Interpreting the coefficients in Linear Models

The dot product has been a recurring character during our Classical Machine Learning journey.

$$\mathbf{\Theta} \cdot \mathbf{x}$$

By examining this expression more closely

- We can gain insight into what $\mathbf{\Theta}$ means
- Understand mathematically why transformation may be necessary
- Gain an appeciation of the "pattern matching" that it embodies

Recall the places in which dot product appears

- Linear Regression
$$\hat{y} = \Theta \cdot \mathbf{x}$$
- Logistic Regression
$$\hat{s} = \Theta \cdot \mathbf{x}$$
for score $\hat{s}$ (which becomes a probablility via $\hat{p} = \sigma(\hat{s})$
- Boundary equation for linearly separable classifiers, e.g., SVM
$$0 = \Theta \cdot \mathbf{x}$$

Consider one feature $\mathbf{x}_j^{(\mathbf{i})}$ for example i.

- A unit increase in $\mathbf{x}_j^{(\mathbf{i})}$
- Holding constant the values for all other features,
- Increases $\Theta \cdot \mathbf{x}^{(\mathbf{i})}$ by Θ_j

Thus

$$\Theta_j = \frac{\partial}{\partial \mathbf{x}_j} \Theta \cdot \mathbf{x}$$

$\boldsymbol{\Theta}_j$ may be interpreted as

- The sensitivity of $\boldsymbol{\Theta} \cdot \mathbf{x}$ to changes in feature j

Numeric features

Consider numeric features $\mathbf{x}_j, \mathbf{x}_{j'}$.

Does

$$\Theta_j > \Theta_{j'}$$

mean that feature j is "more important" than feature j' ?

- No !
- It just means it has a larger impact
- Which can *also* occur if $\mathbf{x}_j, \mathbf{x}_{j'}$ are on different scales

For example consider the equality

$$\mathbf{y} = \Theta \cdot \mathbf{x}$$

- Replacing $\mathbf{x}_j$
- By $\mathbf{x}_{j''} = \mathbf{x}_j * 10$
- Mathemtically results in $\Theta_{j''} = \Theta_j / 10$

Thus, the scale of the parameter is dependent on the scale of the feature.

Unless two features are on the same scale: we cant directly compare their corresponding parameters.

Categorical features

Consider a categorical feature with categories from

$$C = \{c_1, c_2, \ldots\}$$

One Hot Encoding this feature replaces the original feature with $||C||$ binary features

- Is c_1
- Is c_2
- $\vdots$
- Is $c_{||C||}$

Suppose $\mathbf{x}_j$ corresponds to the binary feature

$$\text{Is } c_1$$

Then, by the formula for dot product

- Θ_j is the *increment* to $\Theta \cdot \mathbf{x}$
- Arising from $\mathbf{x}_j^{(\mathbf{i})} = 1$
- Compared to $x_j^{(\mathbf{i})} = 0$

That is:

- Θ_j is how much $\Theta \cdot \mathbf{x}$ increases
- When example i has feature value c_1 rather than any of $\{c_2, \ldots, \}$

We can use this interpretation

- To further emphasize the problem of
- Treating a categorical variables as a number rather than a collection of binary indicator variables

For example, let's revisit the Passenger Class `Pclass` $\in \{1, 2, 3\}$ from the Titanic example.

- As a collection of binary indicator variables, the increment of being in each class is
$$\Theta_{\text{Is 1}}, \Theta_{\text{Is 2}}, \Theta_{\text{Is 3}}$$
- As a numeric variable with parameter value Θ_j
    - Being in Class 3 has three times the effect as being in Class 1

Thus

- As numeric, we imply a particular magnitude with each category
- As binary indicator,the magnitude is determined by the data

Motivating a transformation

Suppose we have a Linear Regression

$$\mathbf{y} = \Theta \cdot \mathbf{x}$$

Now that we know that Θ_j is the impact on $\mathbf{y}$ of a unit increase in $\mathbf{x}_j$

- Do we want to *transform* either $\mathbf{y}$ or $\mathbf{x}$
- So as to make this increment more plausible ?

For example, consider

- $\mathbf{y}$ and $\mathbf{x}$ are time-series of prices, with different scales
- The impact of $\mathbf{y}$ of a unit change in $\mathbf{x}_j$
    - When $\mathbf{x}_j$ is small
    - Is different than when $\mathbf{x}_j$ is large
    - So
$$\mathbf{y} \neq \Theta \cdot \mathbf{x}$$

Now consider transforming $\mathbf{y}, \mathbf{x}$ to $\mathbf{y}', \mathbf{x}'$

- Where $\mathbf{y}'$
  $$= \% \text{ change in } \mathbf{y}$$
- Where $\mathbf{x}'$
  $$= \% \text{ change in } \mathbf{x}$$

That is, $\mathbf{y}', \mathbf{x}'$ are the daily *returns*

It may be the case that a unit increase

- In the daily returns of $\mathbf{x}_j$
- Has the same impact on the daily return of $\mathbf{y}$
- Independent of the magnitude of $\mathbf{x}_j$

So

$$\mathbf{y}' = \Theta' \cdot \mathbf{x}'$$

Transformed targets

At times we may apply transformations to target values rather than just features.

This means that Θ_j is the sensitivity of the *transformed* target.

Recall that Logistic Regression could be formulated as

- Linear Regression of the features
- Verus the *log odds*

$$\log_e \frac{\hat{p}}{1-\hat{p}} = \mathbf{\Theta}^T \mathbf{x}$$

So a unit change in feature $\mathbf{x}_j$ with parameter Θ_j changes the *odds* $\frac{\hat{p}}{1-\hat{p}}$ in a *multiplicative* way

$$\log(\frac{\hat{p}}{1-\hat{p}}) + \Theta_j = \log\left(\frac{\hat{p}}{1-\hat{p}} * \exp\Theta_j\right)$$

-

Examples

- Log transform of target:
  - $\log \mathbf{y} = \Theta_0 + \Theta_1 * \mathbf{x}_1$
  - $\theta_1 = \frac{\partial \log \mathbf{y}}{\partial \mathbf{x}_1} = \%$ change in $\mathbf{y}$ per unit change in $\mathbf{x}_1$
- Log transform of both target and feature:
  - $\log \mathbf{y} = \Theta_0 + \Theta_1 * \log \mathbf{x}_1$
  - $\Theta_1 = \frac{\partial \log \mathbf{y}}{\partial \log \mathbf{x}_1} = \%$ change in y per $\%$ change in $\mathbf{x}_1$
- Standardize feature
  - Transform $\mathbf{x}$ into $z_\mathbf{x} = \frac{\mathbf{x} - \bar{\mathbf{x}}}{\sigma_\mathbf{x}}$
  - $\mathbf{y} = \Theta_0 + \Theta_1 * z_\mathbf{x}$
  - $\Theta_1 = \frac{\partial \log \mathbf{y}}{\partial z_\mathbf{x}}$ change in $\mathbf{y}$ per 1 standard deviation change in $\mathbf{x}$
    - since z is in units of "number of standard deviations"

Remember

- if you transform features in training, you must apply the same transformation to features in test
    - if the transformation is parameterized, the parameters are determined at **train** fit time, not test !
- if you transform the target, the prediction is in different units than the original
    - you can perform the inverse transformation to get a prediction in original units

Bucketing/Binning re-visited

Suppose $\mathbf{x}_j$ is a continous numeric feature (e.g., `Age`).

Some questions to consider

- Is a 1 year increase in age equally relevant for all ages ?
    - If so: numeric
- Is a 1 year increase in age of the same relevance for a senior adult compared to an infant ?
    - If not: consider reducing discrete ages to discrete buckets
    - Is there a linear relationship between target and the center point of the bucket ?
        - If so: bucket featurecan be numeric
        - If not: bucket feature categorical

Interpreting the MNIST classifier: template matching

The Θ produced by a linear classifier can be viewed as templates

- the strength of Θ_j tells you how strongly feature $\mathbf{x}_j$ influences the target

So we can interpret Θ as a "template" for what a model is looking for.

Let's look at the template for

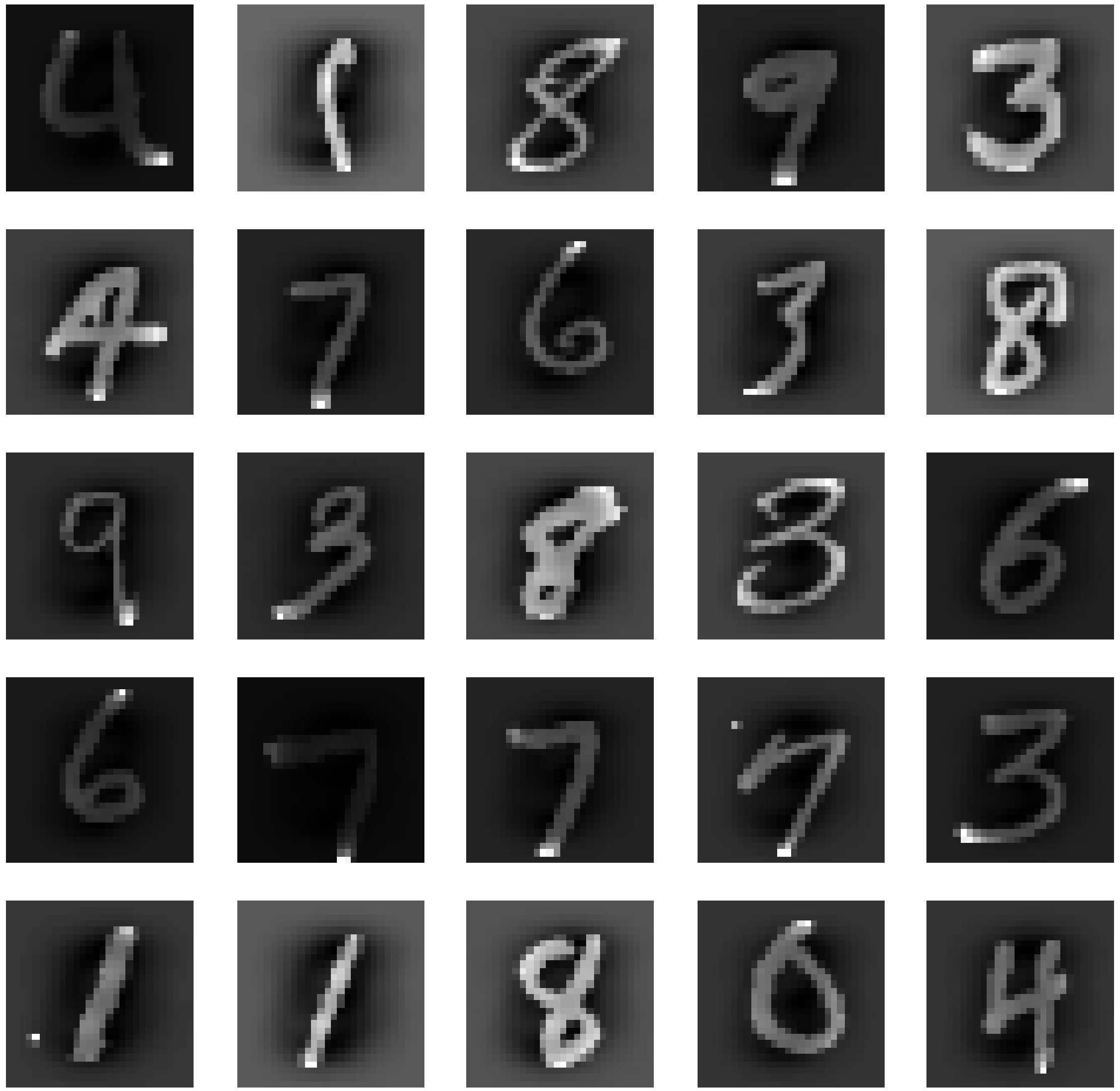
- The 10 separate, single-digit binary MNIST classifiers
- Or similarly: each *row* of Θ for the multinomial 10 class MNIST classifier

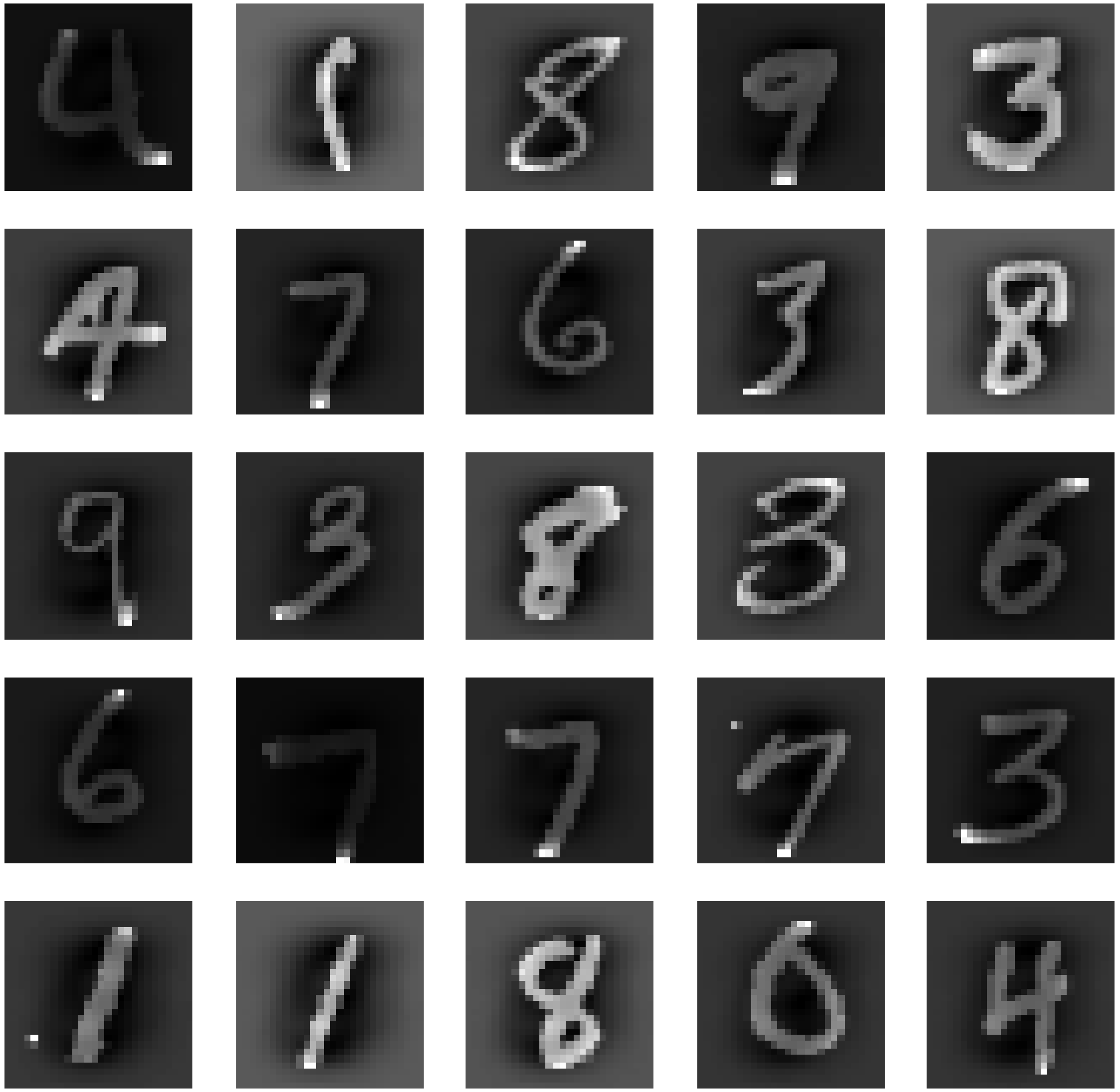
Here's the training data

In [5]:
```
mnh.setup()
mnh.visualize()
```

```
Retrieving MNIST_784 from cache
```

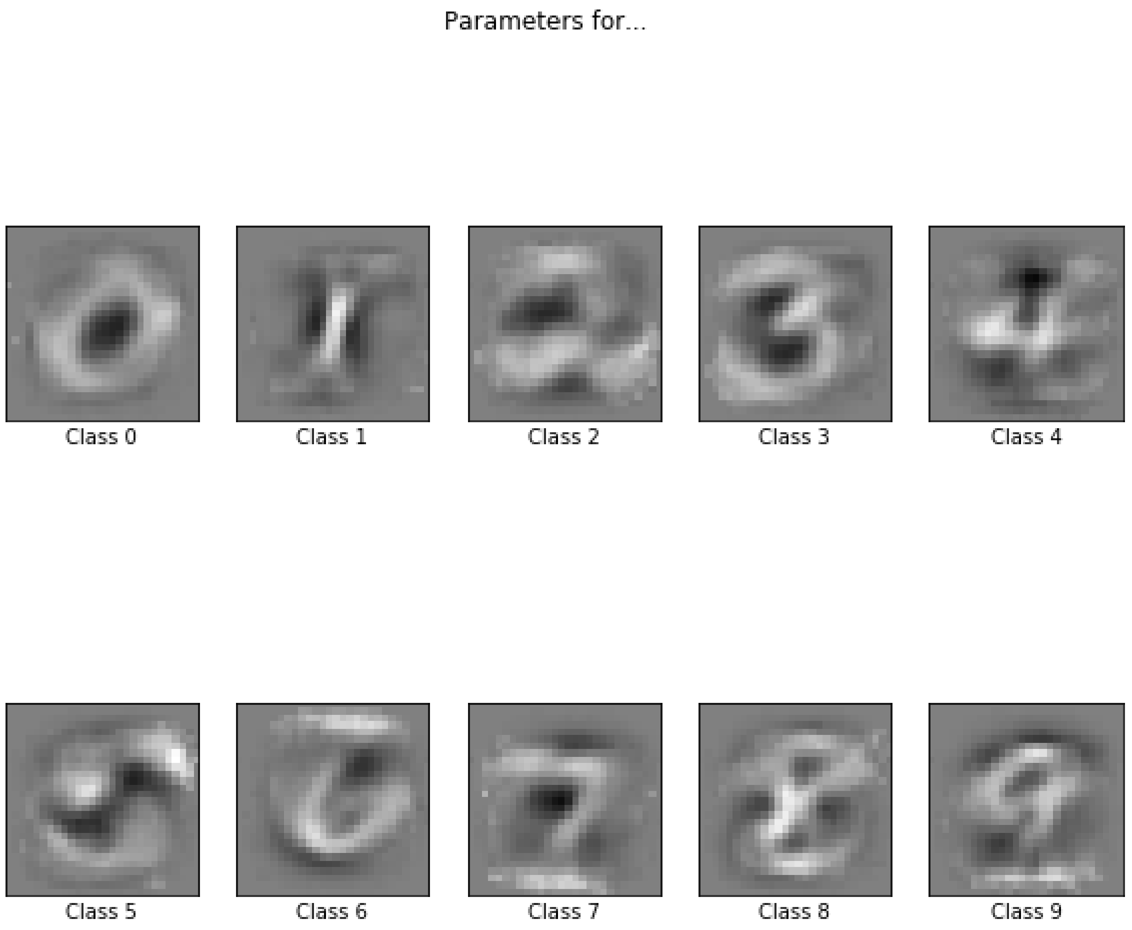
Out[5]:





Let's fit a `LogisticRegression` model and examine the templates (coefficients Θ)

In [9]:
```
_= mnh.fit()
mnist_fig, mnist_ax = mnh.plot_coeff()
```



Recall

- There is one parameter per pixel
- The parameters are ordered in the same way as the linearization of the pixels
    - from (28×28) grid to a vector of 784 numbers.
- We can display the 784 parameters in a (28×28) image to show the intensity of parameter associated with a pixel
- White is high parameter value; Black is low (or negative)

- The template for 0 emphasizes small values (absence of bright pixels) in the center of the image
- The template for 1 emphasizes bright vertical pixels
- The template for 8 emphasizes the absence of bright pixels
    - in the two circles
    - in the pinched waist

You can now imagine how these templates might lead to misclassification

What is the classification of

- a "7" with a strong vertical line in the center (that's what the "1" template tries to match)
- a thin "0" (the "0" template is looking for a large donut)

So interpretation is a very powerful diagnostic tool for both understanding and improving your models.

In [7]:
```
print("Done")
```
```
Done
```