CHAPTER 6

6.B.1 Suppose first that $L \succ L'$. A first application of the independence axiom (in the "only-if" direction in Definition 6.B.4) yields

$$\alpha L + (1 - \alpha)L'' \succsim \alpha L' + (1 - \alpha)L''.$$

If these two compound lotteries were indifferent, then a second application of the independence axiom (in the "if" direction) would yield $L' \succsim L$, which contradicts $L \succ L'$. We must thus have

$$\alpha L + (1 - \alpha)L'' \succ \alpha L' + (1 - \alpha)L''.$$

Suppose conversely that $\alpha L + (1 - \alpha)L'' \succ \alpha L' + (1 - \alpha)L''$, then, by the independence axiom, $L \succsim L'$. If these two simple lotteries were indifferent, then the independence axiom would imply

$$\alpha L' + (1 - \alpha)L'' \succsim \alpha L + (1 - \alpha)L'',$$

a contradiction. We must thus have $L \succ L'$.

Suppose next that $L \sim L'$, then $L \succsim L'$ and $L' \succsim L$. Hence by applying the independence axiom twice (in the "only if" direction), we obtain

$$\alpha L + (1 - \alpha)L'' \sim \alpha L' + (1 - \alpha)L''.$$

Conversely, we can show that if $\alpha L + (1 - \alpha)L'' \sim \alpha L' + (1 - \alpha)L''$, then $L \sim L'$.

For the last part of the exercise, suppose that $L \succ L'$ and $L'' \succ L'''$, then, by the independence axiom and the first assertion of this exercise,

$$\alpha L + (1 - \alpha)L'' \succ \alpha L' + (1 - \alpha)L''$$

and

$$\alpha L' + (1 - \alpha)L'' \succ \alpha L' + (1 - \alpha)L'''.$$

Thus, by the transitivity of $\succ$ (Proposition 1.B.1(i)),

$$\alpha L + (1 - \alpha)L'' \succ \alpha L' + (1 - \alpha)L'''.$$

6.B.2 Assume that the preference relation $\succsim$ is represented by an v.N-M expected utility function $U(L) = \sum_n u_n p_n$ for every $L = (p_1,\dots,p_N) \in \mathcal{L}$. Let $L = (p_1,\dots,p_N) \in \mathcal{L}$, $L' = (p'_1,\dots,p'_N) \in \mathcal{L}$, $L'' = (p''_1,\dots,p''_N) \in \mathcal{L}$, and $\alpha \in (0,1)$. Then $L \succsim L'$ if and only if $\sum_n u_n p_n \geq \sum_n u_n p'_n$. This inequality is equivalent to

$$\alpha(\sum_n u_n p_n) + (1 - \alpha)(\sum_n u_n p''_n) \geq \alpha(\sum_n u_n p'_n) + (1 - \alpha)\,(\sum_n u_n p''_n).$$

This latter inequality holds if and only if $\alpha L + (1 - \alpha)L'' \succsim \alpha L' + (1 - \alpha)L''$. Hence $L \succsim L'$ if and only if $\alpha L + (1 - \alpha)L'' \succsim \alpha L' + (1 - \alpha)L''$. Thus the independence axiom holds.

6.B.3 Since the set C of outcomes is finite, there are best and worst outcomes in C. Let $\bar{L}$ be the lottery that yields a particular best outcome with probability one and $\underline{L}$ be the lottery that yields a particular worst outcome with probability one. We shall now prove that $\bar{L} \succsim L \succsim \underline{L}$ for every $L \in \mathcal{L}$ by applying the following lemma:

Lemma: Let $L_0, L_1, \dots, L_K$ be $(1 + K)$ lotteries and $(\alpha_1,\dots,\alpha_K) \geq 0$ be probabilities with $\sum_{k=1}^K \alpha_k = 1$. If $L_k \succsim L_0$ for every k, then $\sum_{k=1}^K \alpha_k L_k \succsim L_0$. If $L_0 \succsim L_k$ for every k, then $L_0 \succsim \sum_{k=1}^K \alpha_k L_k$.

Proof of Lemma: We shall prove this lemma by induction on K. If K = 1, there is nothing to prove. So let K > 1 and suppose that the lemma is true for K - 1. Assume that $L_k \succsim L_0$ for every k. By the definition of a compound lottery,

$$\sum_{k=1}^K \alpha_k L_k = (1 - \alpha_K)\sum_{k=1}^{K-1} \frac{\alpha_k}{1 - \alpha_K} L_k + \alpha_K L_K.$$

By the induction hypothesis, $\sum_{k=1}^{K-1} \frac{\alpha_k}{1 - \alpha_K} L_k \succsim L_0$. Hence, as our first application of the independence axiom, we obtain

$$(1 - \alpha_K)\sum_{k=1}^{K-1} \frac{\alpha_k}{1 - \alpha_K} L_k + \alpha_K L_K \succsim (1 - \alpha_K)L_0 + \alpha_K L_K$$

Applying the axiom once again, we obtain

$$(1 - \alpha_K)L_0 + \alpha_K L_K \succsim (1 - \alpha_K)L_0 + \alpha_K L_0 = L_0.$$

Hence, by the transitivity, $\sum_{k=1}^{K} \alpha_k L_k \succsim L_0$. The first statement is thus verified. The case of $L_0 \succsim L_k$ can similarly be verified.

Now, for each n, let L^n be the lottery that yields outcome n with probability one. Then $\bar{L} \succsim L^n$ because both of them can be identified with sure outcomes. Let $L = (p_1, \ldots, p_N)$ be any lottery, then $L = \sum_n p_n L^n$. Thus, $\bar{L} \succsim L$ the above lemma. The same argument can be used to prove that $L \succsim \underline{L}$.

6.B.4 [First printing errata: On the the 11th and the 12th line of the exercise, the phrase "the lottery of B with probability q and D with probability 1 - q" should be "the lottery of A with probability q and D with probability 1 - q". Also, in the description of Criterion 2, the phrase "an unnecessary evacuation in 5%" should be "an unnecessary evacuation in 15%".]

(a) We can choose an assign utility levels (u_A, u_B, u_C, u_D) so that $u_A = 1$ and $u_D = 0$ as a normalization (Proposition 6.B.2). Then $u_B = p \cdot 1 + (1 - p) \cdot 0 = p$ and $u_C = q \cdot 1 + (1 - q) \cdot 0 = q$.

(b) The probability distribution under Criterion 1 is

$$(p_A, p_B, p_C, p_D) = (0.891,\ 0.099,\ 0.009,\ 0.001).$$

The probability distribution under Criterion 2 is

$$(p_A, p_B, p_C, p_D) = (0.8415,\ 0.1485,\ 0.0095,\ 0.0005).$$

The expected utility under Criterion 1 is thus $0.891 + 0.099p + 0.009q$. The expected utility under Criterion 2 is thus $0.8415 + 0.1485p + 0.0095q$. Hence the agency would prefer Criterion 1 if and only if $99 > 99p + q$, and it would prefer Criterion 2 if and only if $99 < 99p + q$.

6.B.5 (a) This follows from Exercise 6.B.1.

(b) The equivalence of the betweenness axiom and straight indifference curves can be established in the same way as in the part of Section 6.B on pp. 175-176 that explains how the independence axiom implies straight indifference curves. (Note that the argument there does not use the fully fledged independence axiom; as it is concerned with two indifferent lotteries, the betweenness axiom suffices.) Those straight lines need not be parallel, because the betweenness axiom imposes restrictions only on straight indifferent curves and nothing on the relative positions of different indifference lines. In fact, the argument for Figure 6.B.5(c) is not applicable to the betweenness axiom.

(c) Any preference represented by straight, but not parallel indifference curves satisfies the betweenness axiom but does not satisfy the independence axiom. Hence the former is weaker than the latter.

(d) Here is an example of a preference relation and its indifference map that satisfies the betweenness axiom and yields the choice of the Allais paradox.

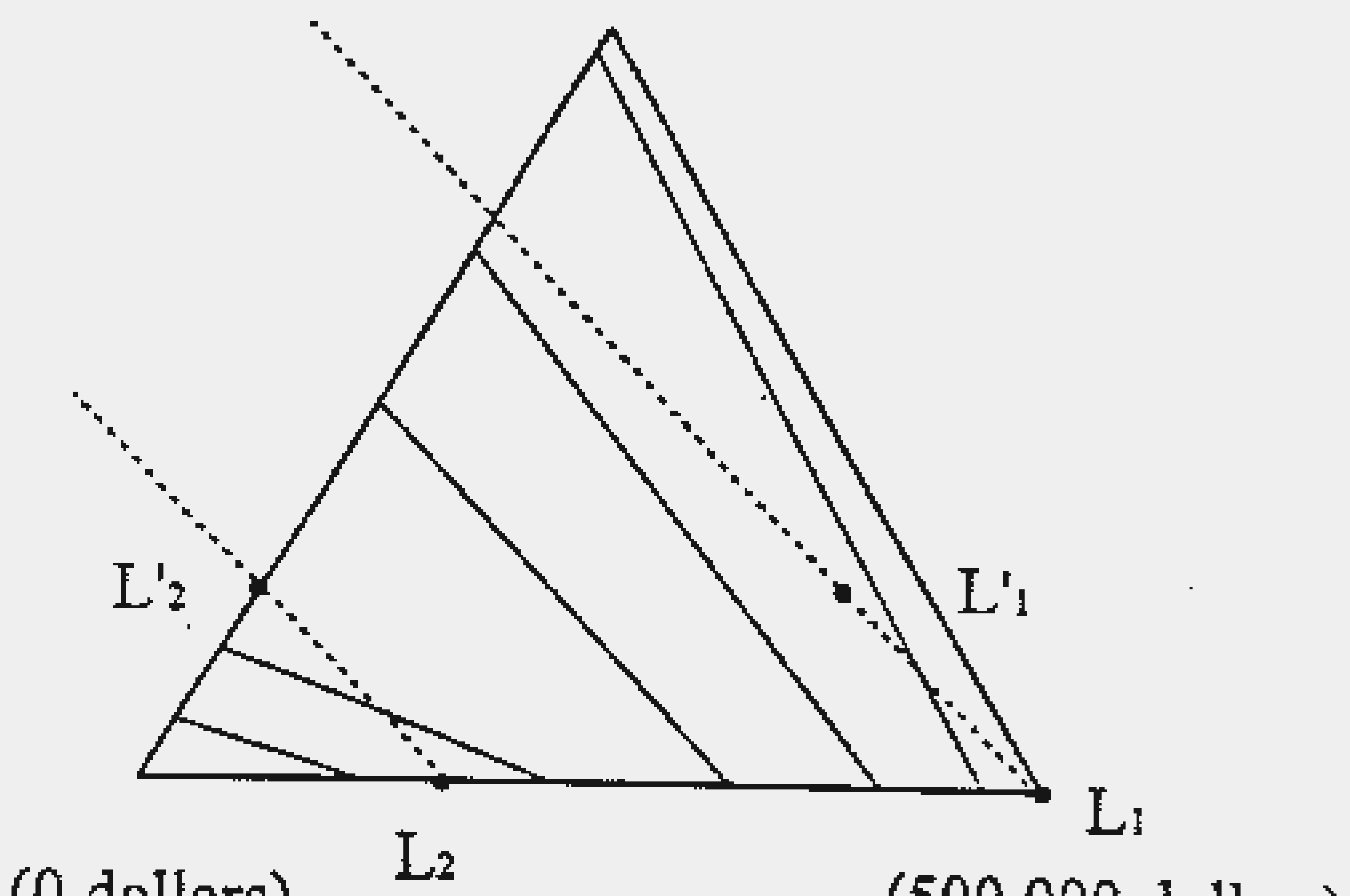

Figure 6.B.5(d)

6.B.6 Define $C = \{(u_1(a),\ldots,u_N(a)) \in \mathbb{R}^N: a \in A\}$, then

$$U(p) = \text{Max}\{p\cdot c \in \mathbb{R}: c \in C\} = -\text{Min}\{p\cdot c \in \mathbb{R}: c \in -C\}.$$

Hence $U(\cdot)$ is equal to $-\mu_{-C}(\cdot)$, the support function (Section 3.F) of $-C$ multiplied by -1, where the domain of the support function is restricted to the simplex $\{p \in \mathbb{R}^N_+: \sum_n p_n = 1\}$. Since any support function is concave, $U(\cdot)$ is convex. (A more direct proof is possible, which is essentially the same as the proof of concavity of support functions in Section 3.F.)

As an example of a nonlinear Bernoulli utility function, consider $A = B = \{1,2\}$ and define $u_1(1) = u_2(2) = 1$ and $u_2(1) = u_1(2) = 0$. Let $L = (p_1,p_2)$, then $U(L) = \text{Max}\ \{p_1,p_2\}$. (This is essentially the same as Example 6.B.4.)

6.B.7 Since the individual prefers L to L' and is indifferent between L and x_L and between L' and $x_{L'}$, by Proposition 1.B.1(iii), he prefers x_L to $x_{L'}$. By the monotonicity, this this is equivalent to $x_L > x_{L'}$.

6.C.1 If $\alpha = D > 0$ (complete insurance), then

$$- q(1 - \pi)u'(w - \alpha q) + \pi(1 - q)u'(w - D + \alpha(1 - q))$$

$$= - q(1 - \pi)u'(w - Dq) + \pi(1 - q)u'(w - Dq)$$

$$= u'(w - Dq)(\pi(1 - q) - q(1 - \pi)) < 0$$

$$= u'(w - Dq)(\pi - q) < 0$$

by $q > \pi$. Thus the first-order condition is not satisfied at $\alpha = D$. Hence the individual will not insure completely.

6.C.2 (a) Let $F(\cdot)$ be a distribution function, then

$$\int u(x)dF(x) = \int(\beta x^2 + \gamma x)dF(x) = \beta\int x^2 dF(x) + \gamma\int x dF(x)$$

$$= \beta(\text{mean of } F)^2 + \beta(\text{variance of } F) + \gamma(\text{mean of } F).$$

(b) We prove by contradiction that $U(\cdot)$ is not compatible with any Bernoulli utility function. So suppose that there is a Bernoulli utility function $u(\cdot)$ such that $U(F) = \int u(x)dF(x)$ for every distribution function $F(\cdot)$. Let x and y be two amounts of money, $G(\cdot)$ be the distribution that puts probability one at x, and $H(\cdot)$ be the distribution that puts probability one at y. Then

$$u(x) = U(G) = (\text{mean of } G) - (\text{variance of } G) = x - 0 = x,$$

$$u(y) = U(H) = (\text{mean of } H) - (\text{variance of } H) = y - 0 = y.$$

Thus, $x \geq y$ if and only if $u(x) \geq u(y)$. Hence $u(\cdot)$ is strictly monotone. Now let $F_0(\cdot)$ be the distribution that puts probability one on 0 and $F(\cdot)$ be the distribution that puts probability 1/2 on 0 and on $4/r > 0$. Since the mean and the variance of $F_0(\cdot)$ are zero, $U(F_0) = 0$. The strict monotonicity of $u(\cdot)$ thus implies that $U(F) > 0$. However, the mean of $F(\cdot)$ is $2/r$ and the variance is $4/r^2$. Hence $U(F) = 2/r - r(4/r^2) = - 2/r < 0$, which is a contradiction. Hence $U(\cdot)$ is not compatible with any Bernoulli utility function.

An example of two lotteries with the property requested in the exercise was given in the above proof of incompatibility. (Note that if all we need to show were the incompatibility of $U(\cdot)$ and any Bernoulli utility function, the equality $u(x) = x$ obtained above would be sufficient to complete the proof, because this implies the risk neutrality, which contradicts the fact that the variance of $F(\cdot)$ is subtracted in the definition of $U(\cdot)$.)

6.C.3 Suppose first that condition (i) holds. Let $x \in \mathbb{R}$ and $\varepsilon > 0$. Let $F(\cdot)$ be the distribution that puts probability 1/2 on $x - \varepsilon$ and on $x + \varepsilon$, and $F_\varepsilon(\cdot)$ be the distribution that puts probability $1/2 - \pi(x,\varepsilon,u)$ on $x - \varepsilon$ and $1/2 + \pi(x,\varepsilon,u)$ on $x + \varepsilon$. That is,

$$F(z) = \begin{cases} 0 & \text{if} \quad z < x - \varepsilon, \\ 1/2 & \text{if } x - \varepsilon \le z < x + \varepsilon, \\ 1 & \text{if } x + \varepsilon \ge z. \end{cases}$$

$$F_\varepsilon(z) = \begin{cases} 0 & \text{if} \quad z < x - \varepsilon, \\ 1/2 - \pi(x,\varepsilon,u) & \text{if } x - \varepsilon \le z < x + \varepsilon, \\ 1 & \text{if } x + \varepsilon \ge z. \end{cases}$$

Then $\int z dF(z) = x$ and $\int u(z)dF(z) \le u(x) = \int u(z)dF_\varepsilon(z)$ by (i). But

$$\int u(z)dF(z) = (1/2)u(x - \varepsilon) + (1/2)u(x + \varepsilon),$$

$$\int u(z)dF_\varepsilon(z) = (1/2 - \pi(x,\varepsilon,u))u(x - \varepsilon) + (1/2 + \pi(x,\varepsilon,u))u(x + \varepsilon).$$

$$= (1/2)u(x - \varepsilon) + (1/2)u(x + \varepsilon) + \pi(x,\varepsilon,u))(u(x + \varepsilon) - u(x - \varepsilon)).$$

Since $u(x + \varepsilon) - u(x - \varepsilon) > 0$, the above inequality is equivalent to $\pi(x,\varepsilon,u) \ge 0$. Thus (i) implies (iv).

Suppose conversely that condition (iv) holds. Let $y \in \mathbb{R}$, $z \in \mathbb{R}$, and $y > z$. Define $x = (y + z)/2$ and $\varepsilon = (y - z)/2$, then $y = x + \varepsilon$, $z = x - \varepsilon$, and

$$u(x) = (1/2 + \pi(x,\varepsilon,u))u(y) + (1/2 - \pi(x,\varepsilon,u))u(z)$$

$$= (1/2)u(y) + (1/2)u(z) + \pi(x,\varepsilon,u)(u(y) - u(z)).$$

Since $\pi(x,\varepsilon,u) \ge 0$ and $u(y) \ge u(z)$, this implies

$$(1/2)u(y) + (1/2)u(z) \le u(x) = u((1/2)y + (1/2)z).$$

Although we omit the proof, this is sufficient for the concavity of $u(\cdot)$.

Hence (iv) implies (ii). Since the equivalence of (i), (ii), and (iii) have already been established, this completes the proof of the equivalence of all four conditions.

6.C.4 (a) Let $\alpha = (\alpha_1,\ldots,\alpha_N) \in \mathbb{R}^N_+$, $\alpha' = (\alpha'_1,\ldots,\alpha'_N) \in \mathbb{R}^N_+$, and $\alpha \geq \alpha'$, then $\sum_n\alpha_n z_n \geq \sum_n\alpha'_n z_n$ for almost every realization $(z_1,\ldots,z_N)$, because all the returns are nonnegative with probability one. Since $u(\cdot)$ is increasing, this implies that $u(\sum_n\alpha_n z_n) \geq u(\sum_n\alpha'_n z_n)$ with probability one. Hence

$$\int u(\textstyle\sum_n\alpha_n z_n)dF(z_1,\ldots,z_N) \geq \int u(\textstyle\sum_n\alpha'_n z_n)dF(z_1,\ldots,z_N),$$

that is, $U(\alpha) \geq U(\alpha')$.

(b) Let $\alpha = (\alpha_1,\ldots,\alpha_N) \in \mathbb{R}^N_+$, $\alpha' = (\alpha'_1,\ldots,\alpha'_N) \in \mathbb{R}^N_+$, and $\lambda \in [0,1]$, then, by the concavity of $u(\cdot)$,

$$\begin{aligned} u(\textstyle\sum_n(\lambda\alpha_n + (1-\lambda)\alpha'_n)z_n) &= u(\lambda\textstyle\sum_n\alpha_n z_n + (1-\lambda)\textstyle\sum_n\alpha'_n z_n) \\ &\geq \lambda u(\textstyle\sum_n\alpha_n z_n) + (1-\lambda)u(\textstyle\sum_n\alpha'_n z_n) \end{aligned}$$

for almost every realization $(z_1,\ldots,z_n)$. Hence

$$\begin{aligned} &U(\lambda\alpha + (1-\lambda)\alpha') \\ &= \int u(\textstyle\sum_n(\lambda\alpha_n + (1-\lambda)\alpha'_n)z_n)dF(z_1,\ldots,z_N) \\ &\geq \int(\lambda u(\textstyle\sum_n\alpha_n z_n) + (1-\lambda)u(\textstyle\sum_n\alpha'_n z_n))dF(z_1,\ldots,z_N) \\ &= \lambda\int u(\textstyle\sum_n\alpha_n z_n)dF(z_1,\ldots,z_N) + (1-\lambda)\int u(\textstyle\sum_n\alpha'_n z_n))dF(z_1,\ldots,z_N) \\ &= \lambda U(\alpha) + (1-\lambda)U(\alpha'). \end{aligned}$$

(c) Let $(\alpha^m)_{m\in\mathbb{N}}$ be a sequence in $\mathbb{R}^N_+$ converging to $\alpha \in \mathbb{R}^N_+$, then there exists a positive number B such that $\alpha^m \leq (B,\ldots,B)$ for every m. Of course, $U(B,\ldots,B)$ is finite. But this is equivalent to saying that the (measurable) function $z \to u(\sum_n Bz_n)$ is integrable. Since $u(\cdot)$ is monotone and all the returns are nonnegative with probability one, $u(\sum_n\alpha^m_n z_n) \leq u(\sum_n Bz_n)$ for every m and for every realization $(z_1,\ldots,z_N)$. Moreover, since $u(\cdot)$ is continuous, $u(\sum_n\alpha^m_n z_n)$

converges to $u(\sum_n \alpha_n z_n)$ for almost every realization $(z_1,\ldots,z_N)$. Hence, by Lebesgue's dominated convergence theorem,

$$\int u(\textstyle\sum_n \alpha_n^m x_n)dF(x_1,\ldots,x_N) \to \int u(\sum_n \alpha_n x_n)dF(x_1,\ldots,x_N).$$

That is, $U(\alpha^m) \to U(\alpha)$.

6.C.5 (a) Let $x \in \mathbb{R}^L_+$, $y \in \mathbb{R}^L_+$ and $\lambda \in [0,1]$. In analogy with expression (6.C.1) the value $\lambda u(x) + (1 - \lambda)u(y)$ can be considered as the expected utility from the lottery that yields x with probability λ and y with probability $1 - \lambda$. On the other hand, the value $u(\lambda x + (1 - \lambda)y)$ is the utility from consuming the mean $\lambda x + (1 - \lambda)y$ of the lottery with probability one. The concavity of $u(\cdot)$ would then imply that consuming the mean bundle of the L commodities with probability one is at least as good as entering into the lottery. But this is the defining property of risk aversion in Definition 6.C.1.

(b) [First printing errata: The Bernoulli utility function $u(\cdot)$ for wealth should be denoted by another symbol, say $\tilde{u}(\cdot)$, to avoid confusion with the original utility function $u(\cdot)$ defined on $\mathbb{R}^L_+$.] Let $p \gg 0$ be a fixed price vector, w and w' be two wealth levels, and $\lambda \in [0,1]$. Denote the demand function by $x(\cdot)$ and let $x = x(p,w)$ and $x' = x(p,w')$, then $p\cdot(\lambda x + (1 - \lambda)x') \le \lambda w + (1 - \lambda)w'$. Thus $u(\lambda x + (1 - \lambda)x') \le \tilde{u}(\lambda w + (1 - \lambda)w')$. If $u(\cdot)$ is concave, then

$$u(\lambda x + (1 - \lambda)x') \ge \lambda u(x) + (1 - \lambda)u(x') = \lambda\tilde{u}(w) + (1 - \lambda)\tilde{u}(w').$$

Hence $\tilde{u}(\lambda w + (1 - \lambda)w') \ge \lambda\tilde{u}(w) + (1 - \lambda)\tilde{u}(w')$. Thus $\tilde{u}(\cdot)$ also exhibits risk aversion.

The following interpretation can be given to this result. Although, in the text, we are mainly concerned with the cases where outcomes are monetary amounts, in many cases in economic theory, utilities do not directly come from money, but from physical commodities. It is therefore desirable to *derive* risk aversion of Bernoulli utility functions for money from the properties of the underlying utility function for the commodities. The above result says that, if an individual has a risk-averse utility function for commodities, the his Bernoulli utility functions exhibits risk aversion.

(c) We shall give an example with the properties stated in the exercise. Let $L = 2$. Define $u\colon \mathbb{R}^2_+ \to \mathbb{R}$ by $u(x) = \sqrt{\text{Max}\{x_1, x_2\}}$, then $u(\cdot)$ is not concave. Now consider the price vector $p = (1,2)$, then, for each $w \geq 0$, $x(p,w) = (w,0)$. Hence $\tilde{u}(w) = \sqrt{w}$, which is concave and exhibits risk aversion. The lesson from this example is that, in order to obtain the risk aversion of $\tilde{u}(\cdot)$ for a fixed price vector, all that matters is the risk attitude along the wealth expansion path.

6.C.6 (a) Suppose that condition (ii) is true and let $F(\cdot)$ be any distribution function, then

$$\psi(u_1(c(F,u_2))) = u_2(c(F,u_2)) = \int u_2(x)dF(x).$$

Since $\psi(\cdot)$ is concave,

$$\int u_2(x)dF(x) = \int \psi(u_1(x))dF(x) \leq \psi(\int u_1(x)dF(x)).$$

Thus $\psi(u_1(c(F,u_2))) \leq \psi(\int u_1(x)dF(x))$. Since $\psi(\cdot)$ is increasing, this implies that $u_1(c(F,u_2)) \leq \int u_1(x)dF(x)$. Since $\int u_1(x)dF(x) = u_1(c(F,u_1))$, this implies that $u_1(c(F,u_2)) \leq u_1(c(F,u_1))$. Since $u_1(\cdot)$ is increasing, we obtain $c(F,u_2) \leq c(F,u_1)$. Condition (iii) is thus established.

Conversely, suppose that (iii) is true. Let $x \in \mathbb{R}$, $y \in \mathbb{R}$, and $\lambda \in [0,1]$. We shall prove that

$$\psi(\lambda u_1(x) + (1 - \lambda)u_1(y)) \geq \lambda\psi(u_1(x)) + (1 - \lambda)\psi(u_1(y)).$$

Let $F(\cdot)$ be the distribution function that puts probability λ on x and probability $1 - \lambda$ on y. Then, $\lambda u_1(x) + (1 - \lambda)u_1(y) = u_1(c(F,u_1))$ and hence $\psi(\lambda u_1(x) + (1 - \lambda)u_1(y)) = u_2(c(F,u_1))$. On the other hand, by the definition,

$$\lambda\psi(u_1(x)) + (1 - \lambda)\psi(u_1(y)) = \lambda u_2(x) + (1 - \lambda)u_2(y) = u_2(c(F,u_2)).$$

By (iii) and the increasingness of $u_2(\cdot)$, we obtain

$$\psi(\lambda u_1(x) + (1 - \lambda)u_1(y)) \geq \lambda\psi(u_1(x)) + (1 - \lambda)\psi(u_1(y)).$$

(b) Suppose first that condition (iii) holds. If $\int u_2(x)dF(x) \geq u_2(\bar{x})$, then $u_2(c(F,u_2)) \geq u_2(\bar{x})$. Thus $c(F,u_2) \geq \bar{x}$. By condition (iii), $c(F,u_1) \geq \bar{x}$. Hence $u_1(c(F,u_1)) \geq u_1(\bar{x})$, or $\int u_1(x)dF(x) \geq u_1(\bar{x})$. Thus condition (v) holds.

Suppose next that (v) holds, then $\int u_1(x)dF(x) \geq u_1(c(F,u_2))$. Since $\int u_1(x)dF(x) = u_1(c(F,u_1))$, we have $u_1(c(F,u_1)) \geq u_1(c(F,u_2))$ and hence $c(F,u_1) \geq c(F,u_2)$.

6.C.7 Suppose first that condition (iii) holds. Let $x \in \mathbb{R}$ and $\varepsilon > 0$. Denote by $F(\cdot)$ the distribution function that puts probability $1/2 - \pi(x,\varepsilon,u_2)$ on $x - \varepsilon$ and $1/2 + \pi(x,\varepsilon,u_2)$ on $x + \varepsilon$. That is,

$$F(z) = \begin{cases} 0 & \text{if} \quad z < x - \varepsilon, \\ 1/2 - \pi(x,\varepsilon,u_2) & \text{if} \quad x - \varepsilon \leq z < x + \varepsilon, \\ 1 & \text{if} \quad x + \varepsilon \geq z. \end{cases}$$

Then $c(F,u_2) = x$. By (iii), $c(F,u_1) \geq x$. Thus $u_1(c(F,u_1)) \geq u_1(x)$. But here, we have

$$\begin{aligned} & u_1(c(F,u_1)) \\ &= (1/2 - \pi(x,\varepsilon,u_2))u_1(x - \varepsilon) + (1/2 + \pi(x,\varepsilon,u_2))u_1(x + \varepsilon) \\ &= (1/2)u_1(x - \varepsilon) + (1/2)u_1(x + \varepsilon) + \pi(x,\varepsilon,u_2)(u_1(x + \varepsilon) - u_1(x - \varepsilon)) \end{aligned}$$

and

$$\begin{aligned} & u_1(x) \\ &= (1/2 - \pi(x,\varepsilon,u_1))u_1(x - \varepsilon) + (1/2 + \pi(x,\varepsilon,u_1))u_1(x + \varepsilon) \end{aligned}$$

$= (1/2)u_1(x - \varepsilon) + (1/2)u_1(x + \varepsilon) + \pi(x,\varepsilon,u_1)(u_1(x + \varepsilon) - u_1(x - \varepsilon)).$

Thus the last inequality is equivalent to $\pi(x,\varepsilon,u_2) \geq \pi(x,\varepsilon,u_1)$. Hence condition (iv) holds.

Suppose now that condition (iv) holds. Since $\pi(x,0,u_1) = \pi(x,0,u_2) = 0$, (iv) implies that $\partial\pi(x,0,u_2)/\partial\varepsilon \geq \partial\pi(x,0,u_2)/\partial\varepsilon$. Since $r_A(x,u_1) = 4\partial\pi(x,0,u_1)/\partial\varepsilon$ and $r_A(x,u_2) = 4\partial\pi(x,0,u_2)/\partial\varepsilon$, (i) follows.

6.C.8 Let w_1 and w_2 be two wealth levels such that $w_1 > w_2$ and define $u_1(z) = u(w_1 + z)$ and $u_2(z) = u(w_2 + z)$, then $u_2(\cdot)$ is a concave transformation of $u_1(\cdot)$ by Proposition 6.C.3. It was shown in Example 6.C.2 continued that the demand for the risky asset of $u_1(\cdot)$ is greater than that of $u_2(\cdot)$. This means that the demand for the risky asset of $u(\cdot)$ is greater at wealth level w_1 than at w_2.

6.C.9 [<u>First printing errata</u>: The function $u(\cdot)$ on the left-hand side of the equality on the fifth line should be denoted by a different symbol, because, on the right-hand side, $u(\cdot)$ is used for the utility function on the first period.]

(a) The first-order condition for the first problem is $u'(w - x_0) = v'(x_0)$. For the second problem, let's first define a function $\phi(\cdot)$ by

$$\phi(x) = u(w - x) + E[v(x + y)].$$

Then $\phi'(x) = - u'(w - x) + E[v'(x + y)]$ and $\phi''(x) = u''(w - x) + E[v''(x + y)]$. Note also that $\phi'(x^*) = 0$ and $\phi''(x) \leq 0$ for every x, which implies that if $\phi'(x) > 0$, then $x^* > x$. Now, since $E[v'(x_0 + y)] > v'(x_0)$,

$$\phi'(x_0) = - u'(w - x_0) + E[v'(x_0 + y)] = - v'(x_0) + E[v'(x_0 + y)] > 0.$$

Hence $x^* > x_0$.

(b) Define two functions $\eta_1(\cdot)$ and $\eta_2(\cdot)$ by $\eta_1(x) = -v_1'(x)$ and $\eta_2(x) = -v_2'(x)$. Then $\eta_1(\cdot)$ and $\eta_2(\cdot)$ are increasing and the coefficients of absolute prudence of $v_1(\cdot)$ and of $v_2(\cdot)$ are equal to the coefficients of absolute risk aversion of $\eta_1(\cdot)$ and of $\eta_2(\cdot)$. Thus, if the coefficient of absolute prudence of $v_1(\cdot)$ is not larger than that of $v_2(\cdot)$, then the coefficient of absolute risk aversion of $\eta_1(\cdot)$ is not larger than that of $\eta_2(\cdot)$. Moreover, since $E[v_1'(x_0 + y)] > v_1'(x_0)$, we have $E[\eta_1(x_0 + y)] < \eta_1(x_0)$. Thus, by applying Proposition 6.C.2 to $\eta_1(\cdot)$ and $\eta_2(\cdot)$, we obtain $E[\eta_2(x_0 + y)] < \eta_2(x_0)$. Hence $E[v_2'(z_0 + y)] > v_2'(z_0)$.

The implication of this fact to part (a) is that, if the coefficient of absolute prudence of the first is not larger than that of the second, and if the risk y induces the first individual to save more, then it also induces the second to do so. Hence coefficients of absolute prudence measure how much individuals are willing to save when faced with a risk in the future.

(c) If $v'''(x) > 0$, then $\eta''(x) = -v'(x) < 0$ and hence $\eta(\cdot)$ exhibits risk aversion. Thus $E[\eta(x + y)] < \eta(x)$, that is, $E[v_1'(x + y)] > v_1'(x)$.

(d) Since

$$r_A'(x,v) = -\frac{v'''(x)v'(x) - v''(x)^2}{v'(x)^2} = \frac{v''(x)}{v'(x)}\left(-\frac{v'''(x)}{v''(x)} + \frac{v''(x)}{v'(x)}\right) < 0,$$

the assertion follows.

6.C.10 Throughout this answer, we let x_1 and x_2 be two fixed wealth levels such that $x_1 > x_2$ and define $u_1(z) = u(x_1 + z)$ and $u_2(z) = u(x_2 + z)$. It is sufficient to prove that each of the five conditions of Proposition 6.C.3 is equivalent to its counterpart of Proposition 6.C.2.

Since $r_A(z,u_1) = r_A(x_1 + z, u)$ and $r_A(z,u_2) = r_A(x_2 + z, u)$, property (i) of Proposition 6.C.3 is equivalent to (i) of Proposition 6.C.2.

Property (ii) of Proposition 6.C.3 is nothing but a restatement of (ii) of Proposition 6.C.2.

As for property (iii), note that

$$\int u_1(z)dF(z) = \int u(x_1 + z)dF(z) = u(c_{x_1}) = u((c_{x_1} - x_1) + x_1) = u_1(c_{x_1} - x_1)$$

and likewise for $u_2(\cdot)$. Thus the certainty equivalent for $u_1(\cdot)$ is smaller than that for $u_2(\cdot)$ if and only if $c_{x_1} - x_1 < c_{x_2} - x_2$. Thus property (iii) of Proposition 6.C.3 is equivalent to (iii) of Proposition 6.C.2.

As for property (iv), since

$$u(x_1) = (1/2 - \pi(x_1,\varepsilon,u))u(x_1 - \varepsilon) + (1/2 + \pi(x_1,\varepsilon,u))u(x_1 + \varepsilon),$$

we have

$$u_1(0) = (1/2 - \pi(x_1,\varepsilon,u))u_1(-\varepsilon) + (1/2 + \pi(x_1,\varepsilon,u))u_1(\varepsilon).$$

Hence $\pi(x_1,\varepsilon,u) = \pi(0,\varepsilon,u_1)$. Similarly, $\pi(x_2,\varepsilon,u) = \pi(0,\varepsilon,u_2)$. Hence (iv) of Proposition 6.C.3 is equivalent to (iv) of Proposition 6.C.2.

Note that $\int u(x_1 + z)dF(z) \geq u(x_1)$ if and only if $\int u_1(z)dF(z) \geq u_1(0)$, and likewise for $u_2(\cdot)$. Thus property (v) of Proposition 6.C.3 is equivalent to (v) of Proposition 6.C.2.

6.C.11 For any wealth level x, denote by $\gamma(x)$ the optimal proportion of x invested in the risky asset. We shall give a direct proof that if the coefficient of relative risk aversion is increasing, then $\gamma'(x) < 0$; along the same line of proof, we can show that if it is decreasing, then $\gamma'(x) > 0$. As shown in Exercise 6.C.2, $\gamma(x)$ is positive and satisfies the following first-order condition for every x:

$$\int u'((1 - \gamma(x) + \gamma(x)z)x)(z - 1)x\,dF(z) = 0.$$

Hence

$$\gamma'(x) = \frac{-\int u''((1-\gamma(x)+\gamma(x)z)x)(1-\gamma(x)+\gamma(x)z)(z-1)x\,dF(z)}{\int u''((1-\gamma(x)+\gamma(x)z)x)(z-1)^2x^2\,dF(z)}.$$

Since the denominator is negative, it is sufficient to show that the numerator is positive.

By the definition of the coefficient of relative risk aversion,

$$- u''((1-\gamma(x)+\gamma(x)z)x)(1-\gamma(x)+\gamma(x)z)x$$
$$= r_R((1-\gamma(x)+\gamma(x)z)x)u'((1-\gamma(x)+\gamma(x)z)x)$$

for every realization z. Note also that if $z > 1$, then $(1-\gamma(x)+\gamma(x)z)x > x$ by $\gamma(x) > 0$. Since the coefficient of relative risk aversion is increasing, this implies that $r_R((1-\gamma(x)+\gamma(x)z)x) > r_R(x)$. Hence

$$- u''((1-\gamma(x)+\gamma(x)z)x)(1-\gamma(x)+\gamma(x)z)x$$
$$> r_R(x)u'((1-\gamma(x)+\gamma(x)z)x).$$

By $z - 1 > 0$,

$$- u''((1-\gamma(x)+\gamma(x)z)x)(1-\gamma(x)+\gamma(x)z)x(z-1)$$
$$> r_R(x)u'((1-\gamma(x)+\gamma(x)z)x)(z-1).$$

We can similarly show that this last inequality also holds for every $z < 1$. Therefore,

$$- \int u''((1-\gamma(x)+\gamma(x)z)x)(1-\gamma(x)+\gamma(x)z)x(z-1)dF(z)$$
$$> \int r_R(x)u'((1-\gamma(x)+\gamma(x)z)x)(z-1)dF(z)$$
$$= r_R(x)\int u'((1-\gamma(x)+\gamma(x)z)x)(z-1)dF(z) = 0$$

by the first-order condition.

6.C.12 (a) [<u>First printing errata</u>: The coefficient β should be positive if $\rho < 1$ and negative if $\rho > 1$. This makes $u(\cdot)$ increasing.] It is easy to check that, if $u(x) = \beta x^{1-\rho} + \gamma$ with $\rho \neq 1$ and $\gamma \in \mathbb{R}$, then $u(\cdot)$ exhibits constant relative risk aversion ρ. Suppose conversely that $u(\cdot)$ exhibits constant risk aversion ρ, then $u''(x)/u'(x) = -\rho/x$. Thus $\ln(u'(x)) = -\rho \ln x + c_1$ for some

$c_1 \in \mathbb{R}$. Thus $u'(x) = (\exp c_1)x^{-\rho}$. Hence $u(x) = (\exp c_1)x^{1-\rho}/(1-\rho) + c_2$ for some $c_2 \in \mathbb{R}$. Letting $\beta = (\exp c_1)/(1-\rho)$ and $\gamma = c_2$, we complete the proof.

(b) It is easy to check that, if $u(x) = \beta \ln x + \gamma$ with $\beta > 0$ and $\gamma \in \mathbb{R}$, then $u(\cdot)$ exhibits constant relative risk aversion one. The other direction can be shown in the same way as in (a).

(c) By L'hopital's rule,

$$\lim_{\rho \to 1} [(x^{1-\rho} - 1)/(1-\rho)] = \lim_{\rho \to 1} (-\ln x)x^{1-\rho}/(-1) = \ln x.$$

6.C.13 Let $\pi(\cdot)$ be the profit function and $F(\cdot)$ be the distribution function of the random price. Since $\pi(\cdot)$ is convex, $\int \pi(p)dF(p) \geq \pi(\int p dF(p))$ by Jensen's inequality. But the left-hand side is the expected payoff from the uncertain prices and the right-hand side is the utility of the expected price vector. Thus the firm prefers the uncertain prices.

6.C.14 Define a function $g(\cdot)$ by $g(\alpha) = k\alpha + v(u^{-1}(\alpha))$, then $g(u(x)) = ku(x) + v(x) = u^*(x)$. It is thus sufficient to show that $g(\cdot)$ is concave. For this, in turn, it is sufficient to prove that $(v \circ u^{-1})(\cdot)$ is concave.

Let $\alpha, \beta \in \mathbb{R}$ and $\lambda \in [0,1]$. Since $u(\cdot)$ is increasing and concave, $u^{-1}(\cdot)$ is convex. Thus

$$u^{-1}(\lambda\alpha + (1-\lambda)\beta) \leq \lambda u^{-1}(\alpha) + (1-\lambda)u^{-1}(\beta).$$

Since $v(\cdot)$ is nonincreasing, this implies

$$v(u^{-1}(\lambda\alpha + (1-\lambda)\beta)) \geq v(\lambda u^{-1}(\alpha) + (1-\lambda)u^{-1}(\beta)).$$

Since $v(\cdot)$ is concave,

$$v(\lambda u^{-1}(\alpha) + (1-\lambda)u^{-1}(\beta)) \geq \lambda v(u^{-1}(\alpha)) + (1-\lambda)v(u^{-1}(\beta)).$$

Thus,

$$v(u^{-1}(\lambda\alpha + (1 - \lambda)\beta)) \ge \lambda v(u^{-1}(\alpha)) + (1 - \lambda)v(u^{-1}(\beta)).$$

or, equivalently,

$$(v\circ u^{-1})(\lambda\alpha + (1 - \lambda)\beta) \ge \lambda(v\circ u^{-1})(\alpha) + (1 - \lambda)(v\circ u^{-1})(\beta).$$

(b) [First printing errata: The entire interval $[0, +\infty]$ should be $[0, +\infty)$.] Suppose that we have $u^*(x) = ku(x) + v(x)$ for a non-constant $v(\cdot)$. Since $v(\cdot)$ is decreasing and concave, $v(x + 1) - v(x)$ is negative and decreasing with x. On the other hand, since $u(\cdot)$ is increasing, concave, and bounded above, $u(x + 1) - u(x)$ is positive and decreasing, and converges to zero. Since

$$u^*(x + 1) - u^*(x) = k(u(x + 1) - u(x)) + (v(x + 1) - v(x)),$$

$u^*(x + 1) - u^*(x)$ is negative for any sufficiently large x. That is, $u^*(\cdot)$ is not increasing around such x. But this is a contradiction to the assumption that $u^*(\cdot)$ is increasing. Thus, if $u(\cdot)$ is bounded, then there is no non-constant $v(\cdot)$ such that $u^*(x) = ku(x) + v(x)$ for all $x \in [0, +\infty)$.

(c) By (a) and (b), it is sufficient to find $u(\cdot)$ and $u^*(\cdot)$ such that $u^*(\cdot)$ is more risk averse (in the Arrow-Pratt sense) than $u(\cdot)$ and $u(\cdot)$ is bounded. Define $u(x) = -\exp(-\alpha x)$ and $u^*(x) = -\exp(-\beta x)$, where $0 < \alpha < \beta$. By Example 6.C.4, $u(\cdot)$ and $u^*(\cdot)$ exhibit constant absolute risk aversion with coefficients α and β. Hence, $u^*(\cdot)$ is more risk averse than $u(\cdot)$, but, since $u(x) < 0$ for all x, $u^*(\cdot)$ is not strongly more risk averse than $u(\cdot)$.

6.C.15 Throughout this answer, we assume that $a \neq b$, because, otherwise, there would be no uncertainty involved in the payment of the second asset.

(a) If $\text{Min}\,\{a,b\} \ge 1$, the risky asset pays at least as high a return as the riskless asset at both states, and a strictly higher return at one of them. Then all the wealth is invested to the risky asset. Thus, $\text{Min}\,\{a,b\} < 1$ is a necessary condition for the demand for the riskless asset to be strictly positive.

(b) If $\pi a + (1 - \pi)b \leq 1$, then the expected return does not exceed the payments of the riskless asset and hence the risk-averse decision maker does not demand the risky asset at all. Thus, $\pi a + (1 - \pi)b > 1$ is a necessary condition for the demand for the risky asset to be strictly positive.

In the following answers, we assume that the demands for both assets are always positive.

(c) Since the prices of the two assets are equal to one, their marginal utilities must be equal. Thus

$$\pi u'(x_1 + x_2 a) + (1 - \pi)u'(x_1 + x_2 b) = \pi a u'(x_1 + x_2 a) + (1 - \pi)b u'(x_1 + x_2 b).$$

That is,

$$\pi(1 - a)u'(x_1 + x_2 a) + (1 - \pi)(1 - b)u'(x_1 + x_2 b) = 0.$$

This and $x_1 + x_2 = 1$ constitute the first-order condition.

(d) Taking b as constant, define

$$\phi(a,\pi,x_1) = \pi(1 - a)u'(x_1 + (1 - x_1)a) + (1 - \pi)(1 - b)u'(x_1 + (1 - x_1)b),$$

then

$$\partial\phi/\partial a = -\pi u'(x_1 + (1 - x_1)a) + \pi(1 - a)(1 - x_1)u''(x_1 + (1 - x_1)a) < 0,$$

$$\partial\phi/\partial x_1 = \pi(1 - a)^2 u''(x_1 + (1 - x_1)a) + (1 - \pi)(1 - b)^2 u''(x_1 + (1 - x_1)b) < 0.$$

Thus, by the implicit function theorem (Theorem M.E.1),

$$dx_1/da = -\frac{\partial\phi/\partial a}{\partial\phi/\partial x_1} < 0.$$

(e) It follows from the condition of (b) that $b > 1$, that is, that a is the worse outcome of the risky asset. Thus, if the probability π of the worse outcome is increased, then it is anticipated that the demand for the riskless asset is increased.

(f) Since $b > 1$,

$$\partial\phi/\partial\pi = (1 - a)u'(x + (1 - x)a) - (1 - b)u'(x + (1 - x)b)$$

$$= (1 - a)u'(x + (1 - x)a) + (b - 1)u'(x + (1 - x)b) > 0,$$

because $a < 1 < b$. Thus $dx/d\pi = -\dfrac{\partial\phi/\partial\pi}{\partial\phi/\partial x} > 0$, as anticipated.

6.C.16 Throughout the answer, we assume that $u(\cdot)$ is continuous, so that the maximum and the minimum are attained.

(a) If the individual owns the lottery, his random wealth is $(w + G, w + B)$. Thus the minimal selling price R_s is defined by

$$pu(w + G) + (1 - p)u(w + B) = u(w + R_s).$$

(b) If he buys the lottery at price R, his random wealth is $(w - R + G, w - R + B)$. The maximal buying price R_b is defined by

$$pu(w - R_b + G) + (1 - p)u(w - R_b + B) = u(w).$$

(c) In general, these two prices are different. However, if $u(\cdot)$ exhibits constant absolute risk aversion, then they are the same. In fact, the above two equations can be restated as $c_w = w + R_s$ and $c_{w-R_b} = w$, where c_w and c_{w-R_b} are defined as in (iii) of Proposition 6.C.3. According to the proposition, the constant absolute risk aversion implies that

$$w - c_w = (w - R_b) - c_{w-R_b}.$$

This is equivalent to $R_s = R_b$.

(d) By a direct calculation,

$$R_s = 5[(7 - 4\sqrt{3})p^2 + (4\sqrt{3} - 6)p + 1],$$

and R_b is one of the solutions to the quadratic equation

$$(1 - 2p^2)R_b^2 - 10(2p^3 + 7p^2 - 8p + 1)R_b - 25(23p^2 - 54p + 29) = 0.$$

6.C.17 According to Exercise 6.C.12, if $u(\cdot)$ exhibits constant relative risk aversion ρ, then $u(x) = \beta x^{1-\rho} + \gamma$ or $u(x) = \beta \ln(x) + \gamma$. In this answer, we assume $u(x) = \beta x^{1-\rho}$. The case of $\beta \ln(x) + \gamma$ can be proven by the same argument. Let's first consider the portfolio problem of the individual in period t = 1, after a realization of the random return has generated wealth level w_1. Denoting the distribution function of the return by $F(\cdot)$, his problem is

$$\text{Max}_{0 \le \alpha_1 \le 1} \int u(((1 - \alpha_1)R + \alpha_1 x_2)w_1)dF(x_2).$$

As discussed in Example 6.C.2 continued (and also in Exercise 6.C.11), we can show that the solution does not depend on the value of w_1. Denote the solution by α^*. If he chooses portfolio α_0 at t = 0, then his random wealth at t = 1 is $w_1 = ((1 - \alpha_0)R + \alpha_0 x_1)w_0$. Given the solution α^* at t = 1, his problem in period t = 0 is

$$\text{Max}_{0 \le \alpha_1 \le 1} \int\int u(((1 - \alpha^*)R + \alpha^* x_2)((1 - \alpha_0)R + \alpha_0 x_1)w_0)dF(x_2)dF(x_1).$$

Since the distributions of x_1 and x_2 are independent and $u(x) = \beta x^{1-\rho}$, we can rewrite the objective function as

$$[\int((1 - \alpha^*)R + \alpha^* x_2)^{1-\rho}dF(x_2)][\int u(((1 - \alpha_0)R + \alpha_0 x_1)w_0)dF(x_1)].$$

Since the first integral does not depend on the choice of α_0, the solution is again $\alpha_0 = \alpha^*$. This completes the proof.

For the case of a utility function exhibits constant absolute risk aversions, the absolute amounts of wealth invested on the risky asset may vary over the two periods t = 0,1, but those in period t = 1 do not depend on the realization of x_1. To see this, let $u(x) = -\beta e^{-\rho}$. The individual's problem at t = 1 is

$$\text{Max}_{0\le\alpha_1\le w} \int u((w_1 - \alpha_1)R + \alpha_1 x_2)dF(x_2).$$

The solution turns out to be independent of the value of w_1, and hence of x_0. Denote the solution by α^*. If he chooses portfolio α_0 at $t = 0$, then his random wealth at $t = 1$ is $w_1 = (w_0 - \alpha_0)R + \alpha_0 x_1$. Given the solution α^* at $t = 1$, his problem in period $t = 0$ is

$$\text{Max}_{0\le\alpha_1\le 1} \int\int u(((w_0 - \alpha_0)R + \alpha_0 x_1 - \alpha^*)R + \alpha^* x_2)dF(x_2)dF(x_1).$$

Since the distributions of x_1 and x_2 are independent and $u(x) = -\beta e^{-\rho}$, we can rewrite the objective function as

$$[\int \exp(-\alpha^* R + \alpha^* x_2)dF(x_2)][\int -\beta\exp(-((w_0 - \alpha_0)R + \alpha_0 x_1)R\rho)dF(x_1)].$$

Since the first integral does not depend on the choice of α_0, the solution of this maximization problem is the same as the solution of the problem of maximizing

$$\int -\beta\exp(-((w_0 - \alpha_0)R + \alpha_0 x_1)R\rho)dF(x_1).$$

But the latter is the same as what the consumer would choose at $t = 1$ if his coefficient of absolute risk aversion is equal to $R\rho$.

Now, if $R = 1$, then the consumer invests a constant absolute amount of wealth over two periods. Thus, their proportions out of the total wealths are larger if the total wealths are smaller. So, the proportion α_1/w_1 now depends on w_1 and hence on the realization x_1. Hence the proportions can no longer be constant.

6.C.18 (a) A direct calculation shows that the coefficient of absolute risk aversion at $w = 5$ is 0.1. Exercise 6.C.12(a) shows that the coefficient of relative risk aversion is 0.5, which is constant over w.

(b) By a direct calculation, the certainty equivalent is 9 and the probability premium is $(\sqrt{10} - 3)/2$.

(c) By a direct calculation, the certainty equivalent is 25 and the probability premium is $(\sqrt{26} - 5)/2$.

For each of these two lotteries, the difference between the mean of the lottery and the certainty equivalent is equal to one. However, the probability premium for the first lottery is larger. This is because $u(\cdot)$ exhibits constant relative risk aversion and hence decreasing absolute risk aversion.

6.C.19 Foe each n, denote by β_n the wealth invested in risky asset n. The wealth invested in the riskless asset is then $w - \sum_n \beta_n$. If the individual takes portfolio $\beta = (\beta_1,\ldots,\beta_N) \in \mathbb{R}^N$, then his random consumption is $x = (w - \sum_n \beta_n)r + \sum_n \beta_n z_n$, where z_n denotes the random return of asset n. By linearity of normal distributions, x is a normal distribution with mean $(w - \sum_n \beta_n)r + \sum_n \beta_n \mu_n$ and variance $\beta\cdot V\beta$. The expected utility from x is $E[-\exp(-\alpha x)]$. But this is equal to the value, multiplied by -1, at $-\alpha$ of the moment-generating function of the normal distribution with mean $(w - \sum_n \beta_n)r + \sum_n \beta_n \mu_n$ and variance $\beta\cdot V\beta$. Therefore,

$$E[-\exp(-\alpha x)] = -\exp[((w - \textstyle\sum_n \beta_n)r + \sum_n \beta_n \mu_n)(-\alpha) - (\beta\cdot V\beta)(-\alpha)^2/2].$$

By applying the monotone transformation $u \to (-1/\alpha)\ln(-u)$ to this utility function, we obtain

$$((w - \textstyle\sum_n \beta_n)r + \sum_n \beta_n \mu_n) + (\beta\cdot V\beta)\alpha/2.$$

The first-order condition for a maximum of this objective function with respect to β gives the optimal portfolio $\beta^* = \alpha^{-1}V^{-1}(\mu - re)$, where e is the vector of $\mathbb{R}^N$ whose components are all equal to one.

6.C.20 For each $\varepsilon \geq 0$, let $F_\varepsilon(\cdot)$ be the distribution function of the lottery that pays x + ε with probability 1/2 and x − ε with probability 1/2. Then, $c(F_\varepsilon,u)$ is defined as the solution to the equation

$$(1/2)u(x + \varepsilon) + (1/2)u(x - \varepsilon) - u(c) = 0$$

with respect to c. Hence, by the implicit function theorem (Theorem M.E.1), $c(F_\varepsilon,u)$ is a differentiable function of ε and

$$(1/2)u'(x + \varepsilon) - (1/2)u'(x - \varepsilon) - u'(c(F_\varepsilon,u))(\partial c(F_\varepsilon,u)/\partial\varepsilon) = 0.$$

By putting $\varepsilon = 0$, we obtain $\partial c(F_0,u)/\partial\varepsilon = 0$. Also, by further differentiating the left-hand side of this equality with respect to ε, we obtain

$$\begin{aligned}&(1/2)u''(x + \varepsilon) + (1/2)u''(x - \varepsilon)\\ &- u''(c(F_\varepsilon,u))(\partial c(F_\varepsilon,u)/\partial\varepsilon)^2 - u'(c(F_\varepsilon,u))(\partial^2 c(F_\varepsilon,u)/\partial\varepsilon^2) = 0.\end{aligned}$$

Thus, by putting $\varepsilon = 0$ and substituting $\partial c(F_0,u)/\partial\varepsilon = 0$, we obtain

$$u''(x) - u'(c(F_\varepsilon,u))(\partial^2 c(F_\varepsilon,u)/\partial\varepsilon^2) = 0.$$

Thus $\partial^2 c(F_\varepsilon,u)/\partial\varepsilon^2 = -r_A(x)$.

6.D.1 Let $L = (p_1,p_2,p_3)$ and $L' = (p_1',p_2',p_3')$ be two lotteries and $F(\cdot)$ and $G(\cdot)$ be their distribution functions.

(a) If a Bernoulli utility function is increasing, then there exists p $\in$ (0,1) such that the decision maker is indifferent between the sure outcome of \$2 and the lottery that pays \$1 with probability p and \$3 with probability 1 − p. Thus, the indifference line that goes through the \$2-vertex must hit some point on the (\$1,\$3)-face (excluding the vertices) and all indifference lines must be parallel to it. Conversely, this condition implies that the Bernoulli utility function increasing. By varying p vary from 0 to 1, we can identify the area of the lotteries that are above all indifference curves going through L. The area is shaded in the following figure:

6.D.2 [First printing errata: The phrase "the mean of x under $F(\cdot)$, $\int x dF(x)$, exceeds that under $G(\cdot)$, $\int x dG(x)$" should be "the mean of x under $G(\cdot)$, $\int x dG(x)$, cannot exceed that under $F(\cdot)$, $\int x dF(x)$". That is, the equality of the two means should be allowed.] For the first assertion, simply put $u(x) = x$ and apply Definition 6.D.1. As for the second, let $p \in (0,1/2)$ and consider the following two distributions:

$$F(z) = \begin{cases} 0 & \text{if } \quad z < 0, \\ p & \text{if } 0 \le z < 2, \\ 1 & \text{if } 2 \le z, \end{cases}$$

$$G(z) = \begin{cases} 0 & \text{if } z < 1, \\ 1 & \text{if } 1 \le z. \end{cases}$$

Then $F(1/2) = p > 0 = G(1/2)$ and $\int x dF(x) = 2(1 - p) > 1 = \int x dG(x)$. Hence $F(\cdot)$ does not first-order stochastically dominate $G(\cdot)$, but the mean of $F(\cdot)$ is larger than that of $G(\cdot)$.

6.D.3 Any elementary increase in risk from a distribution $F(\cdot)$ is a mean-preserving spread of $F(\cdot)$. In Example 6.D.2, we saw that any mean-preserving spread of $F(\cdot)$ is second-order stochastically dominated by $F(\cdot)$. Hence the assertion follows.

6.D.4 Let $L = (p_1,p_2,p_3)$ and $L' = (p'_1,p'_2,p'_3)$ be two lotteries.

(a) By a direct calculation, the means of L and L' are $2 - p_1 + p_3$ and $2 - p'_1 + p'_3$. Thus the two lotteries have an equal mean if and only if $p_1 - p_3 = p'_1 - p'_3$. Hence they have an equal mean if and only if they are both on a segment that is parallel to the segment connecting the \$2-vertex and the middle point of the (\$1,\$3)-face, as depicted below:

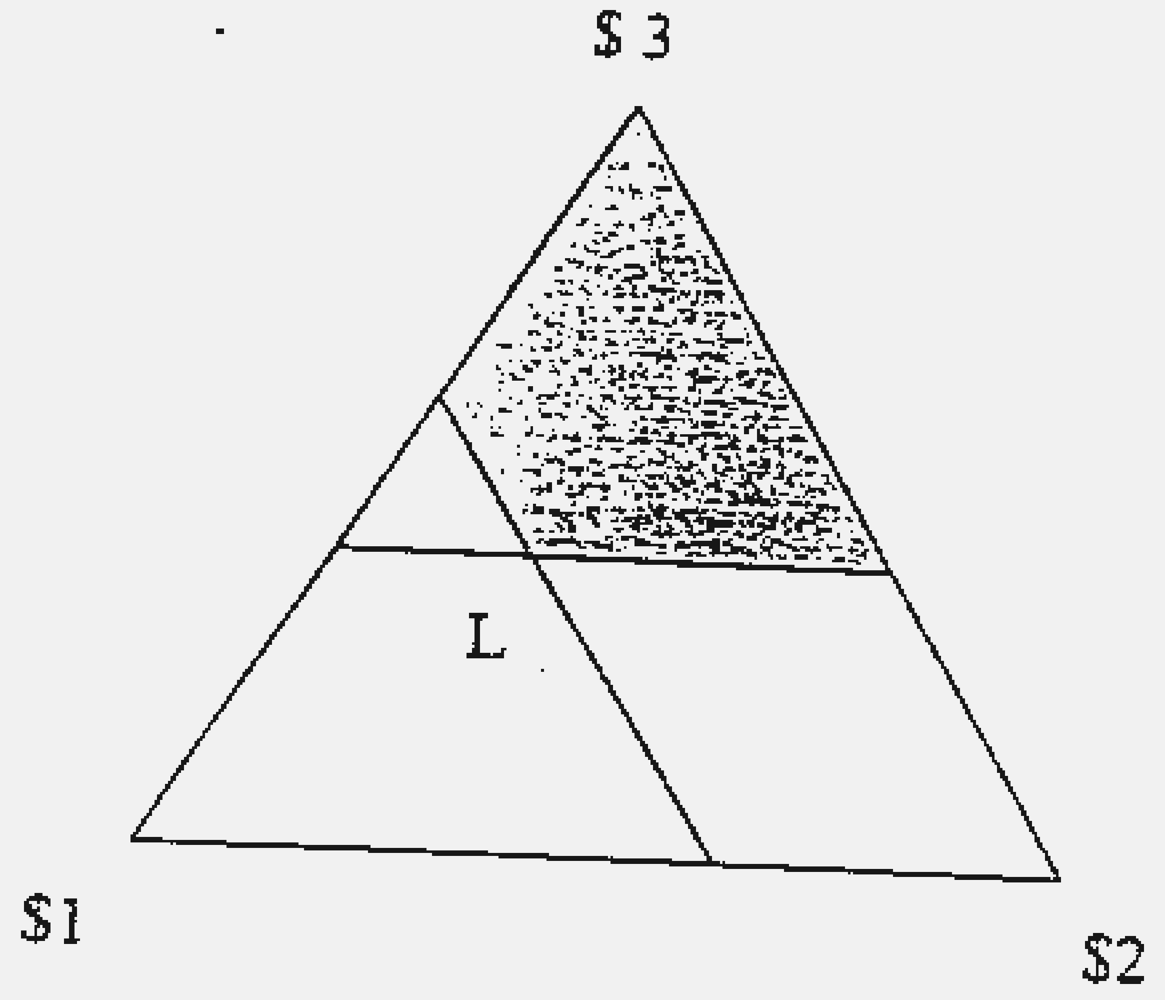

Figure 6.D.1(a)

Thus, $G(\cdot)$ first-order stochastically dominates $F(\cdot)$ if and only if L' is located above the segment that goes through L and is parallel to the (\$1,\$2)-face and also above the segment that goes through L and parallel to the (\$2,\$3)-face.

(b) The distribution $G(\cdot)$ first-order stochastically dominates $F(\cdot)$ if and only if $p_1 \geq p'_1$ and $p_1 + p_2 \geq p'_1 + p'_2$. Since the second inequality is equivalent to $p_3 \leq p'_3$, $G(\cdot)$ first-order stochastically dominates $F(\cdot)$ if and only if L' is located in the shaded area in the figure below:

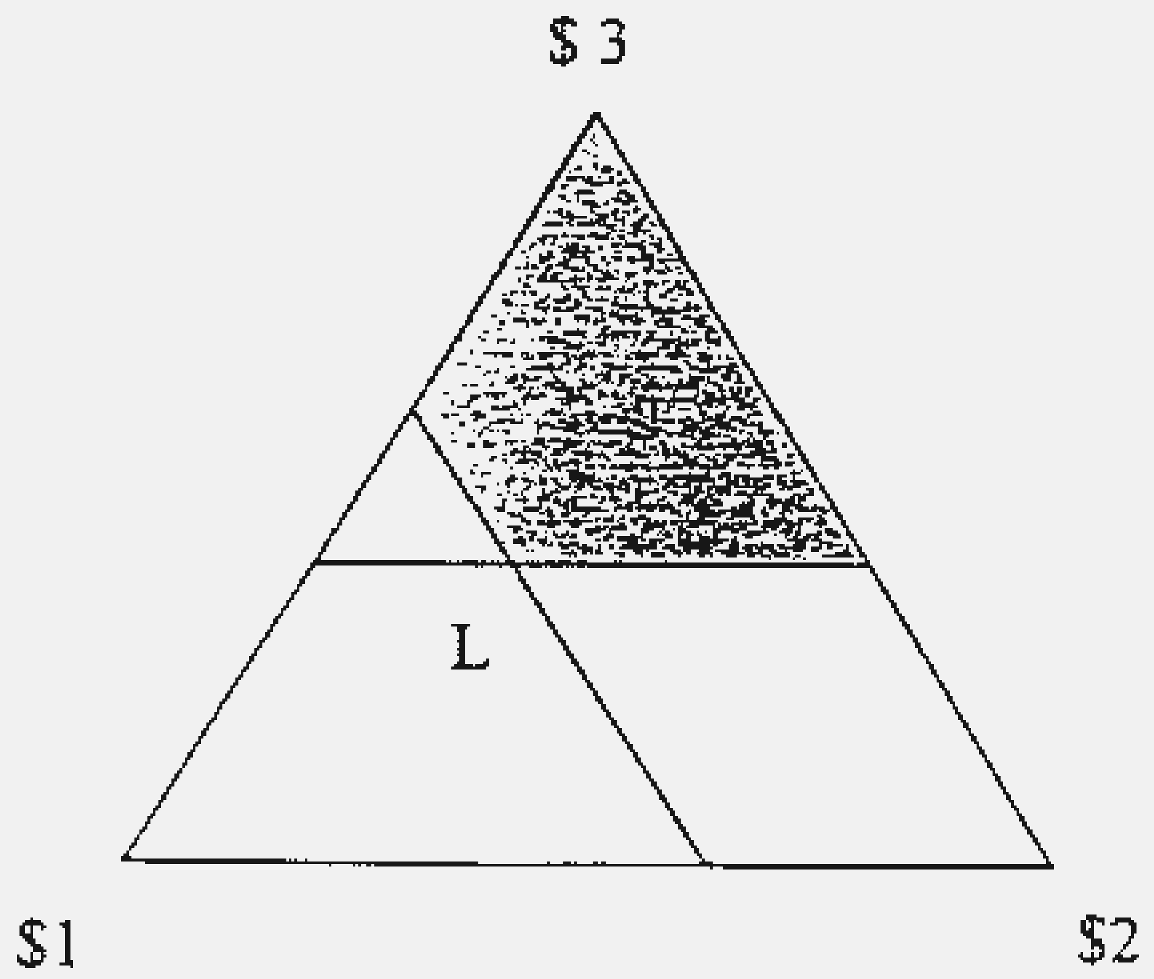

Figure 6.D.1(b)

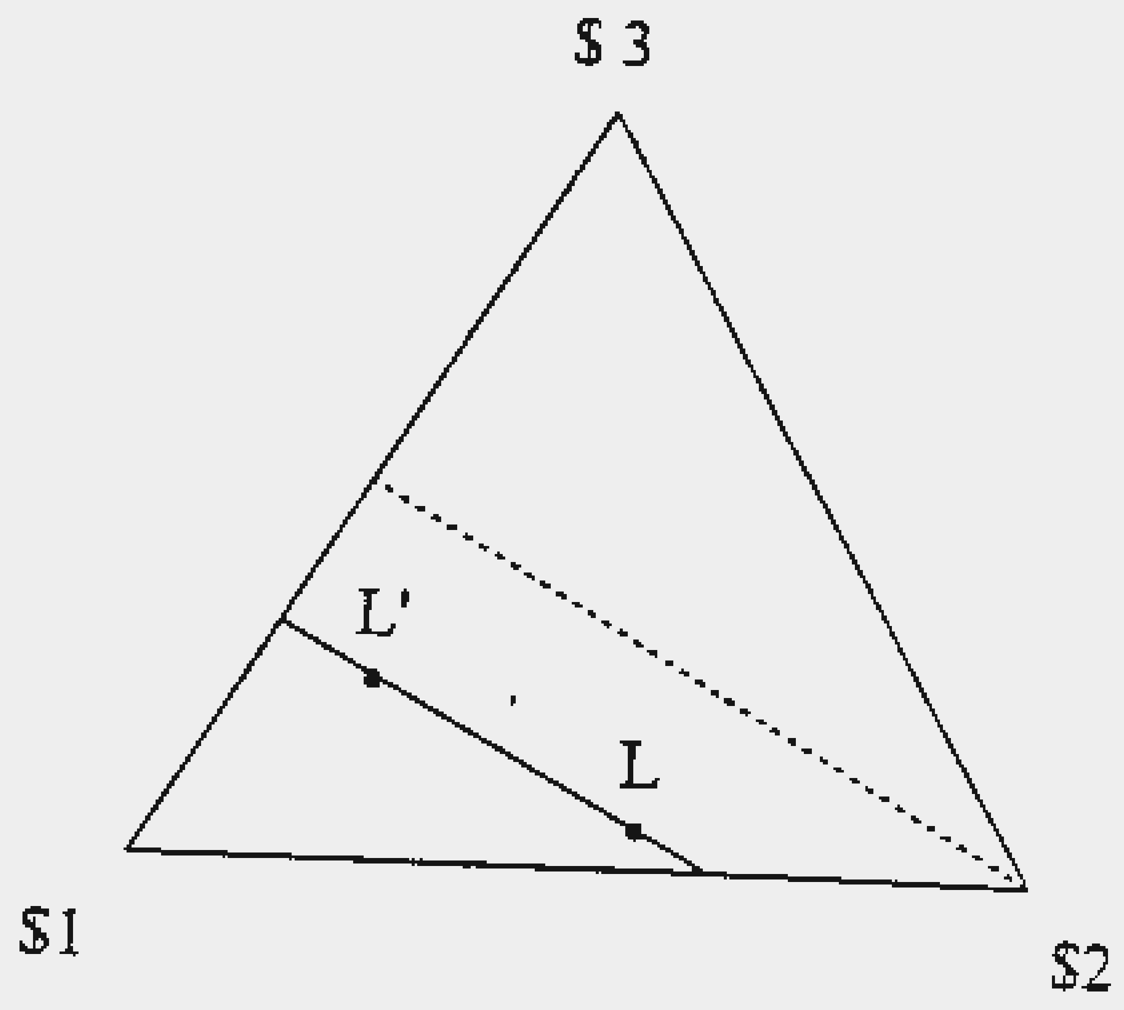

Figure 6.D.4(a)

(b) If the decision-maker exhibits risk aversion, then he prefers getting \$2 with probability one to the lottery yielding \$1 with probability 1/2 and \$3 with probability 1/2. Hence the indifference lines are steeper than the segment connecting the \$2-vertex and the middle point of the (\$1,\$3)-face. Hence, when L and L' have an equal mean, L is preferred to L' if and only if L is located on the right of L'. Therefore, L second-order stochastically dominates L' if and only if L is located on the right of L', as depicted in the figure below:

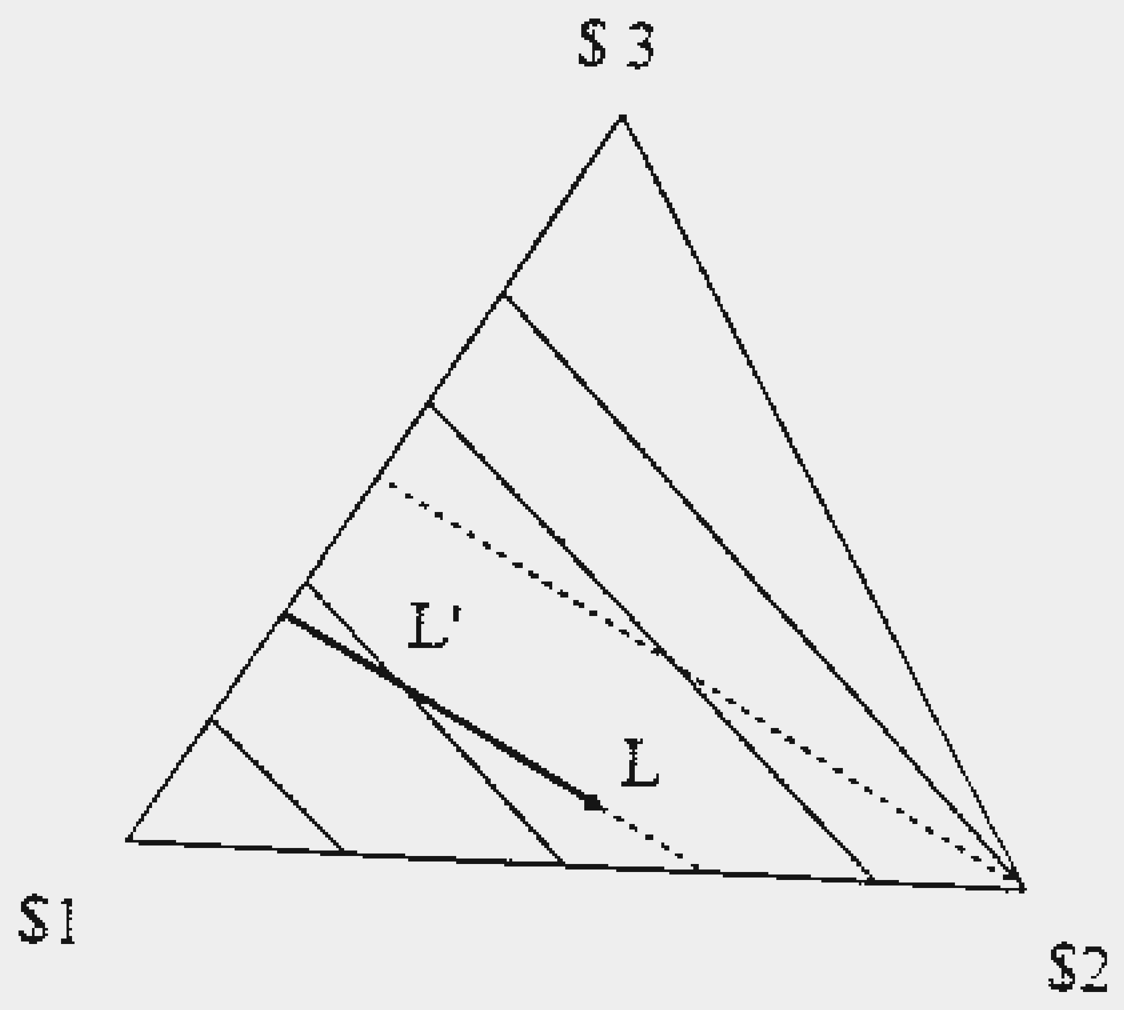

Figure 6.D.4(b)

(c) The distribution of L' is a mean preserving spread of that of L if and only if they are both on a segment that is parallel to the segment connecting the \$2-vertex and the middle point of the (\$1,\$3)-face, and L' is closer to the (\$1,\$3)-face than L. This is depicted below:

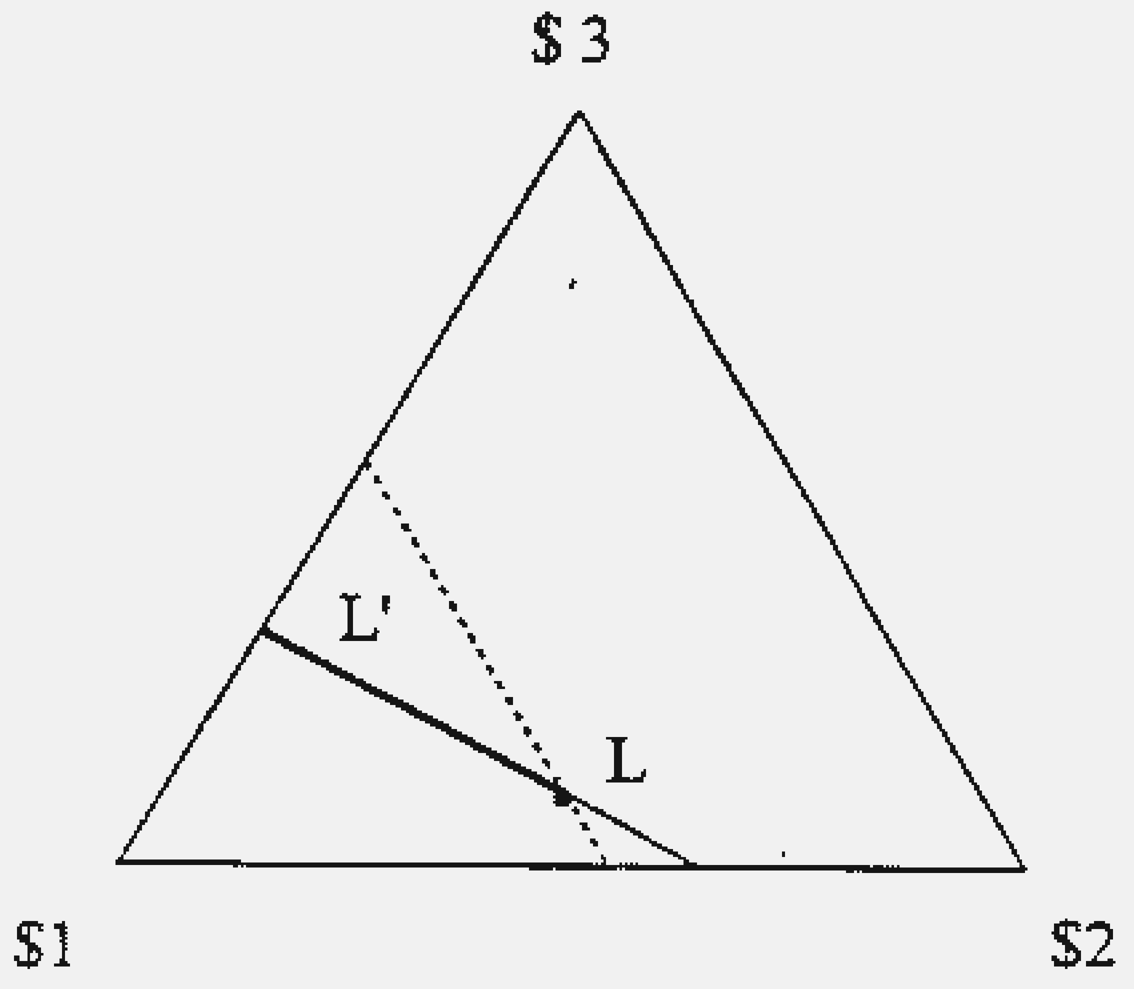

Figure 6.D.4(c)

(d) Inequality (6.D.1) holds if and only if $p_1' \geq p_1$ and $p_1' + (p_1' + p_2') \geq p_1 + (p_1 + p_2)$. But, since L and L' are assumed to have an equal mean, $p_1' - p_1 = p_3' - p_3$ and hence these two inequalities are equivalent to $p_1' \geq p_1$ alone. Thus, (6.D.1) holds if and only if L is located in the right of L', as depicted below:

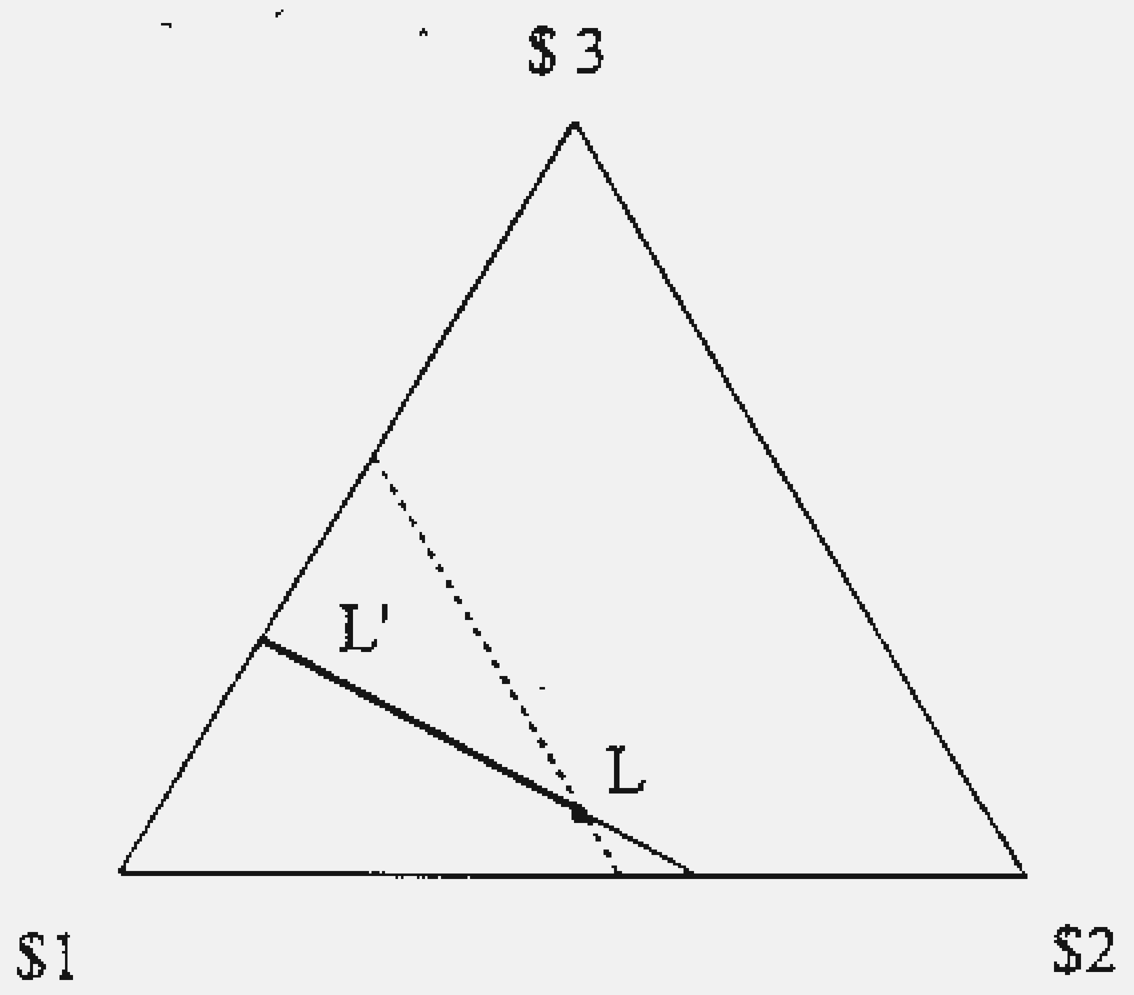

Figure 6.D.4(d)

6.E.1 Denote by $R(x,x')$ the expected regret associated with lottery x relative to x', and similarly for the other lotteries. A direct calculation yields:

$$R(x,x') = 2/3, \ R(x',x) = \sqrt{3}/3,$$

$$R(x',x'') = (\sqrt{2} + 1)/3, \ R(x'',x') = \sqrt{5}/3,$$

$$R(x'',x) = (\sqrt{2} + 1)/3, \ R(x,x'') = \sqrt{2}/3.$$

Thus, x' is preferred to x, x" is preferred to x', but x is preferred to x".

6.E.2 (a) Denote the probability of state s by π_s and the expected utility from the contingent commodity vector (x_1,x_2) by $U(x_1,x_2)$, then $U(x_1,x_2) = \pi_1 u(x_1) + \pi_2(1 - \pi)u(x_2)$. Since $u(\cdot)$ is concave by the assumption of risk aversion, $U(\cdot)$ is also concave. Thus the preference ordering on (x_1,x_2) is convex.

(b) According to Exercise 6.C.5(a), the concavity of $U(\cdot)$ implies the risk aversion for the lotteries on (x_1,x_2).

(c) By the additive separability of $U(\cdot)$ and Exercise 3.G.4(c), both x_1 and x_2 are normal goods.

6.E.3 Since $g^*(s) = 1 + \alpha(g(s) - 1)$ for every s, we have

$$g^*(s) > g(s) \text{ if } g(s) < 1;$$

$$g^*(s) = g(s) \text{ if } g(s) = 1;$$

$$g^*(s) < g(s) \text{ if } g(s) > 1.$$

Thus $G^*(x) \le G(x)$ for every $x < 1$ and $G^*(x) \ge G(x)$ for every $x > 1$. Since $G(\cdot)$ and $G^*(\cdot)$ are continuous from the right, we have $G^*(1) \ge G(1)$. Hence property (6.D.2) holds and thus $G^*(\cdot)$ second-order stochastically dominates $G(\cdot)$ weakly. (If $g(s) \neq 1$ for some s, then $G^*(x) < G(x)$ for some $x < 1$ and

$G^*(x) > G(x)$ for some $x > 1$. Hence, in this case, $G^*(\cdot)$ second-order stochastically dominates $G(\cdot)$ strictly.)

6.F.1 We shall first prove the uniqueness of the utility function on money up to origin and scale. Suppose that two utility function $u(\cdot)$ and $\hat{u}(\cdot)$ satisfy the condition of the theorem. Since the state preferences $\succsim_s$ are represented by both $\int(\pi_s u(x_s) + \beta_s)dF_s(x_s)$ and $\int(\hat{\pi}_s \hat{u}(x_s) + \hat{\beta}_s)dF_s(x_s)$, by applying Proposition 6.B.2 to the set of all the lotteries in some state s, we know that $\pi_s u(\cdot) + \beta_s$ and $\hat{\pi}_s \hat{u}(\cdot) + \hat{\beta}_s$ are the same up to origin and scale. Hence so are $u(\cdot)$ and $\hat{u}(\cdot)$.

It remains to verify the uniqueness of subjective probability. Suppose that both $\sum_s \pi_s(\int u(x_s)dF_s(x_s))$ and $\sum_s \hat{\pi}_s(\int \hat{u}(x_s)dF_s(x_s))$ represents the same preference relation on $\mathcal{L}$. Now that we have shown that $u(\cdot)$ and $\hat{u}(\cdot)$ are the same up to origin and scale, without loss of generality, we can assume that $u(\cdot) = \hat{u}(\cdot)$. We can normalize $u(\cdot)$ so that $u(0) = 0$ and $u(1) = 1$. Note here that if a distribution function $F_s(\cdot)$ puts probability p_s on 1 and probability $1 - p_s$ on 0, then the expected utility is p_s. Thus, by choosing p_s suitably for each s, any point in $[0,1]^S$ can be represented in the form

$$(\int u(x_1)dF_1(x_1),\ldots,\int u(x_S)dF_S(x_S)).$$

Hence, if $(\pi_1,\ldots,\pi_S) \neq (\hat{\pi}_1,\ldots,\hat{\pi}_S)$, then there would exist $(F_1,\ldots,F_S) \in \mathcal{L}$ and $(F_1',\ldots,F_S') \in \mathcal{L}$ such that

$$\sum_s \pi_s(\int u_s(x_s)dF_s(x_s)) > \sum_s \pi_s(\int u_s(x_s)dF_s'(x_s)),$$

$$\sum_s \hat{\pi}_s(\int u_s(x_s)dF_s(x_s)) < \sum_s \hat{\pi}_s(\int u_s(x_s)dF_s'(x_s)).$$

This contradicts the assumption that they represent the same preference. Thus $(\pi_1,\ldots,\pi_S) = (\hat{\pi}_1,\ldots,\hat{\pi}_S)$.

6.F.3 (a) If $P = \{\pi\}$, then $U_W(H) = \pi$ and $U_B(H) = 1 - \pi$. Hence they are determined from the expected utility $\pi u(1000) + (1 - \pi)u(0)$. Moreover, $U_W(R) > U_W(H)$ if and only if $0.49 > \pi$. But this is equivalent to $0.51 < 1 - \pi$, which is, in turn, equivalent to $U_B(R) < U_B(H)$.

(b) We have $U_W(R) > U_W(H)$ if and only if $0.49 > \text{Min } P$. We have $U_B(R) > U_B(H)$ if and only if $0.51 > \text{Min}\{1 - \pi: \pi \in P\}$, which is equivalent to $0.49 < \text{Max } P$. Hence $\text{Min } P < 0.49 < \text{Max } P$ if and only if $U_W(R) > U_W(H)$ and $U_B(R) > U_B(H)$.