Problem Set 3 solutions

Additional Problem 1. Consider an economy with two goods, good l and a numeraire good, 2 consumers, $i = 1, 2$, and 2 firms, $j = 1, 2$. Let $x_i \in \mathbb{R}_+$ denote consumer $i's$ consumption of good l, and $m_i \in \mathbb{R}$ her consumption of the numeraire. The utility function of each consumer i is given by:

$$\begin{aligned} u_i(m_i, x_i) &= m_i + 1 - \frac{(1-x_i)^2}{2}, \text{ if } 0 \leq x_i \leq 1 \\ &= m_i + 1, \text{ if } x_i > 1. \end{aligned}$$

Each firm j produces $q_j \geq 0$ units of good l using amount $c_j(q_j)$ of the numeraire where

$$c_j(q_j) = (q_j)^2.$$

Each consumer has an initial endowment of 50 units of the numeraire and owns $\frac{1}{2}$ share of each firm.

(a) Derive the competitive equilibrium.
(b) Write down the set of all the efficient allocations in this economy.

Answer.
(a) Let $p > 0$ be the price of good l. We first derive the demand curve for good l. Note that consuming more than 1 unit of good l cannot be utility-maximizing for each consumer. Each consumer i's marginal utility from good l is $1 - x_i$ when $x_i \in [0, 1]$. Hence his demand is

$$x_i(p) = 1 - p \text{ if } p \in (0, 1].$$

The aggregate demand for good l is

$$x(p) = 2 - 2p \text{ if } p \in (0, 1].$$

Next we find the supply. Profit maximization implies that each firm's marginal cost is equal to the price (corner solution is not possible since $c_1'(0) = c_2'(0) = 0 < p$). That is, $p = 2q_1 = 2q_2$. Hence

$$q_1(p) = q_2(p) = \frac{p}{2}$$

and the aggregate supply for good l is

$$q(p) = p.$$

Equate supply and demand for good l:

$$2 - 2p = p \quad \Rightarrow p^* = \frac{2}{3}.$$

Given the competitive equilibrium price p^*, we have $x_i^* = \frac{1}{3}$ for each i, and $q_j^* = \frac{1}{3}$ for each j.

Each firm's profit is $\frac{2}{3} \cdot \frac{1}{3} - (\frac{1}{3})^2 = \frac{1}{9}$. Then each consumer i receives a profit of $\frac{1}{9}$ from the two firms. His expenditure on good l is $\frac{2}{9}$. Since i is also endowed with 50 units of the numeraire, we have $m_i^* = 50 + \frac{1}{9} - \frac{2}{9} = 49\frac{8}{9}$.

Therefore, the competitive equilibrium consists of the allocation $(x_1^* = \frac{1}{3}, x_2^* = \frac{1}{3}, q_1^* = \frac{1}{3}, q_2^* = \frac{1}{3}, m_1^* = 49\frac{8}{9}, m_2^* = 49\frac{8}{9})$ and the price $p^* = \frac{2}{3}$.

(b) Given that $m_1^* + m_2^* = 99\frac{7}{9}$, the set of efficient allocations is

$$\left\{(x_1, x_2, q_1, q_2, m_1, m_2) : x_1 = x_2 = q_1 = q_2 = \frac{1}{3}, m_1 + m_2 = 99\frac{7}{9}\right\}.$$

Additional Problem 2.

(a) Suppose that the preference relation $\succeq$ on the set of lotteries $\mathscr{L}$ is complete. Show that $\succeq$ satisfies the independence axiom if and only if the following property holds:

For all $L_1, L_2, L_3 \in \mathscr{L}$ *and* $\alpha \in (0,1)$, $L_1 \succ L_2$ *if and only if* $\alpha L_1 + (1-\alpha)L_3 \succ \alpha L_2 + (1-\alpha)L_3$.

(b) Show that if $\succeq$ on $\mathscr{L}$ satisfies the independence axiom, then the following property holds:

For all $L_1, L_2, L_3 \in \mathscr{L}$ *and* $\alpha \in (0,1)$, $L_1 \sim L_2$ *if and only if* $\alpha L_1 + (1-\alpha)L_3 \sim \alpha L_2 + (1-\alpha)L_3$.

Answer.

(a). "only if" part. Consider any $L_1, L_2, L_3 \in \mathscr{L}$ and $\alpha \in (0,1)$, we have

$$\begin{aligned} L_1 \nsucc L_2 &\Leftrightarrow L_2 \succeq L_1 \\ &\Leftrightarrow \alpha L_2 + (1-\alpha)L_3 \succeq \alpha L_1 + (1-\alpha)L_3 \\ &\Leftrightarrow \alpha L_1 + (1-\alpha)L_3 \nsucc \alpha L_2 + (1-\alpha)L_3 \end{aligned}$$

where the second equivalence relation follows from the independence axiom, the first and the third equivalence relation follow from the completeness of $\succeq$ as well as the definition of $\succ$.

Then, clearly,

$$L_1 \nsucc L_2 \Leftrightarrow \alpha L_1 + (1-\alpha)L_3 \nsucc \alpha L_2 + (1-\alpha)L_3$$

implies

$$L_1 \succ L_2 \Leftrightarrow \alpha L_1 + (1-\alpha)L_3 \succ \alpha L_2 + (1-\alpha)L_3$$

"if" part. The proof is similar. Consider any $L_1, L_2, L_3 \in \mathscr{L}$ and $\alpha \in (0,1)$, we have

$$\begin{aligned} L_1 \nsucceq L_2 &\Leftrightarrow L_2 \succ L_1 \\ &\Leftrightarrow \alpha L_2 + (1-\alpha)L_3 \succ \alpha L_1 + (1-\alpha)L_3 \\ &\Leftrightarrow \alpha L_1 + (1-\alpha)L_3 \nsucceq \alpha L_2 + (1-\alpha)L_3 \end{aligned}$$

where the second equivalence relation follows from the property stated in part (a), the first and the third equivalence relation follow from the completeness of $\succeq$ as well as the definition of $\succ$.

Then, clearly,

$$L_1 \nsucceq L_2 \Leftrightarrow \alpha L_1 + (1-\alpha)L_3 \nsucceq \alpha L_2 + (1-\alpha)L_3$$

implies

$$L_1 \succeq L_2 \Leftrightarrow \alpha L_1 + (1-\alpha)L_3 \succeq \alpha L_2 + (1-\alpha)L_3$$

(b). If $L_1 \sim L_2$, then $L_1 \succeq L_2$ and $L_2 \succeq L_1$. By the independence axiom,

$$L_1 \succeq L_2 \Rightarrow \alpha L_1 + (1-\alpha)L_3 \succeq \alpha L_2 + (1-\alpha)L_3$$

$$L_2 \succeq L_1 \Rightarrow \alpha L_2 + (1-\alpha)L_3 \succeq \alpha L_1 + (1-\alpha)L_3$$

Therefore, $\alpha L_1 + (1-\alpha)L_3 \sim \alpha L_2 + (1-\alpha)L_3$.

On the other hand, if $\alpha L_1 + (1-\alpha)L_3 \sim \alpha L_2 + (1-\alpha)L_3$, then $\alpha L_1 + (1-\alpha)L_3 \succeq \alpha L_2 + (1-\alpha)L_3$ and $\alpha L_2 + (1-\alpha)L_3 \succeq \alpha L_1 + (1-\alpha)L_3$. By the independence axiom,

$$\alpha L_1 + (1-\alpha)L_3 \succeq \alpha L_2 + (1-\alpha)L_3 \Rightarrow L_1 \succeq L_2$$

$$\alpha L_2 + (1-\alpha)L_3 \succeq \alpha L_1 + (1-\alpha)L_3 \Rightarrow L_2 \succeq L_1$$

Therefore $L_1 \sim L_2$.