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1. Basic Properties of weighted scalars

Consider a set of scalars $\boldsymbol{\phi} = \{\phi^{(1)}, \dots, \phi^{(N)}\}$, with weights $w_{(i)}$ for the ith particle.

The weights are normalized such that:

$$\sum_{i=1}^{N} w_{(i)} = 1$$

The mean:

$$\widetilde{\phi} = \sum_{i=1}^{N} w_{(i)} \phi^{(i)}$$

The variance:

$$\widetilde{\phi''^2} = \sum_{i=1}^{N} w_{(i)} (\phi^{(i)} - \widetilde{\phi})^2$$

2. Implementations of weighted samples

2.1 Uniform weighted samples

For any i, let $w_{(i)} = 1/N$.

2.2 Uniform distributed samples

Given ϕ^{min} and ϕ^{max}, let ϕ uniformly distributed in $[\phi^{min}, \phi^{max}]$

$$\phi^{(i)} = \frac{i-1}{N-1}(\phi^{max} - \phi^{min}) + \phi^{min}, \; i = 1, \dots, N$$

And for the ith particle, its weight $w_{(i)} = f(\phi^{(i)})$, where $f(\phi)$ is the probability density function (PDF) of $\boldsymbol{\phi}$.

2.3 Random distributed samples

Let ϕ randomly distributed in $[\phi^{min}, \phi^{max}]$, for the sorted particles $\phi^{(1)}, \dots, \phi^{(N)}$, set the particle weights as:

$$w_{(i)} = \Delta\phi_{(i)} \frac{\partial F(\phi)}{\partial \phi}$$

where $F(\phi)$ is the cumm density function (CDF) of ϕ, $\Delta\phi_{(i)} = \phi^{(i+1)} - \phi^{(i)}$ is the interval of ϕ_i.

3. Mixing models with weighted particles

3.1 IEM

IEM(Interaction by Exchange with the Mean) model let scalars mix towards the mean.

(1) Mixing rule:

$$\delta\phi^{(i)} = -\frac{1}{2}\Omega_\phi \delta t(\phi^{(i)} - \widetilde{\phi})$$

(2) Change of the mean:

$$\begin{aligned}
\delta\widetilde{\phi} &= \sum_{i=1}^{N} -w_{(i)}\Omega_\phi \delta t(\phi^{(i)} - \widetilde{\phi}) \\
&= -\frac{1}{2}\Omega_\phi \delta t \left(\sum_{i=1}^{N} w_{(i)}\phi^{(i)} - \widetilde{\phi}\sum_{i=1}^{N} w_{(i)}\right) \\
&= 0
\end{aligned}$$

(3) Change of the variance:

$$\begin{aligned}
\delta\widetilde{\phi''^2} &= \sum_{i=1}^{N} w_{(i)} \left[\left(\phi^{(i)} - \frac{1}{2}\Omega_\phi \delta t(\phi^{(i)} - \widetilde{\phi}) - \widetilde{\phi}\right)^2 - (\phi^{(i)} - \widetilde{\phi})^2\right] \\
&= \sum_{i=1}^{N} w_{(i)} \left[\left(-\Omega_\phi \delta t + \frac{1}{4}\Omega_\phi^2 \delta t^2\right)(\phi^{(i)} - \widetilde{\phi})^2\right] \\
&\approx -\Omega_\phi \delta t \widetilde{\phi''^2}
\end{aligned}$$

3.2 MC

MC (Modified Curl's) model let mix happen in pair-wise format.

(1) Mixing rule: for particle pair (p, q), mixing with random ratio $\alpha \in U[0, 1]$

$$\delta\phi^{(p)} = -\alpha\left(\phi^{(p)} - \widetilde{\phi^{(p,q)}}\right)$$

$$\delta\phi^{(q)} = -\alpha\left(\phi^{(q)} - \widetilde{\phi^{(p,q)}}\right)$$

where $\widetilde{\phi^{(p,q)}} = (w_{(p)}\phi^{(p)} + w_{(q)}\phi^{(q)})/(w_{(p)} + w_{(q)})$ is the weight mean of particle p and q.

(2) Change of the mean: the mean does not change, since pair mixing exchanges the same value between each pair.

(3) Change of the variance: the variance decay from single pair mixing is:

$$\begin{aligned}
\delta_{p,q}\widetilde{\phi''^2} =& w_{(p)}\left[-2\alpha\left(\phi^{(p)} - \widetilde{\phi^{(p,q)}}\right)(\phi^{(p)} - \widetilde{\phi}) + \alpha^2\left(\phi^{(p)} - \widetilde{\phi^{(p,q)}}\right)^2\right] \\
& + w_{(q)}\left[-2\alpha\left(\phi^{(q)} - \widetilde{\phi^{(p,q)}}\right)(\phi^{(q)} - \widetilde{\phi}) + \alpha^2\left(\phi^{(q)} - \widetilde{\phi^{(p,q)}}\right)^2\right] \\
=& \left[-2\alpha w_{(p)}\left(\phi^{(p)} - \widetilde{\phi^{(p,q)}}\right)\phi^{(q)} + \alpha^2 w_{(p)}\left(\phi^{(p)} - \widetilde{\phi^{(p,q)}}\right)^2\right] \\
& + \left[-2\alpha w_{(q)}\left(\phi^{(q)} - \widetilde{\phi^{(p,q)}}\right)\phi^{(q)} + \alpha^2 w_{(q)}\left(\phi^{(q)} - \widetilde{\phi^{(p,q)}}\right)^2\right] \\
=& \left[-2\alpha\frac{w_{(p)}w_{(q)}}{w_{(p)} + w_{(q)}}(\phi^{(p)} - \phi^{(q)})\phi^{(p)} + \alpha^2 w_{(p)}\left(\frac{w_{(q)}}{w_{(p)} + w_{(q)}}\right)^2(\phi^{(p)} - \phi^{(q)})^2\right] \\
& + \left[-2\alpha\frac{w_{(q)}w_{(p)}}{w_{(p)} + w_{(q)}}(\phi^{(q)} - \phi^{(p)})\phi^{(q)} + \alpha^2 w_{(q)}\left(\frac{w_{(p)}}{w_{(p)} + w_{(q)}}\right)^2(\phi^{(q)} - \phi^{(p)})^2\right] \\
=& (-2\alpha + \alpha^2)\frac{w_{(p)}w_{(q)}}{w_{(p)} + w_{(q)}}(\phi^{(q)} - \phi^{(p)})^2
\end{aligned}$$

For uniform weighted samples, $w_{(i)} = 1/N$, the expected variance decay of single mixing pair is:

$$\begin{aligned}
\left\langle \delta_{p,q}\widetilde{\phi''^2} \right\rangle &= \langle -2\alpha + \alpha^2 \rangle \left\langle \frac{w_{(p)}w_{(q)}}{w_{(p)} + w_{(q)}} (\phi^{(q)} - \phi^{(p)})^2 \right\rangle \\
&= -\frac{2}{3} \sum_p \frac{1}{N} \sum_q \frac{1}{N} \frac{1}{2N} (\phi^{(q)} - \phi^{(p)})^2 \\
&= -\frac{1}{3N} \left[\widetilde{\phi^2} + \widetilde{\phi^2} - 2\left(\widetilde{\phi}\right)^2 \right] \\
&= -\frac{2}{3N} \widetilde{\phi''^2}
\end{aligned}$$

For general cases, the expected variance decay of single mixing pair is:

$$\begin{aligned}
\left\langle \delta_{p,q}\widetilde{\phi''^2} \right\rangle &= \langle -2\alpha + \alpha^2 \rangle \left\langle \frac{w_{(p)}w_{(q)}}{w_{(p)} + w_{(q)}} (\phi^{(q)} - \phi^{(p)})^2 \right\rangle \\
&= -\frac{2}{3} \sum_p \frac{1}{N} \frac{w_{(p)} + \widetilde{w}}{2\widetilde{w}} \sum_q \frac{1}{N} \frac{w_{(p)} + w_{(q)}}{w_{(p)} + \widetilde{w}} \frac{w_{(p)}w_{(q)}}{w_{(p)} + w_{(q)}} (\phi^{(q)} - \phi^{(p)})^2 \\
&= -\frac{1}{3N} \sum_p w_{(p)} \sum_q w_{(q)} (\phi^{(q)} - \phi^{(p)})^2 \\
&= -\frac{1}{3N} \left[\left(\sum_q w_{(q)}\right) \sum_p w_{(p)} (\phi^{(p)})^2 - 2\left(\sum_p w_{(p)}\phi^{(p)}\right)\left(\sum_q w_{(q)}\phi^{(q)}\right) + \left(\sum_p w_{(p)}\right) \sum_q w_{(q)} (\phi^{(q)})^2 \right] \\
&= -\frac{1}{3N} \left[\widetilde{\phi^2} + \widetilde{\phi^2} - 2\left(\widetilde{\phi}\right)^2 \right] \\
&= -\frac{2}{3N} \widetilde{\phi''^2}
\end{aligned}$$

This means that, we need to generate the samples by:

1. Randomly choose p with probability $(w_{(p)} + \widetilde{w})/(w_{max} + \widetilde{w})$
2. Randomly choose q with probability $(w_{(p)} + w_{(q)})/(w_{(p)} + w_{max})$

After mixing for $N_{pair} = \frac{3}{2} N \Omega_\phi \delta t$ pairs, the expected variance decay is achieved.

3.3 KerM

To include the localness *(the physics concept that scalars tend to mixing with each other that is near to itself in the composition space)*, we can assume a kernel function $k(d)$ that allows randomly selected pair $\phi^{(p)}, \phi^{(q)}$ mix with probability $P_{\text{mix } p,q} = k(d_{p,q})$, where **$d_{p,q}$ is the distance of p, q measured in some reference space**. The kernel function is chosen to be a Gaussian radial basis function (RBF), as:

$$k(d_{p,q}) = \exp\left(-\frac{d_{p,q}^2}{4\sigma_k^2} \right)$$

where σ_k is the kernel size in the reference space, and being the only model parameter that controls the level of localness.

(1) Mixing rule: When select a single pair $\phi^{(p)}$ and $\phi^{(q)}$ at time t, we have:

- With probability $P = k(d_{p,q})$, mix $\phi^{(p)}$ and $\phi^{(q)}$ with MC rule:

$$\delta\phi^{(p)} = -\alpha \left(\phi^{(p)} - \widetilde{\phi^{(p,q)}} \right)$$
$$\delta\phi^{(q)} = -\alpha \left(\phi^{(q)} - \widetilde{\phi^{(p,q)}} \right)$$

- With probability $1 - P$, do not change $\phi^{(p)}$ and $\phi^{(q)}$:

$$\delta\phi^{(p)} = 0$$
$$\delta\phi^{(q)} = 0$$

(2) Change of the mean: the mean does not change, since pair mixing exchanges the same value between each pair.

(3) Change of the variance:

As the mixing format is the same with MC, once pair (p, q) are selected to mix, the expected variance decay should be:

$$\begin{aligned}\left\langle \delta_{p,q}\widetilde{\phi''^2} \right\rangle &= -\frac{2}{3}\left\langle k(d_{p,q}) \cdot \frac{w_{(p)}w_{(q)}}{w_{(p)}+w_{(q)}}(\phi^{(q)} - \phi^{(p)})^2 + (1 - k(d_p, q)) \cdot 0 \right\rangle \\ &= -\frac{2}{3}\left\langle k(d_{p,q}) \cdot \frac{w_{(p)}w_{(q)}}{w_{(p)}+w_{(q)}}(\phi^{(q)} - \phi^{(p)})^2 \right\rangle\end{aligned}$$

By defining a variance decay coefficient C_{eff} as the variance decay efficiency against MC model:

$$C_{eff} = \frac{\left\langle \frac{w_{(p)}w_{(q)}}{w_{(p)}+w_{(q)}}(\phi^{(q)} - \phi^{(p)})^2 \right\rangle}{\left\langle k(d_{p,q}) \cdot \frac{w_{(p)}w_{(q)}}{w_{(p)}+w_{(q)}}(\phi^{(q)} - \phi^{(p)})^2 \right\rangle} = \frac{\frac{1}{N}\widetilde{\phi''^2}}{\left\langle k(d_{p,q}) \cdot \frac{w_{(p)}w_{(q)}}{w_{(p)}+w_{(q)}}(\phi^{(q)} - \phi^{(p)})^2 \right\rangle}$$

Then the expected variance decay:

$$\left\langle \delta_{p,q}\widetilde{\phi''^2} \right\rangle = -\frac{1}{C_{eff}}\frac{2}{3N}\widetilde{\phi''^2}$$

So the number of pairs for mixing is $N_{pair} = \frac{3}{2}C_{eff}N\Omega_\phi\delta t$.

4. Appendix

4.1 The variance decay view from Gaussian Distribution $\boldsymbol{\pi}_\phi \sim \mathcal{N}(\mu, \sigma^2)$

Consider Curl's model that pair-wise mixes ϕ_p and ϕ_q to their mean: (without account of weights here)

$$\phi_p(t + \delta t) = \phi_q(t + \delta t) = \frac{1}{2}(\phi_p(t) + \phi_q(t))$$

The variance decay is:

$$\begin{aligned}Var(t + \delta t) - Var(t) &= \frac{1}{N}\left[\left(\phi_p(t + \delta t) - \tilde{\phi}\right)^2 + \left(\phi_q(t + \delta t) - \tilde{\phi}\right)^2 - \left(\phi_p(t) - \tilde{\phi}\right)^2 - \left(\phi_p(t) - \tilde{\phi}\right)^2\right] \\ &= \frac{1}{N}\left[\phi_p^2(t + \delta t) + \phi_q^2(t + \delta t) - \phi_p^2(t) - \phi_q^2(t)\right] \\ &= \frac{1}{2N}\left[(\phi_p(t) + \phi_q(t))^2 - \phi_p^2(t) - \phi_q^2(t)\right] \\ &= -\frac{1}{2N}(\phi_p(t) - \phi_q(t))^2\end{aligned}$$

Denote the difference in $\phi_p(t)$ and $\phi_q(t)$ as an Eulerian distance $D = ||\phi_p(t) - \phi_q(t)||_2$, then $Var(t + \delta t) - Var(t) = -\frac{1}{2}D^2$.

If $\boldsymbol{\pi}_\phi$ follows a Standard Gaussian Distribution that $\boldsymbol{\pi}_\phi \sim \mathcal{N}(0, 1)$, which has probability distribution function (PDF):

$$f_\phi(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)$$

Thus, as detailed in arXiv:1508.02238, the distribution of D follows the PDF of

$$f_D(x) = \frac{1}{\sqrt{\pi}}\exp\left(-\frac{1}{4}x^2\right)$$

More generally, if $\boldsymbol{\pi}_\phi$ follows a general gaussian distribution that $\boldsymbol{\pi}_\phi \sim \mathcal{N}(\mu, \sigma^2)$, which has the PDF:

$$f_\phi(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right)$$

Then we can get the distribution of D by first obtain the CDF of random variable D as:

$$
\begin{aligned}
F_D(x) &= \int_{-\infty}^{+\infty}\int_{p+x}^{p-x} \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(p-\mu)^2}{2\sigma^2}\right)\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(q-\mu)^2}{2\sigma^2}\right)dpdq \\
&= \int_{-\infty}^{+\infty} \frac{1}{2\sigma\sqrt{2\pi}}\exp\left(-\frac{(p-\mu)^2}{2\sigma^2}\right)\left[\mathrm{erf}\left(\frac{p-\mu+x}{\sqrt{2\sigma^2}}\right)-\mathrm{erf}\left(\frac{p-\mu-x}{\sqrt{2\sigma^2}}\right)\right]dp
\end{aligned}
$$

So the PDF of D is now:

$$
\begin{aligned}
f_D(x) &= \frac{\partial}{\partial x}[F_D(x)] \\
&= \frac{\partial}{\partial x}\left\{\int_{-\infty}^{+\infty} \frac{1}{2\sigma\sqrt{2\pi}}\exp\left(-\frac{(p-\mu)^2}{2\sigma^2}\right)\left[\mathrm{erf}\left(\frac{p-\mu+x}{\sqrt{2\sigma^2}}\right)-\mathrm{erf}\left(\frac{p-\mu-x}{\sqrt{2\sigma^2}}\right)\right]dp\right\} \\
&= \int_{-\infty}^{+\infty} \frac{1}{2\sigma\sqrt{2\pi}}\exp\left(-\frac{(p-\mu)^2}{2\sigma^2}\right)\frac{\partial}{\partial x}\left[\mathrm{erf}\left(\frac{p-\mu+x}{\sqrt{2\sigma^2}}\right)-\mathrm{erf}\left(\frac{p-\mu-x}{\sqrt{2\sigma^2}}\right)\right]dp \\
&= \int_{-\infty}^{+\infty} \frac{1}{2\sigma\sqrt{2\pi}}\exp\left(-\frac{(p-\mu)^2}{2\sigma^2}\right)\frac{2}{\sqrt{\pi}}\frac{1}{\sqrt{2\sigma^2}}\left[\exp\left(\frac{(p-\mu+x)^2}{2\sigma^2}\right)+\exp\left(\frac{(p-\mu-x)^2}{2\sigma^2}\right)\right]dp \\
&= \frac{1}{\sqrt{\pi}\sigma}\exp\left(-\frac{x^2}{4\sigma^2}\right)
\end{aligned}
$$

Therefore, in the general case that $\boldsymbol{\pi}_\phi \sim \mathcal{N}(\mu, \sigma^2)$, the mean variance decay should be:

$$
\begin{aligned}
E\left[Var(t+\delta t) - Var(t)\right] &= E\left[-\frac{1}{2N}D^2\right] \\
&= \int_0^\infty -\frac{1}{2N}D^2\frac{1}{\sqrt{\pi}\sigma}\exp\left(-\frac{1}{4\sigma^2}D^2\right)dD \\
&= \frac{1}{N}\left[\frac{\sigma D}{\sqrt{\pi}}\exp\left(-\frac{D^2}{4\sigma^2}\right)+\sigma^2\mathrm{erf}\left(\frac{D}{2\sigma}\right)+C\Big|_0^\infty\right] \\
&= -\frac{1}{N}\sigma^2
\end{aligned}
$$

Which makes the Curl model satisfy the exponentially variance decay rate with $N_{pairs} = \Omega_\phi N\delta t$

$$
\frac{d\widetilde{\phi''^2}}{dt} = -\Omega_\phi\widetilde{\phi''^2}
$$

4.2 KerM with Eulerian distance as the reference variable

Consider using Eulerian distance $D_{p,q}$ as the reference distance $d_{p,q}$ in KerM model.

Generally, if $\boldsymbol{\pi}_\phi$ follows a general gaussian distribution that $\boldsymbol{\pi}_\phi \sim \mathcal{N}(\mu, \sigma^2)$, which has the PDF:

$$
f_\phi(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)
$$

Then $D_{p,q}$ follows the distribution of

$$
f_D(x) = \frac{1}{\sqrt{\pi}\sigma}\exp\left(-\frac{x^2}{4\sigma^2}\right)
$$

Then for the selected gaussian kernel function $k(d_{p,q}) = \exp\left(-\frac{d_{p,q}^2}{4\sigma_k^2}\right)$, one can calculate the mean variance decay as:

$$
\begin{aligned}
E[Var(t+\delta t) - Var(t)] &= \frac{2}{3}E\left[-\frac{1}{2N}k(D)D^2\right] \\
&= \frac{2}{3N}\int_0^\infty -\frac{1}{2}\exp\left(-\frac{D^2}{4\sigma_k^2}\right)D^2\frac{1}{\sqrt{\pi}\sigma}\exp\left(-\frac{D^2}{4\sigma^2}\right)dD \\
&= -\frac{2}{3N}\frac{1}{\sigma\left(\frac{1}{\sigma^2}+\frac{1}{\sigma_k^2}\right)^{3/2}} \\
&= -\frac{2}{3N}\sigma^2\left(1+\frac{\sigma^2}{\sigma_k^2}\right)^{-3/2}
\end{aligned}
$$

It's easy to observe that when the kernel size σ_k is much larger than original length scale of the PDF of ϕ, $\sigma_k \gg \sigma$, this model will perform the same as Modified Curl Model:

$$\frac{d\sigma^2}{d\delta t} = -\frac{2}{3N}\sigma^2\left(1+\frac{\sigma^2}{\sigma_k^2}\right)^{-3/2} \approx -\frac{2}{3N}\sigma^2$$

And when σ_k is comparable with σ, this model will decay slower than Modified Curl Model, due to its local mixing.

When the mixing scale is much smaller than data scale $\sigma_k \ll \sigma$, then the mixing will proceed with $-\frac{1}{2}$-law:

$$\frac{d\sigma^2}{d\delta t} = -\frac{2}{3N}\sigma^2\left(1+\frac{\sigma^2}{\sigma_k^2}\right)^{-3/2} \approx -\frac{2}{3N}\sigma_k^3(\sigma^2)^{-1/2}$$