Overreaction in Macroeconomic Expectations
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Appendix: For Online Publication Only

A. Proofs

Proposition 1. The data generating process is $x_t = \rho x_{t-1} + u_t$, where $u_t \sim \mathcal{N}(0, \sigma_u^2)$ i.i.d. over time and $\rho > 0$. Forecaster i observes a noisy signal $s_t^i = x_t + \epsilon_t^i$, where $\epsilon_t^i \sim \mathcal{N}(0, \sigma_\epsilon^2)$ is i.i.d. analyst specific noise. Rational expectations are obtained iteratively:

$$f(x_t|S_t^i) = f(x_t|S_{t-1}^i)\frac{f(s_t^i|x_t)}{f(s_t^i)}$$

The rational estimate thus follows $f(x_t|S_t^i) \sim \mathcal{N}\left(x_{t|t}^i, \frac{\Sigma_{t|t-1}\sigma_\epsilon^2}{\Sigma_{t|t-1}+\sigma_\epsilon^2}\right)$ with

$$x_{t|t}^i = x_{t|t-1}^i + \frac{\Sigma_{t|t-1}}{\Sigma_{t|t-1}+\sigma_\epsilon^2}\left(s_t^i - x_{t|t-1}^i\right),$$

where $\Sigma_{t|t-1}$ is the variance of the prior $f(x_t|S_{t-1}^i)$. The variance of $f(x_{t+1}|S_t^i)$ is:

$$\Sigma_{t+1|t} \equiv var_t(\rho x_t + u_{t+1}) = \rho^2\frac{\Sigma_{t|t-1}\sigma_\epsilon^2}{\Sigma_{t|t-1}+\sigma_\epsilon^2} + \sigma_u^2,$$

where the steady state variance $\Sigma = \Sigma_{t+1|t} = \Sigma_{t|t-1}$ is equal to:

$$\Sigma = \frac{-(1-\rho^2)\sigma_\epsilon^2 + \sigma_u^2 + \sqrt{[(1-\rho^2)\sigma_\epsilon^2 - \sigma_u^2]^2 + 4\sigma_\epsilon^2\sigma_u^2}}{2}$$

Beliefs about the current state are then described by $f(x_t|S_t^i) \sim \mathcal{N}\left(x_{t|t}^i, \frac{\Sigma\sigma_\epsilon^2}{\Sigma+\sigma_\epsilon^2}\right)$, where:

$$x_{t|t}^i = x_{t|t-1}^i + \frac{\Sigma}{\Sigma+\sigma_\epsilon^2}\left(s_t^i - x_{t|t-1}^i\right)$$

Note that there is a discontinuity at $\rho = 0$ for contemporaneous beliefs in steady state. When shocks have zero persistence, steady state beliefs are constant and described by $f(x_t|S_t^i) \sim \mathcal{N}(0, \Sigma)$ where the steady state variance is $\Sigma = \sigma_u^2$. In particular, the contemporaneous Kalman gain is zero.[1] This also implies that for $\rho = 0$ there are no diagnosticity distortions, because $f(x_t|S_t^i) = f\left(x_t|S_{t-1}^i \cup \{x_{t|t-1}^i\}\right)$, so that $f^\theta(x_t|S_t^i) = f(x_t|S_t^i)$.

Let us now construct diagnostic expectations for $\rho > 0$. For $s_t^i = x_{t|t-1}^i$ we have $x_{t|t}^i = x_{t|t-1}^i = \rho x_{t-1|t-1}^i$, so that $f\left(x_t|S_{t-1}^i \cup \{x_{t|t-1}^i\}\right) \sim \mathcal{N}\left(\rho x_{t-1|t-1}^i, \frac{\Sigma\sigma_\epsilon^2}{\Sigma+\sigma_\epsilon^2}\right)$. In light of the definition of diagnostic expectations in Equation (7), we have that the diagnostic distribution $f^\theta(x_t|S_t^i)$ fulfils:

[1] Expectations for future realizations satisfy $x_{t+h|t}^i = \rho^h x_{t|t-1}^i + \rho^h\frac{\Sigma}{\Sigma+\sigma_\epsilon^2}\left(s_t^i - x_{t|t-1}^i\right)$. The Kalman gain, and therefore the expectations, are now continuous at $\rho = 0$.

$$\ln f^{\theta}\left(x_t|S_t^i\right) \propto -\frac{\left(x_t - x_{t|t}^i\right)^2}{2\frac{\Sigma\sigma_\epsilon^2}{\Sigma+\sigma_\epsilon^2}} - \theta\frac{\left(x_t - x_{t|t}^i\right)^2 - \left(x_t - x_{t|t-1}^i\right)^2}{2\frac{\Sigma\sigma_\epsilon^2}{\Sigma+\sigma_\epsilon^2}}$$

$$= -\frac{1}{2\frac{\Sigma\sigma_\epsilon^2}{\Sigma+\sigma_\epsilon^2}}\left[x_t^2 - 2x_t\left(x_{t|t}^i + \theta\left(x_{t|t}^i - x_{t|t-1}^i\right)\right) + \left(x_{t|t}^i\right)^2(1+\theta) - \theta\left(x_{t|t-1}^i\right)^2\right]$$

Given the normalization $\int f^{\theta}\left(x|S_t^i\right)dx = 1$, we find $f^{\theta}\left(x_t|S_t^i\right) \sim \mathcal{N}\left(x_{t|t}^{i,\theta}, \frac{\Sigma\sigma_\epsilon^2}{\Sigma+\sigma_\epsilon^2}\right)$ with $x_{t|t}^{i,\theta} = x_{t|t}^i + \theta\left(x_{t|t}^i - x_{t|t-1}^i\right)$. Using the definition of the Kalman filter $x_{t|t}^i$ we can write:

$$x_{t|t}^{i,\theta} = x_{t|t-1}^i + (1+\theta)\frac{\Sigma}{\Sigma+\sigma_\epsilon^2}\left(s_t^i - x_{t|t-1}^i\right). \blacksquare$$

Proposition 2. Denote by $K = \Sigma/(\Sigma + \sigma_\epsilon^2)$ the contemporaneous Kalman gain for $\rho > 0$. The rational consensus estimate for the current state is then equal to $\int x_{t|t}^i\, di \equiv x_{t|t} = x_{t|t-1} + K\left(x_t - x_{t|t-1}\right)$. The consensus forecast error under rationality is then equal to $x_t - x_{t|t} = \frac{1-K}{K}\left(x_{t|t} - x_{t|t-1}\right)$. The diagnostic filter for an individual analyst is equal to $x_{t|t}^{i,\theta} = x_{t|t}^i + \theta\left(x_{t|t}^i - x_{t|t-1}^i\right)$, which implies a consensus equation $x_{t|t}^{\theta} = x_{t|t} + \theta\left(x_{t|t} - x_{t|t-1}\right)$. We thus have:

$$x_t - x_{t|t}^{\theta} = \left(\frac{1-K}{K} - \theta\right)\left(x_{t|t} - x_{t|t-1}\right).$$

Note, in addition, that the diagnostic consensus forecast revision is equal to:

$$x_{t|t}^{\theta} - x_{t|t-1}^{\theta} = (1+\theta)\left(x_{t|t} - x_{t|t-1}\right) - \theta\rho\left(x_{t-1|t-1} - x_{t-1|t-2}\right).$$

Therefore, the consensus CG coefficient is given by:

$$\beta = \frac{cov\left(x_{t+h} - x_{t+h|t}^{\theta}, x_{t+h|t}^{\theta} - x_{t+h|t-1}^{\theta}\right)}{var\left(x_{t+h|t}^{\theta} - x_{t+h|t-1}^{\theta}\right)}$$

$$= \left(\frac{1-K}{K} - \theta\right)\cdot\frac{cov\left[x_{t|t} - x_{t|t-1}, (1+\theta)\left(x_{t|t} - x_{t|t-1}\right) - \theta\rho\left(x_{t-1|t-1} - x_{t-1|t-2}\right)\right]}{var\left[(1+\theta)\left(x_{t|t} - x_{t|t-1}\right) - \theta\rho\left(x_{t-1|t-1} - x_{t-1|t-2}\right)\right]}.$$

Where we have that:

$$cov\left[x_{t|t} - x_{t|t-1}, (1+\theta)\left(x_{t|t} - x_{t|t-1}\right) - \theta\rho\left(x_{t-1|t-1} - x_{t-1|t-2}\right)\right]$$
$$= (1+\theta)var\left(x_{t|t} - x_{t|t-1}\right) - \theta\rho cov\left(x_{t|t} - x_{t|t-1}, x_{t-1|t-1} - x_{t-1|t-2}\right),$$

and

$$var\left[(1+\theta)\left(x_{t|t} - x_{t|t-1}\right) - \theta\rho\left(x_{t-1|t-1} - x_{t-1|t-2}\right)\right]$$
$$= \left[(1+\theta)^2 + \theta^2\rho^2\right]var\left(x_{t|t} - x_{t|t-1}\right)$$
$$- 2\theta(1+\theta)\rho cov\left(x_{t|t} - x_{t|t-1}, x_{t-1|t-1} - x_{t-1|t-2}\right).$$

To compute the covariance between adjacent rational revisions, note that $x_{t|t} = x_{t|t-1} + K\left(x_t - x_{t|t-1}\right)$ and $x_{t|t-1} = x_{t|t-2} + K\left(\rho x_{t-1} - x_{t|t-2}\right)$ imply that:

$$x_{t|t} - x_{t|t-1} = (1-K)\rho\left(x_{t-1|t-1} - x_{t-1|t-2}\right) + Ku_t.$$

As a result,

$$cov\left(x_{t|t} - x_{t|t-1}, x_{t-1|t-1} - x_{t-1|t-2}\right) = (1-K)\rho \cdot var\left(x_{t|t} - x_{t|t-1}\right)$$

Therefore:

$$\beta = \left(\frac{1-K}{K} - \theta\right) \cdot \frac{(1+\theta) - \theta\rho^2(1-K)}{[(1+\theta)^2 + \theta^2\rho^2] - 2\theta(1+\theta)\rho^2(1-K)},$$

which is positive if and only if $1 - K > \theta K$, namely, $\theta < \sigma_\epsilon^2/\Sigma$.

Consider individual level forecasts. The coefficient (at the individual level) of regressing forecast error on forecast revision is equal to:

$$\beta^p = \frac{cov\left(x_{t+h} - x_{t+h|t}^{i,\theta}, x_{t+h|t}^{i,\theta} - x_{t+h|t-1}^{i,\theta}\right)}{var\left(x_{t+h|t}^{i,\theta} - x_{t+h|t-1}^{i,\theta}\right)} =$$

$$\frac{cov\left(x_{t|t} - x_{t|t}^{i,\theta}, x_{t|t}^{i,\theta} - x_{t|t-1}^{i,\theta}\right)}{var\left(x_{t|t}^{i,\theta} - x_{t|t-1}^{i,\theta}\right)}$$

where $x_{t|t}^{i,\theta} - x_{t|t-1}^{i,\theta} = (1+\theta)\left(x_{t|t}^{i} - x_{t|t-1}^{i}\right) - \theta\rho\left(x_{t-1|t-1}^{i} - x_{t-1|t-2}^{i}\right)$. Because at the individual level $cov\left(x_{t|t}^{i} - x_{t|t-1}^{i}, x_{t|t-1}^{i} - x_{t|t-2}^{i}\right) = 0$, we immediately have that:

$$\beta^p = -\frac{\theta(1+\theta)}{(1+\theta)^2 + \rho^2\theta^2}.$$

Overreaction is larger (β^p is more negative) for series with lower persistence. Intuitively, when persistence is low, rational beliefs respond less to news (the denominator $var\left(x_{t|t}^{i,\theta} - x_{t|t-1}^{i,\theta}\right)$ is smaller) and there is more scope for overreaction.

For completeness, consider the case of $\rho = 0$. In this case, all forecasters hold the same beliefs, which are independent of their idiosynchratic signals s_t^i. Thus, consensus and individual forecasts are the same, $x_{t|t}^{i,\theta} = x_{t|t}^{\theta}$. Moreover, these forecasts are not revised (as under the rational benchmark) so that the CG coefficients are zero. Thus, because contemporaneous beliefs are discontinuous at $\rho = 0$, so are the CG coefficients.

Finally, we extend the analysis to the case where the degree of diagnosticity varies across forecasters, so that forecaster i's beliefs are given by

$$x_{t|t}^{i,\theta} = x_{t|t}^{i} + \theta^i\left(x_{t|t}^{i} - x_{t|t-1}^{i}\right)$$

Consider first consensus beliefs. We have:

$$\frac{1}{I}\sum_i x_{t|t}^{i,\theta} = x_{t|t} + \frac{1}{I}\sum_i \theta^i\left(x_{t|t}^{i} - x_{t|t-1}^{i}\right)$$

where I denotes the number of forecasters. Because the revision $x_{t|t}^{i} - x_{t|t-1}^{i}$ is uncorrelated with θ^i, then for large I the consensus becomes $x_{t|t} + \theta\left(x_{t|t} - x_{t|t-1}\right)$ as in the case of homogeneous forecasters. As a consequence, Equation (12) goes through.

Consider now the effect of pooling heterogeneous forecasters on the individual level CG coefficient. To do so, write $FR_t^{i,\theta} = x_{t|t}^{i,\theta} - x_{t|t-1}^{i,\theta}$ and $FE_t^{i,\theta} = x_t - x_{t|t}^{i,\theta}$, where $t = 1, \dots T$, as well as $FR^{\theta} = \frac{1}{I}\frac{1}{T}\sum_i \sum_t FR_t^{i,\theta}$ and $FE^{\theta} = \frac{1}{I}\frac{1}{T}\sum_i \sum_t FE_t^{i,\theta}$. In a pooled estimation, we have:

$$\beta_1^p = \frac{\sum_i \sum_t \left(FR_t^{i,\theta} - FR^{\theta}\right)\left(FE_t^{i,\theta} - FE^{\theta}\right)}{\sum_i \sum_t \left(FR_t^{i,\theta} - FR^{\theta}\right)^2}$$

Because the series of shocks is uncorrelated with forecaster heterogeneity, this can be written as:

$$\beta_1^p = \frac{\sum_i \beta_1^i var_t\left(FR_t^{i,\theta}\right)}{var\left(FR^{i,\theta}\right) + \sum_i var_t\left(FR_t^{i,\theta}\right)} + \frac{cov\left(FR^{i,\theta}, FE^{i,\theta}\right)}{var\left(FR^{i,\theta}\right) + \sum_i var_t\left(FR_t^{i,\theta}\right)}$$

where β_1^i is the coefficient of the CG regression on forecaster i, and $FR^{i,\theta}, FE^{i,\theta}$ are the (time) average forecast error and forecast revision of forecaster i.

Clearly, in the case of homogeneous forecasters, this coefficient is unbiased. However, under heterogeneity two forces bias the coefficient upwards, towards zero, provided forecasters differ in their average forecast revision, $var\left(FR^{i,\theta}\right) > 0$. When this is the case, then the first term, which pools the individual β_1^is, is dampened below a weighted average of the latter. Heterogeneity in the size of forecast revisions directly dampens the role of individual overreaction in the pooled estimate because the pooled variance is now larger than the sum of individual variances. Second, to the extent that they are positively correlated, heterogeneity in forecast revisions and errors also pushes up the pooled coefficient. This is the usual heterogeneity mechanism whereby forecasters who are more optimistic make both more positive mistakes and more positive revisions, leading to a spurious positive correlation between revision and error in the pooled sample.

Thus, in general forecaster heterogeneity biases the pooled estimates against our predictions. Equivalently, to find negative coefficients in a pooled estimate it is necessary that (sufficiently many) forecasters overreact and have negative β_1^i.■

Corollary 1. Denote by p_i the precision of the private signal, by p the precision of the public signal, by p_f the precision of the lagged rational forecast $x_{t|t-1}^i$. The diagnostic filter at time t is:

$$x_{t|t}^{i,\theta} = x_{t|t-1}^i + (1+\theta)\frac{p_i}{p_i + p + p_f}\left(s_t^i - x_{t|t-1}^i\right) + (1+\theta)\frac{p}{p_i + p + p_f}\left(s_t - x_{t|t-1}^i\right).$$

The precision p_f of the forecast depends on the sum of the precisions $(p_i + p)$ and hence stays constant as we vary the relative precision of the public versus private signal.

Denote the Kalman gains as $K_1 = \frac{p_i}{p_i+p+p_f}$ and $K_2 = \frac{p}{p_i+p+p_f}$, and $K = K_1 + K_2$. The consensus Kalman filter can then be written as $x_{t|t} = x_{t|t-1} + K\left(x_t - x_{t|t-1}\right) + K_2 v_t$, while the diagnostic filter can be written as $x_{t|t}^{\theta} = x_{t|t} + \theta\left(x_{t|t} - x_{t|t-1}\right)$. The consensus coefficient is then:

$$\frac{cov\left(x_{t+h} - x_{t+h|t}^{\theta}, x_{t+h|t}^{\theta} - x_{t+h|t-1}^{\theta}\right)}{var\left(x_{t+h|t}^{\theta} - x_{t+h|t-1}^{\theta}\right)} = \frac{\rho^{2h} cov\left(x_t - x_{t|t}^{\theta}, x_{t|t}^{\theta} - x_{t|t-1}^{\theta}\right)}{\rho^{2h} var\left(x_{t|t}^{\theta} - x_{t|t-1}^{\theta}\right)}.$$

Consider first the numerator. Denote by $FR_t \equiv x_{t|t} - x_{t|t-1}$ the revision of the rational forecast of x_t between t and $t-1$. Then:

$$x_t - x_{t|t}^{\theta} = \left(\frac{1-K}{K} - \theta\right) FR_t - \frac{K_2}{K} v_t,$$

$$x_{t|t}^{\theta} - x_{t|t-1}^{\theta} = (1+\theta) FR_t - \theta\rho FR_{t-1}.$$

The difference between $x_{t|t} = x_{t|t-1} + K\left(x_t - x_{t|t-1}\right) + K_2 v_t$ and $x_{t|t-1} = x_{t|t-2} + K\left(\rho x_{t-1} - x_{t|t-2}\right) + K_2 \rho v_{t-1}$ reads:

$$FR_t = (1-K)\rho FR_{t-1} + K u_t + K_2(v_t - \rho v_{t-1}),$$

which in turn implies:

$$cov(FR_t, FR_{t-1}) = (1-K)\rho \cdot var(FR_t) - \rho K_2^2 \sigma_v^2. \qquad (A.1)$$

It is also immediate to find that:

$$var(FR_t) = \frac{K^2\sigma_u^2 + [(1+\rho^2) - 2\rho^2(1-K)]K_2^2\sigma_v^2}{1 - [(1-K)\rho]^2}.$$

The numerator of the CG coefficient is then equal to:

$$cov\left(x_t - x_{t|t}^{\theta}, x_{t|t}^{\theta} - x_{t|t-1}^{\theta}\right) = \left(\frac{1-K}{K} - \theta\right) cov[FR_t, (1+\theta)FR_t - \theta\rho FR_{t-1}] - \frac{K_2}{K}(1+\theta)K_2\sigma_v^2$$

$$= \left(\frac{1-K}{K} - \theta\right)\left[[1 + \theta - \theta\rho^2(1-K)]var(FR_t) + \theta\rho^2 K_2^2\sigma_v^2\right] - \frac{(1+\theta)K_2^2\sigma_v^2}{K} \qquad (A.2)$$

The denominator of the CG coefficient equals:

$$var\left(x_{t|t}^{\theta} - x_{t|t-1}^{\theta}\right) = var[(1+\theta)FR_t - \theta\rho FR_{t-1}]$$

$$= [(1+\theta)^2 + \theta^2\rho^2]var(FR_t) - 2\theta(1+\theta)\rho cov(FR_t, FR_{t-1})$$

which implies that:

$$\frac{var\left(x_{t|t}^{\theta} - x_{t|t-1}^{\theta}\right)}{[(1+\theta)^2 + \theta^2\rho^2]} + \frac{2\theta(1+\theta)\rho}{[(1+\theta)^2 + \theta^2\rho^2]} cov(FR_t, FR_{t-1}) = var(FR_t). \qquad (A.3)$$

Putting $(A.3)$ together with $(A.1)$ one obtains:

$$cov(FR_t, FR_{t-1}) =$$

$$= \frac{(1-K)\rho var\left(x_{t|t}^{\theta} - x_{t|t-1}^{\theta}\right)}{\left[1 - \frac{2\theta(1-K)(1+\theta)\rho^2}{[(1+\theta)^2 + \theta^2\rho^2]}\right][(1+\theta)^2 + \theta^2\rho^2]} - \frac{\rho K_2^2\sigma_v^2}{\left[1 - \frac{2\theta(1-K)(1+\theta)\rho^2}{[(1+\theta)^2 + \theta^2\rho^2]}\right]} \qquad (A.4)$$

Using Equations $(A.2)$ and $(A.4)$ we find:

$$cov\left(x_t - x_{t|t}^{\theta}, x_{t|t}^{\theta} - x_{t|t-1}^{\theta}\right)$$

$$= \left(\frac{1-K}{K} - \theta\right)\left[(1+\theta)\frac{var\left(x_{t|t}^{\theta} - x_{t|t-1}^{\theta}\right)}{(1+\theta)^2 + \theta^2\rho^2}\right.$$

$$\left. + \theta\rho\left(\frac{2(1+\theta)^2}{(1+\theta)^2 + \theta^2\rho^2} - 1\right) cov(FR_t, FR_{t-1})\right] - \frac{(1+\theta)K_2^2\sigma_v^2}{K} =$$

$$= \beta_\infty var\left(x^\theta_{t|t} - x^\theta_{t|t-1}\right) - K_2^2\sigma_v^2 \left[\frac{\theta\rho^2\left(\frac{1-K}{K}-\theta\right)\left(\frac{2(1+\theta)^2}{(1+\theta)^2+\theta^2\rho^2}-1\right)}{\left[1-\frac{2\theta(1-K)(1+\theta)\rho^2}{(1+\theta)^2+\theta^2\rho^2}\right]} + \frac{(1+\theta)}{K}\right],$$

where β_∞ is the consensus coefficient obtained when the public signal is fully uninformative, namely $\sigma_u^2 \to \infty$ and thus $K_2 \to 0$. On the other hand using equation (A.3) this can be rewritten as:

$$var\left(x^\theta_{t|t} - x^\theta_{t|t-1}\right) = \frac{[(1+\theta)^2+\theta^2\rho^2 - 2\theta(1+\theta)(1-K)\rho^2]K^2\sigma_u^2}{1-[(1-K)\rho]^2} + AK_2^2\sigma_v^2,$$

where A is a suitable positive coefficient. The CG coefficient is then equal to:

$$\frac{cov\left(x_t - x^\theta_{t|t}, x^\theta_{t|t} - x^\theta_{t|t-1}\right)}{var\left(x^\theta_{t|t} - x^\theta_{t|t-1}\right)} = \beta_\infty - \frac{\left[\frac{\theta\rho^2\left(\frac{1-K}{K}-\theta\right)\left(\frac{2(1+\theta)^2}{(1+\theta)^2+\theta^2\rho^2}-1\right)}{1-\frac{2\theta(1-K)(1+\theta)\rho^2}{(1+\theta)^2+\theta^2\rho^2}} + \frac{(1+\theta)}{K}\right]K_2^2\sigma_v^2}{\frac{[(1+\theta)^2+\theta^2\rho^2-2\theta(1+\theta)(1-K)\rho^2]K^2\sigma_u^2}{1-[(1-K)\rho]^2} + AK_2^2\sigma_v^2}.$$

For given total informativeness K, the above expression falls in the precision of the public signal, namely as K_2^2 grows, if and only if:

$$\left[\frac{\theta\rho^2\left(\frac{1-K}{K}-\theta\right)\left(\frac{2(1+\theta)^2}{(1+\theta)^2+\theta^2\rho^2}-1\right)}{1-\frac{2\theta(1-K)(1+\theta)\rho^2}{(1+\theta)^2+\theta^2\rho^2}} + \frac{(1+\theta)}{K}\right] > 0.$$

A sufficient condition for this to hold is that $\left(\frac{1-K}{K} - \theta\right) > 0$, which is equivalent to $\beta_\infty > 0$.

■

Lemma A.1 *Suppose that forecasters observe individual signals as well as the lagged variable. Formally, they receive a signal vector* $\left(s_t^i, y_t^i\right)$ *given by:*

$$\begin{cases} s_t^i = x_t + \epsilon_t^i \\ y_t = x_{t-1} \end{cases}$$

Then the individual CG coefficient is given by Equation (12), and there exists a positive threshold θ^* *such that the consensus CG coefficient is positive for* $\theta \in [0, \theta^*)$ *and negative for* $\theta > \theta^*$.

Proof Consider first updating under rational expectations. After observing $\left(s_{t-1}^i, y_{t-1}^i\right)$ at $t-1$, forecaster i's belief about x_{t-1} is normal with mean

$$x^i_{t-1|t-1} = \rho x_{t-2} + \frac{\sigma_u^2}{\sigma_u^2+\sigma_\epsilon^2}\left(s_{t-1}^i - \rho x_{t-2}\right)$$

and variance σ_ϵ^2 (because uncertainty about x_{t-1} is restricted to uncertainty about u_t which can be written $u_t = s_t^i - \rho x_{t-1} - \epsilon_t^i$). In fact, under rational expectations beliefs are invariant over the timing of the signal, and can be easily derived in the specification where the individual signal about the current state s_{t-1}^i follows the fully revealing signal about the lagged state x_{t-2}.

Consider now diagnostic expectations, in which the believed probability of a current realization x_t is distorted by its representativeness relative to news at t. From Equation (6), we have:

$$R(x_t) = \frac{f\left(x_t | (s_t^i, y_t)\right)}{f\left(x_t | (\rho x_{t-1|t-1}^i, x_{t-1|t-1}^i)\right)}$$

Here $(\rho x_{t-1|t-1}^i, x_{t-1|t-1}^i)$ is the signal at t that is expected at $t-1$, and the expression highlights the fact that, because the lagged state is fully revealed, forecasters optimally ignore any previous signals. $R(x_t)$ compares two normal distributions characterized by the same variance, namely σ_ϵ^2, but different means, namely $x_{t|t}^i$ and $\rho x_{t-1|t-1}^i$. Diagnostic beliefs $f^\theta\left(x_t | (s_t^i, y_t^i)\right)$, defined by Equation (7), are then normally distributed with variance σ_ϵ^2 and mean:

$$x_{t|t}^{i,\theta} = x_{t|t}^i + \theta\left(x_{t|t}^i - \rho x_{t-1|t-1}^i\right) = \rho x_{t-1} + K\left(s_t^i - \rho x_{t-1}\right) + \theta\left[K\left(s_t^i - \rho x_{t-1}\right) + \rho\left(x_{t-1} - x_{t-1|t-1}^i\right)\right]$$

with $K = \frac{\sigma_u^2}{\sigma_u^2 + \sigma_\epsilon^2}$. Relative to rationality, there are now two distortions: the second and third terms exhibit the diagnostic Kalman filter which captures overreaction to the signal s_{t-1}^i relative to expectations conditional on the true lagged state ρx_{t-1}; the last term captures overreaction to surprise about the lagged state itself. The relative weights of the two distortions are given by the respective impact on the signals on beliefs, K and ρ.

Thus, expectations are too optimistic provided

$$K\left(s_t^i - \rho x_{t-1}\right) + \rho\left(x_{t-1} - x_{t-1|t-1}^i\right) > 0$$

This can be rewritten:

$$K\left(u_t + \epsilon_t^i\right) + \rho\left(u_{t-1} - K\left(u_{t-1} + \epsilon_{t-1}^i\right)\right) > 0$$

Thus, overoptimism at t depends on the sequence of shocks at $t-1$ and t. In particular, because ϵ_t^i is mean zero, this condition is more likely to hold when the process has received two positive fundamental shocks, $u_{t-1}, u_t > 0$. In contrast, if a good shock follows a bad shock, overreaction to the latter is dampened by the realization that the lagged state was not as good as expected. The same intuition holds for consensus forecasts, for which we find:

$$x_{t|t}^\theta = \rho x_{t-1} + K(x_t - \rho x_{t-1}) + \theta\left[K u_t + (1-K)\rho u_{t-1}\right]$$

We now derive the Coibion-Gorodnichenko coefficients. Consider first the consensus specification. We have:

$$x_{t|t}^\theta = x_{t|t} + \theta\left(x_{t|t} - x_{t|t-1}\right)$$

where $x_{t|t} = \rho x_{t-1} + K(x_t - \rho x_{t-1})$. Using $x_t = \frac{1}{K}\left(x_{t|t} - \rho x_{t-1}\right) + \rho x_{t-1}$ and

$$x_{t|t}^\theta = \rho x_{t-1} + (1+\theta)K(x_t - \rho x_{t-1}) + \theta\rho(1-K)(x_{t-1} - \rho x_{t-2})$$

the forecast error reads:

$$x_t - x_{t|t}^\theta = \left(1 - K(1+\theta)\right)(x_t - \rho x_{t-1}) - \theta\rho(1-K)(x_{t-1} - \rho x_{t-2})$$

The forecast revision is:

$$x_{t|t}^{\theta} - x_{t|t-1}^{\theta} = \rho[1+\theta(1-K)-(1+\theta)K](x_{t-1}-\rho x_{t-2}) + (1+\theta)K(x_t-\rho x_{t-1}) - \theta\rho^2(1-K)(x_{t-2}-\rho x_{t-3})$$

So the consensus coefficient is:

$$\frac{cov\left(x_t - x_{t|t}^{\theta}, x_{t|t}^{\theta} - x_{t|t-1}^{\theta}\right)}{var\left(x_{t|t}^{\theta} - x_{t|t-1}^{\theta}\right)} = \frac{(1-(1+\theta)K)(1+\theta)K - \theta\rho^2(1-K)[1+\theta(1-K)-(1+\theta)K]}{(\rho[1+\theta(1-K)-(1+\theta)K])^2 + (1+\theta)^2K^2 + \theta^2\rho^4(1-K)^2}$$

This is positive if and only if

$$(1-K)K + \theta[K(1-2K) - \rho^2(1-K)^2] - \theta^2[K^2 + \rho^2(1-K)(1-2K)] > 0$$

This holds for $\theta \in [0, \theta^*)$ (since the quadratic coefficient is positive) where

$$\theta^* = \frac{-[K(1-2K)-\rho^2(1-K)^2] + \sqrt{[K(1-2K)-\rho^2(1-K)^2]^2 - 4(1-K)K[K^2+\rho^2(1-K)(1-2K)]}}{2[K^2+\rho^2(1-K)(1-2K)]}$$

Consider now the individual level forecast. The forecast revision reads:

$$x_{t|t}^{i,\theta} - x_{t|t-1}^{i,\theta} = (1+\theta)\left(x_{t|t}^{i} - \rho x_{t-1|t-1}^{i}\right) - \theta\rho\left(x_{t-1|t-1}^{i} - \rho x_{t-2|t-2}^{i}\right)$$

The forecast error reads: $x_t - x_{t|t}^{i,\theta} = x_t - x_{t|t}^{i} - \theta\left(x_{t|t}^{i} - \rho x_{t-1|t-1}^{i}\right)$

So:

$$cov\left(x_t - x_{t|t}^{i,\theta}, x_{t|t}^{i,\theta} - x_{t|t-1}^{i,\theta}\right) = -\theta(1+\theta)var\left(x_{t|t}^{i} - x_{t|t-1}^{i}\right)$$

since $cov\left(x_t - x_{t|t}^{i}, x_{t|t}^{i} - \rho x_{t-1|t-1}^{i}\right) = 0$ by definition of the Kalman filter and similarly $cov\left(x_t - x_{t|t}^{i}, x_{t-1|t-1}^{i} - \rho x_{t-2|t-2}^{i}\right) = 0$. Moreover:

$$var\left(x_{t|t}^{i,\theta} - x_{t|t-1}^{i,\theta}\right) = [(1+\theta)^2 + (\theta\rho)^2]var\left(x_{t|t}^{i} - x_{t|t-1}^{i}\right)$$

So the coefficient is:

$$\frac{cov\left(x_{t|t}^{i} - x_{t|t}^{i,\theta}, x_{t|t}^{i,\theta} - x_{t|t-1}^{i,\theta}\right)}{var\left(x_{t|t}^{i,\theta} - x_{t|t-1}^{i,\theta}\right)} = -\frac{\theta(1+\theta)}{(1+\theta)^2 + (\theta\rho)^2}$$

As in the baseline case of Proposition 2. ∎