Lecture One: New Keynesian DSGE models

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Dynamic Stochastic General Equilibrium (DSGE) models

- Partial equilibrium (PE): an individual market with a downward sloping demand curve and an upward sloping supply curve.
- Walrasian general equilibrium (GE): prices adjust to equate supply and demand in every market simultaneously.
- Dynamic: current equilibrium affected by past values and expected values
- Stochastic: economic shocks are uncertain, not deterministic.

What drives economic fluctuations?

- Real Business Cycle theory
  - The main source is technological changes (supply-side shocks)
  - Demand shocks take (little) effects through labor supply.
- Old Keynesian theory
  - Excess supply in labor market $\rightarrow$ Labor market does not clear.
  - Demand shocks take effects through labor demand.
- New Keyensian theory:
  - Evolve from old Keynesian theory but ensure all market clears.
  - Emphasize the role of nominal rigidities.

Overview of This Lecture

- Simple model with only two sectors: households and firms
- Monetary policy implications
- Extended model with physical capital and habit persistence

This lecture note is based on paper "Christiano, Lawrence J. & Trabandt, Mathias & Walentin, Karl, 2010. "DSGE Models for Monetary Policy Analysis," Handbook of Monetary Economics."

Household I

- Households are identical and live for infinite periods.
- The representative household consumes C_t, supplies labor hours H_t to maximize the expected sum of the discounted utility flows,

$$U_t = E_t \sum_{j=0}^{\infty} \beta^j (\ln C_{t+j} - \frac{H_{t+j}^{1+\phi}}{1+\phi}) \tag{1}$$

  E_t is the expectation operator, $\beta \in (0,1)$ is the discount factor, $\phi \geq 0$ is the inverse elasticity of labor supply.

- In each period, the household faces a budget constraint,

$$\frac{B_{t-1}R_{t-1}}{P_t} + w_t H_t + T_t \geq C_t + \frac{B_t}{P_t} \tag{2}$$

  B_t denotes risk-free savings, R_t denotes the gross interest rate, w_t is the real wage rate, T_t is a lump sum transfer (firm dividends and government transfers)

Household II

- Denote λ_t as the Lagrange multiplier on the budget constraint
- Rewrite the household's objective as follows,

$$\max_{C_t, H_t, B_t} E_t \sum_{j=0}^{\infty} \beta^j (\ln C_{t+j} - \frac{H_{t+j}^{1+\phi}}{1+\phi})$$

$$+ \sum_{j=0}^{\infty} \beta^j \lambda_{t+j} (\frac{B_{t+j-1} R_{t+j-1}}{P_{t+j}} + w_{t+j} H_{t+j} + T_{t+j} - C_{t+j} - \frac{B_{t+j}}{P_{t+j}})$$

$$= \max_{C_t, H_t, B_t} E_t \sum_{j=0}^{\infty} \beta^j (\ln C_{t+j} - \frac{H_{t+j}^{1+\phi}}{1+\phi})$$

$$+ \lambda_t (\frac{B_{t-1} R_{t-1}}{P_t} + w_t H_t + T_t - C_t - \frac{B_t}{P_t})$$

$$+ \beta \lambda_{t+1} (\frac{B_{t+1-1} R_{t+1-1}}{P_{t+1}} + w_{t+1} H_{t+1} + T_{t+1} - C_{t+1} - \frac{B_{t+1}}{P_{t+1}}) + \ldots$$

Household III

- The first order condition with respect to consumption and labor hours are given by,

$$C_t : \frac{1}{C_t} - \lambda_t = 0$$
$$H_t : -H_t^{\phi} + w_t \lambda_t = 0 \Rightarrow C_t H_t^{\phi} = w_t.$$

- The first order condition with respect to savings is given by,

$$B_t : -\lambda_t \frac{1}{P_t} + \beta E_t \lambda_{t+1} \frac{R_t}{P_{t+1}} = 0 \Rightarrow \frac{1}{C_t} = \beta E_t \frac{1}{C_{t+1}} \frac{R_t}{\pi_{t+1}}$$

Note: The household's saving decision is a intertemporal problem because B_t not only appears in the budget constraint for the current period but also appears in the budget constraint for the next period.

Firm

- **Wholesale goods producers** produce homogeneous wholesale goods using labor and intermediate input (e.g. upstream firms, production department).
- **Retail goods producers** produce differentiated retail goods using homogeneous wholesale goods and make pricing decisions (e.g. downstream firms, market and sales department)
- **Aggregate goods producers** produce homogeneous aggregate goods using retail goods as inputs (e.g. supermarkets, consumers).

Firm: Aggregate goods producer

- Assume that aggregate goods are produced using differentiated retail goods with a Dixit-Stigliz production function,

$$Y_t = \left(\int_0^1 Y_{it}^{\frac{1}{\lambda_f}} di \right)^{\lambda_f} \tag{3}$$

  where $\lambda_f > 1$ is a parameter associated with the elasticity of substitution among retail goods.
  Note: If $\lambda_f = 1$, then $Y_t = \int_0^1 Y_{it} di$, retail goods are homogeneous and perfectly competitive with each other.

- The representative aggregate goods producer maximizes the profit,

$$\max P_t Y_t - \int_0^1 P_{it} Y_{it} di \tag{4}$$

  where P_t denotes the price for aggregate goods, and P_{it} denotes the price for retail goods i.

Firm: Aggregate goods producer

- The first order condition for Y_{it} gives,

$$Y_{it} = Y_t \left(\frac{P_{it}}{P_t} \right)^{-\frac{\lambda_f}{\lambda_f - 1}} \tag{5}$$

  - The above equation is the demand curve faced by the retailers: the demand for Y_{it} decreases with P_{it}.
  - The elasticity of substitution among retail goods equals: $\frac{\lambda_f}{\lambda_f - 1}$. As λ_f decreases and converges to 1, the elasticity of substitution among retail goods increases and converges to $+\infty$. Retailers are more competitive and the demand for Y_{it} is more elastic to price changes.

- Substituting (5) into (3), we have,

$$P_t = \left(\int_0^1 P_{it}^{-\frac{1}{\lambda_f - 1}} di \right)^{-(\lambda_f - 1)} \tag{6}$$

Firm: Wholesale goods producer

- Wholesale goods Y_t^w are produced using labor H_t and material input I_t under the Cobb Douglas production function, which is given by,

$$Y_t^w = Z_t H_t^{\gamma} I_t^{1-\gamma} \tag{7}$$

where Z_t denotes the aggregate productivity. I_t denotes material, converted one for one from Y_t. $0 < \gamma \leq 1$ denotes the production share of labor.
Note: If $\gamma = 1$, then material inputs are not used in the production of wholesale goods.

- Assume that wholesale producers face working capital constraint: have to pay a fraction ψ of production costs before production.
  - Wholesale goods producers borrow $\psi(w_t H_t + I_t)P_t$ at the gross interest rate R_t at the beginning of the period t.
  - By the end of period t, they make the loan repayments and the rest of the production cost $(1-\psi)(w_t H_t + I_t)P_t$.

Firm: Wholesale goods producer

- The wholesale goods producer choose Y_t^w, H_t and I_t to maximize the profit,

$$\max\left[s_t Y_t^w - (w_t H_t + I_t)(1-\nu_t)(1-\psi+\psi R_t)\right]P_t \tag{8}$$

where s_t denotes the real wholesale price (relative to the price of aggregate goods), ν_t denotes the production subsidy.

- The first order conditions are given by,

$$H_t : \frac{\gamma s_t Y_t^w}{H_t} - w_t(1-\nu_t)(1-\psi+\psi R_t) = 0 \tag{9}$$

$$I_t : \frac{(1-\gamma) s_t Y_t^w}{I_t} - (1-\nu_t)(1-\psi+\psi R_t) = 0. \tag{10}$$

- The above two equations implies that,

$$\frac{w_t H_t}{I_t} = \frac{\gamma}{1-\gamma}. \tag{11}$$

Note: γ is not only the production share of labor hours, but also the cost share of labor hours.

Firm: Wholesale goods producer

- Substituting the wholesale producers' optimal production decision into the production function, s_t is given by the marginal cost,

$$s_t = (1 - \nu_t)\left(\frac{1}{1-\gamma}\right)^{1-\gamma}\left(\frac{w_t}{\gamma z_t^{\frac{1}{\gamma}}}\right)^{\gamma}(1 - \psi + \psi R_t) \qquad (12)$$

- Consider a central planner that maximizes household welfare. In this efficient equilibrium, social cost of production equals its real value:

$$\left(\frac{1}{1-\gamma}\right)^{1-\gamma}\left(\frac{w_t}{\gamma z_t^{\frac{1}{\gamma}}}\right)^{\gamma} = 1$$

  Note: Under efficient equilibrium, social cost of production equals for wholesales goods, retail goods and aggregate goods.

- In the market equilibrium, retail good producers' pricing decision implies that $\lambda_f s_t = 1$ in the steady state where $\pi_t = 1$.
- To ensure that market equilibrium achieves efficient allocation in the steady state, ν_t satisfies $\lambda_f(1-\nu)(1-\psi+\psi\bar{R}) = 1$

Firm: Retail goods producer

- Assume that retail goods producers produce differentiated retail goods using homogeneous wholesale goods as input. They then sell these retail goods to the aggregate goods producers.
- In the absence of price setting frictions, the retailer i chooses Y_{it} and P_{it} to maximize its profit,

$$\max P_{it} Y_{it} - s_t P_t Y_{it}$$

subject to the demand curve (5).

$$Y_{it} = Y_t \left(\frac{P_{it}}{P_t} \right)^{-\frac{\lambda_f}{\lambda_f - 1}}$$

- The first order condition for the price of retail goods P_{it} is given by,

$$P_{it} = \lambda_f P_t s_t.$$

- Retailers charge a premium under monopolistic competition $\lambda_f > 1$

Firm: Retail goods producer

- Price setting frictions (Calvo (1983)): Individual retailer i can set its price $P_{it} = \tilde{P}_t$ optimally with only probability $1 - \xi_p$. With probability ξ_p, the retailer must keep price constant $P_{it} = P_{it-1}$.($\tilde{P}_t$ denotes the optimal price.)
- **Note:** $\tilde{P}_t \neq \lambda_f P_t s_t$ because the retailer have to consider the case of being stuck with $\tilde{P}_t$ in the future.
- The retailer maximize the sum of the current profit as well as the expected future profits in the case of being stuck with $\tilde{P}_t$, subject to the demand curve (5),

$$\max_{\tilde{P}_t} E_t \sum_{j=0}^{\infty} (\xi_p \beta)^j \lambda_{t+j} \frac{1}{P_{t+j}} \left[\tilde{P}_t Y_{i,t+j} - P_{t+j} s_{t+j} Y_{i,t+j} \right]$$

$$s.t. \quad Y_{i,t+j} = Y_{t+j} \left(\frac{\tilde{P}_t}{P_{t+j}} \right)^{-\frac{\lambda_f}{\lambda_f - 1}}$$

where $\beta^j \lambda_{t+j}$ is the discount factor derived from the household's optimizing decisions.

Firm: Retail goods producer

- The first order condition for $\tilde{P}_t$ is given by,

$$\tilde{P}_t = \frac{E_t \sum_{j=0}^{\infty} (\beta \xi_p)^j \lambda_{t+j} \left(\frac{1}{P_{t+j}} \right)^{\frac{-1}{\lambda_f - 1}} Y_{t+j} \lambda_f s_{t+j} P_{t+j}}{E_t \sum_{j=0}^{\infty} (\beta \xi_p)^j \lambda_{t+j} \left(\frac{1}{P_{t+j}} \right)^{\frac{-1}{\lambda_f - 1}} Y_{t+j}} \tag{13}$$

  - The above equation suggests that the optimal price $\tilde{P}_t$ is weighted average of expected future marginal costs with mark-up.
  - The retailer put more weights on future marginal costs if the stucking probability ξ_p is higher, or if the future output is expected to be larger.
  - **Note:** When $\xi_p = 0$, there is no price-setting friction: $\tilde{P}_t = \lambda_f s_t P_t$.

Firm: Retail goods producer

- Let's rewrite (6) as follows,

$$P_t = \left(\int_0^1 P_{it}^{-\frac{1}{\lambda_f - 1}} di \right)^{-(\lambda_f - 1)} \tag{14}$$

$$\Rightarrow \; P_t = \left[\xi P_{t-1}^{-\frac{1}{\lambda_f - 1}} + (1 - \xi) \tilde{P}_t^{-\frac{1}{\lambda_f - 1}} \right]^{-(\lambda_f - 1)} \tag{15}$$

$$\Rightarrow \; \tilde{p}_t = \frac{\tilde{P}_t}{P_t} = \left[\frac{1 - \xi_p \pi_t^{\frac{1}{\lambda_f - 1}}}{1 - \xi_p} \right]^{-(\lambda_f - 1)} \tag{16}$$

The economic intuition here is straightforward.

- When you observe inflation $\pi_t > 1$, then $\tilde{p}_t > 1$, suggesting that firms that could optimally set price will set a higher price relative to firms that must keep price constant, leading to resource mis-allocation.

Quantify the mis-allocation

- How much of aggregate goods Y_t is produced given the quantity of wholesale output Y_t^w?

$$Y_t^w = \int_0^1 Y_{it} di = Y_t \int_0^1 \left(\frac{P_{it}}{P_t}\right)^{-\frac{\lambda_f}{\lambda_f - 1}} di = Y_t P_t^{\frac{\lambda_f}{\lambda_f - 1}} \int_0^1 (P_{it})^{-\frac{\lambda_f}{\lambda_f - 1}} di$$

- Yun (1996) construct a measure of price dispersion:

$$P_t^* = \left[\int_0^1 (P_{it})^{-\frac{\lambda_f}{\lambda_f - 1}} di\right]^{-\frac{\lambda_f - 1}{\lambda_f}} \Rightarrow Y_t = \left(\frac{P_t^*}{P_t}\right)^{\frac{\lambda_f}{\lambda_f - 1}} Y_t^w = p_t^* Y_t^w.$$

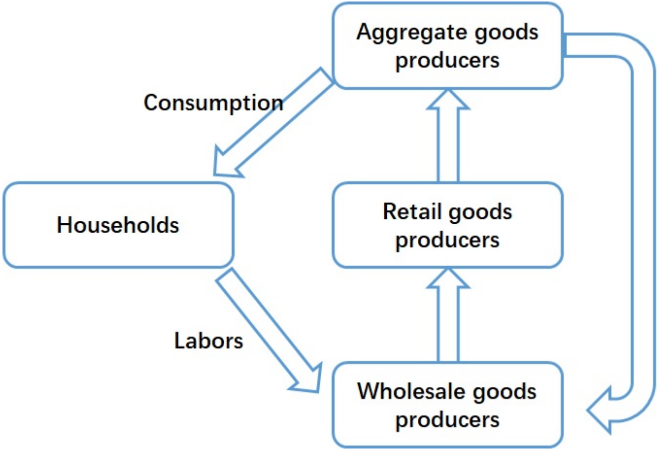
- Using (16), $p_t^* \equiv \left(\frac{P_t^*}{P_t}\right)^{\frac{\lambda_f}{\lambda_f - 1}}$ follows,

$$p_t^* = \left[(1 - \xi_p)\left(\frac{1 - \xi_p \pi_t^{\frac{1}{\lambda_f - 1}}}{1 - \xi_p}\right)^{\lambda_f} + \xi_p \frac{\pi_t^{\frac{\lambda_f}{\lambda_f - 1}}}{p_{t-1}^*}\right]^{-1} \leq 1. \qquad (17)$$

- Only when $p_{t-1}^* = 1$ and $\pi_t = 1$, efficient allocation: $p_t^* = 1$.

Market clearing condition

- Aggregate goods market clears: $C_t + I_t = Y_t$.



What is log linearization?

- The equilibrium of the model is defined as a set of variables x_t that satisfies,

$$E_t f(x_{t-1}, x_t, x_{t+1}) = 0$$

where $f(\cdot)$ are the equilibrium conditions (including agents' optimal conditions and market clearing conditions) of the model.

- The steady-state equilibrium of the model is an equilibrium that satisfies $x_t = x_{t-1} = x_{t+1} = \bar{x}$ and,

$$f(\bar{x}, \bar{x}, \bar{x}) = 0$$

- For each variable x_t, we denote $\hat{x}_t$ as the log deviation of x_t from its steady state level $\bar{x}$: $\hat{x}_t \equiv \ln x_t - \ln \bar{x}$.
- First order Taylor expansion of $\hat{x}_t$ with respect to x_t around $\bar{x}$ gives,

$$\hat{x}_t = \ln x_t - \ln \bar{x} \approx (\ln \bar{x} - \ln \bar{x}) + \frac{1}{\bar{x}}(x_t - \bar{x}) = \frac{x_t - \bar{x}}{\bar{x}}$$

What is log linearization?

- Similarly, first order Taylor expansion of $f(x_{t-1}, x_t, x_{t+1})$ with respect to x_t around $\bar{x}$ gives,

$$f(x_{t-1}, x_t, x_{t+1}) = f(\bar{x}, \bar{x}, \bar{x}) + f_1(\bar{x}, \bar{x}, \bar{x})(x_{t-1} - \bar{x}) \\ + f_2(\bar{x}, \bar{x}, \bar{x})(x_t - \bar{x}) + f_3(\bar{x}, \bar{x}, \bar{x})(x_{t+1} - \bar{x}).$$

- Recall that $f(\bar{x}, \bar{x}, \bar{x}) = 0$ and $\hat{x}_t = \frac{x_t - \bar{x}}{\bar{x}}$
- So log linearizing the model $f(x_{t-1}, x_t, x_{t+1}) = 0$ gives,

$$f_1(\bar{x}, \bar{x}, \bar{x})\bar{x}\hat{x}_{t-1} + f_2(\bar{x}, \bar{x}, \bar{x})\bar{x}\hat{x}_t + f_3(\bar{x}, \bar{x}, \bar{x})\bar{x}\hat{x}_{t+1} = 0.$$

Why log linearization?

- Linearization is nice because we know how to work with linear difference equations.
- Putting things in percentage terms (that is the log part) is nice because it provides natural interpretations of the units (i.e. everything is in percentage terms).
- Linearization can lead to large bias when the shock is really large. Even when the shock is small, higher order terms could have important implications on the long term means and linearization makes them go away.

Why log linearization?

- For example, assume that

$$x_t = \epsilon_t + \epsilon_t^2$$

where ϵ_t is a small shock. $E(\epsilon_t) = \mu$, $E(\epsilon_t^2) = \sigma^2$, and $\epsilon_t^3 \approx 0$.

  - Under the second order approximation (which is also the true case),

$$E(x_t) = E(\epsilon_t + \epsilon_t^2) = \mu + \sigma^2. \quad E(x_t^2) = E(\epsilon_t^2 + 2\epsilon_t^3 + \epsilon_t^4) \approx E(\epsilon_t^2) = \sigma^2.$$

  - Under the first order approximation, $x_t = \epsilon_t$. Then,

$$E(x_t) = E(\epsilon_t) = \mu. \quad E(x_t^2) = E(\epsilon_t^2) = \sigma^2.$$

  - The first order approximation of x_t offers unbiased estimate for its volatility (which is important for the short run), but biased estimate for its expected mean, which is usually important if you are interested in long run implications.

Phillips curve (the aggregate supply equation)

- Using the optimal price-setting condition (13), we have

$$\left(1-\beta\xi_p\right)^{-1}\hat{\tilde{p}}_t = \hat{s}_t + \left(1-\beta\xi_p\right)^{-1}\beta\xi_p E_t\left(\hat{\tilde{p}}_{t+1} + \hat{\pi}_{t+1}\right)$$

Denote $\tilde{p}_t \equiv \frac{\tilde{P}_t}{P_t}$ as the real optimal price, so that, $\hat{\tilde{p}}_t \equiv \hat{\tilde{P}}_t - \hat{P}_t$. And $\tilde{P}_t$, s_t^n and P_t denotes the nominal optimal retail price, the nominal wholesale price $s_t P_t$, and the nominal price for aggregate goods.

- Recall that the real optimal price $\tilde{p}_t$ is related to the inflation rate π_t as follows,

$$\tilde{p}_t = \left[\frac{1-\xi_p \pi_t^{\frac{1}{\lambda_f - 1}}}{1-\xi_p}\right]^{-(\lambda_f - 1)}$$

$$\Rightarrow \hat{\tilde{p}}_t = \frac{\xi_p}{1-\xi_p}\hat{\pi}_t$$

Phillips curve (the aggregate supply equation)

- Then after some algebra,

$$\frac{\xi_p}{(1-\beta\xi_p)(1-\xi_p)}\hat{\pi}_t = \hat{s}_t + E_t \frac{\beta\xi_p}{(1-\beta\xi_p)(1-\xi_p)}\hat{\pi_{t+1}}$$

$$\Rightarrow \hat{\pi}_t = \frac{(1-\beta\xi_p)(1-\xi_p)}{\xi_p}\hat{s}_t + \beta E_t \hat{\pi_{t+1}}$$

The above equations imply that, current inflation π_t is determined by the optimal price $\tilde{p}_t$, which is in turn determined by the marginal production cost s_t and future expected inflation $E_t\hat{\pi_{t+1}}$.

Phillips curve (the aggregate supply equation)

- Now, what determines s_t?
  Using the wholesale firm's production conditions and the household's labor supply, we have

$$\hat{s}_t = \gamma(1+\phi)\hat{x}_t + \frac{\psi R}{1-\psi+\psi R}\hat{R}_t$$

where $\hat{x}_t = \hat{\tilde{y}}_t = \hat{\tilde{c}}_t$, where $\tilde{C}_t = \frac{C_t}{z_t^{1/\gamma}}$, $\tilde{Y}_t = \frac{Y_t}{z_t^{1/\gamma}}$.

- The output gap $\hat{x}_t$ is the percent deviation of actual output y_t from potential output $z_t^{1/\gamma}\bar{y}$ ($\bar{z}=1$).

$$\hat{\tilde{y}}_t = \frac{\tilde{y}_t - \bar{y}}{\bar{y}} = \frac{y_t - z_t^{1/\gamma}\bar{y}}{z_t^{1/\gamma}\bar{y}}$$

- The equality $\hat{\tilde{y}}_t = \hat{\tilde{c}}_t$ do not hold in more generalized models.

Phillips curve (the aggregate supply equation)

- Now, what determines s_t?
  Using the wholesale firm's production conditions and the household's labor supply, we have

$$\hat{s}_t = \gamma(1+\phi)\hat{x}_t + \frac{\psi R}{1-\psi+\psi R}\hat{R}_t$$

  where $\hat{x}_t = \hat{\tilde{y}}_t = \hat{\tilde{c}}_t$, where $\tilde{C}_t = \frac{C_t}{z_t^{1/\gamma}}$, $\tilde{Y}_t = \frac{Y_t}{z_t^{1/\gamma}}$.

- How a rise in output gap $\hat{x}_t$ raises marginal cost $\hat{s}_t$?
  Output gap $\hat{x}_t \uparrow \Rightarrow$ demand for labor $H_t \uparrow \Rightarrow$ wage $w_t \uparrow \Rightarrow s_t \uparrow$

- A rise in the labor share $\gamma \uparrow$ strengthens the transmission from wage w_t to marginal cost s_t.

- A rise in the inverse elaticity of labor supply $\phi \uparrow$ strengthens the transmission from labor demand H_t to wage w_t.

- If $\psi > 0$, R_t rises could drive up s_t.

Euler Equation (the aggregate demand equation)

$$\frac{1}{C_t} = \beta E_t \frac{1}{C_{t+1}} \frac{R_t}{\pi_{t+1}}$$
$$\Rightarrow \hat{x}_t = E_t \left[\hat{x_{t+1}} - (\hat{R}_t - \hat{\pi_{t+1}} - \hat{R_t}^*) \right].$$

- $R_t^* \equiv E_t \frac{1}{\beta} (\frac{z_{t+1}}{z_t})^{1/\gamma}$ is the real risk-free interest rate in the absence of nominal rigidities ($\xi_p = 0$).
- The Euler equation implies that the output gap $\hat{x}_t$ increases with future expected output gap $E_t \hat{x_{t+1}}$ but decreases with interest rate gap: $\hat{R}_t - \hat{\pi_{t+1}} - \hat{R_t}^*$.

Monetary policy

- The market clearing condition of the bond (loan) market is given by,

$$B_t = B_{gt} + \psi P_t(w_t H_t + I_t).$$

where B_{gt} denotes the supply of government bonds in the market. The left hand of the equation is the supply of funds from the household. The right hand of the equation is the demand of funds from the government and the firms.

- We assume the monetary authority sets the interest rate by adjusting the supply of government bonds B_{gt} such that

$$\hat{R}_t = r_\pi E_t \hat{\pi_{t+1}} + r_x \hat{x}_t.$$

Taylor principle

- We now examine if the implementation of a Taylor rule can achieve low and stable inflation. The evolution of the economy is determined by the following three equations.

$$\hat{\pi}_t = \kappa_p \left[\gamma(1+\phi)\hat{x}_t + \frac{\psi R}{1-\psi+\psi R}\hat{R}_t\right] + \beta E_t \pi_{t+1} \quad (18)$$

$$\hat{x}_t = E_t\left[\hat{x_{t+1}} - (\hat{R}_t - \hat{\pi_{t+1}} - \hat{R}_t^*)\right] \quad (19)$$

$$\hat{R}_t = r_\pi E_t \hat{\pi_{t+1}} + r_x \hat{x}_t \quad (20)$$

where $\kappa_p \equiv \frac{(1-\beta\xi_p)(1-\xi_p)}{\xi_p}$, ϕ is the inverse elasticity of labor supply, ψ is the fraction of production cost to be paid in advance, and γ is the share of labor in the production function.

Taylor principle: Benchmark case

- We first consider a benchmark case with no material inputs and no working capital ($\gamma = 1$ *and* $\psi = 0$).
- We focus on the deterministic case (i.e. no stochastic shocks).
  - Phillips curve: inflation is a function of present and future output gaps,

$$
\begin{aligned}
\hat{\pi}_t &= \kappa_p (1+\phi)\hat{x}_t + \beta E_t \hat{\pi_{t+1}} \\
&= \kappa_p (1+\phi)\hat{x}_t + \beta\kappa_p (1+\phi)\hat{x_{t+1}} + \beta^2\kappa_p (1+\phi)\hat{x_{t+2}} + \cdots
\end{aligned}
$$

  - Euler equation: output gap is function of long-term real interest rates,

$$
\begin{aligned}
\hat{x}_t &= E_t \hat{x_{t+1}} - \left(\hat{R}_t - \hat{\pi_{t+1}}\right) \\
&= -\left(\hat{R}_t - \hat{\pi_{t+1}}\right) - \left(\hat{R_{t+1}} - \hat{\pi_{t+2}}\right) - \left(\hat{R_{t+2}} - \hat{\pi_{t+3}}\right) - \cdots
\end{aligned}
$$

Taylor principle: Benchmark case

For simplicity, we assume that, $r_x = 0$ and examine how the value of r_π affects the stability of the equilibrium.

- If $r_\pi > 1$,

$$\text{A rise in } E_t(\pi_{t+1}) \Rightarrow \hat{R}_t - E_t(\hat{\pi_{t+1}}) \uparrow \Rightarrow \hat{x}_t \downarrow \Rightarrow \hat{\pi}_t \downarrow.$$

- If $r_\pi < 1$,

$$\text{A rise in } E_t(\pi_{t+1}) \text{ is self-fulfilling} \Rightarrow \hat{R}_t - E_t(\hat{\pi_{t+1}}) \downarrow \Rightarrow \hat{x}_t \uparrow \Rightarrow \hat{\pi}_t \uparrow.$$

- How to prove if $\hat{\pi}_t = 0$ is a stable equilibrium?

Taylor principle: Benchmark case

Maintain the assumption that $r_x = 0$ and then reduce the model to a single second-order difference equation in inflation π_t,

$$\hat{\pi}_t + \left[\kappa_p \gamma (1+\phi)(r_\pi - 1) - (\kappa_p \alpha_\psi r_\pi + \beta) - 1\right] \hat{\pi_{t+1}} + (\kappa_p \alpha_\psi r_\pi + \beta) \hat{\pi_{t+2}} = 0$$

- The general set of solutions to this difference equation is given by: $\hat{\pi}_t = \alpha_0 \lambda_1^t + \alpha_1 \lambda_2^t$, where λ_1,λ_2 are roots of

  $$1 + \left[\kappa_p \gamma (1+\phi)(r_\pi - 1) - (\kappa_p \alpha_\psi r_\pi + \beta) - 1\right] \lambda + (\kappa_p \alpha_\psi r_\pi + \beta) \lambda^2 = 0$$

- Only when $|\lambda_1| > 1$, $|\lambda_2| > 1$, then $\hat{\pi}_t = \alpha_0 = \alpha_1 = 0$ is the unique equilibrium. Otherwise, there are many solutions to the equilibrium conditions. In this case, inflation is self-fulfilling.

Taylor principle: Alternative case

We now consider the case with working capital ($\psi > 0$),

$$\text{A rise in } E_t\left(\pi_{t+1}\right) \Rightarrow \hat{R}_t \uparrow \Rightarrow \hat{\pi}_t \uparrow$$

Whether the implementing the Taylor rule $r_\pi > 1$ still ensure a unique equilibrium becomes ambiguous in the presence of working capital ? ($\beta = 0.99, \xi_p = 0.75, r_\pi = 1.5$, see Figure 1)

- Inflation is more likely to be self-fulfilling when the fraction of working capital ψ is higher.
- The equilibrium becomes more likely to be indeterminent if the labor share is lower ($\gamma \downarrow$) or the inverse elasticity of labor supply is lower ($\phi \downarrow$). Recall how $r_\pi > 1$ ensures a stable inflation,

$$\text{A rise in } E_t\left(\pi_{t+1}\right) \Rightarrow \hat{R}_t - E_t\left(\hat{\pi_{t+1}}\right) \uparrow \Rightarrow \hat{x}_t \downarrow \Rightarrow \hat{\pi}_t \downarrow$$

The last step is weakened if γ or ϕ becomes smaller.

Taylor principle: Alternative case

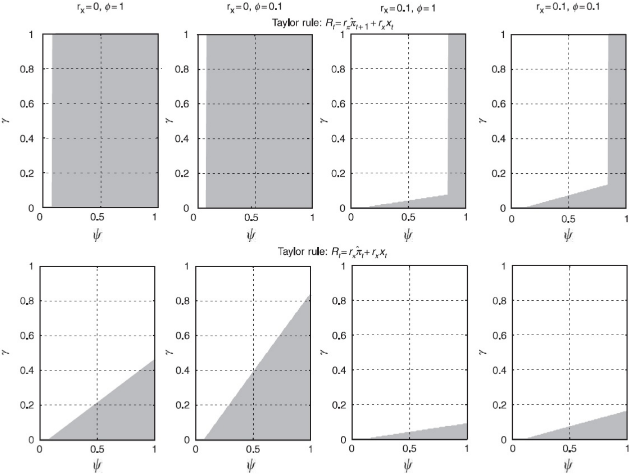
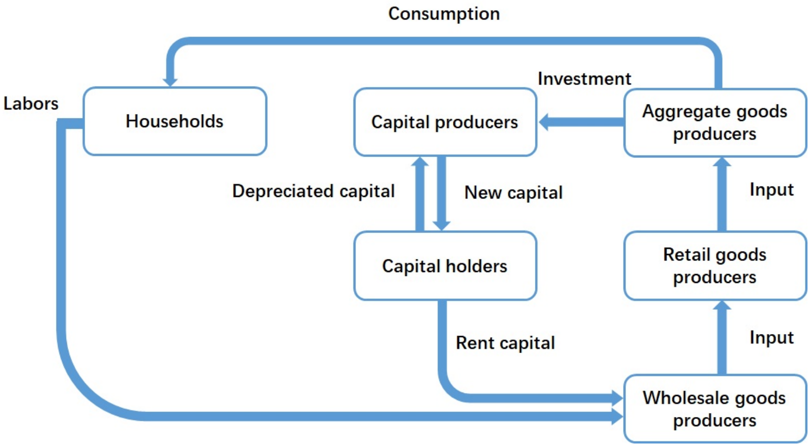


Figure 1 Indeterminacy regions for model with working capital channel and materials inputs. Note: Gray area is region of indeterminacy and white area is region of determinacy.

Introducing capital

- Capital stock can be used for future production, and how much capital to hold is an intertemporal decision.
- Three types of agents:
  - **Capital producers**: produce capital and sell it to the capital holders.
  - **Capital holders**: purchase capital from capital producers, hold it and rent it in the next period.
  - **Capital renters**: rent capital and hire labor to make the production.

A New Keyensian DSGE model with physical capital



Capital renters (wholesale goods producers)

- In this model, wholesale goods producers rent capital and hire labor to make the production. Their production function becomes,

$$Y_t^w = Z_t K_t^\alpha H_t^{1-\alpha} \tag{21}$$

where K_t represents the amount of capital stock rented by wholesale goods producers.$0 < \alpha < 1$ is the production share of capital.

- Wholesale goods producers choose capital K_t and labor hours H_t to maximize their profits,

$$\max s_t Y_t^w - w_t H_t - r_t^k K_t$$

where r_t^k denotes the real capital rental rate.

- The optimization condition for rented capital K_t is given by,

$$K_t : \alpha \times \frac{s_t Y_t^w}{K_t} - r_t^k = 0$$

The above equation implies that producers' demand for capital falls if the capital rental rate r_t^k becomes higher.

Capital holders I

- Denote $\bar{K}_t$ as their capital holdings at the end of period t. Then the amount of capital that capital holders purchase is given by,

$$\bar{K}_t - (1-\delta)\bar{K}_{t-1}$$

  where δ denotes the capital depreciation rate and $\bar{K}_{t-1}$ denotes the capital holdings by the end of period $t-1$.

- In period t, capital holders rent a fraction u_t of their existing capital holdings $\bar{K}_{t-1}$ to the wholesale goods producers at the real rental rate r_t^k, where u_t denotes the capital utilization rate. The capital holders also incurs a capital utilization cost $a(u_t)$. The real rental revenues is then given by,

$$r_t^k u_t \bar{K}_{t-1} - a(u_t)\bar{K}_{t-1}$$

- The capital holders maximize their discounted sum of capital flows,

$$max \sum_{t=0}^{\infty} \beta^t \lambda_t \left\{ \left[r_t^k u_t \bar{K}_{t-1} - a(u_t)\bar{K}_{t-1} \right] - q_t^k \left[\bar{K}_t - (1-\delta)\bar{K}_{t-1} \right] \right\}$$

  where q_t^k denotes the real capital price.

Capital holders II

- The capital holders' optimization condition for $\bar{K}_t$ is given by,

$$\bar{K}_t : -\lambda_t q_t^k + E_t \beta \lambda_{t+1} \{[r_{t+1}^k u_{t+1} - a(u_{t+1})] + (1-\delta) q_{t+1}^k\} = 0$$

$\Rightarrow$

$$E_t \frac{[r_{t+1}^k u_{t+1} - a(u_{t+1})] + (1-\delta) q_{t+1}^k}{q_t^k} = E_t \frac{\lambda_t}{\beta \lambda_{t+1}} = E_t \frac{R_t}{\pi_{t+1}}$$

where the left hand side of the equation is the real return to capital of the capital holders, and the right hand side of the equation is the real deposit interest rate.

Capital holders III

- The capital holders' optimization condition for u_t is given by,

$$u_t : r_t^k = a'\left(u_t\right)$$

About capital utilization cost $a\left(u_t\right)$:
1) $a(1) = 0$, $u = 1$ in steady state.
2) $a'(u) > 0$, $a''(u) > 0$.
3) Why introduce $a(u)$?
    - Make supply of capital services elastic.
    - Explain the small response of inflation to monetary policy shock in the data: In any model, prices are heavily influenced by costs. Costs in turn are influenced by the supply elasticity of production factors. If factors can respond rapidly to changes in costs (i.e. supply of factors is elastic), then inflation will not change much after a monetary policy shock.
    - Monetary policy shock ϵ_t: a residual in the Taylor rule:

$$\hat{R}_t = r_\pi E_t \hat{\pi_{t+1}} + r_x \hat{x}_t + \epsilon_t$$

Capital producers I

- Capital producers produce capital and sell it to the capital holders. Assume that capital producers use I_t units of final goods to produce $\left[1 - S\left(\frac{I_t}{I_{t-1}}\right)\right] I_t$ units of physical capital, where $S\left(\frac{I_t}{I_{t-1}}\right) \geqslant 0$ is capital adjustment cost.
- Some commonly used assumption on $S(\frac{I_t}{I_{t-1}})$:
  1) $S(1) = S'(1) = 0$.
  2) $S''(\cdot) > 0$ is constant.
  e.g. $S\left(\frac{I_t}{I_{t-1}}\right) = \frac{\Omega_k}{2}\left(\frac{I_t}{I_{t-1}} - 1\right)^2$
- Capital producers also maximize their discounted sum of profits,

$$max \sum_{t=0}^{\infty} \beta^t \lambda_t \left\{ \left[1 - S\left(\frac{I_t}{I_{t-1}}\right)\right] I_t q_t^k - I_t \right\}$$

Capital producers II

- The optimal condition for capital investment is then given by,

$$I_t : \left\{ \left[1 - S\left(\frac{I_t}{I_{t-1}}\right) - S'\left(\frac{I_t}{I_{t-1}}\right)\left(\frac{I_t}{I_{t-1}}\right) \right] q_t^k - 1 \right\} \lambda_t + \beta \lambda_{t+1} S'\left(\frac{I_{t+1}}{I_t}\right)\left(\frac{I_{t+1}}{I_t}\right)^2 q_{t+1}^k = 0 \tag{22}$$

Note: 1) Capital investment decision and capital holding decision are different.
2) If $S\left(\frac{I_t}{I_{t-1}}\right) = 0 \Rightarrow q_t^k = 1$ real capital price is constant.
3) Log linearizing the above equation gives,

$$\hat{q}_t^k = S''(\hat{I}_t - \hat{I}_{t-1}) - S''\beta(\hat{I}_{t+1} - \hat{I}_t). \tag{23}$$

- Consider an increase in I_t. How will it affect the capital price q_t^k?
a) $\frac{I_t}{I_{t-1}} \uparrow \Rightarrow$ current capital adjustment cost $\uparrow \Rightarrow$ current desire to invest $I_t \downarrow \Rightarrow q_t^k \uparrow$.
b) $E_t \frac{I_{t+1}}{I_t} \downarrow \Rightarrow$ future capital adjustment cost $\downarrow \Rightarrow$ current desire to invest $I_t \downarrow \Rightarrow q_t^k \uparrow$.

Capital producers III

- Capital adjustment cost explains the empirical evidence that, after an expansionary monetary policy shock (i.e. unexpected cut in R_t), a) real interest rate falls. Optimal capital holding decision implies,

$$\frac{R_t}{\pi_{t+1}} \downarrow \Rightarrow \frac{r^k_{t+1} + (1-\delta) q^k_{t+1}}{q^k_t} \downarrow$$

  Without the capital adjustment cost ($S(\cdot) = 0$), then the real capital price is constant $q^k_t = 1$. Since $1 - \delta$ is about 0.9 in annual terms, and R/π is around 1.03 per annum, any reasonable drop in the return on capital requires a very large percentage drop in the real capital rent r^k_{t+1}, and therefore a very unreasonably large expansion in capital and investment.
  b) investment responds in a hump-shaped pattern. Without investment adjustment cost, capital producers will want to make all the investments right away. In this case, the biggest response of investment I_t occurs in the period of the monetary policy shock. With investment adjustment cost, investment expansion in the period of shock boosts investment in subsequent periods.

Persistence in habit I

- We could also introduce persistent in habit in the household sector. Assume that the household's utility flow becomes

$$ln\left(C_t - bC_{t-1}\right) - \frac{H_t^{1+\phi}}{1+\phi}$$

where b denotes the consumption persistence parameter $0 \leq b < 1$.

- The household's problem and their optimal consumption decision becomes

$$max\ E_t \sum_{j=0}^{\infty} \beta^j \left[ln\left(C_{t+j} - bC_{t+j-1}\right) - \frac{H_{t+j}^{1+\phi}}{1+\phi} \right] 0 \leqslant b < 1$$

$$s.t. C_t + \frac{B_t}{P_t} = \frac{B_{t-1}R_{t-1}}{P_t} + w_t H_t + T_t$$

$$C_t : E_t \frac{1}{C_t - bC_{t-1}} - \frac{\beta b}{C_{t+1} - bC_t} = \lambda_t$$

Persistence in habit II

- $b > 0$ explains the empirical evidence that, after a monetary policy shock, real interest rate falls persistently and consumption responds in a hump-shaped pattern.
  Recall Euler equation

$$\beta E_t \frac{\lambda_{t+1}}{\lambda_t} \frac{R_t}{\pi_{t+1}} = 1$$

  - Persistent falls in real interest rates $\frac{R_t}{\pi_{t+1}}$ requires the Lagrange multiplier on the budget constraint λ_t to be increasing.
  - If $b = 0$, $\lambda_t = \frac{1}{C_t}$, C_t is decreasing after an immediate increase in the initial period, inconsistent with the evidence.
  - If $b = 1$, $\lambda_t = \frac{1}{\Delta C_t} - \frac{\beta}{\Delta C_{t+1}}$. ΔC_t is decreasing but C_t can be hump-shaped, consistent with the evidence.
  - Non-separability between C and H is also a solution. e.g. $U(C_t, H_t) = C_t^{\gamma} H_t^{1-\gamma}$.