Quantitative Macroeconomics HW2

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1 Question 1: Computing Transitions in a Representative Agent Economy

Consider the following closed optimal growth economy populated by a large number of identical infinitely lived households that maximises:

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{1}$$

over consumption and leisure $u(c_t) = \ln c_t$, subject to:

$$c_t + i_t = y_t \tag{2}$$

$$y_t = k_t^{1-\theta}(zh_t)^{\theta} \tag{3}$$

$$i_t = k_{t+1} - (1-\delta)k_t \tag{4}$$

Set labour share to $\theta = 0.67$. Also, start with, set $h_t = 0.31\,\forall t$. Population does not grow.

1. **Compute the steady state. Choose z to match an annual capital-output ratio of 4, and an investment-output ratio** 0.25

   First, we need develop some equations, and work out some variables. We can define some relations between this variables in the steady-state as:

   (a) $\frac{K}{L}$ is the capital-output ratio.

   (b) $y = i + c$ if we normalize $y = 1$ we get the consumption in the steady-state i.e $c = 1 - i \rightarrow c = 1 - 0.25 \rightarrow c = 0.75$.

   (c) $i = k_{t+1} - (1-\delta)K_t$ since in the steady-state $k^* = k_t = k_{t+1}$, we get $\delta = \frac{i}{k}$, then we have from above that $k^* = 4$ and $i = 0.25$, thus $\delta = \frac{0.25}{4} = 0.0625$.

   (d) Now, we clear from equation 2 z_t.

   So, $\frac{1}{k_t^{1-\theta}} = z^{\theta} h_t^{\theta} \rightarrow z^{\theta} = \frac{1}{k_t^{1-\theta} h_t^{\theta}}$, then if we power by $1/\theta$, finally we arrive to

   $z = \left(\frac{1}{k_t^{1-\theta} h_t^{\theta}} \right)^{1/\theta} = 1.62$

   We maximised the problem of RA, in this economy by constructing the Lagrangian:

$$\mathcal{L}c_t, k_{t+1}, \lambda_t) = \mathbf{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t u(c_t) \right\} - \lambda_t (c_t + k_{t+1} - k_t^{1-\theta}(zh_t)^\theta - (1-\delta)k_t)$$

So, we take the first derivative on c_t,c_{t+1} and k_{t+1} and we get:

$$\frac{\partial \mathcal{L}}{\partial c_t} = 0 \rightarrow \beta^t u'(c_t) = \lambda_t \tag{5}$$

$$\frac{\partial \mathcal{L}}{\partial c_{t+1}} = 0 \rightarrow \beta^{t+1} t u'(c_{t+1}) = \lambda_{t+1} \tag{6}$$

$$\frac{\partial \mathcal{L}}{k_{t+1}} = 0 \rightarrow \lambda_t = \lambda_{t+1}((1-\theta)k_{t+1}^{-\theta}(zh_t)^\theta + (1-\delta)) \tag{7}$$

Using equation 5 and 6 into equation 7, we can obtain the Euler equation:

$$u'(c_t) = \beta u'(c_{t+1})((1-\theta)k_{t+1}^{-\theta}(zh_t)^\theta + (1-\delta)) \tag{8}$$

Imposing the steady state to 8 and clearing for k, we get:

$$k_{ss} = hz \left[\frac{\beta(1-\theta)}{1-\beta(1-\delta)} \right]^{\frac{1}{\theta}} \tag{9}$$

Now, We take the Euler equation 8 and in steady state, we clear the parameter beta, so:

$$\frac{1}{\beta} = (1-\theta)k_{ss}^{-\theta}(zh_t)^\theta + 1 - \delta \rightarrow \beta \approx 0.98$$

This part coincides with the part made in Python.

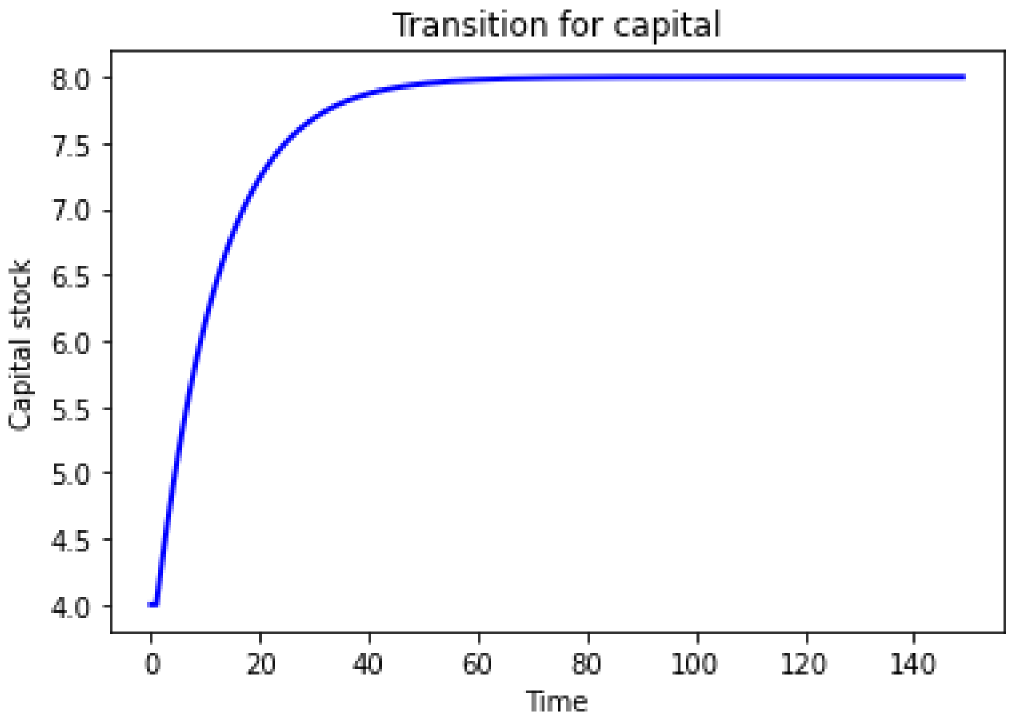
2. **Double permanently the productivity parameter z and solve for the new steady state.**

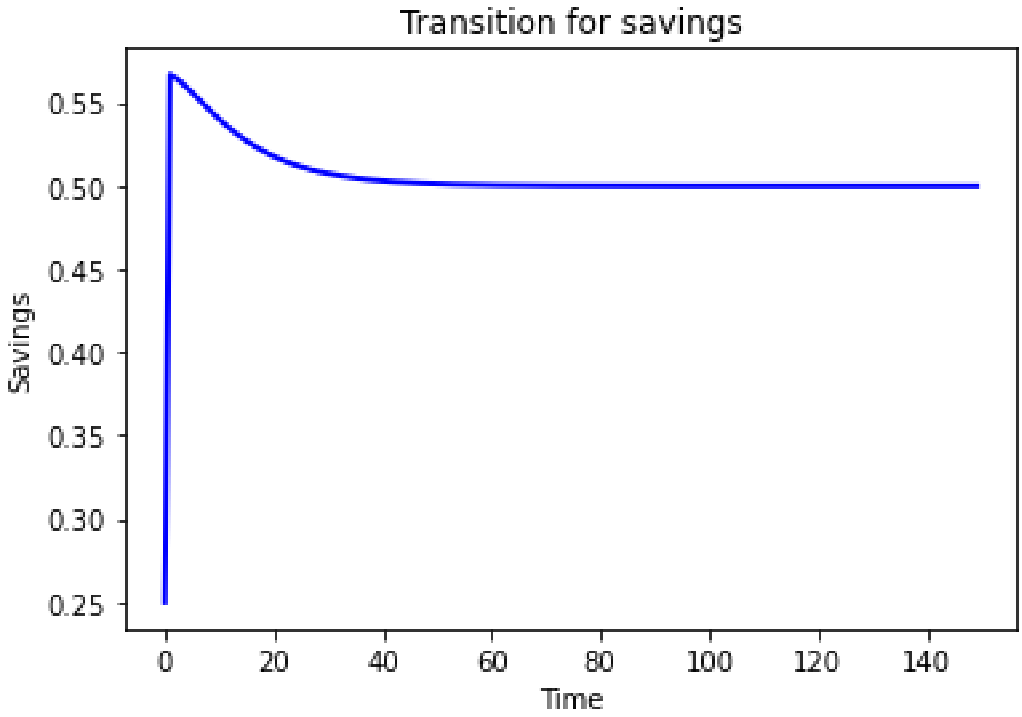
   We take the same parameters calculated before $(\delta, \beta, \theta, h_t)$. We suppose a shock in productivity, that is twice as big as the current one, thus $z \approx 3.26$. Now we can compute again the new steady state but using Python. The new values are: $c \approx 1.5, i \approx 0.5, y \approx 2, k \approx 8$.

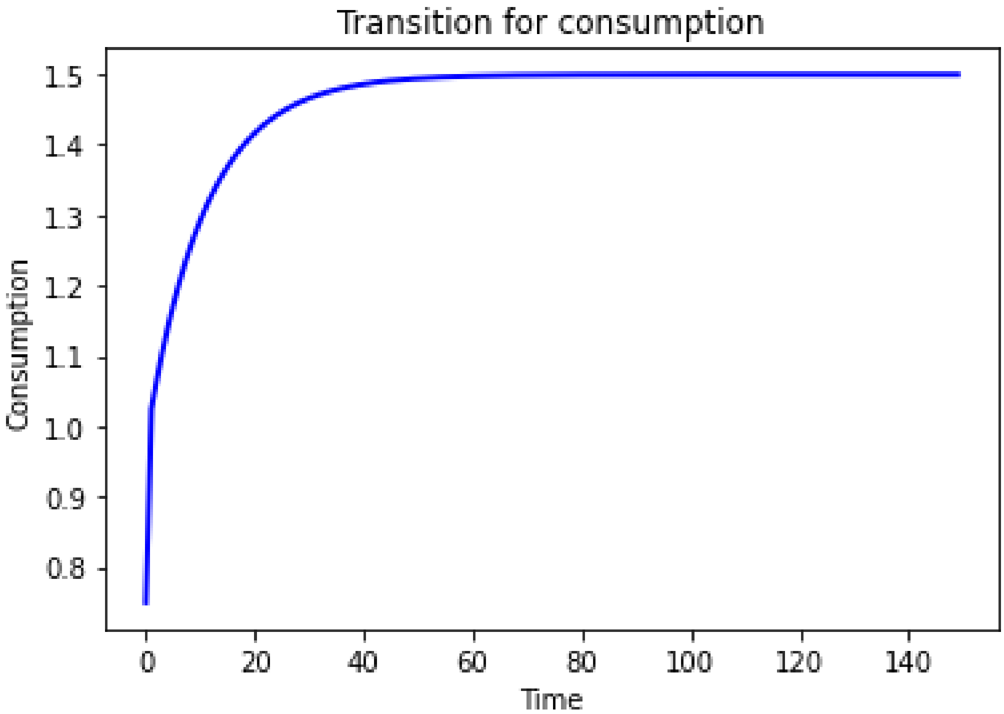
   All variables have doubled its values compared to before the shock.

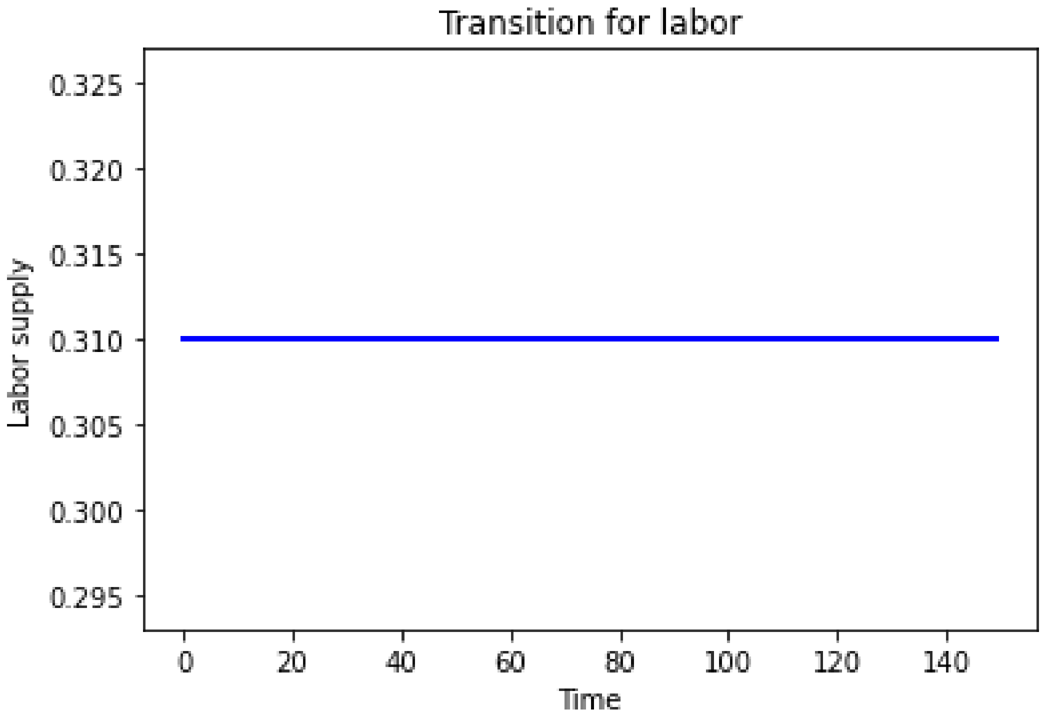
3. **Compute the transition from the first to the second steady state and report the time-path, for saving, consumption, labour and output.**

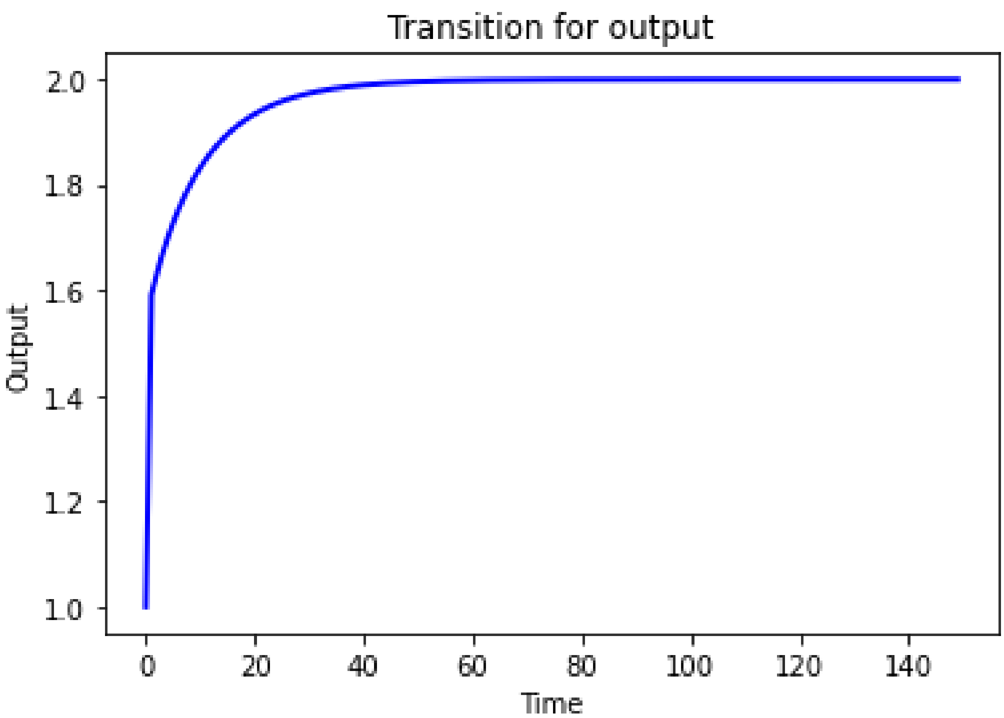
   We need to solve period by period the transitions to the new steady state, we create a loop that recorded the root of system of equations and that hold the Euler equations and capital law for each periods. We set up an "initial" Euler equation, a end Euler equation and an Euler equation for all periods between. Then finally, we get a sequences of capitals converging towards steady state.




Transition for savings
Savings
0.55
0.50
0.45
0.40
0.35
0.30
0.25
0
20
40
60
80
100
120
140
Time


Transition for consumption
Consumption
1.5
1.4
1.3
1.2
1.1
1.0
0.9
0.8
0
20
40
60
80
100
120
140
Time

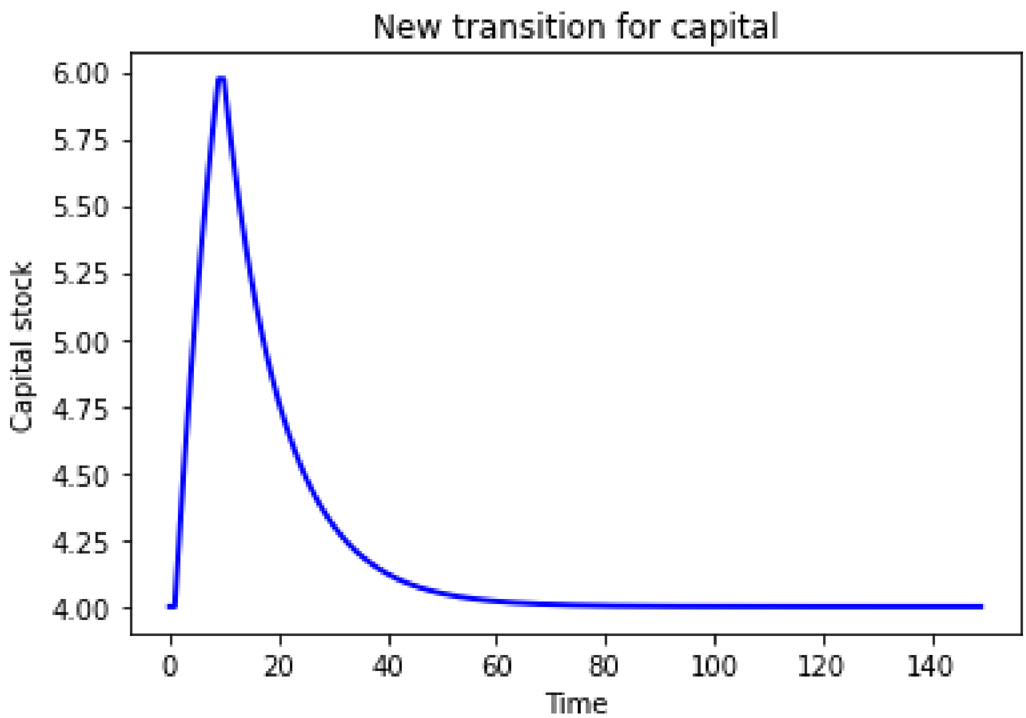


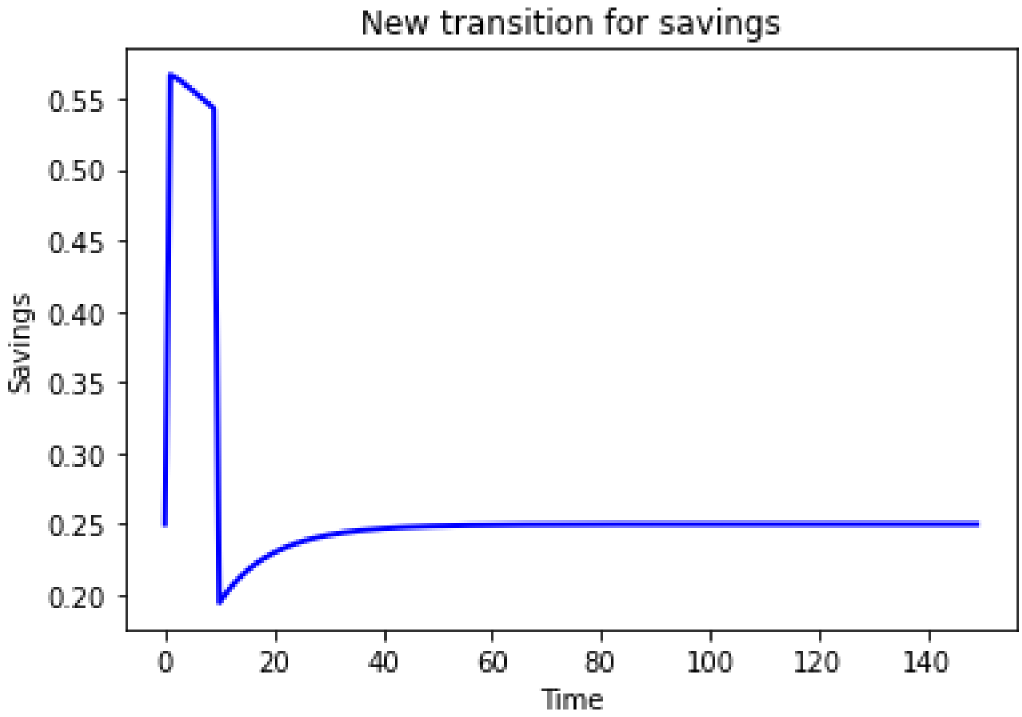


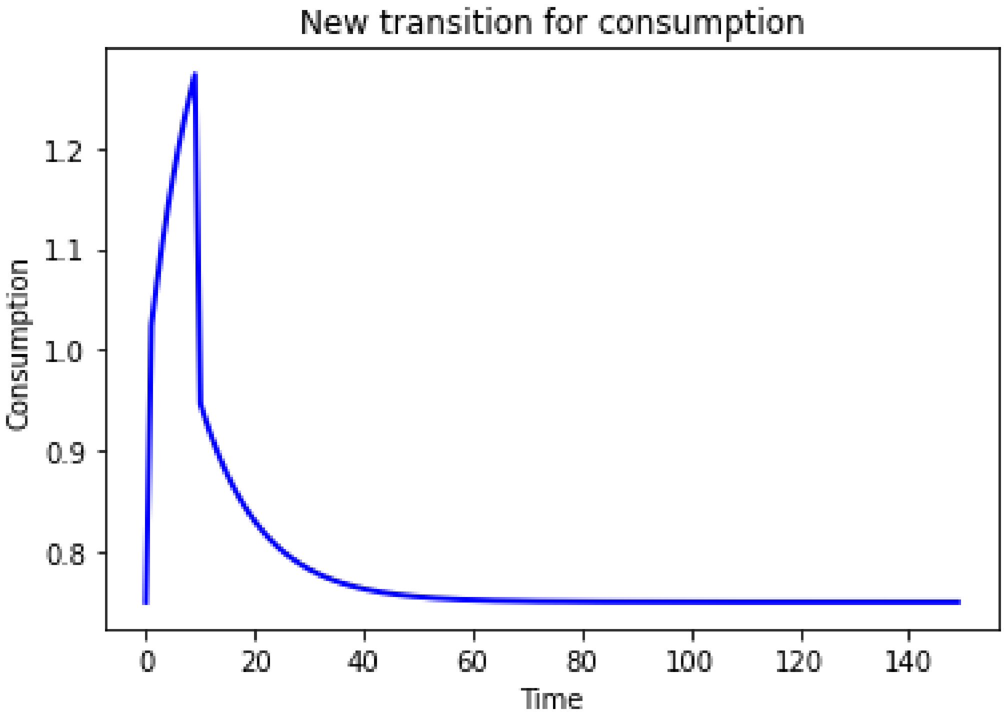
We can see in ??, ??, ??, ??, ??, all variables converge to the new steady state about 47 periods, the labour demand is always constant by assumption.Since the production function is a Cobb Douglas, the variables are complements so when one of them increase the other take the same way. Consumption towards a new steady state, this rise is due to the wealth effect. Investment has a high jump but after that start to decrease while the consumption rise. That is normal by definition of the restrictions of the problem. The output also increase although labour is constant but capital increases, and that leads to higher productivity capital output.

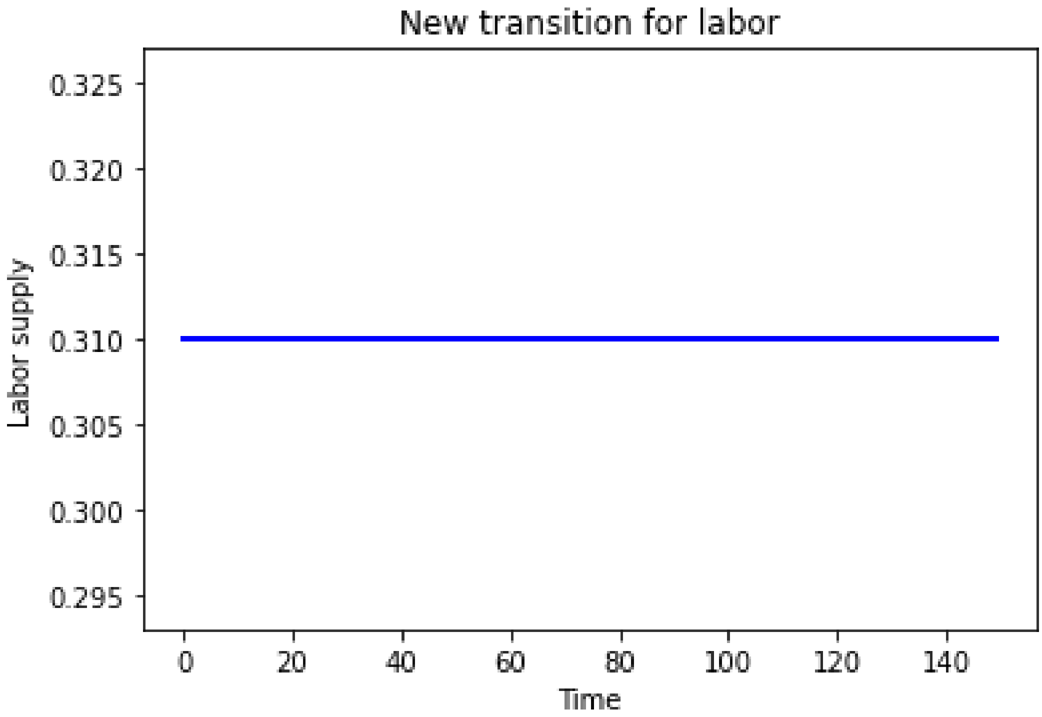
4. **Unexpected shocks. Let the agents believe productivity z_t doubles once and for all periods. However, after 10 periods, surprise the economy by cutting the productivity z_t backs to original value. Compute the transition for savings, consumption, labour and output.**

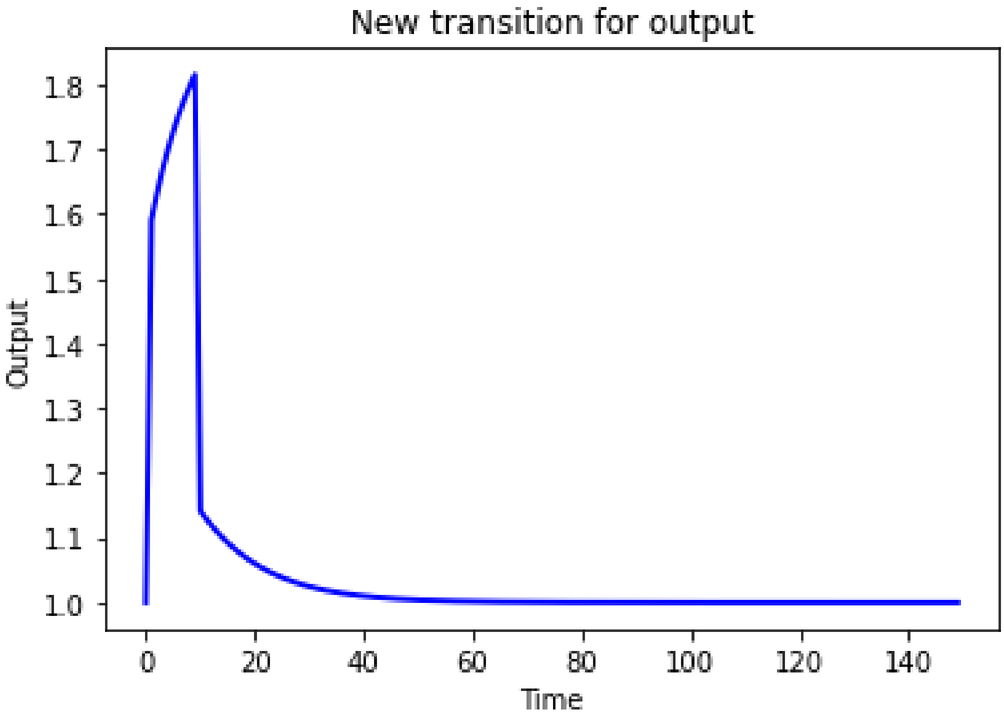
   Since the shock take place in the tenth period, we have the same as before, so in period 10 we introduced the shock in productivity, z come back to the initial value. The results we find in the following graphs.In this case, it is clear that all variables were following a path of convergence similar to the one in the previous question. All variables react in that same period with a significant increase, and then slowly adjust back to the initial steady state values.




New transition for savings
Savings
0.55
0.50
0.45
0.40
0.35
0.30
0.25
0.20
0
20
40
60
80
100
120
140
Time


New transition for consumption
Consumption
1.2
1.1
1.0
0.9
0.8
0
20
40
60
80
100
120
140
Time





5. **Bonus Question: Labour Choice Allow for elastic labour supply. That is, let preferences be**

$$u(c_t, 1-h_t) = \ln c_t - k\frac{h_t^{1+\frac{1}{v}}}{1+\frac{1}{v}} \tag{10}$$

and recompute the transition as posed in Question 1.

We solve for the following maximize problem, since the labour now is endogenous, we need to put into the objective function, also we need a new budget constraint for labour market clearing conditions, so we have:

$$\begin{aligned}
\max_{c,h} \quad & \sum_{t=0}^{\infty} \beta^t u(c_t, 1-ht) \qquad \text{subject to} \quad c_t + i_t = y_t \\
& y_t = k_t^{1-\theta}(zh_t)^{\theta} \\
& i_t = k_{t+1} - (1-\delta)k_t \\
& h_t + l_t = 1
\end{aligned}$$

The lagrangian associated is:

$$\mathcal{L}(c_t, c_t+1, h_t, k_{t+1}, \lambda) = \sum_{t=0}^{\infty} \beta^t u(c_t, 1-ht) -$$

$\lambda_t(c_t + k_{t+1} - k^{1-\theta}(zh_t)^{\theta} - (1-\delta)k_t) - \lambda_{t+1}(h_t + l_t - 1)$

So the FOC's:

$$\frac{\partial \mathcal{L}}{\partial c_t} = \beta^t u'(c_t) = \lambda_t \tag{11}$$

$$\frac{\partial \mathcal{L}}{\partial c_{t+1}} = \beta^{t+1} u'(c_{t+1}) = \lambda_{t+1} \tag{12}$$

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}} = \lambda_t = \lambda_{t+1}\left[(1-\theta)k_t^{-\theta}(zh_t)^{\theta} - (1-\delta)\right] \tag{13}$$

$$\frac{\partial \mathcal{L}}{\partial h_t} = -\beta_t^{t1/v} + \lambda_t k_t^{1-\theta} z^{\theta} h_t^{\theta-1}\theta - \lambda_{t+1} \tag{14}$$

From equation 1, 2 and 3 we get the Euler equation:

$$u'(c_t) = \beta u'(c_{t+1} \left[(1-\theta) k_t^{-\theta} (z h_t)^{\theta} + (1-\delta) \right] \tag{15}$$

From equation 1,2 and 4, we get:

$$\beta u'(c_{t+1}) + \kappa h_t^{1/v} = u'(c_t) \theta k_t^{1-\theta} z^{\theta} h_t^{\theta-1} \tag{16}$$

I try to compute the a and b, the results are in the python pdf, I had to fix z in order to obtain some results.

2 Question 2: Solve the optimal COVID-19 lockdown model posed in the slides.

The social planner problem is given by:

$$\max_{H_f, H_{nf}} \quad \left(A_f H_f^{\frac{\rho-1}{\rho}} + c(TW) A_{nf} H_{nf}^{\frac{\rho-1}{\rho}} \right)^{\frac{\rho}{\rho-1}} - \kappa_f H_f - \kappa_{nf} H_{nf} - \omega D$$

$$\text{subject to} \quad H_f + H_{nf} \leq N$$

$$D = (1-\gamma)\beta(HC)\frac{i_0 H_f^2}{N}$$

Replacing D into the objective function, we can build the lagrangian:

$$\text{L}(\text{H}_f, H_{nf}, \lambda) = \left(A_f H_f^{\frac{\rho-1}{\rho}} + c(TW) A_{nf} H_{nf}^{\frac{\rho-1}{\rho}} \right)^{\frac{\rho}{\rho-1}} - \kappa_f H_f - \kappa_{nf} H_{nf} - \omega \left((1-\gamma)\beta(HC)\frac{i_0 H_f^2}{N} \right) - \lambda(H_f + H_{nf} - N)$$

We take the derivative w.r.t. H_f and $H_n f$

$$\frac{\partial \mathcal{L}}{\partial H_f} = \left(A_f H_f^{\frac{\rho-1}{\rho}} + c(TW) A_{nf} H_{nf}^{\frac{\rho-1}{\rho}} \right)^{\frac{1}{\rho-1}} A_f H_f^{\frac{-1}{\rho}} = \kappa_f + 2\omega(1-\gamma)\beta(HC)\frac{i_0 H_f}{N} \tag{17}$$

$$\frac{\partial \mathcal{L}}{\partial H_f} = \left(A_f H_f^{\frac{\rho-1}{\rho}} + c(TW) A_{nf} H_{nf}^{\frac{\rho-1}{\rho}} \right)^{\frac{1}{\rho-1}} A_{nf} H_{nf}^{\frac{-1}{\rho}} c(TW) = \kappa_{nf} \tag{18}$$

1. **Show your results for a continuum of combinations of the $\in [0,1]$ parameter (vertical axis) and the $c(TW) \in [0,1]$ parameter (hztal axis). That is, pot for each pair of β and $c(TW)$ the optimal allocations of $H, H_f, H_{nf}, H_f/H$, output welfare, amount of infections and deaths. Note that if $H = N$ there is no lockdown, so pay attention to the potential non-binding constraint $H < N$. Discuss your results.**

   You may want to use the following parameters: $A_f = A_{nf} = 1; \rho = 1.1, k_f = k_{nf} = 0.2, \omega = 20, \gamma = 0.9, i_0 = 0.2$ **and** $N = 1$

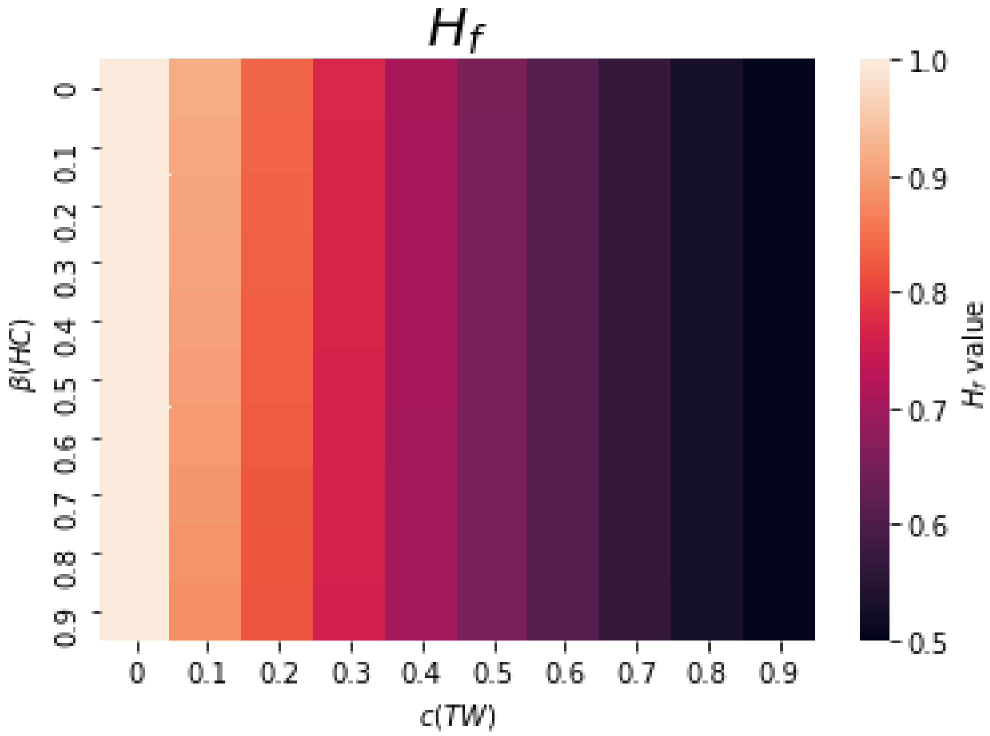
   First, we introduce a little of the variables of the model, so we can understand better what say us the graphs.

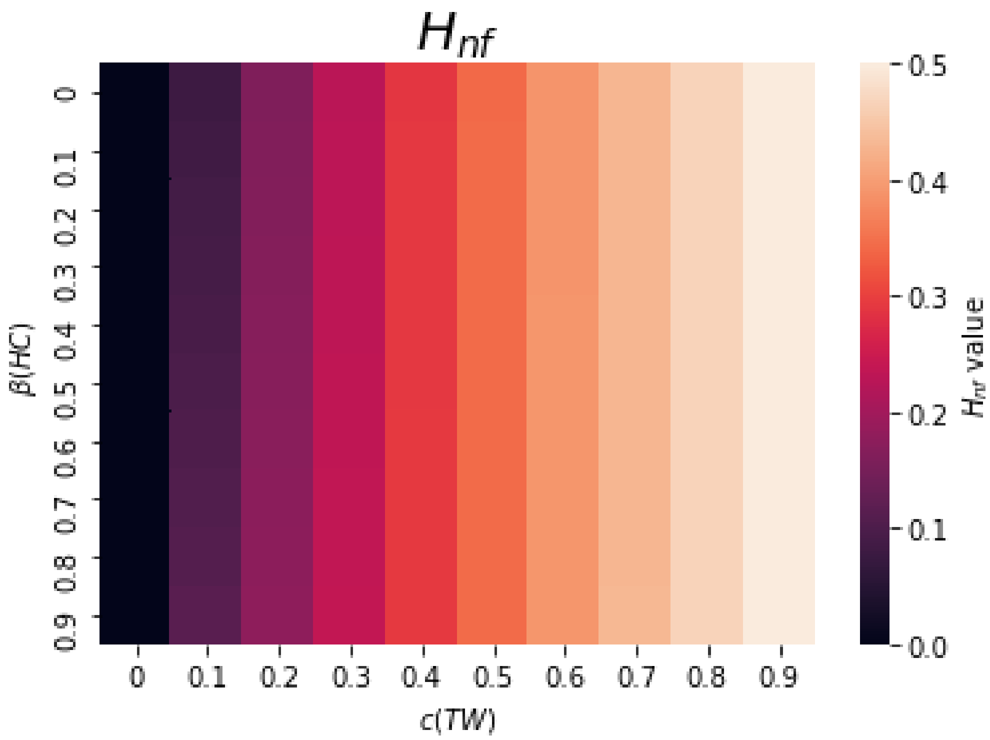
   - Production Technology: H_f is the aggregate hours at the workplace, in the other hand H_{nf} which denotes the aggregate hours teleworking, $c(TW)$ is a factor that captures a productivity loss associated with teleworking.

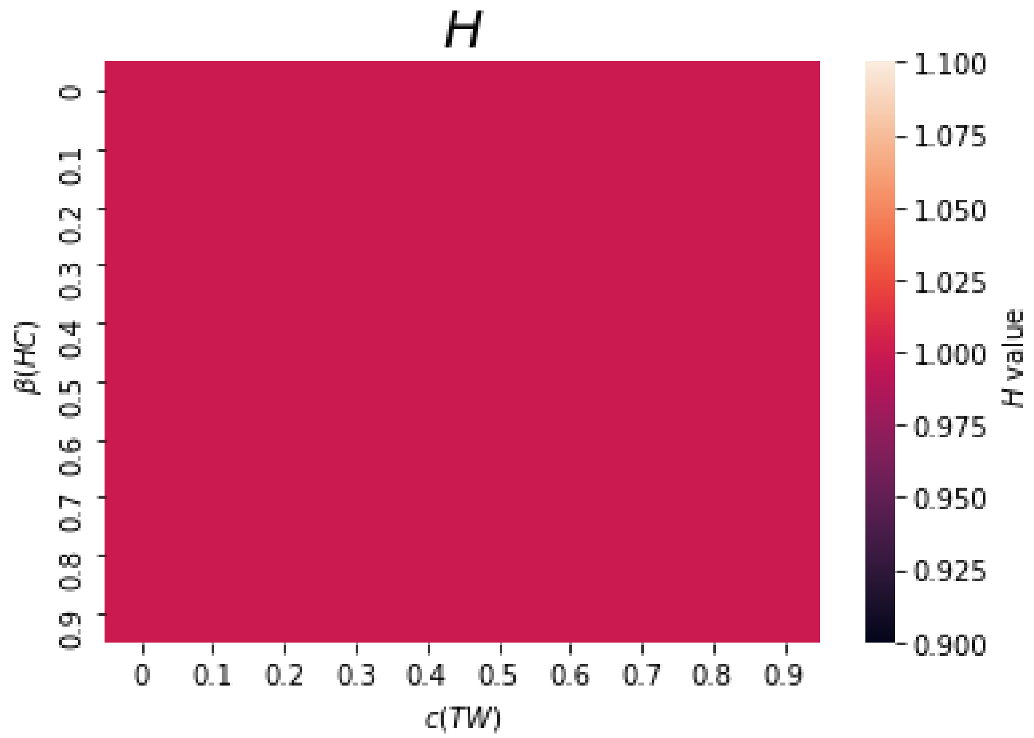
   - Contagion Process:

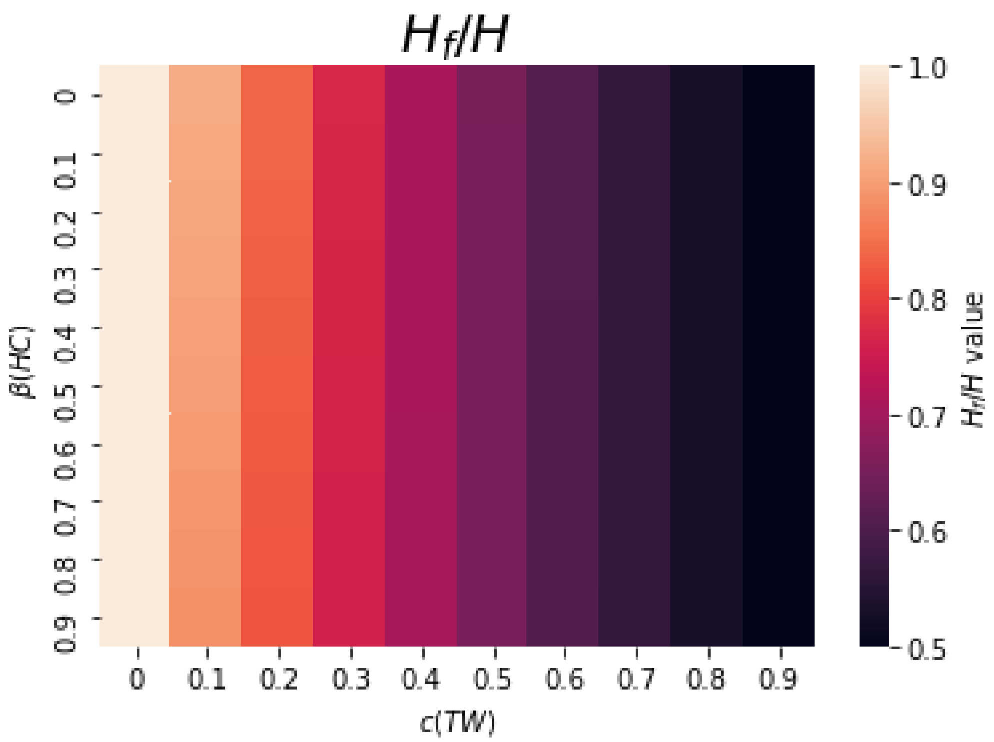
     The number of infections depends on the *human contact* that is not implicitly in the equation, this only happens at workplace, so I is the unconditional infection rate, β is the conditional infection rate which depends on the extend of human contact. D tell us which of these infections translate into death.

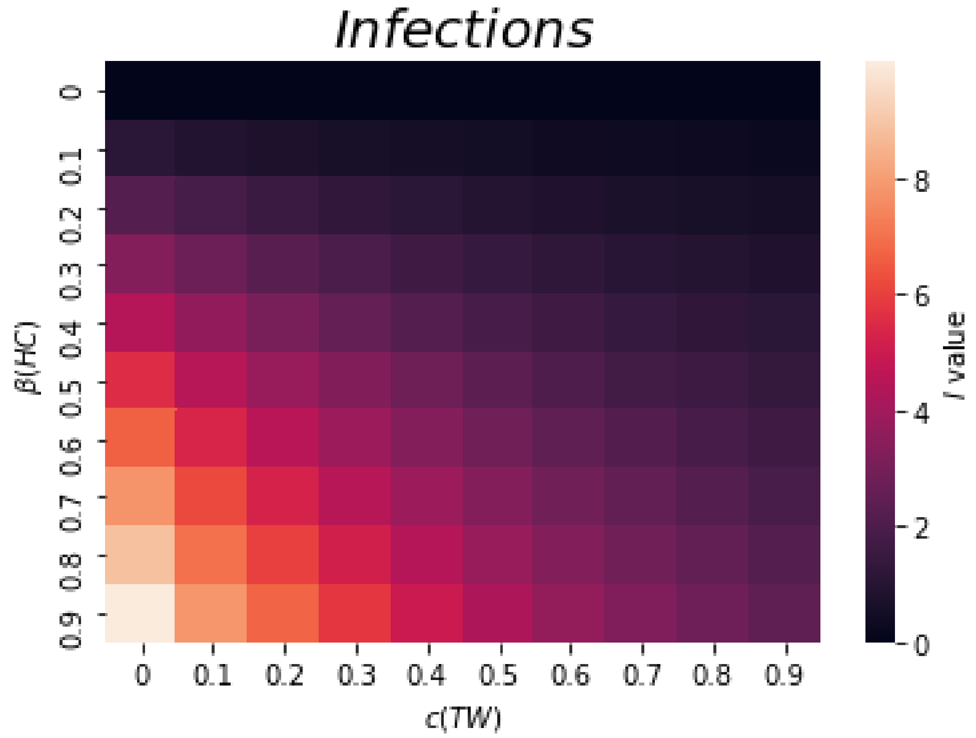
     The results are in the followings graphs:

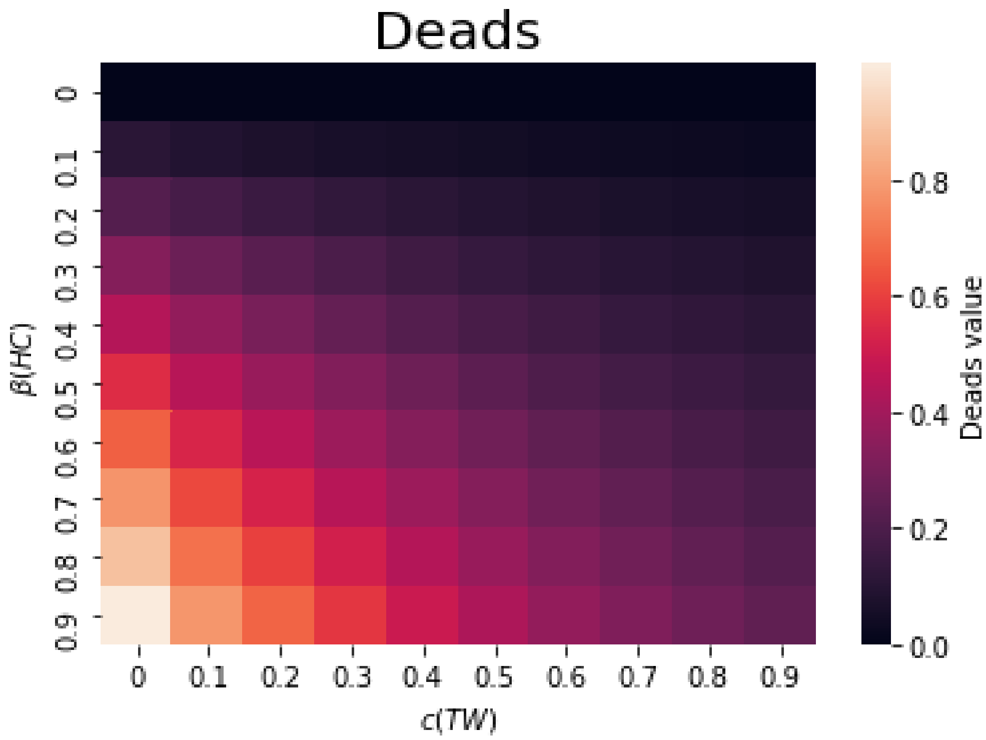

H_f
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
$\beta(HC)$
$c(TW)$
1.0
0.9
0.8
0.7
0.6
0.5
H_f value

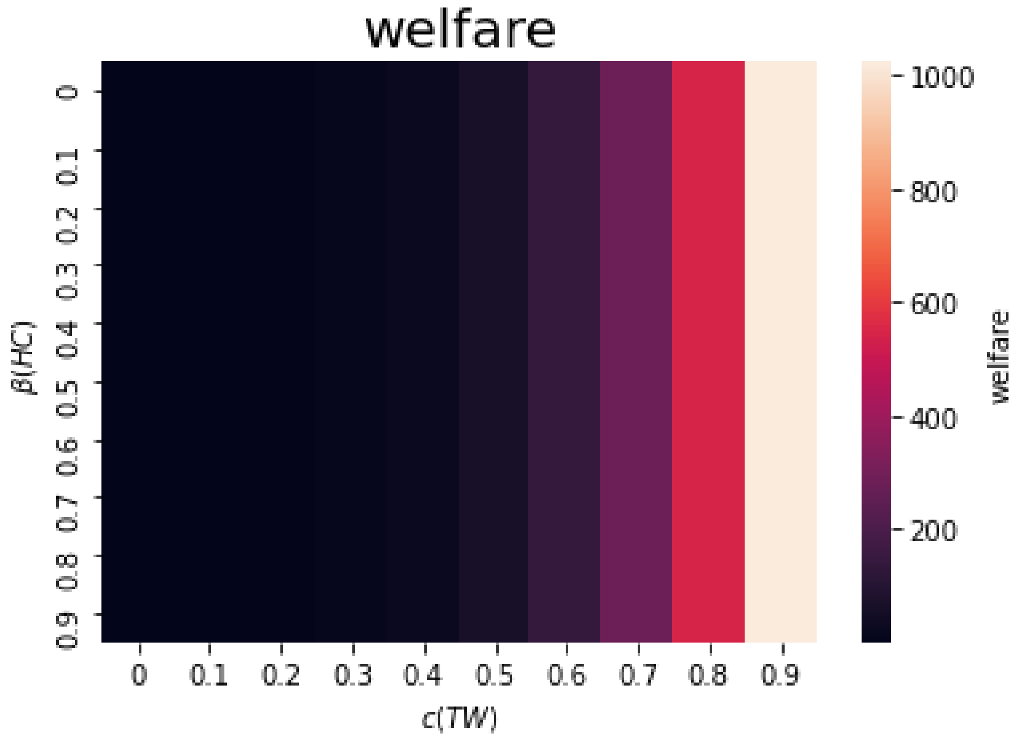

H_{nf}
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
$\beta(HC)$
$c(TW)$
0.5
0.4
0.3
0.2
0.1
0.0
H_{nf} value

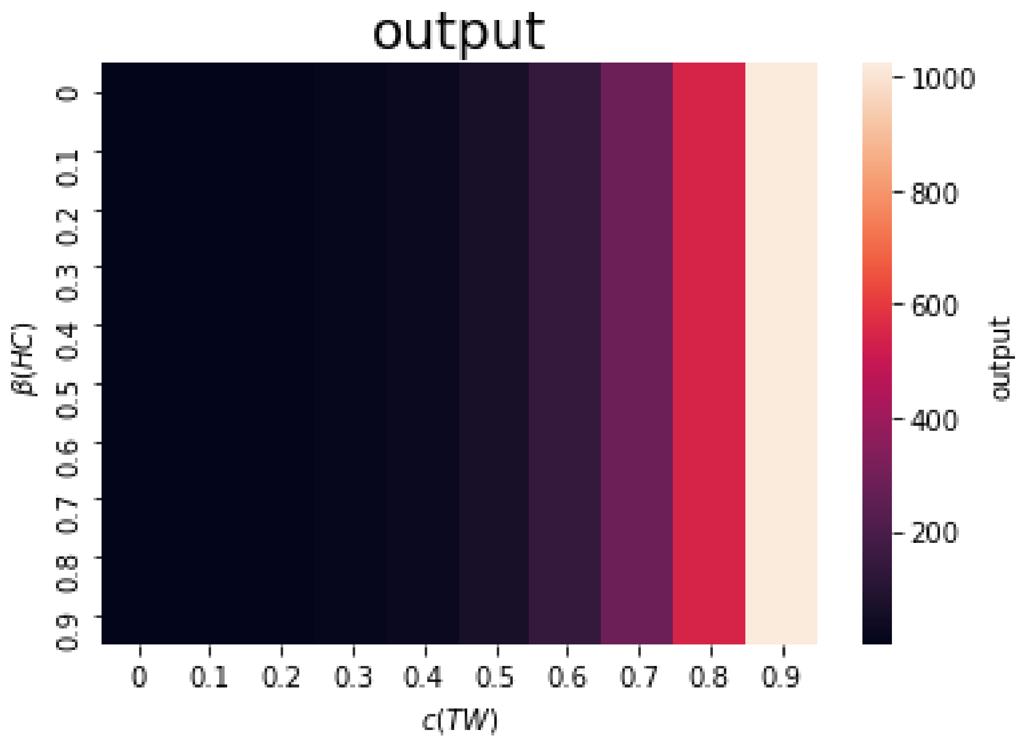

H
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
$\beta(HC)$
$c(TW)$
1.100
1.075
1.050
1.025
1.000
0.975
0.950
0.925
0.900
H value


H_f/H
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
$\beta(HC)$
$c(TW)$
1.0
0.9
0.8
0.7
0.6
0.5
H_f/H value


Infections
β(HC)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
c(TW)
I value
8
6
4
2
0


Deads
β(HC)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
c(TW)
Deads value
0.8
0.6
0.4
0.2
0.0



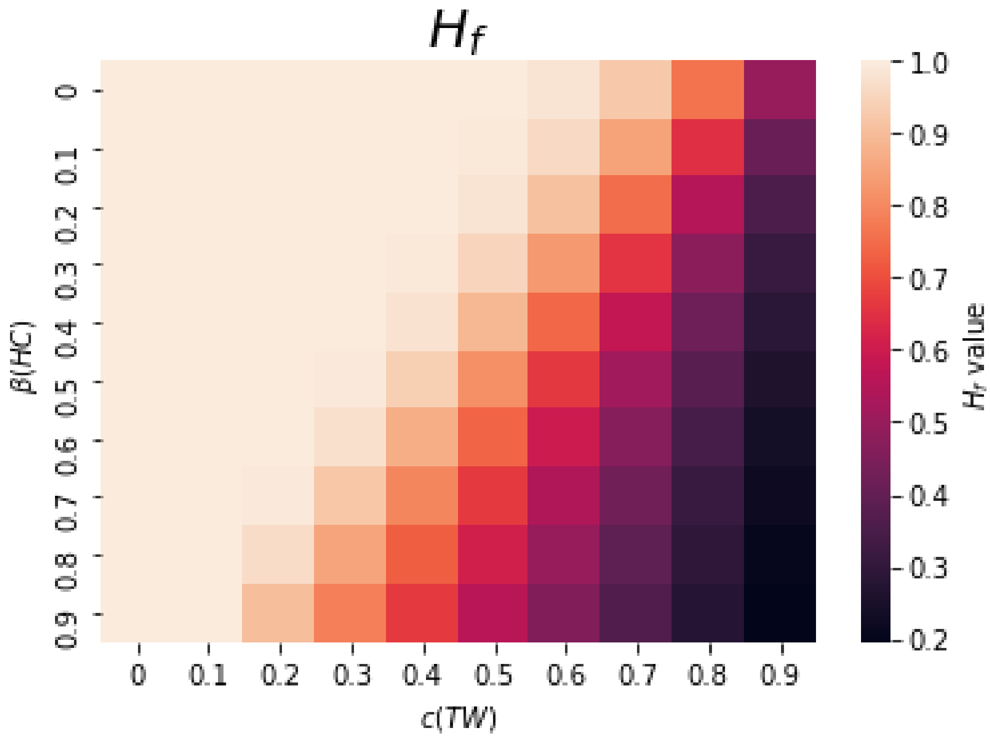


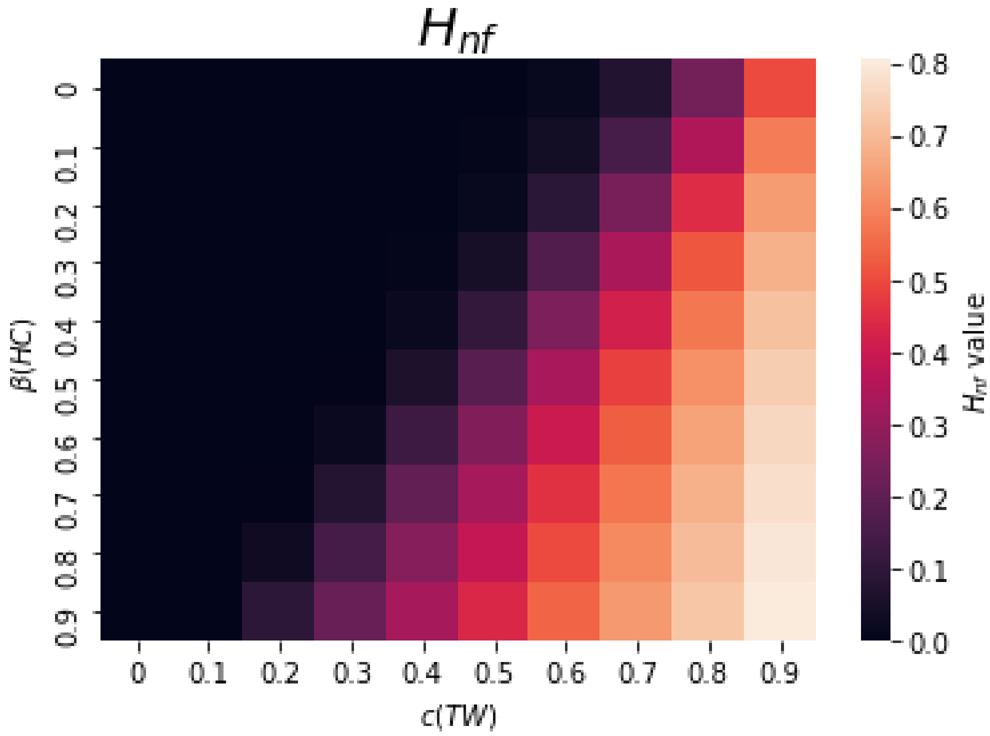
The total labour employment is binding for both β and c, In the graphs H_f and H_{nf}, we see how it is expected that when the parameters associated with each of the variables of teleworking and office work change, the other moves in the opposite direction. In the infections graph we can see that when people are teleworking the probability of infection is lower, since by assuming the problem the risk is zero, as well as the risk of death, it is logical that when people telework, to control the risk of infection, it makes the number of deaths fall, this has been contrasted with the reality lived where confinement was the most effective measure. Social welfare affects and is correlated with teleworking if it is very productive and takes on very high values.

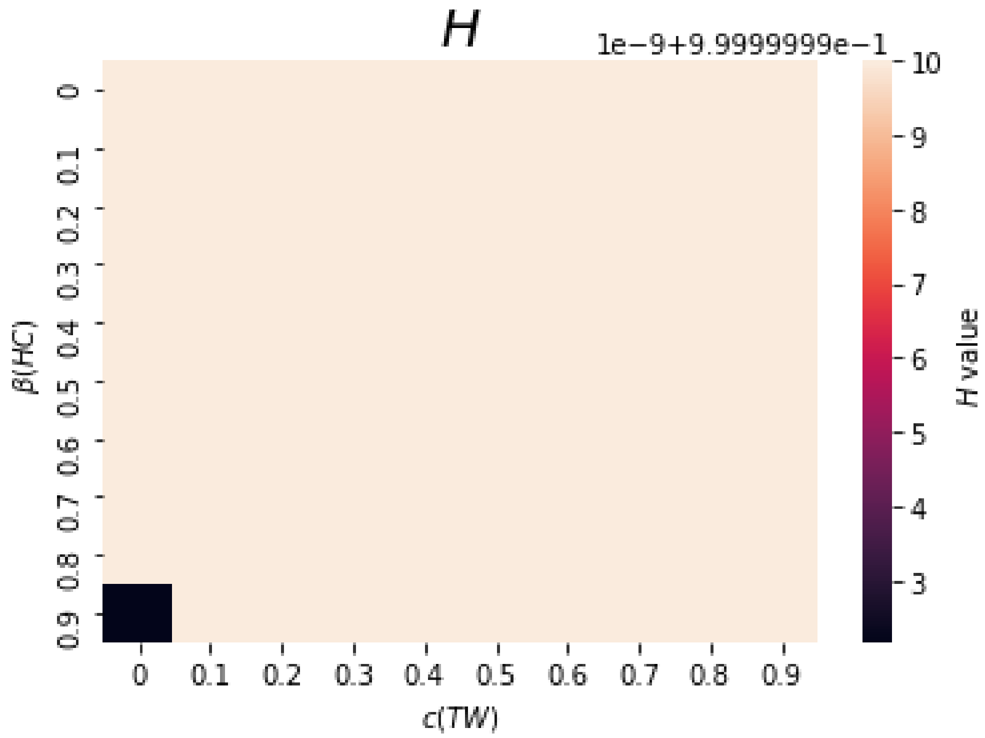
2. **What happens to yours results when you increases (decrease) ρ or ω**

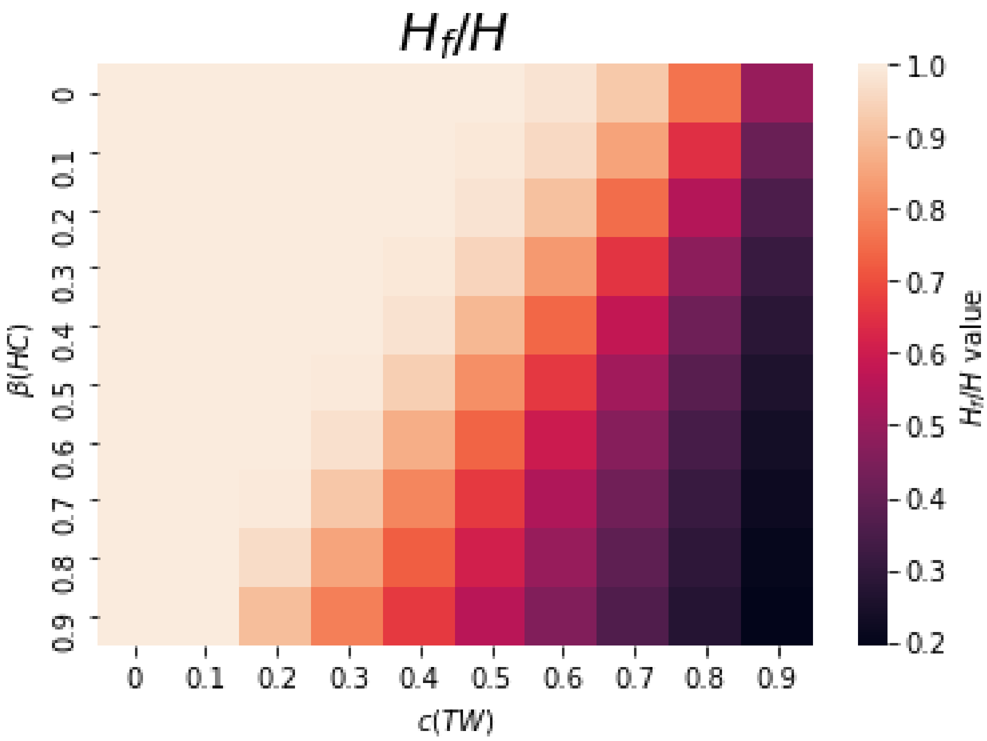
The parameter ρ is the elasticity of substitution between work at place and telework, so we changed this parameter increasing until 8.5. When we increase this parameter the workers have more facilities to change telework by job at place and vice-versa. The result are completely different w.r.t. the part a). We can do the interpretation in the same way as before, the output is the following:

- $\rho = 8.5$

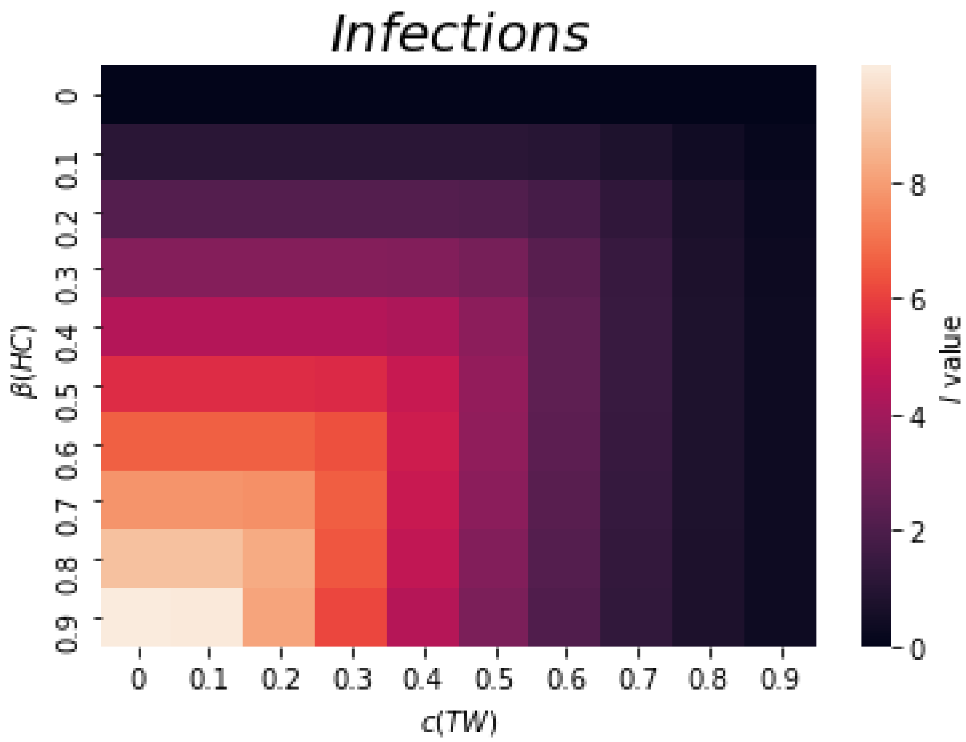



H_{nf}
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
$\beta(HC)$
$c(TW)$
H_{nf} value
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8

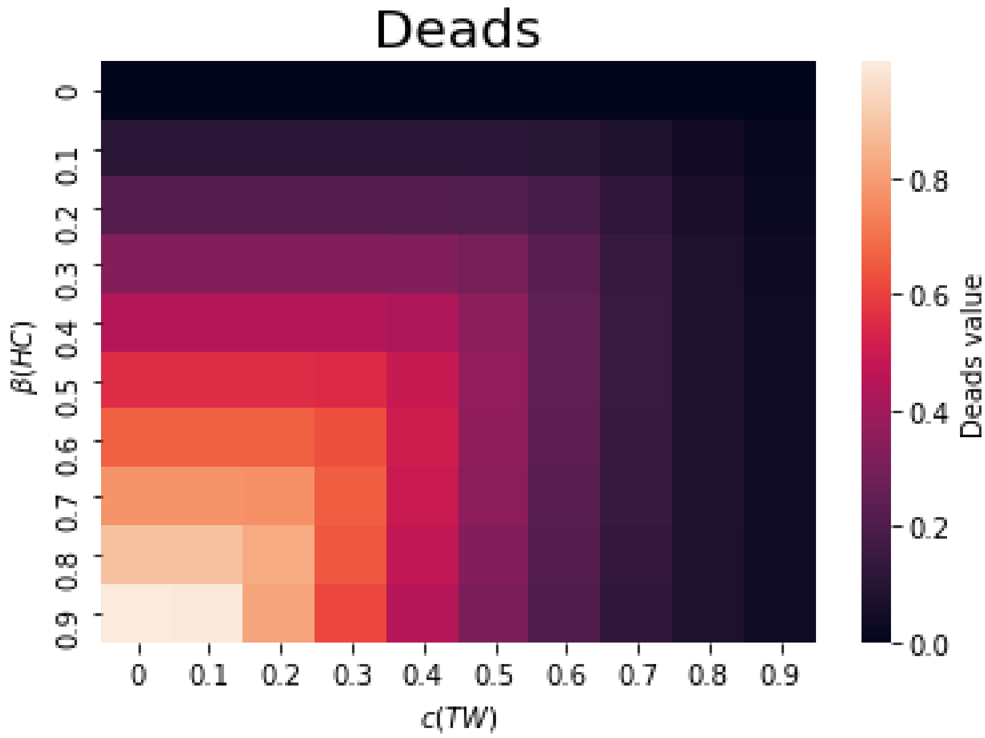

H
1e−9+9.9999999e−1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
$\beta(HC)$
$c(TW)$
H value
3
4
5
6
7
8
9
10

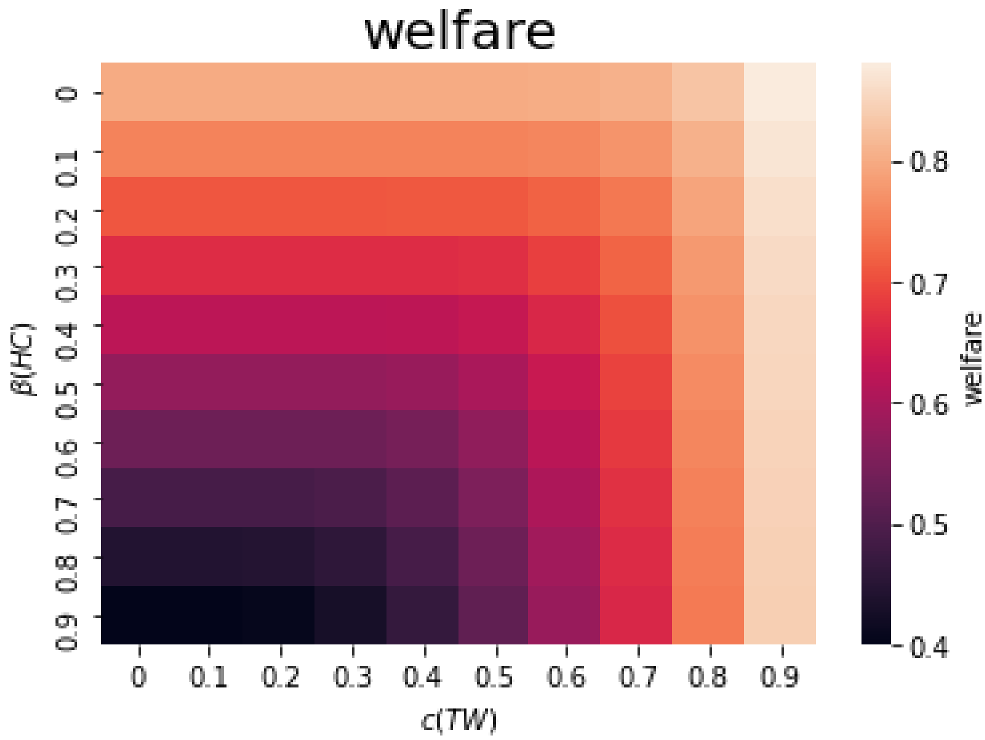


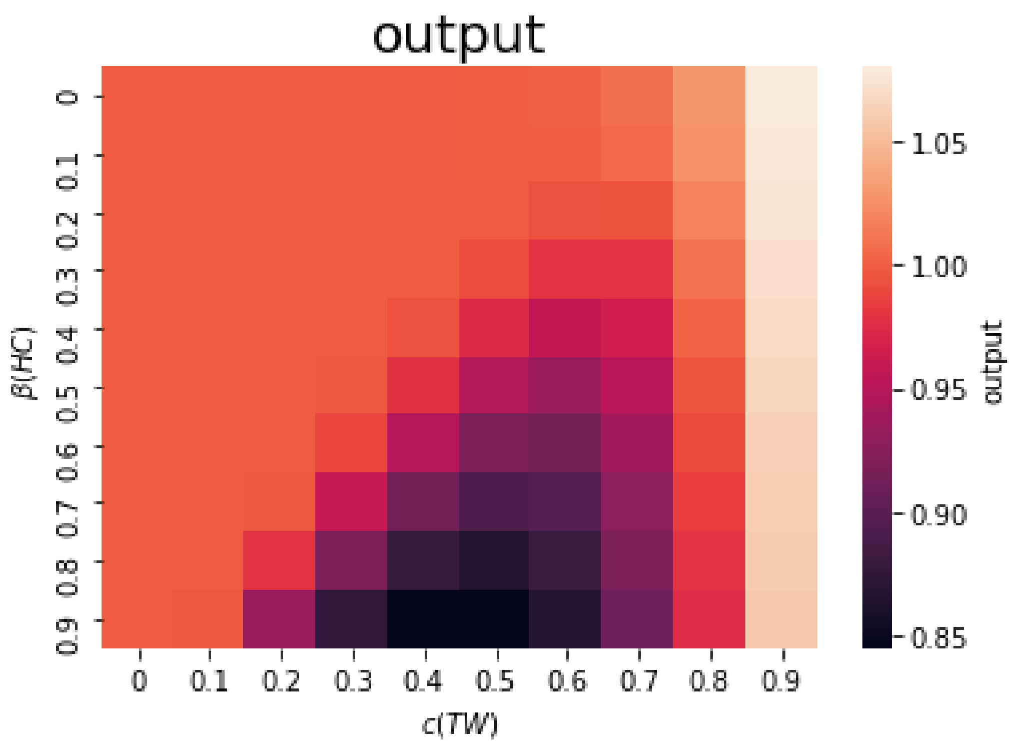
H_f/H
$\beta(HC)$
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
$c(TW)$
H_f/H value
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2



Infections
$\beta(HC)$
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
$c(TW)$
I value
8
6
4
2
0


Deads
β(HC)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
c(TW)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
Deads value
0.0
0.2
0.4
0.6
0.8

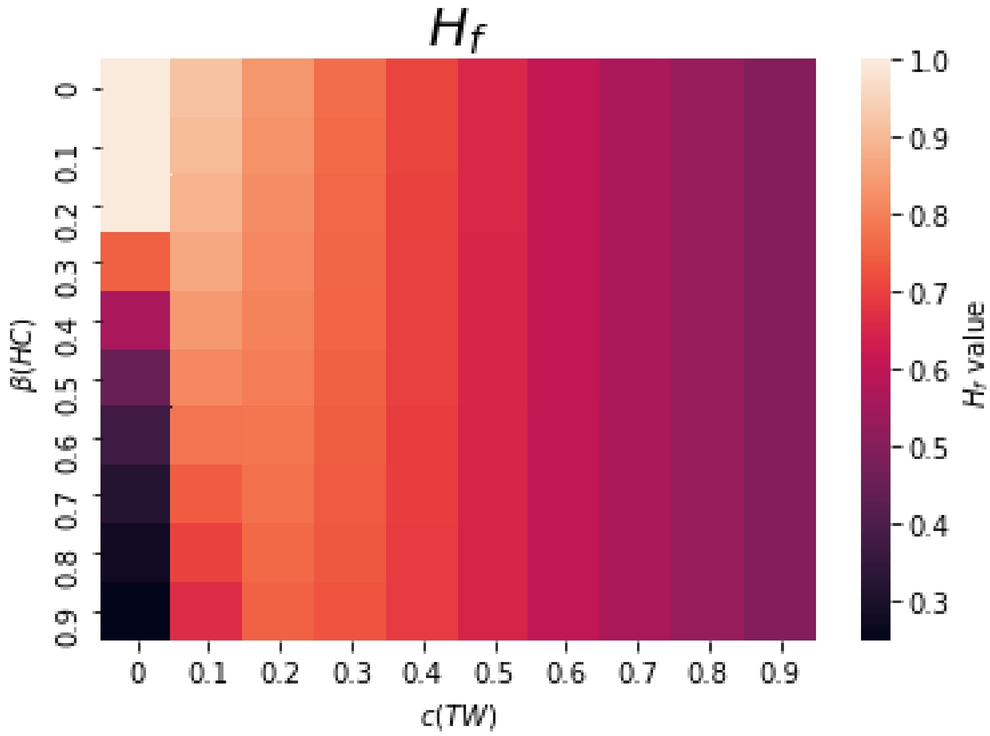

welfare
β(HC)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
c(TW)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
welfare
0.4
0.5
0.6
0.7
0.8

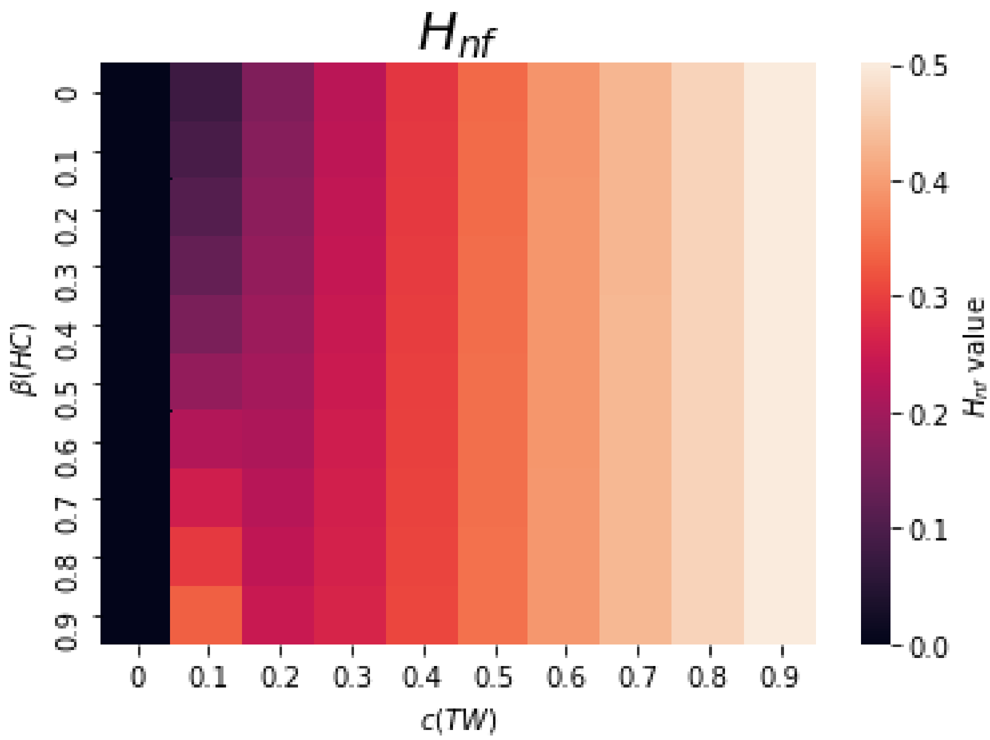


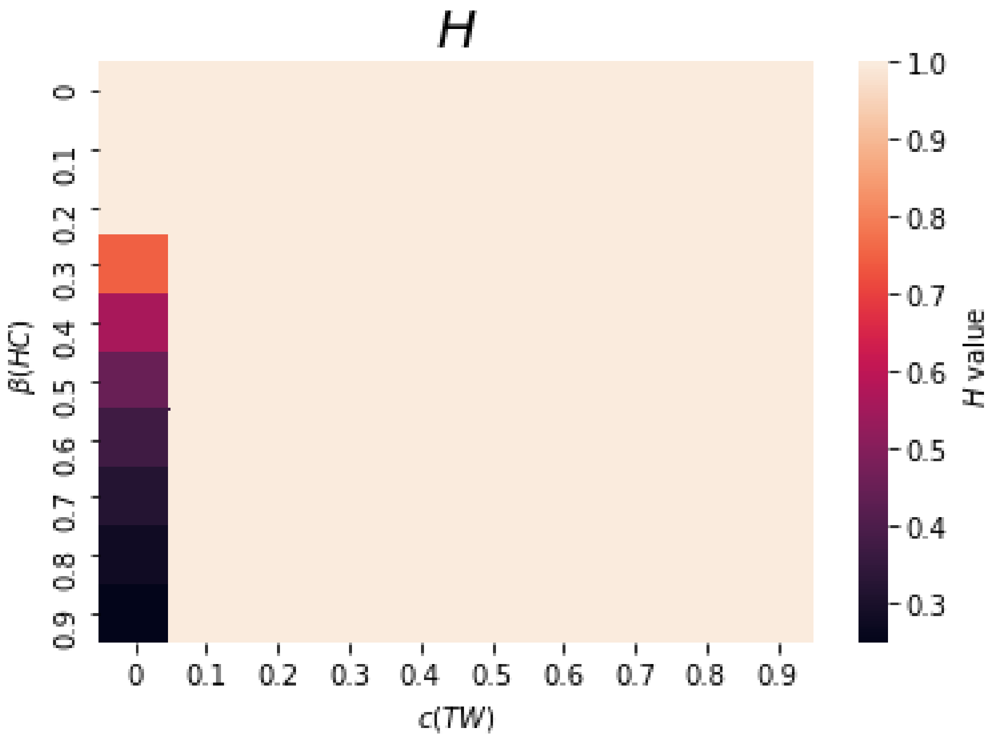
Now the graphs H_f and H_{nf} have changed completely since now the high value get when people work at job, that means telework has a lower productivity, then the people only can telework if they have higher productivity and this is not fulfilled, as we see in output. As a consequence of this we observe that the number of infections has the highest value, due to the parameter $\beta(CH)$, this also makes that the number of deaths increases considerably due to the higher risk of infection, the cost in welfare is very high as we can see in its graph, since the correlation between deaths, infections and welfare makes that some bad data affect it. The increase in output is significant.

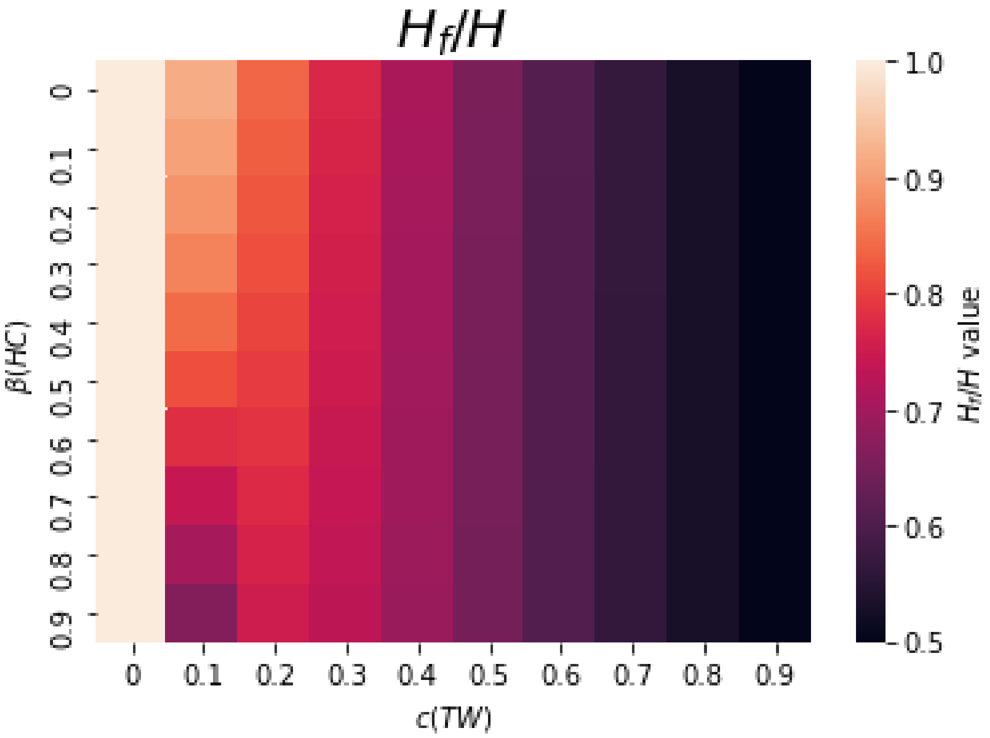
- $\omega = 80$

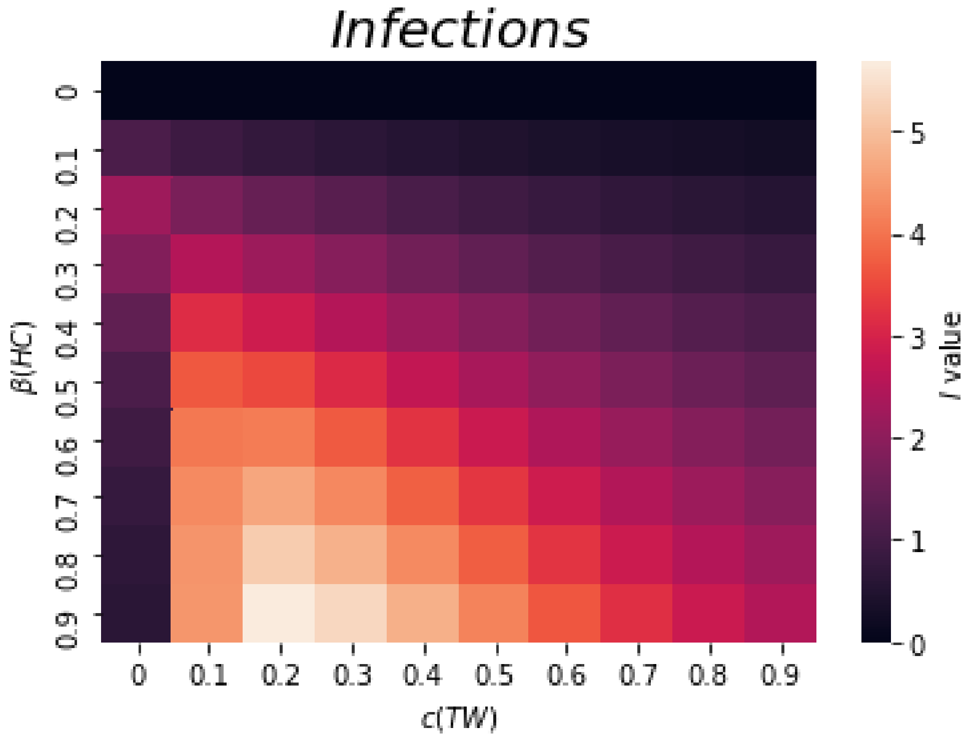
When we increase omega, high conditional infection rate, welfare is practically linked to deaths, because when the rate of loss of labour productivity peaks with telework, people still get the highest level of welfare even with the loss of efficiency. The conditional infection rate say us the same, life now seems to be the driving force behind the actions of individuals, no matter what. even in the H_f chart the people working in the office barely have a range where they achieve great utility, this is due to the weight that has gained ω. NOtice also that the number of infections and deaths decrease as a consequence the workers are at home.

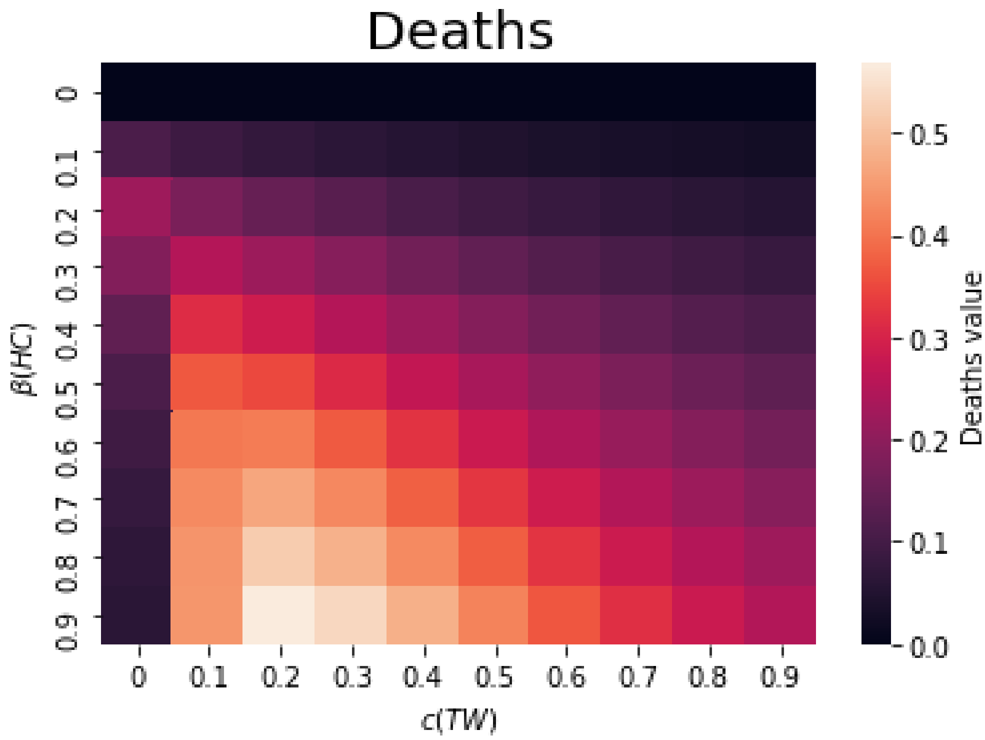

H_f
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
$\beta(HC)$
$c(TW)$
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
H_f value

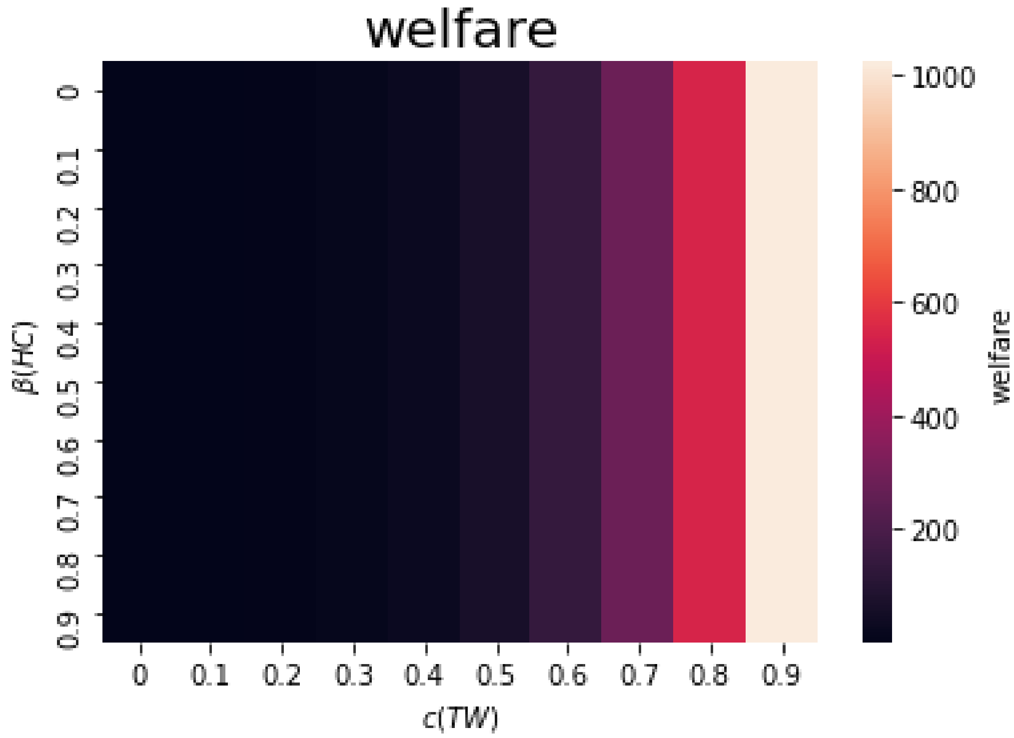

H_{nf}
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
$\beta(HC)$
$c(TW)$
0.5
0.4
0.3
0.2
0.1
0.0
H_{nf} value

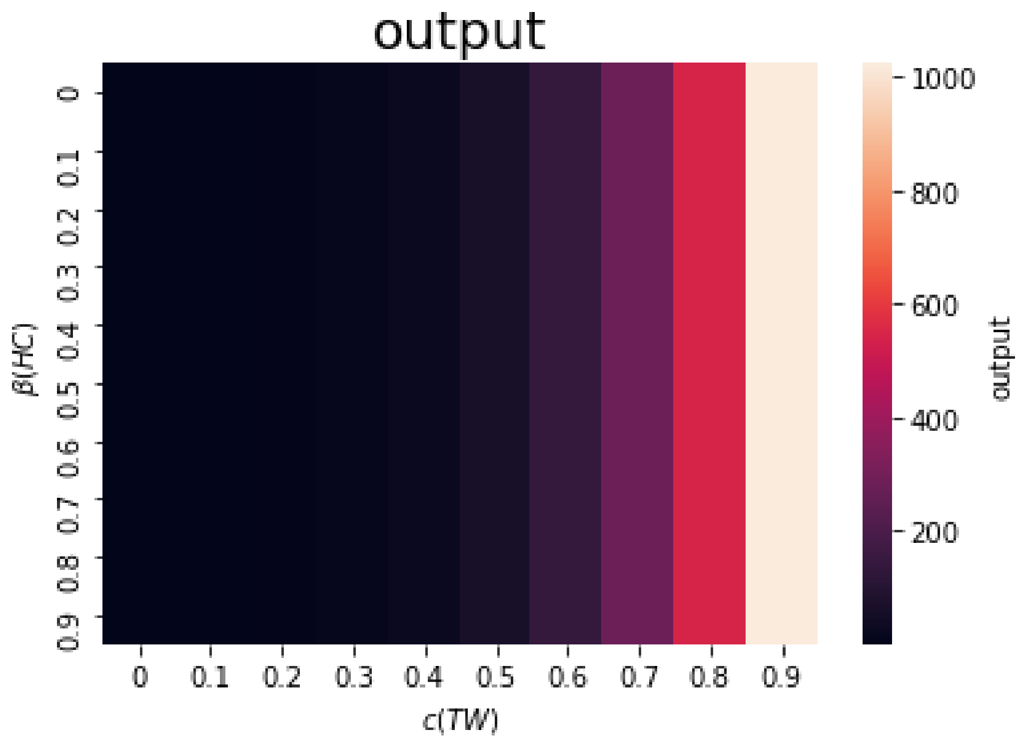

H
$\beta(HC)$
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
$c(TW)$
H value
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3


H_f/H
$\beta(HC)$
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
$c(TW)$
H_f/H value
1.0
0.9
0.8
0.7
0.6
0.5


Infections
β(HC)
c(TW)
I value
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
2
3
4
5


Deaths
β(HC)
c(TW)
Deaths value
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
0.0


welfare
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
β(HC)
c(TW)
1000
800
600
400
200
welfare


output
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
β(HC)
c(TW)
1000
800
600
400
200
output