In[ ]:=
```
f[x_, y_] := Sin[x^2 - y^2];
f[x, y] /. {x → 0, y → Sqrt[π / 4]}
```

Out[ ]= $-\frac{1}{\sqrt{2}}$

In[ ]:=
```
Clear[f, x, y];
f = Sin[x^2 - y^2];
f /. {x → 0, y → Sqrt[π / 4]}
```

Out[ ]= $-\frac{1}{\sqrt{2}}$

In[ ]:=
```
f /. {x → 1 - π, y → 1 + π}
```

Out[ ]= $\mathrm{Sin}\left[(1-\pi)^2 - (1+\pi)^2\right]$

In[ ]:=
```
N[Sin[(1 - π)^2 - (1 + π)^2]]
```

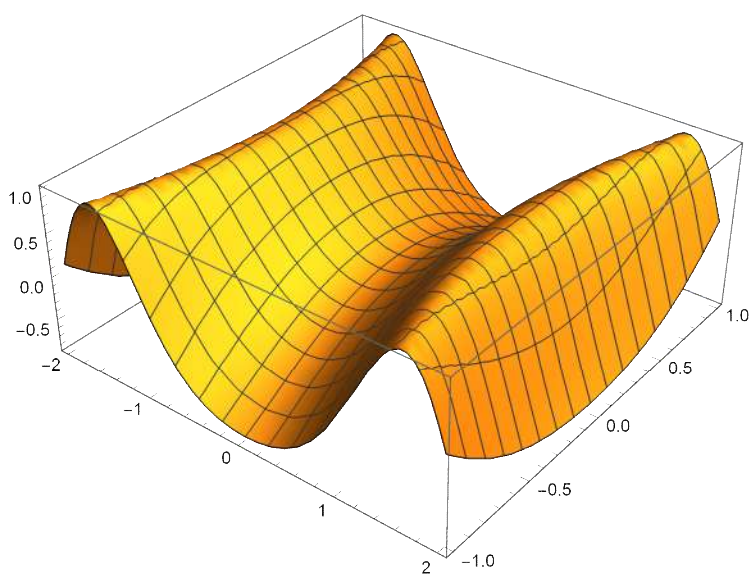
Out[ ]= 4.89859×10^{-16}

In[ ]:=
```
Simplify[%]
```

Out[ ]= 4.89859×10^{-16}

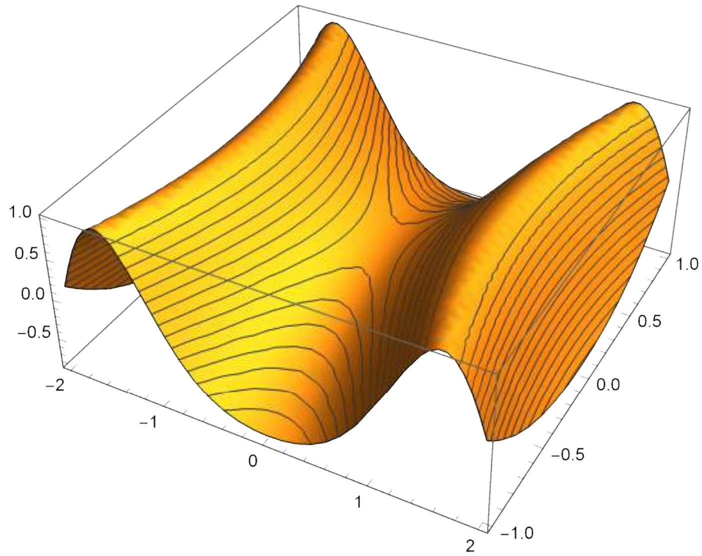
In[ ]:=
```
Clear[f, x, y, z];
f = Sin[x^2 - y^2];
Plot3D[f, {x, -2, 2}, {y, -1, 1}]
```

Out[ ]=



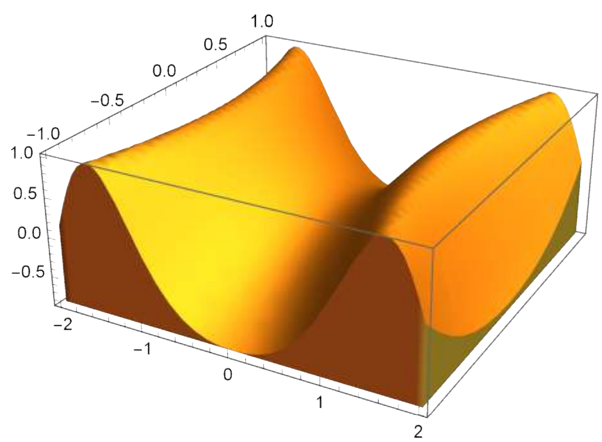
In[ ]:=
```
Clear[f, x, y, z];
f = Sin[x^2 - y^2];
Plot3D[f, {x, -2, 2}, {y, -1, 1}, PlotTheme → "ZMesh"]
```

Out[ ]=

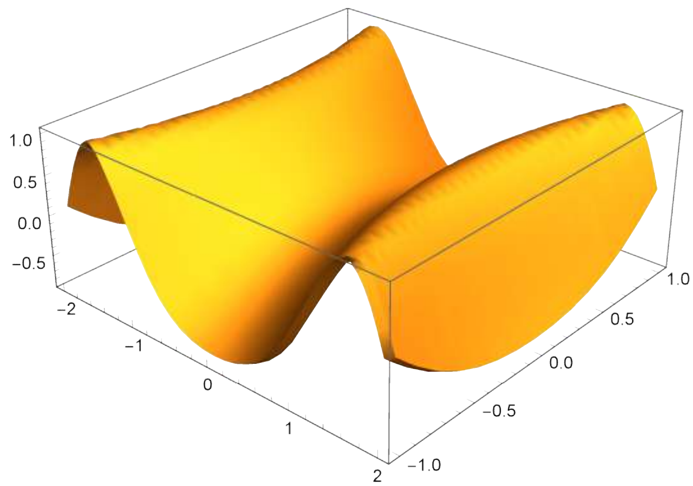


In[ ]:=
```
Plot3D[f, {x, -2, 2}, {y, -1, 1}, PlotTheme → "FilledSurface"]
```

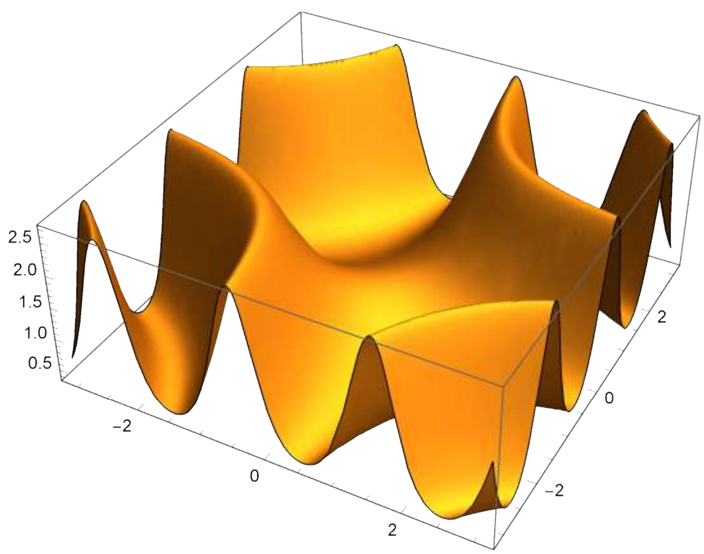
Out[ ]=



In[ ]:= `Plot3D[f, {x, -2, 2}, {y, -1, 1}, PlotTheme → "ThickSurface"]`

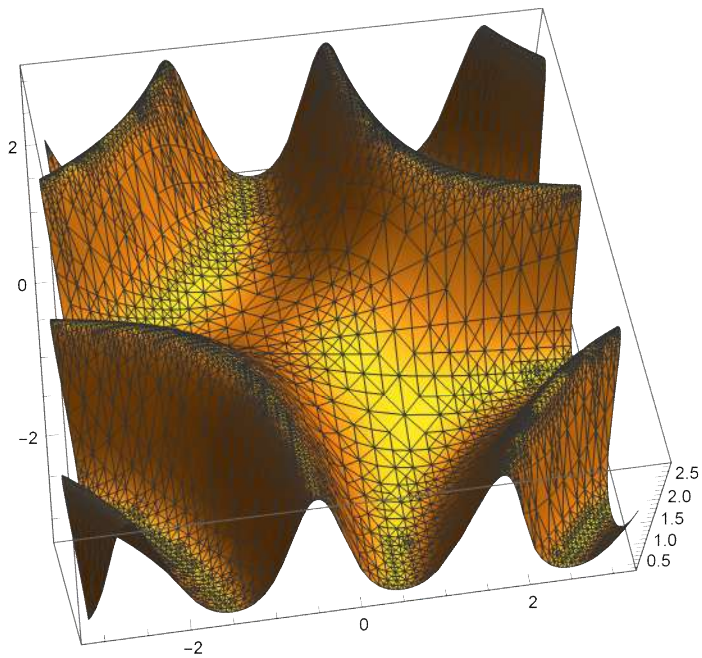
Out[ ]=


In[ ]:= `Plot3D[e^Sin[x * y], {x, -π, π}, {y, -π, π}, Mesh → None, MaxRecursion → 4]`

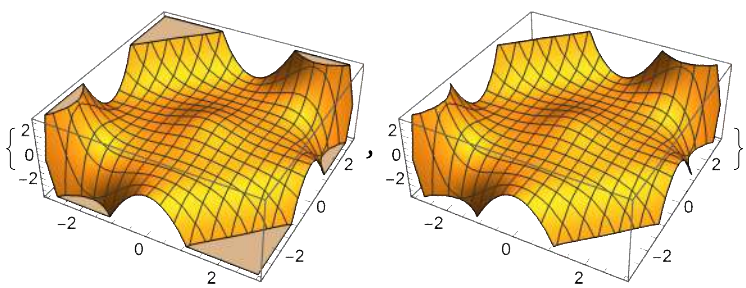
Out[ ]=


In[ ]:=
```
Plot3D[ⅇ^Sin[x * y], {x, -π, π}, {y, -π, π}, Mesh → All, MaxRecursion → 4]
```

Out[ ]=

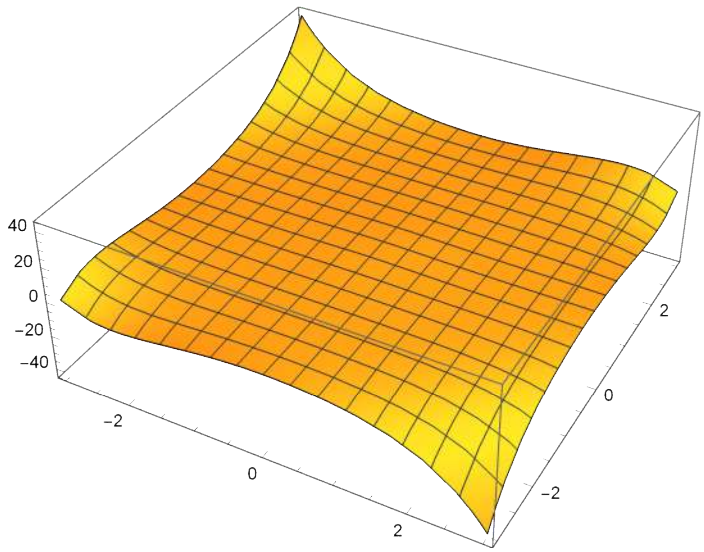


In[ ]:=
```
Table[Plot3D[((x^2 * y^5) - (x^5 * y^2)) / 100 + ⅇ^-(x^2 + y^2),
  {x, -3, 3}, {y, -3, 3}, ClippingStyle → k], {k, {Automatic, None}}]
```

Out[ ]= {  }

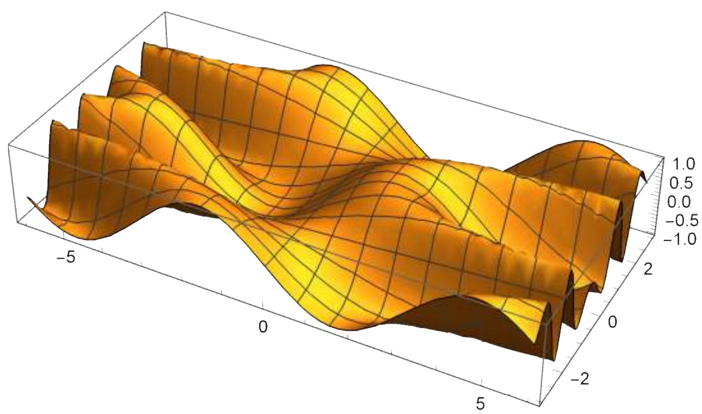
In[ ]:=
```
Plot3D[((x^2 * y^5) - (x^5 * y^2)) / 100 + ⅇ^-(x^2 + y^2),
  {x, -3, 3}, {y, -3, 3}, PlotRange → All]
```

Out[ ]=



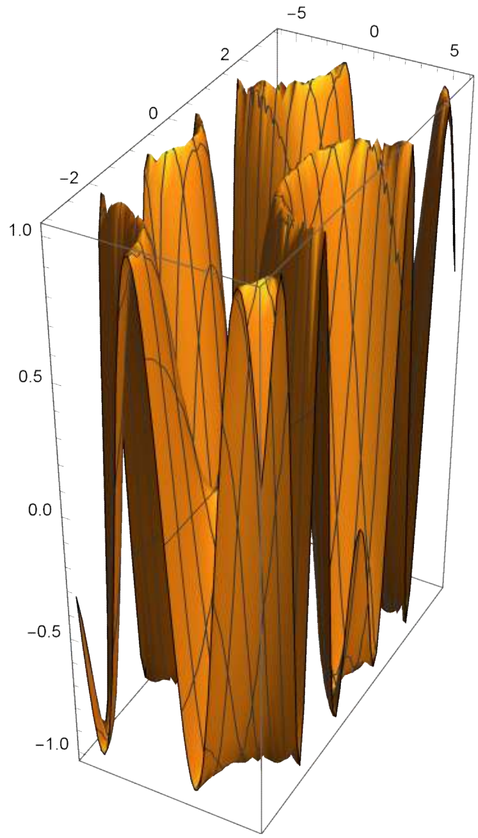
In[ ]:=
```
Plot3D[Sin[x * Cos[y]], {x, -6, 6}, {y, -3, 3}, BoxRatios → Automatic]
```

Out[ ]=



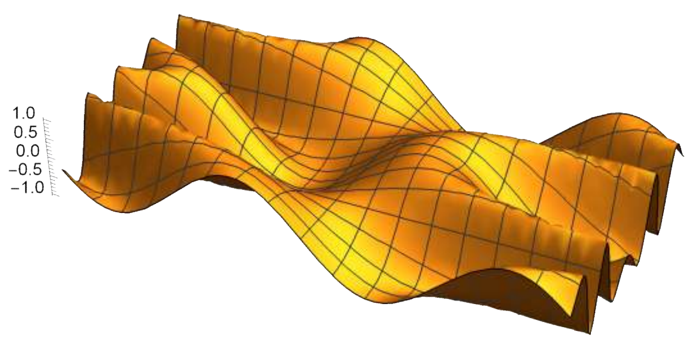
In[ ]:= `Plot3D[Sin[x * Cos[y]], {x, -6, 6}, {y, -3, 3}, BoxRatios → {1, 2, 3}]`

Out[ ]=



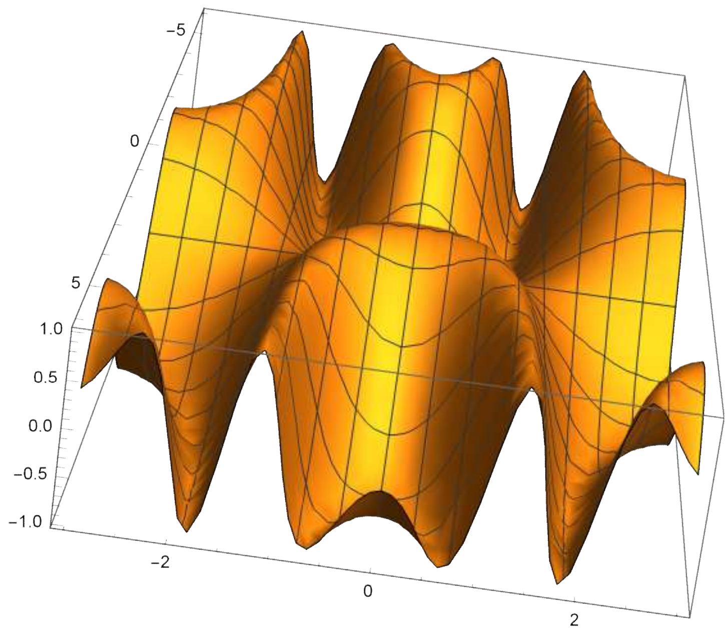
In[ ]:= `Plot3D[Sin[x * Cos[y]], {x, -6, 6}, {y, -3, 3}, BoxRatios → Automatic, Boxed → False,`
`Axes → {False, False, True}, AxesEdge → {Automatic, Automatic, {-1, -1}}]`

Out[ ]=



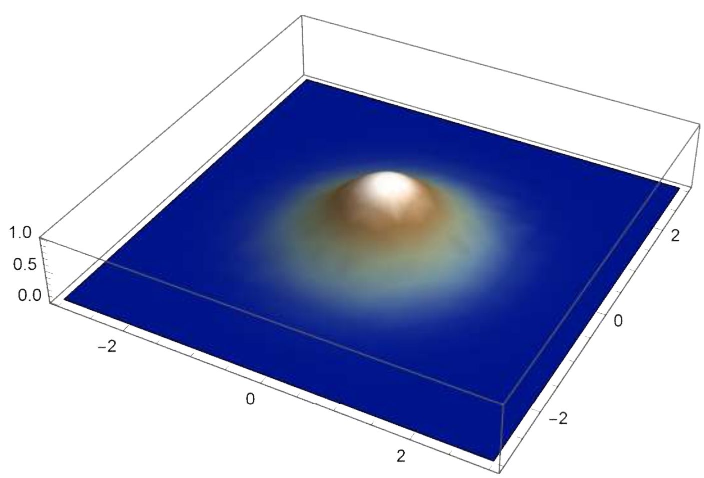
In[ ]:= `Plot3D[Sin[x * Cos[y]], {x, -6, 6}, {y, -3, 3}, BoxRatios → Automatic, ViewPoint → {3, 0, 1}]`

Out[ ]=



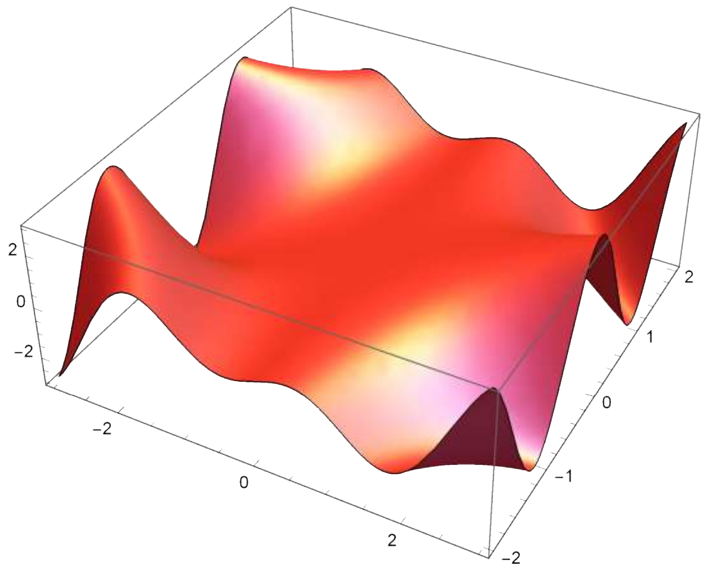
In[ ]:= `Plot3D[e^-(x^2 + y^2), {x, -3, 3}, {y, -3, 3}, BoxRatios → Automatic, ColorFunction → "DarkTerrain", Mesh → None, PlotRange → All]`

Out[ ]=



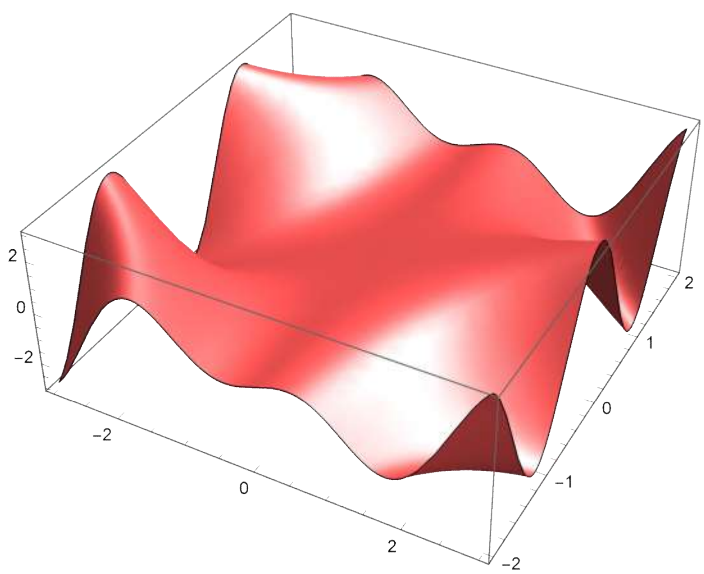
In[ ]:= Plot3D[x * Cos[x * y], {x, -3, 3}, {y, -2, 2}, Mesh → None,
MaxRecursion → 4, PlotStyle → Directive[Lighter[Red], Specularity[White, 20]]]

Out[ ]=



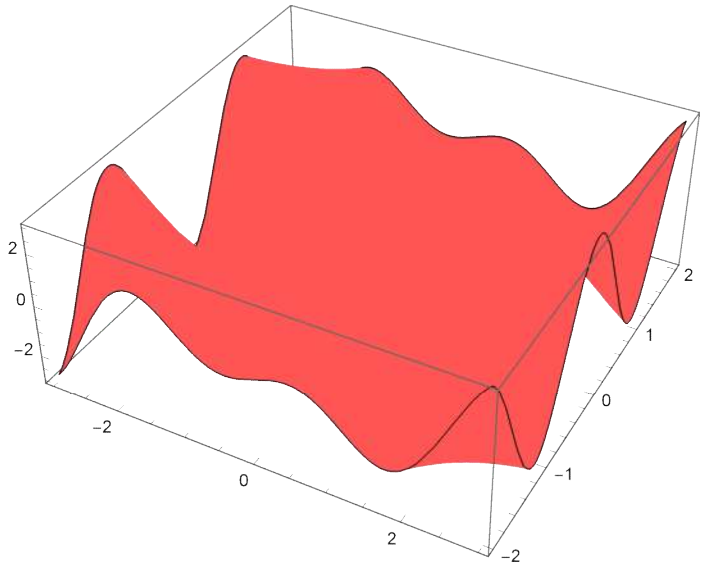
In[ ]:= Plot3D[x * Cos[x * y], {x, -3, 3}, {y, -2, 2}, Mesh → None, MaxRecursion → 4,
PlotStyle → Directive[Lighter[Red], Specularity[White, 20]], Lighting → "Neutral"]

Out[ ]=



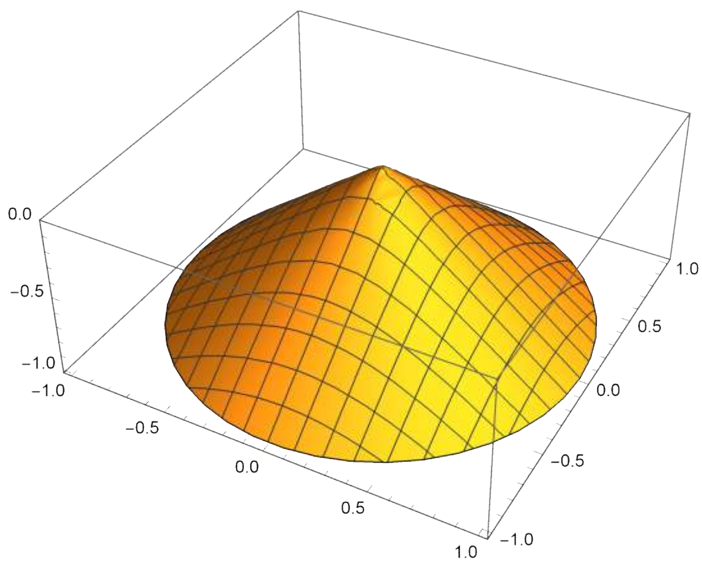
```
In[]:= Plot3D[x * Cos[x * y], {x, -3, 3}, {y, -2, 2}, Mesh → None,
        MaxRecursion → 4, PlotStyle → Directive[Lighter[Red], Specularity[White, 20]],
        Lighting → {{"Ambient", White}}]
```

Out[]=



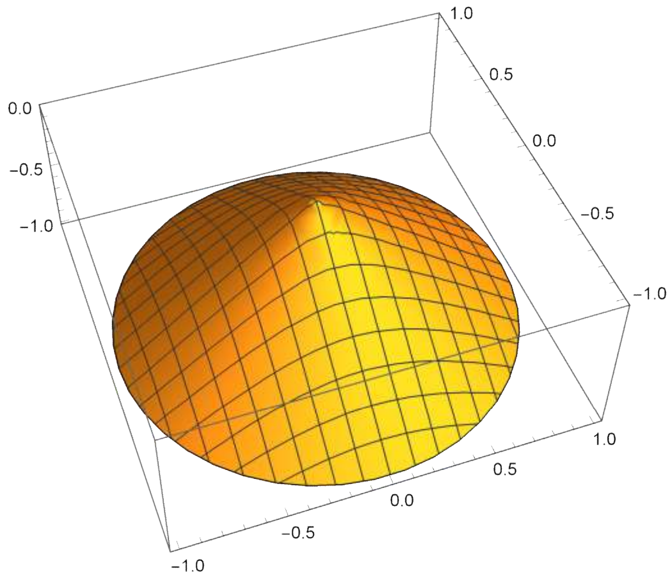

```
In[]:= Plot3D[-Sqrt[x^2 + y^2], {x, y} ∈ Disk[]]
```

Out[]=

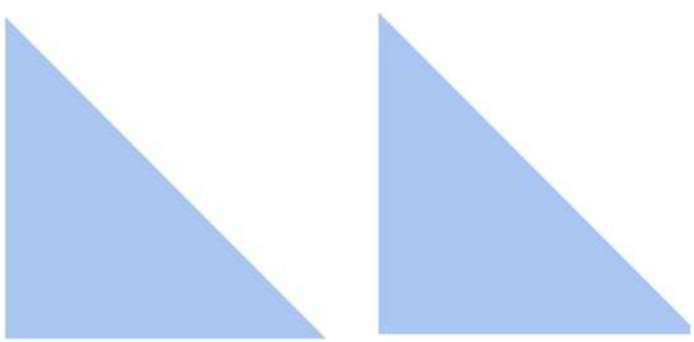


In[ ]:=
```
Plot3D[-Sqrt[x^2 + y^2], {x, y} ∈ Disk[{0, 0}, 1]]
```

Out[ ]=

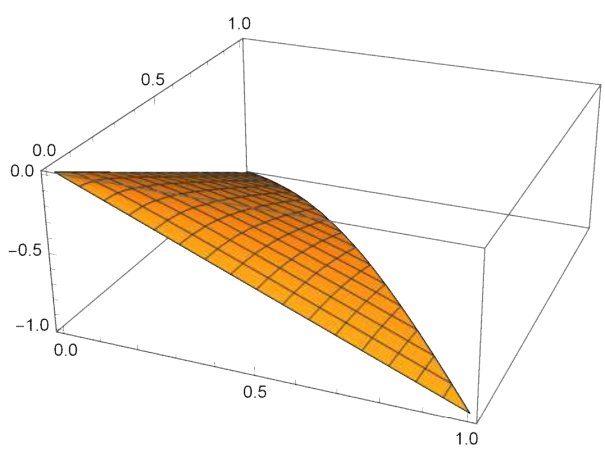


In[ ]:=
```
𝒯 = Triangle[{{0, 0}, {1, 0}, {0, 1}}];
ℐ = ImplicitRegion[x > 0 && y > 0 && 0 < y < 1 - x, {x, y}];
GraphicsRow[{Region[𝒯], Region[ℐ]}]
```



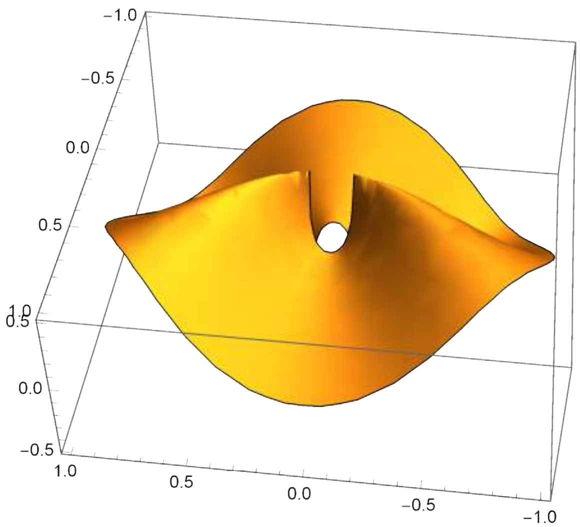
In[ ]:=
```
Plot3D[-Sqrt[x^2 + y^2], {x, y} ∈ 𝒯]
```

Out[ ]=



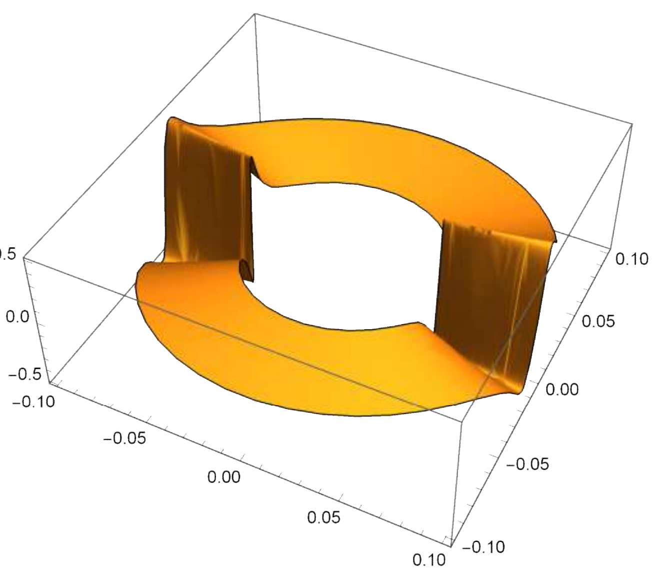
In[ ]:=
```
Clear[𝒯, ℐ]
Plot3D[(x^2 * y) / (x^4 + y^2),
 {x, y} ∈ Annulus[{0, 0}, {.1, 1}], Mesh → None, MaxRecursion → 4]
```

Out[ ]=



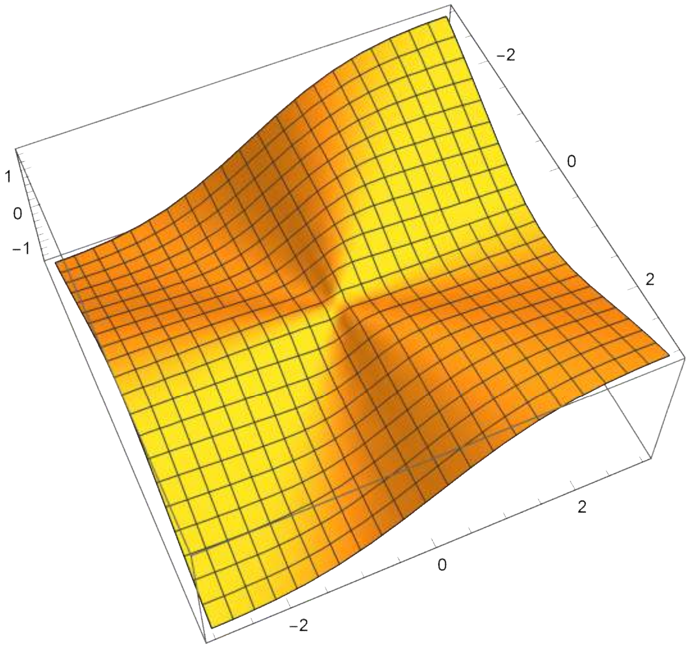
In[ ]:=
```
Plot3D[(x^2 * y) / (x^4 + y^2), {x, y} ∈ Annulus[{0, 0}, .1], Mesh → None, MaxRecursion → 4]
```

Out[ ]=



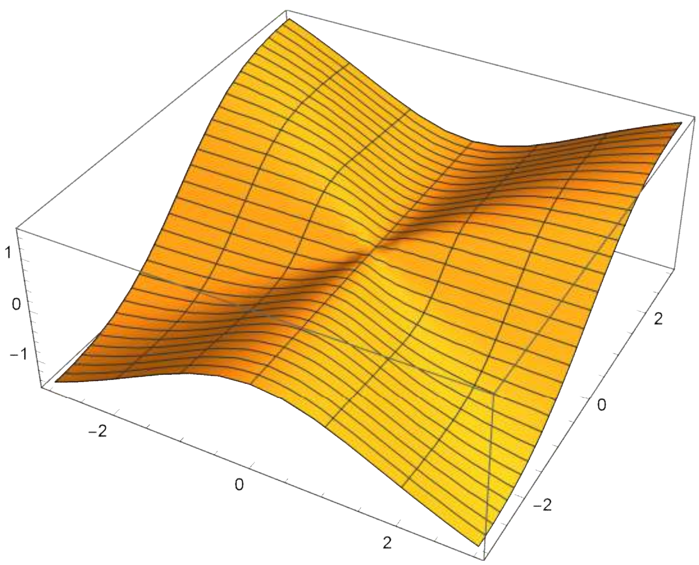
In[]:= `Plot3D[(x^2 * y) / (x^2 + y^2), {x, -3, 3}, {y, -3, 3}, Mesh → 20]`

Out[]=

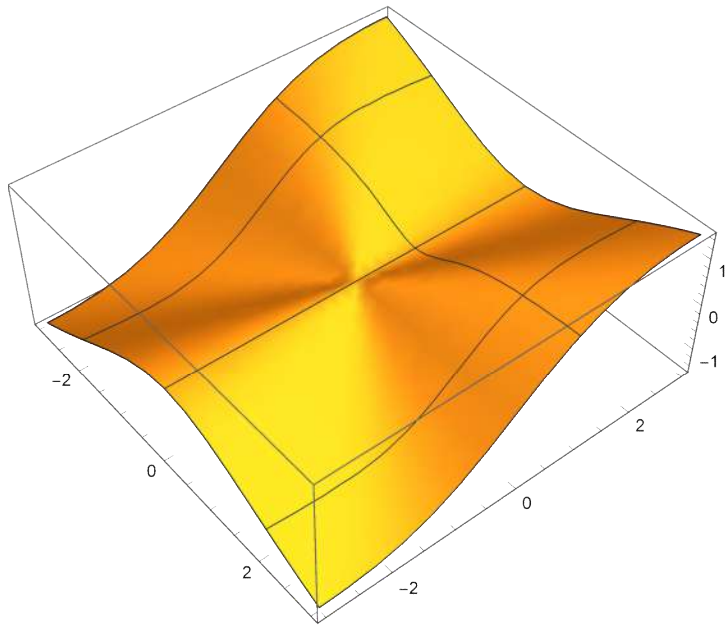


In[]:= `Plot3D[(x^2 * y) / (x^2 + y^2), {x, -3, 3}, {y, -3, 3}, Mesh → {5, 30}]`

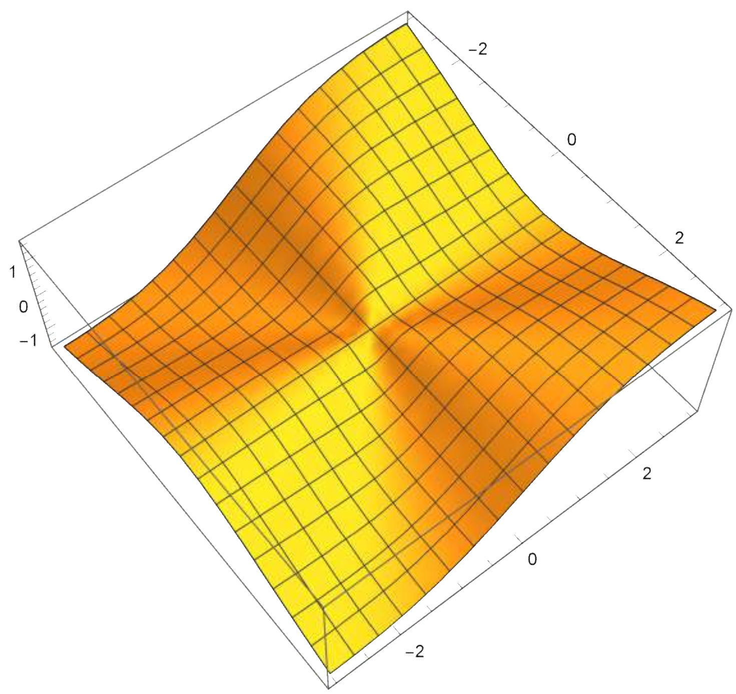
Out[]=



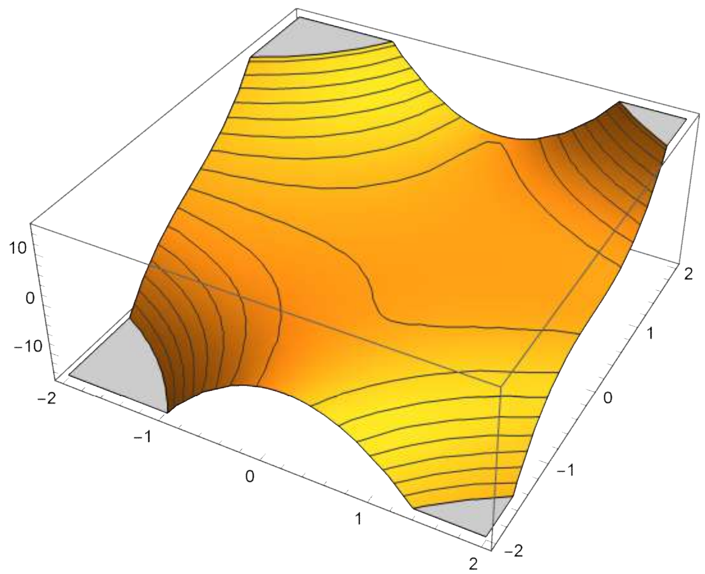
In[ ]:= `Plot3D[(x^2 * y) / (x^2 + y^2), {x, -3, 3}, {y, -3, 3}, Mesh → {{-2, 0, 2}, {1}}]`

Out[ ]=


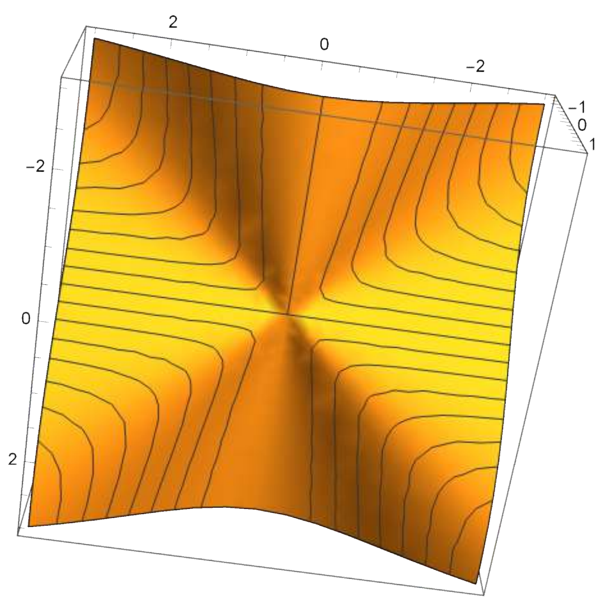
In[ ]:= `Plot3D[(x^2 * y) / (x^2 + y^2), {x, -3, 3}, {y, -3, 3}, MeshFunctions → {#1 &, #2 &}]`

Out[ ]=


In[ ]:= **Plot3D[x^2 * y^3 + (x - 1)^2 * y, {x, -2, 2}, {y, -2, 2}, MeshFunctions → {#3 &}]**

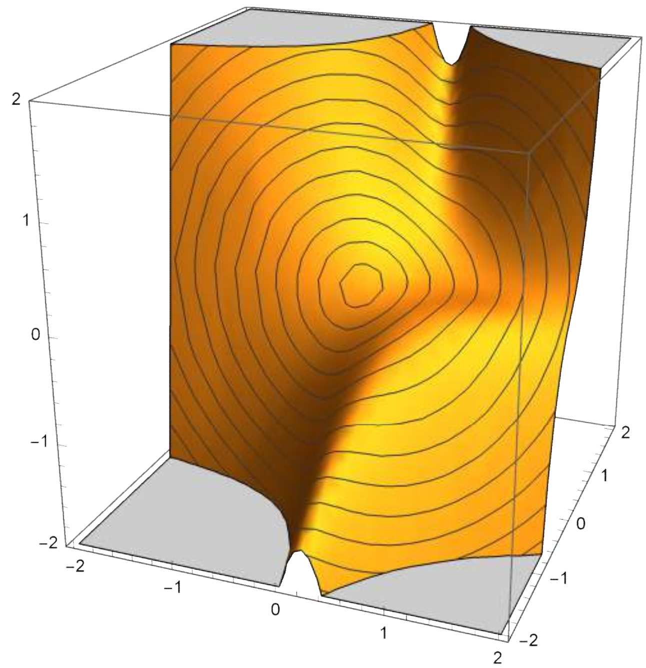
Out[ ]=


In[ ]:= **Plot3D[(x^2 * y) / (x^2 + y^2), {x, -3, 3}, {y, -3, 3}, MeshFunctions → {#3 &}]**

Out[ ]=


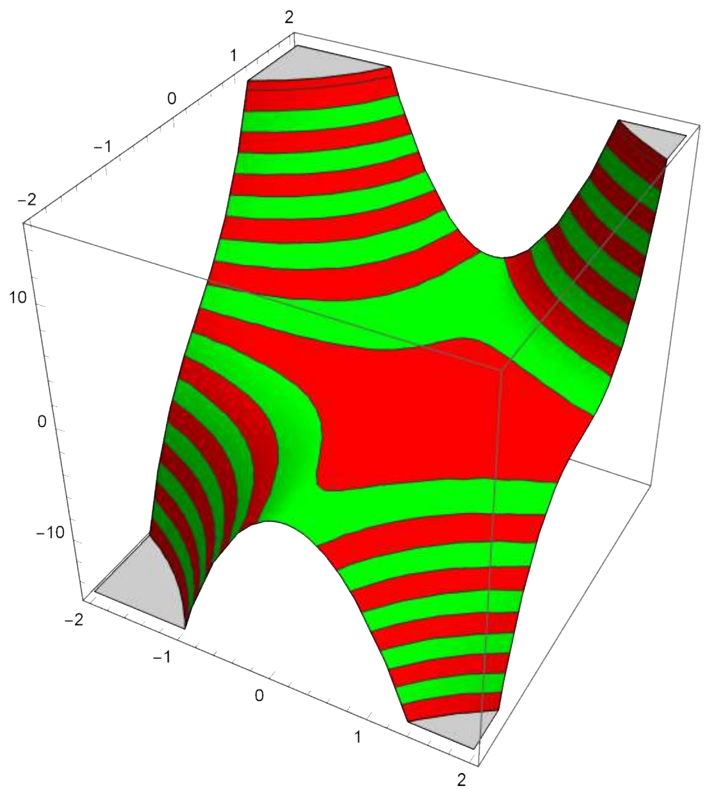
In[ ]:=
```
Plot3D[x^2 * y^3 + (x - 1)^2 * y, {x, -2, 2}, {y, -2, 2},
  MeshFunctions → {Norm[{#1, #2, #3}] &}, PlotRange → 2, BoxRatios → 1]
```

Out[ ]=



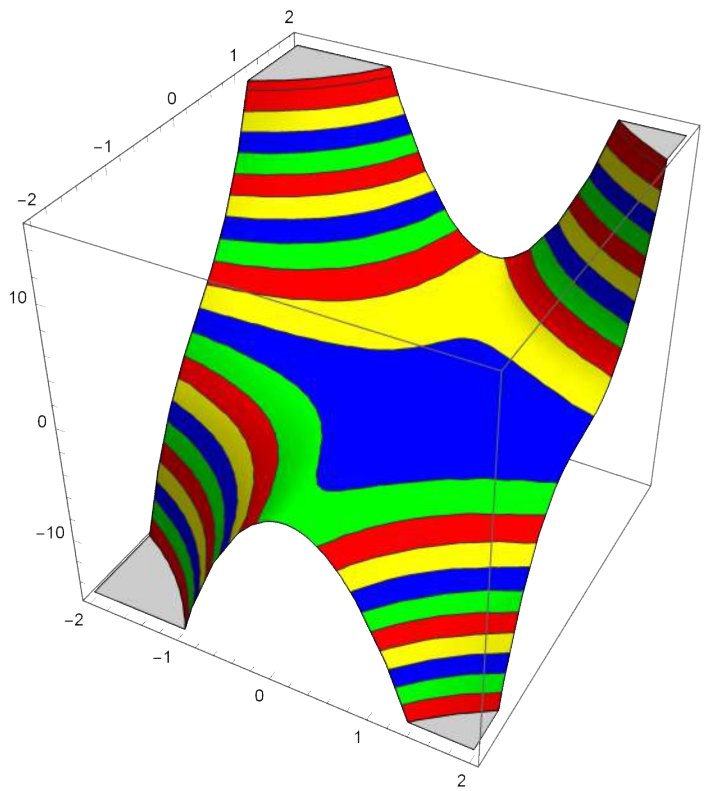
In[ ]:=
```
Plot3D[x^2 * y^3 + (x - 1)^2 * y, {x, -2, 2}, {y, -2, 2}, MeshFunctions → {#3 &},
  Mesh → 20, BoxRatios → 1, MeshShading → {Red, Green}, Lighting → "Neutral"]
```

Out[ ]=



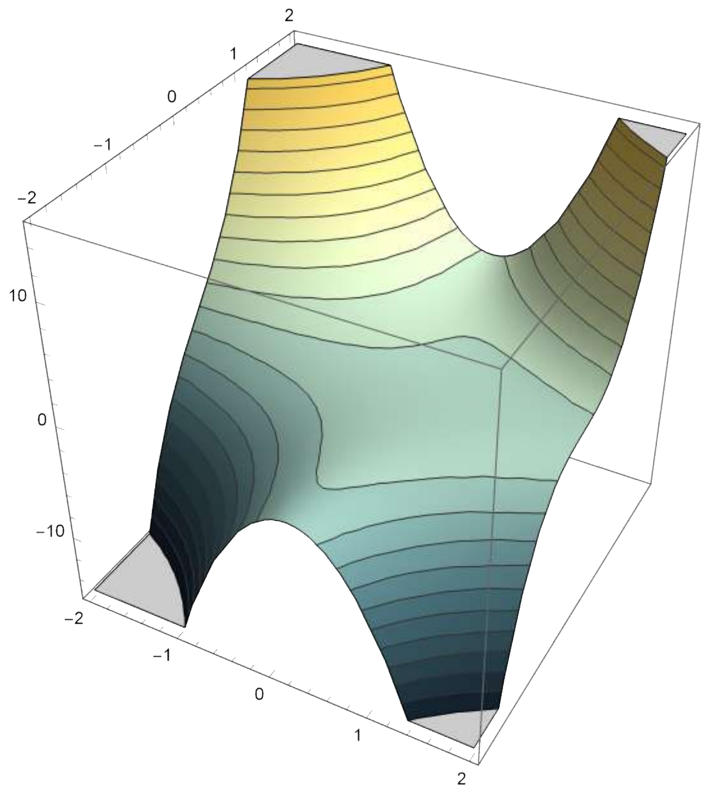
In[ ]:=
```
Plot3D[x^2 * y^3 + (x - 1) ^2 * y, {x, -2, 2}, {y, -2, 2}, MeshFunctions → {#3 &}, Mesh → 20,
  BoxRatios → 1, MeshShading → {Red, Green, Blue, Yellow}, Lighting → "Neutral"]
```

Out[ ]=



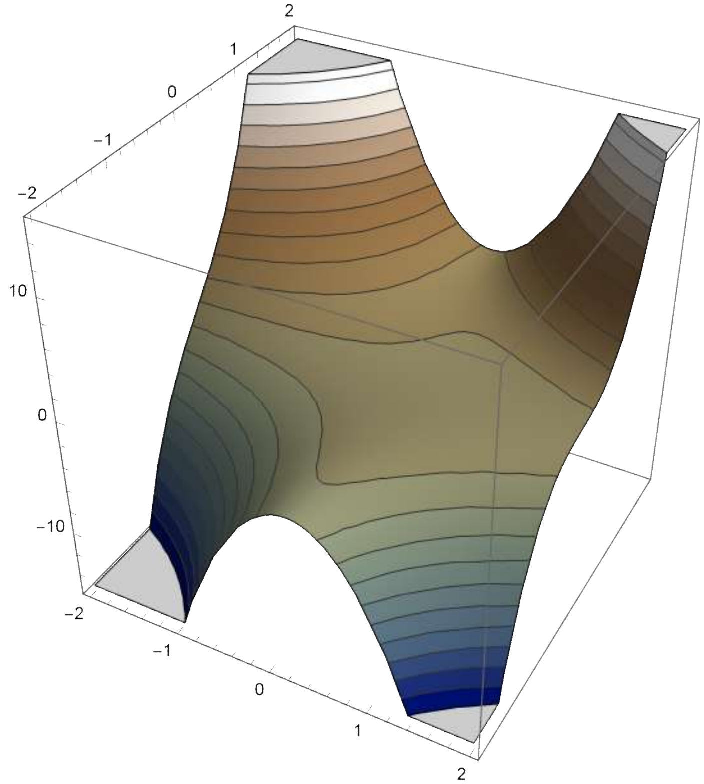
In[ ]:=
```
Plot3D[x^2 * y^3 + (x - 1) ^2 * y, {x, -2, 2}, {y, -2, 2},
  MeshFunctions → {#3 &}, Mesh → 20, BoxRatios → 1, MeshShading →
   Table[ColorData["StarryNightColors"][t], {t, 0, 1, 1 / 20}], Lighting → "Neutral"]
```

Out[ ]=



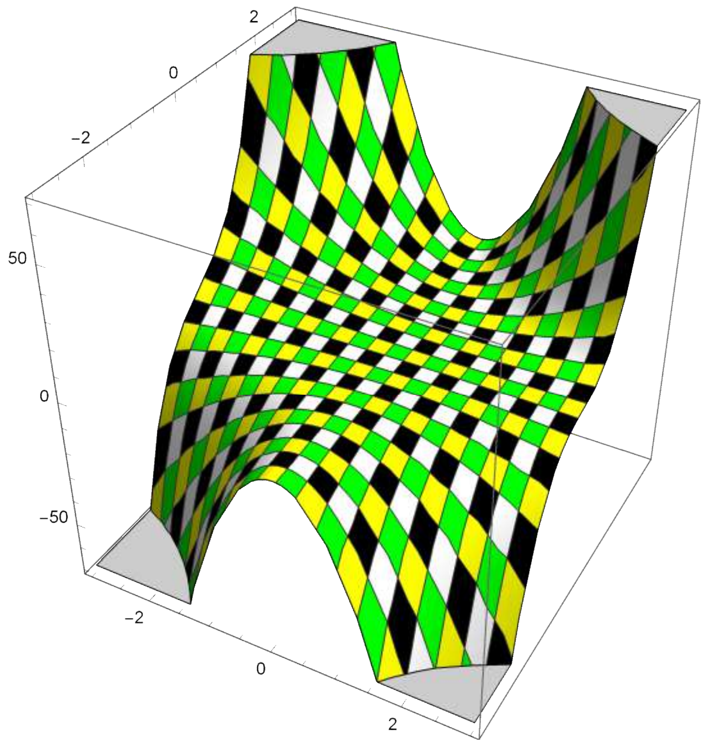
In[ ]:=
```
Plot3D[x^2 * y^3 + (x - 1) ^2 * y, {x, -2, 2},
  {y, -2, 2}, MeshFunctions → {#3 &}, Mesh → 20, BoxRatios → 1,
  MeshShading → Table[ColorData["DarkTerrain"][t], {t, 0, 1, 1 / 20}], Lighting → "Neutral"]
```

Out[ ]=



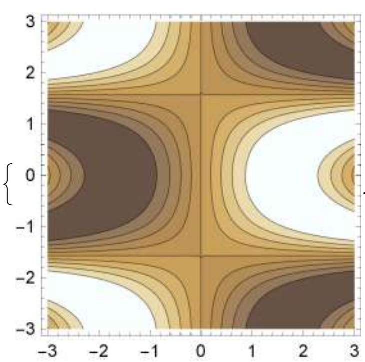
In[ ]:=
```
Plot3D[x^2 * y^3 + (x - 1) ^2 * y, {x, -3, 3}, {y, -3, 3}, Mesh → 20, BoxRatios → 1,
  MeshShading → {{Yellow, Green}, {Black, White}}, Lighting → "Neutral"]
```

Out[ ]=

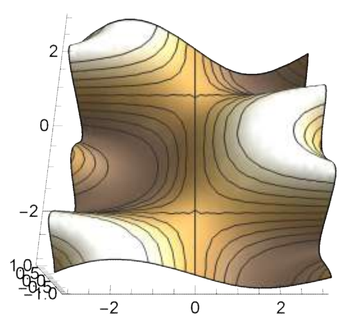


In[ ]:=
```
{ContourPlot[Sin[x * Cos[y]], {x, -3, 3},
   {y, -3, 3}, Contours → 9, ColorFunction → "CoffeeTones"],
  Plot3D[Sin[x * Cos[y]], {x, -3, 3}, {y, -3, 3}, MeshFunctions → {#3 &}, Mesh → 9,
   ColorFunction → "CoffeeTones", ViewPoint → {0, -1, 2}, Boxed → False]}
```

Out[ ]= {



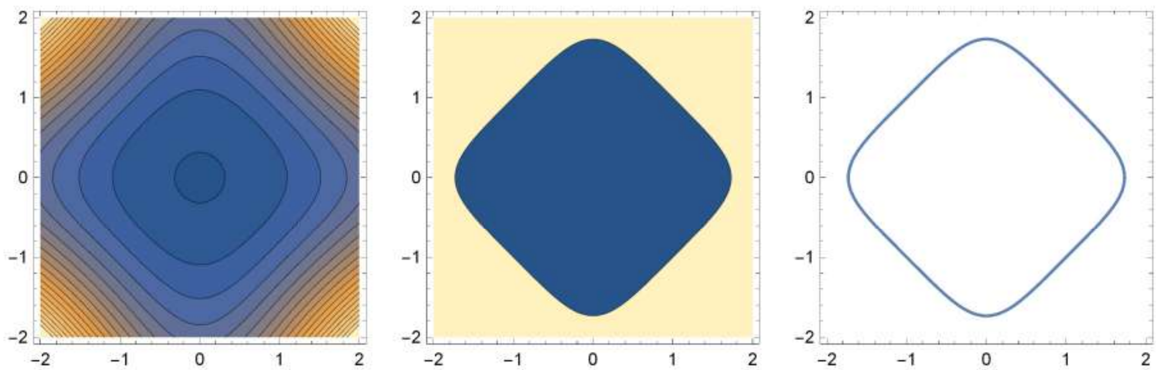
,



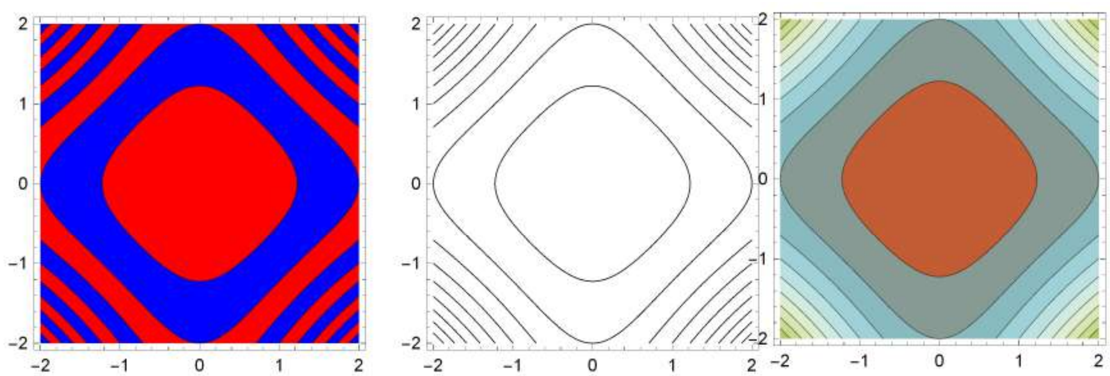
}

In[ ]:=
```
GraphicsRow[{ContourPlot[(1 + x^2) * (1 + y^2), {x, -2, 2}, {y, -2, 2}, Contours → 20],
  ContourPlot[(1 + x^2) * (1 + y^2), {x, -2, 2}, {y, -2, 2}, Contours → {4}],
  ContourPlot[(1 + x^2) * (1 + y^2) == 4, {x, -2, 2}, {y, -2, 2}]}]
```

Out[ ]=

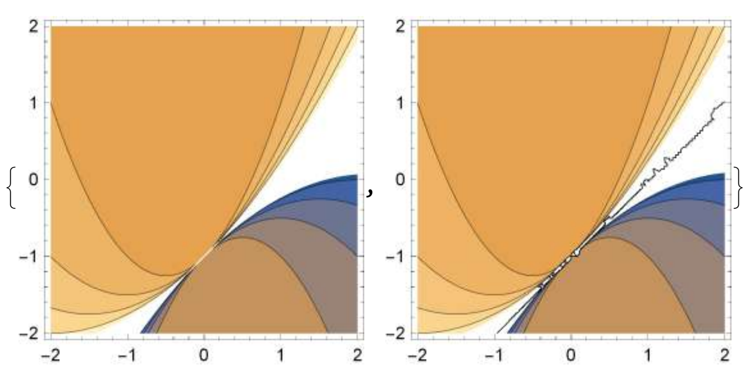


In[ ]:=
```
GraphicsRow[
 {ContourPlot[(1 + x^2) * (1 + y^2), {x, -2, 2}, {y, -2, 2}, ContourShading → {Red, Blue}],
  ContourPlot[(1 + x^2) * (1 + y^2), {x, -2, 2}, {y, -2, 2}, ContourShading → None],
  ContourPlot[(1 + x^2) * (1 + y^2), {x, -2, 2},
   {y, -2, 2}, ColorFunction → "IslandColors"]}]
```



In[ ]:=
```
{ContourPlot[x^2 / (1 - x + y), {x, -2, 2}, {y, -2, 2}, PlotPoints → 60], ContourPlot[
  x^2 / (1 - x + y), {x, -2, 2}, {y, -2, 2}, Exclusions → "Discontinuities"]} // Quiet
```

Out[ ]=



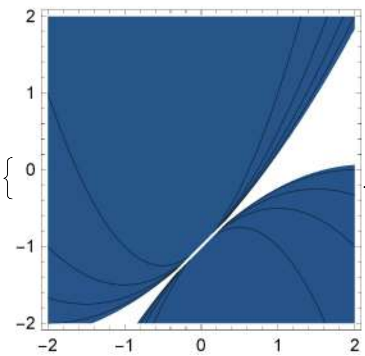
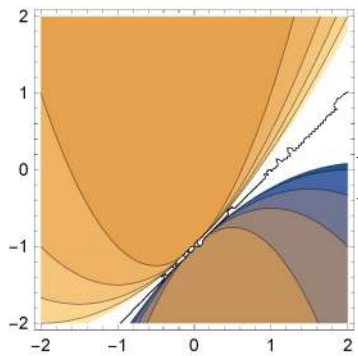
In[ ]:=
```
{ContourPlot[x^2 / (1 - x + y), {x, -2, 2}, {y, -2, 2}],
 ContourPlot[x^2 / (1 - x + y), {x, -2, 2}, {y, -2, 2}, Exclusions → "Discontinuities"]}
```

Power: Infinite expression $\frac{1}{0.}$ encountered.

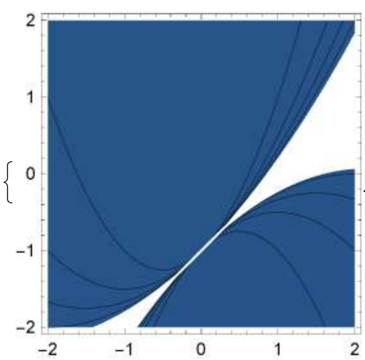
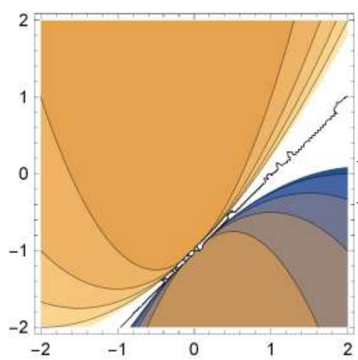
Power: Infinite expression $\frac{1}{0.}$ encountered.

Power: Infinite expression $\frac{1}{0.}$ encountered.

General: Further output of Power::infy will be suppressed during this calculation.

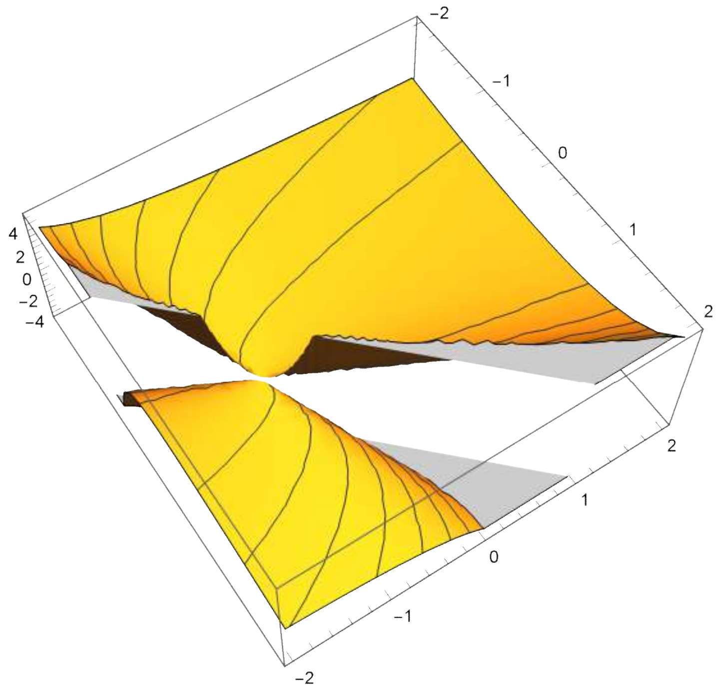
Out[ ]= { ,  }

In[ ]:=
```
{ContourPlot[x^2 / (1 - x + y), {x, -2, 2}, {y, -2, 2}], ContourPlot[
   x^2 / (1 - x + y), {x, -2, 2}, {y, -2, 2}, Exclusions → "Discontinuities"]} // Quiet
```

Out[ ]= { ,  }

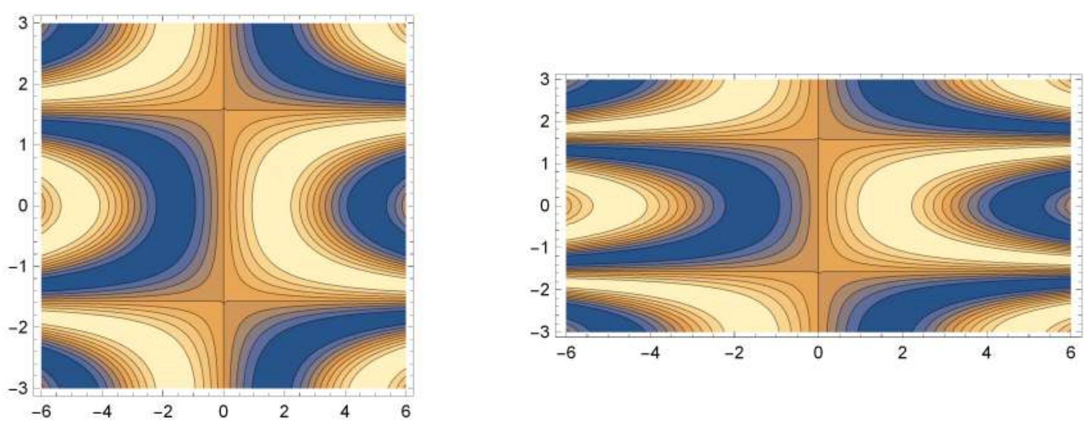
In[ ]:= `Plot3D[x^2 / (1 - x + y), {x, -2, 2}, {y, -2, 2}, PlotTheme → "ZMesh"]`

Out[ ]=

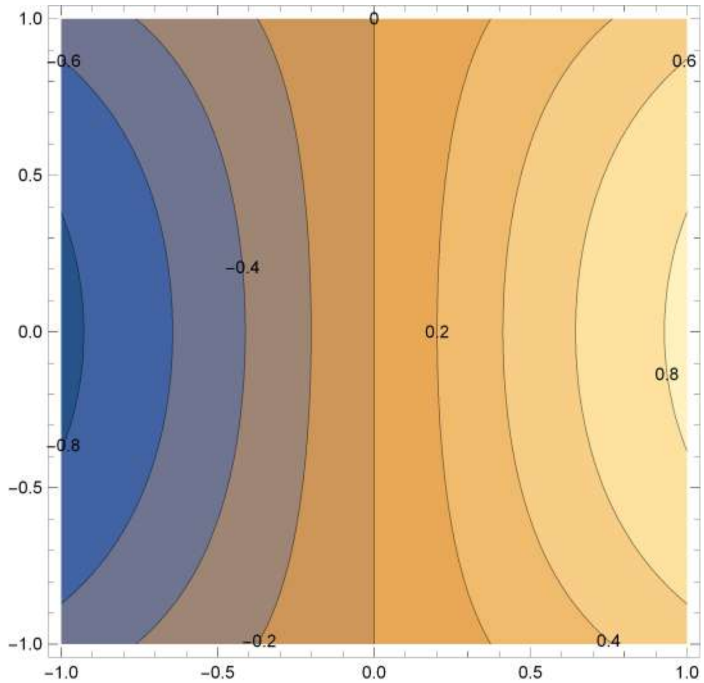


In[ ]:= `GraphicsRow[{ContourPlot[Sin[x * Cos[y]], {x, -6, 6}, {y, -3, 3}],`
`ContourPlot[Sin[x * Cos[y]], {x, -6, 6}, {y, -3, 3}, AspectRatio → Automatic]}]`

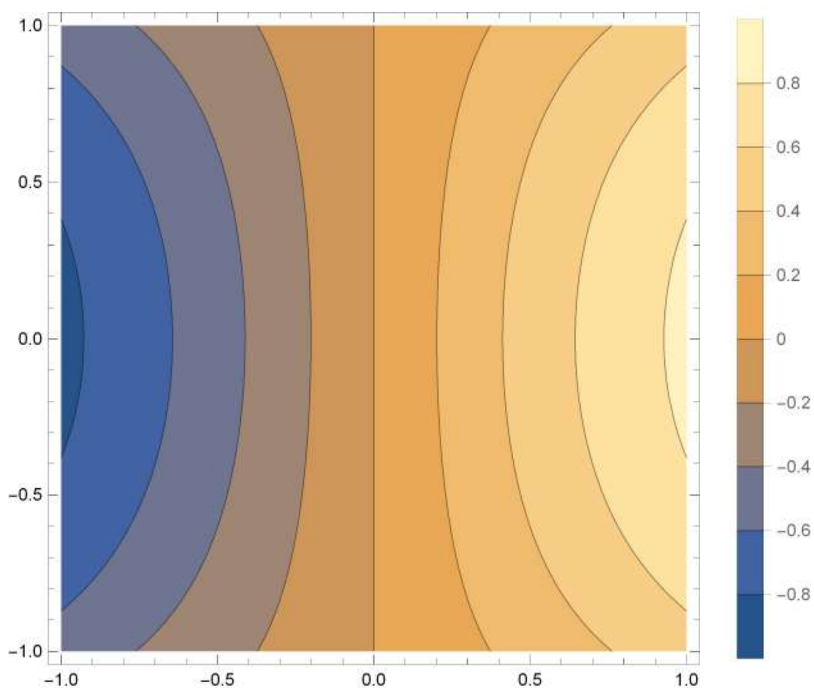
Out[ ]=



In[ ]:= **ContourPlot[Sin[x * Cos[y]], {x, -1, 1}, {y, -1, 1}, ContourLabels → All]**

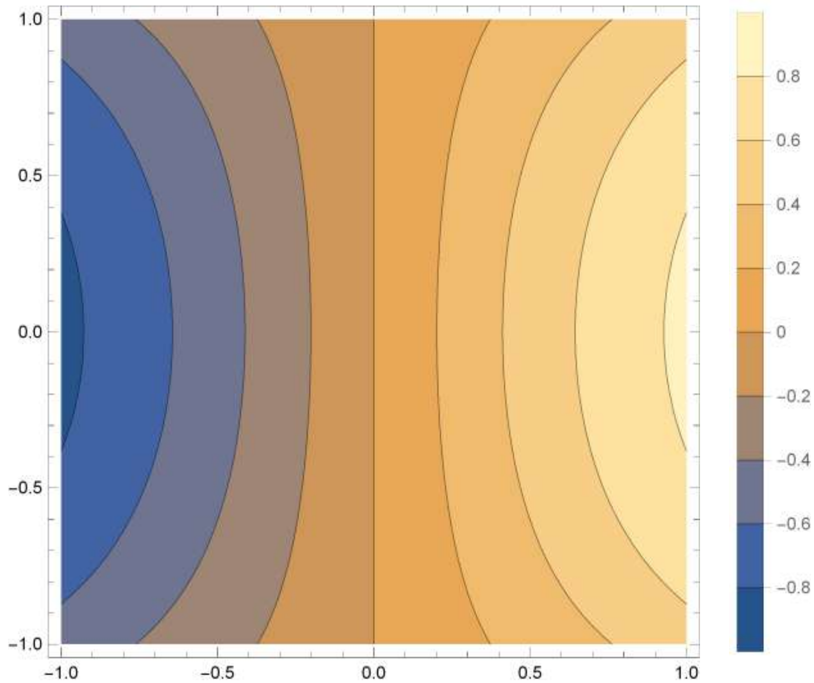
Out[ ]=


In[ ]:= **ContourPlot[Sin[x * Cos[y]], {x, -1, 1}, {y, -1, 1}, PlotLegends → All]**

Out[ ]=


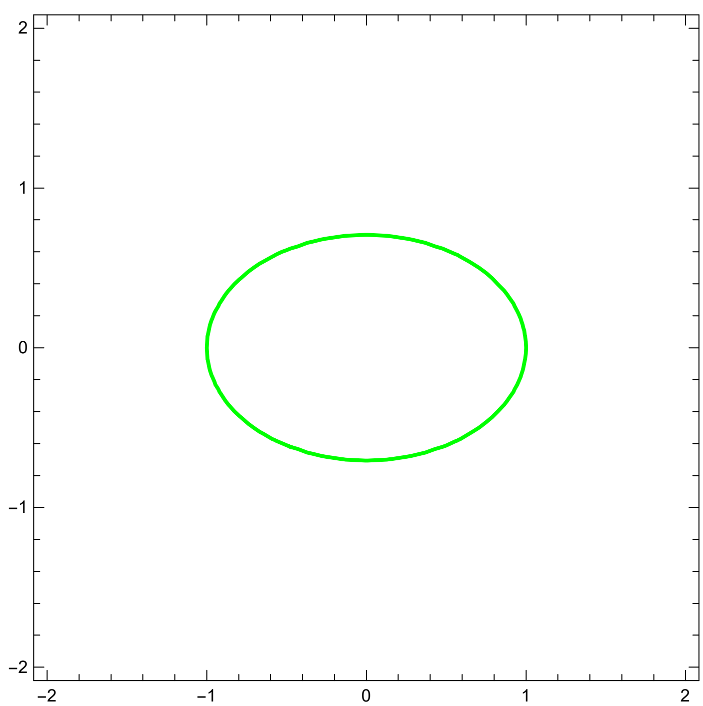
In[ ]:= `ContourPlot[Sin[x * Cos[y]], {x, -1, 1}, {y, -1, 1}, PlotLegends → Automatic]`

Out[ ]=

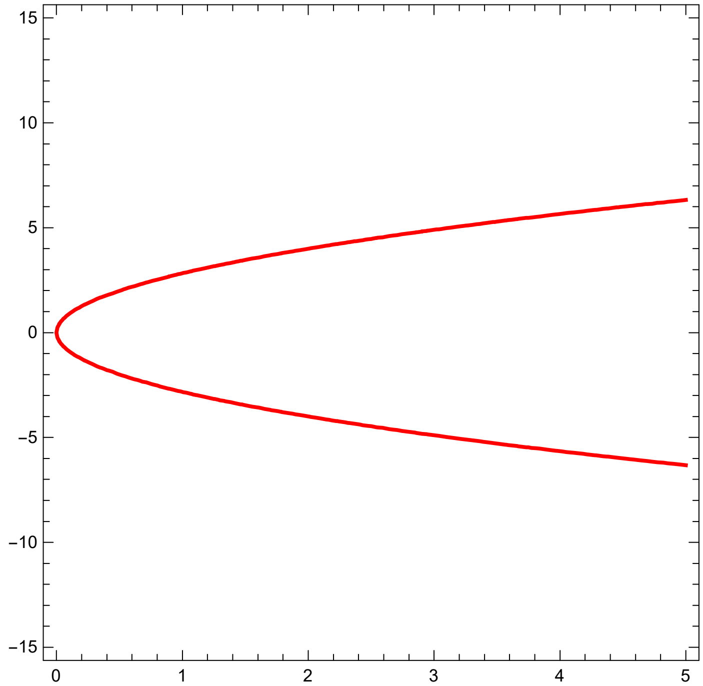


In[ ]:= `ellipse = ContourPlot[x^2 + 2 y^2 == 1,`
`{x, -2, 2}, {y, -2, 2}, ContourStyle → Directive[Thick, Green]]`

Out[ ]=



In[ ]:=
```
parabola =
 ContourPlot[y^2 == 8 x, {x, 0, 5}, {y, -15, 15}, ContourStyle → Directive[Thick, Red]]
```

Out[ ]=


In[ ]:=
```
? ellipse
```

Out[ ]=

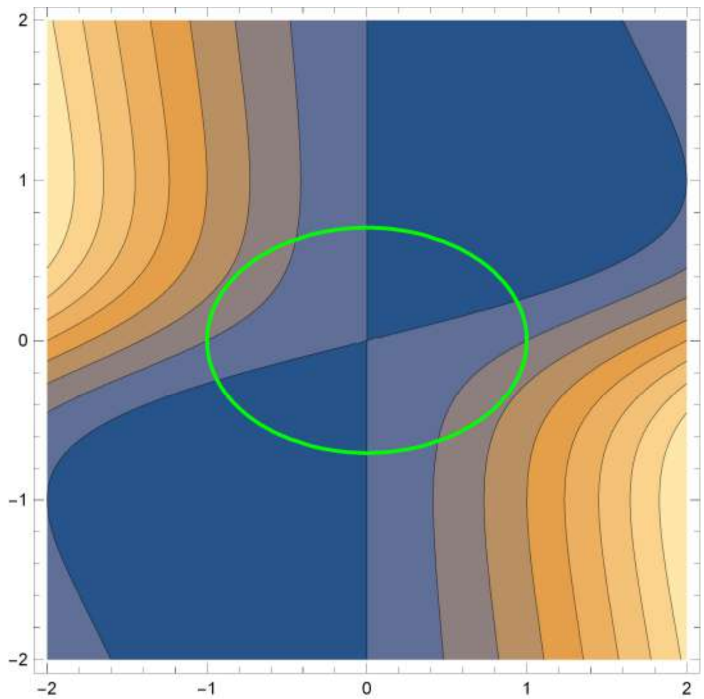
Symbol

Global`ellipse

Full Name Global`ellipse

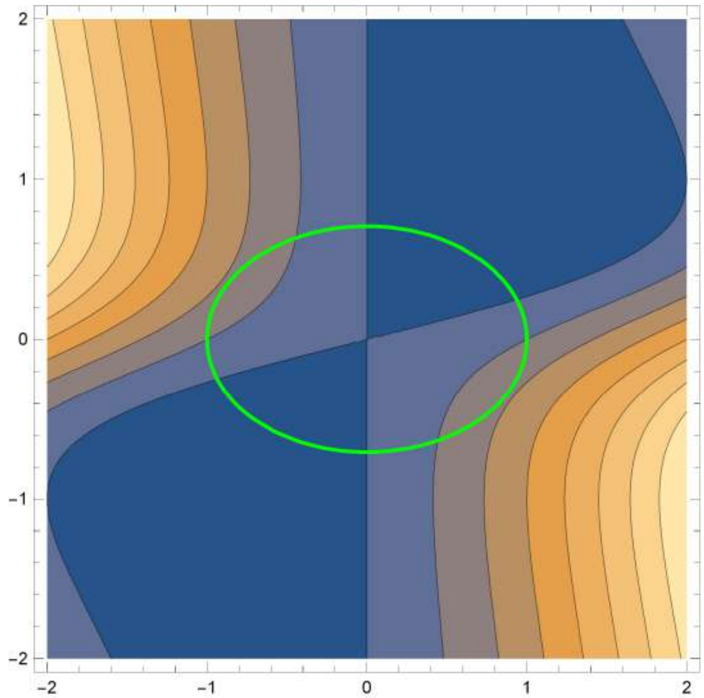
In[ ]:=
```
Clear[ellipse, parabola];
ellipse = ContourPlot[x^2 + 2 y^2 == 1,
   {x, -2, 2}, {y, -2, 2}, ContourStyle → Directive[Thick, Green]];
function = ContourPlot[x^2 - (4 x * y / (y^2 + 1)), {x, -2, 2}, {y, -2, 2}];
Show[function, ellipse]
```

Out[ ]=



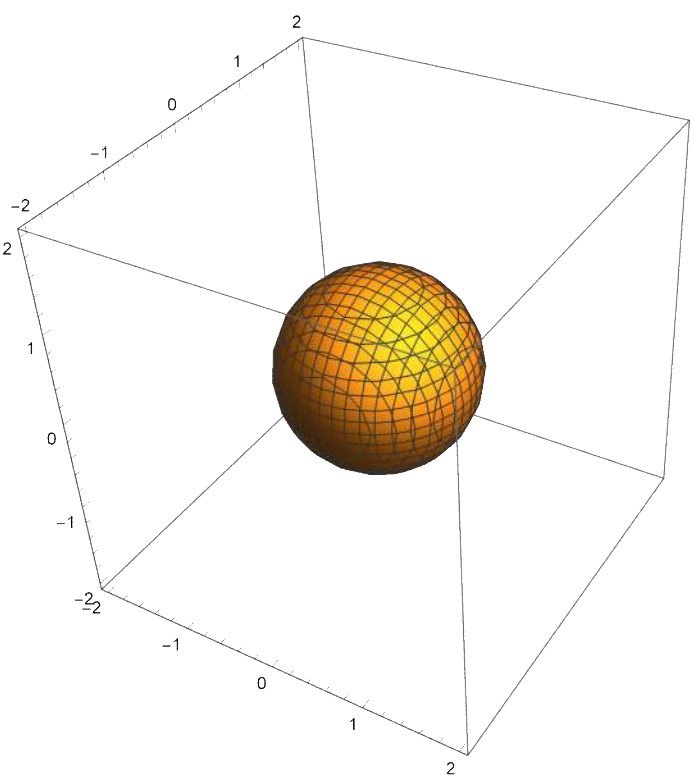
In[ ]:=
```
ContourPlot[x^2 - (4 x * y / (y^2 + 1)), {x, -2, 2},
 {y, -2, 2}, MeshFunctions → {Function[{x, y}, x^2 + 2 y^2]},
 Mesh → {{1}}, MeshStyle → Directive[Thick, Green]]
```

Out[ ]=



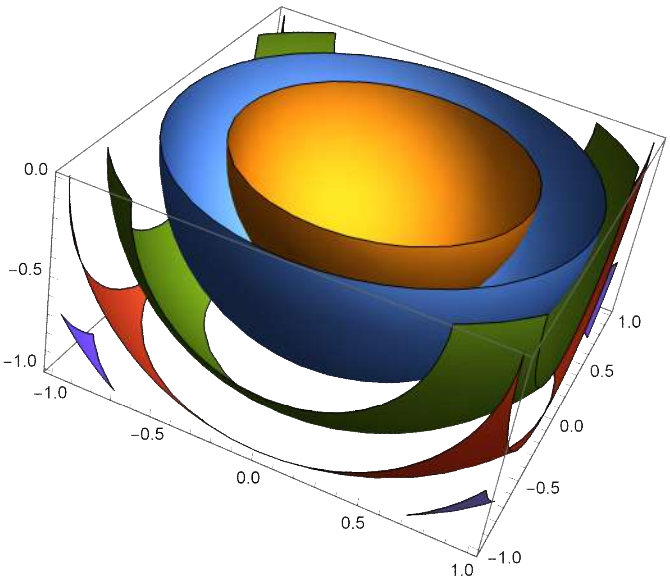
In[ ]:=
```
ContourPlot3D[x^2 + y^2 + z^2 == 1, {x, -2, 2}, {y, -2, 2}, {z, -2, 2}]
```

Out[ ]=



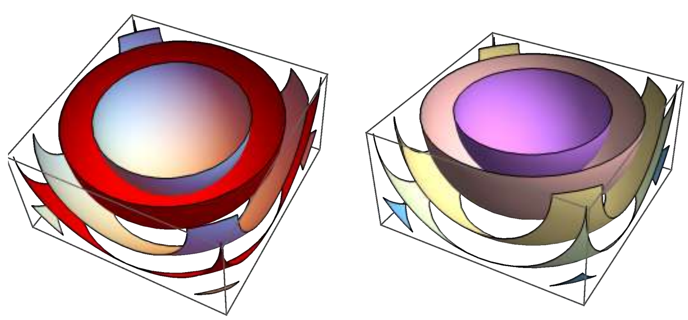
In[ ]:=
```
ContourPlot3D[x^2 + y^2 + z^2, {x, -1, 1}, {y, -1, 1},
  {z, -1, 0}, BoxRatios → {2, 2, 1}, Mesh → None, Contours → 5]
```

Out[ ]=



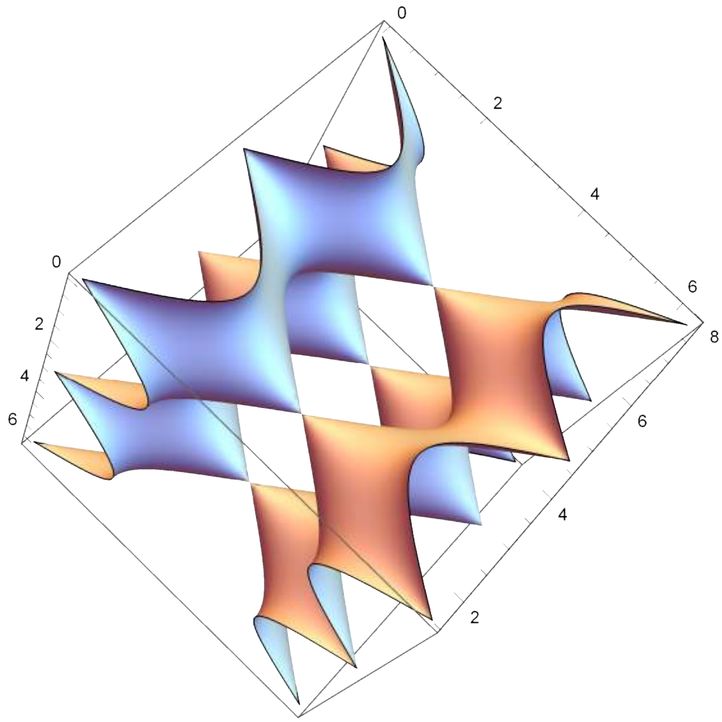
In[ ]:=
```
GraphicsRow[{ContourPlot3D[x^2 + y^2 + z^2, {x, -1, 1}, {y, -1, 1}, {z, -1, 0}, Contours → 5,
   Mesh → None, Axes → False, BoxRatios → {2, 2, 1}, ContourStyle → {White, Red}],
  ContourPlot3D[x^2 + y^2 + z^2, {x, -1, 1}, {y, -1, 1}, {z, -1, 0}, Contours → 5,
   Mesh → None, Axes → False, BoxRatios → {2, 2, 1}, ColorFunction → "Pastel"]}]
```

Out[ ]=

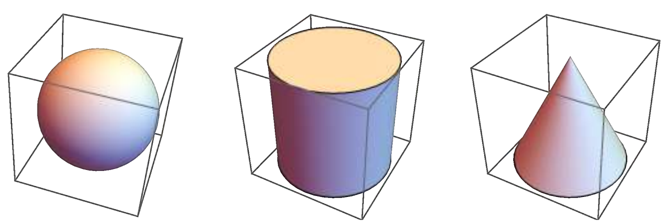


In[ ]:=
```
ContourPlot3D[Cos[x]^2 + Sin[y]^2 == 1 + Cos[z], {x, 0, 2π}, {y, π/2, 5π/2},
  {z, 0, 2π}, Mesh → None, ContourStyle → Directive[White, Specularity[White, 10]]]
```

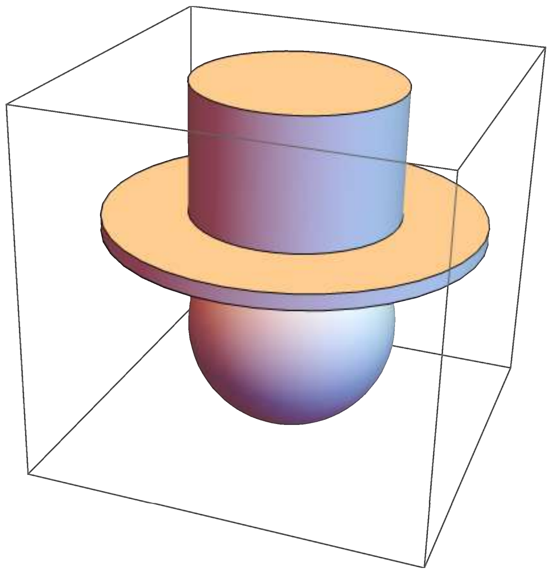
Out[ ]=



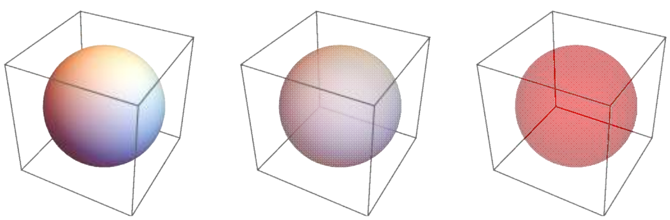
In[ ]:=
```
GraphicsRow[{Graphics3D[Sphere[]], Graphics3D[Cylinder[]], Graphics3D[Cone[]]}]
```



In[ ]:=
```
Graphics3D[{Sphere[{0, 0, 0}, 2],
  Cylinder[{{0, 0, 2}, {0, 0, 2.3}}, 3.6], Cylinder[{{0, 0, 2.3}, {0, 0, 4.8}}, 2]}]
```

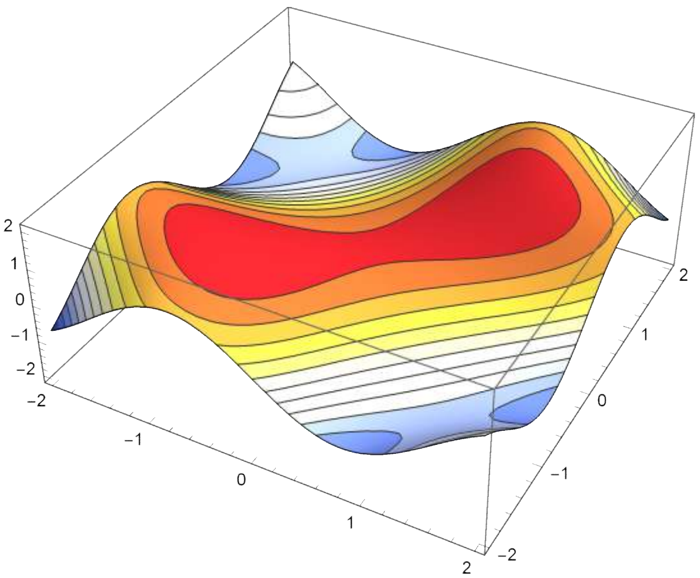
Out[ ]=


In[ ]:=
```
GraphicsRow[{Graphics3D[Sphere[]], Graphics3D[{Opacity[0.6], Sphere[]}],
  Graphics3D[{Directive[Red, Opacity[0.3]], Sphere[]}]}]
```

Out[ ]=


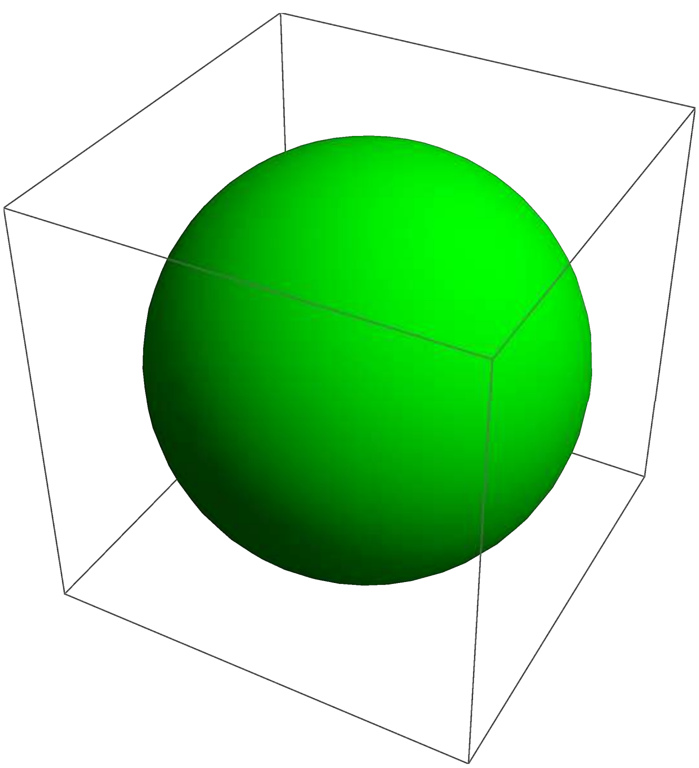
In[ ]:=
```
Clear[function];
function = Plot3D[Sin[x * y], {x, -2, 2}, {y, -2, 2},
  PlotRange → {-2.2, 2.2}, Lighting → "Neutral", PlotPoints → 40,
  Mesh → 12, MeshFunctions → {Function[{x, y, z}, x^2 + y^2 + (z - 2)^2]},
  MeshShading → Table[ColorData["TemperatureMap"][1 - k], {k, 0, 1, 1 / 12}]]
```

Out[ ]=

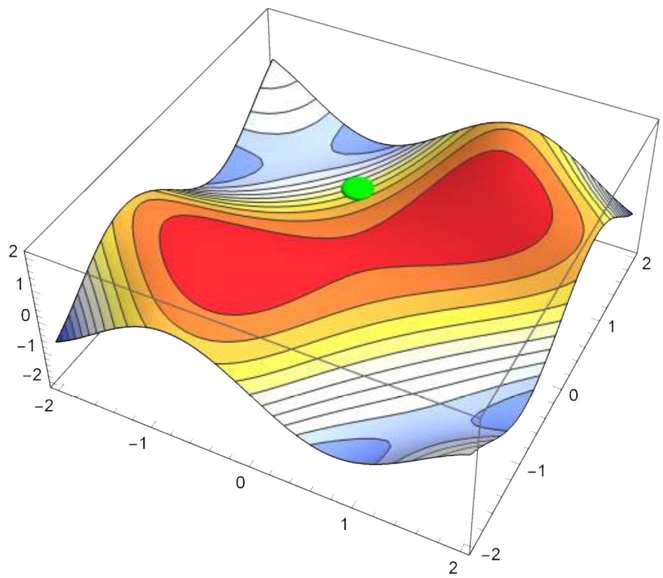


In[ ]:=
```
point = Graphics3D[{Directive[Thick, Green], Sphere[{0, 0, 2}, 0.15]}]
```

Out[ ]=



In[ ]:= `Show[function, point]`

Out[ ]=


In[ ]:=
```
Clear[f, x, y, z];
f = (y - 2 x + 3) / (x - 1);
Limit[f, {x, y} → {0, 0}]
```

Out[ ]= -3

In[ ]:= `Limit[1 / (x^2 + y^2), {x, y} → {0, 0}]`

Out[ ]= ∞

In[ ]:= `Limit[f, {x, y} → {1, -1}]`

Out[ ]= Indeterminate

In[ ]:= `Limit[f, {x → 1, y → -1}]`

Out[ ]= -2

In[ ]:=
```
Clear[f, x, y, z];
f = Sin[x^2 - y^2];
D[f, x]
```

Out[ ]= $2\,x\,\mathrm{Cos}\left[x^2 - y^2\right]$

In[ ]:= `D[f, y]`

Out[ ]= $-2\,y\,\mathrm{Cos}\left[x^2 - y^2\right]$

In[ ]:= `% /. {x → 0, y → Sqrt[π]}`

Out[ ]= $2\sqrt{\pi}$

In[ ]:= `D[f, y] /. {x → 0, y → Sqrt[π]}`

Out[ ]= $2\sqrt{\pi}$

In[ ]:= ∂_x **Sin[x^2 - y^2]**

Out[ ]= $2\,x\,\mathrm{Cos}\left[x^2 - y^2\right]$

In[ ]:= ∂_x **x^2 + 2 x * y**

Out[ ]= $2\,x + 2\,x\,y$

In[ ]:= ∂_x **(x^2 + 2 x * y)**

Out[ ]= $2\,x + 2\,y$

In[ ]:= **? f**

Out[ ]= Missing[UnknownSymbol, f]

In[ ]:= **? f**

Out[ ]=

Symbol
Global`f
Full Name Global`f

In[ ]:=
```
Clear[f, x, y, z];
f = Sin[x^2 - y^2];
D[f, {x, 2}]
```

Out[ ]= $2\,\mathrm{Cos}\left[x^2 - y^2\right] - 4\,x^2\,\mathrm{Sin}\left[x^2 - y^2\right]$

In[ ]:= **D[f, x, y]**

Out[ ]= $4\,x\,y\,\mathrm{Sin}\left[x^2 - y^2\right]$

In[ ]:= **D[f, y, x]**

Out[ ]= $4\,x\,y\,\mathrm{Sin}\left[x^2 - y^2\right]$

In[ ]:= $\partial_{x,x}$ **f**

Out[ ]= $2\,\mathrm{Cos}\left[x^2 - y^2\right] - 4\,x^2\,\mathrm{Sin}\left[x^2 - y^2\right]$

In[ ]:= $\partial_{x,y}$ **f**

Out[ ]= $4\,x\,y\,\mathrm{Sin}\left[x^2 - y^2\right]$

In[ ]:= **D[f, {x, 3}, {y, 3}]**

Out[ ]= $-12\,x\,\left(8\,y^3\,\mathrm{Cos}\left[x^2 - y^2\right] - 12\,y\,\mathrm{Sin}\left[x^2 - y^2\right]\right) - 8\,x^3\,\left(-12\,y\,\mathrm{Cos}\left[x^2 - y^2\right] - 8\,y^3\,\mathrm{Sin}\left[x^2 - y^2\right]\right)$

In[ ]:= **Grad[f, {x, y}]**

Out[ ]= $\left\{2\,x\,\mathrm{Cos}\left[x^2 - y^2\right],\ -2\,y\,\mathrm{Cos}\left[x^2 - y^2\right]\right\}$

```
In[ ]:= Grad[x^2 * y^3, {x, y}]
```

Out[ ]= $\left\{2\,x\,y^3,\ 3\,x^2\,y^2\right\}$

```
In[ ]:= Grad[(x^2 * y^3 * z^4) / l^5, {x, y, z, l}]
```

Out[ ]= $\left\{\frac{2\,x\,y^3\,z^4}{l^5},\ \frac{3\,x^2\,y^2\,z^4}{l^5},\ \frac{4\,x^2\,y^3\,z^3}{l^5},\ -\frac{5\,x^2\,y^3\,z^4}{l^6}\right\}$

```
In[ ]:= Grad[f, {x, y}] /. {x → 0, y → Sqrt[π]}
```

Out[ ]= $\left\{0,\ 2\sqrt{\pi}\right\}$

```
In[ ]:= Grad[x^2 * y^3, {x, y}].Normalize[{3, -1}]
```

Out[ ]= $-\frac{3\,x^2\,y^2}{\sqrt{10}} + 3\sqrt{\frac{2}{5}}\,x\,y^3$

```
In[ ]:= Grad[x^2 * y^3, {x, y}].{3, -1} / Norm[{3, -1}]
```

Out[ ]= $\frac{-3\,x^2\,y^2 + 6\,x\,y^3}{\sqrt{10}}$

```
In[ ]:= Grad[x^2 * y^3, {x, y}].Normalize[{3, -1}] /. {x → 2, y → 3}
```

Out[ ]= $108\sqrt{\frac{2}{5}}$

```
In[ ]:= Grad[x^2 * y^3, {x, y}].{3, -1} / Norm[{3, -1}] /. {x → 2, y → 3}
```

Out[ ]= $108\sqrt{\frac{2}{5}}$

```
In[ ]:= Clear[f, x, y, z];
        f = x^2 * y^3;
        Dt[f]
```

Out[ ]= $2\,x\,y^3\,\mathrm{Dt}[x] + 3\,x^2\,y^2\,\mathrm{Dt}[y]$

```
In[ ]:= Dt[f] /. {Dt[x] → 0.03, Dt[y] → -0.01, x → 2, y → 3}
```

Out[ ]= 2.16

```
In[ ]:= Maximize[-85 + 16 x - 4 x^2 - 4 y - 4 y^2 + 40 z - 4 z^2, {x, y, z}]
```

Out[ ]= $\left\{32,\ \left\{x \to 2,\ y \to -\frac{1}{2},\ z \to 5\right\}\right\}$

In[ ]:=
```
Minimize[-85 + 16 x - 4 x^2 - 4 y - 4 y^2 + 40 z - 4 z^2, {x, y, z}]
```

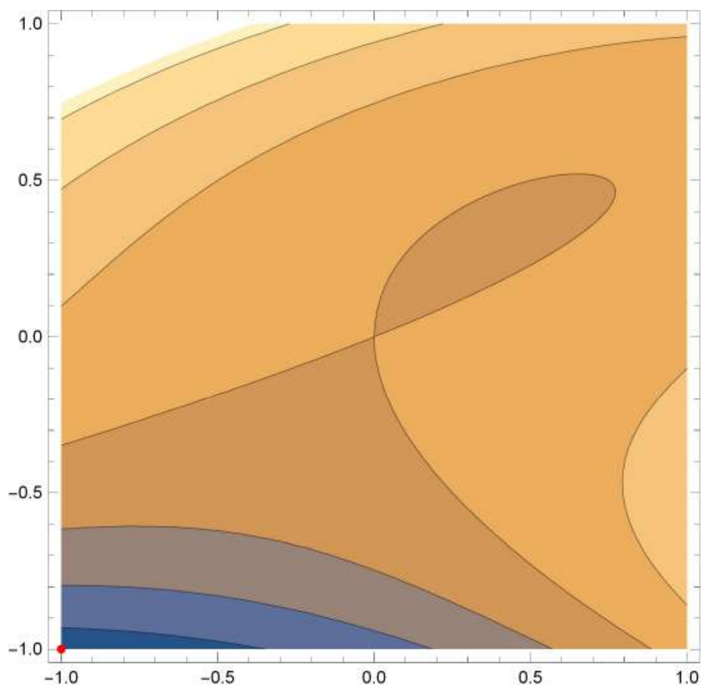
Minimize: The minimum is not attained at any point satisfying the given constraints.

Out[ ]= $\left\{-\infty, \left\{x \to -\infty, y \to -\frac{12}{5}, z \to -\frac{1}{2}\right\}\right\}$

In[ ]:=
```
Clear[f, x, y, z];
f = 12 y^3 + 4 x^2 - 10 x * y;
Minimize[{f, -1 ≤ x ≤ 1 && -1 ≤ y ≤ 1}, {x, y}]
```

Out[ ]= $\{-18, \{x \to -1, y \to -1\}\}$

In[ ]:=
```
ContourPlot[f, {x, -1, 1}, {y, -1, 1},
 Epilog → {Red, PointSize[Medium], Point[{x, y} /. Last[%]]}]
```

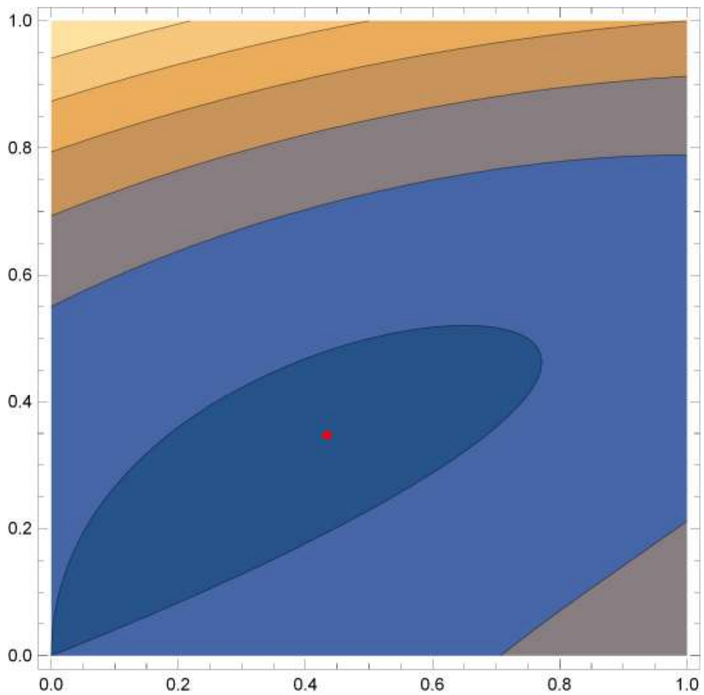
Out[ ]=


In[ ]:=
```
f = 12 y^3 + 4 x^2 - 10 x * y;
Minimize[{f, 0 ≤ x ≤ 1 && 0 ≤ y ≤ 1}, {x, y}]
```

Out[ ]= $\left\{-\frac{15625}{62208}, \left\{x \to \frac{125}{288}, y \to \frac{25}{72}\right\}\right\}$

In[ ]:=
```
ContourPlot[f, {x, 0, 1}, {y, 0, 1},
 Epilog → {Red, PointSize[Medium], Point[{x, y} /. Last[%]]}]
```

Out[ ]=



In[ ]:=
```
Clear[f, x, y, z];
f = 12 y^3 + 4 x^2 - 10 x * y;
crPts = Solve[Grad[f, {x, y}] == {0, 0}, {x, y}]
```

Out[ ]= $\left\{\{x \to 0, y \to 0\}, \left\{x \to \frac{125}{288}, y \to \frac{25}{72}\right\}\right\}$

In[ ]:= $(\partial_{x,x} f) * (\partial_{y,y} f) - (\partial_{x,y} f)$^2 /. crPts // N

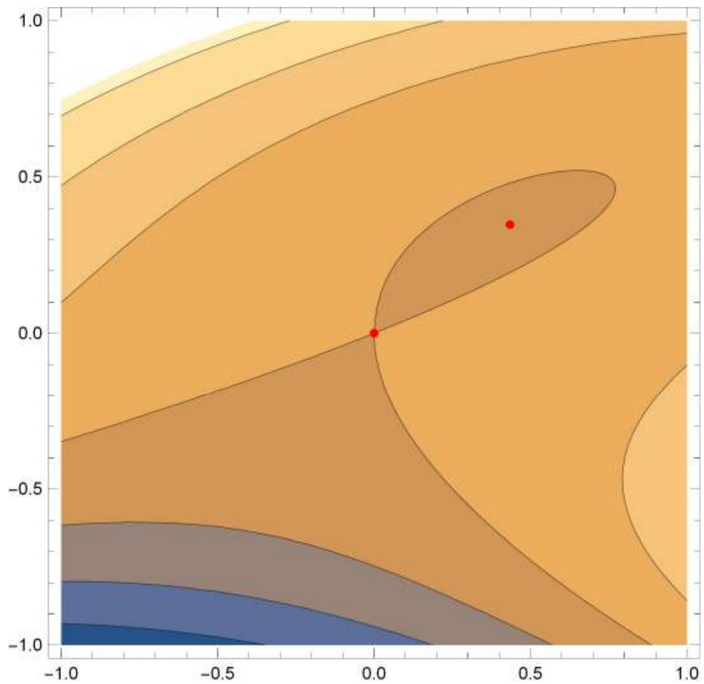
Out[ ]= {-100., 100.}

In[ ]:= $\partial_{x,x} f$ /. crPts[[2]] // N

Out[ ]= 8.

```
In[ ]:= ContourPlot[f, {x, -1, 1}, {y, -1, 1},
          Epilog → {Red, PointSize[Medium], Point[{x, y} /. crPts]}]
```

Out[ ]=




```
In[ ]:= f /. crPts
```

Out[ ]= $\left\{0, -\frac{15625}{62208}\right\}$

```
Clear[crPts, f, x, y, z]
```

```
In[ ]:= Clear[crPts, f, x, y, z];
        f = x^3 + 3 x * y^2 - 15 x^2 - 15 y^2 + 72 x;
        crPts = Solve[Grad[f, {x, y}] == {0, 0}, {x, y}]
```

```
Out[ ]= {{x → 4, y → 0}, {x → 5, y → -1}, {x → 5, y → 1}, {x → 6, y → 0}}
```

In[ ]:= $(\partial_{x,x} f) * (\partial_{y,y} f) - (\partial_{x,y} f)$ ^2 /. crPts // N

```
Out[ ]= {36., -36., -36., 36.}
```

In[ ]:= $\partial_{x,x} f$ /. crPts[[1]] // N

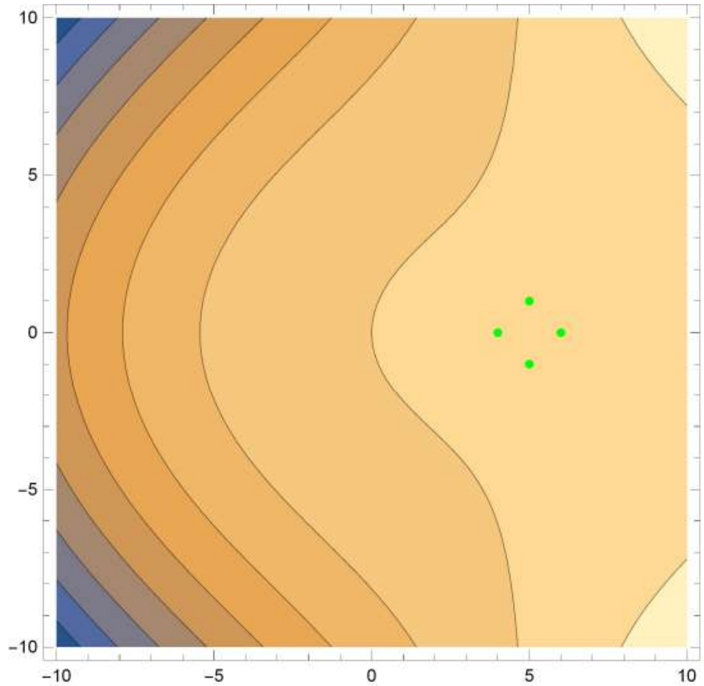
```
Out[ ]= -6.
```

In[ ]:= $\partial_{x,x} f$ /. crPts[[4]] // N

```
Out[ ]= 6.
```

In[ ]:=
```
ContourPlot[f, {x, -10, 10}, {y, -10, 10},
 Epilog → {Green, PointSize[Medium], Point[{x, y} /. crPts]}]
```

Out[ ]=



In[ ]:=
```
f /. crPts
```

Out[ ]= {112, 110, 110, 108}

In[ ]:=
```
Clear[crPts, f, x, y, z]
```

In[ ]:=
```
Solve[x^4 - 1 == 0, x]
```

Out[ ]= {{x → -1}, {x → -i}, {x → i}, {x → 1}}

In[ ]:=
```
Solve[x^4 - 1 == 0, x, Reals]
```

Out[ ]= {{x → -1}, {x → 1}}

In[ ]:=
```
Reduce[x^4 - 1 == 0, x]
```

Out[ ]= x == -1 || x == -i || x == i || x == 1

In[ ]:=
```
Reduce[x^4 - 1 == 0, x, Reals]
```

Out[ ]= x == -1 || x == 1

In[ ]:=
```
NSolve[x^3 + 3 x + 5 == 0, x]
```

Out[ ]= {{x → -1.15417}, {x → 0.577086 - 1.99977 i}, {x → 0.577086 + 1.99977 i}}

In[ ]:=
```
f = Sin[x * Cos[y]];
Solve[Grad[f, {x, y}] == {0, 0}, {x, y}]
```

Solve: Inconsistent or redundant transcendental equation. After reduction, the bad equation is ArcCos[Cos[x Cos[y]]] == 0.

Solve: Inconsistent or redundant transcendental equation. After reduction, the bad equation is ArcCos[Cos[x Cos[y]]] == 0.

Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.

Solve: Equations may not give solutions for all "solve" variables.

Out[ ]=
$$\left\{\left\{x\to 0,\ y\to -\frac{\pi}{2}\right\},\ \left\{x\to 0,\ y\to \frac{\pi}{2}\right\},\ \left\{x\to -\frac{\pi}{2},\ y\to 0\right\},\ \left\{x\to -\frac{\pi}{2},\ y\to -\pi\right\},\ \left\{x\to -\frac{\pi}{2},\ y\to \pi\right\},\ \left\{x\to \frac{\pi}{2},\ y\to 0\right\},\ \left\{x\to \frac{\pi}{2},\ y\to -\pi\right\},\ \left\{x\to \frac{\pi}{2},\ y\to \pi\right\}\right\}$$

In[ ]:=
```
f = Sin[x * Cos[y]];
Solve[{Grad[f, {x, y}] == {0, 0}, -3 ≤ x ≤ 3, -3 ≤ y ≤ 3}, {x, y}, Reals]
```

Solve: Equations may not give solutions for all "solve" variables.

Out[ ]=
$$\left\{\left\{y\to \text{ConditionalExpression}\left[-\text{ArcCos}\left[-\frac{\pi}{2\,x}\right],\ -3\le x\le -\frac{\pi}{2}\ ||\ -\frac{1}{2}\pi\,\text{Sec}[3]\le x\le 3\right]\right\},\right.$$
$$\left\{y\to \text{ConditionalExpression}\left[\text{ArcCos}\left[-\frac{\pi}{2\,x}\right],\ -3\le x\le -\frac{\pi}{2}\ ||\ -\frac{1}{2}\pi\,\text{Sec}[3]\le x\le 3\right]\right\},$$
$$\left\{y\to \text{ConditionalExpression}\left[-\text{ArcCos}\left[\frac{\pi}{2\,x}\right],\ -3\le x\le \frac{1}{2}\pi\,\text{Sec}[3]\ ||\ \frac{\pi}{2}\le x\le 3\right]\right\},$$
$$\left\{y\to \text{ConditionalExpression}\left[\text{ArcCos}\left[\frac{\pi}{2\,x}\right],\ -3\le x\le \frac{1}{2}\pi\,\text{Sec}[3]\ ||\ \frac{\pi}{2}\le x\le 3\right]\right\},$$
$$\left.\left\{x\to 0,\ y\to -\frac{\pi}{2}\right\},\ \left\{x\to 0,\ y\to \frac{\pi}{2}\right\}\right\}$$

In[ ]:=
$$(\partial_{x,x}f) * (\partial_{y,y}f) - (\partial_{x,y}f)\text{^}2\ /.\ \{\{x\to 0,\ y\to -\pi/2\},\ \{x\to 0,\ y\to \pi/2\}\}$$

Out[ ]= {-1, -1}

In[ ]:=
```
Clear[f, x, y, z, λ];
f = 4 x * y;
g = 4 x^2 + y^2 - 8;
sols = Solve[Grad[f - λ * g, {x, y, λ}] == {0, 0, 0}, {x, y, λ}, Reals]
```

Out[ ]=
$$\{\{x\to -1,\ y\to -2,\ \lambda\to 1\},\ \{x\to -1,\ y\to 2,\ \lambda\to -1\},\ \{x\to 1,\ y\to -2,\ \lambda\to -1\},\ \{x\to 1,\ y\to 2,\ \lambda\to 1\}\}$$

In[ ]:=
```
{x, y} /. sols
```

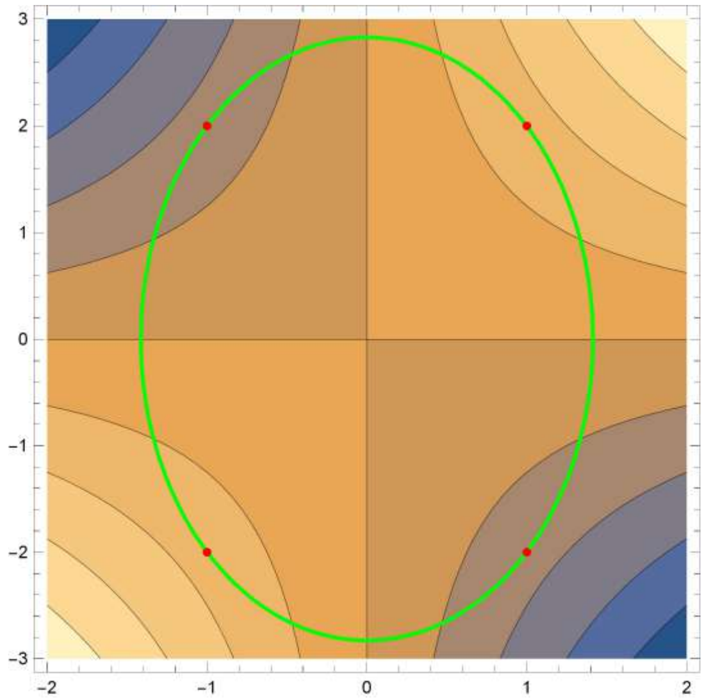
Out[ ]= {{-1, -2}, {-1, 2}, {1, -2}, {1, 2}}

In[ ]:=
```
f /. sols
```

Out[ ]= {8, -8, -8, 8}

In[ ]:=
```
Show[ContourPlot[f, {x, -2, 2}, {y, -3, 3}],
 ContourPlot[g == 0, {x, -2, 2}, {y, -3, 3}, ContourStyle → Directive[Thick, Green]],
 Epilog → {Red, PointSize[Medium], Point[{x, y} /. sols]}]
```

Out[ ]=



In[ ]:=
```
Maximize[{f, g == 0}, {x, y}]
```

Out[ ]= $\{8, \{x \to -1, y \to -2\}\}$

In[ ]:=
```
Minimize[{f, g == 0}, {x, y}]
```

Out[ ]= $\{-8, \{x \to -1, y \to 2\}\}$

In[ ]:=
```
Clear[f, g, x, y, z, λ, sols];
```

In[ ]:=
```
f = x^2 + y^2 + z^2;
g = 2 x + 3 y - z - 5;
sols = Solve[Grad[f - λ * g, {x, y, z, λ}] == {0, 0, 0, 0}, {x, y, z, λ}, Reals]
```

Out[ ]= $\left\{\left\{x \to \frac{5}{7}, y \to \frac{15}{14}, z \to -\frac{5}{14}, \lambda \to \frac{5}{7}\right\}\right\}$

In[ ]:=
```
{x, y, z} /. sols
```

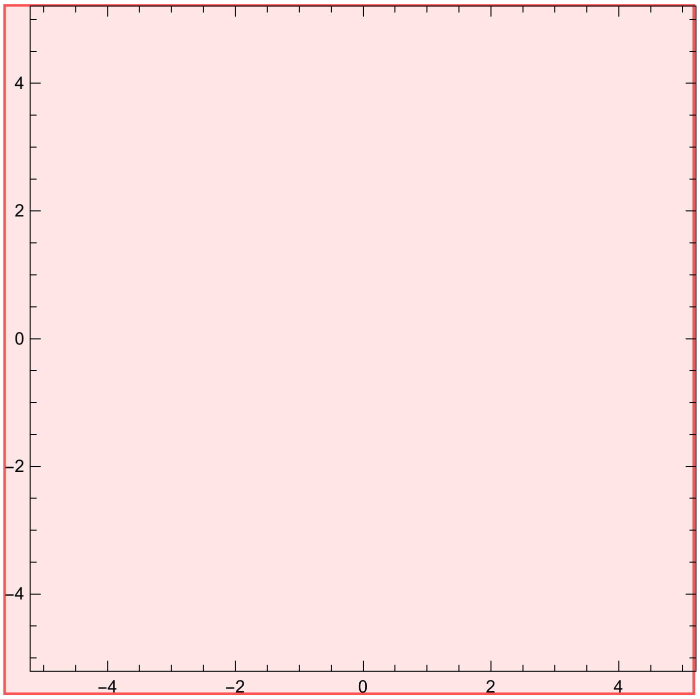
Out[ ]= $\left\{\left\{\frac{5}{7}, \frac{15}{14}, -\frac{5}{14}\right\}\right\}$

In[ ]:=
```
f /. sols
```

Out[ ]= $\left\{\frac{25}{14}\right\}$

In[ ]:=
```
Show[ContourPlot[f, {x, -5, 5}, {y, -5, 5}],
 ContourPlot[g == 0, {x, -5, 5}, {y, -5, 5}, ContourStyle → Directive[Red, Thick]],
 Epilog → {Green, PointSize[Medium], Point[{x, y, z} /. sols]}]
```

Out[ ]=



In[ ]:=
```
Maximize[{f, g == 0}, {x, y, z}]
```

Maximize: The maximum is not attained at any point satisfying the given constraints.

Out[ ]= {∞, {x → Indeterminate, y → Indeterminate, z → Indeterminate}}

In[ ]:=
```
Minimize[{f, g == 0}, {x, y, z}]
```

Out[ ]= $\left\{\frac{25}{14}, \left\{x \to \frac{5}{7}, y \to \frac{15}{14}, z \to -\frac{5}{14}\right\}\right\}$

In[ ]:=
```
Clear[f, g, x, y, z, sols, λ];
```

In[ ]:=
```
Integrate[5 - x^2 * y^2, {y, 1, 3}, {x, 0, 2}]
```

Out[ ]= $-\frac{28}{9}$

In[ ]:=
```
Integrate[5 - x^2 * y^2, y, x]
```

Out[ ]= $5\,x\,y - \frac{x^3\,y^3}{9}$

$$\left(\int_0^2 (5 - x\text{^}2 * y\text{^}2)\, dx\right)$$

In[ ]:= $\int_1^3 \left(\int_0^2 (5 - x^2 * y^2)\, dx\right) dy$

Out[ ]= $-\frac{28}{9}$

$\left(\int (5 - x^2 * y^2)\, dx\right)$

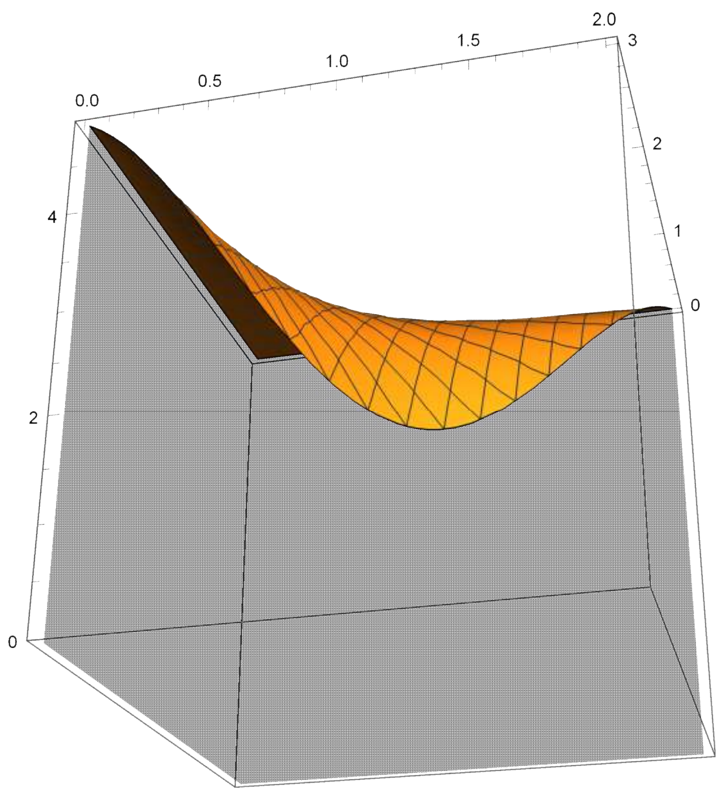
In[ ]:= $\int \left(\int (5 - x^2 * y^2)\, dx\right) dy$

Out[ ]= $5\, x\, y - \frac{x^3\, y^3}{9}$

In[ ]:= `Integrate[5 - x^2 * y^2, {x, y} ∈ Triangle[{{0, 0}, {3, 0}, {0, 2}}]]`

Out[ ]= $\frac{69}{5}$

In[ ]:=
```
Plot3D[5 - x^2 * y^2, {x, y} ∈ Triangle[{{0, 0}, {3, 0}, {0, 2}}],
 Filling → Axis, PlotRange → {0, 5}, BoxRatios → {1, 1, 1}]
```

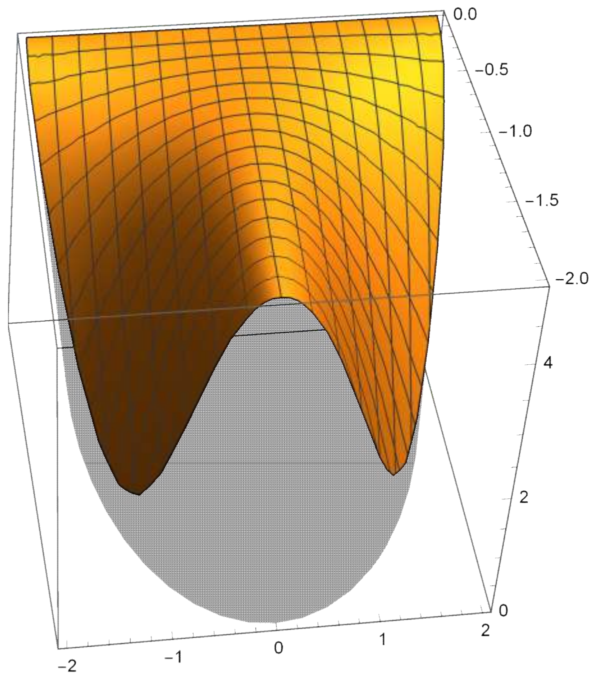
Out[ ]= 

In[ ]:= `Integrate[5 - x^2 * y^2, {x, y} ∈ Disk[{0, 0}, 2, {π, 2 π}]]`

Out[ ]= $\frac{26\, \pi}{3}$

In[ ]:=
```
Plot3D[5 - x^2 * y^2, {x, y} ∈ Disk[{0, 0}, 2, {π, 2 π}],
  Filling → Axis, PlotRange → {0, 5}, BoxRatios → {1, 1, 1}]
```

Out[ ]=



In[ ]:=
```
CylindricalDecomposition[{x, y} ∈ Disk[{0, 0}, 2, {π, 2 π}], {x, y}]
```

Out[ ]= $(x == -2 \,\&\&\, y == 0) \,||\, \left(-2 < x < 2 \,\&\&\, -\sqrt{4 - x^2} \le y \le 0\right) \,||\, (x == 2 \,\&\&\, y == 0)$

$$\left(\int_{-\text{Sqrt}[4-x^2]}^{0} (5 - x^2 * y^2)\, dy\right)$$

In[ ]:= $$\int_{-2}^{2} \left(\int_{-\text{Sqrt}[4-x^2]}^{0} (5 - x^2 * y^2)\, dy\right) dx$$

Out[ ]= $$\frac{26\pi}{3}$$

In[ ]:=
```
CylindricalDecomposition[{x, y, z} ∈ Ball[{0, 0, 0}, 2], {x, y, z}]
```

Out[ ]= $(x == -2 \,\&\&\, y == 0 \,\&\&\, z == 0) \,||\, \left(-2 < x < 2 \,\&\&\, \left(\left(y == -\sqrt{4 - x^2} \,\&\&\, z == -\sqrt{4 - x^2 - y^2}\right) \,||\, \left(-\sqrt{4 - x^2} < y < \sqrt{4 - x^2} \,\&\&\, -\sqrt{4 - x^2 - y^2} \le z \le \sqrt{4 - x^2 - y^2}\right) \,||\, \left(y == \sqrt{4 - x^2} \,\&\&\, z == -\sqrt{4 - x^2 - y^2}\right)\right)\right) \,||\, (x == 2 \,\&\&\, y == 0 \,\&\&\, z == 0)$

$$\left(\int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} (x * y - z^2)\, dz\right)$$

$$\left(\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left(\int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} (x * y - z^2)\, dz\right) dy\right)$$

In[ ]:= $\int_{-2}^{2}\left(\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\left(\int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}}(x*y-z\text{^}2)\,dz\right)dy\right)dx$

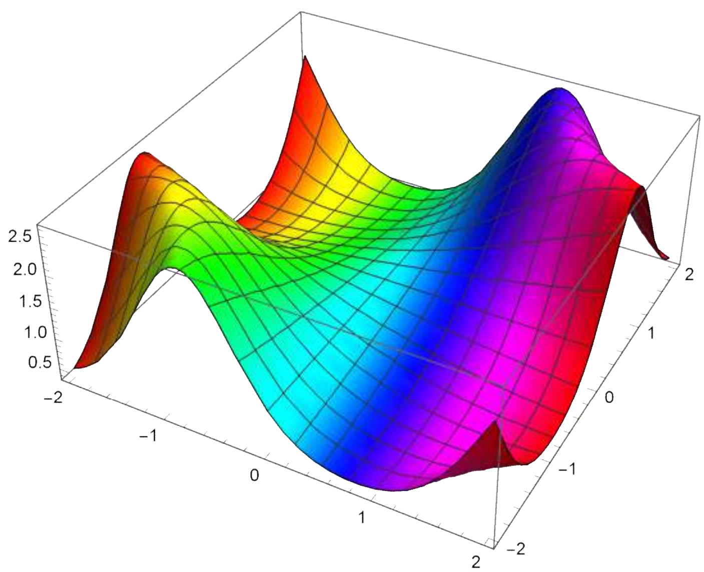
Out[ ]= $-\frac{128\pi}{15}$

EXERCISE

1 (a)

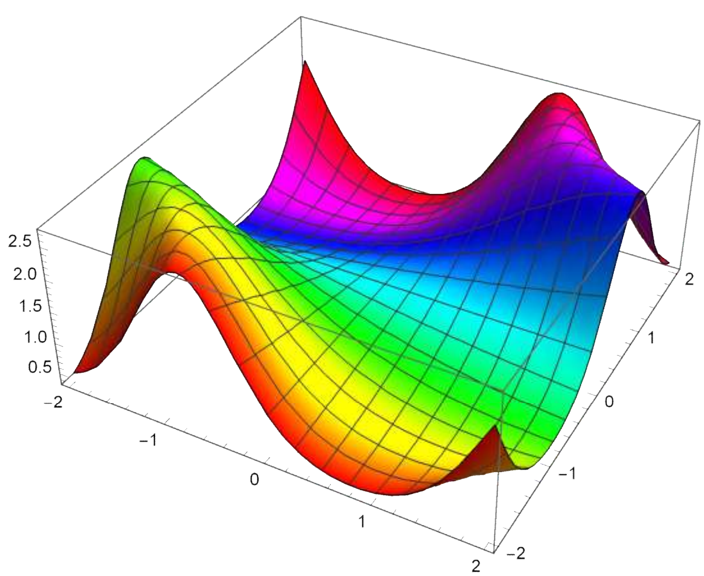
```
In[ ]:= Plot3D[e^Sin[x * y], {x, -2, 2}, {y, -2, 2}, ColorFunction → (Hue[#1] &)]
```

Out[ ]=

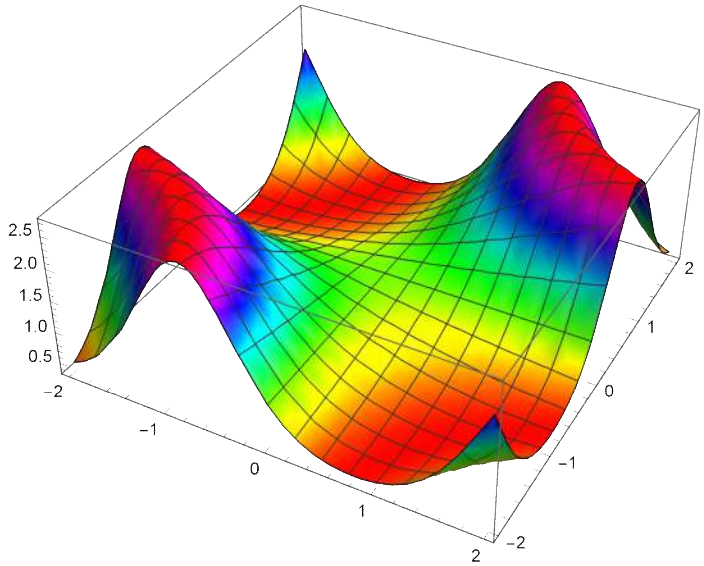



```
In[ ]:= Plot3D[e^Sin[x * y], {x, -2, 2}, {y, -2, 2}, ColorFunction → (Hue[#2] &)]
```

Out[ ]=

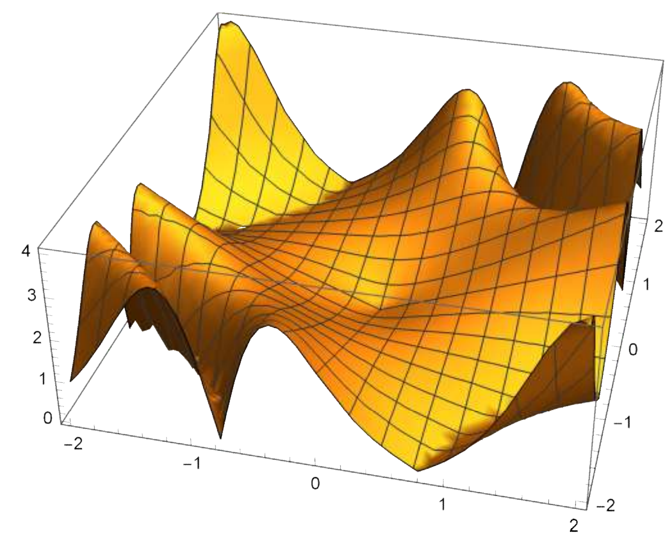


In[]:=
```
Plot3D[e^Sin[x * y], {x, -2, 2}, {y, -2, 2}, ColorFunction → (Hue[#3] &)]
```

Out[]=


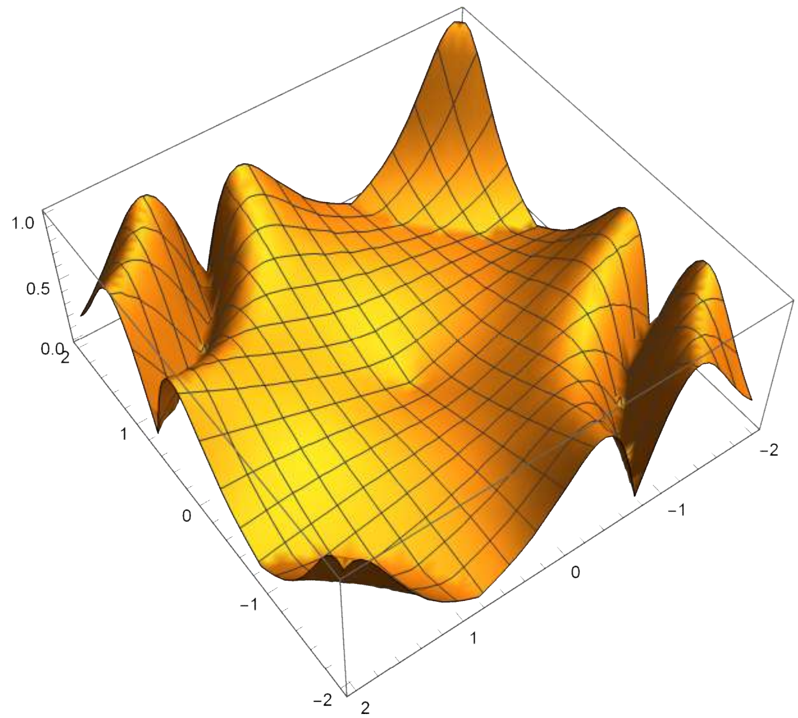
(b)

In[]:=
```
Plot3D[(e^Sin[x * y]) * Sqrt[(x * Cos[x * y])^2 + (y * Cos[x * y])^2],
  {x, -2, 2}, {y, -2, 2}, PlotRange → All]
```

Out[]=


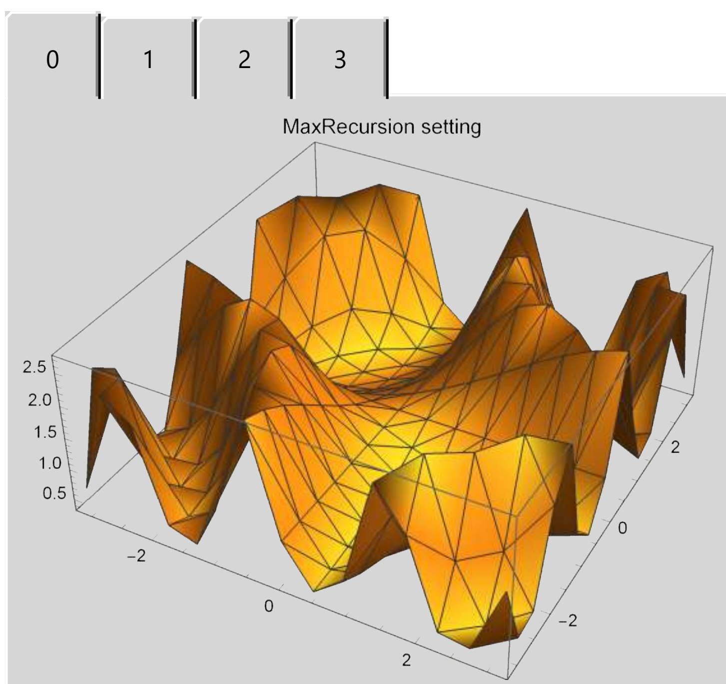
(c)

In[ ]:=
```
Plot3D[Rescale[(ⅇ^Sin[x * y]) * Sqrt[(x * Cos[x * y])^2 + (y * Cos[x * y])^2], {0, 4}],
  {x, -2, 2}, {y, -2, 2}, PlotRange → All]
```

Out[ ]=


(2)

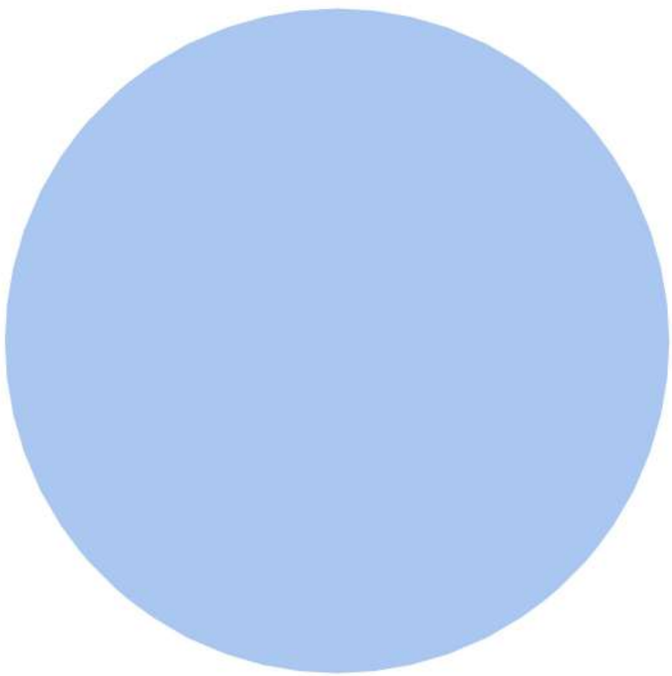
In[ ]:=
```
TabView[Table[a → Plot3D[ⅇ^Sin[x * y], {x, -π, π}, {y, -π, π}, MaxRecursion → a,
    Mesh → All, PlotLabel → "MaxRecursion setting"], {a, 0, 3}], 1]
```

Out[ ]=


(3)

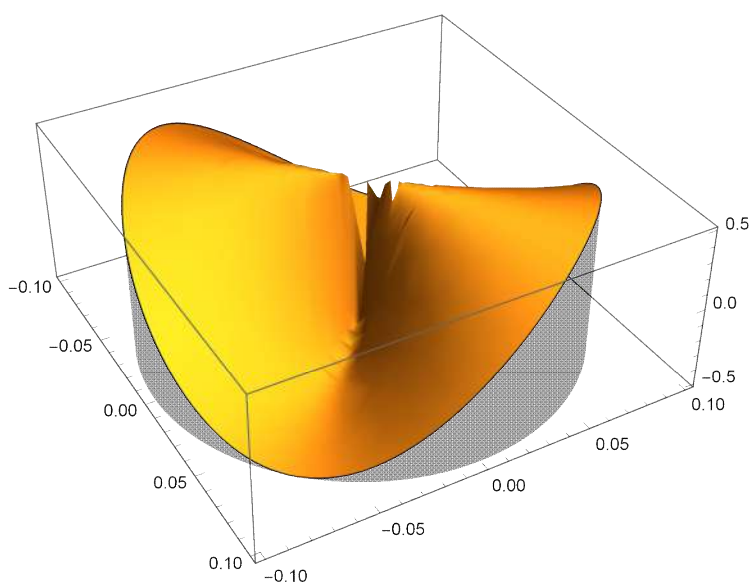
```
In[ ]:= Clear[ℛ, x, y];
ℛ = ImplicitRegion[-.1 ≤ Sqrt[x^2 + y^2] ≤ .1, {x, y}];
Region[ℛ]
```

Out[ ]=



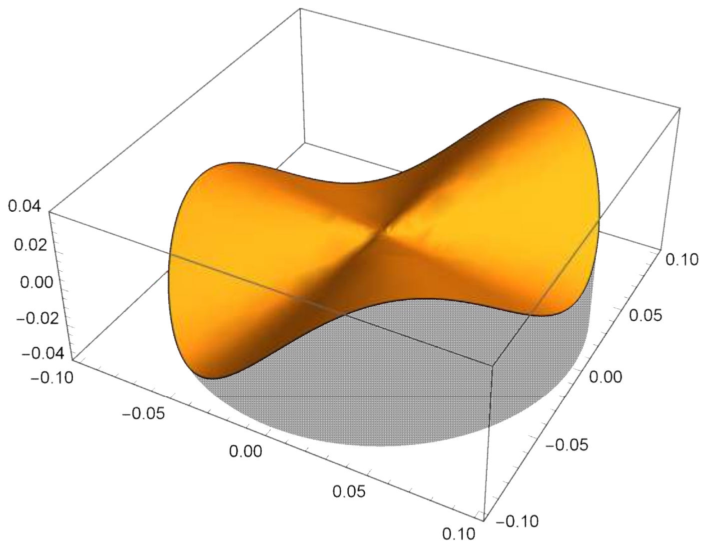
```
In[ ]:= Plot3D[(x * y) / (x^2 + y^2), {x, y} ∈ ℛ, Mesh → None, Filling → Bottom]
```

Out[ ]=



In[ ]:= `Plot3D[(x * y^2) / (x^2 + y^2), {x, y} ∈ ℛ, Mesh → None, Filling → Bottom]`

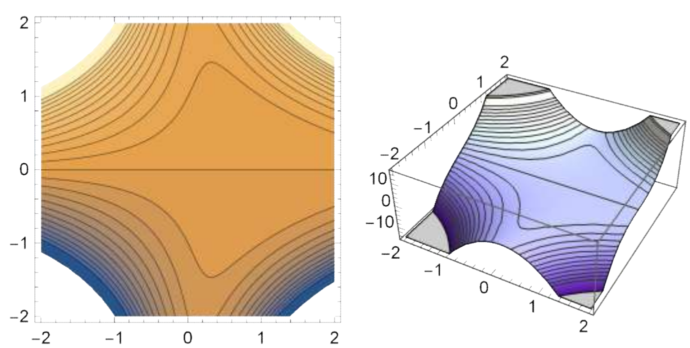
Out[ ]=



In[ ]:= `Clear[ℛ]`

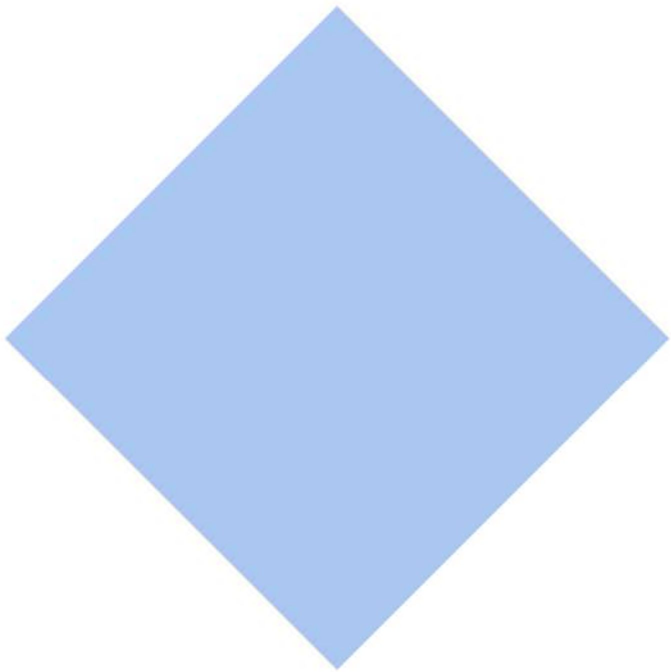
In[ ]:=
```
GraphicsRow[
 {ContourPlot[x^2 * y^3 + ((x - 1)^2) * y, {x, -2, 2}, {y, -2, 2}, Contours → Range[-12, 12]],
  Plot3D[x^2 * y^3 + ((x - 1)^2) * y, {x, -2, 2}, {y, -2, 2}, MeshFunctions → {#3 &},
   Mesh → {Range[- 12, 12]}, ColorFunction → "LakeColors"]}]
```

Out[ ]=



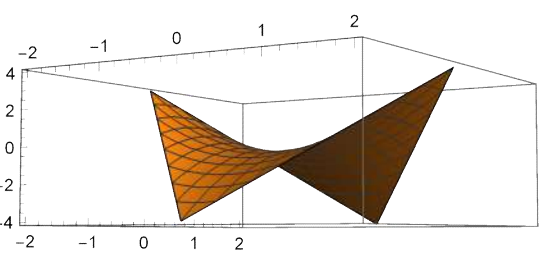
In[ ]:=
```
Clear[x, y, ℛ];
ℛ = ImplicitRegion[y > x - 2 && y < x + 2 && y < -x + 2 && y > -x - 2, {x, y}];
Region[ℛ]
```

Out[ ]=



In[ ]:=
```
Plot3D[x^2 - y^2, {x, y} ∈ ℛ]
```

Out[ ]=



In[ ]:=
```
Integrate[x^2 - y^2, {x, y} ∈ ℛ]
```

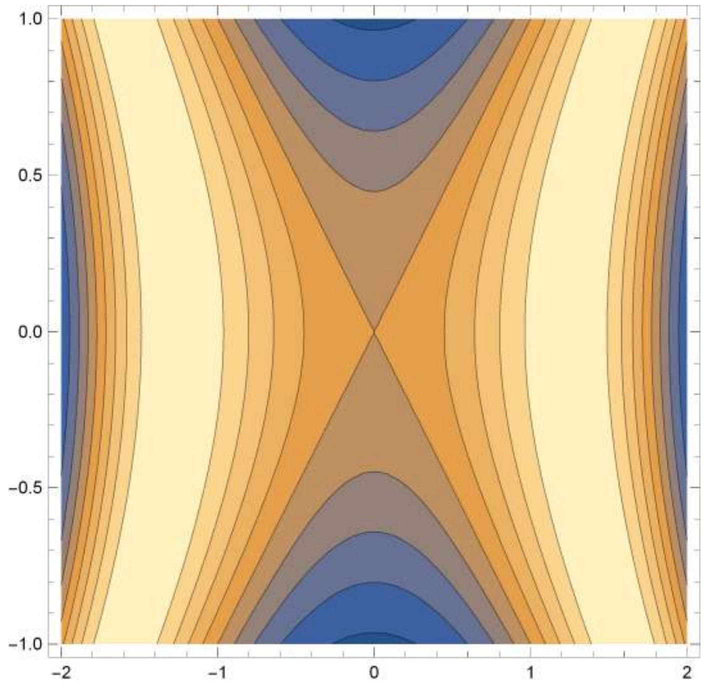
Out[ ]= 0

In[ ]:=
```
Clear[ℛ]
```

(7)

In[ ]:=
```
Clear[f, x, y];
f = Sin[x^2 - y^2];
ContourPlot[f, {x, -2, 2}, {y, -1, 1}]
```

Out[ ]=



In[ ]:=
```
crPts1 = Reduce[{Grad[f, {y, x}] == {0, 0}, -2 ≤ x ≤ 2, -1 ≤ y ≤ 1}, {y, x}]
```

Out[ ]=
$$(y == 0 \,\&\&\, x == 0) \;||\; \left(-1 \le y \le 1 \,\&\&\, \left(x == -\frac{\sqrt{\pi + 2\,y^2}}{\sqrt{2}} \;||\; x == \frac{\sqrt{\pi + 2\,y^2}}{\sqrt{2}}\right)\right)$$

In[ ]:=
```
crPts2 = Reduce[{Grad[f, {x, y}] == {0, 0}, -2 ≤ x ≤ 2, -1 ≤ y ≤ 1}, {x, y}]
```

Out[ ]=
$$(x == 0 \,\&\&\, y == 0) \;||\; \left(-\sqrt{\frac{2+\pi}{2}} \le x < -\sqrt{\frac{\pi}{2}} \,\&\&\, \left(y == -\frac{\sqrt{-\pi + 2\,x^2}}{\sqrt{2}} \;||\; y == \frac{\sqrt{-\pi + 2\,x^2}}{\sqrt{2}}\right)\right) \;||$$
$$\left(x == -\sqrt{\frac{\pi}{2}} \,\&\&\, y == 0\right) \;||\; \left(x == \sqrt{\frac{\pi}{2}} \,\&\&\, y == 0\right) \;||$$
$$\left(\sqrt{\frac{\pi}{2}} < x \le \sqrt{\frac{2+\pi}{2}} \,\&\&\, \left(y == -\frac{\sqrt{-\pi + 2\,x^2}}{\sqrt{2}} \;||\; y == \frac{\sqrt{-\pi + 2\,x^2}}{\sqrt{2}}\right)\right)$$

In[ ]:=
```
Clear[f, crPts, crPts1, crPts2]
```

(8)

(a)

In[ ]:=
```
f = x * Cos[x * y];
crPts = Reduce[Grad[f, {x, y}] == {0, 0}, {x, y}]
```

Out[ ]= False

In[ ]:=
```
ContourPlot[f, {x, -3, 3}, {y, -2, 2},
 MeshFunctions → {Function[{x, y, z}, ∂x f], Function[{x, y, z}, ∂y f]},
 MeshStyle → {Directive[Thick, Dotted], Directive[Thick, Yellow]}, Mesh → {{0}, {0}}]
```

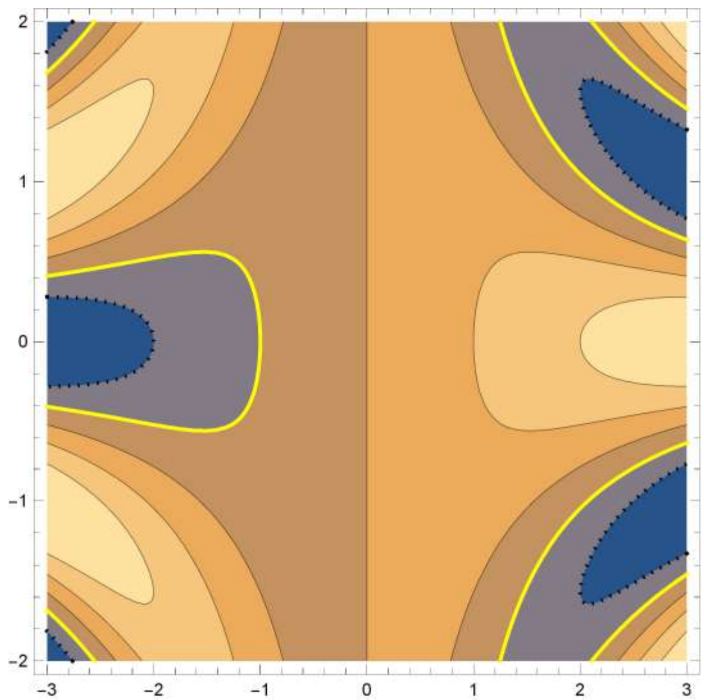
General: −3. is not a valid variable.

MeshFunctions: MeshFunctions−>Function[{x, y, z}, ∂_x f] must be a pure function or a list of pure functions.

General: −2. is not a valid variable.

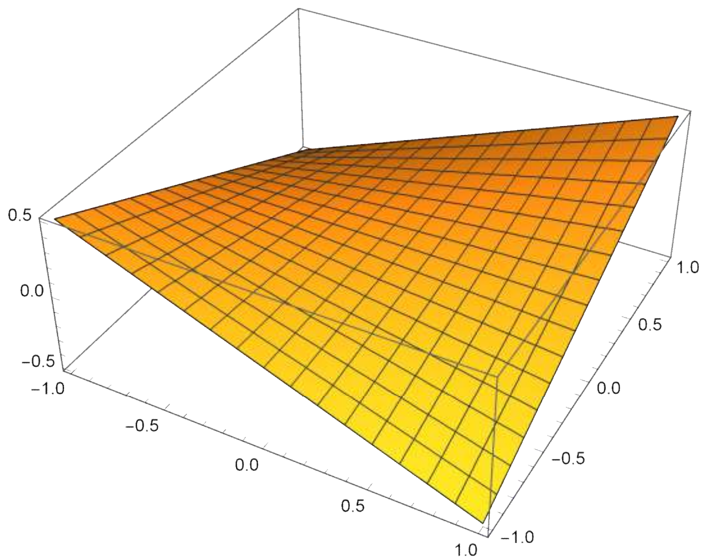
MeshFunctions: MeshFunctions−>Function[{x, y, z}, ∂_y f] must be a pure function or a list of pure functions.

Out[ ]=

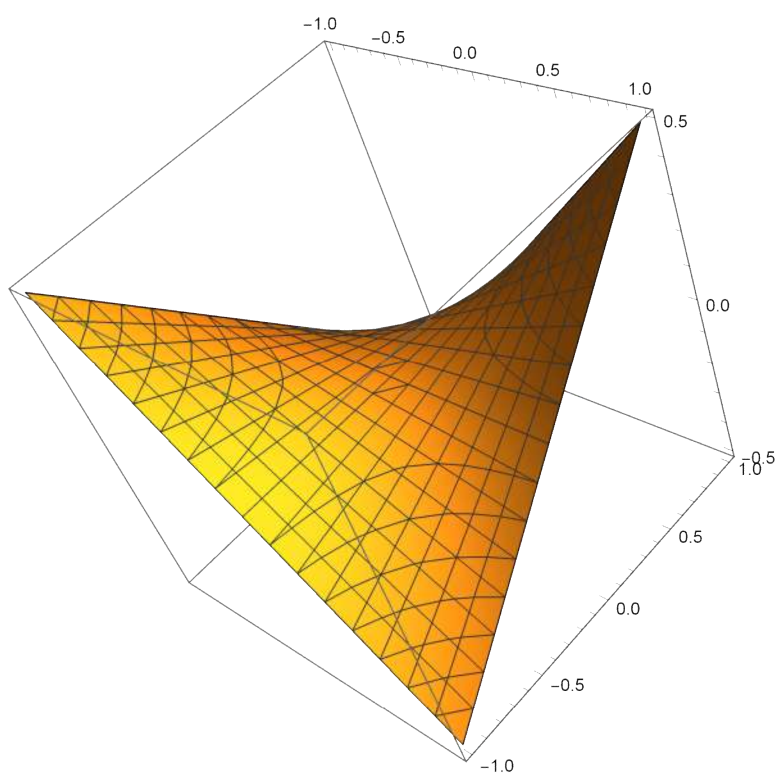


(9)

In[ ]:= **Plot3D[(1 / 2) (x * y), {x, -1, 1}, {y, -1, 1}]**

Out[ ]=


In[ ]:= **ContourPlot3D[z == (1 / 2) (x * y), {x, -1, 1}, {y, -1, 1}, {z, -0.5, 0.5}]**

Out[ ]=


In[ ]:= **Clear[f, g, x, y, z, λ]**

(10)

In[ ]:=
```
f = x^(1/3) * y^(2/3);
g = 40 x + 50 y - 10000;
sols = Solve[Grad[f - λ * g, {x, y, λ}] == {0, 0, 0}, {x, y, λ}, Reals]
```

Out[ ]= $\left\{\left\{x \to \frac{250}{3},\ y \to \frac{400}{3},\ \lambda \to \frac{1}{30 \times 5^{2/3}}\right\}\right\}$

In[ ]:=
```
{x, y} /. sols
```

Out[ ]= $\left\{\left\{\frac{250}{3}, \frac{400}{3}\right\}\right\}$

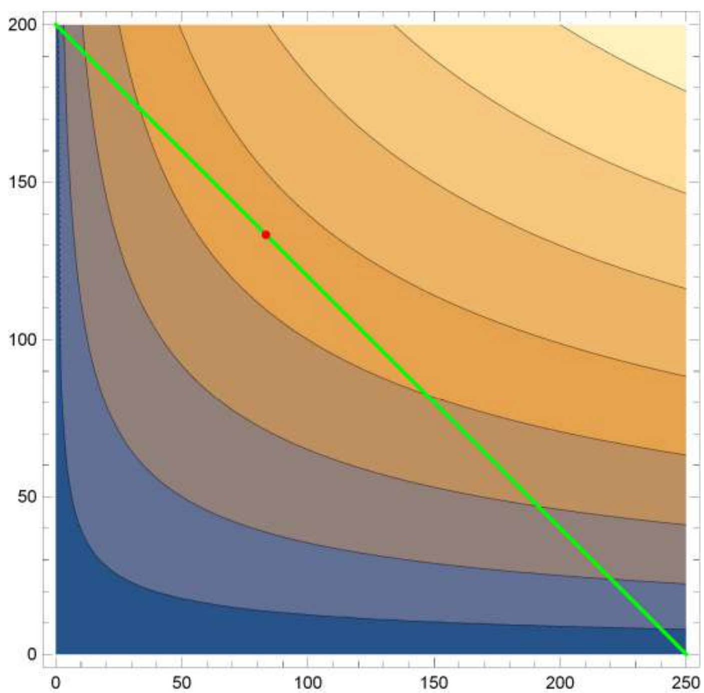
In[ ]:=
```
f /. sols
```

Out[ ]= $\left\{\frac{200 \times 5^{1/3}}{3}\right\}$

In[ ]:=
```
Maximize[{f, g == 0}, {x, y}]
```

Out[ ]= $\left\{-\ -114....,\ \left\{x \to \frac{250}{3},\ y \to \frac{400}{3}\right\}\right\}$

In[ ]:=
```
Show[ContourPlot[f, {x, 0, 250}, {y, 0, 200}],
 ContourPlot[g == 0, {x, 0, 250}, {y, 0, 200}, ContourStyle → Directive[Thick, Green]],
 Epilog → {Red, PointSize[Medium], Point[{x, y} /. sols]}]
```

Out[ ]=


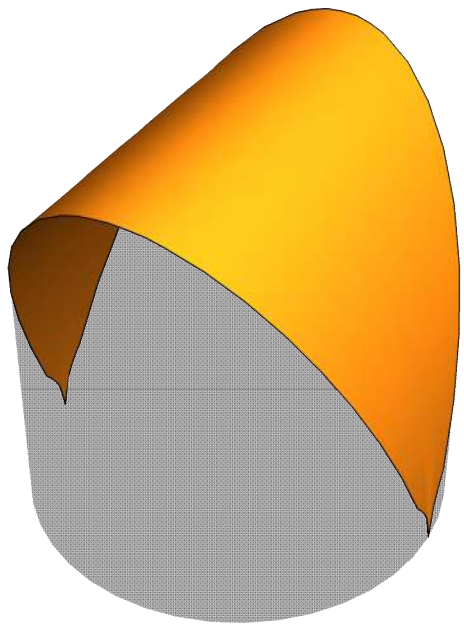
In[ ]:=
```
Clear[f, g, x, y, z, λ]
```

(12)

(a)

In[ ]:=
```
Plot3D[2 Sqrt[1 - x^2], {x, y} ∈ Disk[], Filling → Bottom,
  BoxRatios → Automatic, Boxed → False, Axes → False, Mesh → None]
```

Out[ ]=



In[ ]:=
```
Integrate[2 Sqrt[1 - x^2], {x, y} ∈ Disk[]]
```

Out[ ]= $\frac{16}{3}$

(14)

In[ ]:=
```
RegionPlot3D[z^2 + (Sqrt[x^2 + y^2] - 3)^2 ≤ 1, {x, -2, 4}, {y, 0, 4},
 {z, -1, 1}, MeshFunctions → {Function[{x, y, z}, Norm[{x, y, z}]]},
 MeshShading → Table[ColorData["TemperatureMap"][1 - k], {k, 0, 1, .1}],
 Mesh → 10, BoxRatios → Automatic, Lighting → "Neutral"]
```

Out[ ]=