A SUPPORTING THEORY AND PROOFS

A.1 Loss of Smoothness in Elf

We provide a formula to quantify the **smoothness loss** after conversion:

$$\hat{q} = \min(v_X.q, v_Y.q), \tag{7}$$

$$S(v_X, v_Y) = \text{CBL}\left(\text{abs}((v_X \times 10^{-\hat{q}}) - (v_Y \times 10^{-\hat{q}}))\right), \tag{8}$$

The smoothness loss is defined as:

$$\text{Loss} = S(\hat{v}_X, \hat{v}_Y) - S(v_X, v_Y). \tag{9}$$

Certain algorithms, such as Elf, introduce precision-related errors that disrupt smoothness. Specifically, for Elf, the loss can be expressed as:

$$\text{Loss(Elf)} = S(v_X - \delta_{v_X}, v_Y - \delta_{v_Y}) - S(v_X, v_Y). \tag{10}$$

Here, δ_{v_X} denotes the discrepancy between the value after erasure and the original value, which may be represented as:

$$\delta_{v_X} = sign \times 2^{(exp-bias)} \times fraction_{(\text{lower } g(v_X) \text{ bits})}, \tag{11}$$

where the lower $g(v_X)$ bits of the original value get erased.

This concept, originally defined by the Elf authors, can be described in the language of this paper as:

$$g(v_X) = 52 - (\lceil(-q)log_2(10)\rceil + exp - bias). \tag{12}$$

When XOR operations are applied, precision mismatches occur. Specifically, when $g(v_X) \neq g(v_Y)$, it follows that $\delta_{v_X} \neq \delta_{v_Y}$. As a result, the Elf algorithm can disrupt up to abs $\left(g(v_X) - g(v_Y)\right)$ bits of already-erased CBL.

Although Elf attempts to combine redundancy elimination and smoothness exploitation, examples illustrate that performing XOR after erasure leads to poor results. Reversing the order (i.e., performing erasure after XOR) is equally problematic. Moreover, precision becomes uncontrollable after XOR operations, making such optimizations challenging to implement effectively.

A.2 Zero Loss of DECIMAL XOR Converter

We formally prove that the DECIMAL XOR converter incurs no loss of smoothness.

Lemma 5 (Zero Loss of DECIMAL XOR)**.** *The smoothness loss of DECIMAL XOR satisfies:* $\forall \underline{v_X}, v_Y \in \mathbb{R}, Loss(\underline{v_X} \diamond v_Y) = 0$.

Proof. According to the preconditions defined in Section 4.2.1, the DECIMAL XOR operation produces:

$$\hat{v}_X = \underline{v_X} \diamond v_Y = v_X - \alpha, \quad \hat{v}_Y = v_Y - \alpha, \tag{13}$$

where α denotes the shared prefix between v_X and v_Y.

Let $\hat{q} = \min(v_X.q, v_Y.q)$. Next, we compute the smoothness metric $S(\hat{v}_X, \hat{v}_Y)$,

$$\begin{aligned} S(\hat{v}_X, \hat{v}_Y) &= \text{CBL}\left(\text{abs}((\hat{v}_X \times 10^{-\hat{q}}) - (\hat{v}_Y \times 10^{-\hat{q}}))\right) \\ &= \text{CBL}\left(\text{abs}((v_X \times 10^{-\hat{q}} + \gamma) - (v_Y \times 10^{-\hat{q}} + \gamma))\right), \end{aligned}$$

where $\gamma = -\alpha \times 10^{-\hat{q}}$.

Since the γ cancels out due to the shared prefix, we have:

$$S(\hat{v}_X, \hat{v}_Y) = \text{CBL}\left(\text{abs}((v_X \times 10^{-\hat{q}}) - (v_Y \times 10^{-\hat{q}}))\right) = S(v_X, v_Y). \tag{14}$$

Thus, the smoothness loss is:

$$\text{Loss}(v_X \diamond v_Y) = S(\hat{v}_X, \hat{v}_Y) - S(v_X, v_Y) = 0. \tag{15}$$

This completes the proof. □

A.3 Proof of Fixed Bit Allocation for Unsigned Binary Suffix

In Section 4.3, we introduce Lemma 4, which asserts that for any $\beta_i \in \mathbb{Z}$, a fixed allocation of $\bar{\ell}_i$ bits achieves better compression than variable allocation of ℓ_i. Specifically, it holds that:

$$\mathbb{E}[(4 + \bar{\ell}_i)] < \mathbb{E}[(6 + \ell_i)].$$

The detailed proof is provided below:

Proof. We know the condition:

$$\begin{gathered} \delta = o_i - q_i \in \mathbb{N}, \quad \text{abs}(\beta_i) \in [10^{\delta-1}, 10^{\delta}), \\ \ell_i = \lceil \log_2(\text{abs}(\beta_i) + 1) \rceil, \quad \bar{\ell}_i = \lceil \log_2(10^{\delta}) \rceil. \end{gathered}$$

We aim to prove:

$$\mathbb{E}[(\bar{\ell}_i + 4)] < \mathbb{E}[(6 + \ell_i)] \iff \mathbb{E}[(\bar{\ell}_i - \ell_i - 2)] < 0.$$

We divide the range of $\text{abs}(\beta_i)$ by powers of 2. Let $j \in \mathbb{N}$ be the smallest integer such that $2^j > 10^{\delta-1}$, implying:

$$2^{j-1} \leq \mathbf{10}^{\delta-1} < 2^j < 2^{j+1} < 2^{j+2} < \mathbf{10}^{\delta} < 2^{j+4}.$$

We now analyze the relationship between 2^{j+3} and 10^{δ}, which leads to two cases:

Case (1): $2^{j+3} > 10^{\delta}$, with probability $\mathbb{P}_1$. In this case, the fixed allocation is $\bar{\ell}_i = \lceil \log_2(10^{\delta}) \rceil = j + 3$, while the variable allocation ℓ_i is given by:

$$\ell_i = \begin{cases} j & \text{if } \text{abs}(\beta_i) \in [10^{\delta-1}, 2^j), \\ j+1 & \text{if } \text{abs}(\beta_i) \in [2^j, 2^{j+1}), \\ j+2 & \text{if } \text{abs}(\beta_i) \in [2^{j+1}, 2^{j+2}), \\ j+3 & \text{if } \text{abs}(\beta_i) \in [2^{j+2}, 10^{\delta}). \end{cases}$$

To calculate the expectation for this case, we have: $\mathbb{E}_1 = \mathbb{E}(\bar{\ell}_i + 4 \mid 2^{j+3} > 10^{\delta}) - \mathbb{E}(\ell_i + 6 \mid 2^{j+3} > 10^{\delta}) = j + 1 - \mathbb{E}(\ell_i \mid 2^{j+3} > 10^{\delta})$.

Now, let's calculate $\mathbb{E}(\ell_i \mid 2^{j+3} > 10^{\delta})$. This can be expressed as:

$$\begin{aligned} \mathbb{E}(\ell_i \mid 2^{j+3} > 10^{\delta}) = &\frac{(j+3)(10^{\delta} - 2^{j+2})}{10^{\delta} - 10^{\delta-1}} + \frac{(j+2)(2^{j+2} - 2^{j+1})}{10^{\delta} - 10^{\delta-1}} \\ &+ \frac{(j+1)(2^{j+1} - 2^j)}{10^{\delta} - 10^{\delta-1}} + \frac{j(2^j - 10^{\delta-1})}{10^{\delta} - 10^{\delta-1}}. \end{aligned}$$

This expression can be further simplified as follows:

$$\begin{aligned} \mathbb{E}(\ell_i \mid 2^{j+3} > 10^{\delta}) &= j + \frac{3(10^{\delta} - 2^{j+2}) + 2(2^{j+2} - 2^{j+1}) + (2^{j+1} - 2^j)}{10^{\delta} - 10^{\delta-1}} \\ &= j + \frac{3 \times 10^{\delta} - 2^{j+2} - 2^{j+1} - 2^j}{10^{\delta} - 10^{\delta-1}} \\ &= j + 3 - \frac{2^{j+2} + 2^{j+1} + 2^j - 3 \times 10^{\delta-1}}{10^{\delta} - 10^{\delta-1}}. \end{aligned}$$

Next, we approximate the terms as follows:

$$\mathbb{E}(\ell_i \mid 2^{j+3} > 10^{\delta}) > j + 3 - \frac{2^{j+2} + 2^{j+1} + 2^j - 3 \times 2^{j-1}}{2^{j+2} - 2^j}.$$

After simplifying the above expression, we obtain:

$$\mathbb{E}(\ell_i \mid 2^{j+3} > 10^{\delta}) > j + \frac{7}{6}.$$

Thus, the expectation simplifies to:

$$\mathbb{E}_1 = j + 1 - \mathbb{E}(\ell_i \mid 2^{j+3} > 10^{\delta}) < -\frac{1}{6}.$$

We now calculate the probability $\mathbb{P}_1$ for case (1):

$$\mathbb{P}_1\{2^{j+3} > 10^{\delta}\} = \mathbb{P}\{(j+3)\log_{10} 2 > \delta\}.$$

According to the condition $2^j > 10^{\delta-1}$, we have:

$$j \log_{10} 2 + 1 > \delta.$$

Similarly, from $2^{j-1} \le 10^{\delta-1}$, this implies:

$$(j-1)\log_{10} 2 + 1 \le \delta.$$

Therefore, the value of δ lies within the range:

$$\delta \in \big[(j-1)\log_{10} 2 + 1, j\log_{10} 2 + 1\big).$$

In summary, we can compute the possibility as follows:

$$\mathbb{P}\{(j+3)\log_{10} 2 > \delta\} = \frac{(j+3)\log_{10} 2 - (j-1)\log_{10} 2 - 1}{\log_{10} 2}$$

$$= \frac{4\log_{10} 2 - 1}{\log_{10} 2} \implies \mathbb{P}_1\{2^{j+3} > 10^{\delta}\} = 4 - \log_2 10 \approx 0.6781.$$

Case (2): $2^{j+3} \le 10^{\delta}$, with the probability given by:

$$\mathbb{P}_2\{2^{j+3} \le 10^{\delta}\} = 1 - \mathbb{P}_1 \approx 0.3219.$$

It is straightforward to see that for $j \ge 0$, $2^{j+3} \ne 10^{\delta}$, and thus we can replace the $2^{j+3} \le 10^{\delta}$ with a strict inequality $2^{j+3} < 10^{\delta}$.

In this case, the fixed allocation is $\bar{\ell}_i = \lceil \log_2(10^{\delta}) \rceil = j + 4$, while the variable allocation ℓ_i is given by:

$$\ell_i = \begin{cases} j & \text{if } \operatorname{abs}(\beta_i) \in [10^{\delta-1}, 2^j), \\ j+1 & \text{if } \operatorname{abs}(\beta_i) \in [2^j, 2^{j+1}), \\ j+2 & \text{if } \operatorname{abs}(\beta_i) \in [2^{j+1}, 2^{j+2}), \\ j+3 & \text{if } \operatorname{abs}(\beta_i) \in [2^{j+2}, 2^{j+3}), \\ j+4 & \text{if } \operatorname{abs}(\beta_i) \in [2^{j+3}, 10^{\delta}). \end{cases}$$

We first compute the expectation as follows: $\mathbb{E}_2 = \mathbb{E}(\bar{\ell}_i + 4 \mid 2^{j+3} < 10^{\delta}) - \mathbb{E}(\ell_i + 6 \mid 2^{j+3} < 10^{\delta}) = j + 2 - \mathbb{E}(\ell_i \mid 2^{j+3} < 10^{\delta})$.

Next, we proceed to calculate the expectation $\mathbb{E}(\ell_i \mid 2^{j+3} < 10^{\delta})$ as follows:

$$\mathbb{E}(\ell_i \mid 2^{j+3} < 10^{\delta}) = j + \frac{4 \times 10^{\delta} - 2^{j+3} - 2^{j+2} - 2^{j+1} - 2^j}{10^{\delta} - 10^{\delta-1}}$$

$$= j + 4 - \frac{2^{j+3} + 2^{j+2} + 2^{j+1} + 2^j - 4 \times 10^{\delta-1}}{10^{\delta} - 10^{\delta-1}}.$$

After approximating the terms, we obtain:

$$\mathbb{E}(\ell_i \mid 2^{j+3} < 10^{\delta}) > j + 4 - \frac{2^{j+3} + 2^{j+2} + 2^{j+1} + 2^j - 4 \times 2^{j-1}}{2^{j+3} - 2^j} = j + \frac{15}{7}.$$

Thus, the expectation simplifies to:

$$\mathbb{E}_2 = j + 2 - \mathbb{E}(\ell_i | 2^{j+3} < 10^{\delta}) < -\frac{1}{7}.$$

Finally, we compute the overall expectation:

$$\mathbb{E}[(\bar{\ell}_i - \ell_i - 2)] = \mathbb{E}_1 \times \mathbb{P}_1 + \mathbb{E}_2 \times \mathbb{P}_2 < -\frac{1}{6} \times \mathbb{P}_1 - \frac{1}{7} \times \mathbb{P}_2 \approx -0.159 < 0.$$

□