Discrete-time model of the Leveraged ETF returns

The year of pandemic 2020 - 2021

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Abstract

Last year was an extremely tough year for the market worldwide. The Chicago Board Options Exchange's CBOE Volatility Index (VIX), which based on SP 500 options reached the highest point for the past five year. The average of VIX was extremely high in the history. The other notable peak was in 2008 during the financial crisis which Avellaneda and Zhang's paper was produced soon after. We can observe some similar behaviors of the leveraged ETFs, such as the under performances of the leveraged ETFs. In this study, we will use the discrete time model in the original paper to produce the returns of 20 double-long and double-short leveraged ETFs from 2020 - 2021. Moreover, the estimation of instantaneous variance in the discrete time model cannot be evaluated in the continuous manner. However, we can reduce the range for the evaluation of the variance to produce a more accurate output for the returns.

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1 Introduction

In 2009 Avellaneda and Zhang published the paper call "Path-dependence of Leveraged ETF returns". In the paper, they present the discrete-time and continuous-time model for the LETF returns and compared the modelled returns with true returns of LETFs for 2008 — the year of financial crisis.

In the light of this model, we will replicate the returns from 2020-2021 — the year of COVID 19 pandemic. We expect some similar performances of the LETFs in 2008 and 2020 since the volatility of both years are extremely high in the history. In section 1, we will use the ETF prices from Yahoo Finance to examine the performances of one double-long, one double-short and one triple-long LETF. As an expectation of the model, we see when the volatility is high, the double-short LETF exhibits dominant under-performance if the return is not large.

In section 3, we will briefly describe the discrete-time and continuous-time model, and the relation to stochastic differential equation. In section 4, we will present the empirical results for 20 double-long and double-short LETFs. At the end, we will introduce a short discussion on the estimation of instantaneous variance and how it affects the accuracy of the discrete-time model.

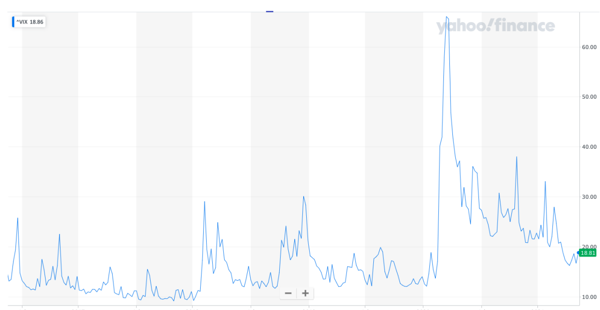

Figure 1: Chicago Board Options Exchange's CBOE Volatility Index for the past five years. VIX is based on S&P 500 options. Sourced from Yahoo Finance.

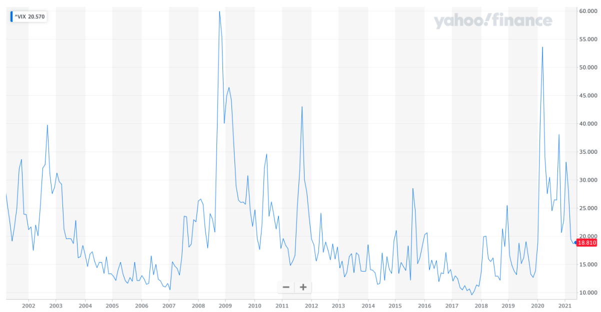

Figure 2: Chicago Board Options Exchange's CBOE Volatility Index from 2001 to 2021. Sourced from Yahoo Finance.

2 60 Day Returns of LETF vs. Underlying ETF

In this section, we will examine the return of LETF together with the underlying ETF. The method is the following. From the downloaded daily price of the LETF and the underlying ETF, we compute the 60 day returns according to

$$\text{60 Days Return} = \frac{\text{Price}(ETF_t) - \text{Price}(ETF_{t-60})}{\text{Price}(ETF_{t-60})}.$$

Then, in the logarithmic plot, we compute

$$\beta * ln(ETF_t/ETF_{t-60}) \approx ln(LETF_t/LETF_{t-60}).$$

The return of LETF can be modelled by the formula.

$$\frac{L_t}{L_0} \approx \Big(\frac{S_t}{S_0}\Big)^{\beta} exp\Big\{\frac{\beta - \beta^2}{2} V_t + \beta H_t + ((1-\beta)r - f)t\Big\},$$

where L and S are the price of the LETF and underlying ETF, resp., and V_t is the realized variance. H_t is the total cost for borrowing the components of the underlying ETF, i.e. $\sum \lambda_i \Delta t$.

According to the return model above, we have a good intuition on the return of the leveraged ETFs. Now, we consider the term corresponding to the variance for different ETFs.

$$\frac{\beta - \beta^2}{2}\bigg|_{\beta=-2} = \frac{(-2) - (-2)^2}{2} = -3,$$

$$\frac{\beta - \beta^2}{2}\bigg|_{\beta=2} = \frac{2 - 2^2}{2} = -1,$$

$$\left.\frac{\beta - \beta^2}{2}\right|_{\beta=3} = \frac{3 - 3^2}{2} = -3.$$

This is obvious that the returns of both bearish and bullish leveraged ETFs have negative dependence on the realized variance. Moreover, the double-short and triple-long leveraged ETF have stronger dependence on Vt.

Then, we will have a reasonable expectation of the return for the leveraged ETFs. The returns of LETFs are mostly under-performed, particularly when the volatility is large and returns are small. On the other hand, when the volatility is small but returns are high, we expect outperforms. Here, the volatility is the variance for a certain period of time.

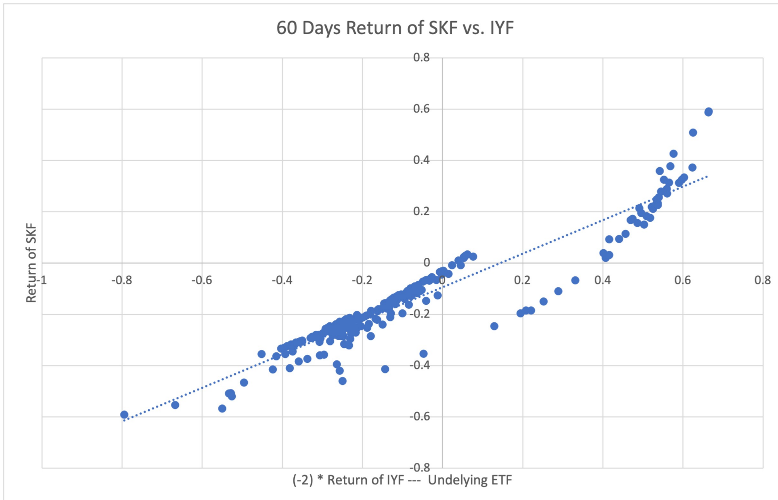

Figure 3: 60 Day Returns of LETF SKY vs. the Underlying ETF IYF. $\beta = -2$ The 60 day return is gain from March 1, 2020 to March 1, 2021, during the year of COVID pandemic. The initial price is on Jan. 1 2020.

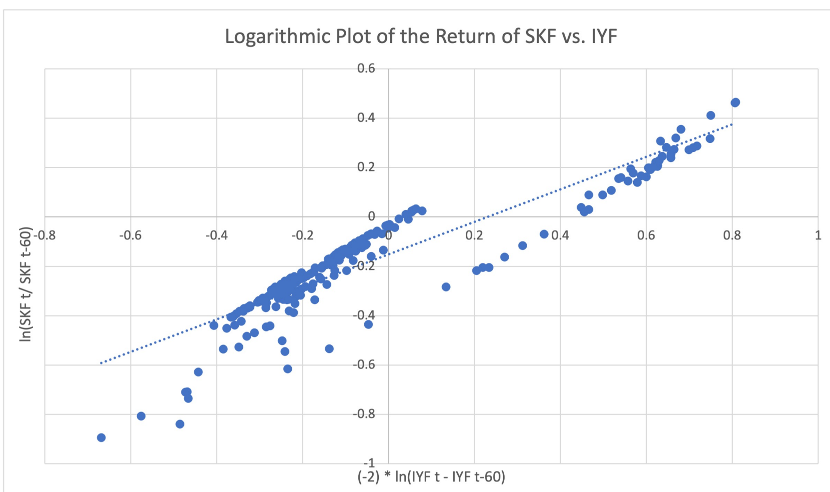

Figure 4: Logarithmic Plot of 60 Day Returns of LETF IYF vs. the Underlying ETF SKF. Same data as in Figure 1

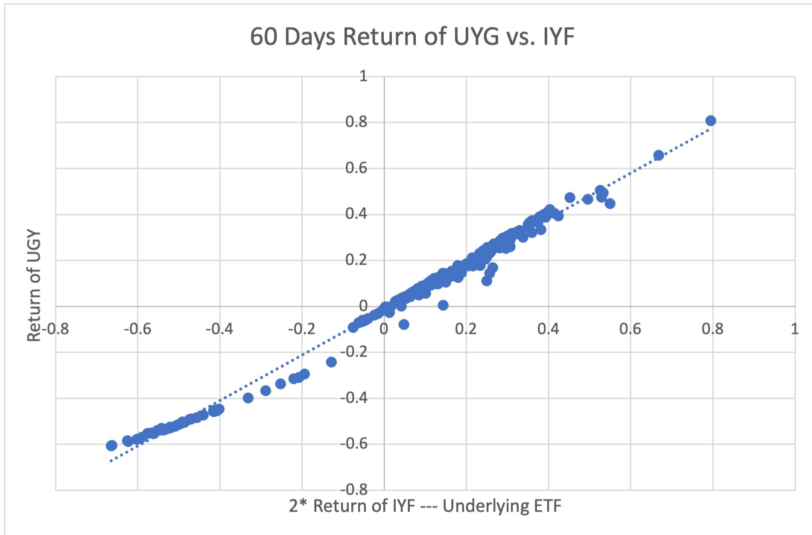

Figure 5: 60 Day Returns of LETF UYG vs. the Underlying ETF IYF. $\beta = 2$ The 60 day return is gain from March 1, 2020 to March 1, 2021, during the year of COVID pandemic. The initial price is on Jan. 1 2020.

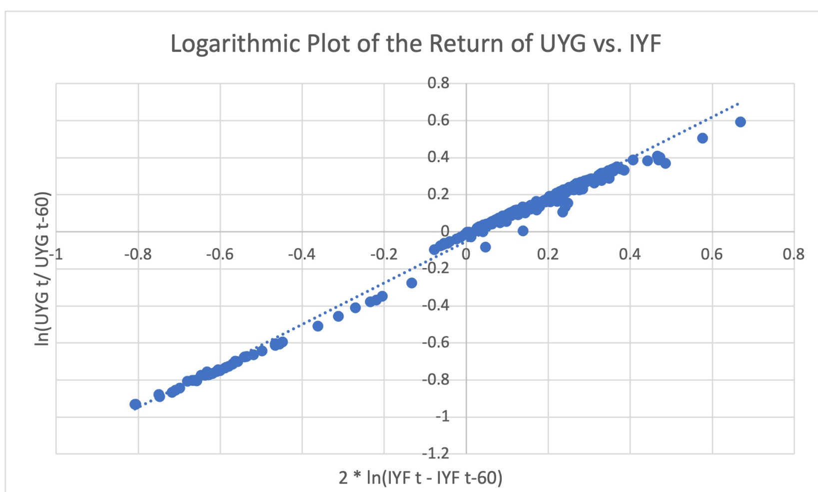

Figure 6: Logarithmic Plot of 60 Day Returns of LETF IYF vs. the Underlying ETF UYG. Same data as in Figure 1

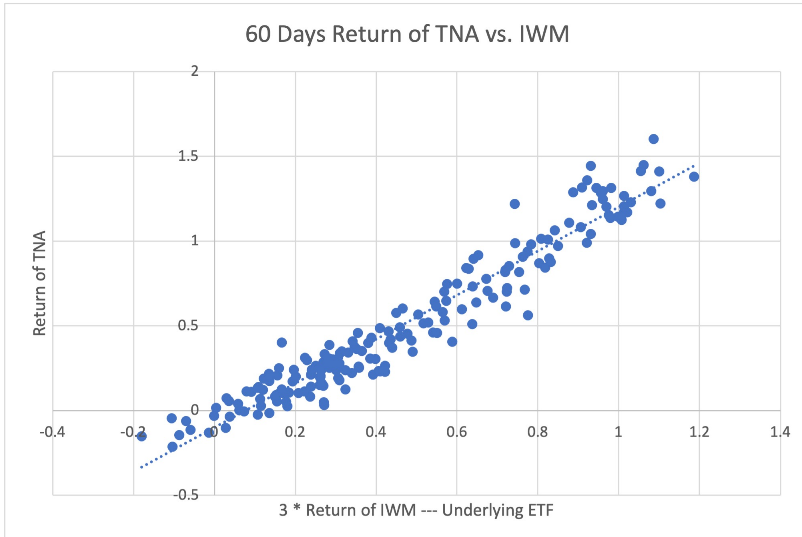

Figure 7: 60 Day Returns of LETF IWM vs. the Underlying ETF TNA. $\beta = 3$ The 60 day return is gain from March 1, 2020 to March 1, 2021, during the year of COVID pandemic. The initial price is on Jan. 1 2020.

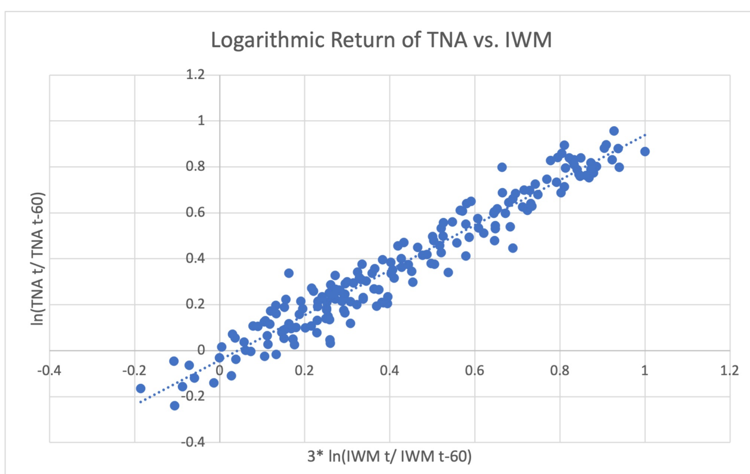

Figure 8: Logarithmic Plot of 60 Day Returns of LETF IWM vs. the Underlying ETF TNA. Same data as in Figure 1

3 Estimation of Leveraged ETF returns

3.1 Discrete-Time

Consider the model described in section 2,

$$\frac{L_t}{L_0} \approx \left(\frac{S_t}{S_0}\right)^{\beta} exp\left\{\frac{\beta - \beta^2}{2} V_t + \beta H_t + ((1-\beta)r - f)t\right\}.$$

Through out the whole project, we will assume $H_t = 0$. The expense ratio f are mostly $\approx 0.95\%$ for double-short and double long ETFs. The interest rate r is obtain from the 3-month LIBOR rate by Federal Reserve Bank for the year of 2020-2021. The factors that affect the returns are

- realized Variance V_t
- leverage ratio β
- interest rate r
- the management fee f

Note that here, we only apply the discrete-time model for V_t. Now, we will briefly summarize the method to get the model.

For bullish ($\beta > 1$) ETFs, the daily return of the ith day is given by

$$R_i^L = \beta R_i^S - \beta r \Delta t - f\Delta + r\Delta t = \beta R_i^S - ((\beta - 1)r + f)\Delta t.$$

For bearish ($\beta \leq$ -1) ETFs, the daily return of the ith day is given by the similar formula, but with λ_i,

$$R_i^L = \beta R_i^S - \beta(r - \lambda_t)\Delta t - f\Delta + r\Delta t = \beta R_i^S - ((\beta - 1)r + f - \beta\lambda_i)\Delta t.$$

Now, since we assume $\lambda_i = 0$, the two formulae yield the same.

Moreover, the price up to time t is calculated by a well-know formula

$$S_t = S_s \prod_{i=1}^{N} (1 + R_i^S)$$

$$L_t = S_s \prod_{i=1}^{N} (1 + R_i^L)$$

And, the realized variance is given by $NVar(R_i^S)$, i.e.,

$$V_t = \sum_{i=1}^{N} (R_i^S - \bar{R_i^S})^2.$$

3.2 Continuous-time

In the continuous-time model, we assume that the price for the underlying ETF is a Wiener process. Then, the continuous time model is related to the stochastic differential equation (SDE).

$$dS_t/S_t = \sigma_t dW_t + \mu_t dt.$$

In terms of the leveraged ETF and underlying ETF, for both bullish and bearish LETF, ($\lambda_t = 0$)

$$dL_t/L_t = \beta dS_t/S_t - (\beta - 1)r + f)dt.$$

3.3 Derive the Approximation

This is in the appendix of the original paper. We evaluate the difference

$$ln(\frac{L_t}{L_0}) - \beta ln(\frac{S_t}{S_0}),$$

by Taylor expansion. Then, drop the higher order terms, for example, $(\Delta t)^{3/2}$.

4 Empirical Results and Tracking Error of the LETFs

4.1 Methods

The tracking error is given by

$$\epsilon(t) \approx \frac{L_t}{L_0} - \Big(\frac{S_t}{S_0}\Big)^\beta exp\Big\{\frac{\beta - \beta^2}{2} V_t + \beta H_t + ((1-\beta)r - f)t\Big\}.$$

In this section, we will consider the error and standard deviation for the double-long and double-short leveraged ETFs. The instantaneous variance is calculated based on 5-day variance of the return. In both of the table, we use the price from May. 15, 2020 to May. 15, 2021. The results are calculated in Excel. The leveraged ETFs we will consider in this section are summarized in the following table.

Double-Leveraged ETFs Table [1]

[1] The expanse ratio is collected by Ruichen Xu. A lot of thanks!

Underlying ETF	Expanse Ratio f (%)	Proshares Ultra	Proshares Ultra Short
QQQ	0.95	QLD	QID
DIA	0.95	DDM	DXD
SPY	0.91	SSO	SDS
IJH	0.95	MVV	MZZ
IJR	0.95	SAA	SDD
IWM	0.95	UWM	TWM
IYC	0.95	UCC	SCC
IYF	0.95	UYG	SKF
IYH	0.95	RXL	RXD
IYJ	0.95	UXI	SIJ

4.2 Proshares Ultra Short $\beta = -2$

Double-Leveraged Ultra Short ETFs ($\beta = -2$)

Underlying ETF	Tracking Error (%)	STD(%)	Leveraged ETF
QQQ	-0.057	0.17	QID
DIA	-0.031	0.14	DXD
SPY	-0.029	0.11	SDS
IJH	-0.051	0.25	MZZ
IJR	-0.081	0.31	SDD
IWM	-0.072	0.19	TWM
IYC	-0.038	3.22	SCC
IYF	-0.065	3.81	SKF
IYH	-0.032	0.37	RXD
IYJ	-0.049	0.51	SIJ

4.3 Proshares Ultra $\beta = 2$

Double-Leveraged Ultra Long ETFs ($\beta = 2$)

Underlying ETF	Tracking Error (%)	STD(%)	Leveraged ETF
QQQ	-0.069	0.15	QLD
DIA	-0.033	3.21	DDM
SPY	-0.042	3.12	SSO
IJH	-0.063	0.15	MVV
IJR	-0.093	0.35	SAA
IWM	-0.083	0.16	UWM
IYC	-0.039	3.01	UCC
IYF	-0.076	3.80	UYG
IYH	-0.032	0.03	RXL
IYJ	-0.049	0.33	UXI

5 Conclusion and Consequences

In this project, we examine the returns and average tracking errors of double-long and double-short leveraged ETFs. 20 double-long LETFs and 20 double-short ETFs are studies. The discrete time model for the returns of the LETFs described in section 3 produces a tracking error on the scale of 0.05% for the past year of the ETF returns. This is a reasonable result for discrete time model. Hence, we have verified that the discrete time model is valid for use.

Consequences for the investors are the following. Remember in section 1. We presented that most of the leveraged ETFs exhibits high returns when the volatility is low. On the other hand, the returns of the leveraged ETFs will underperform when the volatility is high if the price does not change significantly. If the investors have a solid model to predict the volatility ahead, then the investors will have a strong tool to direct the investment. The returns of the leveraged ETFs can also the predict as well.

6 Open Question

Estimation of instantaneous volatility

It is quite intuitive that if we can compute the instantaneous more continuously, then the error for the discrete time model can be reduced more. The question is what is a suitable way to estimate the instantaneous variance.

In this project, we use variance of the five-day returns to calculate the instantaneous volatility. However, in the more continuous time model, we can use a better estimation for the instantaneous volatility given by Danyliv and Bland in 2019. [2] For example, a model can be built to track the variance by minute or even second (even if I don't know if this is actually feasible).

References

[1] Marco Avellaneda and Stanley Zhang. *Path-dependence of Leveraged ETF returns.* SSRN, 2009.

[2] Oleh Danyliv and Bruce Bland. *An Instantaneous Market Volatility Estimation.* SSRN, 2019.

[2]https://papers.ssrn.com/sol3/papers.cfm?abstract$_i d = 3434093$