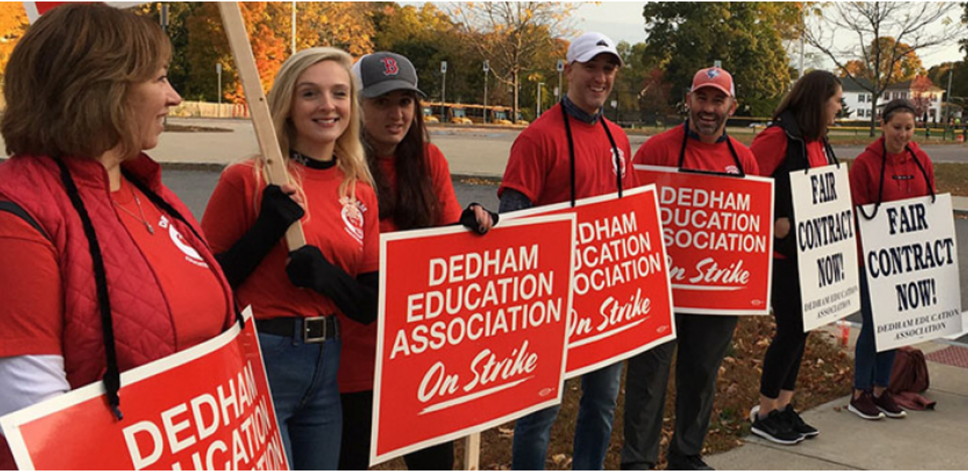
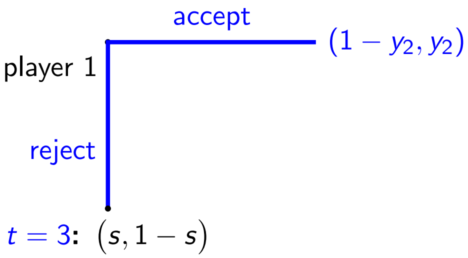
- For finite T, we can solve the game by backward induction.
- This is commonly known as the Rubinstein bargaining game.

The consequence of no deal



The case of $T = 3$



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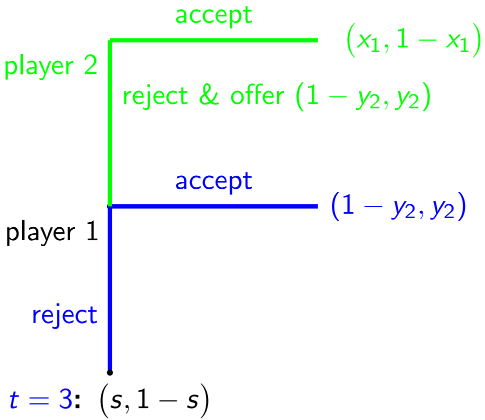

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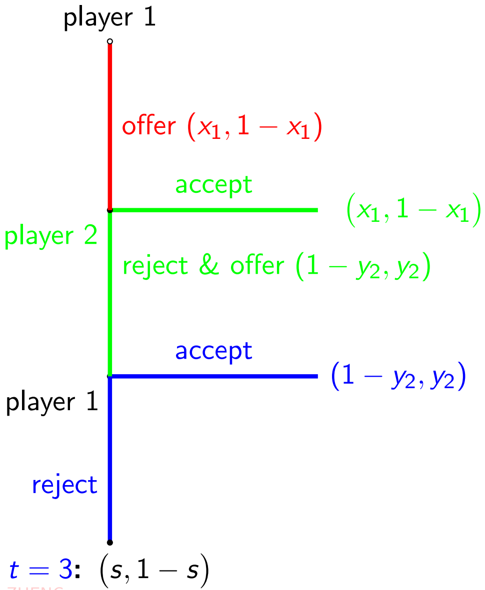

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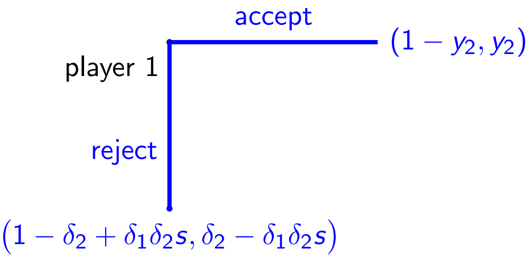
- Hence, in equilibrium player 1 proposes $(1 - \delta_2 + \delta_1 \delta_2 s, \delta_2 - \delta_1 \delta_2 s)$ in $T = 1$ and player 2 accepts.

The case of $T = 5$



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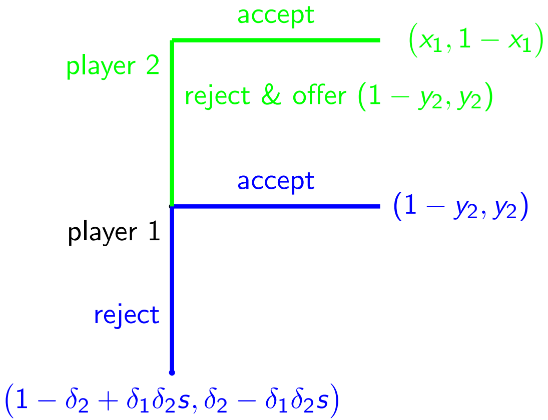

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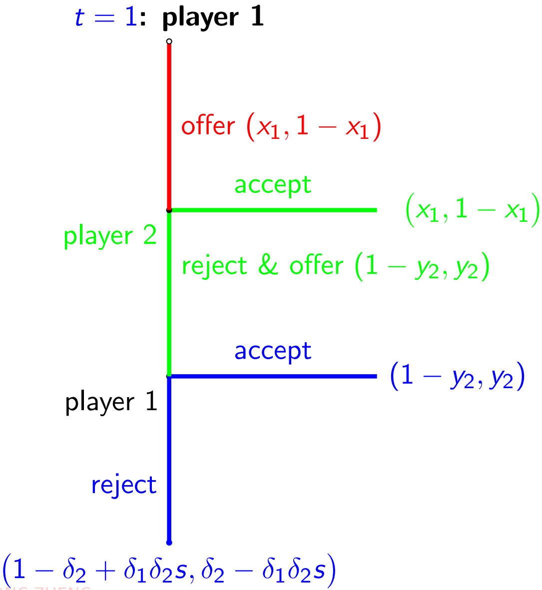

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- Substituting the new breakdown payoff into the equilibrium for $T = 3$ gives the first period offer:

$$\begin{aligned} x_1 &= 1 - \delta_2 + \delta_1\delta_2\left(1 - \delta_2 - \delta_1\delta_2 s\right) \\ &= \left(1 - \delta_2\right)\left(1 + \delta_1\delta_2\right) + \left(\delta_1\delta_2\right)^2 s, \end{aligned}$$

$$x_2 = 1 - \left(1 - \delta_2\right)\left(1 + \delta_1\delta_2\right) - \left(\delta_1\delta_2\right)^2 s.$$

The general case

- In general, when $T = 2n + 1$, we have player 1's equilibrium share

$$x_1^*(2n+1) = (1-\delta_2)\sum_{i=1}^{n}(\delta_1\delta_2)^{i-1} + (\delta_1\delta_2)^n s.$$

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- So, in this case, player 1 offers in period 1

$$x_1^* (2n + 2) = 1 - \delta_2 (1 - \delta_1) \sum_{i=1}^{n} (\delta_1 \delta_2)^{i-1} - \delta_2 (\delta_1 \delta_2)^n s.$$

The limit case

- We can take limit to see how increasing T affects the result.
- As T goes to infinity,

$$\lim_T x_1^*(T) \equiv x_1^* = \frac{1-\delta_2}{1-\delta_1\delta_2};$$
$$\lim_T x_2^*(T) \equiv x_2^* = \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}.$$

- Note that the limit is the same whether T is odd or even.

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- Let $(y_1^*(T), y_2^*(T))$ denote the equilibrium division if player 2 proposes in the first period.

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- Note that

$$x_2^* = \delta_2 y_2^* \text{ and } y_1^* = \delta_1 x_1^*.$$

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- Note that the principle only works for subgame perfect equilibrium in perfect information games. It is not true for Nash equilibrium, and it is not true for SPNE in games of imperfect information.

Main result on infinite bargaining

- Theorem: In the Rubinstein bargaining game with infinite horizon, there is a unique subgame perfect equilibrium where in every odd period, player 1 proposes (x_1^*, x_2^*) and player 2 accepts any $x_2 \geq x_2^*$, and in every even period player 2 proposes (y_1^*, y_2^*) and player 1 accepts any $y_1 \geq y_1^*$.

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- If player 1 proposes $x_2 < x_2^*$, then player 2 will rejects the offer and player 1 will obtain

$$y_1^* = \delta_1 x_1^* < x_1^*$$

in the next period, making him worse off.

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- The equilibrium is unique.