Lecture 4: Kinematics:

Forward and Inverse Kinematics

- Kinematic Chains

Lecture 4: Kinematics:

Forward and Inverse Kinematics

- Kinematic Chains
- The Denavit-Hartenberg Convention

Lecture 4: Kinematics:

Forward and Inverse Kinematics

- Kinematic Chains
- The Denavit-Hartenberg Convention
- Inverse Kinematics

Kinematic Chains

Basic Assumptions and Terminology:

- A robot manipulator is composed of a set of links connected together by joints;

Kinematic Chains

Basic Assumptions and Terminology:

- A robot manipulator is composed of a set of links connected together by joints;
- Joints can be either
  - revolute joint (a rotation by an angle about fixed axis)
  - prismatic joint (a displacement along a single axis)
  - more complicated joints (of 2 or 3 degrees of freedom) are represented as combinations of the simplest ones

Kinematic Chains

Basic Assumptions and Terminology:

- A robot manipulator is composed of a set of links connected together by joints;
- Joints can be either
  - revolute joint (a rotation by an angle about fixed axis)
  - prismatic joint (a displacement along a single axis)
  - more complicated joints (of 2 or 3 degrees of freedom) are represented as combinations of the simplest ones
- A robot manipulator with n joints will have $(n + 1)$ links. Each joint connects two links;

Kinematic Chains

Basic Assumptions and Terminology:

- A robot manipulator is composed of a set of links connected together by joints;
- Joints can be either
  - revolute joint (a rotation by an angle about fixed axis)
  - prismatic joint (a displacement along a single axis)
  - more complicated joints (of 2 or 3 degrees of freedom) are represented as combinations of the simplest ones
- A robot manipulator with n joints will have $(n+1)$ links. Each joint connects two links;
- We number joints from 1 to n, and links from 0 to n. So that joint i connects links $(i-1)$ and i;

Kinematic Chains

Basic Assumptions and Terminology:

- A robot manipulator is composed of a set of links connected together by joints;
- Joints can be either
  - revolute joint (a rotation by an angle about fixed axis)
  - prismatic joint (a displacement along a single axis)
  - more complicated joints (of 2 or 3 degrees of freedom) are represented as combinations of the simplest ones
- A robot manipulator with n joints will have $(n+1)$ links. Each joint connects two links;
- We number joints from 1 to n, and links from 0 to n. So that joint i connects links $(i-1)$ and i;
- The location of joint i is fixed with respect to the link $(i-1)$;

Kinematic Chains

Basic Assumptions and Terminology:

- When joint i is actuated, the link i moves. Hence the link $\mathbf{0}$ is fixed;

Kinematic Chains

Basic Assumptions and Terminology:

- When joint i is actuated, the link i moves. Hence the link $\mathbf{0}$ is fixed;
- With the i^{th} joint, we associate joint variable

$$q_i = \begin{cases} \theta_i & \text{if joint } i \text{ is revolute} \\ d_i & \text{if joint } i \text{ is prismatic} \end{cases}$$

Kinematic Chains

Basic Assumptions and Terminology:

- When joint i is actuated, the link i moves. Hence the link $\mathbf{0}$ is fixed;
- With the i^{th} joint, we associate joint variable

$$q_i = \begin{cases} \theta_i & \text{if joint } i \text{ is revolute} \\ d_i & \text{if joint } i \text{ is prismatic} \end{cases}$$

- For each link we attached rigidly the coordinate frame, $o_i x_i y_i z_i$ for the link i;

Kinematic Chains

Basic Assumptions and Terminology:

- When joint i is actuated, the link i moves. Hence the link $\mathbf{0}$ is fixed;
- With the i^{th} joint, we associate joint variable

$$q_i = \begin{cases} \theta_i & \text{if joint } i \text{ is revolute} \\ d_i & \text{if joint } i \text{ is prismatic} \end{cases}$$

- For each link we attached rigidly the coordinate frame, $o_i x_i y_i z_i$ for the link i;
- When joint i is actuated, the link i and its frame experience a motion;

Kinematic Chains

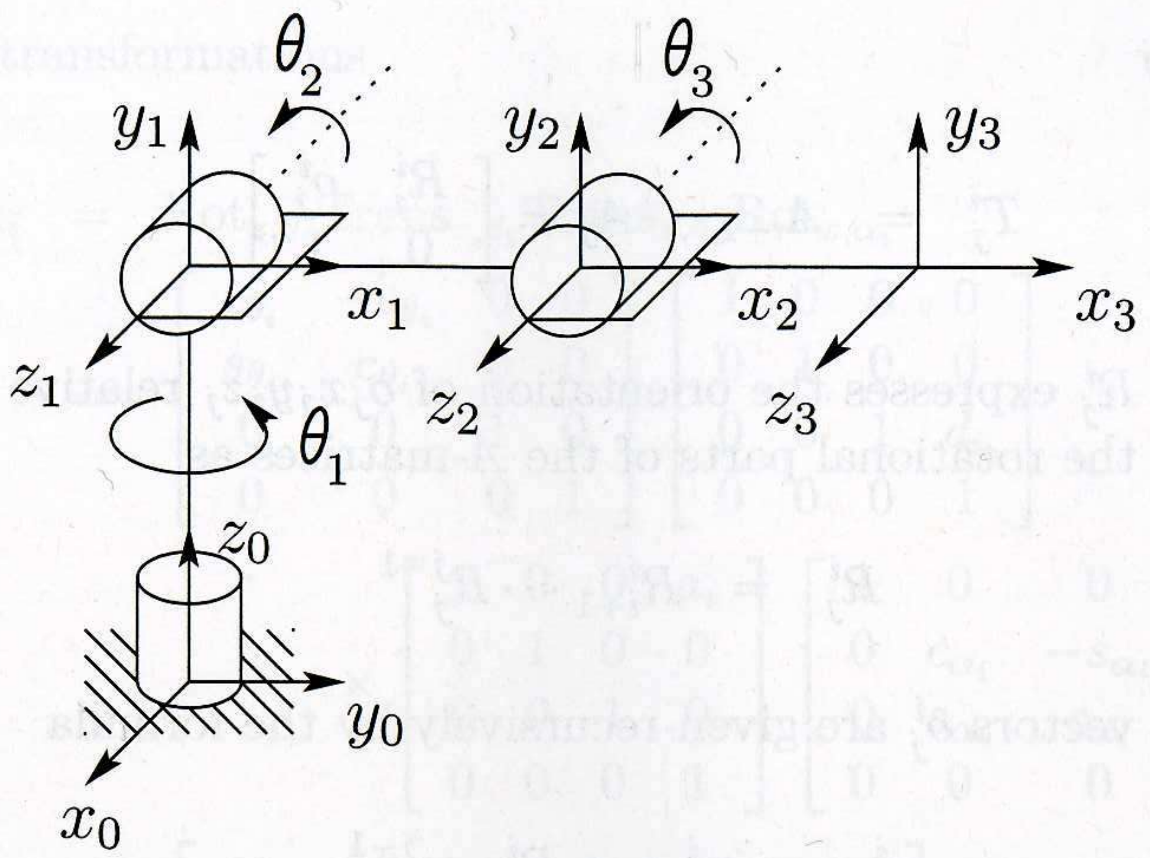
Basic Assumptions and Terminology:

- When joint i is actuated, the link i moves. Hence the link $\mathbf{0}$ is fixed;
- With the i^{th} joint, we associate joint variable

$$q_i = \begin{cases} \theta_i & \text{if joint } i \text{ is revolute} \\ d_i & \text{if joint } i \text{ is prismatic} \end{cases}$$

- For each link we attached rigidly the coordinate frame, $o_i x_i y_i z_i$ for the link i;
- When joint i is actuated, the link i and its frame experience a motion;
- The frame $o_0 x_0 y_0 z_0$ attached to the base is referred to as inertia frame

Kinematic Chains



Coordinate frames attached to elbow manipulator

Kinematic Chains

Basic Assumptions and Terminology:

- Suppose A_i is the homogeneous transformation that gives
  - position
  - orientation

  of frame $o_i x_i y_i z_i$ with respect to frame $o_{i-1} x_{i-1} y_{i-1} z_{i-1}$;

Kinematic Chains

Basic Assumptions and Terminology:

- Suppose A_i is the homogeneous transformation that gives
  - position
  - orientation

  of frame $o_i x_i y_i z_i$ with respect to frame $o_{i-1} x_{i-1} y_{i-1} z_{i-1}$;
- The matrix A_i is changing as robot configuration changes;

Kinematic Chains

Basic Assumptions and Terminology:

- Suppose A_i is the homogeneous transformation that gives
  - position
  - orientation

  of frame $o_i x_i y_i z_i$ with respect to frame $o_{i-1} x_{i-1} y_{i-1} z_{i-1}$;
- The matrix A_i is changing as robot configuration changes;
- Due to the assumptions $A_i = A_i(q_i)$, i.e. it is the function of a scalar variable;

Kinematic Chains

Basic Assumptions and Terminology:

- Suppose A_i is the homogeneous transformation that gives
  - position
  - orientation

  of frame $o_i x_i y_i z_i$ with respect to frame $o_{i-1} x_{i-1} y_{i-1} z_{i-1}$;
- The matrix A_i is changing as robot configuration changes;
- Due to the assumptions $A_i = A_i(q_i)$, i.e. it is the function of a scalar variable;
- Homogeneous transformation that expresses the position and orientation of $o_j x_j y_j z_j$ with respect to $o_i x_i y_i z_i$

$$T_j^i = \begin{cases} A_{i+1} A_{i+2} \cdots A_{j-1} A_j, & \text{if } i < j \\ I, & \text{if } i = j \end{cases}, \qquad T_j^i = (T_i^j)^{-1}, \text{ if } i > j$$

is called a transformation matrix

Kinematic Chains

If the position and orientation of the end-effector with respect to the inertia frame are

$$o_n^0, \qquad R_n^0$$

Then the position and orientation of the end-effector in inertia frame are given by homogeneous transformation

$$T_n^0 = A_1(q_1)A_2(q_2)\cdots A_{n-1}(q_{n-1})A_n(q_n) = \begin{bmatrix} R_n^0 & o_n^0 \\ 0 & 1 \end{bmatrix}$$

with
$$A_i(q_i) = \begin{bmatrix} R_i^{i-1} & o_i^{i-1} \\ 0 & 1 \end{bmatrix}$$

Kinematic Chains

If the position and orientation of the end-effector with respect to the inertia frame are

$$o_n^0, \qquad R_n^0$$

Then the position and orientation of the end-effector in inertia frame are given by homogeneous transformation

$$T_n^0 = A_1(q_1)A_2(q_2)\cdots A_{n-1}(q_{n-1})A_n(q_n) = \begin{bmatrix} R_n^0 & o_n^0 \\ 0 & 1 \end{bmatrix}$$

with
$$A_i(q_i) = \begin{bmatrix} R_i^{i-1} & o_i^{i-1} \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad T_j^i = A_{i+1}A_{i+2}\cdots A_{j-1}A_j = \begin{bmatrix} R_j^i & o_j^i \\ 0 & 1 \end{bmatrix}$$

with
$$R_j^i = R_{i+1}^i \cdots R_j^{j-1}, \quad o_j^i = o_{j-1}^i + R_{j-1}^i o_{j-1}^j$$

Lecture 4: Kinematics:
Forward and Inverse Kinematics

- Kinematic Chains
- The Denavit-Hartenberg Convention
- Inverse Kinematics

DH Convention:

The idea is to represent each homogeneous transform A_i as a product

$$A_i = \mathrm{Rot}_{z,\theta_i} \cdot \mathrm{Trans}_{z,d_i} \cdot \mathrm{Trans}_{x,a_i} \cdot \mathrm{Rot}_{x,\alpha_i}$$

$$= \left[\begin{array}{ccc|c} c_{\theta_i} & -s_{\theta_i} & 0 & 0 \\ s_{\theta_i} & c_{\theta_i} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & c_{\alpha_i} & -s_{\alpha_i} & 0 \\ 0 & s_{\alpha_i} & c_{\alpha_i} & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

DH Convention:

The idea is to represent each homogeneous transform A_i as a product

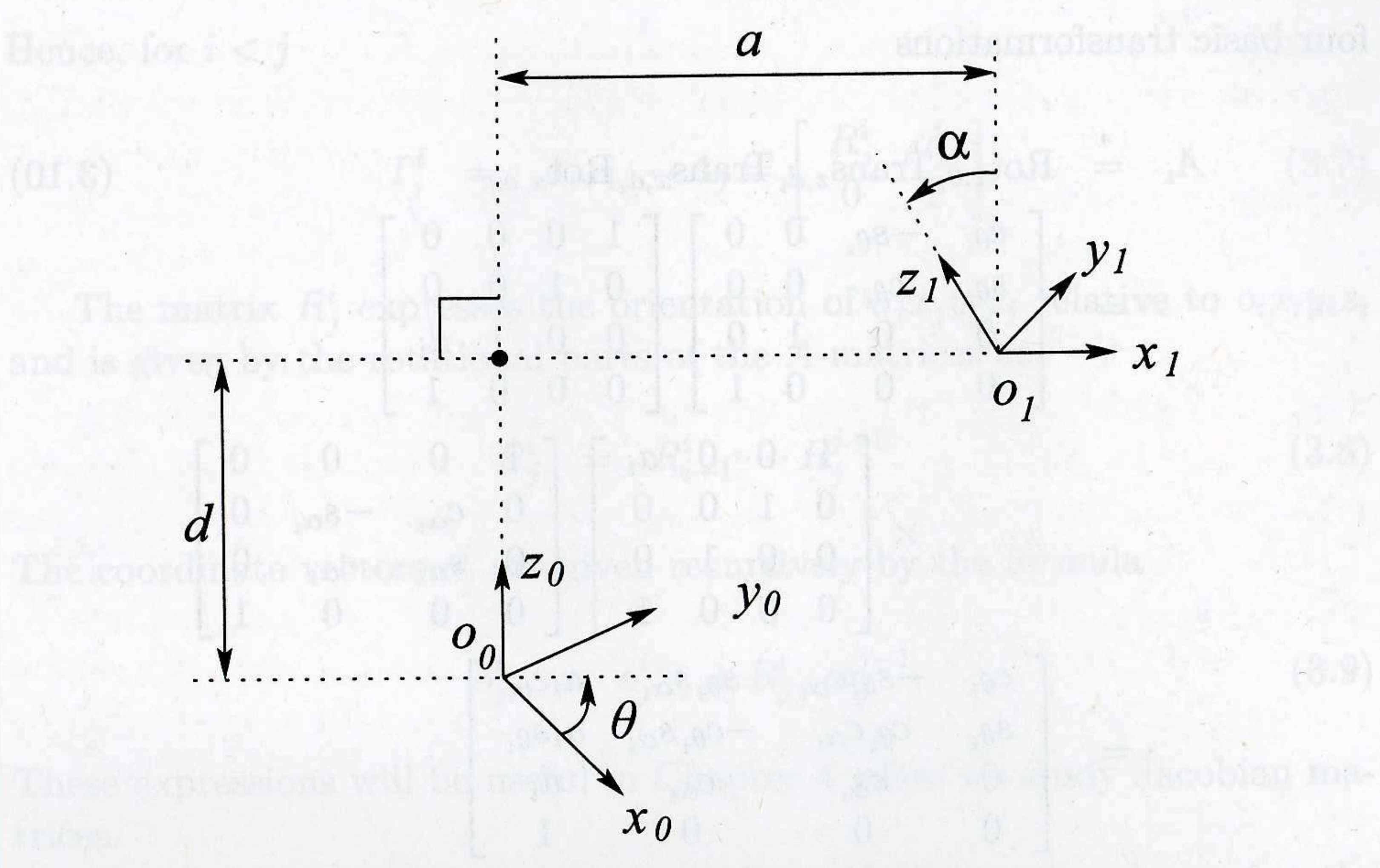
$$A_i = \text{Rot}_{z,\theta_i} \cdot \text{Trans}_{z,d_i} \cdot \text{Trans}_{x,a_i} \cdot \text{Rot}_{x,\alpha_i}$$

$$= \left[\begin{array}{ccc|c} c_{\theta_i} & -s_{\theta_i} & 0 & 0 \\ s_{\theta_i} & c_{\theta_i} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & c_{\alpha_i} & -s_{\alpha_i} & 0 \\ 0 & s_{\alpha_i} & c_{\alpha_i} & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

The parameters of transform are known as

- a_i: link length
- α_i: link twist
- d_i: link offset
- θ_i: link angle

Conditions for Existence 4 Parameters:



DH1: The axis x_1 is perpendicular to the axis z_0

DH2: The axis x_1 intersects the axis z_0

Assigning Frames Following DH-Convention:

Given a robot manipulator with

- n revolute and/or prismatic joints
- $(n + 1)$ links

The task is to define coordinate frames for each link so that transformations between frames can be written in DH-convention

Assigning Frames Following DH-Convention:

Given a robot manipulator with

- n revolute and/or prismatic joints
- $(n+1)$ links

The task is to define coordinate frames for each link so that transformations between frames can be written in DH-convention

---

The algorithm of assigning $(n+1)$ frames for $(n+1)$ links

- treats separately first n-frames and the last one (end-effector frame)
- is recursive in first part, so that it is generic

Assigning First n-Frames

Step 1 (Choice of z-axises):

- Choose z_0-axis along the actuation line of the 1^{st}-link;

Assigning First n-Frames

Step 1 (Choice of z-axises):

- Choose z_0-axis along the actuation line of the 1^{st}-link;
- Choose z_1-axis along the actuation line of the 2^{nd}-link;

Assigning First n-Frames

Step 1 (Choice of z-axises):

- Choose z_0-axis along the actuation line of the 1^{st}-link;
- Choose z_1-axis along the actuation line of the 2^{nd}-link;
- ...
- Choose $z_{(n-1)}$-axis along the actuation line of the n^{th}-link

Assigning First n-Frames

Step 1 (Choice of z-axises):

- Choose z_0-axis along the actuation line of the 1^{st}-link;
- Choose z_1-axis along the actuation line of the 2^{nd}-link;
- ...
- Choose $z_{(n-1)}$-axis along the actuation line of the n^{th}-link

---

We need to finish the job and assign

- point on each of z_i-axis that will be the origin of the i^{th}-frame
- x_i-axis for each frame so that two DH-conditions hold

  DH1: The axis x_1 is perpendicular to the axis z_0

  DH2: The axis x_1 intersects the axis z_0
- y_i-axis for each frame

Assigning First n-Frames

Step 2 (Choice of x-axises):

- Suppose that we have chosen the $(i-1)^{th}$-frame and need to proceed with the i^{th}-frame

Assigning First n-Frames

Step 2 (Choice of x-axises):

- Suppose that we have chosen the $(i-1)^{th}$-frame and need to proceed with the i^{th}-frame
- For the i^{th}-frame, the z_i axis is already fixed

Assigning First n-Frames

Step 2 (Choice of x-axises):

- Suppose that we have chosen the $(i-1)^{th}$-frame and need to proceed with the i^{th}-frame
- For the i^{th}-frame, the z_i axis is already fixed
- To meet conditions DH1-DH2
  the x_i-axis should intersects z_{i-1} and $x_i \perp z_{i-1}$ and $x_i \perp z_i$.
  Is it possible?

Assigning First n-Frames

Step 2 (Choice of x-axises):

- Suppose that we have chosen the $(i-1)^{th}$-frame and need to proceed with the i^{th}-frame
- For the i^{th}-frame, the z_i axis is already fixed
- To meet conditions DH1-DH2
  the x_i-axis should intersects z_{i-1} and $x_i \perp z_{i-1}$ and $x_i \perp z_i$.
  Is it possible?
- There are 3 cases:
  - z_i and z_{i-1} are not coplanar
  - z_i and z_{i-1} are parallel
  - z_i and z_{i-1} intersect

Assigning First n-Frames

Step 2 (Choice of x-axises):

- Suppose that we have chosen the $(i-1)^{th}$-frame and need to proceed with the i^{th}-frame
- For the i^{th}-frame, the z_i axis is already fixed
- To meet conditions DH1-DH2
  the x_i-axis should intersects z_{i-1} and $x_i \perp z_{i-1}$ and $x_i \perp z_i$.
  Is it possible?
- There are 3 cases:
  - z_i and z_{i-1} are not coplanar
  - z_i and z_{i-1} are parallel
  - z_i and z_{i-1} intersect
- For all 3 cases it is possible!

Assigning First n-Frames

Step 2 (Choice of x-axises):

If z_i and z_{i-1} are not coplanar, then there is the common perpendicular for both lines!

Assigning First n-Frames

Step 2 (Choice of x-axises):

If z_i and z_{i-1} are not coplanar, then there is the common perpendicular for both lines!

It will define new origin o_i and the x_i-axis for the i^{th}-frame

Assigning First n-Frames

Step 2 (Choice of x-axises):

If z_i and z_{i-1} are not coplanar, then there is the common perpendicular for both lines!

It will define new origin o_i and the x_i-axis for the i^{th}-frame

---

If z_i and z_{i-1} are parallel, then there are many common perpendiculars for both lines!

Assigning First n-Frames

Step 2 (Choice of x-axises):

If z_i and z_{i-1} are not coplanar, then there is the common perpendicular for both lines!

It will define new origin o_i and the x_i-axis for the i^{th}-frame

---

If z_i and z_{i-1} are parallel, then there are many common perpendiculars for both lines!

One of them will define new origin o_i and the x_i-axis for the i^{th}-frame

Assigning First n-Frames

Step 2 (Choice of x-axises):

If z_i and z_{i-1} are not coplanar, then there is the common perpendicular for both lines!

It will define new origin o_i and the x_i-axis for the i^{th}-frame

---

If z_i and z_{i-1} are parallel, then there are many common perpendiculars for both lines!

One of them will define new origin o_i and the x_i-axis for the i^{th}-frame

---

If z_i and z_{i-1} intersect, then there is a vector orthogonal to the plane formed by z_i and z_{i-1}!

Assigning First n-Frames

Step 2 (Choice of x-axises):

If z_i and z_{i-1} are not coplanar, then there is the common perpendicular for both lines!

It will define new origin o_i and the x_i-axis for the i^{th}-frame

If z_i and z_{i-1} are parallel, then there are many common perpendiculars for both lines!

One of them will define new origin o_i and the x_i-axis for the i^{th}-frame

If z_i and z_{i-1} intersect, then there is a vector orthogonal to the plane formed by z_i and z_{i-1}!

The point of intersection can be new origin o_i and the x_i-axis for the i^{th}-frame is the orthogonal to this plane

Assigning First n-Frames

Step 3 (Choice of y-axises):

If we have already chosen the vectors z_i, x_i and the point o_i for the i^{th}-frame, y_i can be assigned by

Assigning First n-Frames

Step 3 (Choice of y-axises):

If we have already chosen the vectors z_i, x_i and the point o_i for the i^{th}-frame, y_i can be assigned by

cross-product operation: $\vec{y_i} = \vec{z_i} \times \vec{x_i}$

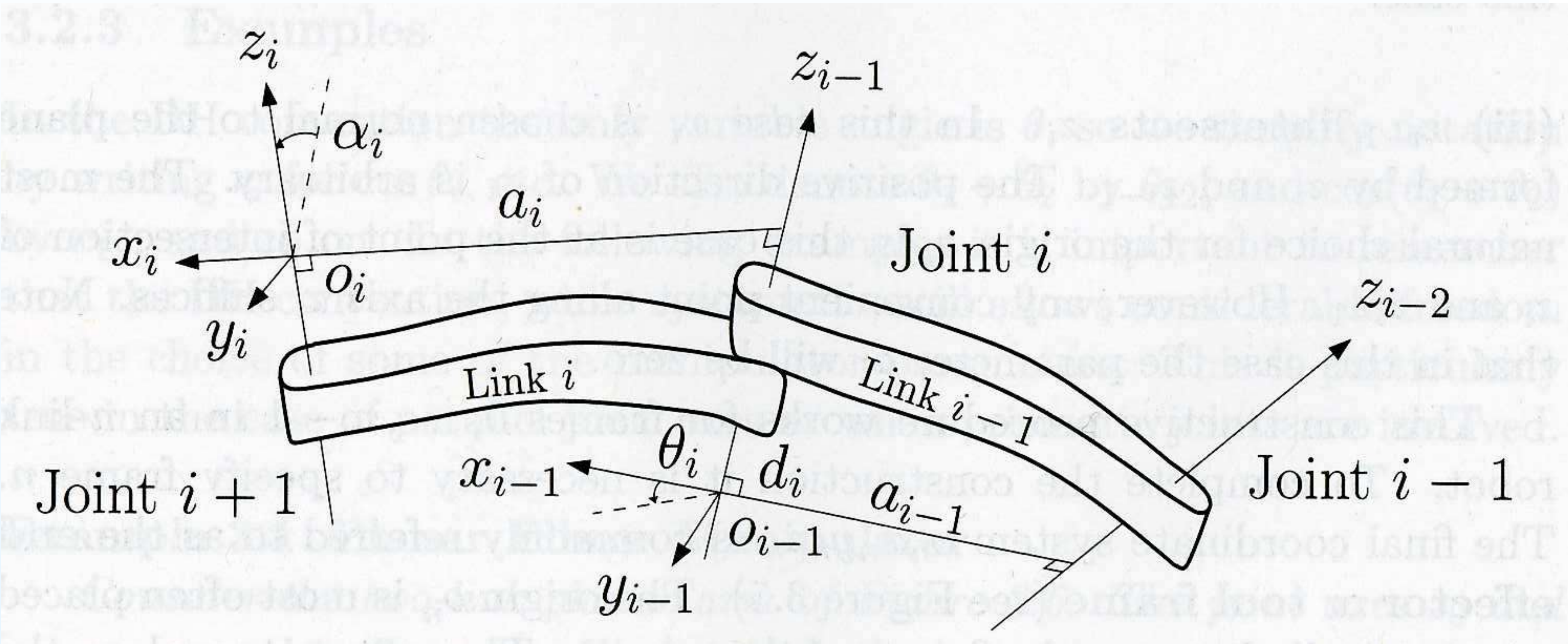
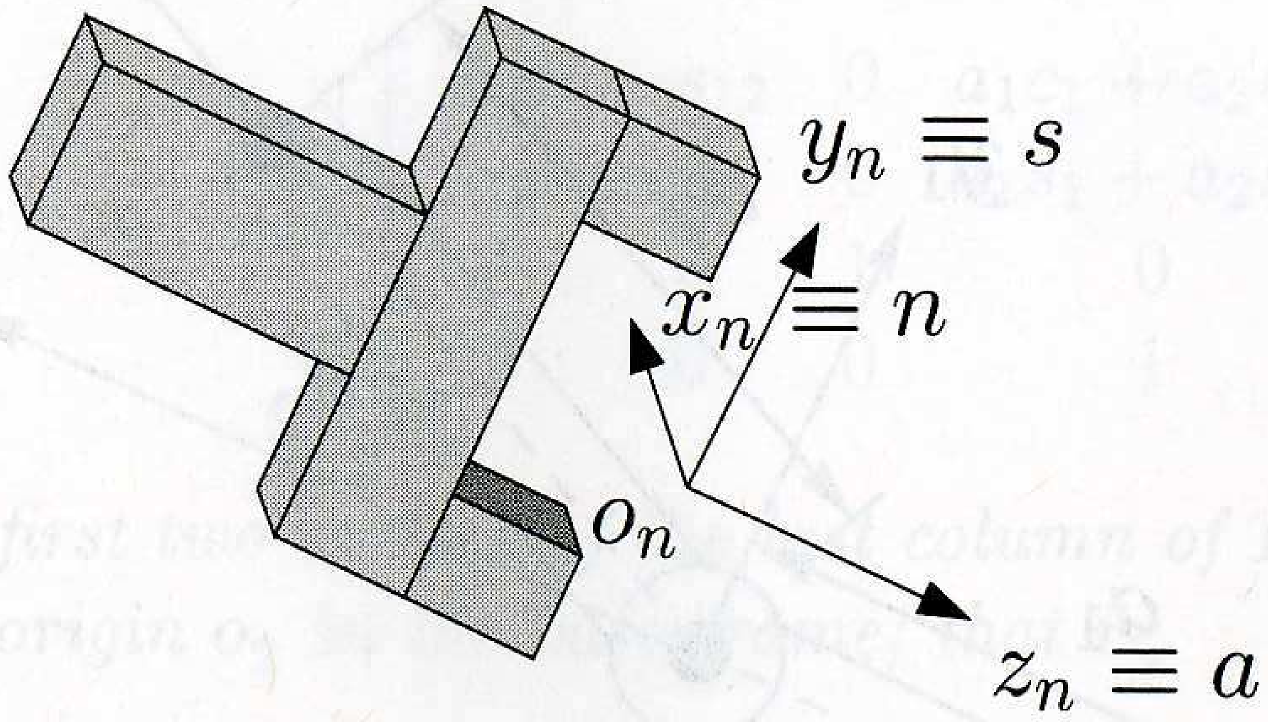


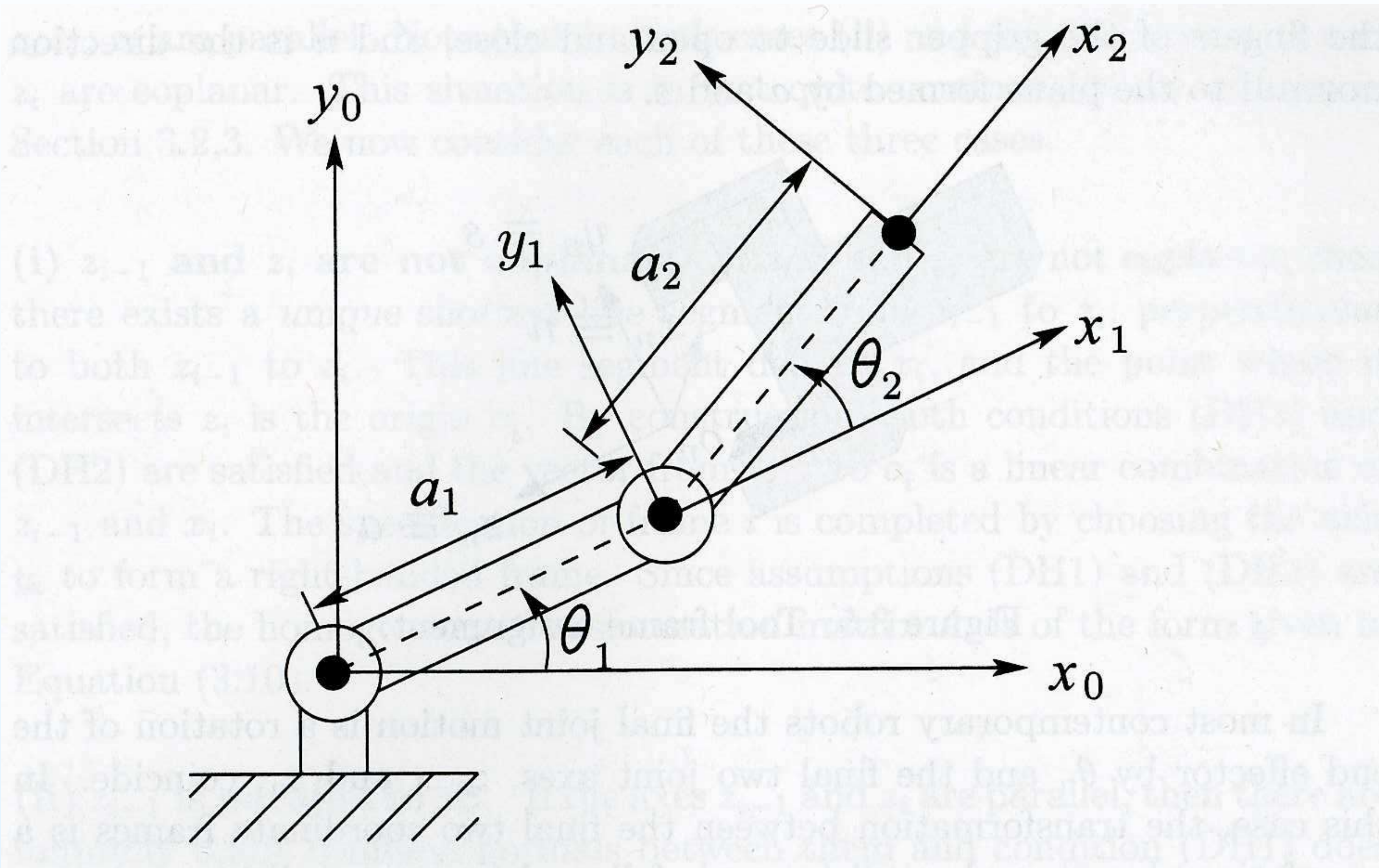
Illustration of DH-frame assignment

Assigning the Last Frame for the End-Effector

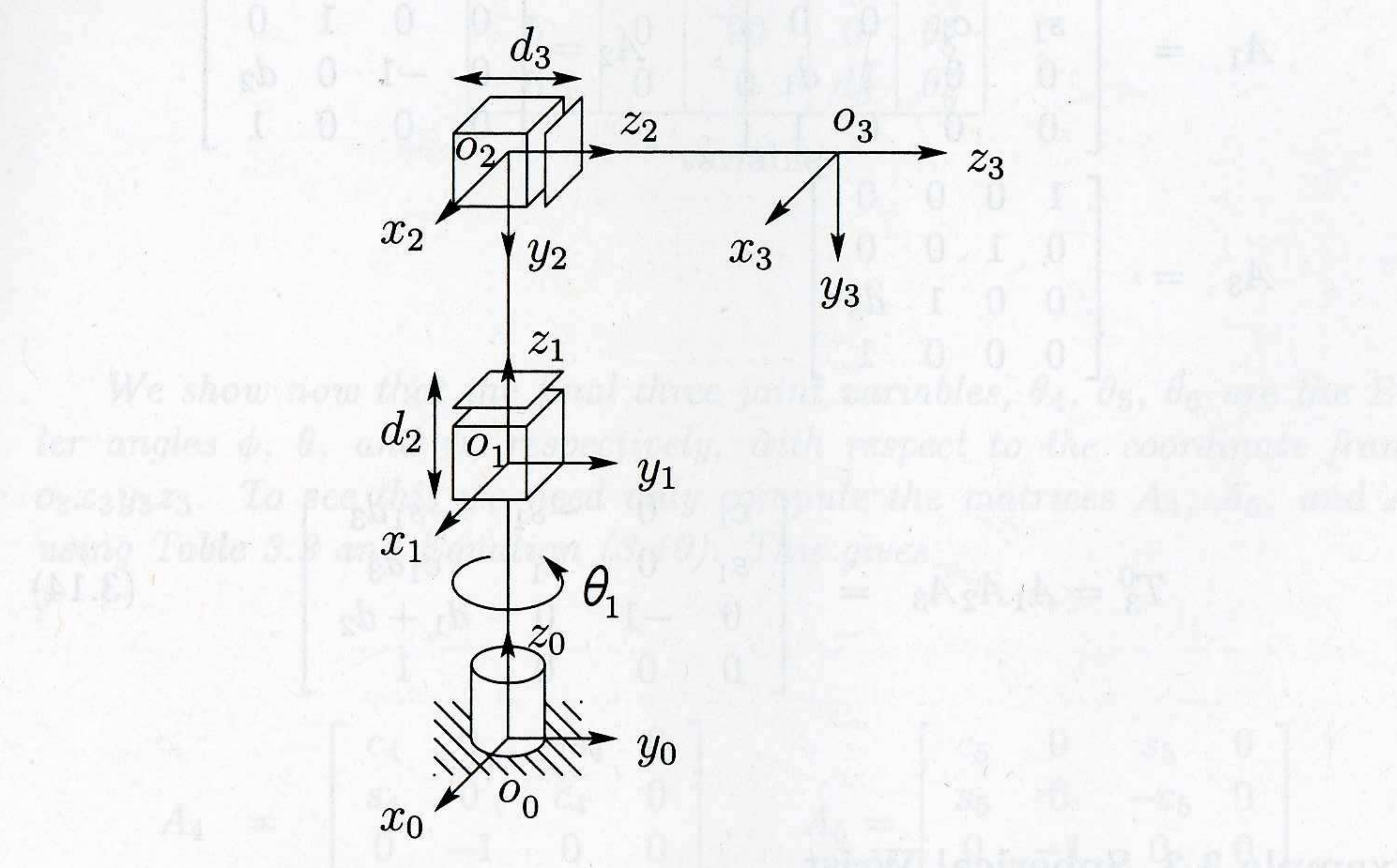


For most robots z_{n-1} and z_n coincide. So that the transformation between two frames is

- translation by d_n along z_{n-1}-axis
- rotation by θ_n about z_n-axis



Example 3.1: Planar two-link manipulator



Example 3.2: Three-link cylindrical manipulator