Problem Set 1

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Problem 1

Consider a new model of preferences, the PI-model. The primitives of this model are two binary relations, P and I, defined on X, where P is interpreted as the "strictly better than" relation, and I is interpreted as the "indifference" relation. We impose three conditions on P and I in this model: (1) for any $x \in X$, xIx and $x\bar{P}x$; (2) for any $x, y \in X$ with $x \neq y$, exactly one of the following three is true: xPy, yPx and xIy; (3) both P and I are transitive. Based on the construction in this model, prove the following results.

(a) I is symmetric.

(b) If xPy and yIz, then xPz; If xIy and yPz, then xPz.

(c) The PI-model is equivalent to the $\succeq$-model.

Answer to problem 1

(a) I is symmetric.

Consider any $x, y \in X$ with xIy, hence by the second condition of this model, both xPy and yPx are not true. Then consider a similar argument, if both yPx and xPy are not true, we have yIx for all $x, y \in X$. Therefore, for any $x, y \in X$ with $x \neq y$, xIy implies yIx, I is symmetric.

(b) If xPy and yIz, then xPz; If xIy and yPz, then xPz.

(1) If xPy and yIz, then xPz.

Suppose xIz, as I is transitive and yIz, we have zIy, contradiction. Suppose zPx, as P is transitive and xPy, we have zPy, contradiction. Hence by the second condition of this model, we have xPz.

(2) If xIy and yPz, then xPz.

Suppose xIz, as I is transitive and xIy, we have zIy, contradiction. Suppose zPx, as P is transitive and yPz, we have yPx, contradiction. Hence by the second condition of this model, we have xPz.

(c) The PI-model is equivalent to the $\succeq$-model.

Proposition:

(1) Given the complete and transitive $\succeq$, define two new binary relations, P' and I' as follows: for any $x, y \in X$, $xP'y$ if $x \succeq y$ and $y \not\succeq x$, $xI'y$ if $x \succeq y$

and $y \succeq x$. Then P' and I' satisfy the three conditions above.
(2) Given the three conditions on P and I, define a new binary relation $\succeq'$ as followers: for any $x, y \in X$, $x \succeq' y$ if xPy or xIy. Then $\succeq'$ is completeness and transitivity.

Proof:
(1) $\succeq$-model $\rightarrow$ PI-model

Condition(1):

For any $x, y \in X$ with $x = y$, by the construction of I' and P' and the completeness of $\succeq$, $xI'x$ and $x\bar{P}'x$.

Condition(2):

If $xP'y$, then $x \succeq y$ and $y \not\succeq x$, obviously both $yP'x$ and $xI'y$ are not true. So by a similar argument, it can be shown that only one of $xP'y$, $yP'x$ and $xI'y$ is true.

Condition(3):

Consider any $x, y, z \in X$ with $xP'y$ and $yP'z$. By the definition of P' and the transitivity and the negatively transitivity of $\succeq$, we have $xP'z$. Consider any $x, y, z \in X$ with $xI'y$ and $yI'z$. By the definition of I' and the transitivity of $\succeq$, we have $xI'z$.

(2) PI-model $\rightarrow$ $\succeq$-model

Completeness:

For any $x, y \in X$, by the definition of $\succeq'$ and the second condition of PI-model, we have $x \succeq' y$ or $y \succeq' x$.

Transitivity:

Consider any $x, y, z \in X$with $x \succeq' y$ and $y \succeq' z$. By the definition $\succeq'$, we have xPyor xIy and yPz or yIz. Then by the transitivity of P and I and second result above, we have $x \succeq' z$.

Problem 2

Let C be a choice correspondence defined on the domain $\mathscr{D}$. Assume that for any $A, B \in \mathscr{D}$ with $A \cap B \neq \emptyset$, $A \cap B \in \mathscr{D}$. Show that if C satisfies Sen's properties α and β, then C satisfies the weak axiom of revealed preference.

Answer to problem 2

Assume that C satisfies Sen's properties α and β while C does not satisfy WARP. If C does not satisfy WARP, it means that if for some $A \in \mathscr{D}$ with $x, y \in A$, $x \in C(A)$ and $y \notin C(A)$, there exists $y \in C(B)$ for some $B \in \mathscr{D}$ with $x, y \in B$.

Let $\{x, y\} \subseteq A \cap B \subseteq A$, $x \in C(A)$ and $y \notin C(A)$, by Sens properties α, $x \in C(A \cap B)$. And since there exists $y \in C(B)$ and $y \in A \cap B \subseteq B$, then by Sens properties α, we have $y \in C(A \cap B)$. Thus, we have both $x, y \in C(A \cap B)$. As we also know that $A \cap B \subseteq A$ and $x \in C(A)$, then by Sens properties β, we have $y \in C(A)$, contradiction.

Hence when C satisfies Sen's properties α and β, it must satisfy WARP.

Problem 3

Let $\succeq$ be a preference relation defined on a finite set X, and $\succ$ is the asymmetric component of $\succeq$. Notice that $\succeq$ is not assumed to be rational. We say $\succ$ is *acyclic* if there does not exist a list $(x_1, x_2, \ldots, x_{n1}, x_n)$ such that $x_k \in X$ for each $k \in 1, 2, \ldots, n$, $n \geq 2$, and $x_1 \succ x_2 \succ \ldots \succ x_{n1} \succ x_n \succ x_1$. For any$A \subseteq X$, let

$$C_\succ(A) = \{x \in A : \text{there does not exist } y \in A \text{ such that } y \succ x\}.$$

Prove the following results.
(a) $C_\succ(A) \neq \phi$ for all non-empty $A \subseteq X$ if and only if $\succ$ is *acyclic*.
(b) Assume $\succ$ is *acyclic*. $C_\succ$ satisfies Sen's property α, but may not satisfy property β.

Answer to problem 3

(a) $C_\succ(A) \neq \phi$ for all non-empty $A \subseteq X$ if and only if $\succ$ is *acyclic*.

(1) $C_\succ(A) \neq \phi$ for all non-empty $A \subseteq X \rightarrow \succ$ is *acyclic*

Assume to the contrary, $\succ$ is not *acyclic*, which means there exists a list $(x_1, x_2, \ldots, x_{n1}, x_n)$ such that $x_k \succ x_{k+1} (k \in 1, 2, \ldots, n-1)$ and $x_n \succ x_1$. That is to say, for every $x_k \in X$, there always exists $y \succ x$, hence$C_\succ(A) = \phi$, contradiction. Thus,$\succ$ must be *acyclic*.

(2) $\succ$ is *acyclic* $\rightarrow C_\succ(A) \neq \phi$ for all non-empty $A \subseteq X$

Assume to the contrary, $\exists$ a non-empty $A \subseteq X$, $C_\succ(A) = \phi$. Consider any $x \in A$. Since $x \notin C_\succ(A)$, there exists $y \in A$ such that $y \succ x$. Let

$A = \{x_1\}$ and $x_1 \succ x_2 \succ \cdots \succ x_k (k \geq 2)$, if there exists $y \in A$ such that $y \succ x$, we have $x_1 \succ x_1$, which contradicts to the presumption that $\succ$ is *acyclic*. Thus, $C_{\succ}(A) \neq \phi$ for all non-empty $A \subseteq X$.

(b) Assume $\succ$ is *acyclic*. $C_{\succ}$ satisfies Sen's property α, but may not satisfy property β.

(1) Sen's property α

Define $A \subseteq B \in \mathscr{D}$, since $\succ$ is *acyclic*, we have $C_{\succ}(B) \neq \phi$, so that $x \in C_{\succ}(B)$. By the definition of $C_{\succ}(B)$, there does not exist $y \in B$ such that $y \succ x$. Since $A \subseteq B$, it also means there does not exist $y \in A$ such that $y \succ x$. Hence $x \in C_{\succ}(A)$, $C_{\succ}$ satisfies Sen's property α.

(2) Sen's property β

Let $\mathscr{D} = \{x_1, x_2, x_3\}$, since $\succ$ is *acyclic*, we can assume that $x_1 \succ x_2 \succ x_3$, $x_3 \not\succ x_1$ and $A = \{x_1, x_3\} \subseteq B = \{x_1, x_2, x_3\} \in \mathscr{D}$. Notice that $\succ$ is not assumed to be transitive, we don't have $x_1 \succ x_3$. Thus we have $C_{\succ}(A) = \{x_1, x_3\}$ and $C_{\succ}(B) = \{x_1\}$.

Problem 4

Show that if a choice correspondence C (defined on the domain $\mathscr{D}$) can be rationalized, then it satisfies the *path-invariance* property: for any $B_1, B_2 \in \mathscr{D}$ such that $B_1 \cup B_2 \in \mathscr{D}$ and $C(B_1) \cup C(B_2) \in \mathscr{D}$, we have $C(B_1 \cup B_2) = C(C(B_1) \cup C(B_2))$.

Answer to problem 4

Obviously, $C(C(B_1) \cup C(B_2)) \subseteq C(B_1 \cup B_2)$. Then we only need to prove that $C(B_1 \cup B_2) \subseteq C(C(B_1) \cup C(B_2))$, that is to say, we want to for that for any $x \in C(B_1 \cup B_2)$, $x \in C(C(B_1) \cup C(B_2))$.

If C can be rationalized, then there exists rational $\succeq$ such that $C = C_{\succeq}$. We know that $C_{\succeq}$ satisfies WARP, hence Sen's property α. Since $x \in C(B_1 \cup B_2)$, $B_1 \subseteq (B_1 \cup B_2)$, $B_2 \subseteq (B_1 \cup B_2)$ and Sen's property α, we have$x \in C(B_1)$ and $x \in C(B_2)$. Since $x \in C(B_1) \subseteq B_1$ and $x \in C(B_2) \subseteq B_2$, we have $x \in C(B_1) \cup C(B_2) \subseteq B_1 \cup B_2$. According to Sen's property α, $x \in C(B_1) \cup C(B_2) \subseteq B_1 \cup B_2$ and $x \in C(B_1 \cup B_2)$, we have $x \in C(C(B_1) \cup C(B_2))$. Hence $C(B_1 \cup B_2) \subseteq C(C(B_1) \cup C(B_2))$.

Combining both $C(C(B_1) \cup C(B_2)) \subseteq C(B_1 \cup B_2)$ and $C(B_1 \cup B_2) \subseteq C(C(B_1) \cup C(B_2))$, we have $C(B_1 \cup B_2) = C(C(B_1) \cup C(B_2))$. Q.E.D.