From Math to Program

Neural Networks have the flavor of a Functional Program

- A Sequential Model computes the compostion of per-layer functions
- Layer l is computing a function $\mathbf{y}_{(l)} = F_{(l)}$

$$F_{(l)}(\mathbf{y}_{(l-1)}; \mathbf{W}_{(l)}) = \mathbf{y}_{(l)}$$

$$F_{(l)} : \mathcal{R}^{||\mathbf{y}_{(l-1)}||} \mapsto \mathcal{R}^{||\mathbf{y}_{(l)}||}$$

If we expand $F_{(l)}$, we see that it is the l-fold composition of functions $F_{(1)}, \ldots, F_{(l)}$

$$\begin{aligned}
\mathbf{y}_{(l)} &= F_{(l)}(\mathbf{y}_{(l-1)}; \mathbf{W}_{(l)}) \\
&= F_{(l)}(\ F_{(l-1)}(\mathbf{y}_{(l-2)};\ \mathbf{W}_{(l-1)});\ \mathbf{W}_{(l)}\ ) \\
&= F_{(l)}(\ F_{(l-1)}(\ F_{(l-2)}(\mathbf{y}_{(l-3)};\ \mathbf{W}_{(l-2)});\ \mathbf{W}_{(l-1)}\ ); \mathbf{W}_{(l)}\ ) \\
&= \vdots
\end{aligned}$$

It turns out that it is not too difficult to endow a Neural Network with familiar *imperative* programming constructs

- `if` statement
- `switch/case` statement

This is sometimes called *Neural Programming*.

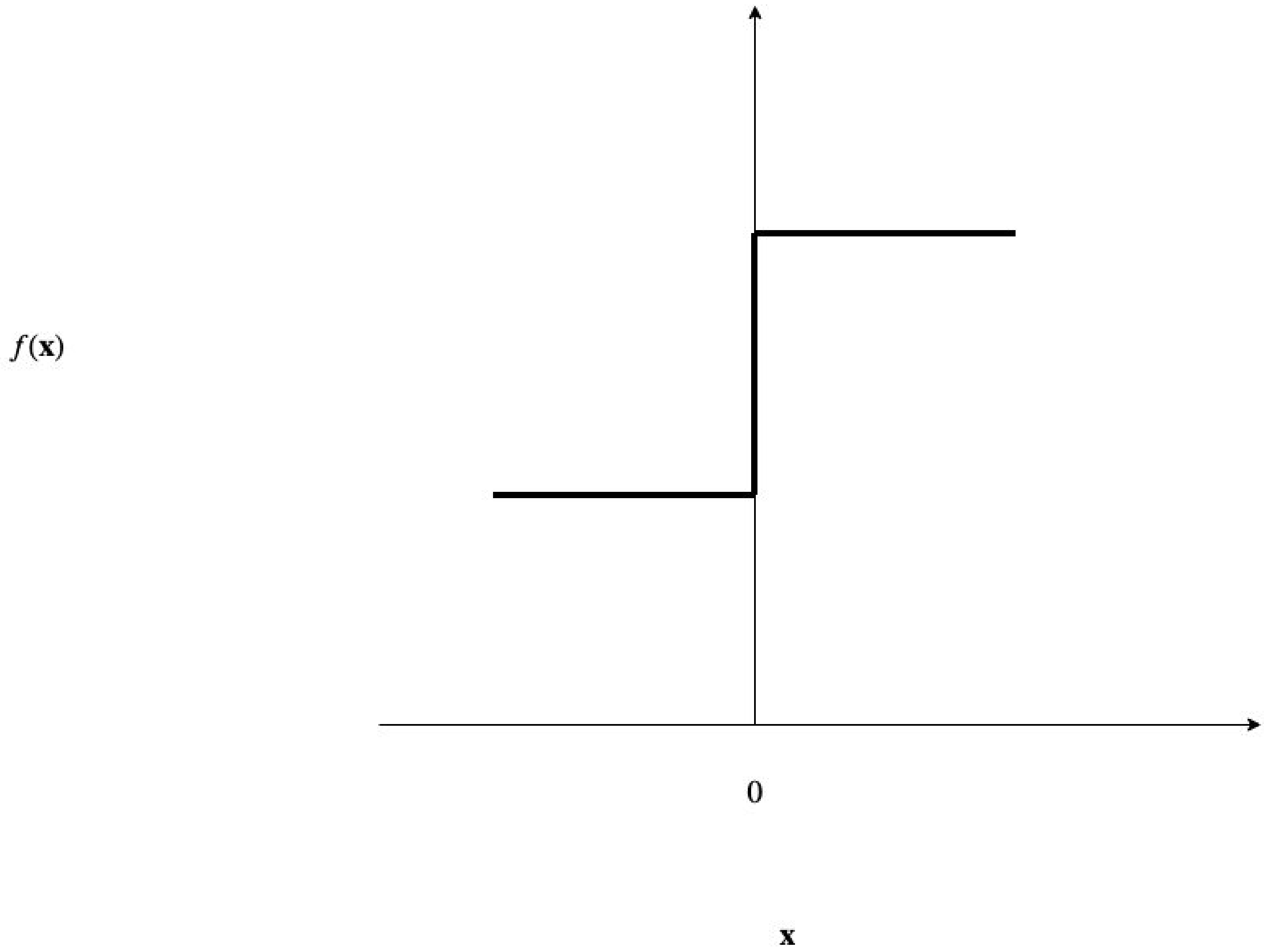
Although interesting in its own right, we introduce this topic as an introduction to more advanced recurrent layer types.

Binary switches

When we introduced Neural Networks, we argued that their power derived from the ability of Activation Functions

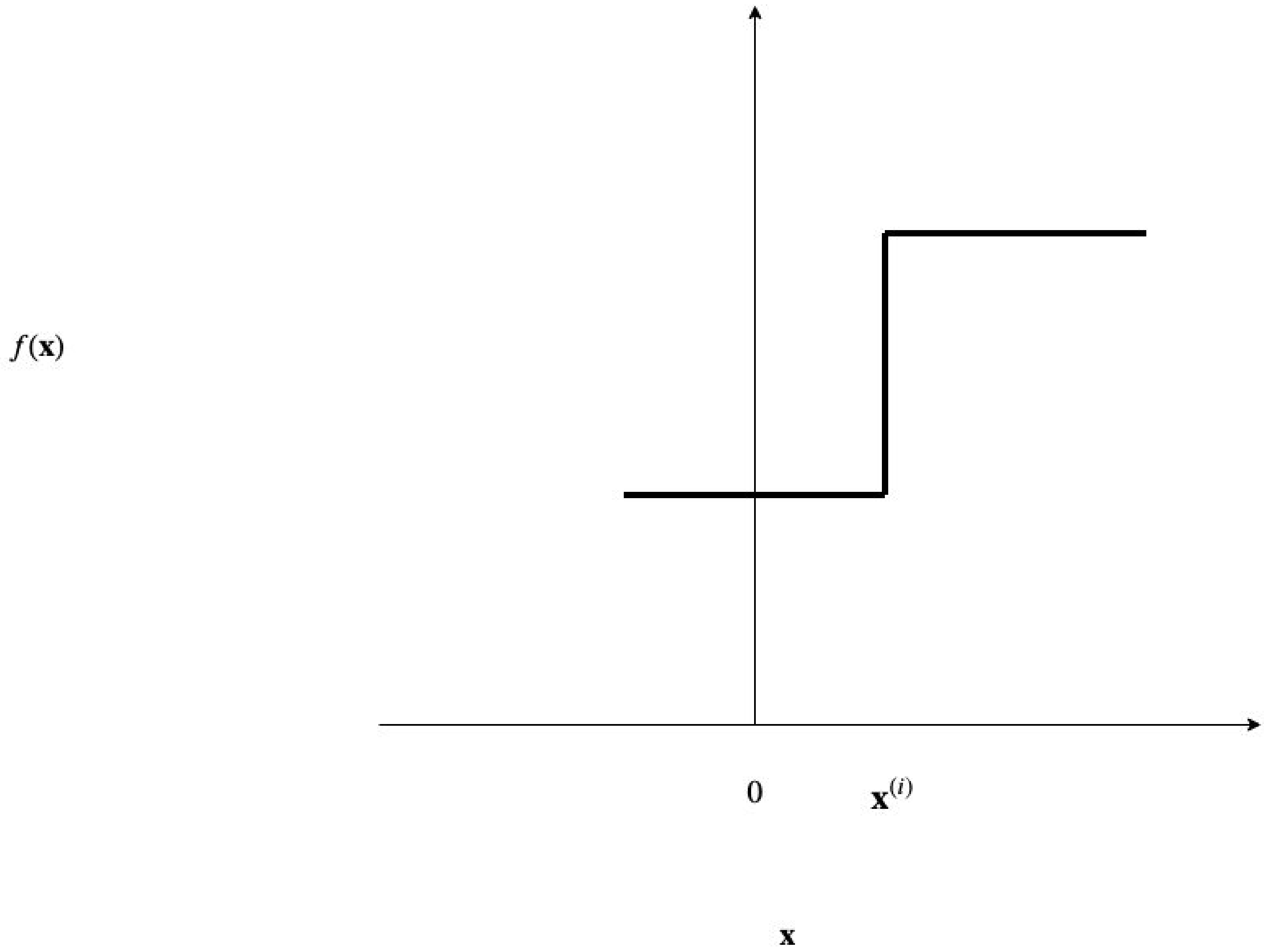
- To act like binary "switches"
- Converting the scalar value computed by the dot product
- Into a True/False answer
- To the question: "Is a particular feature present" ?

Step function: binary switch with threshold 0



By varying the threshold/bias, we can control the region in which the switch is "active"

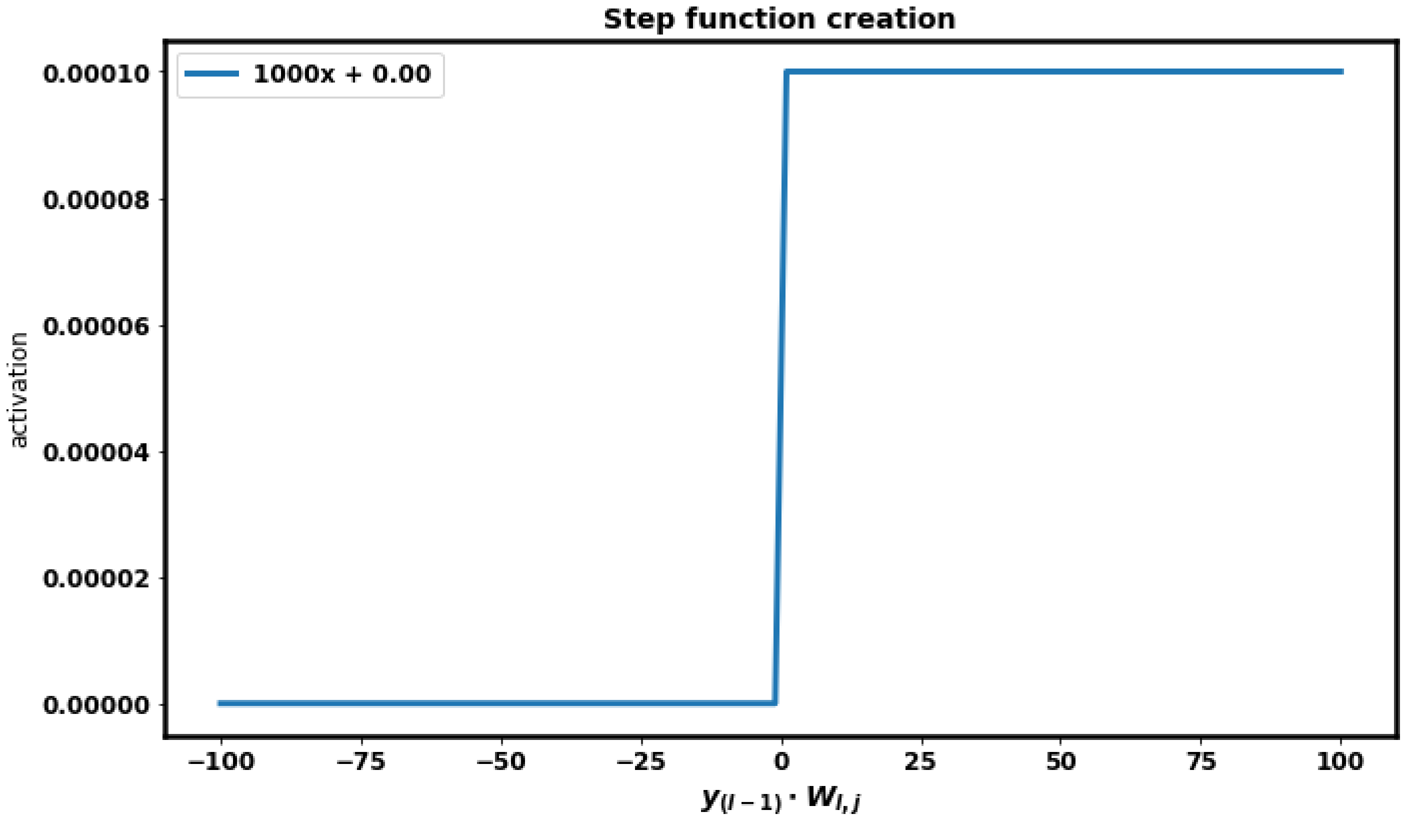
Step function: binary switch with threshold - x^(i)



And, in fact, we showed (Universal_Function_Approximator.ipynb) how to construct a very precise approximation of a binary switch:

In [5]:

```
fig, ax = nnh.step_fn_plot()
```



Neurons as statements

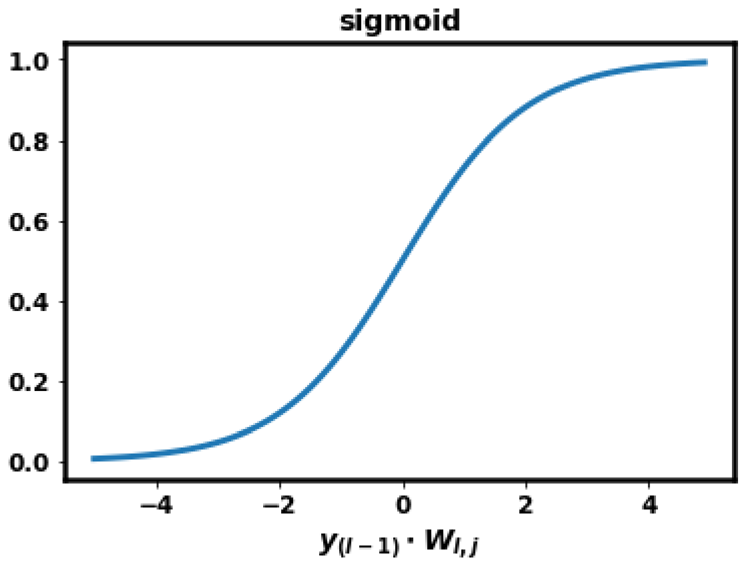
With the ability to implement a binary switch

- We can construct Neural Networks
- With elements that look like primitive statements of a programming language

Rather than building a true step function

- We will settle for the approximation offered by the Sigmoid function σ

In [6]:
```
_= nnh.sigmoid_fn_plot()
```



This is more than laziness or convenience

- The step function is **not** differentiable
- The sigmoid function **is** differentiable

Recall that Gradient Descent is the tool we use to train Neural Networks

- Hence it is important that our functions be differentiable !

”If” statements - Gates

Suppose we want a Neural Network to

- Compute a (vector) output $\mathbf{y}$
- That takes on vector value T if some condition g is `True`
- And F otherwise.

This would be trivial in any programming language having an `if` statement:

```
if (g):
    y = T
else:
    y = F
```

Let's show how to construct the `if` statement with just a little arithmetic.

Suppose scalar $g \in \{0, 1\}$ was the value output by a switch.

Then

$$\mathbf{y} = (g * \mathbf{T}) + (1 - g) * \mathbf{F}$$

does the trick.

In general, we tend to compute vectors rather than scalars.

Let

- $\mathbf{g}, \mathbf{y}$ be vectors of equal length
- $\mathbf{T}, \mathbf{F}$ be vectors of equal length (not necessarily the same as $\mathbf{g}, \mathbf{y}$)
  - So elements of $\mathbf{y}$ have length $||\mathbf{T}|| = ||\mathbf{F}||$

We will construct a "vector" `if` statement

- Making a conditional choice for *each element* of $\mathbf{y}$, independently.
$$\mathbf{y}_j = (\mathbf{g}_j * \mathbf{T}) + (1 - \mathbf{g}_j) * \mathbf{F}$$

Letting

- $\otimes$ denote element-wise vector multiplication (*Hadamard product*)
- $\sigma(\ldots)$ be a sigmoid approximation of a binary switch

The following product (almost) does the trick

$$\mathbf{g} = \sigma(\ldots)$$
$$\mathbf{y} = \mathbf{g} \otimes \mathbf{T} + (1 - \mathbf{g}) \otimes \mathbf{F}$$

It is only "almost"

- Because the sigmoid only takes a value in the range $[0, 1]$
- Rather than exactly either 0 or 1

What we have is

- A continuous (soft) decision $\mathbf{g}$.
- That creates a vector `if`
- Whose elements are *mixtures* of $\mathbf{T}$ and $\mathbf{F}$

This is the price we pay for having $\mathbf{g}$ be differentiable !

Note that the individual elements of vector $\mathbf{y}$ are independent

- $\mathbf{y}_j$ is influenced only by $\mathbf{g}_j$
- The synthetic features represented by $\mathbf{y}$ are not dependent on one another.
- Most importantly: the derivatives of each feature are independent

"Switch/Case" statements

We can easily generalize from a two-case `if` to a `switch/case` statement with $||\mathbf{C}||$ cases.

Suppose we need to set $\mathbf{y}$ to one value from among multiple choices in $\mathbf{C}$

$$\mathbf{g} = \mathrm{softmax}(\ldots)$$
$$\mathbf{y} = \mathbf{g} \otimes \mathbf{C}$$

The *softmax* function

- Was introduced in Multinomial Classification
- Computes a vector (of length $||C||$) values
- With each element being in the range $[0, 1]$
- And summing to 1

We refer to $\mathbf{g}$ as a *mask* for $\mathbf{C}$.

The `if` statement is a special case of the `switch/case` statement where

$$\mathbf{C} = \begin{bmatrix} \mathbf{T} \\ \mathbf{F} \end{bmatrix}$$

Conclusion

We wanted to show that, in concept

- We could create the logic of a simple imperative program
- Using the machinery of Neural Networks

The only catch was

- We cannot use true binary logic (hard decisions)
- All choices are *soft*
- In order to preserve differentiability
- Which is necessary for training with Gradient Descent

This background will facilitate our explanation of more advanced Layer types.

In [7]:
```
print("Done")
```

```
Done
```