MICROECONOMIC THEORY II

Bingyong Zheng

Email: bingyongzheng@gmail.com

Spring 2021

Exchange economy: endogenous variables

- Exogenous variables

Exchange economy: endogenous variables

- Exogenous variables
  - Consumers $i \in \{1, 2, \ldots, I\}$

Exchange economy: endogenous variables

- Exogenous variables
  - Consumers $i \in \{1, 2, \ldots, I\}$
  - Goods $l \in \{1, 2, \ldots, L\}$

Exchange economy: endogenous variables

- Exogenous variables
  - ➢ Consumers $i \in \{1, 2, \ldots, I\}$
  - ➢ Goods $l \in \{1, 2, \ldots, L\}$
  - ➢ Endowments

$$\omega = [\omega_1, \ldots, \omega_I] = \begin{bmatrix} \omega_{11} \cdots & \cdots \omega_{1I} \\ . & . \\ . & . \\ \omega_{L1} \cdots & \cdots \omega_{LI} \end{bmatrix}_{L \times I}$$

Exchange economy: endogenous variables

- Exogenous variables
  - ➢ Consumers $i \in \{1, 2, \ldots, I\}$
  - ➢ Goods $l \in \{1, 2, \ldots, L\}$
  - ➢ Endowments

$$\omega = [\omega_1, \ldots, \omega_I] = \begin{bmatrix} \omega_{11} \cdots & \cdots \omega_{1I} \\ \cdot & \cdot \\ \cdot & \cdot \\ \omega_{L1} \cdots & \cdots \omega_{LI} \end{bmatrix}_{L \times I}$$

  - ➢ Preferences

$$\{\succeq\}_{i=1}^{I} = \{\succeq_1, \succeq_2, \ldots, \succeq_I\}$$

Endogenous variables

- Consumption

$$X = [X_1, \ldots, X_I] = \begin{bmatrix} x_{11} \cdots & \cdots x_{1I} \\ \cdot & \cdot \\ \cdot & \cdot \\ x_{L1} \cdots & \cdots x_{LI} \end{bmatrix}_{L \times I}$$

Endogenous variables

- Consumption

$$X = [X_1, \ldots, X_I] = \begin{bmatrix} x_{11} \cdots & \cdots x_{1I} \\ \cdot & \cdot \\ \cdot & \cdot \\ x_{L1} \cdots & \cdots x_{LI} \end{bmatrix}_{L \times I}$$

- Consumption allocation, X, consumption bundle for agent i, X_i

Endogenous variables

- Consumption

$$X = [X_1, \ldots, X_I] = \begin{bmatrix} x_{11} \cdots & \cdots x_{1I} \\ \cdot & \cdot \\ \cdot & \cdot \\ x_{L1} \cdots & \cdots x_{LI} \end{bmatrix}_{L \times I}$$

- Consumption allocation, X, consumption bundle for agent i, X_i
- Prices

Implicit assumptions

- Private ownership:
  Every portion of every good is owned by exactly one person and that person has the exclusive right to use it in consumption and exchange (or production).

Implicit assumptions

- Private ownership:
  Every portion of every good is owned by exactly one person and that person has the exclusive right to use it in consumption and exchange (or production).
- No externality

Implicit assumptions

- Private ownership:
  Every portion of every good is owned by exactly one person and that person has the exclusive right to use it in consumption and exchange (or production).
- No externality
- Complete information or symmetric information

Two concepts

- Pareto efficiency

Two concepts

- Pareto efficiency
  - Definition: X is feasible if $\sum_i X_i \leq \sum_i \omega_i$

Two concepts

- Pareto efficiency
  - Definition: X is feasible if $\sum_i X_i \leq \sum_i \omega_i$
  - Definition: A feasible allocation X is (Pareto) efficient if there exists no feasible allocation X' such that $\forall i$

$$X'_i \succeq_i X_i$$

and $\exists i$, $X'_i \succ_i X_i$

Two concepts

- Pareto efficiency
  - Definition: X is feasible if $\sum_i X_i \leq \sum_i \omega_i$
  - Definition: A feasible allocation X is (Pareto) efficient if there exists no feasible allocation X' such that $\forall i$

$$X'_i \succeq_i X_i$$

    and $\exists i$, $X'_i \succ_i X_i$
- Walrasian equilibrium: (X^*, P) is an equilibrium if

Two concepts

- Pareto efficiency
  - Definition: X is feasible if $\sum_i X_i \leq \sum_i \omega_i$
  - Definition: A feasible allocation X is (Pareto) efficient if there exists no feasible allocation X' such that $\forall i$

    $$X'_i \succeq_i X_i$$

    and $\exists i$, $X'_i \succ_i X_i$
- Walrasian equilibrium: (X^*, P) is an equilibrium if
  - $\forall i$, X_i^* is maximal for $\succeq_i$ in $\{X_i | PX_i^* \leq P\omega_i\}$;

Two concepts

- Pareto efficiency
  - Definition: X is feasible if $\sum_i X_i \leq \sum_i \omega_i$
  - Definition: A feasible allocation X is (Pareto) efficient if there exists no feasible allocation X' such that $\forall i$

    $$X'_i \succeq_i X_i$$

    and $\exists i$, $X'_i \succ_i X_i$
- Walrasian equilibrium: (X^*, P) is an equilibrium if
  - $\forall i$, X_i^* is maximal for $\succeq_i$ in $\{X_i | PX_i^* \leq P\omega_i\}$;
  - $\sum_i X_i^* = \sum_i \omega_i$.

Some definitions

- $\succeq$ on X is monotonic if $x \in X$ and $y \gg x$ implies $y \succ x$.

Some definitions

- $\succeq$ on X is monotonic if $x \in X$ and $y \gg x$ implies $y \succ x$.
- It is strongly monotonic if $y \geq x$ and $y \neq x$ imply that $y \succ x$.

Some definitions

- $\succeq$ on X is monotonic if $x \in X$ and $y \gg x$ implies $y \succ x$.
- It is strongly monotonic if $y \geq x$ and $y \neq x$ imply that $y \succ x$.
- $\succeq$ is convex if $\forall x \in X$,

$$y \succeq x, z \succeq x \implies \forall\ \alpha \in [0, 1], \quad \alpha y + (1 - \alpha) z \succeq x.$$

Some definitions

- $\succeq$ on X is monotonic if $x \in X$ and $y \gg x$ implies $y \succ x$.
- It is strongly monotonic if $y \geq x$ and $y \neq x$ imply that $y \succ x$.
- $\succeq$ is convex if $\forall x \in X$,

$$y \succeq x, z \succeq x \Longrightarrow \forall\ \alpha \in [0,1], \quad \alpha y + (1-\alpha)z \succeq x.$$

- $\succeq$ is strictly convex if $\forall\ x \in X$,

$$y \succeq x, z \succeq x,\ y \neq z \Longrightarrow \forall\ \alpha \in (0,1), \quad \alpha y + (1-\alpha)z \succ x.$$

Pareto efficiency

- Theorem: Suppose $X^* \gg 0$, and that $\forall i$, $\succeq_i$ is represented by a concave u_i which is twice continuously differentiable, strongly monotonic around X_i^*. The following are equivalent

Pareto efficiency

- Theorem: Suppose $X^* \gg 0$, and that $\forall i$, $\succeq_i$ is represented by a concave u_i which is twice continuously differentiable, strongly monotonic around X_i^*. The following are equivalent
  - X^* is (Pareto) efficient;

Pareto efficiency

- Theorem: Suppose $X^* \gg 0$, and that $\forall i$, $\succeq_i$ is represented by a concave u_i which is twice continuously differentiable, strongly monotonic around X_i^*. The following are equivalent
  - X^* is (Pareto) efficient;
  - $\forall i$,
$$X^* \in arg\max\{u_i(X_i)|X \geq 0, \sum_i X_i \leq \sum_i \omega_i,$$
$$(\forall h \neq i)\ u_h(X_h) \geq u_h(X_h^*)\}$$

Pareto efficiency

- Theorem: Suppose $X^* \gg 0$, and that $\forall i$, $\succeq_i$ is represented by a concave u_i which is twice continuously differentiable, strongly monotonic around X_i^*. The following are equivalent
  - X^* is (Pareto) efficient;
  - $\forall i$,
  $$X^* \in arg\max\{u_i(X_i) | X \geq 0, \sum_i X_i \leq \sum_i \omega_i,$$
  $$(\forall h \neq i)\ u_h(X_h) \geq u_h(X_h^*)\}$$
  - $\exists q = (q_1, \ldots, q_L) \in \mathbb{R}^L_{++}$, shadow prices; $\exists (s_1, \ldots, S_I) \in \mathbb{R}^I_{++}$; $\forall i$,
  $$s_i Du_i(X_i^*) = q, \qquad \sum_i X_i = \sum_i \omega_i.$$

Pareto efficiency

- Theorem: Suppose $X^* \gg 0$, and that $\forall i$, $\succeq_i$ is represented by a concave u_i which is twice continuously differentiable, strongly monotonic around X_i^*. The following are equivalent
  - ➢ X^* is (Pareto) efficient;
  - ➢ $\forall i$,
$$X^* \in arg\max\{u_i(X_i)|X \geq 0, \sum_i X_i \leq \sum_i \omega_i,$$
$$(\forall h \neq i)\ u_h(X_h) \geq u_h(X_h^*)\}$$
  - ➢ $\exists q = (q_1, \ldots, q_L) \in \mathbb{R}^L_{++}$, shadow prices; $\exists (s_1, \ldots, S_I) \in \mathbb{R}^I_{++}$; $\forall i$,
$$s_i Du_i(X_i^*) = q, \qquad \sum_i X_i = \sum_i \omega_i.$$
  - ➢ $\forall i \in \{1, 2, \ldots, I-1\}$ and $\forall l \in \{1, 2, \ldots, L-1\}$,
$$MRS_i^{l,l+1} = MRS_{i+1}^{l,l+1}$$
$$\sum_i X_i = \sum_i \omega_i.$$

Two consumer economy

- Consumer A has 7 units of x_1, 3 units of x_2; B has 3 units of x_1, 7 units of x_2.

Two consumer economy

- Consumer A has 7 units of x_1, 3 units of x_2; B has 3 units of x_1, 7 units of x_2.
- They both have same utility function

$$U_A(x_{1A}, x_{2A}) = (x_{1A}x_{2A})^{1/2} \qquad U_B(x_{1B}, x_{2B}) = (x_{1B}x_{2B})^{1/2}.$$

Two consumer economy

- Consumer A has 7 units of x_1, 3 units of x_2; B has 3 units of x_1, 7 units of x_2.
- They both have same utility function

$$U_A(x_{1A}, x_{2A}) = (x_{1A}x_{2A})^{1/2} \qquad U_B(x_{1B}, x_{2B}) = (x_{1B}x_{2B})^{1/2}.$$

- Feasible allocation: any points in the Edgeworth box such that

$$x_{1A} + x_{1B} \leq 10,$$

$$x_{2B} + x_{2B} \leq 10.$$

Two consumer economy

- Consumer A has 7 units of x_1, 3 units of x_2; B has 3 units of x_1, 7 units of x_2.
- They both have same utility function

$$U_A(x_{1A}, x_{2A}) = (x_{1A}x_{2A})^{1/2} \qquad U_B(x_{1B}, x_{2B}) = (x_{1B}x_{2B})^{1/2}.$$

- Feasible allocation: any points in the Edgeworth box such that

$$x_{1A} + x_{1B} \leq 10,$$

$$x_{2B} + x_{2B} \leq 10.$$

- Pareto efficient allocation: X is PE if no feasible X' that can make one better off without hurting others.

Two consumer economy

- Consumer A has 7 units of x_1, 3 units of x_2; B has 3 units of x_1, 7 units of x_2.
- They both have same utility function

$$U_A(x_{1A}, x_{2A}) = (x_{1A}x_{2A})^{1/2} \qquad U_B(x_{1B}, x_{2B}) = (x_{1B}x_{2B})^{1/2}.$$

- Feasible allocation: any points in the Edgeworth box such that

$$x_{1A} + x_{1B} \leq 10,$$

$$x_{2B} + x_{2B} \leq 10.$$

- Pareto efficient allocation: X is PE if no feasible X' that can make one better off without hurting others.
- Contract curve gives all efficient allocation in the Edgeworth box.

One efficient allocation

Contract curve: all P.E.

Another example

Linear preferences: contract curve

- Preferences

$$U_A = x_{1A} + 2x_{2A}, \qquad U_B = 2x_{1B} + x_{2B}$$

Linear preferences: contract curve

- Preferences

$$U_A = x_{1A} + 2x_{2A}, \qquad U_B = 2x_{1B} + x_{2B}$$

- Initial endowment

$$\omega^A = (7, 3), \qquad \omega^B = (3, 7).$$

Linear preference

Leontief preferences

- Preferences

$$U_A = \min(2x_{1A}, x_{2A}), \qquad U_B = \min(2x_{1B}, x_{2B}).$$

Leontief preferences

- Preferences

$$U_A = \min(2x_{1A}, x_{2A}), \qquad U_B = \min(2x_{1B}, x_{2B}).$$

- Initial endowment

$$\omega^A = (7, 3), \qquad \omega^B = (3, 7).$$

Draw the contract curve

Social planner's problem (1)

- Social planner's problem:

$$\max_{X} \sum_{i=1}^{I} s_i u_i(X_i) \ s.t. \ feasibility \ constraint.$$

Social planner's problem (1)

- Social planner's problem:

$$\max_X \sum_{i=1}^{I} s_i u_i(X_i) \; s.t. \; \textit{feasibility constraint}.$$

- This is equivalent to the optimization problem: $(\forall i)$

$$\max_X u_i(X_i) \quad s.t.$$

$$\sum_i X_i = \sum_i \omega_i$$

$$\forall h \neq i \quad u_h(X_h) \geq u_h(X_h^*)$$

Planner's problem (2)

- Take $i = 2$, the objective function

$$\max_{X} u_2(X_2)$$

with $L \times I$ unknowns, subject to

Planner's problem (2)

- Take $i = 2$, the objective function

$$\max_{X} u_2(X_2)$$

with $L \times I$ unknowns, subject to

  - ➢ Feasibility

$$\sum_i X_i = \sum_i \omega_i$$

Planner's problem (2)

- Take $i = 2$, the objective function

$$\max_X u_2(X_2)$$

with $L \times I$ unknowns, subject to

  - ➢ Feasibility
$$\sum_i X_i = \sum_i \omega_i$$
  - ➢ The other $I - 1$ not worse-off
$$\forall h \neq 2 \quad u_h(X_h) \geq u_h(X_h^*)$$

Planner's problem (2)

- Take $i = 2$, the objective function

$$\max_X u_2(X_2)$$

with $L \times I$ unknowns, subject to

  - Feasibility

$$\sum_i X_i = \sum_i \omega_i$$

  - The other $I - 1$ not worse-off

$$\forall h \neq 2 \quad u_h(X_h) \geq u_h(X_h^*)$$

- The Lagrangian

$$\mathcal{L} = u_2(X_2) + \sum_{l=1}^{L} q_l \left[\sum_{i=1}^{I} \omega_{li} - \sum_{i=1}^{I} x_{li} \right] + \sum_{i \neq 2} s_i \left[u_i(X_i) - u_i(X_i^*) \right]$$

Planner's problem (2)

- Take $i = 2$, the objective function

$$\max_X u_2(X_2)$$

with $L \times I$ unknowns, subject to

  - Feasibility

$$\sum_i X_i = \sum_i \omega_i$$

  - The other $I - 1$ not worse-off

$$\forall h \neq 2 \quad u_h(X_h) \geq u_h(X_h^*)$$

- The Lagrangian

$$\mathcal{L} = u_2(X_2) + \sum_{l=1}^{L} q_l \left[\sum_{i=1}^{I} \omega_{li} - \sum_{i=1}^{I} x_{li} \right] + \sum_{i \neq 2} s_i \left[u_i(X_i) - u_i(X_i^*) \right]$$

- First-order condition gives

$$\forall i, \; s_i D u_i(X_i^*) = q$$

Planner's problem

- Suppose $L = 2, I = 3$

Planner's problem

- Suppose $L = 2, I = 3$
- FOC yields

$$D_{x_{11}}\mathcal{L} = -q_1 + s_1 \frac{\partial u_1}{\partial x_{11}} = 0$$

$$D_{x_{21}}\mathcal{L} = -q_2 + s_1 \frac{\partial u_1}{\partial x_{21}} = 0$$

$$D_{x_{12}}\mathcal{L} = -q_1 + \frac{\partial u_2}{\partial x_{12}} = 0$$

$$D_{x_{22}}\mathcal{L} = -q_2 + \frac{\partial u_2}{\partial x_{22}} = 0$$

$$D_{x_{13}}\mathcal{L} = -q_1 + s_3 \frac{\partial u_3}{\partial x_{13}} = 0$$

$$D_{x_{23}}\mathcal{L} = -q_2 + s_3 \frac{\partial u_3}{\partial x_{23}} = 0$$

Planner's problem

- Suppose $L = 2, I = 3$
- FOC yields

$$D_{x_{11}}\mathcal{L} = -q_1 + s_1 \frac{\partial u_1}{\partial x_{11}} = 0$$

$$D_{x_{21}}\mathcal{L} = -q_2 + s_1 \frac{\partial u_1}{\partial x_{21}} = 0$$

$$D_{x_{12}}\mathcal{L} = -q_1 + \frac{\partial u_2}{\partial x_{12}} = 0$$

$$D_{x_{22}}\mathcal{L} = -q_2 + \frac{\partial u_2}{\partial x_{22}} = 0$$

$$D_{x_{13}}\mathcal{L} = -q_1 + s_3 \frac{\partial u_3}{\partial x_{13}} = 0$$

$$D_{x_{23}}\mathcal{L} = -q_2 + s_3 \frac{\partial u_3}{\partial x_{23}} = 0$$

- Set $s_2 = 1$, we have

$$s_i D u_i = q.$$

Utility possibility frontier

- A curve that connects all the possible combinations of utilities that could arise at the various economically efficient allocations.

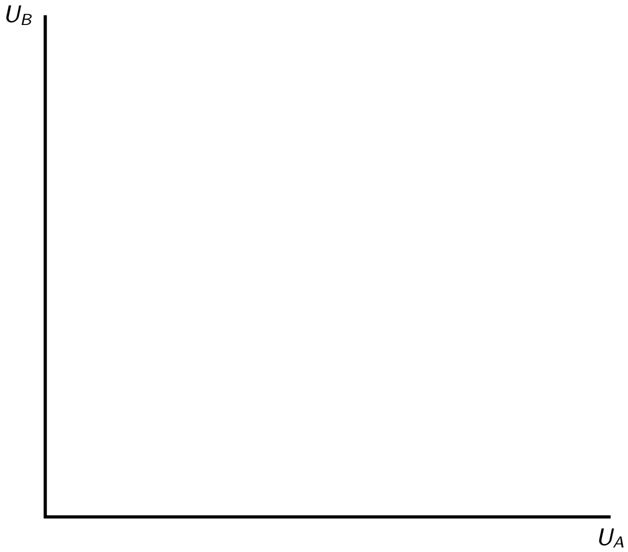
Utility possibility frontier

- A curve that connects all the possible combinations of utilities that could arise at the various economically efficient allocations.
- UPF gives all possible combinations of utilities at P.E. allocations.

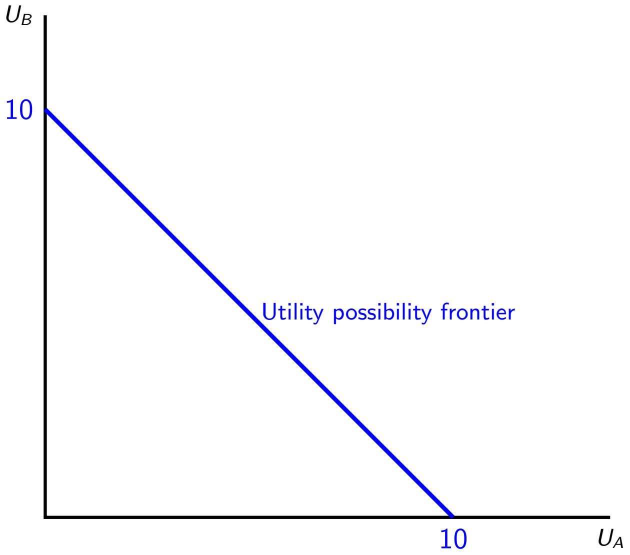
Utility possibility frontier

- A curve that connects all the possible combinations of utilities that could arise at the various economically efficient allocations.
- UPF gives all possible combinations of utilities at P.E. allocations.
- How to find the UPF: identify all PE allocations.

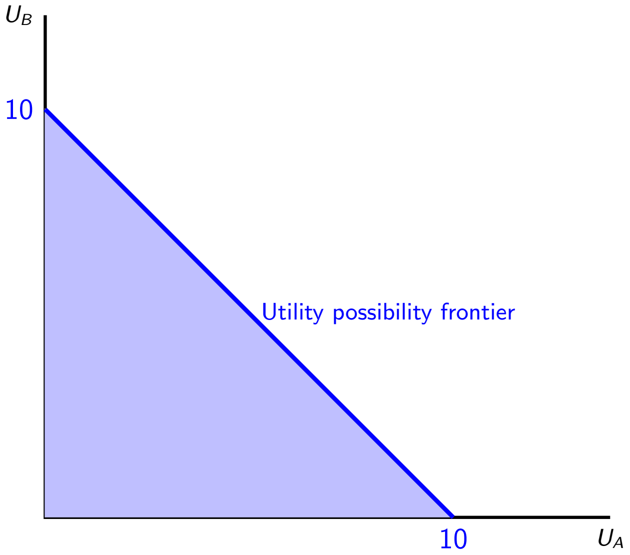
Draw the UPF



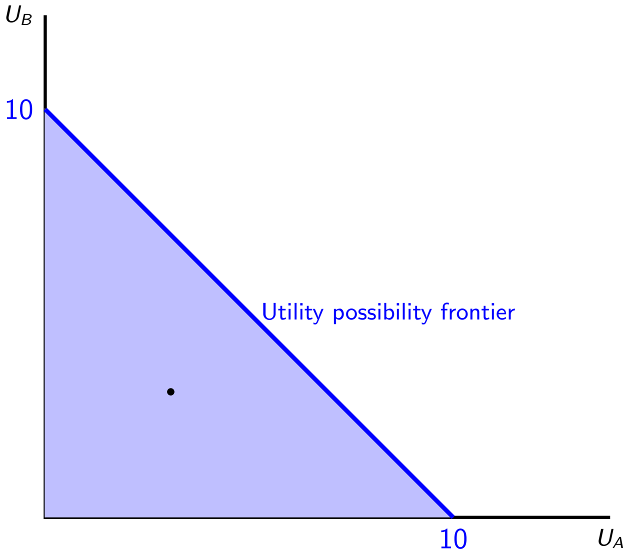
Draw the UPF



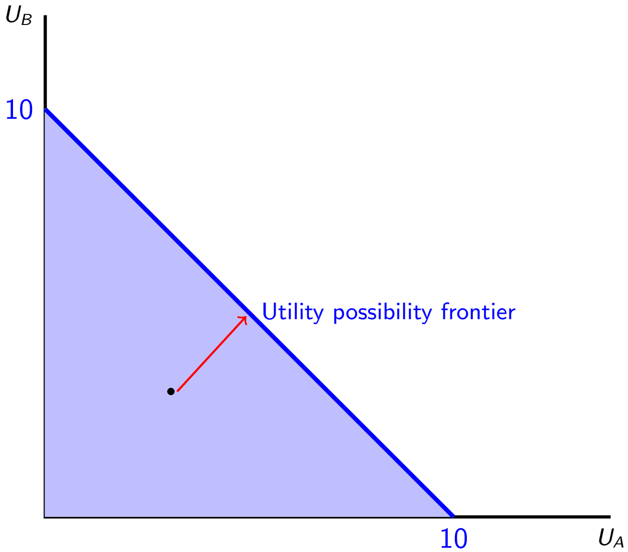
Draw the UPF



Draw the UPF



Draw the UPF



Competitive equilibrium example

- The two-consumers example with Cobb-Douglas preference

$$\omega_A = (7, e) \qquad \omega_B = (3, 7).$$

Competitive equilibrium example

- The two-consumers example with Cobb-Douglas preference

$$\omega_A = (7, e) \qquad \omega_B = (3, 7).$$

- Feasible allocation:

$$x_{1A} + x_{1B} \leq 10, \qquad x_{2B} + x_{2B} \leq 10.$$

Competitive equilibrium example

- The two-consumers example with Cobb-Douglas preference

$$\omega_A = (7, e) \qquad \omega_B = (3, 7).$$

- Feasible allocation:

$$x_{1A} + x_{1B} \leq 10, \qquad x_{2B} + x_{2B} \leq 10.$$

- Utility maximizing for consumers A and B,

$$x_{1A} = \frac{m_A}{2P_1},\ x_{2A} = \frac{m_A}{2P_2} \text{ where } \ m_A = 7P_1 + 3P_2$$
$$x_{1B} = \frac{m_B}{2P_1},\ x_{2B} = \frac{m_B}{2P_2} \text{ where } \ m_B = 3P_2 + 7P_2.$$

Competitive equilibrium example

- The two-consumers example with Cobb-Douglas preference

$$\omega_A = (7, e) \qquad \omega_B = (3, 7).$$

- Feasible allocation:

$$x_{1A} + x_{1B} \leq 10, \qquad x_{2B} + x_{2B} \leq 10.$$

- Utility maximizing for consumers A and B,

$$x_{1A} = \frac{m_A}{2P_1},\; x_{2A} = \frac{m_A}{2P_2} \text{ where } \; m_A = 7P_1 + 3P_2$$

$$x_{1B} = \frac{m_B}{2P_1},\; x_{2B} = \frac{m_B}{2P_2} \text{ where } \; m_B = 3P_2 + 7P_2.$$

- Plugging m_A, m_b into the allocations yields

$$x_{1A} = \frac{7P_1 + 3P_2}{2P_1},\; x_{2A} = \frac{7P_1 + 3P_2}{2P_2},$$

$$x_{1B} = \frac{3P_1 + 7P_2}{2P_1},\; x_{2B} = \frac{3P_1 + 7P_2}{2P_2}.$$

C.E. EXAMPLE CONTINUED

- Market clears

$$5 + \frac{5P_2}{P_1} = 10, \quad 5 + \frac{5P_1}{P_2} = 10.$$

C.E. example continued

- Market clears

$$5+\frac{5P_2}{P_1}=10, \quad 5+\frac{5P_1}{P_2}=10.$$

- The competitive equilibrium (let $P_1=1$)

$$P_1=1,\ P_2=1,\ ,x_{1A}=x_{2A}=5,\ x_{1B}=x_{2B}=5.$$

Equilibrium and P.E. allocations

P.E. allocations

C.E. allocations

1. Exchange efficiency:

$$MRS^A_{1,2} = MRS^B_{1,2}$$

Equilibrium and P.E. allocations

P.E. allocations

C.E. allocations

1. Exchange efficiency:

$$MRS^A_{1,2} = MRS^B_{1,2}$$

2. No resources wasted

$$x_{1A} + x_{1B} = \omega_{1A} + \omega_{1B}$$

$$x_{2A} + x_{2B} = \omega_{2A} + \omega_{2B}$$

Equilibrium and P.E. allocations

P.E. allocations

1. Exchange efficiency:

$$MRS^A_{1,2} = MRS^B_{1,2}$$

2. No resources wasted

$$x_{1A} + x_{1B} = \omega_{1A} + \omega_{1B}$$

$$x_{2A} + x_{2B} = \omega_{2A} + \omega_{2B}$$

C.E. allocations

1. Utility-maximization

(a) $MRS^i_{1,2} = \dfrac{P_1}{P_2} \Longrightarrow$

$MRS^A_{1,2} = MRS^B_{1,2}$

(b) $P_1 x_{1A} + P_2 x_{2A} = m_A$

$P_1 x_{1B} + P_2 x_{2B} = m_B$

Equilibrium and P.E. allocations

P.E. allocations

1. Exchange efficiency:

$$MRS^A_{1,2} = MRS^B_{1,2}$$

2. No resources wasted

$$x_{1A} + x_{1B} = \omega_{1A} + \omega_{1B}$$

$$x_{2A} + x_{2B} = \omega_{2A} + \omega_{2B}$$

C.E. allocations

1. Utility-maximization

   (a) $MRS^i_{1,2} = \dfrac{P_1}{P_2} \Longrightarrow$

   $MRS^A_{1,2} = MRS^B_{1,2}$

   (b) $P_1 x_{1A} + P_2 x_{2A} = m_A$

   $P_1 x_{1B} + P_2 x_{2B} = m_B$

2. Market clears, $j = 1, 2$

$$x_{1A} + x_{1B} = \omega_{1A} + \omega_{1B}$$

$$x_{2A} + x_{2B} = \omega_{2A} + \omega_{2B}$$

Main result on C.E.

- Theorem: Suppose $X^* \gg 0$ and that $\forall i$, $\succeq_i$ is represented by a concave u_i, which is twice continuously differentiable and strongly monotonic around X_i^*, the following are equivalent

Main result on C.E.

- Theorem: Suppose $X^* \gg 0$ and that $\forall i$, $\succeq_i$ is represented by a concave u_i, which is twice continuously differentiable and strongly monotonic around X_i^*, the following are equivalent
  - (X^*, P) is an equilibrium;

Main result on C.E.

- Theorem: Suppose $X^* \gg 0$ and that $\forall i$, $\succeq_i$ is represented by a concave u_i, which is twice continuously differentiable and strongly monotonic around X_i^*, the following are equivalent
  - ➢ (X^*, P) is an equilibrium;
  - ➢ $(\exists\ \lambda_1, \ldots, \lambda_I) \in \mathbb{R}^I_{++}$:

Main result on C.E.

- Theorem: Suppose $X^* \gg 0$ and that $\forall i$, $\succeq_i$ is represented by a concave u_i, which is twice continuously differentiable and strongly monotonic around X_i^*, the following are equivalent
  - (X^*, P) is an equilibrium;
  - $(\exists\ \lambda_1, \ldots, \lambda_I) \in \mathbb{R}^I_{++}$:
    - $\forall i$,

$$Du_i(X_i^*) = \lambda_i P;$$

MAIN RESULT ON C.E.

- Theorem: Suppose $X^* \gg 0$ and that $\forall i$, $\succeq_i$ is represented by a concave u_i, which is twice continuously differentiable and strongly monotonic around X_i^*, the following are equivalent
  - (X^*, P) is an equilibrium;
  - $(\exists\ \lambda_1, \ldots, \lambda_I) \in \mathbb{R}^I_{++}$:
    - $\forall i$,
    $$Du_i(X_i^*) = \lambda_i P;$$
    - Market clears
    $$\sum_i X_i^* = \sum_i \omega_i;$$

Main result on C.E.

- Theorem: Suppose $X^* \gg 0$ and that $\forall i$, $\succeq_i$ is represented by a concave u_i, which is twice continuously differentiable and strongly monotonic around X_i^*, the following are equivalent
  - (X^*, P) is an equilibrium;
  - $(\exists\ \lambda_1, \ldots, \lambda_I) \in \mathbb{R}^I_{++}$:
    - $\forall i$,
      $$Du_i(X_i^*) = \lambda_i P;$$
    - Market clears
      $$\sum_i X_i^* = \sum_i \omega_i;$$
    - For each i
      $$PX_i^* = P\omega_i.$$

First welfare theorem

- FWT: Suppose $\succeq_i$ is locally nonsatiated. Then every equilibrium allocation is efficient. (market is good)

First welfare theorem

- FWT: Suppose $\succeq_i$ is locally nonsatiated. Then every equilibrium allocation is efficient. (market is good)
- In market economy, util-maximization by self-interested consumers will result in Pareto efficiency.

First welfare theorem

- FWT: Suppose $\succeq_i$ is locally nonsatiated. Then every equilibrium allocation is efficient. (market is good)
- In market economy, util-maximization by self-interested consumers will result in Pareto efficiency.
- Competitive market economizes on the information that any one consumer needs to possess.

First welfare theorem

- FWT: Suppose $\succeq_i$ is locally nonsatiated. Then every equilibrium allocation is efficient. (market is good)
- In market economy, util-maximization by self-interested consumers will result in Pareto efficiency.
- Competitive market economizes on the information that any one consumer needs to possess.
- The only thing that any one consumer needs to know to make consumption decisions are the prices of the good

First welfare theorem

- FWT: Suppose $\succeq_i$ is locally nonsatiated. Then every equilibrium allocation is efficient. (market is good)
- In market economy, util-maximization by self-interested consumers will result in Pareto efficiency.
- Competitive market economizes on the information that any one consumer needs to possess.
- The only thing that any one consumer needs to know to make consumption decisions are the prices of the good
- Consumers do not need to know anything about how the goods are produced, or who owns what goods, etc.

First welfare theorem

- FWT: Suppose $\succeq_i$ is locally nonsatiated. Then every equilibrium allocation is efficient. (market is good)
- In market economy, util-maximization by self-interested consumers will result in Pareto efficiency.
- Competitive market economizes on the information that any one consumer needs to possess.
- The only thing that any one consumer needs to know to make consumption decisions are the prices of the good
- Consumers do not need to know anything about how the goods are produced, or who owns what goods, etc.
- If the markets function well enough to determine the competitive price, we are guaranteed an efficient outcome.

First welfare theorem

- FWT: Suppose $\succeq_i$ is locally nonsatiated. Then every equilibrium allocation is efficient. (market is good)
- In market economy, util-maximization by self-interested consumers will result in Pareto efficiency.
- Competitive market economizes on the information that any one consumer needs to possess.
- The only thing that any one consumer needs to know to make consumption decisions are the prices of the good
- Consumers do not need to know anything about how the goods are produced, or who owns what goods, etc.
- If the markets function well enough to determine the competitive price, we are guaranteed an efficient outcome.
- Market dominates other mechanism to allocation resources in an economy.

Local non-satiation

- Local non-satiation: For any consumption bundle x, for all ε, $\exists\ y$ with $||x - y|| < \varepsilon$ such that $y \succ x$.

Local non-satiation

- Local non-satiation: For any consumption bundle x, for all ε, $\exists\ y$ with $||x - y|| < \varepsilon$ such that $y \succ x$.
- Monotonicity implies local nonsatiation.

Local non-satiation

- Local non-satiation: For any consumption bundle x, for all ε, $\exists\ y$ with $||x - y|| < \varepsilon$ such that $y \succ x$.
- Monotonicity implies local nonsatiation.
- Lemma: Suppose $\succeq_i$ is locally nonsatiated, X_i^* is maximal for $\succeq_i$ in $\{X_i | PX_i \leq P\omega_i\}$.

Local non-satiation

- Local non-satiation: For any consumption bundle x, for all ε, $\exists\ y$ with $||x - y|| < \varepsilon$ such that $y \succ x$.
- Monotonicity implies local nonsatiation.
- Lemma: Suppose $\succeq_i$ is locally nonsatiated, X_i^* is maximal for $\succeq_i$ in $\{X_i | PX_i \leq P\omega_i\}$.
  - If $X_i \succeq_i X_i^*$, then $PX_i \geq PX_i^*$;

Local non-satiation

- Local non-satiation: For any consumption bundle x, for all ε, $\exists\ y$ with $||x - y|| < \varepsilon$ such that $y \succ x$.
- Monotonicity implies local nonsatiation.
- Lemma: Suppose $\succeq_i$ is locally nonsatiated, X_i^* is maximal for $\succeq_i$ in $\{X_i | PX_i \leq P\omega_i\}$.
  - If $X_i \succeq_i X_i^*$, then $PX_i \geq PX_i^*$;
  - If $X_i \succ_i X_i^*$, then $PX_i > PX_i^*$.

Proof of the FWT

- Suppose (X^*, P) is an equilibrium, but not efficient.

Proof of the FWT

- Suppose (X^*, P) is an equilibrium, but not efficient.
- There must exist X':

Proof of the FWT

- Suppose (X^*, P) is an equilibrium, but not efficient.
- There must exist X':
  - ➢ Feasible:

$$\sum_i X_i' \leq \sum_i \omega_i \Longrightarrow \sum_i PX_i' \leq \sum_i P\omega_i;$$

Proof of the FWT

- Suppose (X^*, P) is an equilibrium, but not efficient.
- There must exist X':
  - ➢ Feasible:

$$\sum_i X_i' \leq \sum_i \omega_i \Longrightarrow \sum_i PX_i' \leq \sum_i P\omega_i;$$

  - ➢ Pareto improvement:

$$\forall i,\ X_i' \succeq X_i^* \Longrightarrow PX_i' \geq PX_i^*;$$

$$\exists i,\ X_i' \succ X_i^* \Longrightarrow PX_i' > PX_i^*.$$

Proof of the FWT

- Suppose (X^*, P) is an equilibrium, but not efficient.
- There must exist X':
  - Feasible:

$$\sum_i X_i' \leq \sum_i \omega_i \Longrightarrow \sum_i PX_i' \leq \sum_i P\omega_i;$$

  - Pareto improvement:

$$\forall i,\ X_i' \succeq X_i^* \Longrightarrow PX_i' \geq PX_i^*;$$

$$\exists i,\ X_i' \succ X_i^* \Longrightarrow PX_i' > PX_i^*.$$

- However, Pareto improvement implies:

$$\sum_i PX_i' > \sum_i P\omega_i.$$

Contradicts the feasibility constraint.

Second Welfare Theorem

- Second Welfare Theorem: Suppose X^* is an efficient allocation and that an equilibrium exists from X^*. Then X^* is an equilibrium allocation.

Second Welfare Theorem

- Second Welfare Theorem: Suppose X^* is an efficient allocation and that an equilibrium exists from X^*. Then X^* is an equilibrium allocation.
- Under certain conditions, every P.E allocation can be achieved through market.

Second Welfare Theorem

- Second Welfare Theorem: Suppose X^* is an efficient allocation and that an equilibrium exists from X^*. Then X^* is an equilibrium allocation.
- Under certain conditions, every P.E allocation can be achieved through market.
- One condition: preferences are convex.

Second Welfare Theorem

- Second Welfare Theorem: Suppose X^* is an efficient allocation and that an equilibrium exists from X^*. Then X^* is an equilibrium allocation.
- Under certain conditions, every P.E allocation can be achieved through market.
- One condition: preferences are convex.
- SWT implies that problems of distribution and efficiency can be separated:

Second Welfare Theorem

- Second Welfare Theorem: Suppose X^* is an efficient allocation and that an equilibrium exists from X^*. Then X^* is an equilibrium allocation.
- Under certain conditions, every P.E allocation can be achieved through market.
- One condition: preferences are convex.
- SWT implies that problems of distribution and efficiency can be separated:
  - ➢ redistribute endowments of goods to determine how much wealth consumers have; use prices to indicate the relative scarcity of goods.

Second Welfare Theorem

- Second Welfare Theorem: Suppose X^* is an efficient allocation and that an equilibrium exists from X^*. Then X^* is an equilibrium allocation.
- Under certain conditions, every P.E allocation can be achieved through market.
- One condition: preferences are convex.
- SWT implies that problems of distribution and efficiency can be separated:
  - ➢ redistribute endowments of goods to determine how much wealth consumers have; use prices to indicate the relative scarcity of goods.
  - ➢ To achieve efficiency, each consumer must face the true social cost of his or her actions; choices should reflect those cost.

Second Welfare Theorem

- Second Welfare Theorem: Suppose X^* is an efficient allocation and that an equilibrium exists from X^*. Then X^* is an equilibrium allocation.
- Under certain conditions, every P.E allocation can be achieved through market.
- One condition: preferences are convex.
- SWT implies that problems of distribution and efficiency can be separated:
  - ➢ redistribute endowments of goods to determine how much wealth consumers have; use prices to indicate the relative scarcity of goods.
  - ➢ To achieve efficiency, each consumer must face the true social cost of his or her actions; choices should reflect those cost.
  - ➢ In competitive market, this is achieved through consumers' marginal decision to consume more or less given the price, which measures the relative scarcity of the goods.

Second Welfare Theorem

- Second Welfare Theorem: Suppose X^* is an efficient allocation and that an equilibrium exists from X^*. Then X^* is an equilibrium allocation.
- Under certain conditions, every P.E allocation can be achieved through market.
- One condition: preferences are convex.
- SWT implies that problems of distribution and efficiency can be separated:
  - ➢ redistribute endowments of goods to determine how much wealth consumers have; use prices to indicate the relative scarcity of goods.
  - ➢ To achieve efficiency, each consumer must face the true social cost of his or her actions; choices should reflect those cost.
  - ➢ In competitive market, this is achieved through consumers' marginal decision to consume more or less given the price, which measures the relative scarcity of the goods.
- To achieve distribution goal, all that is needed is to transfer the purchasing power of the endowment.

Graphical illustration

Non-convex preferences

Core

- A coalition $S \subseteq \{1, \ldots, I\}$ blocks an allocation X if $\exists\ X'$ such that

CORE

- A coalition $S \subseteq \{1, \dots, I\}$ blocks an allocation X if $\exists$ X' such that
  - $(\forall\ i \in S)$, $X'_i \succeq X_i$;

Core

- A coalition $S \subseteq \{1, \ldots, I\}$ blocks an allocation X if $\exists\ X'$ such that
  - $(\forall\ i \in S)$, $X'_i \succeq X_i$;
  - $(\exists\ i \in S)$, $X'_i \succ X_i$;

Core

- A coalition $S \subseteq \{1, \ldots, I\}$ blocks an allocation X if $\exists\ X'$ such that
  - $(\forall\ i \in S)$, $X'_i \succeq X_i$;
  - $(\exists\ i \in S)$, $X'_i \succ X_i$;
  - $$\sum_{i \in S} X'_i \leq \sum_{i \in S} \omega_i.$$

Core

- A coalition $S \subseteq \{1, \ldots, I\}$ blocks an allocation X if $\exists$ X' such that
    - $(\forall\ i \in S)$, $X'_i \succeq X_i$;
    - $(\exists\ i \in S)$, $X'_i \succ X_i$;
    - $$\sum_{i \in S} X'_i \leq \sum_{i \in S} \omega_i.$$
- The *Core* is the set of unblocked allocations.

Core

- A coalition $S \subseteq \{1, \ldots, I\}$ blocks an allocation X if $\exists$ X' such that
  - $(\forall\ i \in S)$, $X'_i \succeq X_i$;
  - $(\exists\ i \in S)$, $X'_i \succ X_i$;
  - $$\sum_{i \in S} X'_i \leq \sum_{i \in S} \omega_i.$$
- The *Core* is the set of unblocked allocations.
- Observation:

Core

- A coalition $S \subseteq \{1, \ldots, I\}$ blocks an allocation X if $\exists\ X'$ such that
  - $(\forall\ i \in S)$, $X'_i \succeq X_i$;
  - $(\exists\ i \in S)$, $X'_i \succ X_i$;
  - $$\sum_{i \in S} X'_i \leq \sum_{i \in S} \omega_i.$$
- The *Core* is the set of unblocked allocations.
- Observation:
  - An allocation X is unblocked by $S = \{1, \ldots, I\}$ (coalition of the whole) iff X is efficient;

Core

- A coalition $S \subseteq \{1, \ldots, I\}$ blocks an allocation X if $\exists\ X'$ such that
  - ➢ $(\forall\ i \in S)$, $X'_i \succeq X_i$;
  - ➢ $(\exists\ i \in S)$, $X'_i \succ X_i$;
  - ➢ $$\sum_{i \in S} X'_i \leq \sum_{i \in S} \omega_i.$$
- The *Core* is the set of unblocked allocations.
- Observation:
  - ➢ An allocation X is unblocked by $S = \{1, \ldots, I\}$ (coalition of the whole) iff X is efficient;
  - ➢ Inefficient allocations are blocked by $S = \{1, \ldots, I\}$;

Core

- A coalition $S \subseteq \{1, \ldots, I\}$ blocks an allocation X if $\exists\ X'$ such that
  - $(\forall\ i \in S)$, $X'_i \succeq X_i$;
  - $(\exists\ i \in S)$, $X'_i \succ X_i$;
  - $$\sum_{i \in S} X'_i \leq \sum_{i \in S} \omega_i.$$
- The *Core* is the set of unblocked allocations.
- Observation:
  - An allocation X is unblocked by $S = \{1, \ldots, I\}$ (coalition of the whole) iff X is efficient;
  - Inefficient allocations are blocked by $S = \{1, \ldots, I\}$;
  - Equilibrium must be in the core.

Some examples

- Three individual exchange economy

$$U^A = x^{1/2}y^{1/2}, \quad U^B = 2x^{1/2}y^{1/2}, \quad U^C = \min(x, y).$$

Some examples

- Three individual exchange economy

$$U^A = x^{1/2}y^{1/2}, \quad U^B = 2x^{1/2}y^{1/2}, \quad U^C = \min(x, y).$$

- Endowment
$\omega = \begin{bmatrix} 5 & 9 & 1 \\ 5 & 1 & 9 \end{bmatrix}$

Determine core allocations

- Three allocations:
$$X = \begin{bmatrix} 7 & 6 & 2 \\ 4 & 3 & 8 \end{bmatrix} \qquad X' = \begin{bmatrix} 7 & 4 & 4 \\ 7 & 4 & 4 \end{bmatrix}$$
$$X'' = \begin{bmatrix} 4 & 6 & 5 \\ 4 & 6 & 5 \end{bmatrix}$$

Determine core allocations

- Three allocations:
$X = \begin{bmatrix} 7 & 6 & 2 \\ 4 & 3 & 8 \end{bmatrix}$ $\quad X' = \begin{bmatrix} 7 & 4 & 4 \\ 7 & 4 & 4 \end{bmatrix}$
$X'' = \begin{bmatrix} 4 & 6 & 5 \\ 4 & 6 & 5 \end{bmatrix}$
- Are the 3 allocations in the core?

Determine core allocations

- Three allocations:
  $X = \begin{bmatrix} 7 & 6 & 2 \\ 4 & 3 & 8 \end{bmatrix}$ $\quad X' = \begin{bmatrix} 7 & 4 & 4 \\ 7 & 4 & 4 \end{bmatrix}$
  $X'' = \begin{bmatrix} 4 & 6 & 5 \\ 4 & 6 & 5 \end{bmatrix}$
- Are the 3 allocations in the core?
- If not, find a blocking coalition that will block it.

Core and equilibrium

- Theorem: If $\forall \imath$, $\succeq_i$ is locally non-satiated, every equilibrium is in the core.

Core and equilibrium

- Theorem: If $\forall \imath$, $\succeq_i$ is locally non-satiated, every equilibrium is in the core.
- *Proof.* Let (X^*) be an equilibrium, but there exists S and X':

Core and equilibrium

- Theorem: If $\forall \imath$, $\succeq_i$ is locally non-satiated, every equilibrium is in the core.
- *Proof.* Let (X^*) be an equilibrium, but there exists S and X':
  - $\forall\ i \in S$, $X'_i \succeq_i X^*_i$;

Core and equilibrium

- Theorem: If $\forall \imath$, $\succeq_i$ is locally non-satiated, every equilibrium is in the core.
- *Proof.* Let (X^*) be an equilibrium, but there exists S and X':
  - ➢ $\forall\ i \in S$, $X_i' \succeq_i X_i^*$;
  - ➢ $\exists\ i \in S$, $X_i' \succ X_i^*$;

Core and equilibrium

- Theorem: If $\forall \imath$, $\succeq_i$ is locally non-satiated, every equilibrium is in the core.
- *Proof.* Let (X^*) be an equilibrium, but there exists S and X':
  - ➢ $\forall\ i \in S$, $X'_i \succeq_i X^*_i$;
  - ➢ $\exists\ i \in S$, $X'_i \succ X^*_i$;
  - ➢ X' is feasible

$$\sum_{i \in S} X'_i \leq \sum_{i \in S} \omega_i.$$

Core and equilibrium

- Theorem: If $\forall \imath$, $\succeq_i$ is locally non-satiated, every equilibrium is in the core.
- *Proof.* Let (X^*) be an equilibrium, but there exists S and X':
  - ➢ $\forall\ i \in S$, $X_i' \succeq_i X_i^*$;
  - ➢ $\exists\ i \in S$, $X_i' \succ X_i^*$;
  - ➢ X' is feasible

$$\sum_{i \in S} X_i' \leq \sum_{i \in S} \omega_i.$$

- From first two conditions:

$$\forall\ i,\ X_i' \succeq_i X_i^* \Longrightarrow PX_i' \geq P\omega_i$$

$$\exists\ i,\ X_i' \succ X_i^* \Longrightarrow PX_i' > P\omega_i.$$

Core and equilibrium

- Theorem: If $\forall \imath$, $\succeq_i$ is locally non-satiated, every equilibrium is in the core.
- *Proof.* Let (X^*) be an equilibrium, but there exists S and X':
  - ➢ $\forall\ i \in S,\ X'_i \succeq_i X^*_i;$
  - ➢ $\exists\ i \in S,\ X'_i \succ X^*_i;$
  - ➢ X' is feasible
$$\sum_{i \in S} X'_i \leq \sum_{i \in S} \omega_i.$$
- From first two conditions:
$$\forall\ i,\ X'_i \succeq_i X^*_i \implies PX'_i \geq P\omega_i$$
$$\exists\ i,\ X'_i \succ X^*_i \implies PX'_i > P\omega_i.$$
- This implies $\sum_{i \in S} PX'_i > \sum_{i \in S} P\omega_i$.

Core and equilibrium

- Theorem: If $\forall \imath$, $\succeq_i$ is locally non-satiated, every equilibrium is in the core.
- *Proof.* Let (X^*) be an equilibrium, but there exists S and X':
  - ➢ $\forall\ i \in S,\ X'_i \succeq_i X^*_i$;
  - ➢ $\exists\ i \in S,\ X'_i \succ X^*_i$;
  - ➢ X' is feasible
$$\sum_{i \in S} X'_i \leq \sum_{i \in S} \omega_i.$$
- From first two conditions:
$$\forall\ i,\ \ X'_i \succeq_i X^*_i \Longrightarrow PX'_i \geq P\omega_i$$
$$\exists\ i,\ \ X'_i \succ X^*_i \Longrightarrow PX'_i > P\omega_i.$$
- This implies $\sum_{i \in S} PX'_i > \sum_{i \in S} P\omega_i$.
- Contradiction as:
$$\sum_{i \in S} X'_i \leq \sum_{i \in S} \omega_i \Longrightarrow \sum_{i \in S} PX'_i \leq \sum_{i \in S} P\omega_i.$$

Relationship between the 3 concepts

- Efficiency, core and equilibrium: $\{$Equilibrium allocations from $\omega\}\subseteq$ $\{$Core from $\omega\}\subseteq\{$Efficient allocation from $\omega\}$

Relationship between the 3 concepts

- Efficiency, core and equilibrium:
  {Equilibrium allocations from ω}$\subseteq$ {Core from ω}$\subseteq${Efficient allocation from ω}
- P.E. requires no waste of scare resources;

Relationship between the 3 concepts

- Efficiency, core and equilibrium:
  $\{$Equilibrium allocations from $\omega\}\subseteq$ $\{$Core from $\omega\}\subseteq\{$Efficient allocation from $\omega\}$
- P.E. requires no waste of scare resources;
- Core reflects the idea of voluntary exchange;

Relationship between the 3 concepts

- Efficiency, core and equilibrium: $\{$Equilibrium allocations from $\omega\} \subseteq \{$Core from $\omega\} \subseteq \{$Efficient allocation from $\omega\}$
- P.E. requires no waste of scare resources;
- Core reflects the idea of voluntary exchange;
- Equilibrium is achieved through market exchange.

Core of the example

Excess demand function

- Excess demand for i:

$$Z_i(P) = X_i(P, \omega_i) - \omega_i,$$

$X_i(P, \omega_i)$ is the maximal for $\succeq_i$ in $\{X_i | PX_i = P\omega_i\}$

Excess demand function

- Excess demand for i:

$$Z_i(P) = X_i(P, \omega_i) - \omega_i,$$

$X_i(P, \omega_i)$ is the maximal for $\succeq_i$ in $\{X_i | PX_i = P\omega_i\}$

- Aggregate excess demand

$$Z(P) = \sum_i Z_i(P).$$

Excess demand function

- Excess demand for i:

$$Z_i(P) = X_i(P, \omega_i) - \omega_i,$$

$X_i(P, \omega_i)$ is the maximal for $\succeq_i$ in $\{X_i | PX_i = P\omega_i\}$

- Aggregate excess demand

$$Z(P) = \sum_i Z_i(P).$$

- Walras' Law: If for all i, $\succeq_i$ satisfies LNS, then

$$PZ(P) = p(X(P) - \omega) = 0.$$

Excess demand function

- Excess demand for i:

$$Z_i(P) = X_i(P, \omega_i) - \omega_i,$$

$X_i(P, \omega_i)$ is the maximal for $\succeq_i$ in $\{X_i | PX_i = P\omega_i\}$

- Aggregate excess demand

$$Z(P) = \sum_i Z_i(P).$$

- Walras' Law: If for all i, $\succeq_i$ satisfies LNS, then

$$PZ(P) = p(X(P) - \omega) = 0.$$

- Competitive equilibrium: P^* such that $Z(P^*) = 0$.

Example 1: Cobb-Douglas

- Consumers A and B:

$$U_A = x_{1A}x_{2A} \qquad \omega_A = (4, 1)$$
$$U_B = x_{1B}x_{2B} \qquad \omega_B = (1, 4).$$

Example 1: Cobb-Douglas

- Consumers A and B:

$$U_A = x_{1A}x_{2A} \qquad \omega_A = (4, 1)$$
$$U_B = x_{1B}x_{2B} \qquad \omega_B = (1, 4).$$

- Excess demand for A

$$Z_{1A}(P) = \frac{4P_1 + P_2}{2P_1} - 4 = \frac{P_2}{2P_1} - 2;$$
$$Z_{2A}(P) = \frac{2P_1}{P_2} + \frac{1}{2} - 1 = \frac{2P_1}{P_2} - \frac{1}{2}.$$

Example 1: Cobb-Douglas

- Consumers A and B:

$$U_A = x_{1A}x_{2A} \qquad \omega_A = (4,1)$$
$$U_B = x_{1B}x_{2B} \qquad \omega_B = (1,4).$$

- Excess demand for A

$$Z_{1A}(P) = \frac{4P_1 + P_2}{2P_1} - 4 = \frac{P_2}{2P_1} - 2;$$
$$Z_{2A}(P) = \frac{2P_1}{P_2} + \frac{1}{2} - 1 = \frac{2P_1}{P_2} - \frac{1}{2}.$$

- Excess demand for B

$$Z_{1B}(P) = \frac{2P_2}{P_1} - \frac{1}{2}; Z_{2B}(P) = \frac{P_1}{2P_2} - 2.$$

Example 1: Cobb-Douglas

- Consumers A and B:

$$U_A = x_{1A}x_{2A} \qquad \omega_A = (4, 1)$$
$$U_B = x_{1B}x_{2B} \qquad \omega_B = (1, 4).$$

- Excess demand for A

$$Z_{1A}(P) = \frac{4P_1 + P_2}{2P_1} - 4 = \frac{P_2}{2P_1} - 2;$$
$$Z_{2A}(P) = \frac{2P_1}{P_2} + \frac{1}{2} - 1 = \frac{2P_1}{P_2} - \frac{1}{2}.$$

- Excess demand for B

$$Z_{1B}(P) = \frac{2P_2}{P_1} - \frac{1}{2}; Z_{2B}(P) = \frac{P_1}{2P_2} - 2.$$

- Aggregate excess demand

$$Z(P) = \begin{bmatrix} \frac{5P_2}{2P_1} - \frac{5}{2} \\ \frac{5P_1}{2P_2} - \frac{5}{2} \end{bmatrix}$$

Example 1: Cobb-Douglas

- Consumers A and B:

$$U_A = x_{1A}x_{2A} \qquad \omega_A = (4,1)$$
$$U_B = x_{1B}x_{2B} \qquad \omega_B = (1,4).$$

- Excess demand for A

$$Z_{1A}(P) = \frac{4P_1 + P_2}{2P_1} - 4 = \frac{P_2}{2P_1} - 2;$$
$$Z_{2A}(P) = \frac{2P_1}{P_2} + \frac{1}{2} - 1 = \frac{2P_1}{P_2} - \frac{1}{2}.$$

- Excess demand for B

$$Z_{1B}(P) = \frac{2P_2}{P_1} - \frac{1}{2}; Z_{2B}(P) = \frac{P_1}{2P_2} - 2.$$

- Aggregate excess demand
$Z(P) = \begin{bmatrix} \frac{5P_2}{2P_1} - \frac{5}{2} \\ \frac{5P_1}{2P_2} - \frac{5}{2} \end{bmatrix}$
- Walras' Law: $PZ(P) = 0$.

Robinson-Crusoe economy

- One consumer

$$U = x_1^{1/2} + x_2^{1/2}, \qquad \omega = (1, 1).$$

Robinson-Crusoe economy

- One consumer

$$U = x_1^{1/2} + x_2^{1/2}, \qquad \omega = (1,1).$$

- Excess demand
$$Z(P) = \begin{bmatrix} \frac{1+P_2/P_1}{1+P_1/P_2} - 1 \\ \frac{1+P_1/P_2}{1+P_2/P_1} - 1 \end{bmatrix} = \begin{bmatrix} \frac{P_2}{P_1} - 1 \\ \frac{P_1}{P_2} - 1 \end{bmatrix}$$

Robinson-Crusoe economy

- One consumer

$$U = x_1^{1/2} + x_2^{1/2}, \qquad \omega = (1, 1).$$

- Excess demand

$$Z(P) = \begin{bmatrix} \frac{1+P_2/P_1}{1+P_1/P_2} - 1 \\ \frac{1+P_1/P_2}{1+P_2/P_1} - 1 \end{bmatrix} = \begin{bmatrix} \frac{P_2}{P_1} - 1 \\ \frac{P_1}{P_2} - 1 \end{bmatrix}$$

- Walras' Law

$$PZ(P) = 0.$$

Example 3

- Consumers:

$$u_A = \min\{x_{1A}, x_{2A}\}, \qquad u_B = \min\{x_{1B}, x_{2B}\}$$

$$\omega_A = (4, 1), \qquad \omega_B = (1, 4).$$

Example 3

- Consumers:

$$u_A = \min\{x_{1A}, x_{2A}\}, \qquad u_B = \min\{x_{1B}, x_{2B}\}$$

$$\omega_A = (4,1), \qquad \omega_B = (1,4).$$

- Excess demand

$$Z(P) = \left[\begin{array}{c} 0 \\ 0 \end{array} \right]$$

Example 3

- Consumers:

$$u_A = \min\{x_{1A}, x_{2A}\}, \qquad u_B = \min\{x_{1B}, x_{2B}\}$$

$$\omega_A = (4, 1), \qquad \omega_B = (1, 4).$$

- Excess demand

$$Z(P) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- Walras' Law

$$PZ(P) = 0.$$

Example 3

- Consumers:

$$u_A = \min\{x_{1A}, x_{2A}\}, \qquad u_B = \min\{x_{1B}, x_{2B}\}$$

$$\omega_A = (4, 1), \qquad \omega_B = (1, 4).$$

- Excess demand

$$Z(P) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- Walras' Law

$$PZ(P) = 0.$$

- What are the equilibrium prices?

Example 4

- Consumers:

$$u_A = x_{1A}^{1/2} + x_{2A}^{1/2}, \qquad u_B = x_{1B}.$$

$$\omega_A = (0, 1), \qquad \omega_B = (1, 0).$$

Example 4

- Consumers:

$$u_A = x_{1A}^{1/2} + x_{2A}^{1/2}, \qquad u_B = x_{1B}.$$

$$\omega_A = (0,1), \qquad \omega_B = (1,0).$$

- Excess demand

$$Z(P) = \begin{bmatrix} \frac{P_2^2}{P_1P_2+P_1^2} \\ \frac{P_1^2}{P_1P_2+P_1^2} - 1 \end{bmatrix}$$

Example 4

- Consumers:

$$u_A = x_{1A}^{1/2} + x_{2A}^{1/2}, \qquad u_B = x_{1B}.$$

$$\omega_A = (0,1), \qquad \omega_B = (1,0).$$

- Excess demand

$$Z(P) = \begin{bmatrix} \frac{P_2^2}{P_1 P_2 + P_1^2} \\ \frac{P_1^2}{P_1 P_2 + P_1^2} - 1 \end{bmatrix}$$

- Does an equilibrium exist?

On existence of equilibrium

- Definition: A vector $P^* \in \mathbb{R}_{++}$ is called a Walrasian equilibrium if $Z(P^*) = 0$.

On existence of equilibrium

- Definition: A vector $P^* \in \mathbb{R}_{++}$ is called a Walrasian equilibrium if $Z(P^*) = 0$.
- Proposition 17B.2 (MWG): If $\forall i$, $X_i \in \mathbb{R}_+^L$, $\succeq_i$ is continuous, strictly convex and strongly monotonic, $\sum_i \omega_i \gg 0$, then there exists $Z : \mathbb{R}_+^L \to \mathbb{R}$

On existence of equilibrium

- Definition: A vector $P^* \in \mathbb{R}_{++}$ is called a Walrasian equilibrium if $Z(P^*) = 0$.
- Proposition 17B.2 (MWG): If $\forall i$, $X_i \in \mathbb{R}^L_+$, $\succeq_i$ is continuous, strictly convex and strongly monotonic, $\sum_i \omega_i \gg 0$, then there exists $Z : \mathbb{R}^L_+ \rightarrow \mathbb{R}$
    - ➢ Z is homogeneous of degree zero;

On existence of equilibrium

- Definition: A vector $P^* \in \mathbb{R}_{++}$ is called a Walrasian equilibrium if $Z(P^*) = 0$.
- Proposition 17B.2 (MWG): If $\forall i$, $X_i \in \mathbb{R}_+^L$, $\succeq_i$ is continuous, strictly convex and strongly monotonic, $\sum_i \omega_i \gg 0$, then there exists $Z : \mathbb{R}_+^L \to \mathbb{R}$
    - ➢ Z is homogeneous of degree zero;
    - ➢ Walras's Law: $(\forall\ P \in \mathbb{R}_+^L)$, $PZ(P) = 0$;

On existence of equilibrium

- Definition: A vector $P^* \in \mathbb{R}_{++}$ is called a Walrasian equilibrium if $Z(P^*) = 0$.
- Proposition 17B.2 (MWG): If $\forall i$, $X_i \in \mathbb{R}^L_+$, $\succeq_i$ is continuous, strictly convex and strongly monotonic, $\sum_i \omega_i \gg 0$, then there exists $Z : \mathbb{R}^L_+ \rightarrow \mathbb{R}$
  - ➢ Z is homogeneous of degree zero;
  - ➢ Walras's Law: $(\forall\ P \in \mathbb{R}^L_+)$, $PZ(P) = 0$;
  - ➢ $(\exists\ s \in \mathbb{R}_+)$, $(\forall\ l)$ $(\forall\ P)$, $Z_l(P) > -s$;

On existence of equilibrium

- Definition: A vector $P^* \in \mathbb{R}_{++}$ is called a Walrasian equilibrium if $Z(P^*) = 0$.
- Proposition 17B.2 (MWG): If $\forall i$, $X_i \in \mathbb{R}^L_+$, $\succeq_i$ is continuous, strictly convex and strongly monotonic, $\sum_i \omega_i \gg 0$, then there exists $Z : \mathbb{R}^L_+ \to \mathbb{R}$
  - ➢ Z is homogeneous of degree zero;
  - ➢ Walras's Law: $(\forall\ P \in \mathbb{R}^L_+)$, $PZ(P) = 0$;
  - ➢ $(\exists\ s \in \mathbb{R}_+)$, $(\forall\ l)$ $(\forall\ P)$, $Z_l(P) > -s$;
  - ➢ $(\forall (P^n)_{n=1}^{\infty} \to P \neq 0)$ where $P_l = 0$ for some l,

$$\max_l Z_l(P) \to +\infty.$$

On existence of equilibrium

- Definition: A vector $P^* \in \mathbb{R}_{++}$ is called a Walrasian equilibrium if $Z(P^*) = 0$.
- Proposition 17B.2 (MWG): If $\forall i$, $X_i \in \mathbb{R}^L_+$, $\succeq_i$ is continuous, strictly convex and strongly monotonic, $\sum_i \omega_i \gg 0$, then there exists $Z : \mathbb{R}^L_+ \to \mathbb{R}$
  - ➢ Z is homogeneous of degree zero;
  - ➢ Walras's Law: $(\forall\ P \in \mathbb{R}^L_+)$, $PZ(P) = 0$;
  - ➢ $(\exists\ s \in \mathbb{R}_+)$, $(\forall\ l)$ $(\forall\ P)$, $Z_l(P) > -s$;
  - ➢ $(\forall (P^n)_{n=1}^{\infty} \to P \neq 0)$ where $P_l = 0$ for some l,

$$\max_l Z_l(P) \to +\infty.$$

- Proposition 17C.1 (MWG): A Walrasian equilibrium exists in any pure exchange economy in which $\sum_i \omega_i \gg 0$ and $\forall i$, $X_i \in \mathbb{R}^L_+$, $\succeq_i$ is continuous, strictly convex and strongly monotonic.