In[ ]:=
```
c = {{1, 2, -1}, {0, 3, 5}, {0, 4, -1}};
Det[c]
```

Out[ ]= -23

In[ ]:=
```
Clear[c];
m = Array[a##& , {3, 3}];
m // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix}$$

In[ ]:=
```
Det[m]
```

Out[ ]= $-a_{1,3}\,a_{2,2}\,a_{3,1}+a_{1,2}\,a_{2,3}\,a_{3,1}+a_{1,3}\,a_{2,1}\,a_{3,2}-a_{1,1}\,a_{2,3}\,a_{3,2}-a_{1,2}\,a_{2,1}\,a_{3,3}+a_{1,1}\,a_{2,2}\,a_{3,3}$

In[ ]:=
```
Tr[m]
```

Out[ ]= $a_{1,1}+a_{2,2}+a_{3,3}$

EXERCISE

(1)

In[ ]:=
```
Grid[{{1, 2}, {3, 4}}, Dividers → { {1 -> Thick, 3 → Thick}}]
```

Out[ ]=

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$$

In[ ]:=
```
Grid[{{1, 2}, {3, 4}}, Dividers → { {1 -> Thick, 3 → Thick}}] == -2
```

Out[ ]=

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} == -2$$

(2)

In[ ]:=
```
m1 = {{1, 7, 5, 0}, {5, 8, 6, 9}, {2, 1, 6, 4}, {8, 1, 2, 4}};
m2 = ArrayFlatten[{{m1, IdentityMatrix[4]}}];
m2 // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & 7 & 5 & 0 & 1 & 0 & 0 & 0 \\ 5 & 8 & 6 & 9 & 0 & 1 & 0 & 0 \\ 2 & 1 & 6 & 4 & 0 & 0 & 1 & 0 \\ 8 & 1 & 2 & 4 & 0 & 0 & 0 & 1 \end{pmatrix}$$

In[•]:=
```
m3 = RowReduce[m2];
m3 // MatrixForm
```

Out[•]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \frac{46}{993} & -\frac{56}{993} & -\frac{73}{1986} & \frac{325}{1986} \\ 0 & 1 & 0 & 0 & \frac{86}{993} & \frac{68}{993} & -\frac{133}{993} & -\frac{20}{993} \\ 0 & 0 & 1 & 0 & \frac{23}{331} & -\frac{28}{331} & \frac{129}{662} & -\frac{3}{662} \\ 0 & 0 & 0 & 1 & -\frac{148}{993} & \frac{137}{993} & \frac{19}{1986} & -\frac{139}{1986} \end{pmatrix}$$

In[•]:=
```
inv = Take[m3, All, -4];
inv // MatrixForm
```

Out[•]//MatrixForm=

$$\begin{pmatrix} \frac{46}{993} & -\frac{56}{993} & -\frac{73}{1986} & \frac{325}{1986} \\ \frac{86}{993} & \frac{68}{993} & -\frac{133}{993} & -\frac{20}{993} \\ \frac{23}{331} & -\frac{28}{331} & \frac{129}{662} & -\frac{3}{662} \\ -\frac{148}{993} & \frac{137}{993} & \frac{19}{1986} & -\frac{139}{1986} \end{pmatrix}$$

In[•]:=
```
inv.m1 // MatrixForm
```

Out[•]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

7.4 MINORS AND COFACTORS

In[•]:=
```
Clear[m, a]
```

In[•]:=
```
m = Array[a##&, {3, 3}];
m // MatrixForm
```

Out[•]//MatrixForm=

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix}$$

In[•]:=
```
Minors[m] // MatrixForm
```

Out[•]//MatrixForm=

$$\begin{pmatrix} -a_{1,2}\,a_{2,1} + a_{1,1}\,a_{2,2} & -a_{1,3}\,a_{2,1} + a_{1,1}\,a_{2,3} & -a_{1,3}\,a_{2,2} + a_{1,2}\,a_{2,3} \\ -a_{1,2}\,a_{3,1} + a_{1,1}\,a_{3,2} & -a_{1,3}\,a_{3,1} + a_{1,1}\,a_{3,3} & -a_{1,3}\,a_{3,2} + a_{1,2}\,a_{3,3} \\ -a_{2,2}\,a_{3,1} + a_{2,1}\,a_{3,2} & -a_{2,3}\,a_{3,1} + a_{2,1}\,a_{3,3} & -a_{2,3}\,a_{3,2} + a_{2,2}\,a_{3,3} \end{pmatrix}$$

In[•]:=
```
p = {{1, 2, 3}, {4, 5, 6}, {0, 1, 2}};
Minors[p] // MatrixForm
```

Out[•]//MatrixForm=

$$\begin{pmatrix} -3 & -6 & -3 \\ 1 & 2 & 1 \\ 4 & 8 & 4 \end{pmatrix}$$

In[•]:=
```
Clear[p]
```

```
In[•]:= minorsMatrix[m_List?MatrixQ] := Map[Reverse, Minors[m], {0, 1}];

In[•]:= minorsMatrix[m] // MatrixForm
```

Out[•]//MatrixForm=

$$\begin{pmatrix} -a_{2,3}\,a_{3,2}+a_{2,2}\,a_{3,3} & -a_{2,3}\,a_{3,1}+a_{2,1}\,a_{3,3} & -a_{2,2}\,a_{3,1}+a_{2,1}\,a_{3,2} \\ -a_{1,3}\,a_{3,2}+a_{1,2}\,a_{3,3} & -a_{1,3}\,a_{3,1}+a_{1,1}\,a_{3,3} & -a_{1,2}\,a_{3,1}+a_{1,1}\,a_{3,2} \\ -a_{1,3}\,a_{2,2}+a_{1,2}\,a_{2,3} & -a_{1,3}\,a_{2,1}+a_{1,1}\,a_{2,3} & -a_{1,2}\,a_{2,1}+a_{1,1}\,a_{2,2} \end{pmatrix}$$

```
In[•]:= p = {{1, 2, 3}, {4, 5, 6}, {0, 1, 2}};

minorsMatrix[p] // MatrixForm
```

Out[•]//MatrixForm=

$$\begin{pmatrix} 4 & 8 & 4 \\ 1 & 2 & 1 \\ -3 & -6 & -3 \end{pmatrix}$$

```
In[•]:= minorsMatrix[m][[2, 3]]
```

Out[•]= $-a_{1,2}\,a_{3,1}+a_{1,1}\,a_{3,2}$

```
In[•]:= cofactorsMatrix[m_List?MatrixQ] :=
         Table[(-1)^(i + j), {i, Length[m]}, {j, Length[m]}] * minorsMatrix[m]

In[•]:= cofactorsMatrix[m] // MatrixForm
```

Out[•]//MatrixForm=

$$\begin{pmatrix} -a_{2,3}\,a_{3,2}+a_{2,2}\,a_{3,3} & a_{2,3}\,a_{3,1}-a_{2,1}\,a_{3,3} & -a_{2,2}\,a_{3,1}+a_{2,1}\,a_{3,2} \\ a_{1,3}\,a_{3,2}-a_{1,2}\,a_{3,3} & -a_{1,3}\,a_{3,1}+a_{1,1}\,a_{3,3} & a_{1,2}\,a_{3,1}-a_{1,1}\,a_{3,2} \\ -a_{1,3}\,a_{2,2}+a_{1,2}\,a_{2,3} & a_{1,3}\,a_{2,1}-a_{1,1}\,a_{2,3} & -a_{1,2}\,a_{2,1}+a_{1,1}\,a_{2,2} \end{pmatrix}$$

```
In[•]:= cofactorsMatrix[p] // MatrixForm
```

Out[•]//MatrixForm=

$$\begin{pmatrix} 4 & -8 & 4 \\ -1 & 2 & -1 \\ -3 & 6 & -3 \end{pmatrix}$$

```
In[•]:= mat = {{1, 2, 4, 8}, {9, 7, 4, 3}, {9, 8, 0, 2}, {8, 8, 6, 3}};

(1 / Det[mat]) * Transpose[cofactorsMatrix[mat]] // MatrixForm
```

Out[•]//MatrixForm=

$$\begin{pmatrix} -\frac{7}{151} & \frac{82}{151} & -\frac{20}{151} & -\frac{50}{151} \\ \frac{5}{302} & -\frac{94}{151} & \frac{79}{302} & \frac{61}{151} \\ -\frac{19}{604} & \frac{25}{302} & -\frac{119}{604} & \frac{20}{151} \\ \frac{43}{302} & \frac{7}{151} & \frac{15}{302} & -\frac{19}{151} \end{pmatrix}$$

```
In[•]:= Clear[minorsMatrix, cofactorsMatrix]
```

EXERCISE 7.4

(1)

```
In[ ]:= mat1 = {{8, 0, 3, 7}, {9, 4, 2, 9}, {2, 8, 0, 7}, {8, 9, 7, 0}};
        Det[mat1] * Inverse[mat1] // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 434 & -581 & 313 & -20 \\ 203 & -126 & -41 & -51 \\ -757 & 826 & -305 & -96 \\ -356 & 310 & -227 & 64 \end{pmatrix}$$

```
In[ ]:= minorsMatrix[m_List?MatrixQ] := Map[Reverse, Minors[m], {0, 1}];
        cofactorsMatrix[m_List?MatrixQ] :=
          Table[(-1)^(i + j), {i, Length[m]}, {j, Length[m]}] * minorsMatrix[m];
        Transpose[cofactorsMatrix[mat1]] // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 434 & -581 & 313 & -20 \\ 203 & -126 & -41 & -51 \\ -757 & 826 & -305 & -96 \\ -356 & 310 & -227 & 64 \end{pmatrix}$$

```
In[ ]:= Clear[minorsMatrix, cofactorsMatrix, m, a, p]
```

7.5 WORKING WITH LARGE MATRICES

```
In[ ]:= Clear[a, b, c, d]

In[ ]:= s1 = SparseArray[{{1, 1} → a, {2, 3} → b, {5, 2} → c, {6, 7} → d}]
```

Out[ ]= SparseArray[]

Specified elements: 4
Dimensions: {6, 7}
Default: 0
Density: 0.0952
Elements:
{1, 1} → a
{2, 3} → b
{5, 2} → c
{6, 7} → d

```
In[ ]:= s1 // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & d \end{pmatrix}$$

In[ ]:= **s2 = SparseArray[{{1, 1}, {2, 3}, {5, 2}, {6, 7}} → {a, b, c, d}]**

Out[ ]= SparseArray[]

Specified elements: 4
Dimensions: {6, 7}
Default: 0
Density: 0.0952
Elements:
{1, 1} → a
{2, 3} → b
{5, 2} → c
{6, 7} → d

In[ ]:= **s2 // MatrixForm**

Out[ ]//MatrixForm=

$$\begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & d \end{pmatrix}$$

In[ ]:= **s3 = SparseArray[{{1, 1}, {2, 3}, {5, 2}, {6, 7}} → {a, b, c, d}, {8, 10}]**

Out[ ]= SparseArray[]

Specified elements: 4
Dimensions: {8, 10}
Default: 0
Density: 0.05
Elements:
{1, 1} → a
{2, 3} → b
{5, 2} → c
{6, 7} → d

In[ ]:= **s3 // MatrixForm**

Out[ ]//MatrixForm=

$$\begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In[ ]:=
```
s4 = SparseArray[{{1, 1}, {2, 3}, {5, 2}} → {a, b, c}, {5, 5}, 2]
```

Out[ ]= SparseArray[

Specified elements: 3
Dimensions: {5, 5}
Default: 2
Density: 0.12
Elements:
{1, 1} → a
{2, 3} → b
{5, 2} → c
]

In[ ]:=
```
s4 // MatrixForm
```

Out[ ]//MatrixForm=
```
a  2  2  2  2
2  2  b  2  2
2  2  2  2  2
2  2  2  2  2
2  c  2  2  2
```

In[ ]:=
```
Normal[s4]
```

Out[ ]=
```
{{a, 2, 2, 2, 2}, {2, 2, b, 2, 2}, {2, 2, 2, 2, 2}, {2, 2, 2, 2, 2}, {2, c, 2, 2, 2}}
```

In[ ]:=
```
s5 = SparseArray[Table[{2^i, 3 i} → 1, {i, 4}]]
```

Out[ ]= SparseArray[

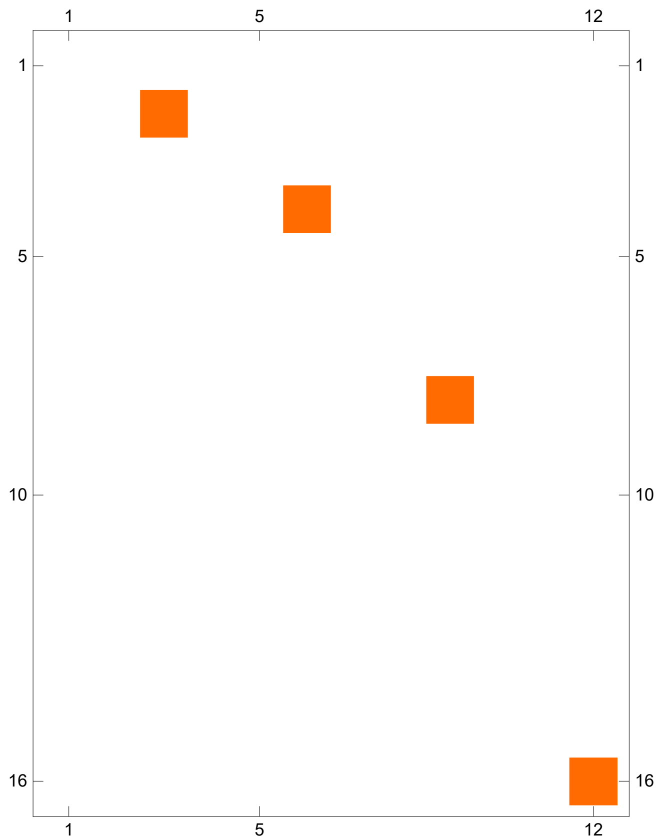
Specified elements: 4
Dimensions: {16, 12}
Default: 0
Density: 0.0208
Elements:
{2, 3} → 1
{4, 6} → 1
{8, 9} → 1
{16, 12} → 1
]

In[ ]:=
```
s5 // MatrixForm
```

Out[ ]//MatrixForm=
```
0  0  0  0  0  0  0  0  0  0  0  0
0  0  1  0  0  0  0  0  0  0  0  0
0  0  0  0  0  0  0  0  0  0  0  0
0  0  0  0  0  1  0  0  0  0  0  0
0  0  0  0  0  0  0  0  0  0  0  0
0  0  0  0  0  0  0  0  0  0  0  0
0  0  0  0  0  0  0  0  0  0  0  0
0  0  0  0  0  0  0  0  1  0  0  0
0  0  0  0  0  0  0  0  0  0  0  0
0  0  0  0  0  0  0  0  0  0  0  0
0  0  0  0  0  0  0  0  0  0  0  0
0  0  0  0  0  0  0  0  0  0  0  0
0  0  0  0  0  0  0  0  0  0  0  0
0  0  0  0  0  0  0  0  0  0  0  0
0  0  0  0  0  0  0  0  0  0  0  0
0  0  0  0  0  0  0  0  0  0  0  1
```

In[ ]:= `MatrixPlot[s5]`

Out[ ]=



In[ ]:= `s6 = SparseArray[Table[{2^i, 3 i, i + 1} → i, {i, 4}]]`

```
SparseArray[  Specified elements: 4
              Dimensions: {16, 12, 5}
              Default: 0
              Density: 0.00417
              Elements:
              {2, 3, 2} → 1
              {4, 6, 3} → 2
              {8, 9, 4} → 3
              {16, 12, 5} → 4  ]
```

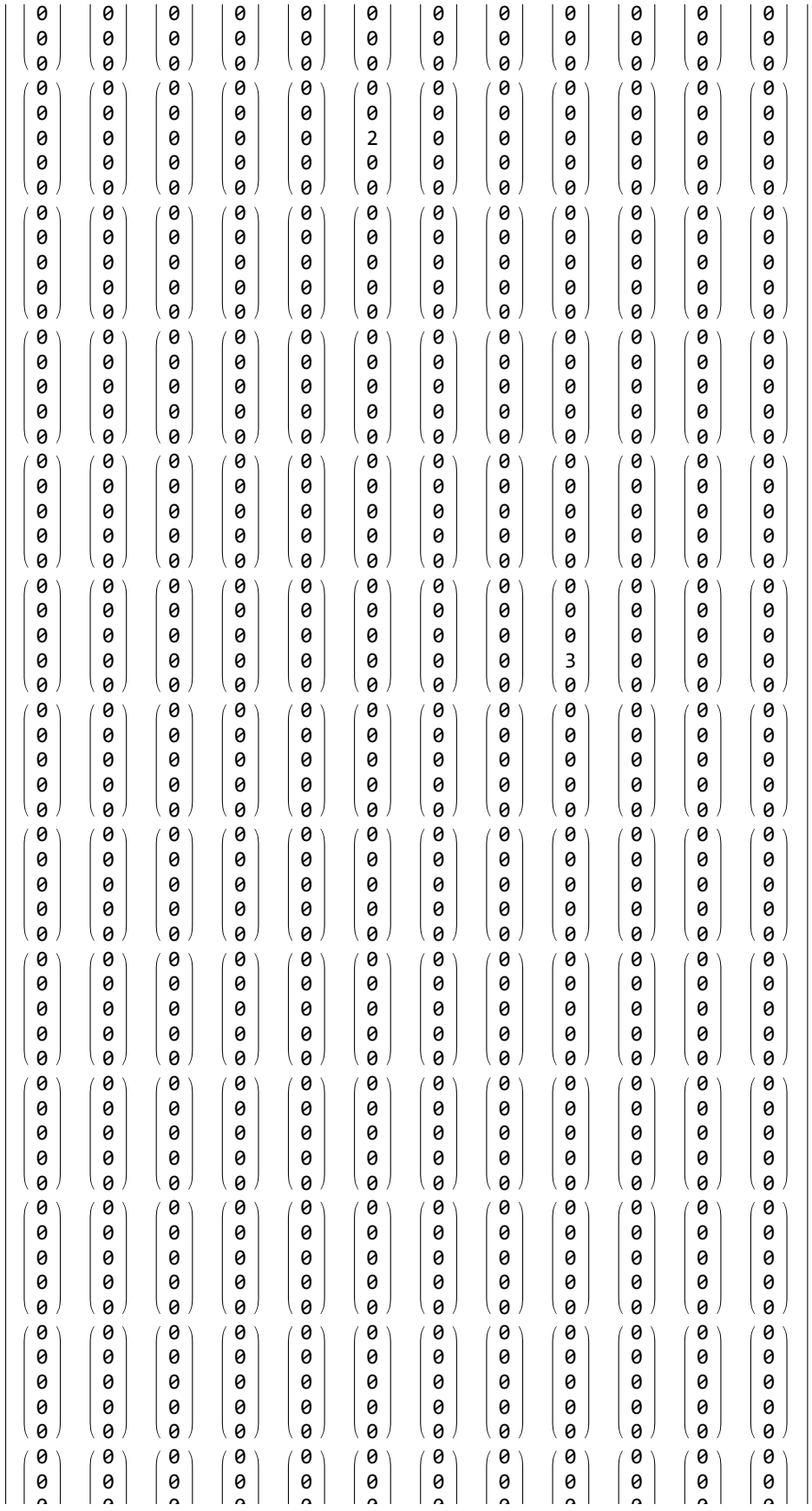
In[ ]:= `s6 // MatrixForm`

Out[ ]//MatrixForm=

```
0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0

0 0 0 0 0 0 0 0 0 0 0 0
0 0 1 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0

0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0
```



```
0  0  0  0  0  0  0  0  0  0  0  0
0  0  0  0  0  0  0  0  0  0  0  0

0  0  0  0  0  0  0  0  0  0  0  0
0  0  0  0  0  0  0  0  0  0  0  0
0  0  0  0  0  0  0  0  0  0  0  0
0  0  0  0  0  0  0  0  0  0  0  0
0  0  0  0  0  0  0  0  0  0  0  4
```

In[∘]:= **ArrayRules[s6]**

Out[∘]= {{2, 3, 2} → 1, {4, 6, 3} → 2, {8, 9, 4} → 3, {16, 12, 5} → 4, {_, _, _} → 0}

In[∘]:= **s7 = SparseArray[{Band[{3, 2}] → 3, Band[{1, 4}] → b}, {6, 6}]**

Out[∘]= SparseArray[Specified elements: 7
Dimensions: {6, 6}]

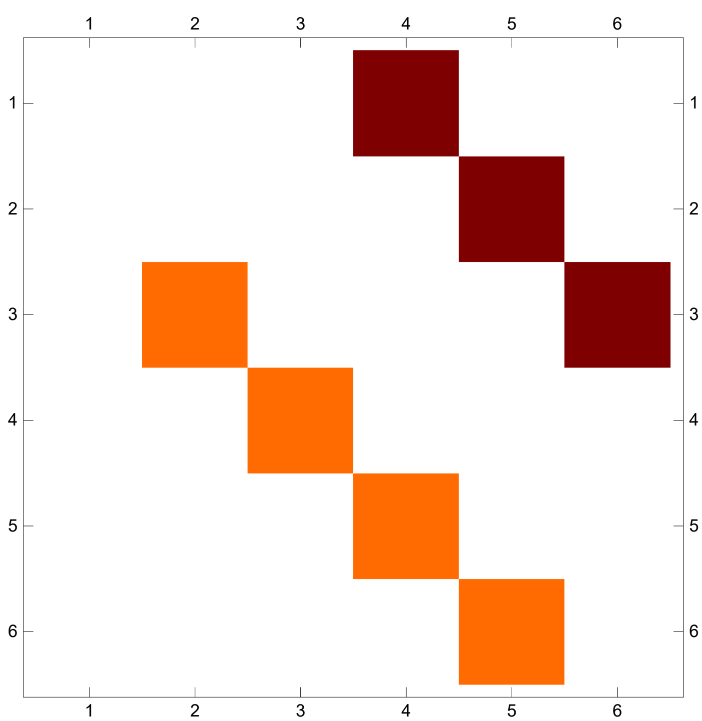
In[∘]:= **s7 // MatrixForm**

Out[∘]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & b & 0 \\ 0 & 3 & 0 & 0 & 0 & b \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \end{pmatrix}$$

In[∘]:= **MatrixPlot[s7]**

Out[∘]=



In[ ]:=
```
s8 = SparseArray[Band[{3, 2}] → {2, 4, 6, 8}, {6, 6}]
```

Out[ ]=
```
SparseArray[ Specified elements: 4
             Dimensions: {6, 6} ]
```

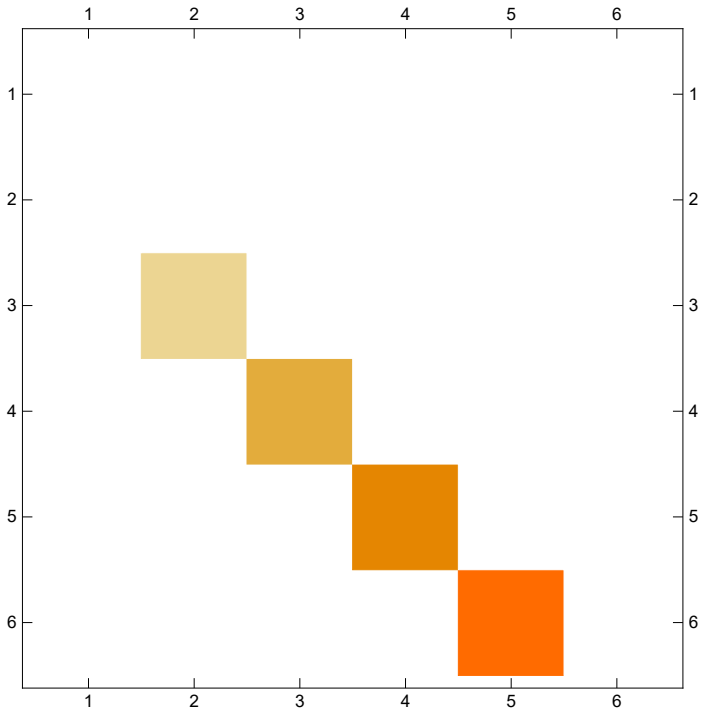
In[ ]:=
```
s8 // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8 & 0 \end{pmatrix}$$

In[ ]:=
```
MatrixPlot[s8]
```

Out[ ]=



EXERCISE 7.5

(1)

In[ ]:=
```
s = SparseArray[Flatten[Table[{i, j} → Abs[i - j], {i, 1, 10}, {j, 1, 10}]]]
```

Out[ ]=
```
SparseArray[ Specified elements: 90
             Dimensions: {10, 10} ]
```

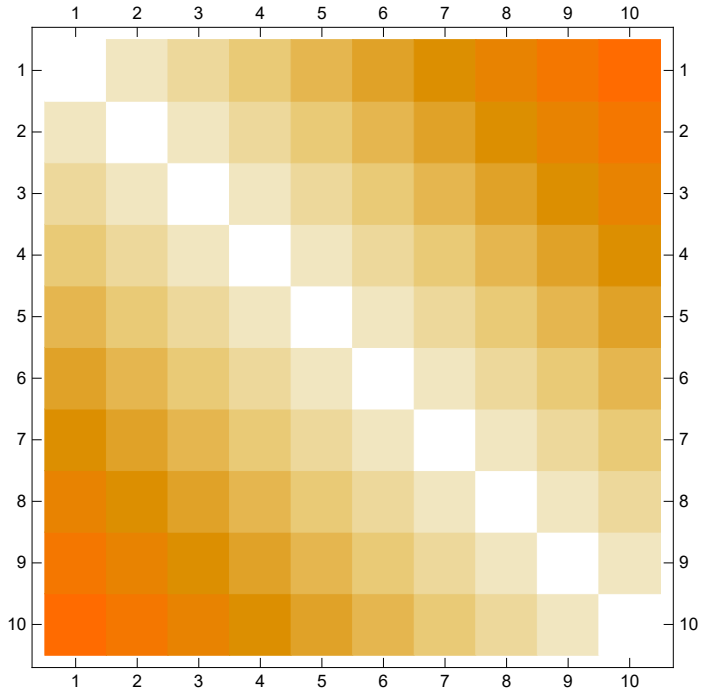
In[ ]:= **s // MatrixForm**

Out[ ]//MatrixForm=

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 2 & 1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 2 & 1 & 0 & 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 & 0 & 1 & 2 & 3 & 4 \\ 6 & 5 & 4 & 3 & 2 & 1 & 0 & 1 & 2 & 3 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 & 1 & 2 \\ 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 & 1 \\ 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \end{pmatrix}$$

In[ ]:= **MatrixPlot[s]**

Out[ ]=



(2)

In[ ]:= **Clear[s]**

In[ ]:= **s = SparseArray[Band[{1, 1}] → Table[{{1, 2}, {3, 4}}, {i, 1, 6}]]**

Out[ ]= SparseArray [Specified elements: 24
Dimensions: {12, 12}]

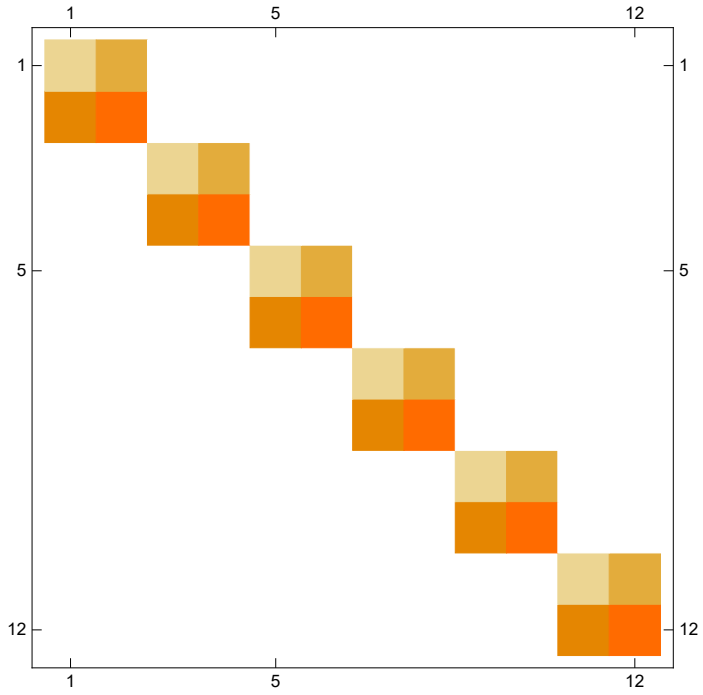
In[ ]:= `s // MatrixForm`

Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 4 \end{pmatrix}$$

In[ ]:= `MatrixPlot[s]`

Out[ ]=



7.6 SOLVING SYSTEM OF LINEAR EQUATIONS

In[ ]:= `Clear[m, x, x1, x2, x3, x4, b]`

In[ ]:= $m = \begin{pmatrix} 1 & 5 & -4 & 1 \\ 3 & 4 & -1 & 2 \\ 3 & 2 & 1 & 5 \\ 0 & -6 & 7 & 1 \end{pmatrix}$; $x = \begin{pmatrix} x1 \\ x2 \\ x3 \\ x4 \end{pmatrix}$; $b = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$;

`m.x == b`

Out[ ]= `{{x1 + 5 x2 - 4 x3 + x4}, {3 x1 + 4 x2 - x3 + 2 x4}, {3 x1 + 2 x2 + x3 + 5 x4}, {-6 x2 + 7 x3 + x4}} == {{1}, {2}, {3}, {4}}`

In[ ]:= `Det[m]`

Out[ ]= 35

```
In[•]:= a = ArrayFlatten[{{m, b}}];
        a // MatrixForm
```

Out[•]//MatrixForm=

$$\begin{pmatrix} 1 & 5 & -4 & 1 & 1 \\ 3 & 4 & -1 & 2 & 2 \\ 3 & 2 & 1 & 5 & 3 \\ 0 & -6 & 7 & 1 & 4 \end{pmatrix}$$

```
In[•]:= RowReduce[a] // MatrixForm
```

Out[•]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -\frac{127}{35} \\ 0 & 1 & 0 & 0 & \frac{141}{35} \\ 0 & 0 & 1 & 0 & \frac{139}{35} \\ 0 & 0 & 0 & 1 & \frac{13}{35} \end{pmatrix}$$

```
In[•]:= LinearSolve[m, b]
```

Out[•]= $\left\{\left\{-\frac{127}{35}\right\}, \left\{\frac{141}{35}\right\}, \left\{\frac{139}{35}\right\}, \left\{\frac{13}{35}\right\}\right\}$

```
In[•]:= Inverse[m].b
```

Out[•]= $\left\{\left\{-\frac{127}{35}\right\}, \left\{\frac{141}{35}\right\}, \left\{\frac{139}{35}\right\}, \left\{\frac{13}{35}\right\}\right\}$

```
In[•]:= Clear[x, b]
```

```
In[•]:= x = {x1, x2, x3, x4}; b = {1, 2, 3, 4};
        m.x == b
```

Out[•]= $\{x1 + 5\,x2 - 4\,x3 + x4,\ 3\,x1 + 4\,x2 - x3 + 2\,x4,\ 3\,x1 + 2\,x2 + x3 + 5\,x4,\ -6\,x2 + 7\,x3 + x4\} == \{1, 2, 3, 4\}$

```
In[•]:= Solve[m.x == b, x]
```

Out[•]= $\left\{\left\{x1 \to -\frac{127}{35},\ x2 \to \frac{141}{35},\ x3 \to \frac{139}{35},\ x4 \to \frac{13}{35}\right\}\right\}$

```
In[•]:= Clear[m, x, x1, x2, x3, x4, b, a]
```

```
In[•]:= m = {{1, 1, 1}, {1, 1, 1}, {1, -1, -1}}; x = {x1, x2, x3}; b = {1, 2, -1};
        m.x == b
```

Out[•]= $\{x1 + x2 + x3,\ x1 + x2 + x3,\ x1 - x2 - x3\} == \{1, 2, -1\}$

```
In[•]:= Solve[m.x == b, x]
```

Out[•]= $\{\}$

```
In[•]:= Clear[m, x, x1, x2, x3, x4, a, b]
```

In[ ]:=
$$m = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & -1 \end{pmatrix}; \quad x = \begin{pmatrix} x1 \\ x2 \\ x3 \end{pmatrix}; \quad b = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix};$$

```
m.x == b
```

Out[ ]=
```
{{x1 + x2 + x3}, {x1 + x2 + x3}, {x1 - x2 - x3}} == {{1}, {2}, {-1}}
```

In[ ]:=
```
a = ArrayFlatten[{{m, b}}];
a // MatrixForm
```

Out[ ]//MatrixForm=
$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & -1 & -1 & -1 \end{pmatrix}$$

In[ ]:=
```
RowReduce[a] // MatrixForm
```

Out[ ]//MatrixForm=
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In[ ]:=
```
LinearSolve[m, b]
```

... LinearSolve: Linear equation encountered that has no solution.

Out[ ]=
```
LinearSolve[{{1, 1, 1}, {1, 1, 1}, {1, -1, -1}}, {{1}, {2}, {-1}}]
```

In[ ]:=
```
Inverse[m].b
```

... Inverse: Matrix {{1, 1, 1}, {1, 1, 1}, {1, –1, –1}} is singular.

Out[ ]=
```
Inverse[{{1, 1, 1}, {1, 1, 1}, {1, -1, -1}}].{{1}, {2}, {-1}}
```

In[ ]:=
```
Clear[m, x, x1, x2, x3, x4, a, b]
```

In[ ]:=
$$m = \begin{pmatrix} 2 & 3 & -4 \\ 4 & 6 & -8 \\ 1 & -1 & -1 \end{pmatrix}; \quad x = \{x1, x2, x3\}; \quad b = \{8, 16, 1\};$$

```
m.x == b
```

Out[ ]=
```
{2 x1 + 3 x2 - 4 x3, 4 x1 + 6 x2 - 8 x3, x1 - x2 - x3} == {8, 16, 1}
```

In[ ]:=
```
Solve[m.x == b, x]
```

... Solve: Equations may not give solutions for all "solve" variables.

Out[ ]=
$$\left\{\left\{x2 \to \frac{4}{7} + \frac{2\,x1}{7},\ x3 \to -\frac{11}{7} + \frac{5\,x1}{7}\right\}\right\}$$

In[ ]:=
```
LinearSolve[m, b]
```

Out[ ]=
$$\left\{\frac{11}{5}, \frac{6}{5}, 0\right\}$$

In[ ]:=
```
Clear[m, x, x1, x2, x3, x4, a, b]
```

In[ ]:=
$$m = \begin{pmatrix} 2 & 3 & -4 \\ 4 & 6 & -8 \\ 1 & -1 & -1 \end{pmatrix};\quad x = \begin{pmatrix} x1 \\ x2 \\ x3 \end{pmatrix};\quad b = \begin{pmatrix} 8 \\ 16 \\ 1 \end{pmatrix};$$

```
m.x == b
```

Out[ ]=
$$\{\{2\,x1 + 3\,x2 - 4\,x3\},\ \{4\,x1 + 6\,x2 - 8\,x3\},\ \{x1 - x2 - x3\}\} == \{\{8\},\ \{16\},\ \{1\}\}$$

In[ ]:=
```
a = ArrayFlatten[{{m, b}}];
a // MatrixForm
```

Out[ ]//MatrixForm=
$$\begin{pmatrix} 2 & 3 & -4 & 8 \\ 4 & 6 & -8 & 16 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

In[ ]:=
```
RowReduce[a] // MatrixForm
```

Out[ ]//MatrixForm=
$$\begin{pmatrix} 1 & 0 & -\frac{7}{5} & \frac{11}{5} \\ 0 & 1 & -\frac{2}{5} & \frac{6}{5} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In[ ]:=
```
Clear[m, x, x1, x2, x3, x4, a, b]
```

In[ ]:=
$$m = \begin{pmatrix} 0 & 2 & 2 & 4 \\ 1 & 0 & -1 & -3 \\ 2 & 3 & 1 & 1 \\ -2 & 1 & 3 & -2 \end{pmatrix};\quad x = \begin{pmatrix} x1 \\ x2 \\ x3 \\ x4 \end{pmatrix};\quad b = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix};$$

```
Det[m]
```

Out[ ]=
```
0
```

In[ ]:=
```
a = ArrayFlatten[{{m, b}}];
a // MatrixForm
```

Out[ ]//MatrixForm=
$$\begin{pmatrix} 0 & 2 & 2 & 4 & 0 \\ 1 & 0 & -1 & -3 & 0 \\ 2 & 3 & 1 & 1 & 0 \\ -2 & 1 & 3 & -2 & 0 \end{pmatrix}$$

In[ ]:=
```
RowReduce[a] // MatrixForm
```

Out[ ]//MatrixForm=
$$\begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In[ ]:=
```
NullSpace[m]
```

Out[ ]=
```
{{1, -1, 1, 0}}
```

EXERCISE 7.6

(1)

In[ ]:=
```
Clear[m, x, x1, x2, x3, x4, a, b]
```

In[ ]:=
```
m = {{1, 2, -3, 4}, {2, -1, 5, 2}, {4, 3, a^2, a + 3}};
RowReduce[m] // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & \frac{57-7\,a+8\,a^2}{5\,\left(1+a^2\right)} \\ 0 & 1 & 0 & \frac{-71+11\,a+6\,a^2}{5\,\left(1+a^2\right)} \\ 0 & 0 & 1 & \frac{-7+a}{1+a^2} \end{pmatrix}$$

(2)

In[ ]:=
```
Clear[m, x, a, b]
```

In[ ]:=
```
m = {{25, 4, -3, 1}, {41, -4, 5, 1}, {53, -2, 7, 1}};
NullSpace[m]
```

Out[ ]= {{-1, 2, 4, 29}}

7.7 VECTOR – SPACES

In[ ]:=
```
v1 = {10, 4, 5};
v2 = {4, 4, 7};
v3 = {8, 1, 0};
b = {1, 2, 3};
m = Transpose[{v1, v2, v3}];
c = LinearSolve[m, b]
```

Out[ ]= $\left\{\frac{3}{2}, -\frac{9}{14}, -\frac{10}{7}\right\}$

In[ ]:=
```
c[[1]] v1 + c[[2]] v2 + c[[3]] v3
```

Out[ ]= {1, 2, 3}

In[ ]:=
```
Clear[m, v1, v2, v3, b, m, c, a]
```

In[ ]:=
```
m = {{10, 4, 8}, {4, 4, 1}, {5, 7, 0}}; b = {{1}, {2}, {3}};
LinearSolve[m, b]
```

Out[ ]= $\left\{\left\{\frac{3}{2}\right\}, \left\{-\frac{9}{14}\right\}, \left\{-\frac{10}{7}\right\}\right\}$

In[ ]:=
```
NullSpace[m]
```

Out[ ]= {}

In[ ]:=
```
Det[m]
```

Out[ ]= 14

In[ ]:=
```
Clear[m, v1, v2, v3, v4, a, b]
```

```
In[•]:= v1 = {2, 1, 15, 10, 6};
        v2 = {2, -5, -3, -2, 6};
        v3 = {0, 5, 15, 10, 0};
        v4 = {2, 6, 18, 8, 6};
        m = {v1, v2, v3, v4};
        RowReduce[m] // MatrixForm
```

Out[•]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & -2 & 3 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

```
In[•]:= Clear[v1, v2, v3, v4, m, a, b, c]
```

```
In[•]:= v1 = {2, 1, 15, 10, 6};
        v2 = {2, -5, -3, -2, 6};
        v3 = {0, 5, 15, 10, 0};
        v4 = {2, 6, 18, 8, 6};
        m = Transpose[{v1, v2, v3, v4}];
        RowReduce[m] // MatrixForm
```

Out[•]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & \frac{5}{6} & 0 \\ 0 & 1 & -\frac{5}{6} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

```
In[•]:= NullSpace[Transpose[{v1, v2, v4}]]
```

```
Out[•]= {}
```

```
In[•]:= NullSpace[m]
```

```
Out[•]= {{-5, 5, 6, 0}}
```

```
In[•]:= Length[NullSpace[m]]
```

```
Out[•]= 1
```

```
In[•]:= MatrixRank[m]
```

```
Out[•]= 3
```

```
In[•]:= mat = {{1, 3, 6}, {9, 7, 4}, {8, 0, 5}};
        NullSpace[mat]
```

```
Out[•]= {}
```

```
In[•]:= Length[NullSpace[mat]]
```

```
Out[•]= 0
```

```
In[•]:= MatrixRank[mat]
```

```
Out[•]= 3
```

In[•]:=
```
mat1 = {{1, 6, 9}, {6, 54, 4}, {8, 89, 5}, {3, 5, -6}};
NullSpace[mat1]
```

Out[•]= {}

In[•]:=
```
MatrixRank[mat1]
```

Out[•]= 3

7.8 EIGENVALUES AND EIGENVECTORS

In[•]:=
```
Clear[m]
```

In[•]:=
```
m = Array[Min, {2, 2}];
m // MatrixForm
```

Out[•]//MatrixForm=

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

In[•]:=
```
{λ1, λ2} = Eigenvalues[m]
```

Out[•]= $\left\{\frac{1}{2}\left(3+\sqrt{5}\right), \frac{1}{2}\left(3-\sqrt{5}\right)\right\}$

In[•]:=
```
{v1, v2} = Eigenvectors[m]
```

Out[•]= $\left\{\left\{\frac{1}{2}\left(-1+\sqrt{5}\right), 1\right\}, \left\{\frac{1}{2}\left(-1-\sqrt{5}\right), 1\right\}\right\}$

In[•]:=
```
m.v1 // Simplify
```

Out[•]= $\left\{\frac{1}{2}\left(1+\sqrt{5}\right), \frac{1}{2}\left(3+\sqrt{5}\right)\right\}$

In[•]:=
```
λ1 * v1 // Simplify
```

Out[•]= $\left\{\frac{1}{2}\left(1+\sqrt{5}\right), \frac{1}{2}\left(3+\sqrt{5}\right)\right\}$

In[•]:=
```
Eigensystem[m]
```

Out[•]= $\left\{\left\{\frac{1}{2}\left(3+\sqrt{5}\right), \frac{1}{2}\left(3-\sqrt{5}\right)\right\}, \left\{\left\{\frac{1}{2}\left(-1+\sqrt{5}\right), 1\right\}, \left\{\frac{1}{2}\left(-1-\sqrt{5}\right), 1\right\}\right\}\right\}$

In[•]:=
```
Eigensystem[N[m]]
```

Out[•]= {{2.61803, 0.381966}, {{0.525731, 0.850651}, {-0.850651, 0.525731}}}

In[•]:=
```
Clear[m]
```

In[•]:=
```
m = Array[Min, {3, 3}];
m // MatrixForm
```

Out[•]//MatrixForm=

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

```
In[•]:= Eigenvalues[m]
```

Out[•]= $\{5.05\ldots, 0.643\ldots, 0.308\ldots\}$

```
In[•]:= Eigenvalues[m, Cubics → True]
```

Out[•]=
$$\left\{2+\frac{7^{2/3}}{\left(\frac{3}{2}\left(9+i\sqrt{3}\right)\right)^{1/3}}+\frac{\left(\frac{7}{2}\left(9+i\sqrt{3}\right)\right)^{1/3}}{3^{2/3}},\right.$$
$$2-\frac{7^{2/3}\left(1-i\sqrt{3}\right)}{2^{2/3}\left(3\left(9+i\sqrt{3}\right)\right)^{1/3}}-\frac{\left(1+i\sqrt{3}\right)\left(\frac{7}{2}\left(9+i\sqrt{3}\right)\right)^{1/3}}{2\times 3^{2/3}},$$
$$\left.2-\frac{7^{2/3}\left(1+i\sqrt{3}\right)}{2^{2/3}\left(3\left(9+i\sqrt{3}\right)\right)^{1/3}}-\frac{\left(1-i\sqrt{3}\right)\left(\frac{7}{2}\left(9+i\sqrt{3}\right)\right)^{1/3}}{2\times 3^{2/3}}\right\}$$

```
In[•]:= Eigenvalues[m] // N
```

Out[•]= {5.04892, 0.643104, 0.307979}

```
In[•]:= Clear[m]
```

```
In[•]:= m = {{2, 1, 0}, {0, 2, 0}, {0, 0, 0}};
        Eigensystem[m]
```

Out[•]= {{2, 2, 0}, {{1, 0, 0}, {0, 0, 0}, {0, 0, 1}}}

```
In[•]:= Clear[m]
```

```
In[•]:= m = {{2, -1, 0}, {-1, 2, 0}, {0, 0, 3}};
        c = Det[λ * IdentityMatrix[3] - m]
```

Out[•]= $-9+15\,\lambda-7\,\lambda^2+\lambda^3$

```
In[•]:= Solve[c == 0, λ]
```

Out[•]= {{λ → 1}, {λ → 3}, {λ → 3}}

```
In[•]:= NullSpace[1 * IdentityMatrix[3] - m]
```

Out[•]= {{1, 1, 0}}

```
In[•]:= NullSpace[3 * IdentityMatrix[3] - m]
```

Out[•]= {{0, 0, 1}, {-1, 1, 0}}

```
In[•]:= Eigensystem[m]
```

Out[•]= {{3, 3, 1}, {{0, 0, 1}, {-1, 1, 0}, {1, 1, 0}}}

```
In[•]:= Clear[m]
```

In[•]:=
$$m = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix};$$

```
{evals, evecs} = Eigensystem[m]
```

Out[•]= {{3, 3, 1}, {{0, 0, 1}, {−1, 1, 0}, {1, 1, 0}}}

In[•]:=
```
d = DiagonalMatrix[evals];
d // MatrixForm
```

Out[•]//MatrixForm=
$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In[•]:=
```
p = Transpose[evecs];
p // MatrixForm
```

Out[•]//MatrixForm=
$$\begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

In[•]:=
```
p.d.Inverse[p] // MatrixForm
```

Out[•]//MatrixForm=
$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

In[•]:=
```
Clear[p, d]
```

In[•]:=
```
{p, d} = JordanDecomposition[m]
```

Out[•]= {{{1, 0, −1}, {1, 0, 1}, {0, 1, 0}}, {{1, 0, 0}, {0, 3, 0}, {0, 0, 3}}}

In[•]:=
```
Map[MatrixForm, %]
```

Out[•]=
$$\left\{\begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}\right\}$$

In[•]:=
```
p.d.Inverse[p] // MatrixForm
```

Out[•]//MatrixForm=
$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

In[•]:=
```
Clear[m, p, d]
```