Spherically Inward Propagating Flame of Type Ia Supernova

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1. Introduction

Supernova represent the catastrophic explosions that mark the end of the life of some stars. The ejected mass is of order 1 to 10 solar masses with bulk velocities ranging from a few thousand to a few tens of thousands of km/s. [1] The traditional single Chandrasekhar mass C–O White Dwarf burning is still considered to be responsible for a large population of type Ia supernova (SN Ia). Specifically, one of the key issues in its modeling is related to the flame acceleration and deflagration-detonation transition (DDT), with flame front instabilities being considered as a possible mechanism in driving the acceleration. The perspective of this proposal is to give a set of solutions of this problem. The milestones of this project are planed to be:

(a) Complete a set of code that can solve the 1D H_2-O_2 flame.

(b) Complete a set of code that can solve the simplest reacting flow in SN Ia conditions.

(c) Try to observe spherical flame acceleration with large Lewis number curvature effect.

(d) Try to observe pulsation in Xing & Zhao's case.

(e) Try to observe deflagration-detonation transition (DDT) in (d).

2. Basic Equations

2.1 Planar flame

- Governing equations for 1-dimensional, Cartesian coordinates, compressible and non-viscous reacting flow are:

$$\frac{\partial \boldsymbol{U}}{\partial t}+\frac{\partial \boldsymbol{F}(\boldsymbol{U})}{\partial x}=\frac{\partial \boldsymbol{D}(\boldsymbol{U})}{\partial x}+\boldsymbol{S}$$

  where $\boldsymbol{U}$ is the conserved variables vector, $\boldsymbol{F}$ is the flux vector, $\boldsymbol{D}$ is the diffusion flux, and $\boldsymbol{S}$ is the source term vector. The vectors are given by:

$$\boldsymbol{U}=\begin{bmatrix}\rho\\\rho u\\\rho E\\\rho Y\end{bmatrix},\quad \boldsymbol{F}(\boldsymbol{U})=\begin{bmatrix}\rho u\\\rho u^2+p\\\rho u(E+p/\rho)\\\rho uY\end{bmatrix},\quad \boldsymbol{D}(\boldsymbol{U})=\begin{bmatrix}0\\0\\\frac{\mu C_p}{Pr}\frac{\partial T}{\partial x}+\frac{\mu Q}{Sc}\frac{\partial Y}{\partial x}\\\frac{\mu}{Sc}\frac{\partial Y}{\partial x}\end{bmatrix},\quad \boldsymbol{S}=\begin{bmatrix}0\\0\\0\\\rho\omega\end{bmatrix}$$

Here ρ is density, u is velocity, E is the total energy per unit mass and Y is the reactant mass fraction, respectively. The total energy is consisted by internal energy, kinetic energy and the "chemical" energy in reactant:

$$\rho E=\rho e+\frac{1}{2}\rho u^2+\rho QY$$

> The viscosity $\mu=\rho\nu$
>
> The Schmidt number $Sc=\nu/D=\mu/\rho D$
>
> The Prandtl number $Pr=\nu/\alpha=\mu c_p/\lambda$
>
> The enthalpy per unit mass $H=E+p/\rho$
>
> The heat value part might need to be put in internal energy part

- **In the scenario of methane flame**, the internal energy per unit mass is given by:

  $$e=\frac{p}{\rho(\gamma-1)}=\frac{R_uT}{\gamma-1}$$

  and the reaction rate is

  $$\omega=-KY\exp\left(-E_a/R_uT\right)$$

- **In the scenario of supernova flame**, the internal energy is described by the equation of state of the relativistic degenerate electron gas [2]:

  $$e(\rho,T)=\frac{3}{4\rho}(3\pi^2)^{\frac{1}{3}}\hbar c_1(\rho N)^{\frac{4}{3}}+\frac{1}{2}N\frac{(3\pi^2)^{\frac{2}{3}}}{3\hbar c_1}\left(\frac{1}{\rho N}\right)^{\frac{1}{3}}(kT)^2$$

  With pressure being $p=\rho e/3$, one can get the form as: (constant $\gamma=4/3$)

  $$e=\frac{p}{\rho(4/3-1)}=\frac{p}{\rho(\gamma-1)}$$

  The reaction rate is adopted from the rate of Carbon-Carbon nuclear reaction [3]:

  $$\omega=AY^2\exp\left(-E_a/T_9^{\frac{1}{3}}\right)$$

  where $T_9=T/10^9$ and $E_a=84.165$ for C12-C12 reaction.

  > N is Carbon's electron density
  > c_1 is the light speed in vacuum
  > $\hbar$ is the reduced Plank constant

2.2 Spherical flame

- In spherical coordinates, the governing equations change to be:

  $$\frac{\partial\boldsymbol{U}}{\partial t}+\frac{1}{r^2}\frac{\partial r^2\boldsymbol{F}(\boldsymbol{U})}{\partial r}=\frac{1}{r^2}\frac{\partial r^2\boldsymbol{D}(\boldsymbol{U})}{\partial r}+\boldsymbol{S}$$

  or in the expanded form:

$$\frac{\partial \boldsymbol{U}}{\partial t}+\frac{\partial \boldsymbol{F}(\boldsymbol{U})}{\partial r}+\frac{2}{r}\boldsymbol{F}(\boldsymbol{U})=\frac{\boldsymbol{D}(\boldsymbol{U})}{\partial r}+\frac{2}{r}\boldsymbol{D}(\boldsymbol{U})+\boldsymbol{S}$$

with the vectors being:

$$\boldsymbol{U}=\begin{bmatrix}\rho\\ \rho u\\ \rho E\\ \rho Y\end{bmatrix},\quad \boldsymbol{F}(\boldsymbol{U})=\begin{bmatrix}\rho u\\ \rho u^2+p\\ \rho uH\\ \rho uY\end{bmatrix},\quad \boldsymbol{D}(\boldsymbol{U})=\begin{bmatrix}0\\ 0\\ \lambda\frac{\partial T}{\partial r}+\rho QD\frac{\partial Y}{\partial r}\\ \rho D\frac{\partial Y}{\partial r}\end{bmatrix},\quad \boldsymbol{S}=\begin{bmatrix}0\\ \frac{2p}{r}\\ 0\\ \rho\omega\end{bmatrix}$$

2.3 Parameters for H_2-O_2 flame

- Reference state ($H_2:O_2=2:1$)
    - pressure $p_0=10^5$ [Pa]
    - molar mass $M=2/3\times 2+1/3\times 16=6.667$ [g/mol]
    - specific gas constant $R_u=R/M=1247.17$ [$J/kg/K$]
    - temperature $T_0=300$ [K]
    - density $\rho_0=p_0/R_uT_0=0.2673$ [kg/m^3]
    - heat capacity ratio $\gamma=1.2$
    - velocity $u_0=\sqrt{R_uT_0}=611.68m/s$ ($S_L\approx 22m/s$)
    - flame thickness $\delta_f\approx 1\times 10^{-4}m$
    - time scale $t_0=\delta_f/u_0=1.63\times 10^{-7}$ [s]
    - heat value $Q\approx 50R_uT_0$ [J/kg]
    - thermal conductivity $\lambda\approx 0.13$ [$W/m/K$]
    - thermal diffusivity $\alpha\approx 1\times 10^{-4}$ [m^2/s]
    - Lewis number $Le\approx 0.3$
    - activation energy $Ea\approx 32R_uT_0$
    - pre-exponential factor $A\approx 2\times 10^8$?

3. Dimensionless Equations

3.1 References variables

Dimensionless variables	Reference values
$\hat\rho = \frac{\rho}{\rho_0}$	$\rho_0 = 3.5 \times 10^{10} kg/m^3$
$\hat u = \frac{u}{u_0}$	$u_0 = 7.871 \times 10^6 m/s, S_L = 466 m/s$ (laminar flame speed)
$r = \frac{\hat r}{\hat r_0}$	$r_0 = \delta_f = \frac{\lambda}{\rho_0 C_p S_t} = 9 \times 10^{-4} m$
$\hat t = \frac{t}{t_0}$	$t_0 = \frac{r_0}{u_0} = 1.143 \times 10^{-10} s$
$\hat T = \frac{T}{T_b - T_0}$	$T_0 = 1 \times 10^8 K; T_b = 3.2 \times 10^9 K$
$\hat p = \frac{p}{p_0}$	$p_0 = \rho_0 u_0^2$
$\hat E = \frac{E}{p_0/\rho_0} = \frac{E}{u_0^2}$	$e(\rho_0, T_b) = 6.1952 \times 10^{13} J/kg$
$\hat Y = \frac{Y}{Y_0}$	$Y_0 = 1$
$\hat\omega = \frac{\omega}{\omega_0}$	$\rho_0 \omega_0 \delta_f = \rho_0 Y_0 S_t$, $\omega_0 = \frac{Y_0 u_0}{r_0} \frac{S_t}{u_0}$
$ \hat \alpha = \frac{\alpha}{u_0 r_0} $	$\hat D = \frac{\hat\alpha}{Le} = \frac{2}{3} \frac{1}{Le}$
$\hat Q = \frac{Q}{u_0^2}$	$\hat Q = 0.904$
$\hat\lambda = \lambda / (\frac{\rho_0 r_0 u_0^3}{T_b - T_0})$	$\hat\lambda = \frac{S_t}{u_0} \frac{C_p (T_b - T_0)}{u_0^2} = 0.14541$

> Laminar flame speed: $S_L = 466 m/s$
>
> Turbulent flame speed: S_t, assumed to be equal to the speed of sound at burnt state

Therefore, the dimensionless equation can be obtained from: (use planar equation for demonstration)

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) &= 0 \\
\frac{\partial (\rho u)}{\partial t} + \frac{\partial}{\partial x}(\rho u^2 + p) &= 0 \\
\frac{\partial \rho E}{\partial t} + \frac{\partial}{\partial x}(\rho u E + up) &= \lambda \frac{\partial^2 T}{\partial x^2} + \rho Q D \frac{\partial^2 Y}{\partial x^2} \\
\frac{\partial \rho Y}{\partial t} + \frac{\partial}{\partial x}(\rho u Y) &= \rho D \frac{\partial^2 Y}{\partial x^2} + \rho \omega
\end{aligned}
$$

$\Rightarrow$

$$
\begin{aligned}
\frac{\rho_0}{t_0}\left[\frac{\partial \hat\rho}{\partial \hat t} + \frac{\partial}{\partial \hat x}(\hat\rho \hat u)\right] &= 0 \\
\frac{\rho_0 u_0}{t_0}\left[\frac{\partial (\hat\rho \hat u)}{\partial \hat t} + \frac{\partial}{\partial \hat x}(\hat\rho \hat u^2 + \hat p)\right] &= 0 \\
\frac{\rho_0 u_0^2}{t_0}\left[\frac{\partial \hat\rho \hat E}{\partial \hat t} + \frac{\partial}{\partial \hat x}(\hat\rho \hat u \hat E + \hat u \hat p)\right] &= \frac{\rho_0 u_0^3 r_0 (T_b - T_0)}{r_0^2 (T_b - T_0)}\left[\hat\lambda \frac{\partial^2 \hat T}{\partial \hat x^2}\right] + \frac{\rho_0 u_0^2 u_0 r_0 Y_0}{r_0^2}\left[\hat\rho \hat Q \hat D \frac{\partial^2 \hat Y}{\partial \hat x^2}\right] \\
\frac{\rho_0 Y_0}{t_0}\left[\frac{\partial \hat\rho \hat Y}{\partial \hat t} + \frac{\partial}{\partial \hat x}(\hat\rho \hat u \hat Y)\right] &= \frac{\rho_0 u_0 r_0 Y_0}{r_0^2}\left[\hat\rho \hat D \frac{\partial^2 \hat Y}{\partial \hat x^2}\right] + \frac{\rho_0 u_0 Y_0}{r_0} \frac{S_t}{u_0}[\hat\rho \hat\omega]
\end{aligned}
$$

$\Rightarrow$

$$\frac{\partial\hat{\rho}}{\partial\hat{t}}+\frac{\partial}{\partial\hat{x}}(\hat{\rho}\hat{u})=0$$

$$\frac{\partial(\hat{\rho}\hat{u})}{\partial\hat{t}}+\frac{\partial}{\partial\hat{x}}(\hat{\rho}\hat{u}^2+\hat{p})=0$$

$$\frac{\partial\hat{\rho}\hat{E}}{\partial\hat{t}}+\frac{\partial}{\partial\hat{x}}(\hat{\rho}\hat{u}\hat{H})=\hat{\lambda}\frac{\partial^2\hat{T}}{\partial\hat{x}^2}+\hat{\rho}\hat{Q}\hat{D}\frac{\partial^2\hat{Y}}{\partial\hat{x}^2}$$

$$\frac{\partial\hat{\rho}\hat{Y}}{\partial\hat{t}}+\frac{\partial}{\partial\hat{x}}(\hat{\rho}\hat{u}\hat{Y})=\hat{\rho}\hat{D}\frac{\partial^2\hat{Y}}{\partial\hat{x}^2}+\frac{S_t}{u_0}\hat{\rho}\hat{\omega}$$

3.2 Dimensionless parameters

- Reference velocity u_0:

  For simplification, the largest internal energy is normalized to be $\hat{e}(\rho,T_b)=e(\rho_0,T_b)/u_0^2=1$. Therefore, one has

$$u_0=\sqrt{e(\rho_0,T_b)}=\sqrt{6.1952\times10^{13}}=7.871\times10^6m/s$$

- Reference length scale $r_0=9\times10^{-4}m$
- Reference time scale $t_0=r_0/u_0=1.143\times10^{-10}s$
- Speed of sound at the burnt state $c=\sqrt{\gamma p/\rho}=\sqrt{\gamma(\gamma-1)e(\rho_0,T_b)}=5.247\times10^6m/s$
- Heat value $\hat{Q}=Q/u_0^2$, with data from Fowler 1975, for $C^{12}+C^{12}\rightarrow Mg^{24}$, $Q=13.931MeV$, thus

$$\begin{aligned}\hat{Q}&=\frac{13.931MeV\times(1.6022\times10^{-13}J/MeV)\times(6.022\times10^{23}mol^{-1})/(0.024kg/mol)}{u_0^2}\\&=\frac{5.6\times10^{13}J/kg}{6.1952\times10^{13}J/kg}=0.904\end{aligned}$$

- The speed ratio $S_t/u_0=2/3$
- Thermal conductivity $\hat{\lambda}$ is related to C_p, while the heat release is related to internal energy:

$$\tilde{C}_p(T_b-T_0)=\int_{T_0}^{T_b}C_pdT=e(\rho_0,T_b)-e(\rho_0,T_0)$$

$$\hat{\lambda}=\frac{S_t}{u_0}\frac{\tilde{C}_p(T_b-T_0)}{u_0^2}=0.14541$$

- The diffusivity D is described by Lewis number $D=\alpha/Le=\lambda/\rho C_pLe$, therefore

$$\hat{D}=\frac{\lambda}{\rho C_pLe}\frac{1}{u_0r_0}=\frac{1}{Le}\frac{S_t}{u_0}\frac{\delta_f}{r_0}=\frac{2}{3}\frac{1}{Le}$$

- The total energy form keeps the same in dimensionless form:

$$\hat{E}=\hat{e}+\frac{1}{2}\hat{u}^2+\hat{Q}\hat{Y},\ \hat{e}=\frac{\hat{p}}{\hat{\rho}(\gamma-1)}$$

  with the equation of state as:

$$p=\rho e(\gamma-1)=\frac{3}{4}(\gamma-1)(3\pi^2)^{\frac{1}{3}}\hbar c_1(\rho N)^{\frac{4}{3}}+\frac{1}{2}(\gamma-1)N\frac{(3\pi^2)^{\frac{2}{3}}}{3\hbar c_1}\left(\frac{\rho^2}{N}\right)^{\frac{1}{3}}(kT)^2$$

$$\begin{aligned}\hat{p}&=\frac{3}{4}\frac{1}{\rho_0u_0^2}(\gamma-1)(3\pi^2)^{\frac{1}{3}}\hbar c_1(\rho_0N)^{\frac{4}{3}}\hat{\rho}^{\frac{4}{3}}+\frac{1}{2}(\gamma-1)N\frac{(3\pi^2)^{\frac{2}{3}}}{3\hbar c_1}\left(\frac{\rho_0{}^2}{N}\right)^{\frac{1}{3}}k^2(T_b-T_0)^2\hat{\rho}^{\frac{2}{3}}\hat{T}^2\\&=C_1\hat{\rho}^{4/3}+C_2\hat{\rho}^{2/3}\hat{T}^2\end{aligned}$$

with $C_1 = 2.60559 \times 10^{-1}, C_2 = 6.82973 \times 10^{-2}$

- The source term equation:

$$\omega_0 = \frac{Y_0 u_0}{r_0} \frac{S_t}{u_0} \approx A Y_0^2 \exp\left(-E_a / T_{b9}^{\frac{1}{3}}\right)$$

$$\hat{\omega} = \frac{\omega}{\omega_0} = \hat{Y}^2 \exp\left(-Ea/T_9^{\frac{1}{3}} + Ea/T_{9b}^{\frac{1}{3}}\right) = \hat{Y}^2 \exp\left(-Ea\left(\frac{T_b - T_0}{10^9}\right)^{-\frac{1}{3}}\left(\hat{T}^{-\frac{1}{3}} - \hat{T}_b^{-\frac{1}{3}}\right)\right)$$

to reduce the term S_t/u_0 in previous equation along with $\hat{\omega}$, denoting

$$\widehat{Ea} = -Ea\left(\frac{T_b - T_0}{10^9}\right)^{-\frac{1}{3}} = 57.7224$$
$$\hat{A} = \frac{S_t}{u_0} \exp\left(Ea\left(\frac{T_b - T_0}{10^9}\right)^{-\frac{1}{3}} \hat{T}_b^{-\frac{1}{3}}\right) = 4.25133 \times 10^{24}$$

then the source term keeps the same in dimensionless form:

$$\hat{\omega} = \hat{A}\hat{Y}^2 \exp\left(-\widehat{Ea}/\hat{T}^{\frac{1}{3}}\right)$$

- And the final dimensionless equations are:

$$\frac{\partial \hat{\rho}}{\partial \hat{t}} + \frac{\partial}{\partial \hat{x}}(\hat{\rho}\hat{u}) = 0$$
$$\frac{\partial (\hat{\rho}\hat{u})}{\partial \hat{t}} + \frac{\partial}{\partial \hat{x}}(\hat{\rho}\hat{u}^2 + \hat{p}) = 0$$
$$\frac{\partial \hat{\rho}\hat{E}}{\partial \hat{t}} + \frac{\partial}{\partial \hat{x}}(\hat{\rho}\hat{u}\hat{H}) = \hat{\lambda}\frac{\partial^2 \hat{T}}{\partial \hat{x}^2} + \hat{\rho}\hat{Q}\hat{D}\frac{\partial^2 \hat{Y}}{\partial \hat{x}^2}$$
$$\frac{\partial \hat{\rho}\hat{Y}}{\partial \hat{t}} + \frac{\partial}{\partial \hat{x}}(\hat{\rho}\hat{u}\hat{Y}) = \hat{\rho}\hat{D}\frac{\partial^2 \hat{Y}}{\partial \hat{x}^2} + \hat{\rho}\hat{\omega}$$
$$\hat{E} = \hat{e} + \frac{1}{2}\hat{u}^2 + \hat{Q}\hat{Y}$$
$$\hat{e} = \frac{\hat{p}}{\hat{\rho}(\gamma - 1)}$$
$$\hat{p} = C_1 \hat{\rho}^{4/3} + C_2 \hat{\rho}^{2/3}\hat{T}^2$$
$$\hat{\omega} = \hat{A}\hat{Y}^2 \exp\left(-\widehat{Ea}/\hat{T}^{\frac{1}{3}}\right)$$

4. Numerical methods

- Discrete methods:
  - Time evolution: 3 order TVD `Runge-Kutta`

$$U^{(1)} = U_n + L(U_n)\Delta t$$
$$U^{(2)} = \frac{3}{4}U_n + \frac{1}{4}(U^{(1)} + L(U^{(1)})\Delta t)$$
$$U_{n+1} = \frac{1}{3}U_n + \frac{2}{3}(U^{(2)} + L(U^{(2)})\Delta t)$$

  - Convection term:

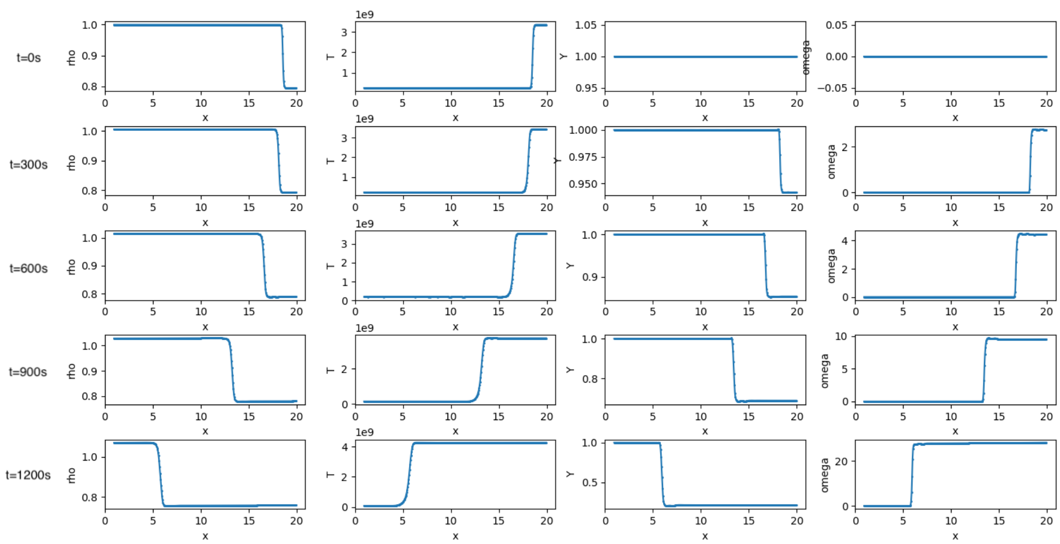
    `Roe` method: solve convection flux by eigen vector
  - Diffusion term: 7 order central difference

$$\frac{\partial m_i}{\partial x} = \frac{1}{\Delta x}\left(\frac{1}{60}m_{i+3} - \frac{9}{60}m_{i+2} + \frac{45}{60}m_{i+1} - \frac{45}{60}m_{i-1} + \frac{9}{60}m_{i-2} - \frac{1}{60}m_{i-3}\right)$$

- Initial conditions:
  - Pressure: whole field the same pressure $\hat{P}(\rho_0, T_0)/\hat{P}_0$
  - Velocity: near zero (can not set zero beacuse of Roe method eigen vector has 1/velocity)
  - Concentration: $Y_0 = 1$
  - Temperature: $T_0 \rightarrow T_b$, $\tanh{(x)}$ profile for about 1 flame thickness area
- Boundary conditions: (cartesian coordinates)
  - Inner boundary conditions:
    - symmetric
  - Outlet boundary conditions
    - symmetric
- Boundary conditions: (spherical coordinates)
  - Inner boundary conditions:
    - conservation ($\int \dot{\rho_{inner}} dV = \int \rho u S dt$)
  - Outer boundary conditions:
    - density: constant
    - velocity: extrapolatation ($m_1 = 2m_0 - m_{-1}$)
    - energy: constant
    - concentration: constant

5. Results

- 1D cartesian coordinates:
  - no reaction acceleration
    - distribution ρ, T, Y, ω of at different time (dimensionless time):



    - flame position and flame velocity (dimensionless time):

Accelerate as exponential function type and propagates with maximum velocity about 0.05 sound speed

- 1D sphercial coordinates:

Appendix

A1. Speed of sound

- Pressure from equation of state:

$$
\begin{aligned}
p = \rho e(\gamma-1) &= \frac{3}{4}(\gamma-1)(3\pi^2)^{\frac{1}{3}}\hbar c_1(\rho N)^{\frac{4}{3}} + \frac{1}{2}(\gamma-1)N\frac{(3\pi^2)^{\frac{2}{3}}}{3\hbar c_1}\left(\frac{\rho^2}{N}\right)^{\frac{1}{3}}(kT)^2 \\
&= C_1\rho^{\frac{4}{3}} + C_2\rho^{\frac{2}{3}}
\end{aligned}
$$

- The theoretical speed of sound is obtained by:

$$
c = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s} = \sqrt{\frac{4}{3}C_1\rho^{\frac{1}{3}} + \frac{2}{3}C_2\rho^{-\frac{1}{3}}} \approx \sqrt{\gamma\frac{p}{\rho}}
$$

A2. Eigenvector of Jacobian

- Eigen vector for `Roe` method:
  $U = [\rho, \rho u, \rho E, \rho Y] = [m_1, m_2, m_3, m_4]$
  $F = [\rho u, \rho u^2 + p, \rho uE + up, \rho uY] = [m_2, m_2^2/m_1 + p,\ (m_3+p)m_2/m_1, m_4m_2/m_1]$
  where $p = (\gamma-1)\rho(E - \frac{1}{2}u^2 - QY)$
  - Solve by `Matlab` symbolic calculation:

```
% definition
syms m1 m2 m3 m4 real
syms Q gamma real
syms m1 m3 m4 gamma Q positive

% declaration
rho = m1;
u = m2/m1;
E = m3/m1;
Y = m4/m1;
e = E-1/2*u*u-Q*Y
p = (gamma-1)*rho*e;
Um = [m1 m2 m3 m4];
Fm = [rho*u, rho*u*u+p; rho*u*E+u*p; rho*u*Y];

% solve
J = jacobian(Fm,Um);
[RM,RD] = eig(J);
[LM,LD] = eig(J');
```

  - The Jacobian matrix is:

$$
\boldsymbol{J} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -u^2 + K_1 & 2u + K_2 & K_3 & K_4 \\ -uH + uK_1 & H + uK_2 & u(1+K_3) & uK_4 \\ -uY & Y & 0 & u \end{bmatrix}
$$

  where $K_1 = (\gamma-1)u^2/2,\ K_2 = (\gamma-1)(-u),\ K_3 = (\gamma-1),\ K_4 = (\gamma-1)(-Q)$
  - Eigenvalue Λ is:

$$\mathbf{\Lambda} = \begin{bmatrix} u-c & & & \\ & u & & \\ & & u & \\ & & & u+c \end{bmatrix}$$

where c is the numerical speed of sound:

$$c = \frac{2}{3}\sqrt{E - \frac{1}{2}u^2 - QY} = \frac{2}{3}\sqrt{\frac{p}{\rho}\frac{1}{\gamma-1}} = \sqrt{\gamma p/\rho}$$

- Right eigen matrix $\boldsymbol{R}$ is: (each column is an eigenvector)

$$\boldsymbol{R} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ u-c & u & 0 & u+c \\ H-uc & u^2/2 & Q & H+uc \\ Y & 0 & 1 & Y \end{bmatrix}$$

- Left eigen matrix $\boldsymbol{L}$ is: (each row is an eigenvector)

$$\boldsymbol{L} = \begin{bmatrix} -\frac{u}{2c} - \frac{K_1}{2c^2} & \frac{1}{2c} + \frac{uK_3}{2c^2} & -\frac{K_3}{2c^2} & -\frac{K_4}{2c^2} \\ -\frac{1}{2} + \frac{K_1}{2c^2} & -\frac{uK_3}{2c^2} & \frac{K_3}{2c^2} & \frac{K_4}{2c^2} \\ -Y & 0 & 0 & 1 \\ -\frac{u}{2c} + \frac{K_1}{2c^2} & \frac{1}{2c} - \frac{uK_3}{2c^2} & \frac{K_3}{2c^2} & \frac{K_4}{2c^2} \end{bmatrix}$$

- To meet the eigen-decomposition criteria and to be used in the eigen-space projection, re-projection step, one can get the eigen-space as: (let $\gamma_1 \equiv \gamma - 1$, then $K_1 = \gamma_1 u^2/2$, $K_3 = \gamma_1$, $K_4 = -\gamma_1 Q$)

$$\boldsymbol{R_F} = \begin{bmatrix} \frac{1}{c} & \frac{1}{c} & 0 & \frac{1}{c} \\ \frac{u}{c} - 1 & \frac{u}{c} & 0 & \frac{u}{c} + 1 \\ \frac{H}{c} - u & \frac{u^2}{2c} & \frac{Q}{c} & \frac{H}{c} + u \\ \frac{Y}{c} & 0 & \frac{1}{c} & \frac{Y}{c} \end{bmatrix}, \; \boldsymbol{R_F^{-1}} = \begin{bmatrix} \frac{u}{2} + \frac{\gamma_1 u^2}{4c} & -\frac{1}{2} - \frac{\gamma_1 u}{2c} & \frac{\gamma_1}{2c} & -\frac{\gamma_1 Q}{2c} \\ c - \frac{\gamma_1 u^2}{2c} & \frac{\gamma_1 u}{c} & -\frac{\gamma_1}{c} & \frac{\gamma_1 Q}{c} \\ -\frac{\gamma_1 u^2 Y}{2c} & \frac{\gamma_1 u Y}{c} & -\frac{\gamma_1 Y}{c} & c + \frac{\gamma_1 QY}{c} \\ -\frac{u}{2} + \frac{\gamma_1 u^2}{4c} & \frac{1}{2} - \frac{\gamma_1 u}{2c} & \frac{\gamma_1}{2c} & -\frac{\gamma_1 Q}{2c} \end{bmatrix}^T$$

which satisfies $\boldsymbol{R_F}\boldsymbol{R_F^{-1}} = \boldsymbol{I}$ and $\boldsymbol{R_F}\mathbf{\Lambda}\boldsymbol{R_F^{-1}} = J$.

- This has been validated in `Matlab` code:

```
syms u real
syms rho E Y real positive
syms Q gamma real positive

e = (E - 1/2*u*u - Q*Y);
p = (gamma-1)*rho*e;
c = sqrt(gamma*p/rho);
H = E + p/rho;

K1 = (gamma-1)*u*u/2;
K2 = (gamma-1)*(-u);
K3 = (gamma-1);
K4 = (gamma-1)*(-Q);

J = [
  [        0,       1,          0,     0];
  [  -u*u+K1, 2*u+K2,          K3,    K4]
  [-u*H+u*K1, H+u*K2,  u*(1+K3), u*K4]
  [     -u*Y,       Y,          0,     u];
```

```
];

D = [
  [u-c, 0, 0,   0];
  [  0, u, 0,   0];
  [  0, 0, u,   0];
  [  0, 0, 0, u+c];
];

RF = [
  [1/c,        1/c,       0,      1/c];
  [u/c-1,      u/c,       0,    u/c+1];
  [H/c-u, u*u/2/c,     Q/c,    H/c+u];
  [Y/c,          0,     1/c,      Y/c];
];

g1 = gamma-1;
c2 = 2*c;
c4 = 4*c;
LF = [
  [ u/2+g1*u*u/c4,   -1/2-g1*u/c2,   g1/c2,     -g1*Q/c2];
  [   c-g1*u*u/c2,         g1*u/c,   -g1/c,       g1*Q/c ];
  [  -Y*g1*u*u/c2,       Y*g1*u/c, -Y*g1/c,   c+Y*g1*Q/c ];
  [-u/2+g1*u*u/c4,    1/2-g1*u/c2,   g1/c2,     -g1*Q/c2];
]';

simplify(RF*LF')        % output is [ 1, 0, 0, 0]
                        %           [ 0, 1, 0, 0]
                        %           [ 0, 0, 1, 0]
                        %           [ 0, 0, 0, 1]

simplify(RF*D*LF' - J)  % output is [ 0, 0, 0, 0]
                        %           [ 0, 0, 0, 0]
                        %           [ 0, 0, 0, 0]
                        %           [ 0, 0, 0, 0]
```

Reference

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