Spherically Inward Propagating Flame of Type Ia Supernova

1. Introduction

Supernova represent the catastrophic explosions that mark the end of the life of some stars. The ejected mass is of order 1 to 10 solar masses with bulk velocities ranging from a few thousand to a few tens of thousands of km/s. [1] The traditional single Chandrasekhar mass C–O White Dwarf burning is still considered to be responsible for a large population of type Ia supernova (SN Ia). Specifically, one of the key issues in its modeling is related to the flame acceleration and deflagration-detonation transition (DDT), with flame front instabilities being considered as a possible mechanism in driving the acceleration. The perspective of this proposal is to give a set of solutions of this problem. The milestones of this project are planed to be:

(a) Complete a set of code that can solve the 1D H_2-O_2 flame.

(b) Complete a set of code that can solve the simplest reacting flow in SN Ia conditions.

(c) Try to observe spherical flame acceleration with large Lewis number curvature effect.

(d) Try to observe pulsation in Xing & Zhao's case.

(e) Try to observe deflagration-detonation transition (DDT) in (d).

2. Basic Equations

2.1 Planar flame

- Governing equations for 1-dimensional, Cartesian coordinates, compressible and non-viscous reacting flow are:

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = \frac{\partial D(U)}{\partial x} + S$$

where U is the conserved variables vector, F is the flux vector, D is the diffusion flux, and S is the source term vector. The vectors are given by:

$$
U = \begin{bmatrix} \rho \\ \rho u \\ \rho E \\ \rho Y \end{bmatrix}, \quad
F(U) = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u(E + p/\rho) \\ \rho u Y \end{bmatrix}, \quad
D(U) = \begin{bmatrix} 0 \\ 0 \\ \frac{\mu C_p}{Pr}\frac{\partial T}{\partial x} + \frac{\mu Q}{Sc}\frac{\partial Y}{\partial x} \\ \frac{\mu}{Sc}\frac{\partial Y}{\partial x} \end{bmatrix}, \quad
S = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \rho \omega \end{bmatrix}
$$

Here ρ is density, u is velocity, E is the total energy per unit mass and Y is the reactant mass fraction, respectively. The total energy is consisted by internal energy, kinetic energy and the "chemical" energy in reactant:

$$
\rho E = \rho e + \frac{1}{2}\rho u^2 + \rho Q Y
$$

The viscosity $\mu = \rho\nu$

The Schmidt number $Sc = \nu/D = \mu/\rho D$

The Prandtl number $Pr = \nu/\alpha = \mu c_p/\lambda$

The enthalpy per unit mass $H = E + p/\rho$

The heat value part might need to be put in internal energy part

- **In the scenario of methane flame**, the internal energy per unit mass is given by:

$$
e = \frac{p}{\rho(\gamma - 1)} = \frac{R_u T}{\gamma - 1}
$$

and the reaction rate is

$$
\omega = -KY\exp\left(-E_a/R_u T\right)
$$

- **In the scenario of supernova flame**, the internal energy is described by the equation of state of the relativistic degenerate electron gas [2]:

$$
e(\rho, T) = \frac{3}{4\rho}(3\pi^2)^{\frac{1}{3}}\hbar c_1(\rho N)^{\frac{4}{3}} + \frac{1}{2}N\frac{(3\pi^2)^{\frac{2}{3}}}{3\hbar c_1}\left(\frac{1}{\rho N}\right)^{\frac{1}{3}}(kT)^2
$$

With pressure being $p = \rho e/3$, one can get the form as: (constant $\gamma = 4/3$)

$$
e = \frac{p}{\rho(4/3 - 1)} = \frac{p}{\rho(\gamma - 1)}
$$

The reaction rate is adopted from the rate of Carbon-Carbon nuclear reaction [3]:

$$
\omega = AY^2\exp\left(-E_a/T_9^{\frac{1}{3}}\right)
$$

where $T_9 = T/10^9$ and $E_a = 84.165$ for C12-C12 reaction.

N is Carbon's electron density
c_1 is the light speed in vacuum
$\hbar$ is the reduced Plank constant

2.2 Spherical flame

- In spherical coordinates, the governing equations change to be:

$$
\frac{\partial U}{\partial t} + \frac{1}{r^2}\frac{\partial r^2 F(U)}{\partial r} = \frac{1}{r^2}\frac{\partial r^2 D(U)}{\partial r} + S
$$

or in the expanded form:

$$\frac{\partial \boldsymbol{U}}{\partial t} + \frac{\partial \boldsymbol{F}(\boldsymbol{U})}{\partial r} + \frac{2}{r}\boldsymbol{F}(\boldsymbol{U}) = \frac{\boldsymbol{D}(\boldsymbol{U})}{\partial r} + \frac{2}{r}\boldsymbol{D}(\boldsymbol{U}) + \boldsymbol{S}$$

with the vectors being:

$$\boldsymbol{U} = \begin{bmatrix} \rho \\ \rho u \\ \rho E \\ \rho Y \end{bmatrix}, \quad \boldsymbol{F}(\boldsymbol{U}) = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u H \\ \rho u Y \end{bmatrix}, \quad \boldsymbol{D}(\boldsymbol{U}) = \begin{bmatrix} 0 \\ 0 \\ \lambda\frac{\partial T}{\partial r} + \rho Q D\frac{\partial Y}{\partial r} \\ \rho D\frac{\partial Y}{\partial r} \end{bmatrix}, \quad \boldsymbol{S} = \begin{bmatrix} 0 \\ \frac{2p}{r} \\ 0 \\ \rho\omega \end{bmatrix}$$

2.3 Parameters for H_2-O_2 flame

- Reference state ($H_2 : O_2 = 2 : 1$), examined in tools/test_h2o2_flame.py
    - pressure $p_0 = 1.013 \times 10^5$ [Pa]
    - specific gas constant $R_u = 692.25$ [$J/kg/K$]
    - temperature $T_0 = 300$ [K]
    - density $\rho_0 = p_0/R_u T_0 = 0.4879$ [kg/m^3]
    - heat capacity ratio $\gamma = 1.4$
    - velocity $u_0 = \sqrt{R_u T_0} = 455.7 m/s$ ($S_L \approx 10m/s$)
    - flame thickness $\delta_f \approx 1mm = 1 \times 10^{-3}m$
    - time scale $t_0 = \delta_f/u_0 = 1.1 \times 10^{-6}$ [s]
    - heat value $Q \approx 36 R_u T_0$ [J/kg] (300K->3076K)
    - thermal conductivity $\lambda \approx 0.3$ [$W/m/K$]
    - thermal diffusivity $\alpha \approx 1.5 \times 10^{-3}$ [m^2/s]
    - Lewis number $Le \approx 1.2$
    - activation energy $Ea \approx 27 R_u T_0$
    - pre-exponential factor $A \approx 2 \times 10^7$?

3. Dimensionless Equations

3.1 References variables

Dimensionless variables	Reference values
$\hat{\rho} = \frac{\rho}{\rho_0}$	$\rho_0 = 3.5 \times 10^{10} kg/m^3$
$\hat{u} = \frac{u}{u_0}$	$u_0 = 7.871 \times 10^6 m/s, S_L = 466 m/s$ (laminar flame speed)
$r = \frac{\hat{r}}{\hat{r}_0}$	$r_0 = \delta_f = \frac{\lambda}{\rho_0 C_p S_t} = 9 \times 10^{-4} m$
$\hat{t} = \frac{t}{t_0}$	$t_0 = \frac{r_0}{u_0} = 1.143 \times 10^{-10} s$
$\hat{T} = \frac{T}{T_b - T_0}$	$T_0 = 1 \times 10^8 K; T_b = 3.2 \times 10^9 K$
$\hat{p} = \frac{p}{p_0}$	$p_0 = \rho_0 u_0^2$
$\hat{E} = \frac{E}{p_0/\rho_0} = \frac{E}{u_0^2}$	$e(\rho_0, T_b) = 6.1952 \times 10^{13} J/kg$
$\hat{Y} = \frac{Y}{Y_0}$	$Y_0 = 1$
$\hat{\omega} = \frac{\omega}{\omega_0}$	$\rho_0 \omega_0 \delta_f = \rho_0 Y_0 S_t, \omega_0 = \frac{Y_0 u_0}{r_0} \frac{S_t}{u_0}$
$ \hat \alpha = \frac{\alpha}{u_0 r_0} $	$\hat{D} = \frac{\hat{\alpha}}{Le} = \frac{2}{3} \frac{1}{Le}$
$\hat{Q} = \frac{Q}{u_0^2}$	$\hat{Q} = 0.904$
$\hat{\lambda} = \lambda / (\frac{\rho_0 r_0 u_0^3}{T_b - T_0})$	$\hat{\lambda} = \frac{S_t}{u_0} \frac{C_p(T_b - T_0)}{u_0^2} = 0.14541$

Laminar flame speed: $S_L = 466 m/s$

Turbulent flame speed: S_t, assumed to be equal to the speed of sound at burnt state

Therefore, the dimensionless equation can be obtained from: (use planar equation for demonstration)

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0$$

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial}{\partial x}(\rho u^2 + p) = 0$$

$$\frac{\partial \rho E}{\partial t} + \frac{\partial}{\partial x}(\rho u E + u p) = \lambda \frac{\partial^2 T}{\partial x^2} + \rho Q D \frac{\partial^2 Y}{\partial x^2}$$

$$\frac{\partial \rho Y}{\partial t} + \frac{\partial}{\partial x}(\rho u Y) = \rho D \frac{\partial^2 Y}{\partial x^2} + \rho \omega$$

$\Rightarrow$

$$\frac{\rho_0}{t_0}\left[\frac{\partial \hat{\rho}}{\partial \hat{t}} + \frac{\partial}{\partial \hat{x}}(\hat{\rho}\hat{u})\right] = 0$$

$$\frac{\rho_0 u_0}{t_0}\left[\frac{\partial(\hat{\rho}\hat{u})}{\partial \hat{t}} + \frac{\partial}{\partial \hat{x}}(\hat{\rho}\hat{u}^2 + \hat{p})\right] = 0$$

$$\frac{\rho_0 u_0^2}{t_0}\left[\frac{\partial \hat{\rho}\hat{E}}{\partial \hat{t}} + \frac{\partial}{\partial \hat{x}}(\hat{\rho}\hat{u}\hat{E} + \hat{u}\hat{p})\right] = \frac{\rho_0 u_0^3 r_0(T_b - T_0)}{r_0^2(T_b - T_0)}\left[\hat{\lambda}\frac{\partial^2 \hat{T}}{\partial \hat{x}^2}\right] + \frac{\rho_0 u_0^2 u_0 r_0 Y_0}{r_0^2}\left[\hat{\rho}\hat{Q}\hat{D}\frac{\partial^2 \hat{Y}}{\partial \hat{x}^2}\right]$$

$$\frac{\rho_0 Y_0}{t_0}\left[\frac{\partial \hat{\rho}\hat{Y}}{\partial \hat{t}} + \frac{\partial}{\partial \hat{x}}(\hat{\rho}\hat{u}\hat{Y})\right] = \frac{\rho_0 u_0 r_0 Y_0}{r_0^2}\left[\hat{\rho}\hat{D}\frac{\partial^2 \hat{Y}}{\partial \hat{x}^2}\right] + \frac{\rho_0 u_0 Y_0}{r_0}\frac{S_t}{u_0}[\hat{\rho}\hat{\omega}]$$

$\Rightarrow$

$$\frac{\partial \hat{\rho}}{\partial \hat{t}} + \frac{\partial}{\partial \hat{x}}(\hat{\rho}\hat{u}) = 0$$

$$\frac{\partial(\hat{\rho}\hat{u})}{\partial \hat{t}} + \frac{\partial}{\partial \hat{x}}(\hat{\rho}\hat{u}^2 + \hat{p}) = 0$$

$$\frac{\partial \hat{\rho}\hat{E}}{\partial \hat{t}} + \frac{\partial}{\partial \hat{x}}(\hat{\rho}\hat{u}\hat{H}) = \hat{\lambda}\frac{\partial^2 \hat{T}}{\partial \hat{x}^2} + \hat{\rho}\hat{Q}\hat{D}\frac{\partial^2 \hat{Y}}{\partial \hat{x}^2}$$

$$\frac{\partial \hat{\rho}\hat{Y}}{\partial \hat{t}} + \frac{\partial}{\partial \hat{x}}(\hat{\rho}\hat{u}\hat{Y}) = \hat{\rho}\hat{D}\frac{\partial^2 \hat{Y}}{\partial \hat{x}^2} + \frac{S_t}{u_0}\hat{\rho}\hat{\omega}$$

3.2 Dimensionless parameters

- Reference velocity u_0:

  For simplification, the largest internal energy is normalized to be $\hat{e}(\rho, T_b) = e(\rho_0, T_b)/u_0^2 = 1$. Therefore, one has

  $$u_0 = \sqrt{e(\rho_0, T_b)} = \sqrt{6.1952 \times 10^{13}} = 7.871 \times 10^6 m/s$$

- Reference length scale $r_0 = 9 \times 10^{-4}m$

- Reference time scale $t_0 = r_0/u_0 = 1.143 \times 10^{-10}s$

- Speed of sound at the burnt state $c = \sqrt{\gamma p/\rho} = \sqrt{\gamma(\gamma-1)e(\rho_0, T_b)} = 5.247 \times 10^6 m/s$

- Heat value $\hat{Q} = Q/u_0^2$, with data from Fowler 1975, for $C^{12} + C^{12} \to Mg^{24}$, $Q = 13.931 MeV$, thus

  $$\hat{Q} = \frac{13.931 MeV \times (1.6022 \times 10^{-13} J/MeV) \times (6.022 \times 10^{23} mol^{-1})/(0.024 kg/mol)}{u_0^2}$$

  $$= \frac{5.6 \times 10^{13} J/kg}{6.1952 \times 10^{13} J/kg} = 0.904$$

- The speed ratio $S_t/u_0 = 2/3$

- Thermal conductivity $\hat{\lambda}$ is related to C_p, while the heat release is related to internal energy:

  $$\tilde{C}_p(T_b - T_0) = \int_{T_0}^{T_b} C_p dT = e(\rho_0, T_b) - e(\rho_0, T_0)$$

  $$\hat{\lambda} = \frac{S_t}{u_0}\frac{\tilde{C}_p(T_b - T_0)}{u_0^2} = 0.14541$$

- The diffusivity D is described by Lewis number $D = \alpha/Le = \lambda/\rho C_p Le$, therefore

  $$\hat{D} = \frac{\lambda}{\rho C_p Le}\frac{1}{u_0 r_0} = \frac{1}{Le}\frac{S_t}{u_0}\frac{\delta_f}{r_0} = \frac{2}{3}\frac{1}{Le}$$

- The total energy form keeps the same in dimensionless form:

  $$\hat{E} = \hat{e} + \frac{1}{2}\hat{u}^2 + \hat{Q}\hat{Y}, \ \hat{e} = \frac{\hat{p}}{\hat{\rho}(\gamma - 1)}$$

  with the equation of state as:

  $$p = \rho e(\gamma - 1) = \frac{3}{4}(\gamma - 1)(3\pi^2)^{\frac{1}{3}}\hbar c_1(\rho N)^{\frac{4}{3}} + \frac{1}{2}(\gamma - 1)N\frac{(3\pi^2)^{\frac{2}{3}}}{3\hbar c_1}\left(\frac{\rho^2}{N}\right)^{\frac{1}{3}}(kT)^2$$

  $$\hat{p} = \frac{3}{4}\frac{1}{\rho_0 u_0^2}(\gamma - 1)(3\pi^2)^{\frac{1}{3}}\hbar c_1(\rho_0 N)^{\frac{4}{3}}\hat{\rho}^{\frac{4}{3}} + \frac{1}{2}(\gamma - 1)N\frac{(3\pi^2)^{\frac{2}{3}}}{3\hbar c_1}\left(\frac{\rho_0^2}{N}\right)^{\frac{1}{3}}k^2(T_b - T_0)^2\hat{\rho}^{\frac{2}{3}}\hat{T}^2$$

  $$= C_1\hat{\rho}^{4/3} + C_2\hat{\rho}^{2/3}\hat{T}^2$$

with $C_1 = 2.60559 \times 10^{-1}$, $C_2 = 6.82973 \times 10^{-2}$

- The source term equation:

$$\omega_0 = \frac{Y_0 u_0}{r_0} \frac{S_t}{u_0} \approx A Y_0^2 \exp\left(-E_a/T_{b9}^{\frac{1}{3}}\right)$$

$$\hat{\omega} = \frac{\omega}{\omega_0} = \hat{Y}^2 \exp\left(-Ea/T_9^{\frac{1}{3}} + Ea/T_{9b}^{\frac{1}{3}}\right) = \hat{Y}^2 \exp\left(-Ea\left(\frac{T_b - T_0}{10^9}\right)^{-\frac{1}{3}}\left(\hat{T}^{-\frac{1}{3}} - \hat{T}_b^{-\frac{1}{3}}\right)\right)$$

to reduce the term S_t/u_0 in previous equation along with $\hat{\omega}$, denoting

$$\widehat{Ea} = -Ea\left(\frac{T_b - T_0}{10^9}\right)^{-\frac{1}{3}} = 57.7224$$

$$\hat{A} = \frac{S_t}{u_0}\exp\left(Ea\left(\frac{T_b - T_0}{10^9}\right)^{-\frac{1}{3}}\hat{T}_b^{-\frac{1}{3}}\right) = 4.25133 \times 10^{24}$$

then the source term keeps the same in dimensionless form:

$$\hat{\omega} = \hat{A}\hat{Y}^2 \exp\left(-\widehat{Ea}/\hat{T}^{\frac{1}{3}}\right)$$

- And the final dimensionless equations are:

$$\frac{\partial \hat{\rho}}{\partial \hat{t}} + \frac{\partial}{\partial \hat{x}}(\hat{\rho}\hat{u}) = 0$$

$$\frac{\partial(\hat{\rho}\hat{u})}{\partial \hat{t}} + \frac{\partial}{\partial \hat{x}}(\hat{\rho}\hat{u}^2 + \hat{p}) = 0$$

$$\frac{\partial \hat{\rho}\hat{E}}{\partial \hat{t}} + \frac{\partial}{\partial \hat{x}}(\hat{\rho}\hat{u}\hat{H}) = \hat{\lambda}\frac{\partial^2 \hat{T}}{\partial \hat{x}^2} + \hat{\rho}\hat{Q}\hat{D}\frac{\partial^2 \hat{Y}}{\partial \hat{x}^2}$$

$$\frac{\partial \hat{\rho}\hat{Y}}{\partial \hat{t}} + \frac{\partial}{\partial \hat{x}}(\hat{\rho}\hat{u}\hat{Y}) = \hat{\rho}\hat{D}\frac{\partial^2 \hat{Y}}{\partial \hat{x}^2} + \hat{\rho}\hat{\omega}$$

$$\hat{E} = \hat{e} + \frac{1}{2}\hat{u}^2 + \hat{Q}\hat{Y}$$

$$\hat{e} = \frac{\hat{p}}{\hat{\rho}(\gamma - 1)}$$

$$\hat{p} = C_1\hat{\rho}^{4/3} + C_2\hat{\rho}^{2/3}\hat{T}^2$$

$$\hat{\omega} = \hat{A}\hat{Y}^2 \exp\left(-\widehat{Ea}/\hat{T}^{\frac{1}{3}}\right)$$

4. Numerical methods

- Discrete methods:
  - Time evolution: 3 order TVD `Runge-Kutta`

$$U^{(1)} = U_n + L(U_n)\Delta t$$

$$U^{(2)} = \frac{3}{4}U_n + \frac{1}{4}(U^{(1)} + L(U^{(1)})\Delta t)$$

$$U_{n+1} = \frac{1}{3}U_n + \frac{2}{3}(U^{(2)} + L(U^{(2)})\Delta t)$$

  - Convection term:

    `Roe` method: solve convection flux by eigen vector

  - Diffusion term: 7 order central difference

$$\frac{\partial m_i}{\partial x} = \frac{1}{\Delta x}\left(\frac{1}{60}m_{i+3} - \frac{9}{60}m_{i+2} + \frac{45}{60}m_{i+1} - \frac{45}{60}m_{i-1} + \frac{9}{60}m_{i-2} - \frac{1}{60}m_{i-3}\right)$$

- Initial conditions:
    - Pressure: whole field the same pressure $\hat{P}(\rho_0, T_0)/\hat{P}_0$
    - Velocity: $u = 0$
    - Concentration: $Y_0 = 1$
    - Temperature: $T_0 \to T_b$, $\tanh(x)$ profile for about 1 flame thickness area
- Boundary conditions: (cartesian coordinates)
    - Inner boundary conditions: `opening / freeflow / inletOutlet`
    - Outlet boundary conditions: `opening / freeflow / inletOutlet`
- Boundary conditions: (spherical coordinates)
    -

5. Results

Appendix

A1. Speed of sound

- Pressure from equation of state:

$$p = \rho e(\gamma - 1) = \frac{3}{4}(\gamma - 1)(3\pi^2)^{\frac{1}{3}}\hbar c_1 (\rho N)^{\frac{4}{3}} + \frac{1}{2}(\gamma - 1)N\frac{(3\pi^2)^{\frac{2}{3}}}{3\hbar c_1}\left(\frac{\rho^2}{N}\right)^{\frac{1}{3}}(kT)^2$$
$$= C_1\rho^{\frac{4}{3}} + C_2\rho^{\frac{2}{3}}$$

- The theoretical speed of sound is obtained by:

$$c = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s} = \sqrt{\frac{4}{3}C_1\rho^{\frac{1}{3}} + \frac{2}{3}C_2\rho^{-\frac{1}{3}}} \approx \sqrt{\gamma\frac{p}{\rho}}$$

A2. Eigenvector of Jacobian

- Eigen vector for `Roe` method:
$U = [\rho, \rho u, \rho E, \rho Y] = [m_1, m_2, m_3, m_4]$
$F = [\rho u, \rho u^2 + p, \rho u E + up, \rho u Y] = [m_2, m_2^2/m_1 + p, \ (m_3 + p)m_2/m_1, m_4 m_2/m_1]$
where $p = (\gamma - 1)\rho(E - \frac{1}{2}u^2 - QY)$
    - Solve by `Matlab` symbolic calculation:

```
% definition
syms m1 m2 m3 m4 real
syms Q gamma real
syms m1 m3 m4 gamma Q positive

% declaration
rho = m1;
u = m2/m1;
E = m3/m1;
Y = m4/m1;
e = E-1/2*u*u-Q*Y
```

```
p = (gamma-1)*rho*e;
Um = [m1 m2 m3 m4];
Fm = [rho*u, rho*u*u+p; rho*u*E+u*p; rho*u*Y];

% solve
J = jacobian(Fm,Um);
[RM,RD] = eig(J);
[LM,LD] = eig(J');
```

o The Jacobian matrix is:

$$
\boldsymbol{J} = \begin{bmatrix}
0 & 1 & 0 & 0 \\
-u^2 + K_1 & 2u + K_2 & K_3 & K_4 \\
-uH + uK_1 & H + uK_2 & u(1 + K_3) & uK_4 \\
-uY & Y & 0 & u
\end{bmatrix}
$$

where $K_1 = (\gamma - 1)u^2/2$, $K_2 = (\gamma - 1)(-u)$, $K_3 = (\gamma - 1)$, $K_4 = (\gamma - 1)(-Q)$

o Eigenvalue Λ is:

$$
\boldsymbol{\Lambda} = \begin{bmatrix}
u - c & & & \\
& u & & \\
& & u & \\
& & & u + c
\end{bmatrix}
$$

where c is the numerical speed of sound:

$$
c = \frac{2}{3}\sqrt{E - \frac{1}{2}u^2 - QY} = \frac{2}{3}\sqrt{\frac{p}{\rho}\frac{1}{\gamma - 1}} = \sqrt{\gamma p/\rho}
$$

o Right eigen matrix $\boldsymbol{R}$ is: (each column is an eigenvector)

$$
\boldsymbol{R} = \begin{bmatrix}
1 & 1 & 0 & 1 \\
u - c & u & 0 & u + c \\
H - uc & u^2/2 & Q & H + uc \\
Y & 0 & 1 & Y
\end{bmatrix}
$$

o Left eigen matrix $\boldsymbol{L}$ is: (each row is an eigenvector)

$$
\boldsymbol{L} = \begin{bmatrix}
-\frac{u}{2c} - \frac{K_1}{2c^2} & \frac{1}{2c} + \frac{uK_3}{2c^2} & -\frac{K_3}{2c^2} & -\frac{K_4}{2c^2} \\
-\frac{1}{2} + \frac{K_1}{2c^2} & -\frac{uK_3}{2c^2} & \frac{K_3}{2c^2} & \frac{K_4}{2c^2} \\
-Y & 0 & 0 & 1 \\
-\frac{u}{2c} + \frac{K_1}{2c^2} & \frac{1}{2c} - \frac{uK_3}{2c^2} & \frac{K_3}{2c^2} & \frac{K_4}{2c^2}
\end{bmatrix}
$$

o To meet the eigen-decomposition criteria and to be used in the eigen-space projection, re-projection step, one can get the eigen-space as: (let $\gamma_1 \equiv \gamma - 1$, then $K_1 = \gamma_1 u^2/2$, $K_3 = \gamma_1$, $K_4 = -\gamma_1 Q$)

$$
\boldsymbol{R_F} = \begin{bmatrix}
\frac{1}{c} & \frac{1}{c} & 0 & \frac{1}{c} \\
\frac{u}{c} - 1 & \frac{u}{c} & 0 & \frac{u}{c} + 1 \\
\frac{H}{c} - u & \frac{u^2}{2c} & \frac{Q}{c} & \frac{H}{c} + u \\
\frac{Y}{c} & 0 & \frac{1}{c} & \frac{Y}{c}
\end{bmatrix}, \quad
\boldsymbol{R_F^{-1}} = \begin{bmatrix}
\frac{u}{2} + \frac{\gamma_1 u^2}{4c} & -\frac{1}{2} - \frac{\gamma_1 u}{2c} & \frac{\gamma_1}{2c} & -\frac{\gamma_1 Q}{2c} \\
c - \frac{\gamma_1 u^2}{2c} & \frac{\gamma_1 u}{c} & -\frac{\gamma_1}{c} & \frac{\gamma_1 Q}{c} \\
-\frac{\gamma_1 u^2 Y}{2c} & \frac{\gamma_1 uY}{c} & -\frac{\gamma_1 Y}{c} & c + \frac{\gamma_1 QY}{c} \\
-\frac{u}{2} + \frac{\gamma_1 u^2}{4c} & \frac{1}{2} - \frac{\gamma_1 u}{2c} & \frac{\gamma_1}{2c} & -\frac{\gamma_1 Q}{2c}
\end{bmatrix}^T
$$

which satisfies $\boldsymbol{R_F R_F^{-1}} = \boldsymbol{I}$ and $\boldsymbol{R_F \Lambda R_F^{-1}} = J$.

o This has been validated in `Matlab` code:

```
syms u real
syms rho E Y real positive
syms Q gamma real positive

e = (E - 1/2*u*u - Q*Y);
p = (gamma-1)*rho*e;
c = sqrt(gamma*p/rho);
H = E + p/rho;

K1 = (gamma-1)*u*u/2;
K2 = (gamma-1)*(-u);
K3 = (gamma-1);
K4 = (gamma-1)*(-Q);

J = [
  [          0,      1,         0,     0];
  [   -u*u+K1, 2*u+K2,         K3,    K4]
  [-u*H+u*K1, H+u*K2,  u*(1+K3), u*K4]
  [      -u*Y,      Y,         0,     u];
];

D = [
  [u-c, 0, 0,    0];
  [  0, u, 0,    0];
  [  0, 0, u,    0];
  [  0, 0, 0, u+c];
];

RF = [
  [1/c,        1/c,       0,      1/c];
  [u/c-1,      u/c,       0,    u/c+1];
  [H/c-u, u*u/2/c,     Q/c,    H/c+u];
  [Y/c,          0,     1/c,     Y/c];
];

g1 = gamma-1;
c2 = 2*c;
c4 = 4*c;
LF = [
  [ u/2+g1*u*u/c4,   -1/2-g1*u/c2,   g1/c2,     -g1*Q/c2];
  [   c-g1*u*u/c2,        g1*u/c,   -g1/c,        g1*Q/c ];
  [  -Y*g1*u*u/c2,      Y*g1*u/c, -Y*g1/c,  c+Y*g1*Q/c ];
  [-u/2+g1*u*u/c4,    1/2-g1*u/c2,   g1/c2,     -g1*Q/c2];
]';

simplify(RF*LF')      % output is [ 1, 0, 0, 0]
                      %           [ 0, 1, 0, 0]
                      %           [ 0, 0, 1, 0]
                      %           [ 0, 0, 0, 1]

simplify(RF*D*LF' - J)  % output is [ 0, 0, 0, 0]
                      %             [ 0, 0, 0, 0]
                      %             [ 0, 0, 0, 0]
                      %             [ 0, 0, 0, 0]
```

Reference

[3] https://en.wikipedia.org/wiki/Divergence#Spherical_coordinates

[4] Landau L D , Lifshitz E M . Statistical Physics, Part 1[J]. Physics Today, 1980.

[5] Woosely. 2011. FLAMES IN TYPE Ia SUPERNOVA: DEFLAGRATION–DETONATION TRANSITION IN THE OXYGEN-BURNING FLAME

[6] http://www.astrophysicsspectator.org/topics/stars/FusionCarbonOxygen.html

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1. Wheeler J C, Harkness R P, Rep. Prog. Phys. 1990, 53:1467-1557 ↩

2. G. Xing, Y. Zhao, et al., Astro. J., 2017, 841:21 ↩

3. Fowler, W. A., Caughlan, G. R., & Zimmerman, B. A. 1975, ARA&A, 13, 69 ↩