Nonlinear Control Design for Slightly Non-minimum Phase Systems: Application to V/STOL Aircraft*†

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Geometric nonlinear control techniques cannot be directly applied to flight control due to the presence of unstable zeros, but an approximate input–output linearization procedure, developed for slightly non-minimum phase nonlinear systems, is shown to be effective for V/STOL aircraft.

Key Words—Aerospace control; nonlinear control systems; linearization techniques; nonlinear non-minimum phase systems.

Abstract—There has been a great deal of excitement recently over the development of a theory for explicitly linearizing the input–output response of a nonlinear system using state feedback. One shortcoming of this theory is the inability to deal with *non-minimum phase* nonlinear systems. Highly maneuverable jet aircraft, such as the V/STOL Harrier, belong to an important class of a *slightly* non-minimum phase nonlinear systems. The non-minimum phase character of aircraft is a result of the small body forces that are produced in the process of generating body moments. In this paper, we show that, while straightforward application of the linearization theory to a non-minimum phase system results in a system with a linear input–output response but unstable internal dynamics, designing a feedback control based on a minimum phase approximation to the true system results in a system with desirable properties such as bounded tracking and asymptotic stability.

1. INTRODUCTION

THE METHOD of input–output linearization provides a natural framework for the design of tracking controllers. This technique has in fact been successfully implemented in several practical applications, such as flight control (Asseo, 1973; Meyer and Cicolani, 1975; Meyer and Cicolani, 1980; Singh and Schy, 1980; Lane and Stengel, 1988) and the control of rigid robots by the so-called computed torque method (Freund, 1975). The theory is now well developed and understood (Isidori, 1989).

One of the major obstacles to the direct application of this theory is the fact that it relies on a nonlinear version of pole–zero cancellation. Of course, the nonlinear pole–zero cancellation implicit in these techniques is only a problem when the cancellation is one involving unstable zero dynamics (introduced in Byrnes and Isidori (1984) and made precise in Isidori and Moog (1989) and Isidori (1987)). In this paper, we focus on this problem with specific emphasis on the aircraft control problem.

While several researchers have applied the methods of nonlinear control to the aircraft problem (see Lane and Stengel (1988) for a nice summary), most have neglected the small moment-to-force coupling without proper justification. This coupling provides dynamic effects that cannot be assumed to be bounded! Due to the fact that we are building a closed loop feedback system, we must carefully analyze the effects of this coupling to guarantee that small changes in this parameter do not result in drastic changes in behavior such as the bifurcation behavior that can result from inertial coupling (Hacker and Oprişiu, 1974) or high angles-of-attack (Mehra *et al.*, 1977). In this paper, we provide rigorous justification for the common practice of ignoring the moment-to-force coupling in the design of the controller.

This paper is organized as follows: Section 2 discusses some modeling issues for aircraft dynamics and presents a simplified planar VTOL

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aircraft that will be used in the main discussion. We then work through the details of both exact and approximate input–output linearization for the simplified aircraft and present illustrative simulations showing the qualitative behavior of this system. In Section 3, we develop the rudiments of a theory for the approximate input–output linearization for slightly non-minimum phase systems.

2. ILLUSTRATIVE EXAMPLE: V/STOL AIRCRAFT

2.1. *Aircraft dynamics*

The complete dynamics of an aircraft, taking into account flexibility of the wings and fuselage, aeroelastic effects, the (internal) dynamics of the engine and control surface actuators, and the multitude of changing variables, are quite complex and somewhat unmanageable for the purposes of control. A useful first approximation is to consider the aircraft as a rigid body upon which a set of forces and moments act.

Then, with r, R, and ω being the aircraft position, orientation (rotation matrix), and angular velocity, respectively, the equations of motion can be written as

$$m\ddot{r} = Rf_a + mg, \tag{1}$$

$$J\dot{\omega}_a = \tau_a - \omega_a \times J\omega_a, \tag{2}$$

$$\dot{R} = \omega \times R, \tag{3}$$

where f_a and τ_a are the force and moment acting on the aircraft expressed in the aircraft reference frame. Here, the a subscript means that a quantity is expressed with respect to the aircraft reference frame.

Depending on the aircraft and its mode of flight, the forces and moments can be generated by aerodynamics (lift, drag, and roll–pitch–yaw moments), by momentum exchange (gross thrust vectoring and reaction controls to generate moments), or a combination of the two. The flight envelope of the aircraft is the set of flight conditions for which the pilot and/or the control system can effect the forces and moments needed to remain in the envelope and achieve the desired task.

While the function mapping the control inputs to the forces and moments is a highly nonlinear state-dependent function, it is useful to note that this function can normally be decomposed as

$$\begin{pmatrix} f_a \\ \tau_a \end{pmatrix} = \mathcal{F}(x) + \mathcal{G}(x)u(x, c), \tag{4}$$

where $x \in \mathbb{R}^n$ denotes the state and $c \in \mathbb{R}^m$ denotes the control input and $\mathcal{F}:\mathbb{R}^n \to \mathbb{R}^6$, $\mathcal{G}:\mathbb{R}^n \to \mathbb{R}^{6 \times m}$, and $u:\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ are (con-

tinuous) functions. In particular, for each x in the function $u(x, \cdot):\mathbb{R}^m \to \mathbb{R}^m$ is one-to-one and hence (algebraically) invertible. The value of the function $u(\cdot, \cdot)$ can often be taken to be the components of the force and moment that the actuators were designed to produce.

As a example, consider the YAV-8B Harrier produced by McDonnell Aircraft Company (McDonnell Douglas Corporation, 1982; McDonnell Aircraft Company, 1983) depicted in Fig. 1 (aircraft frame—A, runway frame—R). The Harrier is a single-seat transonic light attack V/STOL (vertical/short takeoff and landing) aircraft powered by a single turbo-fan engine. Four exhaust nozzles on the turbo-fan engine provide the gross thrust for the aircraft. These nozzles (two on each side of the fuselage) can be simultaneously rotated from the aft position (used for conventional wing-borne flight) forward approximately 100 degrees allowing jet-borne flight and nozzle braking. The throttle and nozzle controls thus provide two degrees of freedom of thrust vectoring within the x–z plane of the aircraft. (If the line of action of the gross thrust does not pass through the aircraft center of mass, then this thrust will also produce a net pitching moment.)

In addition to the conventional aerodynamic control surfaces (aileron, stabilator (*stabil*izer–elev*ator*), and rudder for roll, pitch, and yaw moments, respectively), the Harrier also has a reaction control system (RCS) to provide moment generation during jet-borne and transition flight. Reaction valves in the nose, tail, and wingtips use bleed air from the high-pressure compressor of the engine to produce thrust at these points and therefore moments (and forces) at the aircraft center of mass. The design of the aerodynamic and reaction controls provides
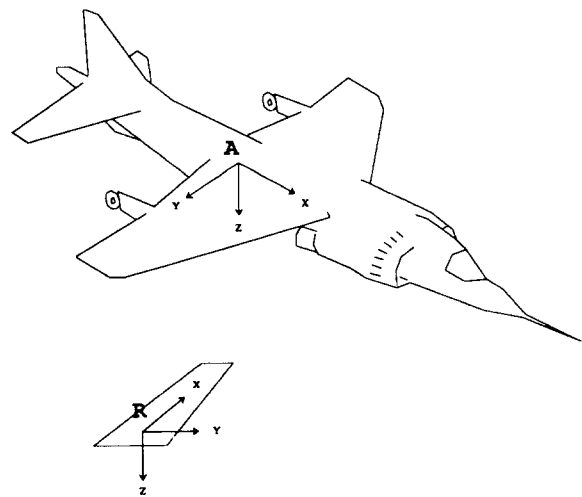

FIG. 1. Aircraft coordinate systems (R—runway, A—aircraft).

complete (three degrees-of-freedom) moment generation throughout the flight envelope of the aircraft. When moments are produced by applying a single force rather than a couple, a nonzero force (proportional to the moment) will be seen at the aircraft center of mass.

Using the throttle, nozzle, roll, pitch, and yaw controls we can produce (within physical limits) any moment and any force in the x-z plane of the aircraft. Therefore, the function $u(\cdot, \cdot)$ for the Harrier can be chosen to map the control inputs to the moment and x-z force on the aircraft (with $u(x, 0) = 0$ so that the force and moment acting on the aircraft with the controls in the zero position are subsumed into $\mathcal{F}(x)$). With this choice of $u(\cdot, \cdot)$, five of the six rows of $\mathcal{G}(x)$ will be the rows of a 5×5 identity matrix. The remaining row will determine the side force f_{ay} and can easily be seen to form a (state-dependent) linear combination of the rolling and yawing moments.

Since the function $u(x, \cdot)$ can be inverted (on its range), we are free to consider u to be the input (control) rather than c. The idea of inverting the algebraic nonlinearities present in the system has been applied to real flight control problems (Meyer and Cicolani, 1975; Meyer and Cicolani, 1980). With these considerations in mind, we see that the dynamics of the aircraft are of the general form

$$\dot{x} = f(x) + \sum_{1}^{m} g_i(x)u_i. \tag{5}$$

The small forces that are produced when moments are commanded result in some important effects. To examine these more closely, consider the geometry of the reaction control system as shown in Fig. 2. Since the roll moment reaction jets create a force that is not perpendicular to the y-axis, the production of a positive rolling moment (to the pilot's right) will also produce a slight acceleration of the aircraft to the left. As we will see, this phenomenon makes the aircraft non-minimum phase.
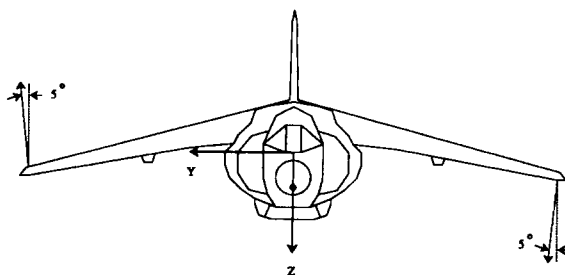
2.2. A simple planar aircraft

For the purpose of illustration, it is particularly useful to consider a simple toy aircraft that has a minimum number of states and inputs but retains many of the features that must be considered when designing control laws for a real aircraft such as the Harrier. Figure 3 shows our prototype PVTOL (planar vertical takeoff and landing) aircraft. This system is the natural restriction of a V/STOL aircraft to jet-borne operation (e.g. hover) in a vertical–lateral plane. The aircraft state is simply the position, x, y, of the aircraft center of mass, the roll angle, θ, of the aircraft, and the corresponding velocities, $\dot{x}$, $\dot{y}$, $\dot{\theta}$. The control inputs, u_1, u_2, are the thrust (directed out the bottom of the aircraft) and the rolling moment. Note that, since this is an illustrative example, we have *not* followed standard variable naming conventions. If desired, one could relabel the system by changing x, y, and θ to $-y$, $-z$, and ϕ, respectively.

We will not explicitly restrict the direction or magnitude of the control inputs. Of course, we would reject any control system design that results in abnormally large control inputs or requires negative thrust (i.e. thrust out the top of the aircraft).

The equations of motion for our PVTOL aircraft are given by

$$\ddot{x} = -\sin \theta u_1 + \epsilon \cos \theta u_2,$$
$$\ddot{y} = \cos \theta u_1 + \epsilon \sin \theta u_2 - 1, \tag{6}$$
$$\ddot{\theta} = u_2.$$

where "-1" is the gravitational acceleration and ϵ is the (small) coefficient giving the coupling between the rolling moment and the lateral acceleration of the aircraft. Note that $\epsilon > 0$ means that applying a (positive) moment to roll left produces an acceleration to the right (positive x). Figure 4 provides a block diagram representation of this dynamical system. In this diagram, the orientation matrix $R(\cdot)$ is now a
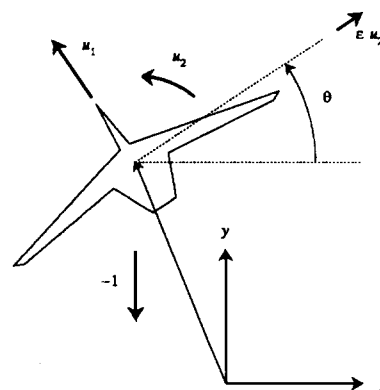

FIG. 2. Reaction control system geometry.

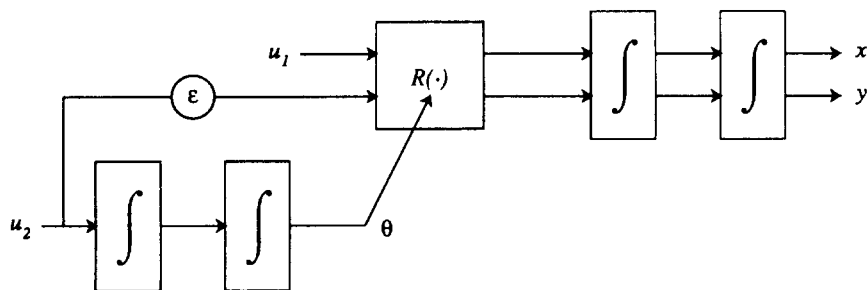FIG. 3. The planar vertical takeoff and landing (PVTOL) aircraft.

FIG. 4. Block diagram of the PVTOL aircraft system.

2×2 rotation matrix depending on the roll angle θ.

The study of this simple planar model provides important insight that extends naturally to the more complicated six degrees-of-freedom aircraft.

2.3. *Exact input–output linearization of the PVTOL aircraft system*

Consider the PVTOL aircraft system given by (6). Since we are interested in controlling the aircraft position, we choose x and y as the outputs to be controlled. We seek a (possibly dynamic) state feedback law of the form

$$u = \alpha(z) + \beta(z)v, \tag{7}$$

such that, for some $\gamma = (\gamma_1, \gamma_2)^T$,

$$
\begin{aligned}
x^{(\gamma_1)} &= v_1, \\
y^{(\gamma_2)} &= v_2.
\end{aligned}
\tag{8}
$$

Here, v is our new input and z is used to denote the entire state of the system (including compensator states, if necessary).

Proceeding in the usual way, we differentiate each output until at least one of the inputs appear. This occurs after differentiating twice and is given by (rewriting the first two equations of (6))

$$
\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \begin{bmatrix} -\sin\theta & \epsilon\cos\theta \\ \cos\theta & \epsilon\sin\theta \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.
\tag{9}
$$

Since the matrix operating on u (the so-called decoupling matrix) is nonsingular (barely—its determinant is $-\epsilon!$), we can linearize (and decouple) the system by choosing the static state feedback law

$$
\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \dfrac{-\sin\theta}{\epsilon} & \dfrac{\cos\theta}{\epsilon} \\ \dfrac{\cos\theta}{\epsilon} & \dfrac{\sin\theta}{\epsilon} \end{bmatrix} \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right).
\tag{10}
$$

The resulting system is

$$
\begin{aligned}
\ddot{x} &= v_1, \\
\ddot{y} &= v_2, \\
\ddot{\theta} &= \frac{1}{\epsilon}(\sin\theta + \cos\theta v_1 + \sin\theta v_2).
\end{aligned}
\tag{11}
$$

This feedback law makes our input–output map linear, but has the unfortunate side-effect of making the dynamics of θ unobservable. In order to guarantee the internal stability of the system, it is not sufficient to look at input–output stability, we must also show that all internal (unobservable) modes of the system are stable as well.

The first step in analyzing the internal stability of the system (11) is to look at the zero dynamics (Byrnes and Isidori, 1984; Isidori and Moog, 1989; Isidori, 1987) of the system. The zero dynamics of a nonlinear system are the internal dynamics of the system subject to the constraint that the outputs (and, therefore, all derivatives of the outputs) are set to zero for all time.

Constraining the outputs and their derivatives to zero by setting $v_1 = v_2 = 0$ (and using appropriate initial conditions), we find the zero dynamics of (11) to be

$$\ddot{\theta} = \frac{1}{\epsilon}\sin\theta. \tag{12}$$

Equation (12) is simply the equation of an undamped pendulum. Figure 5 shows the phase portrait ($\dot{\theta}$ vs θ) of the pendulum (12) with $\epsilon = 1$. The phase portrait for $\epsilon < 0$ is simply a horizontal π-translate of Fig. 5. Thus, for $\epsilon > 0$, the equilibrium point $(\theta, \dot{\theta}) = (0, 0)$ is unstable and the equilibrium point $(\pi, 0)$ is stable but not asymptotically stable and is surrounded by a family of periodic orbits with periods ranging from $2\pi\sqrt{\epsilon}$ to ∞. Outside of these periodic orbits is a family of unbounded trajectories. Thus, depending on the initial condition, the aircraft will either rock from side to side forever or roll continuously in one direction (except at the isolated equilibria).

Nonlinear systems, such as (11), with zero dynamics that are not asymptotically stable are called non-minimum phase. Figure 6 shows the response of the system (11) when (v_1, v_2) is chosen (by a stable feedback law) so that x will track a smooth trajectory from $x = 0$ to $x = 1$ with y remaining at zero. The bottom section of the figure shows snapshots of the PVTOL
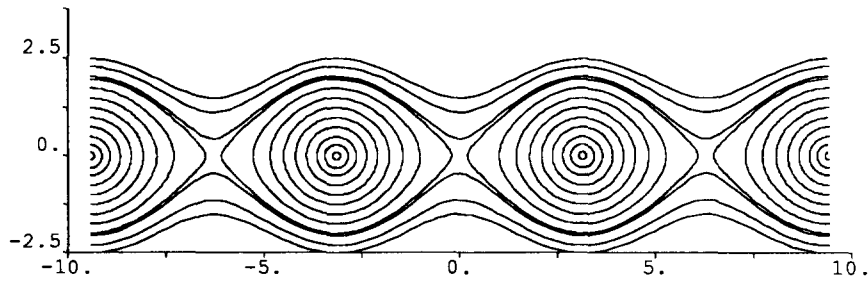
FIG. 5. Phase portrait of an undamped pendulum ($\dot{\theta}$ vs θ, $\epsilon = 1$).

aircraft's position and orientation at 0.2 sec intervals. From the phase portrait of θ (Fig. 6e), we see that the zero dynamics certainly exhibit pendulum-like behavior. Initially, the aircraft rolls left (positive θ) to almost 2π. Then, it rolls right through four revolutions before settling

into a periodic motion about the -3π equilibrium point. Since v_1 and v_2 are zero after $t = 5$, the aircraft continues rocking approximately $\pm\pi$ from the inverted position. Note that, even if the resulting zero dynamics were acceptable, the control law itself is unacceptable
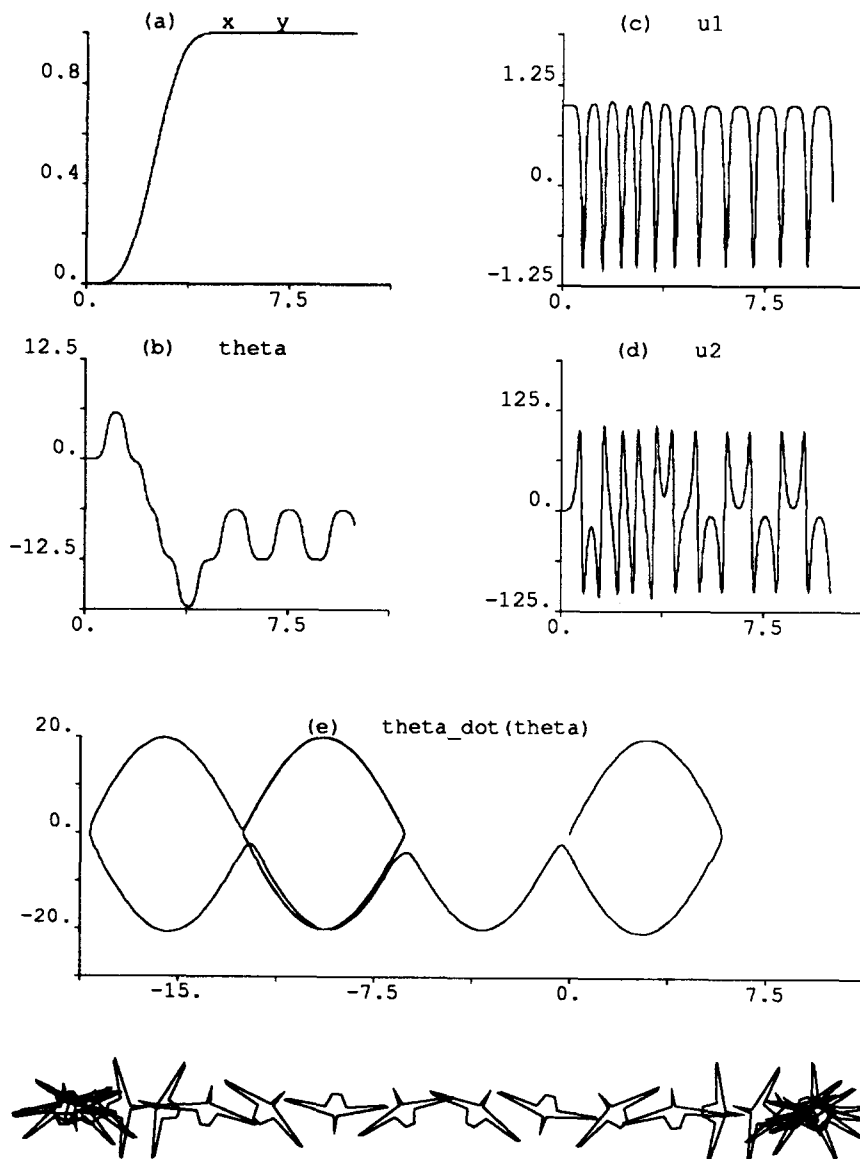

FIG. 6. Response of non-minimum phase system to smooth step input.

since it results in large rapid control inputs and requires the ability to apply negative thrust (i.e. thrust out the top of the aircraft).

From the above analysis and simulations, it is clear that exact input–output linearization of a system such as (6) can produce undesirable results. The source of the problem lies in trying to control modes of the system using inputs that are weakly (ϵ) coupled rather than controlling the system in the way it was designed to be controlled and accepting a performance penalty for the parasitic (ϵ) effects. For our simple PVTOL aircraft, we should control the linear acceleration by vectoring the thrust vector (using moments to control this vectoring) and adjusting its magnitude using the throttle.

2.4. *Approximate input–output linearization of the PVTOL aircraft system using a simplified model*

In the last section we (exactly) linearized input–output map of the PVTOL aircraft system (6). However, due to the small coupling between rolling moments and lateral acceleration, the linearized system had unstable zero dynamics. Thus, while the outputs (the x and y position) can be tracked perfectly, the internal behavior (the aircraft attitude) is not regulated and exhibits unstable behavior.

In this section, we propose controlling the system as if there were no coupling between rolling moments and lateral acceleration (i.e. $\epsilon = 0$). Using this approach to control the true system (6), we expect to see a loss of performance due to the unmodeled dynamics present in the system. In particular, we see that we can guarantee stable asymptotic tracking of constant velocity trajectories and bounded tracking for trajectories with bounded higher order derivatives.

We now model the PVTOL aircraft as ((6) with $\epsilon = 0$)

$$\ddot{x}_m = -\sin \theta u_1,$$

$$\ddot{y}_m = \cos \theta u_1 - 1, \tag{13}$$

$$\ddot{\theta} = u_2,$$

so that there is no coupling between rolling

moments and lateral acceleration. Differentiating the model system outputs, x_m and y_m, we get (analogous to (9))

$$\begin{bmatrix} \ddot{x}_m \\ \ddot{y}_m \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \begin{bmatrix} -\sin \theta & 0 \\ \cos \theta & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \tag{14}$$

Now, however, the matrix multiplying u is singular which implies that there is no static state feedback that will linearize (13). Since u_2 comes into the system (13) through $\ddot{\theta}$, we must differentiate (14) at least two more times. Let u_1 and $\dot{u}_1$ be states (in effect, placing two integrators before the u_1 input) and differentiate (14) twice giving

$$\begin{bmatrix} x_m^{(4)} \\ y_m^{(4)} \end{bmatrix} = \begin{bmatrix} \sin \theta \dot{\theta}^2 u_1 - 2 \cos \theta \dot{\theta} \dot{u}_1 \\ -\cos \theta \dot{\theta}^2 u_1 - 2 \sin \theta \dot{\theta} \dot{u}_1 \end{bmatrix}$$

$$+ \begin{bmatrix} -\sin \theta & -\cos \theta u_1 \\ \cos \theta & -\sin \theta u_1 \end{bmatrix} \begin{bmatrix} \ddot{u}_1 \\ u_2 \end{bmatrix}. \tag{15}$$

The matrix operating on our new inputs, $(\ddot{u}_1, u_2)^T$, has determinant equal to u_1 and therefore is invertible as long as the thrust, u_1, is nonzero. This fact agrees well with our intuition since we know that no amount of rolling will affect the motion of the PVTOL aircraft if there is no thrust to effect an acceleration. Figure 7 shows a block diagram of the model system with u_1 and $\dot{u}_1$ considered as states. Note that each input must go through four integrators to get to the output. Thus, we linearize (13) using the dynamic state feeback law

$$\begin{bmatrix} \ddot{u}_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -\sin \theta & \cos \theta \\ \dfrac{\cos \theta}{u_1} & -\dfrac{\sin \theta}{u_1} \end{bmatrix}$$

$$\times \left(\begin{bmatrix} -\sin \theta \dot{\theta}^2 u_1 + 2 \cos \theta \dot{\theta} \dot{u}_1 \\ \cos \theta \dot{\theta}^2 u_1 + 2 \sin \theta \dot{\theta} \dot{u}_1 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right)$$

$$= \begin{bmatrix} \dot{\theta}^2 u_1 \\ -\dfrac{2\dot{\theta}\dot{u}_1}{u_1} \end{bmatrix}$$

$$+ \begin{bmatrix} -\sin \theta & \cos \theta \\ \dfrac{\cos \theta}{u_1} & -\dfrac{\sin \theta}{u_1} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \tag{16}$$

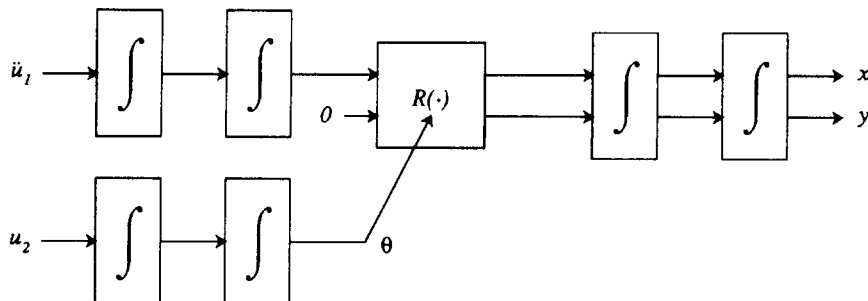Fig. 7. Block diagram of the augmented model PVTOL aircraft system.

resulting in

$$\begin{bmatrix} x_m^{(4)} \\ y_m^{(4)} \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}. \tag{17}$$

Unlike the previous case (equation (11)), the linearized model system does not contain any unobservable (zero) dynamics. Thus, using a stable tracking law for v, we can track an arbitrary trajectory and guarantee that the (model) system will be stable.

Of course, the natural question that comes to mind is: will a control law based on the model system (13) work well when applied to the true system (6)? In the next section, we will show (in a more general setting) that, if ϵ is small enough, then the system will have reasonable properties (such as stability and bounded tracking).

How small is small enough? Figure 8 shows the response of the true system with epsilon ranging from 0 to 0.9 (0.01 is typical during jet-borne flight, i.e. hover, for the Harrier). As in Section 2.3, the desired trajectory is a smooth lateral motion from $x = 0$ to $x = 1$ with the altitude (y) held constant at 0. The figure also shows snapshots of the PVTOL aircraft's position orientation at 0.2 sec intervals for $\epsilon = 0.0$, 0.1 and 0.3. Since the snapshots were taken at uniform intervals, the spacing between successive pictures gives a clue of the aircraft velocity and acceleration. The computer graphics movie of the trajectories provides an even better sense of the system response.

Interestingly, the x response is quite similar to the step response of a non-minimum phase linear system. Note that for ϵ less than approximately 0.6, the oscillations are reasonably damped. Although performance is certainly worse at higher values of ϵ, stability does not appear to be lost until ϵ is in the neighborhood of 0.9. A
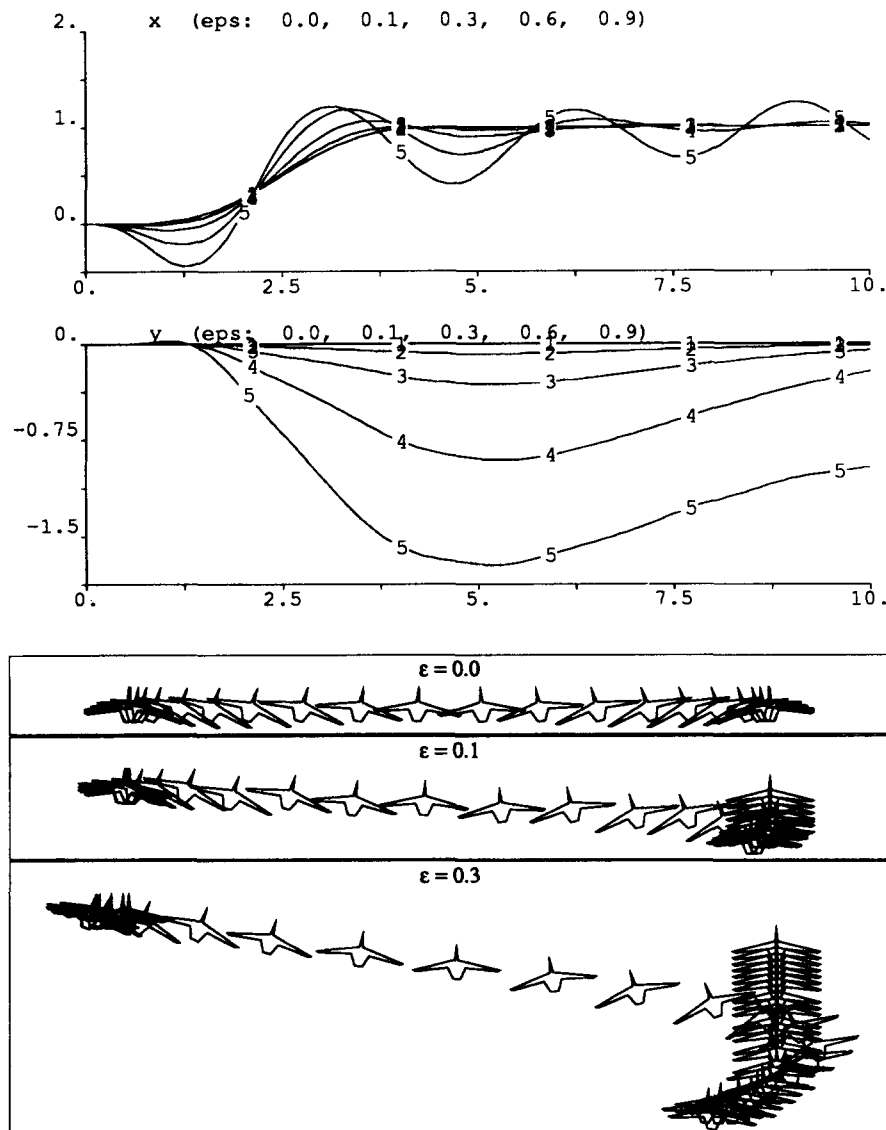

Fig. 8. Response of the true PVTOL aircraft system under the approximate control.

value of 0.9 for ϵ means that the aircraft will experience almost $1g$ (the acceleration of gravity) in the wrong direction when a rolling acceleration of $1\,\mathrm{rad\,sec^{-2}}$ is applied. For the range of ϵ values that will normally be expected, the performance penalty due to approximation is small, almost imperceptible.

Note that, while the PVTOL aircraft system (6) with the approximate control (16) is stable for a large range of ϵ, this control allows the PVTOL aircraft to have a bounded but unacceptable altitude (y) deviation. Since the ground is hard and quite unforgiving and vertical takeoff and landing aircraft are designed to be maneuvered in close proximity to the ground, it is extremely desirable to find a control law that provides exact tracking of altitude if possible. Now, ϵ enters the system dynamics (6) in only one (state-dependent) direction. We therefore expect that one should be able to modify the system (by manipulating the inputs) so that the effects of the ϵ-coupling between rolling moments and aircraft lateral acceleration do not appear in the y-output of the system.

Consider the decoupling matrix of the true PVTOL system (6) given in (9) as

$$\begin{bmatrix} -\sin\theta & \epsilon\cos\theta \\ \cos\theta & \epsilon\sin\theta \end{bmatrix}. \tag{18}$$

To make the y-output independent of ϵ requires that the last row of this decoupling matrix be independent of ϵ. The only legal way to do this is by multiplication on the right (i.e. column operations) by a nonsingular matrix V which corresponds to multiplying the inputs by V^{-1}. In

this case, we see that

$$\begin{bmatrix} -\sin\theta & \epsilon\cos\theta \\ \cos\theta & \epsilon\sin\theta \end{bmatrix}\begin{bmatrix} 1 & -\epsilon\tan\theta \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\sin\theta & \dfrac{\epsilon}{\cos\theta} \\ \cos\theta & 0 \end{bmatrix}, \tag{19}$$

is the desired transformation. Defining new inputs, $\bar{u}$, as

$$\begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{bmatrix} = \begin{bmatrix} 1 & \epsilon\tan\theta \\ 0 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \tag{20}$$

we see that (9) becomes

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$+ \begin{bmatrix} -\sin\theta & \dfrac{\epsilon}{\cos\theta} \\ \cos\theta & 0 \end{bmatrix}\begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{bmatrix}. \tag{21}$$

Following the previous analysis, we set $\epsilon = 0$ and linearize the resulting approximate system using the dynamic feedback law

$$\begin{bmatrix} \ddot{\bar{u}}_1 \\ \bar{u}_2 \end{bmatrix} = \begin{bmatrix} \dot{\theta}^2\bar{u}_1 \\ -\dfrac{2\dot{\theta}\dot{\bar{u}}_1}{\bar{u}_1} \end{bmatrix}$$

$$+ \begin{bmatrix} -\sin\theta & \cos\theta \\ -\dfrac{\cos\theta}{\bar{u}_1} & -\dfrac{\sin\theta}{\bar{u}_1} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}. \tag{22}$$

Note that this control law will approximately linearize the true system. The true system inputs
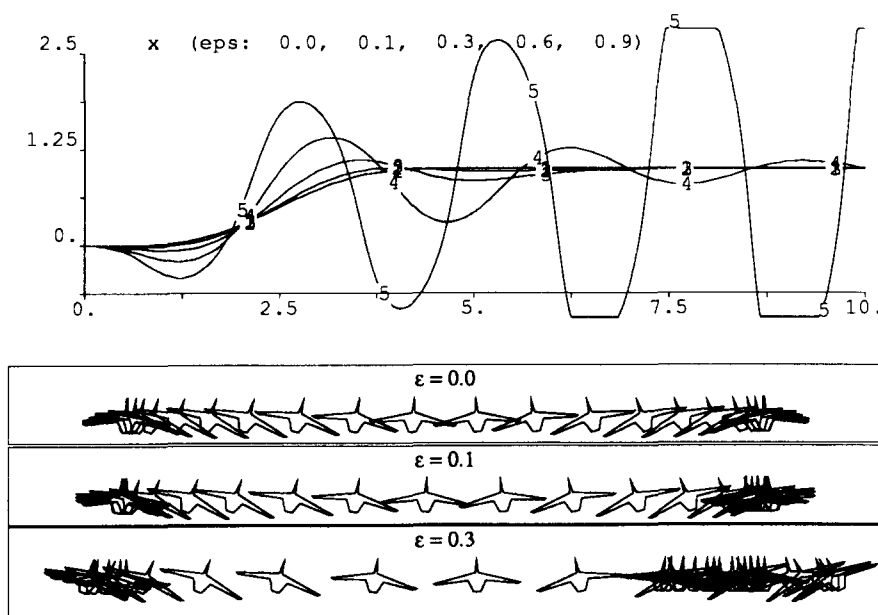

Fig. 9. Response of the true PVTOL aircraft system under the approximate control with input transformation.

are then calculated as

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 & -\epsilon \tan \theta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{bmatrix}. \tag{23}$$

Figure 9 shows the response of the true system using the control law specified by equations (22) and (23) for the same desired trejectory. With this control law, our PVTOL aircraft maintains the altitude as desired and provides stable, bounded lateral (x) tracking for ϵ up to at least 0.6. Note, however, that the system is decidedly unstable for $\epsilon = 0.9$. Since we have forced the error into one direction (i.e. the x-channel), we expect the approximation to be more sensitive to the value of ϵ. In particular, compare the second column of the decoupling matrices of (9) and (21), i.e.

$$\begin{bmatrix} \epsilon \cos \theta \\ \epsilon \sin \theta \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \dfrac{\epsilon}{\cos \theta} \\ 0 \end{bmatrix}. \tag{24}$$

Notice that the first is simply ϵ times a bounded function of θ while the second contains ϵ times an unbounded function of θ (i.e. $1/\cos \theta$). Thus, for (21) with $\epsilon = 0$ to be a good approximation to (21) with non-zero ϵ requires that θ be bounded away from $\pm \pi/2$. This is not a completely unreasonable requirement since most V/STOL aircraft do not have a large enough thrust to weight ratio to maintain level flight with a large roll angle. Since the physical limits of the aircraft usually place constraints on the achievable trajectories, a control law analogous to that defined by (22) and (23) can be used for systems with small ϵ on reasonable trajectories.

3. A FORMAL APPROACH TO THE CONTROL OF SLIGHTLY NON-MINIMUM PHASE SYSTEMS

In this section we will take a more formal approach to the control of systems that are slightly non-minimum phase.

Consider the class of nonlinear systems of the form

$$\left. \begin{array}{l} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{array} \right\}, \tag{25}$$

where $x \in \mathbb{R}^n$, $u, y \in \mathbb{R}^m$, and $f : \mathbb{R}^n \to \mathbb{R}^n$ and the columns of $g : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ are smooth vector fields and $h : \mathbb{R}^n \to \mathbb{R}^m$ is a smooth function with $h(0) = 0$.

In the sequel, we will assume that the origin is an equilibrium point of (25), i.e. $f(0) = 0$, and will consider x in an open neighborhood, U, of the origin, i.e. the analysis will be local. All statements that we make, such as the existence of certain diffeomorphisms, will be assumed merely to hold in U. Also, when we say that a

function is zero, it vanishes on U, and when we say it is non-zero, we mean that it is bounded away from zero on U.

While we will not precisely define slightly non-minimum phase systems, the concept is easy enough to explain. The reader may wish to review the definition of the zero dynamics for nonlinear systems (and the concept of minimum phase) in Isidori and Moog (1989).

3.1. Single-input-single-output (SISO) case

Consider first the single-input-single-output (SISO) case. Suppose that $L_g h(x) = \epsilon \psi(x)$ for some scalar function $\psi(x)$ with $\epsilon > 0$ small. In other words, the relative degree of the system is one, but is very close to being greater than one. Here, $L_g h(x)$ is the Lie derivative of $h(\cdot)$ along $g(\cdot)$ and is defined to be

$$L_g h(x) = \frac{\partial h(x)}{\partial x} g(x). \tag{26}$$

Now, define two systems in normal form (see Byrnes and Isidori (1988)) using the following two sets of local diffeomorphisms of $x \in \mathbb{R}^n$

$$(\xi^T, \eta^T)^T = (\xi_1 := h(x), \eta_1(x), \ldots, \eta_{n-1}(x))^T, \tag{27}$$

and

$$(\xi^T, \bar{\eta}^T)^T$$
$$= (\xi_1 := h(x), \xi_2(x) := L_f h(x), \bar{\eta}_1(x), \ldots, \bar{\eta}_{n-2}(x))^T, \tag{28}$$

with

$$\frac{\partial \eta_i}{\partial x} g(x) = 0, \quad i = 1, \ldots, n-1, \tag{29}$$

and

$$\frac{\partial \bar{\eta}_i}{\partial x} g(x) = 0, \quad i = 1, \ldots, n-2. \tag{30}$$

System 1 (true system).

$$\left. \begin{array}{l} \dot{\xi}_1 = L_f h(x) + L_g h(x)u \\ \dot{\eta} = q(\xi, \eta) \end{array} \right\}. \tag{31}$$

System 2 (approximate system).

$$\left. \begin{array}{l} \dot{\xi}_1 = \xi_2 \\ \dot{\xi}_2 = L_f^2 h(x) + L_g L_f h(x)u \\ \dot{\bar{\eta}} = \bar{q}(\xi, \bar{\eta}) \end{array} \right\}. \tag{32}$$

Note that the system (31) represents the system (25) in normal form and the dynamics of $q(0, \eta)$ represent the zero dynamics of the system (25). System (32) does not represent the system (25), since in the $(\xi, \bar{\eta})$ coordinates of

(28), the dynamics of (25) are given by

$$
\left.\begin{array}{l}
\dot{\xi}_1 = \xi_2 + L_g h(x) u \\
\dot{\xi}_2 = L_f^2 h(x) + L_g L_f h(x) u \\
\dot{\bar{\eta}} = \bar{q}(\xi, \bar{\eta})
\end{array}\right\}. \tag{33}
$$

Informally, we call the system (25) slightly non-minimum phase if the true system (31) (with non-zero ϵ) is non-minimum phase but the approximate system (32) (with $\epsilon = 0$) is minimum phase. Since $L_g h(x) = \epsilon \psi(x)$, we may think of the system (32) as a perturbation of the system (31) ($\equiv$(33)).

Of course, there are two difficulties with exact input–output linearization of (31):

• The input–output linearization requires a large control effort since the linearizing control is

$$
\begin{aligned}
u^*(x) &= \frac{1}{L_g h(x)} (-L_f h(x) + v) \\
&= \frac{-L_f h(x) + v}{\epsilon \psi(x)}.
\end{aligned} \tag{34}
$$

This could present difficulties in the instance that there is saturation at the control inputs.

• If (31) is non-minimum phase, a tracking control law producing a linear input–output response may result in unbounded η states.

Our prescription for the approximate input–output linearization of the system (31) is to use the input–output linearizing control law for the approximate system (32); namely

$$
u_a^* = \frac{1}{L_g L_f h(x)} (-L_f^2 h(x) + v), \tag{35}
$$

where v is chosen depending on the control task. For instance, if y is required to track y_d, we choose v as

$$
v = \ddot{y}_d + \alpha_1(\dot{y}_d - \xi_2) + \alpha_0(y_d - \xi_1) \tag{36}
$$

$$
= \ddot{y}_d + \alpha_1(\dot{y}_d - L_f h(x)) + \alpha_0(y_d - h(x)). \tag{37}
$$

Using (35) and (36) in (33) along with the definitions

$$
\begin{aligned}
e_1 &= \xi_1 - y_d, \\
e_2 &= \xi_2 - \dot{y}_d,
\end{aligned} \tag{38}
$$

yields

$$
\left.\begin{array}{l}
\dot{e}_1 = e_2 + \epsilon \psi(x) u_a^*(x) \\
\dot{e}_2 = -\alpha_1 e_2 - \alpha_0 e_1 \\
\dot{\bar{\eta}} = \bar{q}(\xi, \bar{\eta})
\end{array}\right\}. \tag{39}
$$

As we will see below, exponential stability of the zero dynamics of the approximate system (i.e. $\dot{\bar{\eta}} = \bar{q}(0, \bar{\eta})$) combined with the designed stability of the error system will guarantee overall stability of the system and yield approximate tracking.

The preceding discussion may be generalized to the case when the difference in the relative degrees between the true system and the approximate system is greater than one. For example, if

$$
\begin{aligned}
L_g h(x) &= \epsilon \psi_1(x), \\
L_g L_f h(x) &= \epsilon \psi_2(x), \\
&\vdots \\
L_g L_f^{\gamma-2} h(x) &= \epsilon \psi_{\gamma-1}(x),
\end{aligned} \tag{40}
$$

but $L_g L_f^\gamma h(x)$ is not of order ϵ, we define

$$
\begin{aligned}
(\xi^T, \bar{\eta}^T) &= (h(x), L_f h(x), \ldots, L_f^{\gamma-1} h(x), \bar{\eta}^T)^T \\
&\in \mathbb{R}^n,
\end{aligned} \tag{41}
$$

and note that the true system is

$$
\begin{aligned}
\dot{\xi}_1 &= \xi_2 + \epsilon \psi_1(x) u, \\
&\vdots \\
\dot{\xi}_{\gamma-1} &= \xi_\gamma + \epsilon \psi_{\gamma-1}(x) u, \\
\dot{\xi}_\gamma &= L_f^\gamma h(x) + L_g L_f^{\gamma-1} h(x) u, \\
\dot{\bar{\eta}} &= \bar{q}(\xi, \bar{\eta}).
\end{aligned} \tag{42}
$$

The approximate (minimum phase) system (with $\epsilon = 0$) is given by

$$
\begin{aligned}
\dot{\xi}_1 &= \xi_2, \\
&\vdots \\
\dot{\xi}_{\gamma-1} &= \xi_\gamma, \\
\dot{\xi}_\gamma &= L_f^\gamma h(x) + L_g L_f^{\gamma-1} h(x) u, \\
\dot{\bar{\eta}} &= \bar{q}(\xi, \bar{\eta}).
\end{aligned} \tag{43}
$$

The approximate tracking control law for (43) is

$$
\begin{aligned}
u_a &= \frac{1}{L_g L_f^{\gamma-1} h(x)} (-L_f^\gamma h(x) + y_d^{(\gamma)} + \alpha_{\gamma-1}(y_d^{(\gamma-1)} \\
&\quad - L_f^{\gamma-1} h(x)) + \cdots + \alpha_0(y_d - y)).
\end{aligned} \tag{44}
$$

The following theorem provides a bound for the performance of this control when applied to the true system.

Theorem 3.1. Suppose that

• the zero dynamics of the approximate system (43) are locally exponentially stable and
• the functions $\psi(x) u_a(x)$ are locally Lipschitz continuous.

Then, for ϵ sufficiently small and for desired trajectories with sufficiently small values and derivatives $(y_d, \dot{y}_d, \ldots, y_d^{(\gamma)})$, the states of the system (42) will be bounded and the tracking error

$$
|e_1| := |\xi_1 - y_d| \le k\epsilon, \tag{45}
$$

for some $k < \infty$.

Proof. Define the trajectory error, $e \in \mathbb{R}^{\gamma}$, to be

$$\begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_\gamma \end{bmatrix} = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_\gamma \end{bmatrix} - \begin{bmatrix} y_d \\ \dot{y}_d \\ \vdots \\ y_d^{(\gamma-1)} \end{bmatrix}. \quad (46)$$

Then, the system (42) with the approximate tracking control (44) may be expressed as

$$\begin{bmatrix} \dot{e}_1 \\ \vdots \\ \dot{e}_{\gamma-1} \\ \dot{e}_\gamma \end{bmatrix} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & & 1 \\ -\alpha_0 & -\alpha_1 & \cdots & -\alpha_{\gamma-1} \end{bmatrix}$$

$$\times \begin{bmatrix} e_1 \\ \vdots \\ e_{\gamma-1} \\ e_\gamma \end{bmatrix} + \epsilon \begin{bmatrix} \psi_1(x) \\ \vdots \\ \psi_{\gamma-1}(x) \\ 0 \end{bmatrix} u_a(x), \quad (47)$$

$$\dot{\bar{\eta}} = \bar{q}(\xi, \bar{\eta}),$$

or, compactly,

$$\begin{aligned} \dot{e} &= Ae + \epsilon\psi(x)u_a(x), \\ \dot{\bar{\eta}} &= \bar{q}(\xi, \bar{\eta}). \end{aligned} \quad (48)$$

Since the zero dynamics are assumed to be exponentially stable, a converse Lyapunov theorem implies the existence of a Lyapunov function (see, e.g. Hahn, 1967) $v_2(\bar{\eta})$ for the system

$$\dot{\bar{\eta}} = \bar{q}(0, \bar{\eta}), \quad (49)$$

satisfying

$$\begin{aligned} k_1 |\bar{\eta}|^2 &\leq v_2(\bar{\eta}) \leq k_2 |\bar{\eta}|^2, \\ \frac{\partial v_2}{\partial \bar{\eta}} \bar{q}(0, \bar{\eta}) &\leq -k_3 |\bar{\eta}|^2, \\ \left| \frac{\partial v_2}{\partial \bar{\eta}} \right| &\leq k_4 |\bar{\eta}|, \end{aligned} \quad (50)$$

for some positive constants k_1, k_2, k_3, and k_4.

We first show that e and $\bar{\eta}$ are bounded. To this end, consider as a Lyapunov function for the error system (48)

$$v(e, \bar{\eta}) = e^T Pe + \mu v_2(\bar{\eta}), \quad (51)$$

where $P > 0$ is chosen so that

$$A^T P + PA = -I, \quad (52)$$

(possible since $\dot{e} = Ae$ is stable) and μ is a positive constant to be determined later.

Note that, by assumption, y_d and its first $(\gamma - 1)$ derivatives are bounded,

$$|\xi| \leq |e| + b_d, \quad (53)$$

the functions, $\bar{q}(\xi, \bar{\eta})$ and $\psi(x)u_a(x)$ are locally

Lipschitz with $\psi(0)u_a(0) = 0$,

$$\begin{aligned} |\bar{q}(\xi^1, \bar{\eta}^1) - \bar{q}(\xi^2, \bar{\eta}^2)| \\ \leq l_q(|\xi^1 - \xi^2| + |\bar{\eta}^1 - \bar{\eta}^2|), \quad (54) \end{aligned}$$

$$|2P\psi(x)u_a(x)| \leq l_u |x|, \quad (55)$$

and x is a local diffeomorphism of $(\xi, \bar{\eta})$,

$$|x| \leq l_x(|\xi| + |\bar{\eta}|). \quad (56)$$

Using these bounds and the properties of $v_2(\cdot)$, we have

$$\begin{aligned} \frac{\partial v_2}{\partial \bar{\eta}} \bar{q}(\xi, \bar{\eta}) &= \frac{\partial v_2}{\partial \bar{\eta}} \bar{q}(0, \bar{\eta}) \\ &\quad + \frac{\partial v_2}{\partial \bar{\eta}} (\bar{q}(\xi, \bar{\eta}) - \bar{q}(0, \bar{\eta})) \\ &\leq -k_3 |\bar{\eta}|^2 + k_4 l_q |\bar{\eta}|(|e| + b_d). \quad (57) \end{aligned}$$

Taking the derivative of $v(\cdot, \cdot)$ along the trajectories of (48), we find

$$\begin{aligned} \dot{v} &= -|e|^2 + 2\epsilon e^T P \psi(x) u_a(x) \\ &\quad + \mu \frac{\partial v_2}{\partial \bar{\eta}} \bar{q}(\xi, \bar{\eta}) \\ &\leq -|e|^2 + \epsilon |e| l_u l_x (|e| + b_d + |\bar{\eta}|) \\ &\quad + \mu(-k_3 |\bar{\eta}|^2 + k_4 l_q |\bar{\eta}| (|e| + b_d)) \\ &\leq -\left(\frac{|e|}{2} - \epsilon l_u l_x b_d \right)^2 + (\epsilon l_u l_x b_d)^2 \\ &\quad -\left(\frac{|e|}{2} - (\epsilon l_u l_x + \mu k_4 l_q) |\bar{\eta}| \right)^2 \\ &\quad + (\epsilon l_u l_x + \mu k_4 l_q)^2 |\bar{\eta}|^2 \\ &\quad -\mu k_3 \left(\frac{|\bar{\eta}|}{2} - \frac{k_4 l_q b_d}{k_3} \right)^2 + \mu \frac{(k_4 l_q b_d)^2}{k_3} \\ &\quad -(\tfrac{1}{2} - \epsilon l_u l_x) |e|^2 - \tfrac{3}{4} \mu k_3 |\bar{\eta}|^2 \\ &\leq -(\tfrac{1}{2} - \epsilon l_u l_x) |e|^2 \\ &\quad -(\tfrac{3}{4} \mu k_3 - (\epsilon l_u l_x + \mu k_4 l_q)^2) |\bar{\eta}|^2 \\ &\quad + (\epsilon l_u l_x b_d)^2 + \mu \frac{(k_4 l_q b_d)^2}{k_3}. \quad (58) \end{aligned}$$

Define

$$\mu_0 = \frac{k_3}{4(l_u l_x + k_4 l_q)^2}. \quad (59)$$

Then, for all $\mu \leq \mu_0$ and all $\epsilon \leq \min(\mu, 1/4l_u l_x)$, we have

$$\begin{aligned} \dot{v} &\leq -\frac{|e|^2}{4} - \frac{\mu k_3 |\bar{\eta}|^2}{2} + \frac{\mu(k_4 l_q b_d)^2}{k_3} \\ &\quad + (\epsilon l_u l_x b_d)^2. \quad (60) \end{aligned}$$

Thus, $\dot{v} < 0$ whenever $|\bar{\eta}|$ or $|e|$ is large which implies that $|\bar{\eta}|$ and $|e|$ and, hence, $|\xi|$ and $|x|$, are bounded. The above analysis is valid in a neighborhood of the origin. By choosing b_d sufficiently small and with appropriate initial

conditions, we can guarantee the state will remain in a small neighborhood. Using the boundedness of x and the continuity of $\psi(x)u_a(x)$, we see that

$$\dot{e} = Ae + \epsilon\psi(x)u_a(x), \tag{61}$$

is an exponentially stable linear system driven by an order ϵ input. Thus, we conclude that the tracking error, e, converges to a ball of order ϵ. $\square$

When the control objective is stabilization and the approximate system has no zero dynamics we can do much better. In this case, one can show then that the control law that stabilizes the approximate system also stabilizes the original system.

Suppose that the approximate system has no zero dynamics, i.e.

$$L_g h(x) = \epsilon\psi_1(x),$$
$$L_g L_f h(x) = \epsilon\psi_2(x),$$
$$\vdots \tag{62}$$
$$L_g L_f^{n-2} h(x) = \epsilon\psi_{n-1}(x).$$

Define

$$\xi = (h(x), L_f h(x), \ldots, L_f^{n-1} h(x))^T \in \mathbb{R}^n, \tag{63}$$

and write the approximate system

$$\dot{\xi}_1 = \xi_2,$$
$$\vdots \tag{64}$$
$$\dot{\xi}_n = L_f^n h(x) + L_g L_f^{n-1} h(x)u,$$

and the stabilizing control law

$$u_s(x) = \frac{1}{L_g L_f^{n-1} h(x)}$$
$$\times (-L_f^n h(x) - \alpha_{n-1}\xi_n - \cdots - \alpha_0\xi_1) \tag{65}$$

$$= \frac{1}{L_g L_f^{n-1} h(x)} (-L_f^n h(x)$$
$$- \alpha_{n-1} L_f^{n-1} h(x) - \cdots - \alpha_0 h(x)). \tag{66}$$

The true system in these coordinates is given by

$$\dot{\xi}_1 = \xi_2 + \epsilon\psi_1(x)u,$$
$$\vdots \tag{67}$$
$$\dot{\xi}_{n-1} = \xi_n + \epsilon\psi_{n-1}(x)u,$$
$$\dot{\xi}_n = L_f^n h(x) + L_g L_f^{n-1} h(x)u.$$

Using $u_s(x)$ from (65)) in (67) yields

$$\begin{bmatrix} \dot{\xi}_1 \\ \vdots \\ \dot{\xi}_{n-1} \\ \dot{\xi}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & & 1 \\ -\alpha_0 & -\alpha_1 & \cdots & -\alpha_{n-1} \end{bmatrix}$$
$$\times \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_{n-1} \\ \xi_n \end{bmatrix} + \epsilon \begin{bmatrix} \psi_1(x) \\ \vdots \\ \psi_{n-1}(x) \\ 0 \end{bmatrix} u_s(x). \tag{68}$$

Letting $\psi(x) = (\psi_1(x), \ldots, \psi_{n-1}(x), 0)^T$, we can state the following.

Theorem 3.2. Suppose that $\psi(x)u_s(x)$ is Lipschitz in x and that $\psi(0)u_s(0) = 0$. Then, the system (68) is exponentially stable for ϵ sufficiently small.

Proof. The stabilized system (68) can be compactly written as

$$\dot{\xi} = A\xi + \epsilon\psi(x)u_s(x). \tag{69}$$

Choose as Lyapunov function $v = \xi^T P \xi$ with $A^T P + PA = -I$. Then, using the bounds analogous to (55) and (56), the derivative of v along trajectories of (69) is given by

$$\dot{v} = -|\xi|^2 + 2\epsilon P\psi(x)u_s(x)$$
$$\leq -(1 - \epsilon l_u l_x) |\xi|^2. \tag{70}$$

Thus, for all $\epsilon < \epsilon_0 := 1/l_u l_x$, the system (69) is exponentially stable. $\square$

3.2. Generalization to MIMO systems

We now consider MIMO systems of the form (25) which, for the sake of convenience, we rewrite as

$$\left.\begin{array}{l} \dot{x} = f(x) + g_1(x)u_1 + \cdots + g_m(x)u_m \\ y_1 = h_1(x) \\ \vdots \\ y_m = h_m(x) \end{array}\right\}. \tag{71}$$

Let γ_i be the relative degree of the ith output, i.e. we need to differentiate y_i at least γ_i times before at least one of the inputs appears in the right-hand side. Then, we have

$$y_i^{(\gamma_i)} = L_f^{\gamma_i} h_i + L_{g_1} L_f^{\gamma_i - 1} h_i u_1$$
$$+ \cdots + L_{g_m} L_f^{\gamma_i - 1} h_i u_m, \quad i = 1, \ldots, m. \tag{72}$$

The decoupling matrix is defined to be $A(x) \in \mathbb{R}^{m \times m}$ with

$$A(x) = \begin{bmatrix} L_{g_1} L_f^{\gamma_1 - 1} h_1 & \cdots & L_{g_m} L_f^{\gamma_1 - 1} h_1 \\ \vdots & \ddots & \vdots \\ L_{g_1} L_f^{\gamma_m - 1} h_m & \cdots & L_{g_m} L_f^{\gamma_m - 1} h_m \end{bmatrix}, \tag{73}$$

so that

$$\begin{bmatrix} y_1^{(\gamma_1)} \\ \vdots \\ y_m^{(\gamma_m)} \end{bmatrix} = \begin{bmatrix} L_f^{\gamma_1} h_1 \\ \vdots \\ L_f^{\gamma_m} h_m \end{bmatrix} + A(x) \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}. \quad (74)$$

If the decoupling matrix $A(x)$ is non-singular, the control law

$$u(x) = A(x)^{-1} \left(- \begin{bmatrix} L_f^{\gamma_1} h_1 \\ \vdots \\ L_f^{\gamma_m} h_m \end{bmatrix} + v \right), \quad (75)$$

with $v \in \mathbb{R}^m$ linearizes (and decouples) the system (71) resulting in

$$\begin{bmatrix} y_1^{(\gamma_1)} \\ \vdots \\ y_m^{(\gamma_m)} \end{bmatrix} = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix}. \quad (76)$$

To take up the ideas of Section 3.1, we will first consider the case when $A(x)$ is non-singular but is close to being singular, that is, its smallest singular value is uniformly small for $x \in U$. Definitions of zero dynamics for MIMO systems are considerably more subtle than those for SISO systems and the reader may wish to review them in Isidori and Moog (1989) and Isidori (1987) before proceeding further. Since $A(x)$ is close to being singular, i.e. it is close in norm to a matrix of rank $m - 1$, we may transform $A(x)$ using elementary column operations to get

$$\bar{A}^0(x) = A(x) V^0(x)$$
$$= [\bar{a}_{.1}^0(x) \cdots \bar{a}_{.m-1}^0(x) \epsilon \bar{a}_{.m}^0(x)], \quad (77)$$

where each $\bar{a}_{.i}^0$ is a column of $\bar{A}^0$. This corresponds to redefining the inputs to be

$$\begin{bmatrix} \bar{u}_1^0 \\ \vdots \\ \bar{u}_m^0 \end{bmatrix} = (V^0(x))^{-1} \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}. \quad (78)$$

Now, the normal form of the system (71) is given by defining the following local diffeomorphism of $x \in \mathbb{R}^n$,

$$(\xi^T, \eta^T) = (\xi_1^1 = h_1(x), \dots, \xi_{\gamma_1}^1 = L_f^{\gamma_1 - 1} h_1(x),$$
$$\xi_1^2 = h_2(x), \dots, \xi_{\gamma_2}^2 = L_f^{\gamma_2 - 1} h_2(x),$$
$$\vdots$$
$$\xi_1^m = h_m(x), \dots, \xi_{\gamma_m}^m = L_f^{\gamma_m - 1} h_m(x), \eta^T)$$
$$\quad (79)$$

and noting that

$$\dot{\xi}_1^1 = \xi_2^1$$
$$\vdots$$
$$\dot{\xi}_{\gamma_1}^1 = b_1(\xi, \eta) + \sum_{j=1}^{m-1} \bar{a}_{1j}^0 \bar{u}_j^0 + \epsilon \bar{a}_{1m}^0 \bar{u}_m^0$$
$$\dot{\xi}_1^2 = \xi_2^2 \quad (80)$$
$$\vdots$$
$$\dot{\xi}_{\gamma_m}^m = b_m(\xi, \eta) + \sum_{j=1}^{m-1} \bar{a}_{mj}^0 \bar{u}_j^0 + \epsilon \bar{a}_{mm}^0 \bar{u}_m^0$$
$$\dot{\eta} = q(\xi, \eta) + P(\xi, \eta) \bar{u}^0,$$

where $b_i(\xi, \eta)$ is $L_f^{\gamma_i} h_i(x)$ for $i = 1, \dots, m$ in (ξ, η) coordinates. The zero dynamics of the system are the dynamics of the η coordinates in the subspace $\xi = 0$ with the linearizing control law of (75) (with $v = 0$) substituted, i.e.

$$\dot{\eta} = q(0, \eta) - P(0, \eta)(\bar{A}^0(0, \eta))^{-1} b(0, \eta). \quad (81)$$

We will assume that (71) is non-minimum phase, that is to say that the origin of (81) is not stable.

Now, an approximation to the system is obtained by setting $\epsilon = 0$ in (80). The resultant decoupling matrix is singular and the procedure for linearization by (dynamic) state feedback (the so-called dynamic extension process) proceeds by differentiating (80) and noting that

$$\dot{x} = f(x) + \bar{g}_1^0(x) \bar{u}_1^0 + \cdots + \bar{g}_m^0(x) \bar{u}_m^0, \quad (82)$$

where

$$[\bar{g}_1^0(x) \cdots \bar{g}_m^0(x)] = [g_1(x) \cdots g_m(x)] V^0(x). \quad (83)$$

We then get

$$\begin{bmatrix} y_1^{(\gamma_1 + 1)} \\ \vdots \\ y_{m-1}^{(\gamma_{m-1} + 1)} \\ y_m^{(\gamma_m + 1)} \end{bmatrix} = b^1(x, \bar{u}_1^0, \dots, \bar{u}_{m-1}^0)$$
$$+ A^1(x, \bar{u}_1^0, \dots, \bar{u}_{m-1}^0)$$
$$\times \begin{bmatrix} \dot{\bar{u}}_1^0 \\ \vdots \\ \dot{\bar{u}}_{m-1}^0 \\ \bar{u}_m^0 \end{bmatrix} \quad (84)$$
$$= b^1(x^1) + A^1(x^1) u^1, \quad (85)$$

where

$$u^1 = (\dot{\bar{u}}_1^0, \dots, \dot{\bar{u}}_{m-1}^0, \bar{u}_m^0)^T, \quad (86)$$

is the new input and

$$x^1 = (x^T, \bar{u}_1^0, \dots, \bar{u}_{m-1}^0)^T, \quad (87)$$

is the extended state. Note the appearance of terms of the form $\dot{\bar{u}}_1^0, \dots, \dot{\bar{u}}_{m-1}^0$ in (84). The system (84) is linearizable (and decouplable) if $A^1(x^1)$ is nonsingular. We will assume that the

singular values of A^1 are all of order 1 (i.e. A^1 is uniformly nonsingular) so that (84) is linearizable. The normal form for the approximate system is determined by obtaining a local diffeomorphism of the states x, $\bar{u}_1^0, \ldots, \bar{u}_{m-1}^0$ ($\in \mathbb{R}^{n+m-1}$) given by

$$(\xi^T, \bar{\eta}^T) = (\xi_1^1 = h_1(x), \ldots, \xi_{\gamma_1}^1 = L_f^{\gamma_1-1}h_1(x),$$

$$\xi_{\gamma_1+1}^1 = L_f^{\gamma_1}h_1(x) + \sum_{j=1}^{m-1} \bar{a}_{1j}^0\bar{u}_j^0,$$

$$\xi_1^2 = h_2(x), \ldots, \xi_{\gamma_2}^2 = L_f^{\gamma_2-1}h_2(x),$$

$$\xi_{\gamma_2+1}^2 = L_f^{\gamma_2}h_2(x) + \sum_{j=1}^{m-1} \bar{a}_{2j}^0\bar{u}_j^0,$$

$$\vdots$$

$$\xi_1^m = h_m(x), \ldots, \xi_{\gamma_m}^m = L_f^{\gamma_m-1}h_m(x),$$

$$\xi_{\gamma_m+1}^m = L_f^{\gamma_m}h_m(x) + \sum_{j=1}^{m-1} \bar{a}_{mj}^0\bar{u}_j^0, \eta^T). \quad (88)$$

Note that $\xi \in \mathbb{R}^{\gamma_1+\cdots+\gamma_m+m}$ and $\bar{\eta} \in \mathbb{R}^{n-\gamma_1-\cdots-\gamma_m-1}$ as compared with $\xi \in \mathbb{R}^{\gamma_1+\cdots+\gamma_m}$ and $\eta \in \mathbb{R}^{n-\gamma_1-\cdots-\gamma_m}$. With these coordinates, the true system (71) is given by

$$\left.\begin{array}{c}
\dot{\xi}_1^1 = \xi_2^1 \\
\vdots \\
\dot{\xi}_{\gamma_1}^1 = \xi_{\gamma_1-1}^1 + \epsilon\bar{a}_{1m}^0\bar{u}_m^1 \\
\dot{\xi}_{\gamma_1+1}^1 = b_1^1(\xi, \bar{\eta}) + a_1^1.(\xi, \bar{\eta})u^1 \\
\dot{\xi}_1^2 = \xi_2^2 \\
\vdots \\
\dot{\xi}_{\gamma_m}^m = \xi_{\gamma_m-1}^m + \epsilon\bar{a}_{mm}^0\bar{u}_m^1 \\
\dot{\xi}_{\gamma_m+1}^m = b_m^1(\xi, \bar{\eta}) + a_m^1.(\xi, \bar{\eta})u^1 \\
\dot{\bar{\eta}} = \bar{q}(\xi, \bar{\eta}) + \bar{P}(\xi, \bar{\eta})u^1.
\end{array}\right\} \quad (89)$$

In (89) above, $b_i^1(\xi, \bar{\eta})$ and $a_i^1.(\xi, \bar{\eta})$ are the ith element and row of b^1 and A^1, respectively, in (84) above (in the $\xi, \bar{\eta}$ coordinates). The approximate system used for the design of the linearizing control is obtained from (89) by setting $\epsilon = 0$. The zero dynamics for the approximate system are obtained in the $\xi = 0$ subspace by linearizing the approximate system using

$$u_*^1(\xi, \bar{\eta}) = -(A^1(\xi, \bar{\eta}))^{-1} \begin{bmatrix} b_1^1(\xi, \bar{\eta}) \\ \vdots \\ b_m^1(\xi, \bar{\eta}) \end{bmatrix}, \quad (90)$$

to get

$$\dot{\bar{\eta}} = \bar{q}(0, \bar{\eta}) + \bar{P}(0, \bar{\eta})u_*^1(0, \bar{\eta}). \quad (91)$$

Note that the dimension of $\bar{\eta}$ is one less than the dimension of η in (81). It would appear that we are actually determining the zero dynamics of the approximation to system (71) with dynamic extension—that is to say with integrators appended to the first $m-1$ inputs

$\bar{u}_1^0, \bar{u}_2^0, \ldots, \bar{u}_{m-1}^0$. While this is undoubtedly true, it has been shown in Byrnes and Isidori (Isidori, 1987) that the zero dynamics of systems are unchanged by dynamic extension. Thus, the zero dynamics of (91) are those of the approximation to system (71).

The system (71) is said to be slightly non-minimum phase if the equilibrium $\eta = 0$ of (81) is not asymptotically stable, but the equilibrium $\bar{\eta} = 0$ of (91) is.

It is also easy to see that the preceding discussion may be iterated if it turns out that $A^1(\xi, \bar{\eta})$ has some small singular values. At each stage of the dynamic extension process $m-1$ integrators are added to the dynamics of the system and the act of approximation reduces the dimension of the zero dynamics by one. Also, if at any stage of this dynamic extension process, there are two, three, $\ldots$ singular values of order ϵ, the dynamic extension involves $m-2$, $m-3$, $\ldots$ integrators.

If the objective is tracking, the approximate tracking control law is

$$u_a^1(\xi, \bar{\eta}) = (A^1(\xi, \bar{\eta}))^{-1}\left(-\begin{bmatrix} b_1^1(\xi, \bar{\eta}) \\ \vdots \\ b_m^1(\xi, \bar{\eta}) \end{bmatrix}\right.$$

$$+ \begin{bmatrix} y_{d1}^{(\gamma_1+1)} + \alpha_{\gamma_1}^1(y_{d1}^{(\gamma_1)} - \xi_{\gamma_1+1}^1) \\ + \cdots + \alpha_0^1(y_{d1} - \xi_1^1) \\ \vdots \\ y_{dm}^{(\gamma_m+1)} + \alpha_{\gamma_1}^m(y_{dm}^{(\gamma_m)} - \xi_{\gamma_m+1}^m) \\ + \cdots + \alpha_0^m(y_{dm} - \xi_1^m) \end{bmatrix}\left.\vphantom{\begin{bmatrix}\\\\\\\\\\\end{bmatrix}}\right), \quad (92)$$

with the polynomials

$$s^{\gamma_i+1} + \alpha_{\gamma_i}^i s^{\gamma_i} + \cdots + \alpha_0^i, \quad i = 1, \ldots, m, \quad (93)$$

chosen Hurwitz.

The following theorem is the analog of Theorem 3.1 in terms of providing a bound for the system performance when the control law (92) is applied to the true system (71).

Theorem 3.3. Suppose that

• the zero dynamics (91) of the approximate system are locally exponentially stable and $\bar{q} + \bar{P}u_*^1$ is locally Lipschitz in ξ and $\bar{\eta}$ and
• the functions $\bar{a}_{im}^0 u_m^1$ are locally Lipschitz continuous for $i = 1, \ldots, m$.

Then, for ϵ sufficiently small and for desired trajectories with sufficiently small values and derivatives $(y_{id}, \dot{y}_{id}, \ldots, y_{id}^{(\gamma_i+1)})$, the states of the system (89) are bounded and the tracking errors

satisfy

$$|e_1| = |\xi_1^1 - y_{d1}| \leq k\epsilon,$$
$$|e_2| = |\xi_1^2 - y_{d2}| \leq k\epsilon,$$
$$\vdots$$
$$|e_m| = |\xi_1^m - y_{dm}| \leq k\epsilon,$$

(94)

for some $k < \infty$.

Proof. Similar to that of Theorem 3.1.   □

As in the SISO case, the stronger conclusions of Theorem 3.2 can be stated when the control objective is stabilization and the approximate system has no zero dynamics.

4. CONCLUSION

In this paper, we have described the application of techniques of exact input–output linearization of nonlinear control systems to the flight control of vertical take-off and landing aircraft. We saw that the application of the theory to this example is not straightforward. In particular, the direct application of the theory yielded an undesirable controller. We remedied the situation by neglecting the coupling between the rolling moment input to the aircraft dynamics and the dynamics along the y-axis.

The example of the vertical takeoff and landing aircraft is an example of a system which is slightly non-minimum phase. Thus, the exact lienarization technique resulted in a system which was internally unstable. We generalized the lessons learned from this application to define, informally, slightly non-minimum phase systems and gave methods to linearize them approximately.

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