

# Back propagation through time (BPTT)

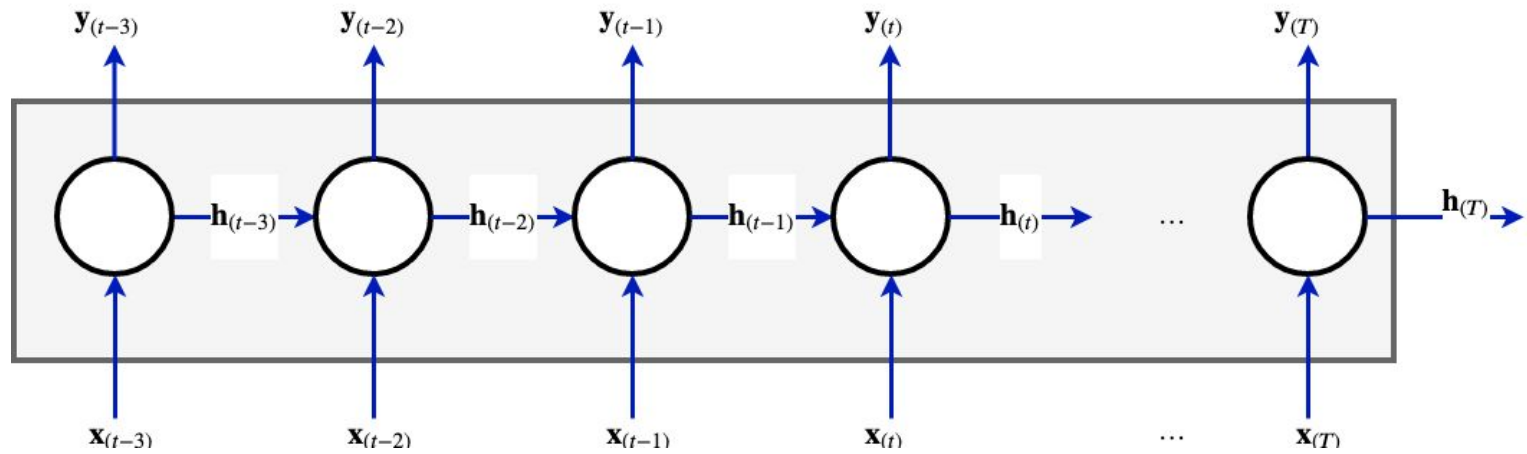
A Recurrent Neural Network (RNN)

- Can be viewed as a loop
- That can be unrolled
- Resulting in a multi-layer network
- One layer per time step

Here are the final layers of an unrolled RNN with input sequence

$$\mathbf{x}_{(1)}, \dots, \mathbf{x}_{(T)}$$

# RNN many to many API



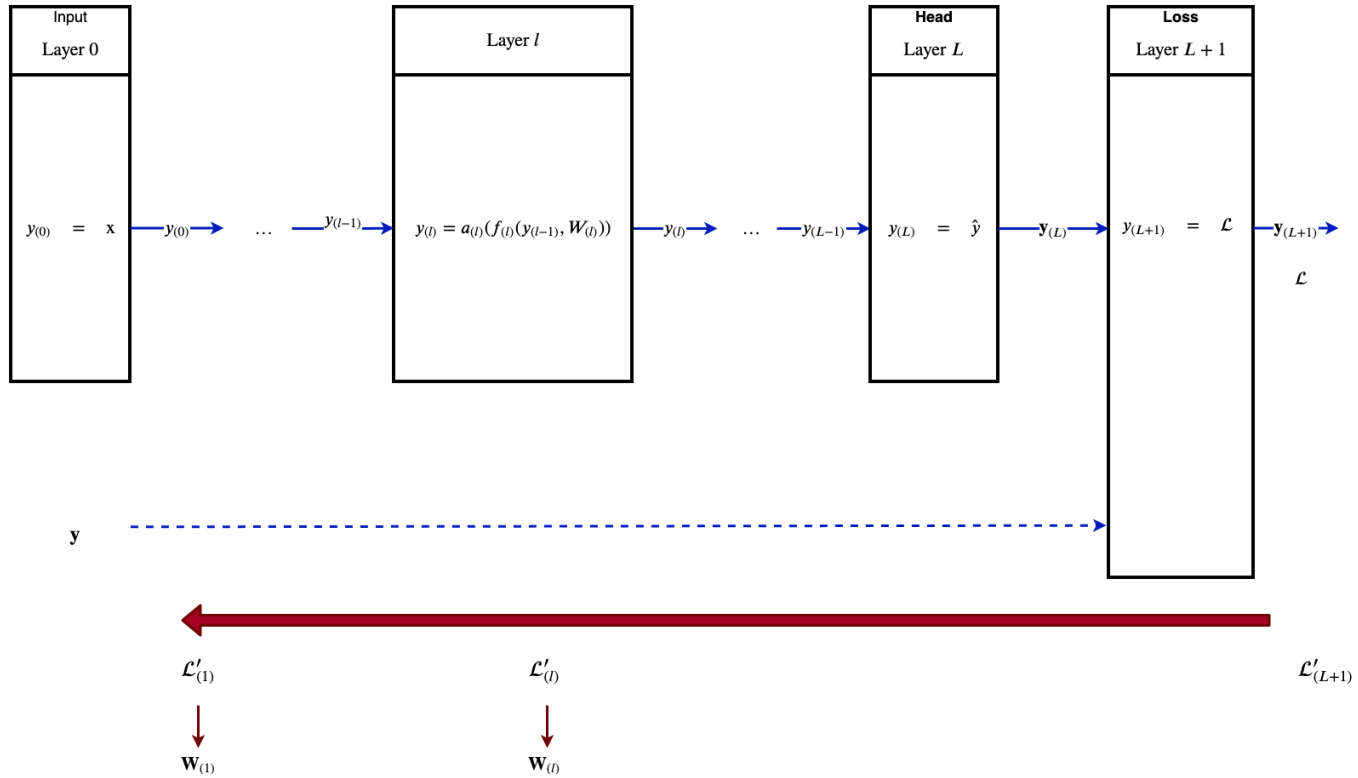
Given enough space: we would continue unrolling on the left to the Input layer

- Resulting in a network with  $T$  unrolled layers
- Plus a Loss layer

To compute the derivatives of the Loss with respect to weights

- We could, in theory, use Back Propagation
- Which is the weight update step of Gradient Descent

## Backward pass: Loss to Weights



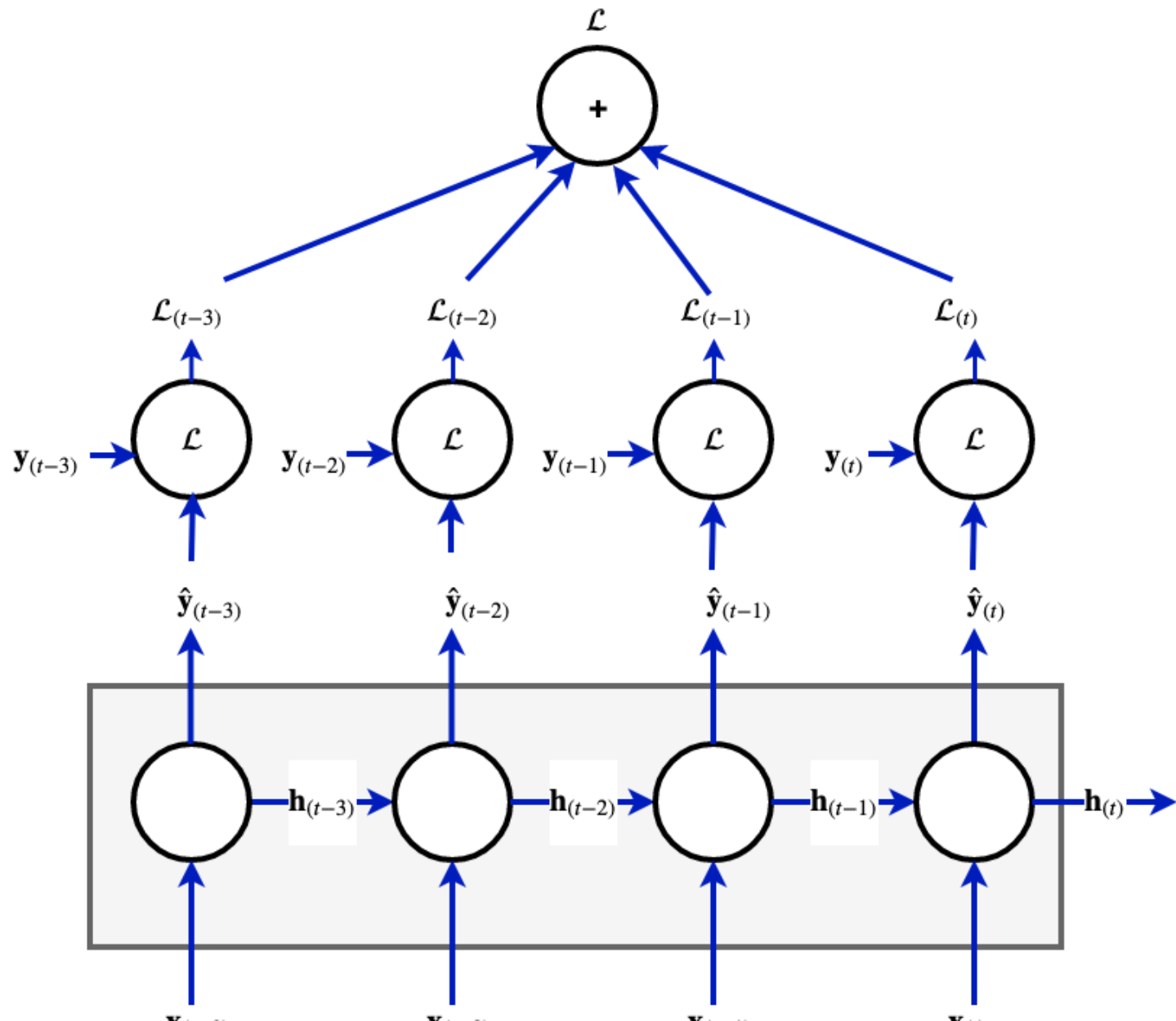
When dealing with unrolled RNN's

- We will index the "unrolled layers" with time steps, denoted by the label  $t$
- Rather than  $l$ , which we use to index layers

This process is called *Back Propagation Through Time* (BPTT).

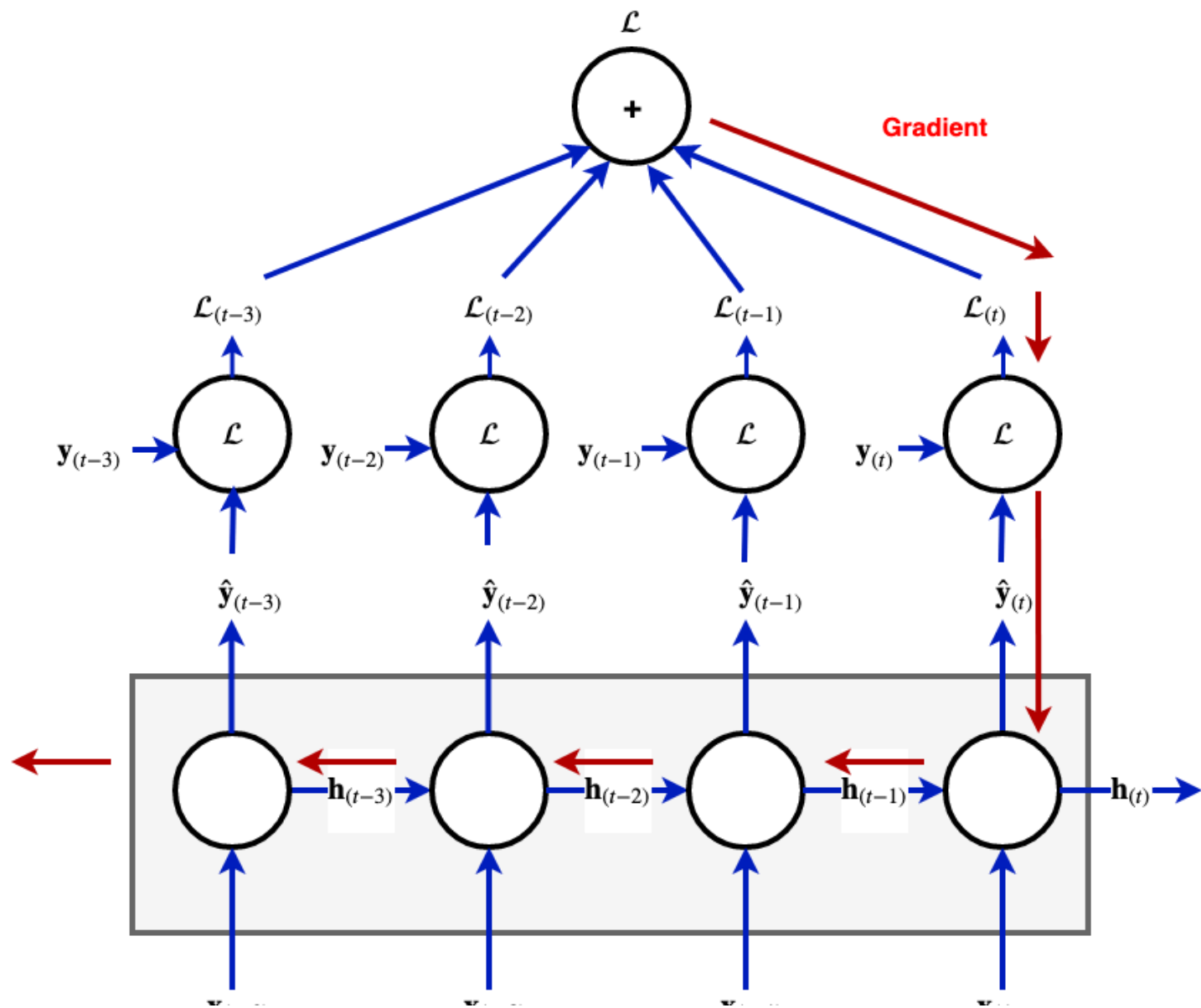
The only special thing to note about BPTT is that the Loss function is more complex

- There is a Loss
- Per example (as in non-recurrent layers)
- **and Per time-step** (unique to recurrent layers)











# Truncated back propagation through time (TBPTT)

An unrolled RNN layer turns into a  $T$  layer network where  $T$  is the number of elements in the input sequence.

For long sequences (large  $T$ ) this may not be practical.

First, there is the computation *time*

- $t$  steps to compute  $\mathcal{L}_{(t)}^{(i)}$ , the loss due to the  $t^{th}$  output  $\mathbf{y}_{(t)}^{(i)}$  of example  $i$
- For each  $1 \leq t \leq T$

Less obvious is the *space* requirement

- As we saw in the module "How a Neural Network Toolkit works"
- We may store information in each layer of the Forward pass (so storage for  $T$  layers)
- To facilitate computation of analytical derivatives on the Backward pass
  - For example: the Multiply layer stored the multiplicands in the forward pass
  - Because they are needed for the derivatives

Moreover, as we shall shortly see

- Derivatives may vanish or explode as we proceed further backwards from the Loss layer to the Input layer

So, in theory, the weights  $\mathbf{W}_{(t)}$  for small  $t$  (close to the input) may not get updated.

- This is certainly a problem in a non-recurrent network
- But is **fatal** in a recurrent layer
- Since there is a **single** weight matrix  $\mathbf{W}$  that is shared across *all time steps*  
$$\mathbf{W}_{(t)} = \mathbf{W} \text{ for all } 1 \leq t \leq T$$



The solution to these difficulties

- Is to *truncate* the unrolled RNN
- To a fixed number of time steps
- From the loss layer backwards
- The truncated graph is a suffix of the fully unrolled graph

This process is known as *Truncated Back Propagation Through Time* (TBPTT).

Note that *truncation only occurs in the backward pass*.

There is *no truncation* of the forward pass of the RNN !

Because the unrolled graph is less than  $T$  steps

- Gradient computation takes fewer steps
- So weight updates can occur more often

The obvious downside to truncation is that

- Gradients are only approximate

But there is a subtle and more impactful difference

- The RNN layer \*cannot capture long-term dependencies\*\*

Suppose we unrolled the layer for only  $\tau$  time steps (the "window" size)

- The loss for the  $t^{th}$  time step ( $\mathcal{L}_{(t)}^{(i)}$ )
- Flows backwards only to steps  
 $(t - \tau + 1), \dots, t$

So the "error signal" from time  $t$  does not affect time steps  $t' < (t - \tau + 1)$

Consider a long sentence or document (sequence of words)

- If the gender of the subject is defined by the early words in the sentence
- An incorrect "prediction" late in the sentence
- May not be able to be corrected

"Z was the first woman who ... **he** said ..."

In other words

- Truncation may affect the ability of an RNN to encode *long-term* dependencies
- Vanishing gradients may cause a similar impact

## TBPTT variant

There are several common ways to decide on how many unrolled time steps to keep.

Let  $t''$  denote the index of the *smallest* time step in the unrolled layer for step  $t$ .

- $t'' = (t - \tau + 1)$



Plain, untruncated BPTT defines

- $t'' = 0$
- Unroll all the way to the Input Layer

$k$ -truncated BPTT defines window size  $\tau = k$

- $t' = \max(0, t - k)$

Subsequence truncated BPTT defines

- $t'' = k * \lfloor t/k \rfloor$

That is, it breaks the sequence into "chunks" of size  $k$

$$\begin{aligned} & \mathbf{x}_{(1)}^{(i)}, \dots, \mathbf{x}_{(k)}^{(i)} \\ & \mathbf{x}_{(k+1)}^{(i)}, \dots, \mathbf{x}_{(2*k)}^{(i)} \\ & \vdots \\ & \mathbf{x}_{((i'*k)+1)}^{(i)}, \dots, \mathbf{x}_{((i'+1)*k)}^{(i)} \\ & \vdots \end{aligned}$$

- Gradients flow *within* chunks
- But *not between* chunks

Subsequence TBPTT is very common as it fits well into the design of current toolkits

See the Deep Dive on [How to deal with long sequences \(RNN Long Sequences.ipynb\)](#) for how to arrange your training examples.

# Calculating gradients in an RNN

There is an important subtlety we have ignored regarding Back Propagation in an unrolled RNN

- There is a **single** weight matrix  $\mathbf{W}$  that is shared across *all time steps*  
 $\mathbf{W}_{(t)} = \mathbf{W}$  for all  $1 \leq t \leq T$

This

- Makes the derivative computation slightly more complex
- Creates an *additional* exposure to the problem of vanishing/exploding gradients

A simple picture will illustrate.

Consider the loss at time step  $t$  of example  $i$

- $\mathcal{L}_{(t)}^{(i)} = L(\hat{\mathbf{y}}_{(t)}^{(i)}, \mathbf{y}_{(t)}^{(i)}; \mathbf{W})$
- The loss is a function of
  - $\hat{\mathbf{y}}_{(t)}^{(i)}$ : The  $t^{th}$  element of the output sequence  $\hat{\mathbf{y}}^{(i)} = \mathbf{y}_{(T)}$  for example  $i$
  - The  $\mathbf{y}_{(t)}^{(i)}$ : The  $t^{th}$  element of the **target** sequence  $\mathbf{y}^{(i)}$  for example  $i$



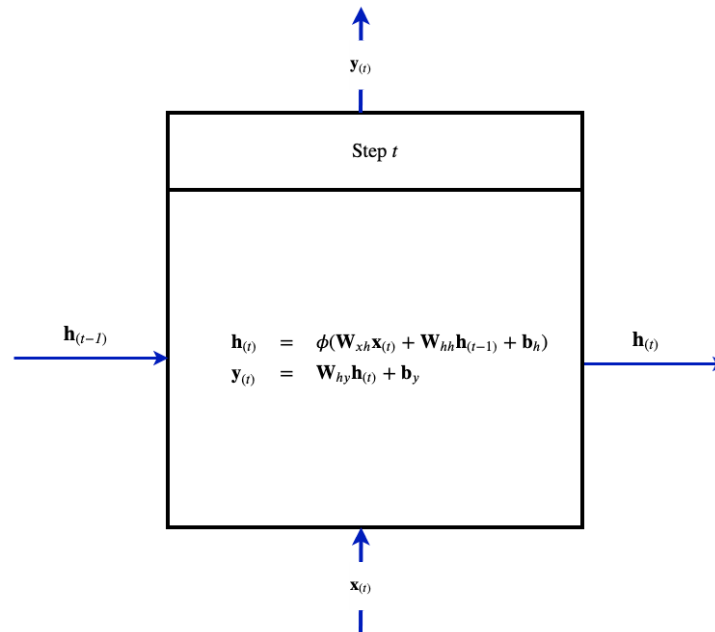
The derivative

$$\frac{\partial \mathcal{L}_{(t)}}{\partial \mathbf{W}}$$

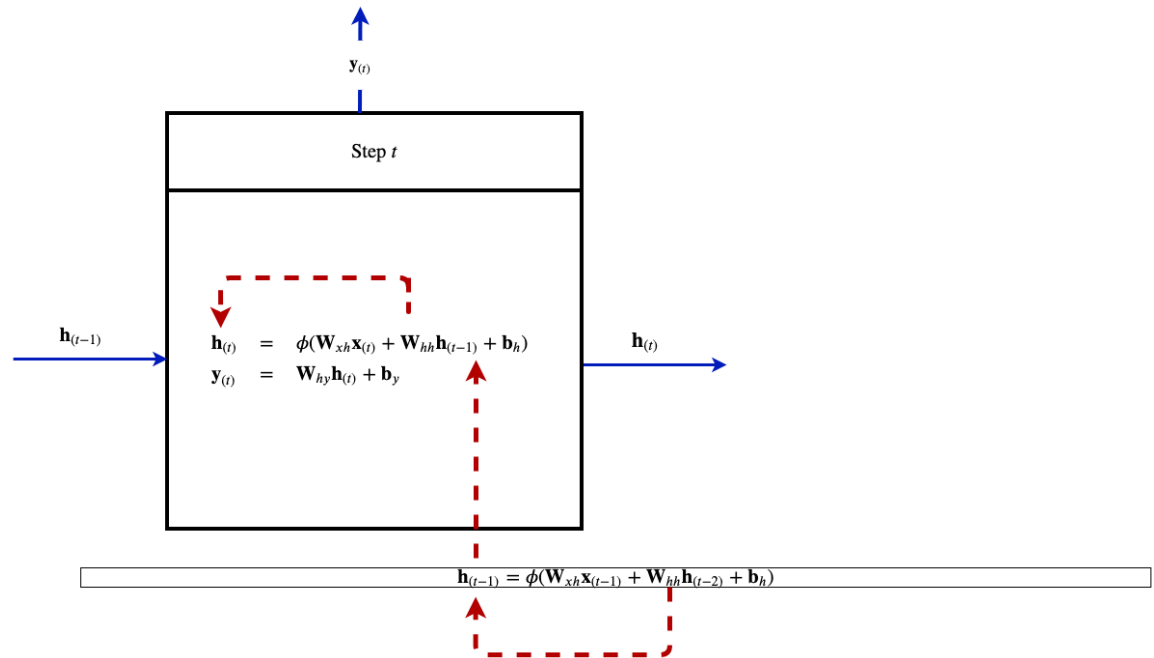
- Which is used in Gradient Descent to update the estimate for weights  $\mathbf{W}$
- Is the sensitivity of  $\mathcal{L}'_{(t)}$  to a change in  $\mathbf{W}$

## RNN Time step

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## RNN multiple dependence on $W$



The red lines show **two** different ways that  $\mathbf{W}$  affects  $\mathbf{h}_{(t)}$

- And thus  $\hat{\mathbf{y}}_{(t)} = \mathbf{W}_{hy}\mathbf{h}_{(t)} + \mathbf{b}_y$
- By its indirect effect on  $\mathbf{h}_{(t)}$  **through**  $\mathbf{h}_{(t-1)}$  (lower line)
- By its direct effect on  $\mathbf{h}_{(t)}$  (upper line)
- Both using the part of  $\mathbf{W}$  denoted by  $\mathbf{W}_{hh}$

So

$$\begin{aligned}\frac{\partial \mathbf{h}_{(t)}^{(i)}}{\partial \mathbf{W}_{hh}} &= \frac{d\mathbf{h}_{(t)}^{(i)}}{d\mathbf{W}_{hh}} + \frac{\partial \mathbf{h}_{(t)}^{(i)}}{\partial \mathbf{h}_{(t-1)}^{(i)}} \frac{\partial \mathbf{h}_{(t-1)}^{(i)}}{\partial \mathbf{W}_{hh}} \\ &= \frac{d(\mathbf{W}_{hh} \mathbf{h}_{(t-1)}^{(i)})}{d\mathbf{W}_{hh}} + \frac{\partial \mathbf{h}_{(t)}^{(i)}}{\partial \mathbf{h}_{(t-1)}^{(i)}} \frac{\partial \mathbf{h}_{(t-1)}^{(i)}}{\partial \mathbf{W}_{hh}}\end{aligned}$$

(Each addend reflect a different path through which  $\mathbf{W}_{hh}$  affects  $\mathbf{h}_{(t)}$ )

So

$$\frac{\partial \mathcal{L}_{(t)}^{(i)}}{\partial \mathbf{W}} = \mathcal{L}'_{(t)} \frac{\partial \mathbf{y}_{(t)}}{\partial \mathbf{W}}$$

and

$$\frac{\partial \mathbf{y}_{(t)}^{(i)}}{\partial \mathbf{W}}$$

*depends* on all time steps from 1 to  $t$ .

Thus, the derivative update for  $\mathbf{W}$  cannot be computed without the gradient (for each time step  $t$ ) flowing all the way back to time step 0.

# Conclusion

Updating the weights of a Recurrent layer appears, at first glance, to be straight forward

- Unroll the loop
- Use ordinary Back Propagation

We have discovered some complexity

- Full unrolling is expensive
- Gradient computation is complicated by shared weights

Fortunately, we have solutions to these complexities.

In [3]: `print("Done")`

Done