Classification

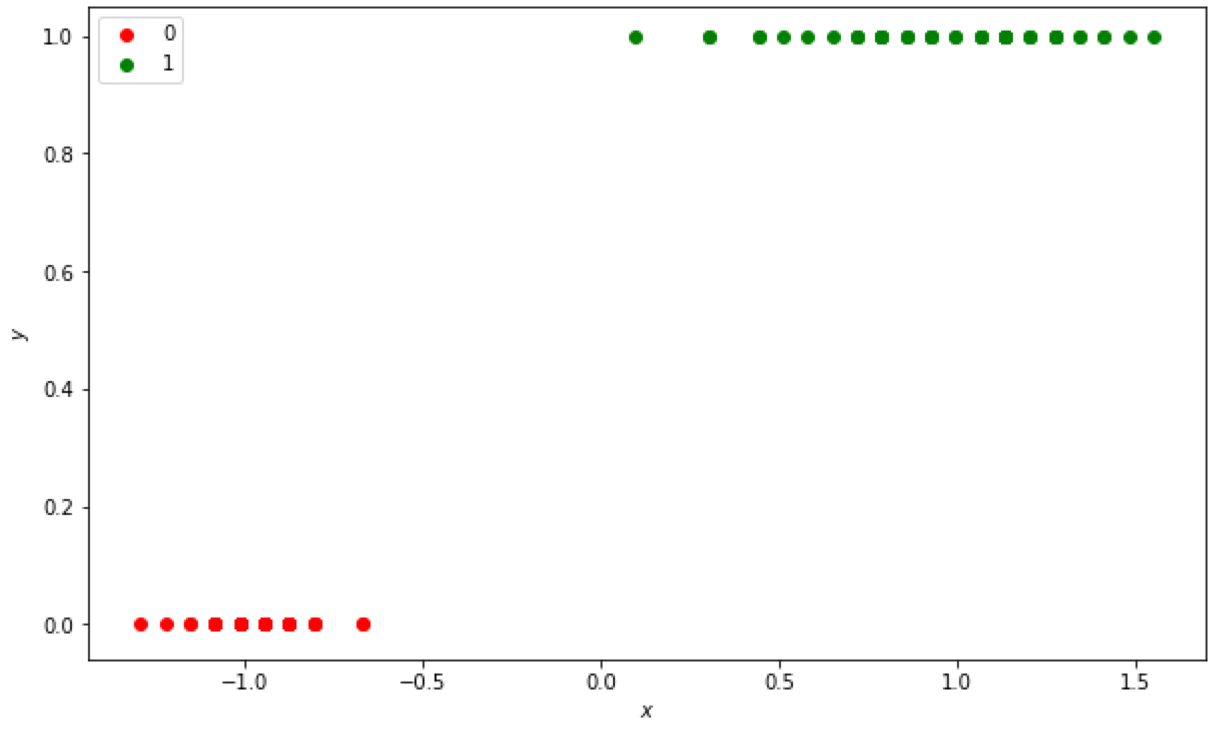
How do we predict a target that is a categorical rather than a number (as in regression) ?

To be concrete: we consider the *Binary Classification* task

- two classes (categories)
- which we refer to as Positive and Negative
- encoded numerically as 1 and 0
- plotted as Green and Red points

Consider examples with a single feature.

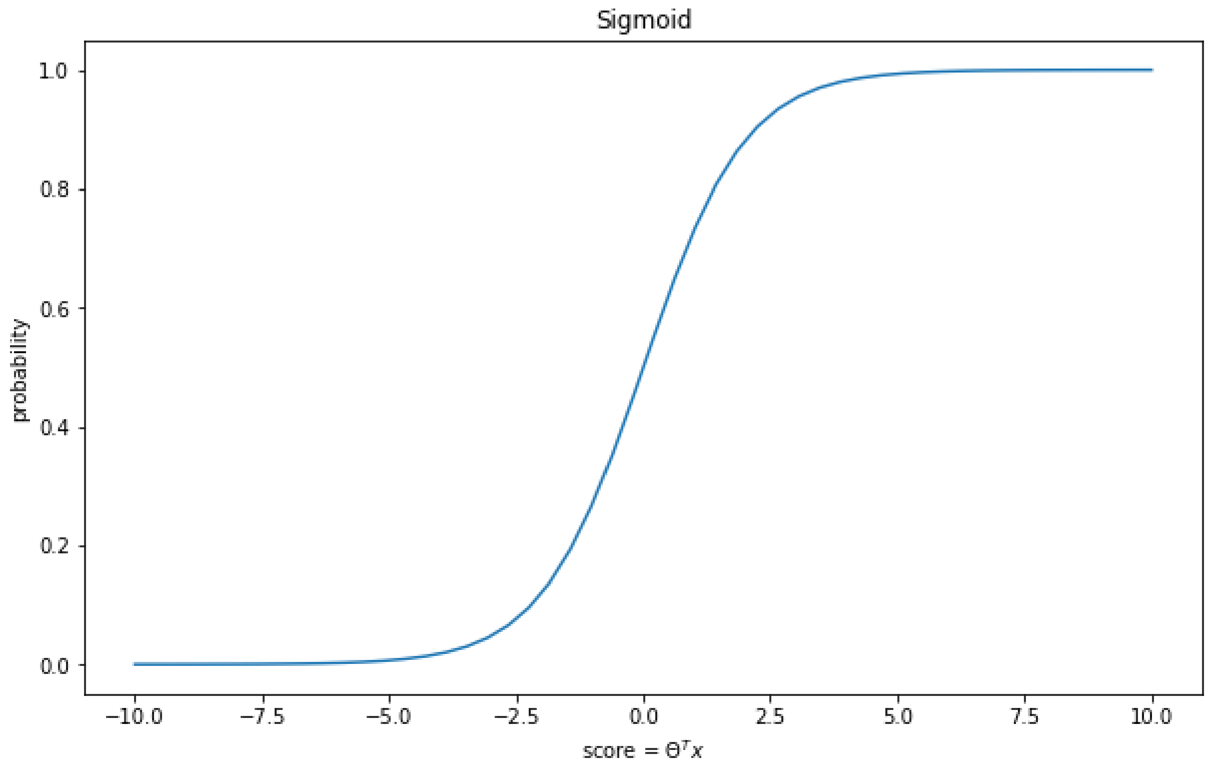
In [4]:
```
X_ls, y_ls = lsh.load_iris()

fig, ax = plt.subplots(figsize=(10,6))
_= lsh.plot_y_vs_x(ax, X_ls[:,0], y_ls)
```



As you can see, fitting a straight line will not do, so straight forward Linear Regression won't suffice.

There is a function, the *sigmoid*, that has the right shape:

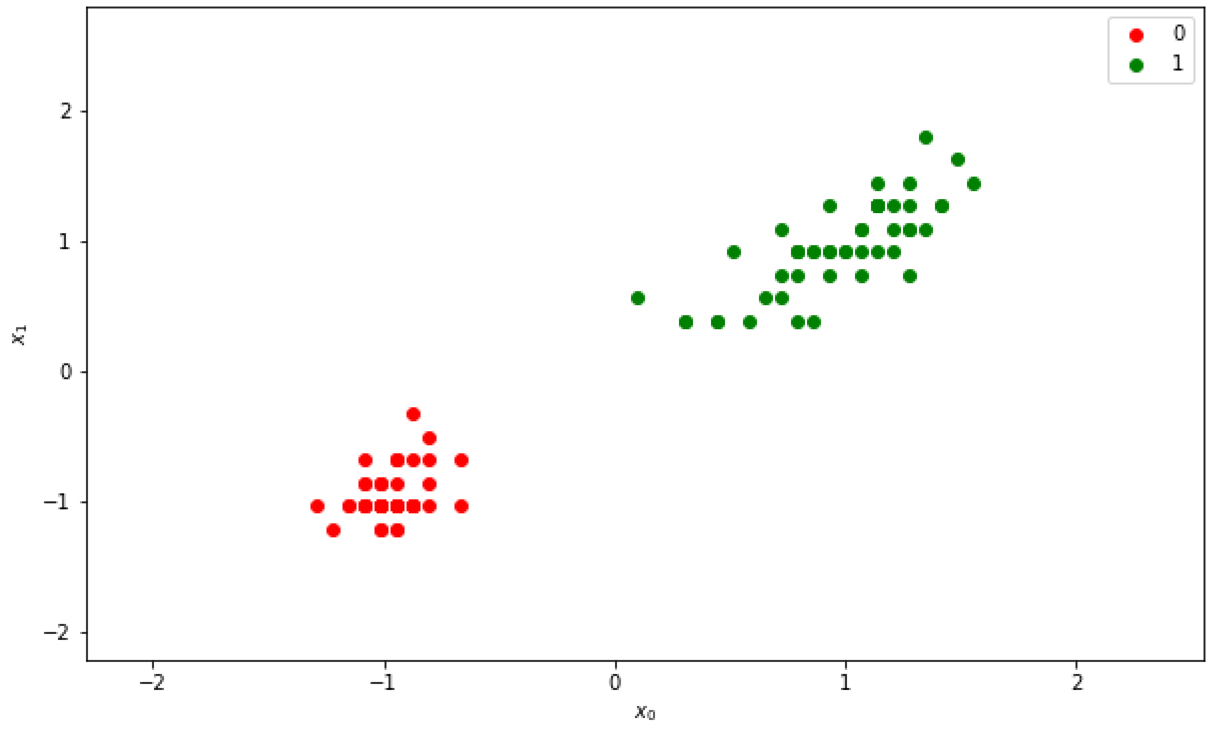
In [5]:
```python
fig, ax = plt.subplots(figsize=(10,6))
_= lsh.plot_sigmoid(ax)
```



This is clearly not linear either, but we will adapt Linear Regression to this functional form.

Here is a example with two features.

In [6]:
```
clf_ls = lsh.fit_LR(X_ls,y_ls)
fig, ax = plt.subplots(figsize=(10,6))
_= lsh.plot(ax, clf_ls, X_ls, y_ls, draw_boundary=False, scores=np.array([]))
```



Is it possible to adapt the Regression task for Classification ?

An obvious idea:

- Use the features $\mathbf{x}^{(\mathbf{i})}$ to compute a "score" (*logit*) $\hat{s}^{(\mathbf{i})}$
- Compare the predicted score to a threshold
- Predict Positive if the score exceeds the threshold; Negative otherwise

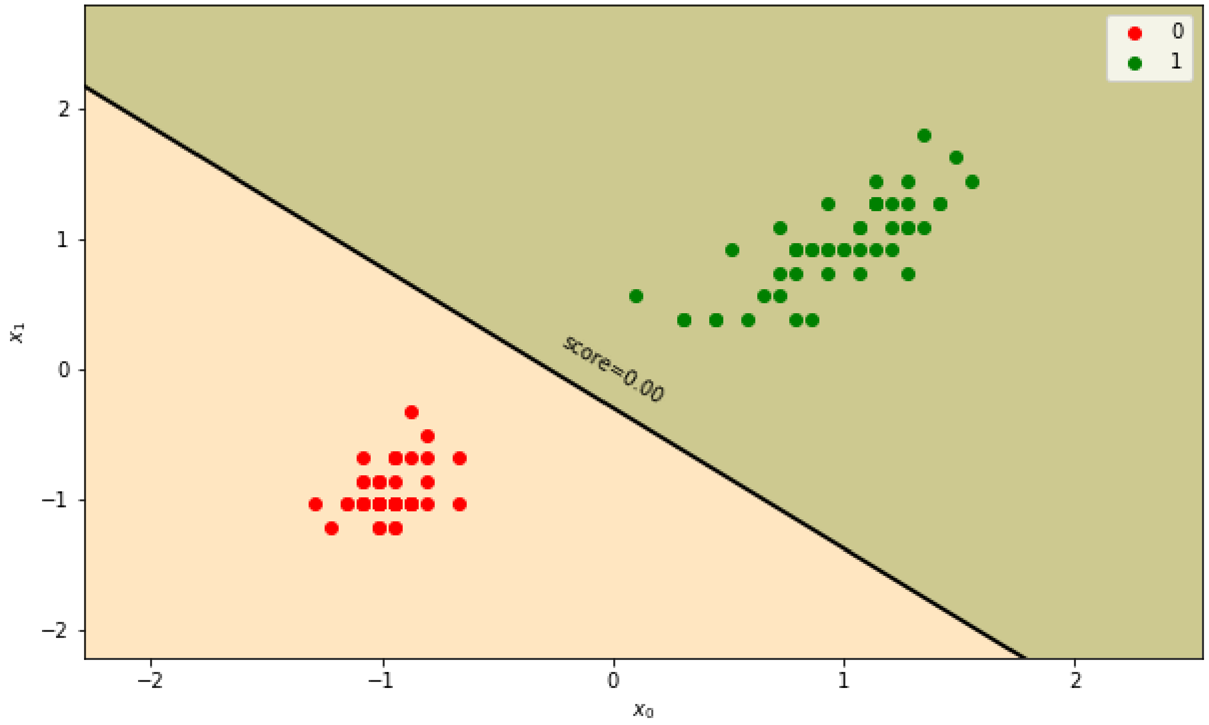
$$\hat{y}^{(\mathbf{i})} = \begin{cases} \text{Negative} & \text{if } \hat{s}^{(\mathbf{i})} < 0 \\ \text{Positive} & \text{if } \hat{s}^{(\mathbf{i})} \geq 0 \end{cases}$$

If the score has the form of a Linear Regression

$$s(\mathbf{x}) = \Theta^T \mathbf{x}$$

then we get something like this

In [7]:
```
fig, ax = plt.subplots(figsize=(10,6))
_= lsh.plot(ax, clf_ls, X_ls, y_ls, draw_prob=False)
```



That is: the score $s(\mathbf{x})$

- is linear in features $\mathbf{x}$
- separates Positive from Negative examples
    - Examples $(\mathbf{x}_0, \mathbf{x}_1)$ withs non-negative scores (i.e, points above the line) get classified as Positive
    - Examples $(\mathbf{x}_0, \mathbf{x}_1)$ withs negative scores (i.e, points below the line) get classified as Negative

If we can successfully classify by this method, the dataset set is *linearly separable*

A classifier for linearly separable data fits a hyperplane (e.g., the line $\hat{s} = 0$) to the training data such that

- examples lying above the plane are classified as Positive
- examples lying below the plane are classified as Negative

$$\boldsymbol{s} = \boldsymbol{\Theta}^T \mathbf{x}$$

Can be interpretted as

- using template matching on the features $\mathbf{x}$ to produce a "score" $\boldsymbol{s} = \boldsymbol{\Theta}^T \mathbf{x}$

Transforming Binary Classification into Linear Regression

How do we fit the scoring function ?

We adapt Linear Regression.

Let's reinterpret the targets/labels $\mathbf{y}^{(\mathbf{i})}$ as a probability $p^{(\mathbf{i})}$

$$p^{(\mathbf{i})} = p(\mathbf{y}^{(\mathbf{i})} = \mathrm{Positive})$$

So

- $\mathbf{y^{(i)}} = \text{Positive}$ is equivalent to $p^{\mathbf{(i)}} = 1$: the target for example i is Positive with 100% probabiity
- $\mathbf{y^{(i)}} = \text{Negative}$ is equivalent to $p^{\mathbf{(i)}} = 0$: the target for example i is Positive with 0% (i.e., is Negative(

So the predicted score $\hat{s}^{(\mathbf{i})}$ being greater than a threshold (e.g. 0) corresponds to $\hat{p}^{(\mathbf{i})} = 1$.

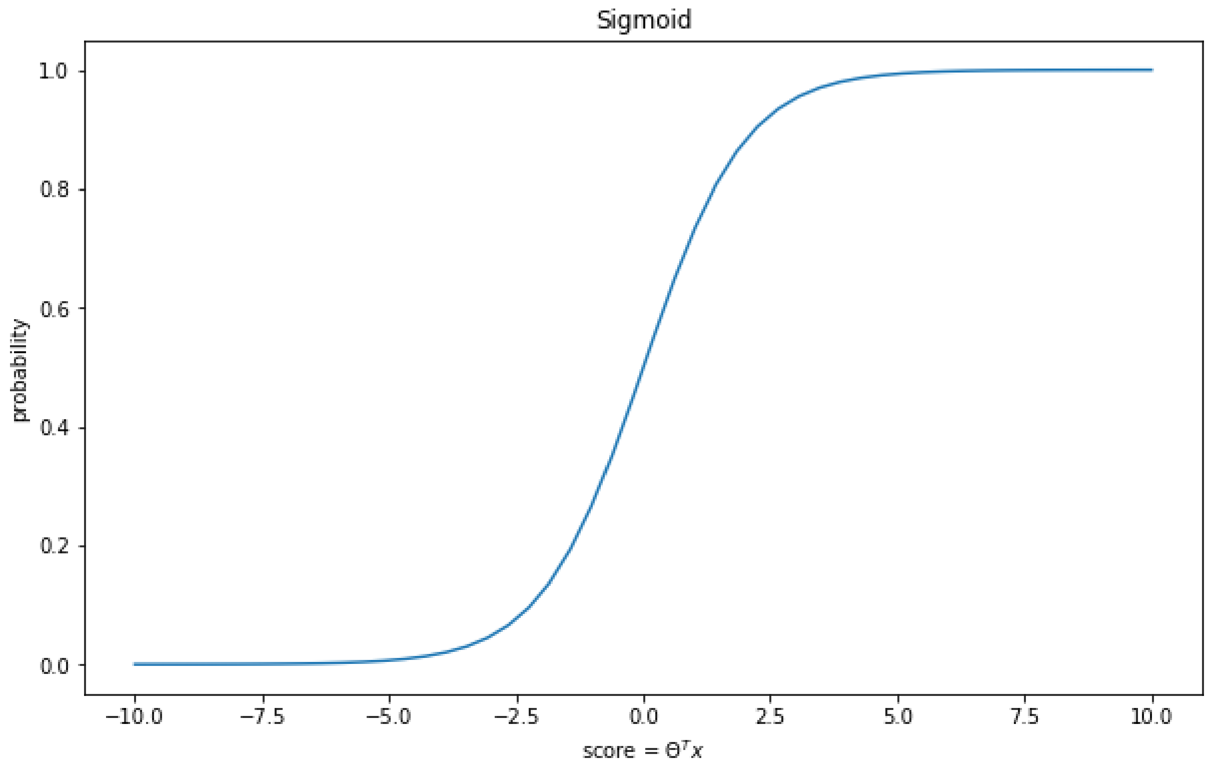
We can go further: map $\hat{s}^{(\mathbf{i})}$, which is continuous, into a continuous probability $\hat{p}^{(\mathbf{i})} \in [0, 1]$.

The *Logistic Function* $\sigma(s)$ transforms a number s (e.g., score) into a probability

$$\hat{p} = \sigma(s) = \frac{1}{1 + e^{-s}}$$

Let's plot the logistic function to gain some intuition:

In [8]:
```
fig, ax = plt.subplots(figsize=(10,6))
_= lsh.plot_sigmoid(ax)
```



As you can see, it acts almost like a binary "switch"

- range is mostly 0 or 1

So this function creates a sharp boundary (measured in probability).

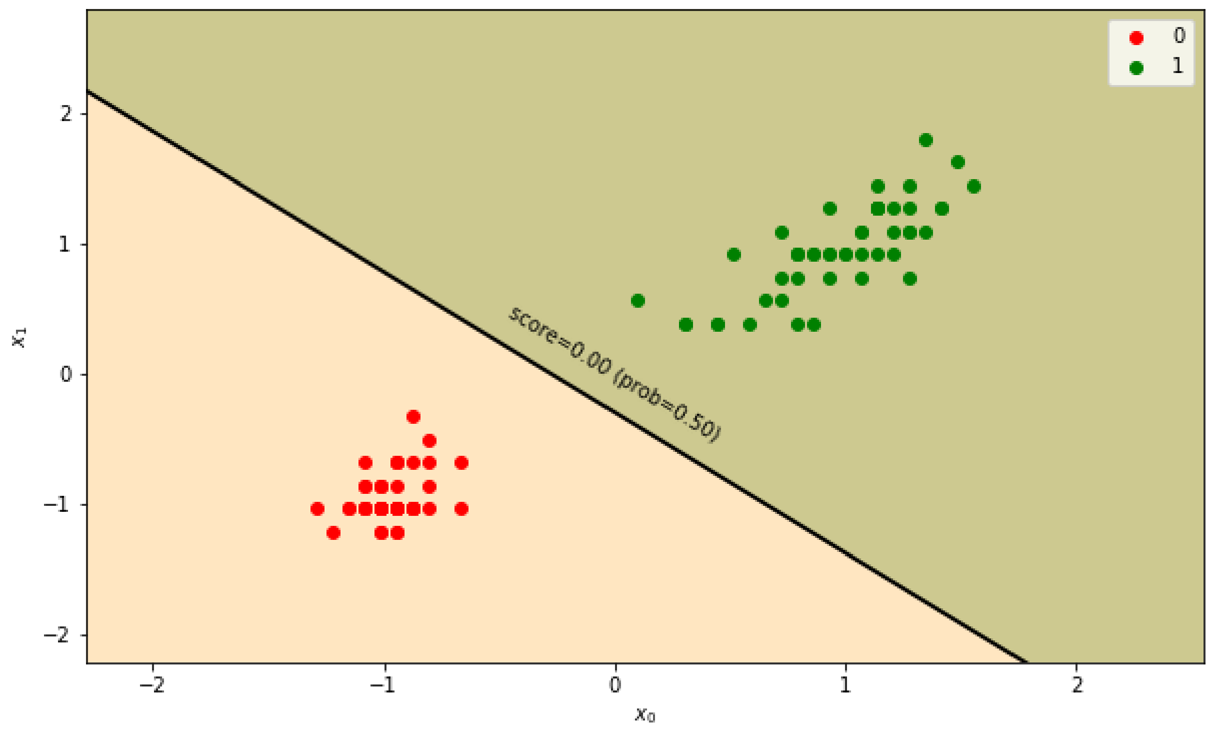
Now that we can convert between scores and probabilities the following two forms of classification are equivalent

$$\hat{y}^{(\mathbf{i})} = \begin{cases} \text{Negative} & \text{if } \hat{s}^{(\mathbf{i})} < 0 \\ \text{Positive} & \text{if } \hat{s}^{(\mathbf{i})} \geq 0 \end{cases}$$

$$\hat{y}^{(\mathbf{i})} = \begin{cases} \text{Negative} & \text{if } \hat{p}^{(\mathbf{i})} < 0.5 \\ \text{Positive} & \text{if } \hat{p}^{(\mathbf{i})} \geq 0.5 \end{cases}$$

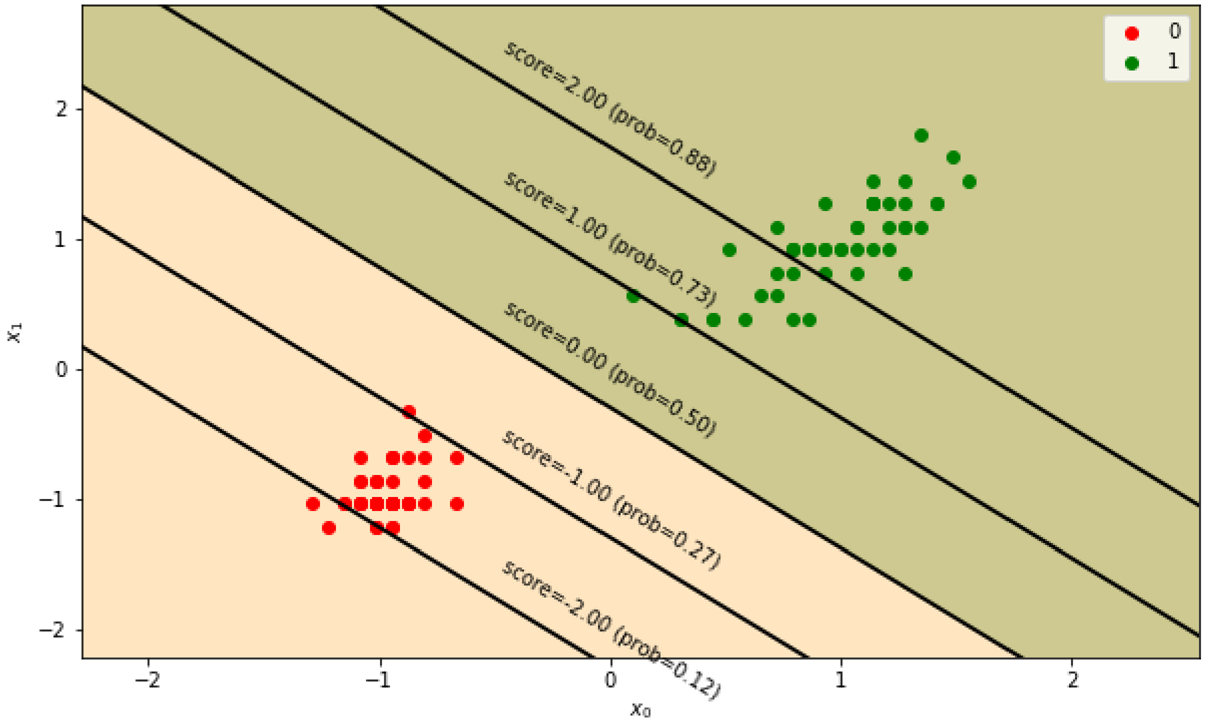
(because $\sigma(0) = 0.5$)

In [9]:
```
fig, ax = plt.subplots(figsize=(10,6))
_= lsh.plot(ax, clf_ls, X_ls, y_ls)
```



One can see the relationship between score and probability by looking at lines of constant score/probability.

In [10]:
```
fig, ax = plt.subplots(figsize=(10,6))
_= lsh.plot(ax, clf_ls, X_ls, y_ls, scores = np.arange(-2, 3,1))

fig.savefig(os.path.join("/tmp",'class_overview_prob_lines.jpg') )
```



- Increasingly positive scores result in increasing probability of Positive
- Increasingly negative scores result in decreasing probability of Positive (and hence increasing probability of Negative)

When the score is infinite, the probability becomes 100% (positive infinity) or 0% (negative infinity)

Logistic Regression

Because we use the Logistic function to map scores to probabilities, this method is called *Logistic Regression*.

To recap:

$$\begin{aligned} s &= \Theta^T \mathbf{x} \\ \hat{p} &= \sigma(s) \end{aligned}$$

$$\hat{y}^{(\mathbf{i})} = \begin{cases} \text{Negative} & \text{if } \hat{p}^{(\mathbf{i})} < 0.5 \\ \text{Positive} & \text{if } \hat{p}^{(\mathbf{i})} \geq 0.5 \end{cases}$$

Preview

The expression for $\hat{p}$

$$\hat{p} = \sigma(\mathbf{\Theta}^T \mathbf{x})$$

which involves

- template matching of features versus template ($\mathbf{\Theta}$)
- convert the score into a probability with the sigmoid function

will reappear in the Deep Learning part of the course.

Of all the functions to "squeeze" score s into the range $[0, 1]$, why choose the Logistic Function ?

Let's invert the relationship induced by the Logistic Function

$$\hat{p} = \sigma(s)$$

between probability $\hat{p}$ and s

$$\begin{aligned}
\frac{\hat{p}}{1-\hat{p}} &= \frac{\frac{1}{1+e^{-s}}}{1-\frac{1}{1+e^{-s}}} \\
&= \frac{\frac{1}{1+e^{-s}}}{\frac{e^{-s}}{1+e^{-s}}} \\
&= e^{s} \\
\log_e \frac{\hat{p}}{1-\hat{p}} &= s
\end{aligned}$$

So, using the logistic function to compute $\hat{p}$ results in

$$\log_e \frac{\hat{p}}{1-\hat{p}} = \Theta^T \mathbf{x}$$

The above equation has the form of Linear Regression where target $\mathbf{y}$ has been transformed to

$$\log_e \frac{\hat{p}}{1-\hat{p}}$$

The term $\frac{\hat{p}}{1-\hat{p}}$ is called the *odds* (of being Positive) so the dependent variable is the *log odds*.

We have thus transformed Binary Classification into Linear Regression.

This introduction glosses over several problems, which we will subsequently address

- the log odds is positive infinity when $p = 1$
- the log odds is negative infinity when $p = 0$

This means that MSE can't be used as a Loss Function for fitting since some residuals are infinite.

```
In [11]: print("Done !")
```

```
Done !
```