

Classification: Loss function

It would be natural to expect the Average Loss to be Accuracy (fraction of correct predictions).

On a per example basis, the corresponding loss $\mathcal{L}^{(i)}$ would be either 1 or 0, depending on correctness.

This is not the case.

Recall the mapping of probability to prediction

$$\hat{y}^{(i)} = \begin{cases} \text{Negative} & \text{if } \hat{p}^{(i)} < 0.5 \\ \text{Positive} & \text{if } \hat{p}^{(i)} \geq 0.5 \end{cases}$$

The prediction for example i changes only when probability $\hat{p}^{(i)}$ crosses the threshold.

Suppose the class for example i is Positive: $\mathbf{y}^{(i)} = \text{Positive}$.

- Is our model "better" when

$\hat{p}^{(i)} \approx 1$ than when $\hat{p}^{(i)} = 0.5$

$\hat{p}^{(i)} = (.5 - \epsilon)$ than when $\hat{p}^{(i)} \approx 0$

- The per-example Accuracy is the same in both comparisons
- But a model with probability $\hat{p}^{(i)}$ as close to $\mathbf{y}^{(i)} = 1$ would seem to better

There is no *degree* or magnitude of inaccuracy

- Two models may have the same Accuracy even though the probabilities of one may be closer to perfect than the other
- In our search for the best Θ , Accuracy won't be a guide

In mathematical terms: we want our Loss function be be continuous and differentiable.

Accuracy (and the per-example analog) satisfies neither.

We will introduce Binary Cross Entropy loss to overcome this difficulty.

Think of Binary Cross Entropy as a continuous analog of Accuracy.

Binary Cross Entropy

Let's encode the Positive labels $y^{(i)}$ with the number 1 and Negative labels with the number 0. The loss for example i will be defined as
$$\text{loss}^{(i)}_{\Theta} = \begin{cases}$$

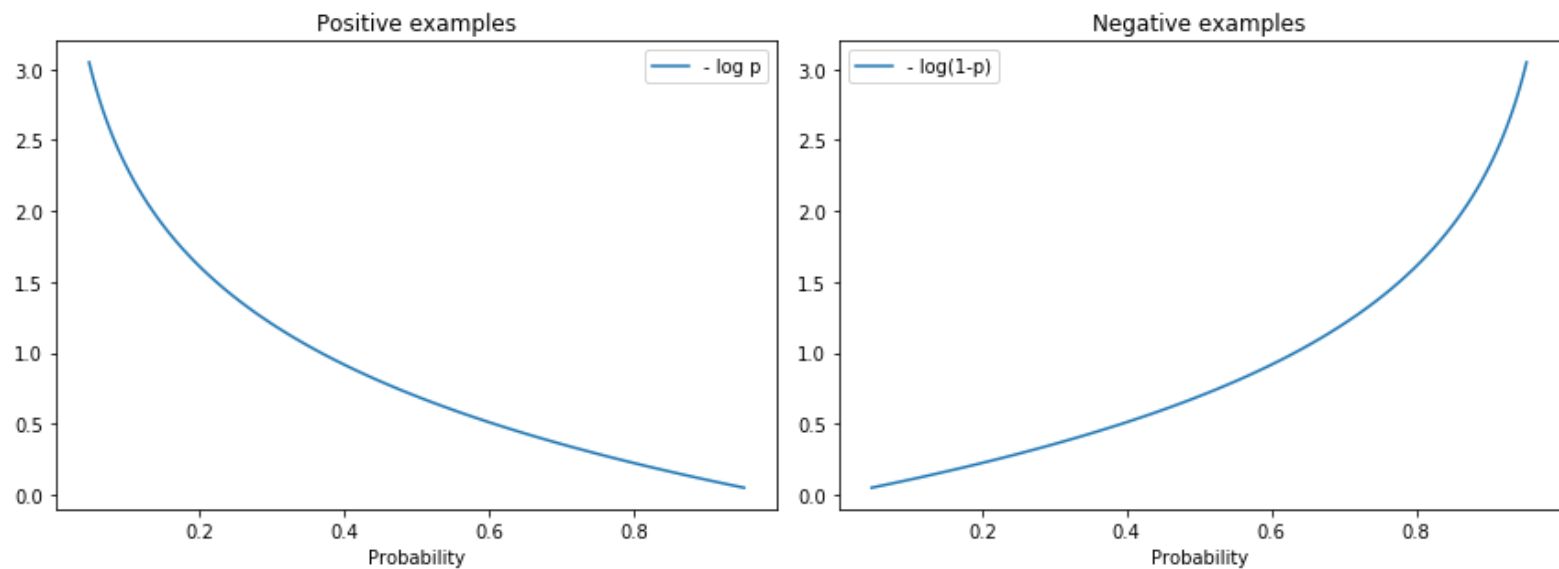
- $\log(\hat{p})$ & $\text{if } y^{(i)} = 1$
- $\log(1-\hat{p})$ & $\text{if } y^{(i)} = 0$ \end{cases}

Note the negative signs:

- the term being negated is a Utility (which we want to maximize)

A plot will give us some intuition.

```
In [4]: svmh.plot_log_p(x_axis="Probability")
```



- For Positive examples: the loss approaches 0 as the predicted probability approaches the correct value (1).
- For Negative examples: the loss approaches 0 as the predicted probability approaches the correct value (0).

In a Deep Dive (after the introduction of a bit of math) we will gain a greater appreciation it's meaning.

For now: be content that Binary Cross Entropy seems to have the right slope and asymptotic behavior.

Because only one of $\mathbf{y}^{(i)}$ and $(1 - \mathbf{y}^{(i)})$ is non zero, we can re-write the two-case statement into a single expression

$$\mathcal{L}_{\theta}^{(i)} = - \left(\mathbf{y}^{(i)} * \log(\hat{p}^{(i)}) + (1 - \mathbf{y}^{(i)}) * \log(1 - \hat{p}^{(i)}) \right)$$

This expression is referred to as *Binary Cross Entropy*; it and the multi-class version will become quite familiar going forward.

The Loss for the entire training set is simply the average (across examples) of the Loss for the example

$$\mathcal{L}_{\Theta} = \frac{1}{m} \sum_{i=1}^m \mathcal{L}_{\Theta}^{(i)}$$

Cost function for Multinomial Classification: Cross Entropy

A Multinomial Classifier (when categories/classes $||C|| > 2$) can be created from multiple Binary Classifiers

- Create a separate Binary Classifier for each $c \in C$
- The classifier for category c attempts to classify
 - Each example with target category of c as Positive
 - All other examples as Negative
- Combine the $||C||$ classifiers to produce a vector \hat{p} of length $||C||$
 - normalize across $c \in C$ to sum to 1
 - \hat{p}_c denotes the normalized value for category c
 - **Notation abuse:** subscripts should be integers, not categories

Both the target \mathbf{y} and the prediction \hat{p} are represented as vectors of length $\|C\|$

- We write \mathbf{y}_c, \hat{p}_c to denote the element of the vector corresponding to category c
- Each vector can be interpreted as a probability distribution, e.g.

$$\forall c \in C : \mathbf{y}_c \geq 0$$

$$\sum_{c \in C} \mathbf{y}_c = 1$$

- \mathbf{y} was created with One Hot Encoding (OHE), so properties satisfied by construction
- \hat{p}_c satisfies the properties by virtue of the normalization of the predictions of the $\|C\|$ binary classifiers

With $\mathbf{y}, \hat{\mathbf{p}}$ encoded as a vectors, per example Binary Cross Entropy can be generalized to $\|C\| \geq 2$ categories:

$$\mathcal{L}_{\Theta}^{(i)} = - \sum_{c=1}^{\|C\|} \left(\mathbf{y}_c^{(i)} * \log(\hat{\mathbf{p}}_c^{(i)}) \right)$$

This is the multinomial analog of Binary Cross Entropy and is called **Cross Entropy**.

Cross Entropy can be interpreted as a measure of the "distance" between distributions \mathbf{y} and \hat{p}

- Minimized when they are identical
- We will use Cross Entropy in the future both as a Loss function and a way of comparing probability distributions

In [5]: `print("Done")`

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