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CHARACTERISTIC-BASED SCHEMES FOR THE EULER EQUATIONS

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Introduction

Computer simulations of fluid flow provide today the sort of detailed information concerning special cases that could previously only be obtained from experiments. The computer is attractive as a replacement for experiments that are difficult, dangerous, or expensive, and as an alternative to experiments that are impossible. Nevertheless, a computer simulation does not have quite the same status as a physical experiment because at present there usually remains some doubt about its accuracy. Even though the computer code may be free of error to the extent that it operates exactly as its author intended, it is seldom possible to give a rigorous proof that these intentions were in all respects correct. Most of the practical codes written to solve complicated problems contain empirical features, sometimes in the form of "adjustable constants" whose values must be "tuned" by appeal to the experiments that the simulations are intended to displace.

A computer code is described as being "robust" if it has the virtue of giving reliable answers to a wide range of problems without needing to be retuned. The ideal code would be one that fully met some declared specification of accuracy and problem range, and whose every line was a necessary contribution to that aim. Few codes yet approach that ideal; a major impediment is that we presently have little idea what properties can be specified without contradiction.

In recent years, however, our understanding of computer codes for a particular class of problems has advanced some way toward completeness. The problems are sufficiently complex that naive numerical techniques can produce disaster, yet sufficiently simple that well-understood physics can

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point the way to greatly improved results. We refer to problems involving unsteady one-dimensional flow of an inviscid compressible gas. The main cause of numerical difficulty in such problems is the occurrence of features that in the inviscid approximation are discontinuous, and even in the presence of viscosity are much too small to be resolved on an affordable computational mesh. The fluid state may jump across shock waves or contact surfaces, and it may have discontinuous derivatives across any characteristic. About a decade ago, it was hotly debated whether such flows could be accurately computed except by methods that specifically allow for discontinuities, tracking their position and solving in their vicinity the appropriate jump relationships rather than the differential equations. Computations that effectively ignored the discontinuities by applying almost the same numerical scheme everywhere in the flow (called "shock-capturing" schemes) met with some practical success, but in regions where discontinuous behavior should have been seen, the numerical solution would display a transition region either of rather shallow slope or, if steep, accompanied by spurious oscillations. It was difficult to counter the argument that one could not expect to find a legitimate solution by solving differential equations in regions where the true solution is not differentiable.

Since then, shock-capturing methods have attained mathematical respectability, partly through the reinterpretation of the numerical equations as expressions of integral rather than differential laws, and partly through the incorporation into the methods of ideas drawn from the theory of characteristics. This paper describes these developments, omitting much detail but aiming nevertheless to give the flavor of the combined physical and mathematical argument that has typified the work. We also discuss the success that has attended the extension of these ideas to multidimensional flow.

Forms of the Euler Equations

We discuss the simplest case. Inviscid flow of a compressible gas in a parallel pipe obeys the following conservation laws for mass, momentum, and energy:

$$\rho_t + (\rho u)_x = 0, \tag{1a}$$

$$(\rho u)_t + (p + \rho u^2)_x = 0, \tag{1b}$$

$$(\rho e)_t + (\rho u e + u p)_x = 0, \tag{1c}$$

where t is time, x distance, ρ density, u velocity, p pressure, and e specific total energy. The system is closed by an equation of state $p = p(\rho, i)$, where i is the specific internal energy ($i = e - \frac{1}{2}u^2$). For an ideal gas whose ratio of

specific heats is γ, we have

$$p = (\gamma - 1)\rho i. \tag{2}$$

A concise notation for Equations (1) is

$$\mathbf{w}_t + \mathbf{F}_x = 0, \tag{3}$$

where the vectors $\mathbf{w}$ and $\mathbf{F}$ are given by

$$\mathbf{w} = \begin{vmatrix} \rho \\ \rho u \\ \rho e \end{vmatrix}, \qquad \mathbf{F} = \begin{vmatrix} \rho u \\ p + \rho u^2 \\ \rho u e + u p \end{vmatrix}. \tag{4}$$

With the equations in this form, it is a simple application of Gauss' divergence theorem to deduce that

$$\oint (\mathbf{w}\, dx - \mathbf{F}\, dt) = 0 \tag{5}$$

around any closed contour in (x, t). Indeed, (5) is a more fundamental statement than (4), since it holds whether or not the flow variables can be differentiated.

We can also derive from Equations (1), by rather straightforward operations, the system of characteristic equations

$$dp - \rho a\, du = 0 \quad \text{along} \quad dx = (u - a)\, dt, \tag{6a}$$

$$dp - a^2\, d\rho = 0 \quad \text{along} \quad dx = u\, dt, \tag{6b}$$

$$dp + \rho a\, du = 0 \quad \text{along} \quad dx = (u + a)\, dt, \tag{6c}$$

in which we introduce the sound speed a, such that

$$a^2 = \frac{\partial p}{\partial \rho} = \frac{\gamma p}{\rho}. \tag{7}$$

Now, although Equations (5) and (6) are (aside from the matter of differentiability) equivalent descriptions of the flow, this equivalence does not leap to the eye. Moreover, an engineer trying to understand some flow problem would select different versions for different purposes. It might be possible to write a computer code that would operate in the same discriminating way, but it would be a complicated business. Most codes, at least of the shock-capturing type, operate everywhere with one form of the equations. Does it matter which one we choose? The most successful codes of recent years are those that somehow contrive to combine explicitly the information inherent in all the different forms.

Conservative Discretizations

Mathematically, the story of the fluid motion is told by a vector-valued function $\mathbf{q}(x,t)$, whose components are any three independent flow variables. Computationally, this is replaced by $\mathbf{q}_i^n$, which is some approximation to $\mathbf{q}(i\Delta x, n\Delta t)$, where for simplicity Δx, Δt are taken to be small constants that define a computing grid of congruent rectangles. The idea that shock-capturing calculations would be best performed if $\mathbf{q}$ were chosen to be $\mathbf{w}$, the vector of conserved variables, is due to Lax (1954). He tackled the basic problem of determining the solution at time level $(n+1)$ from data at level n by creating "interface states" at locations such as $(i+\frac{1}{2})\Delta x$ and proposed schemes of the form

$$\frac{\mathbf{w}_i^{n+1}-\mathbf{w}_i^n}{\Delta t}+\frac{\mathbf{F}_{i+1/2}^n-\mathbf{F}_{i-1/2}^n}{\Delta x}=0. \tag{8}$$

He showed that (8) would be consistent with the differential equations (3) in the limit $\Delta x, \Delta t \to 0$ if $\mathbf{F}_{i+1/2}$ were constructed in any reasonable way from the values of $\mathbf{w}$ in some neighborhood of $i+\frac{1}{2}$. Then, given initial data $\mathbf{w}_i^0$ for all i, we can use Equation (8) to march forward and construct $\mathbf{w}_i^1$, $\mathbf{w}_i^2$, etc.

The most obvious choice is

$$\mathbf{F}_{i+1/2}=\tfrac{1}{2}[\mathbf{F}(\mathbf{w}_i)+\mathbf{F}(\mathbf{w}_{i+1})], \tag{9}$$

but this leads to an unstable scheme, in the sense that after a large number of time steps have been taken, the numerical solution becomes wildly oscillatory instead of remaining close to the true solution (see Wendroff 1966). Lax showed that the modified choice

$$\mathbf{F}_{i+1/2}=\tfrac{1}{2}[\mathbf{F}(\mathbf{w}_i)+\mathbf{F}(\mathbf{w}_{i+1})]-\frac{\Delta x}{2\Delta t}[\mathbf{w}_{i+1}-\mathbf{w}_i] \tag{10}$$

is stable. In numerical experiments simulating the steady motion of a shock wave, he found that a smooth profile formed and propagated with the correct velocity. The theoretical explanation was that any scheme of the form (8) can be rearranged and summed with respect to the index i; thus we have

$$\Delta x \sum_{i_{\min}}^{i_{\max}} \mathbf{w}_i^{n+1}=\Delta x \sum_{i_{\min}}^{i_{\max}} \mathbf{w}_i^n+\Delta t[\mathbf{F}_{i_{\min}-1/2}^n-\mathbf{F}_{i_{\max}+1/2}^n]. \tag{11}$$

In this equation, the left-hand side represents the total amount of mass, momentum, and energy present in the system at the new time, and the first term on the right-hand side represents the same thing at the old time. Within the brackets are the only values of $\mathbf{F}$ to survive the summation;

these represent the fluxes of conserved variables through each end of the tube, together with those pressures that do work on the system. Equation (11) is in fact a correct description of the integral laws (5), and therefore is valid even if shocks, contact surfaces, etc., are present.

The argument becomes even more convincing if the states $\mathbf{w}_i^n$ are not actually regarded as approximations to the state $\mathbf{w}(i\Delta x, n\Delta t)$ but rather as approximations to the average state in an interval. Thus we redefine $\mathbf{w}_i^n$ as

$$\mathbf{w}_i^n = \frac{1}{\Delta x} \int_{(i-1/2)\Delta x}^{(i+1/2)\Delta x} \mathbf{w}(x, n\Delta t)\, dx. \tag{12}$$

In smoothly varying regions of the flow, the two definitions differ only by $O(\Delta x^2)$, but near shocks the difference could be substantial. The advantage of the new definition is that Equation (8) (after multiplication by Δx and Δt) can now itself be regarded as an integral rather than a differential law. Methods based on interpretation (12) are called finite-volume schemes and are almost universally used for shock-capturing codes today.

This development, however, left many problems still to be solved. The scheme defined by (8) and (10) is not particularly accurate, the truncation error being $O(\Delta x, \Delta t)$. Formulas of the same type [i.e. (8)] were devised with truncation errors $O(\Delta x^2, \Delta t^2)$ (Lax & Wendroff 1964) but were found to be prone to local instability near shocks. Such instability can be cured by adding artificial dissipative terms to the equations; this modification has led to successful methods for many applications, in particular where the shock waves eventually become fixed in a steady flow. Steady transonic flows in two and three dimensions are now commonly solved by finite-volume methods with artificial viscosity (Jameson et al. 1981, Rizzi & Eriksson 1984). However, these methods still appear to be unsatisfactory if they are required either to follow rapid transient motion or to achieve the best possible resolution of discontinuities. Later, we show some comparisons between computations made by the "artificial-viscosity" method and the characteristic-based methods to which we now turn.

The Linearized Problem

Many aspects of numerical shock waves can be illustrated very simply. The differential Euler equations, because they are equivalent to the set of ordinary differential equations (6), are of hyperbolic type. The difficulties arising from the need to allow for discontinuous solutions occur with all hyperbolic equations. The simplest example, the linear advection equation

$$u_t + a u_x = 0, \tag{13}$$

has proved to be an amazingly fruitful source of inspiration. [Recent important papers are by Iserles (1981) and by Iserles & Strang (1983).] Here

u is a scalar function and a is a (positive) constant. If a is negative, the problem is mirror imaged. The characteristic form of (13) is

$$du = 0 \quad \text{along} \quad dx = a\,dt, \tag{14}$$

and its general solution is

$$u = f(x - at) \tag{15}$$

for any function f. If f is not differentiable, (15) is called a *weak solution* (Lax 1954). A weak solution that simulates a traveling shock wave arises if f is taken as the Heaviside function. Godunov (1959) studied numerical solutions of (13) having the general linear form

$$u_i^{n+1} = \sum_k c_k u_{i+k}^n. \tag{16}$$

Here u_i^n is an approximation to $u(i\Delta x, n\Delta t)$ in the sense of Equation (12). There are unwelcome limitations on the quality of results provided by such formulas. By expressing the quantities that appear in (16) as Taylor expansions about the point $(i\Delta x, n\Delta t)$, it is found that the method will produce second-order accuracy if

$$\sum_k c_k = 1, \qquad \sum_k k c_k = -v, \qquad \sum_k k^2 c_k = v^2, \tag{17}$$

where $v = a\Delta t/\Delta x$ is the Courant number.

We would also like to impose some qualitative features, and Godunov proposed the requirement that if u_i^0 were a monotonic function of i, then u_i^n ($n = 1, 2, \ldots$, etc.) should also be a monotonic function of i. Constraining the scheme thus to be "monotonicity preserving" would prevent our model shock wave from developing oscillations. The necessary and sufficient condition is that none of the c_k are negative. To see this, we write the first difference of (16) as

$$u_{i+1}^{n+1} - u_i^{n+1} = \sum_k c_k (u_{i+k+1}^n - u_{i+k}^n). \tag{18}$$

The stated condition is sufficient, because if every gradient on the right-hand side is of the same sign, the left-hand side must be of that sign. But if some c_r were negative, consider the monotonic data $u_j^n \equiv 0$ ($j \leqslant r$), $u_j^n \equiv 1$ ($j > r$). Putting $i = 0$ in (18) gives $u_1^{n+1} - u_0^{n+1} = c_r$, so there would be at least one gradient of the wrong sign, which shows that the condition is also necessary. Now let us seek a second-order accurate scheme that is also monotonicity preserving by putting $c_k = (e_k)^2$ in Equations (17). This produces the result

$$\sum (e_k)^2 \sum (k e_k)^2 = \left[\sum (k e_k^2)\right]^2, \tag{19}$$

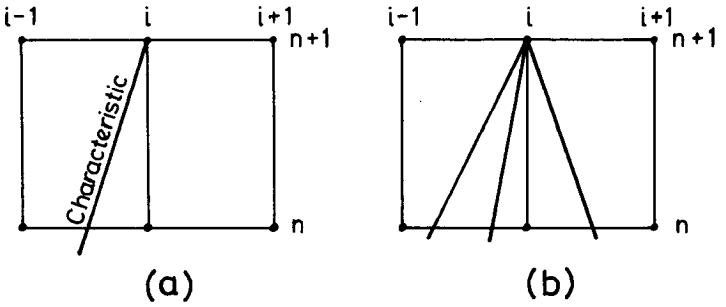

Figure 1 The computing grid in an (x, t) plane, showing the relationship between grid points and characteristic lines.

and since this would violate the Cauchy inequality, we conclude that no such scheme exists.

This result (Godunov's theorem) is a major stumbling block in the development of numerical methods for hyperbolic problems. One strategy for getting around it that has been used by several workers is to start with the most accurate linear scheme that does preserve monotonicity and then add to it nonlinear terms that improve the accuracy. This strategy inevitably involves the characteristic form of the equations. Godunov showed that the monotonicity-preserving scheme with least truncation error is the simple "first-order upwind" scheme

$$u_i^{n+1} = (1-\nu)u_i^n + \nu u_{i-1}^n. \tag{20}$$

This is simply interpreted as the result of tracing back the characteristic from $(i\Delta x, (n+1)\Delta t)$ to find its intersection at $n\Delta t$ in the interval $[(i-1)\Delta x, i\Delta x]$ (see Figure 1*a*) and performing linear interpolation to find the value of u on that characteristic. If the characteristic falls outside the assumed interval, so that the method is based on extrapolation, the computations are again unstable, so we must restrict Δt in such a way that $0 \leqslant \nu \leqslant 1$. If the constant a in (13) is negative, then the corresponding method using data at $i\Delta x, (i+1)\Delta x$ is stable provided $-1 \leqslant \nu \leqslant 0$.

Some Early Upwind Schemes

A method similar to the above for the gasdynamic equations in characteristic form [Equations (6)] had already been proposed by Courant et al. (1952). Their procedure [sometimes called the CIR (Courant-Issacson-Rees) method] is to trace back from $(i\Delta x, (n+1)\Delta t)$ all three characteristic paths (as in Figure 1*b*). Since the problem is nonlinear, the directions of these paths are not known exactly, but to a first approximation they can be taken equal to their known directions at $(i\Delta x, n\Delta t)$. Then each characteristic equation is solved using interpolated data at time $n\Delta t$, in the interval to the

left of i for characteristics with positive speed, and in the interval to the right of i for characteristics with negative speed. This method has two principal drawbacks: Because it does not deal with the integral form of the equations, it cannot convey a shock wave with the proper speed; and it is still only first-order accurate.

Godunov (1959) dealt with the first difficulty by using a highly ingenious device that advances to the next time level by solving a set of Riemann problems. Recall that the Riemann problem for any set of conservation laws such as (3) arises if initial data are prescribed as two semi-infinite states ($\mathbf{w} = \mathbf{w}_L$ for $x < 0$, $\mathbf{w} = \mathbf{w}_R$ for $x > 0$). The solution then consists of centered waves. These are three in number for the Euler equations; the inner one is a contact discontinuity separating states at different temperature, and the outer ones may be shock waves or rarefaction fans. The exact solution of this problem involves only algebraic equations [see Holt (1984) or van Leer (1979)].

To describe Godunov's use of the Riemann problem, let $\mathbf{w}_i^n$ be [as in Equation (12)] the average state over $(i \pm \frac{1}{2})\Delta x$ at $n\Delta t$. Replace the data by an approximate distribution in which the state inside each interval is uniform and equal to $\mathbf{w}_i^n$. For each interface $(i+\frac{1}{2})\Delta x$ we can solve the Riemann problem with $\mathbf{w}_L = \mathbf{w}_i$ and $\mathbf{w}_R = \mathbf{w}_{i+1}$, and this gives an exact solution to the approximate problem, at least for Δt small enough that the waves from neighboring interfaces do not intersect (see Figure 2).

The solution at $(n+1)\Delta t$ can again be approximated by a piecewise-uniform distribution, and then the process can be repeated. From the standpoint of conventional numerical analysis, Godunov's method is distinctly quaint, although there is a shortcut that reduces the numerical work to be done. Since the region $IJKL$ contains an exact solution to the Euler equations, we may apply Equation (5) to it, giving

$$(\mathbf{w}_i^{n+1} - \mathbf{w}_i^n)\Delta x + (\mathbf{F}_{i+1/2}^{n+1/2} - \mathbf{F}_{i-1/2}^{n+1/2})\Delta t = 0. \tag{21}$$

In this equation, $\mathbf{w}_i^n$ is the original uniform distribution and $\mathbf{w}_i^{n+1}$ is the

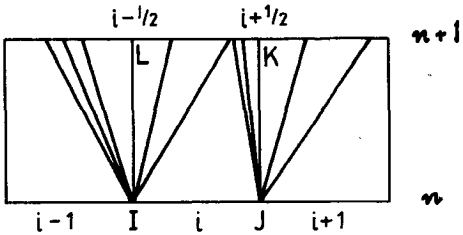

Figure 2 Diagram showing the waves that are supposed to spring from cell interfaces in Godunov's method.

average state at $(n+1)\Delta t$, namely the new uniform distribution after the Riemann solution has been averaged. The values of $\mathbf{F}$ that appear are the flux functions averaged with respect to time, but because the Riemann solution is constant along rays, there is no averaging to do. Equation (21) is Lax's equation (8), with the specific recipe for calculating $\mathbf{F}^n_{i+1/2}$ that it should be the solution at $x = 0$ of the Riemann problem for $\mathbf{w}_i, \mathbf{w}_{i+1}$. It is easy to show that for the simple model problem (13), Godunov's method is identical to the CIR method. Indeed, they remain identical for arbitrary hyperbolic systems of linear equations, but the conservative form of Godunov's method is advantageous when dealing with nonlinear equations. Note also that Godunov's method emphasizes the forward characteristics from level n, rather than the backward characteristics from level $(n+1)$. In this respect, most modern work follows Godunov by analyzing the data, rather than synthesizing the solution. Recent progress consists in performing this analysis by simpler methods than solving Riemann problems exactly, and in exploiting the information to obtain high accuracy without oscillations.

Another closely related method was devised by Glimm (1965), originally as a purely theoretical device for investigating the existence and uniqueness of solutions to hyperbolic problems. There are several versions, but perhaps the simplest uses a staggered grid as shown in Figure 3. The Riemann problem is solved for $\mathbf{w}^n_i, \mathbf{w}^n_{i+1}$, and the solution at $((i+\frac{1}{2})\Delta x, (n+1)\Delta t)$ is taken to be a state chosen at random from the interval LK. [Repetition gives a solution back on the original grid at $(n+2)\Delta t$.] This curious scheme has the following striking property. Suppose $\mathbf{w}_l, \mathbf{w}_r$ are two states such that the solution to the Riemann problem is a shock wave moving to the right with speed s. In addition, suppose we have data such that $\mathbf{w}^n_i = \mathbf{w}_l$ $(i \leqslant k)$ and $\mathbf{w}^n_i = \mathbf{w}_r$ $(i > k)$. For every pair of states except $(k, k+1)$, the solution to the Riemann problem is trivial and will result in

$$\mathbf{w}^{n+1}_{i+1/2} = \mathbf{w}^n_i = \mathbf{w}^n_{i+1}. \tag{22}$$

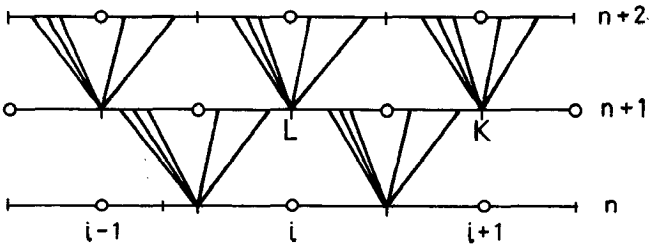

Figure 3 Glimm's method represents the solution at the circles by randomly sampling the Riemann solutions generated at the previous time level.

At $k+\frac{1}{2}$ the Riemann solution will consist of a shock separating $\mathbf{w}_l$ and $\mathbf{w}_r$ [with a probability $\frac{1}{2}(1+s\Delta t/\Delta x)$ that it will be $\mathbf{w}_l$]. Therefore, the solution at $(n+1)\Delta t$ contains only the two states $\mathbf{w}_l, \mathbf{w}_r$, and by induction this will be true for all time. The numerical shock remains perfectly sharp and is translated (somewhat erratically) in the right direction.

The breakthrough that made Glimm's scheme practicable resulted from the realization that one can do better than choose at random. A better choice is

$$\mathbf{w}_{i+1/2}^{n+1} = \mathbf{w}((i+a_n)\Delta x, (n+1)\Delta t), \tag{23}$$

where a_n is the same for all i at given n, and the sequence $a_1, a_2, a_3, \ldots$ is a pseudorandom sequence evenly distributed over $[0, 1]$. This modification, due to Chorin (1976) and further improved by Colella (1982), is now always used in practice. Because of its remarkable shock-capturing property, the method is useful for problems in which shock propagation dominates other events. For recent applications to modeling flow in oil reservoirs, see Concus (1982) and Glimm et al. (1981). Various examples including the flow of detonating mixtures are given by Saito & Glass (1984).

Approximate Riemann Solvers

Since the Riemann problems arising in Godunov's method relate only to an approximation of the data, one might reasonably be satisfied with approximate solutions if these solutions still describe the important nonlinear behavior. Two methods for finding such solutions are described below.

The present author (Roe 1980, 1981) proposed a method exploiting the fact that the Riemann solution for any set of linear conservation laws is easily computed. An equation of the form (3), with $\mathbf{F}$ some linear function of $\mathbf{w}$, can also be written as

$$\mathbf{w}_t + A\mathbf{w}_x = 0, \tag{24}$$

where A is a constant Jacobian matrix $\partial\mathbf{F}/\partial\mathbf{w}$. Given any two states $\mathbf{w}_L, \mathbf{w}_R$, the flux difference can always be uniquely expressed as

$$\mathbf{F}_R - \mathbf{F}_L = \sum \alpha_k \lambda_k \mathbf{e}_k, \tag{25}$$

where the $\{\mathbf{e}_k\}$ are the right eigenvectors of A. Each term in the expansion represents the effect of one wave; α_k is the strength of the kth wave, and λ_k (an eigenvalue of A) is its speed. The required flux at the interface (see Figure 4) could evidently be computed from either

$$\mathbf{F}_{i+1/2}(\mathbf{w}_L, \mathbf{w}_R) = \mathbf{F}_L + \sum{}^{(-)} \alpha_k \lambda_k \mathbf{e}_k \tag{26}$$

or

$$\mathbf{F}_{i+1/2}(\mathbf{w}_L,\mathbf{w}_R) = \mathbf{F}_R - \sum^{(+)} \alpha_k \lambda_k \mathbf{e}_k, \tag{27}$$

where $\sum^{(-)}$ and $\sum^{(+)}$ denote summation over the negative and positive wave speeds, respectively. Taking the average of these expressions, we find

$$\mathbf{F}_{i+1/2}(\mathbf{w}_L, \mathbf{w}_R) = \tfrac{1}{2}(\mathbf{F}_L + \mathbf{F}_R) - \tfrac{1}{2} \sum \alpha_k |\lambda_k| \mathbf{e}_k. \tag{28}$$

For linear problems this method again reproduces the CIR method. To apply (28) to a nonlinear problem, one can define a local linearization by means of a matrix $\tilde{A}(\mathbf{w}_L, \mathbf{w}_R)$ whose eigenvalues and eigenvectors satisfy identically not only (25) but also

$$\mathbf{w}_R - \mathbf{w}_L = \sum_k \alpha_k \mathbf{e}_k. \tag{29}$$

The feature of the method that makes it valuable for nonlinear problems is that it returns the exact solution whenever $\mathbf{w}_L$ and $\mathbf{w}_R$ lie on opposite sides of a shock wave or a contact discontinuity. In that case the Rankine-Hugoniot relationship must hold:

$$\mathbf{F}_R - \mathbf{F}_L = S(\mathbf{w}_R - \mathbf{w}_L), \tag{30}$$

where S is the shock speed. If (25), (29), and (30) are all true, then we must have for all k

$$S\alpha_k = \lambda_k \alpha_k, \tag{31}$$

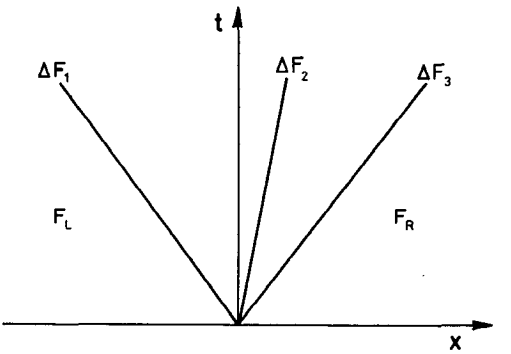

Figure 4 Diagram for computing the interface flux.

which shows that all α_k except one must vanish. Expressions for α_k, $\tilde{\lambda}_k$, and $\tilde{\mathbf{e}}_k$ were given by Roe (1981):

$$\tilde{\mathbf{e}}_1 = \begin{vmatrix} 1 \\ \tilde{u}-\tilde{a} \\ \tilde{h}-\tilde{u}\tilde{a} \end{vmatrix}, \qquad \tilde{\mathbf{e}}_2 = \begin{vmatrix} 1 \\ \tilde{u} \\ \frac{1}{2}\tilde{u}^2 \end{vmatrix}, \qquad \tilde{\mathbf{e}}_3 = \begin{vmatrix} 1 \\ \tilde{u}+\tilde{a} \\ \tilde{h}+\tilde{u}\tilde{a} \end{vmatrix}, \tag{32}$$

$$\tilde{\lambda}_1 = \tilde{u}-\tilde{a}, \qquad \tilde{\lambda}_2 = \tilde{u}, \qquad \tilde{\lambda}_3 = \tilde{u}+\tilde{a}, \tag{33}$$

$$\alpha_1 = \frac{1}{2\tilde{a}^2}[\Delta p - \tilde{\rho}\tilde{a}\Delta u], \qquad \alpha_2 = \frac{1}{\tilde{a}^2}[\tilde{a}^2\Delta\rho - \Delta p],$$

$$\alpha_3 = \frac{1}{2\tilde{a}^2}[\Delta p + \tilde{\rho}\tilde{a}\Delta u], \tag{34}$$

where

$$\tilde{\rho}^2 = \rho_L\rho_R, \tag{35a}$$

$$\tilde{u} = \frac{\rho_L^{1/2}u_L + \rho_R^{1/2}u_R}{\rho_L^{1/2}+\rho_R^{1/2}}, \tag{35b}$$

$$\tilde{h} = \frac{\rho_L^{1/2}h_L + \rho_R^{1/2}h_R}{\rho_L^{1/2}+\rho_R^{1/2}}, \tag{35c}$$

$$\tilde{a}^2 = (\gamma-1)[\tilde{h}-\tfrac{1}{2}\tilde{u}^2]. \tag{35d}$$

Inserting these expressions into (28) gives an explicit expression for $\mathbf{F}_{i+1/2}(\mathbf{w}_L, \mathbf{w}_R)$. It was shown by Roe & Pike (1984) that these particular averages are the only ones having the required properties.

A different technique, which also gives an explicit formula for $\mathbf{F}_{i+1/2}(\mathbf{w}_L, \mathbf{w}_R)$ was devised by Osher (Osher & Solomon 1982). It generalizes earlier work on single conservation laws by Engquist & Osher (1981), which was successfully applied to calculations of transonic potential flow by Goorijan & van Buskirk (1981) and Miller (1983).

The idea is to consider each possible state of the fluid as a point in a phase space (which has three components for one-dimensional flow). A C^1 simple wave in physical space is one in which all changes are brought about by action of the $(u-a)$ characteristics (Courant & Friedrichs 1948). In passing through it, Equations (6b) and (6c) are obeyed. In phase space, a C^1 simple wave path is also a curve along which (6b) and (6c) are obeyed. The tangent to the wave path is such that

$$(d\rho : du : dp) = (\rho : -a : \rho a^2). \tag{36a}$$

We define C^2 and C^3 wave paths in the same way; the tangents follow

respectively the directions

$$(d\rho : du : dp) = (1 : 0 : 0), \tag{36b}$$

$$(d\rho : du : dp) = (\rho : a : \rho a^2). \tag{36c}$$

Taking $\mathbf{w}_L$ and $\mathbf{w}_R$ to be fixed points in phase space, there is a unique path P between them that follows first a C^3 wave path, then a C^2 wave path, and finally a C^1 wave path. The formal expression for the interface flux is

$$\mathbf{F}_{i+1/2}(\mathbf{w}_L, \mathbf{w}_R) = \tfrac{1}{2}(\mathbf{F}_L + \mathbf{F}_R) - \tfrac{1}{2}\int_{\mathbf{w}_L}^{\mathbf{w}_R} |\lambda|\, d\mathbf{w}, \tag{37}$$

where the integral follows the path P and λ is the speed (u, or $u \pm a$) of the wave at each point of the path. Note the formal similarity with (28). Algebraic formulas for the value of this integral are given by Osher & Solomon (1982) and Osher (1982). This method reproduces (by construction) the exact solution to Riemann's problem if only a single expansion wave or contact discontinuity is present. It is not exact if more than one simple wave is present because the wave paths are traversed in the order opposite to that in which they physically occur. This is because the authors find that the seemingly correct ordering does not produce monotonic shock waves.

Since Roe's method does not allow for the finite extent of expansion waves, it can sometimes lead to false results. If an expansion wave contains both left- and right-running characteristics, it is replaced by an "average wave," which runs only in one direction. In the extreme case, when the average wave has zero speed, it makes no contribution to (28) and so no change occurs to the solution. A discontinuity that should be physically unstable is numerically stable. The situation can be remedied by adding an extra term (Harten 1983, Roe & Pike 1984, Roe 1985a). A numerical scheme that can be proved to accept only stable discontinuities is said to be an "entropy scheme," or "E-scheme." Osher (1984) has proved that certain schemes have this property. Interestingly, Godunov's scheme is the limiting member of this class, if $\mathbf{F}_{i+1/2}$ is restricted to depend only on $\mathbf{w}_i$ and $\mathbf{w}_{i+1}$.

Steady shock waves are captured in slightly more compact form by Roe's method. In general the transition zone contains one point (Roe 1981), whereas Osher's method produces two (Osher & Solomon 1982).

Flux-Vector Splitting

The simplest way of introducing upwinding into a conservative scheme seems to be based upon the following mathematical trick. We try to represent the flux vector $\mathbf{F}(\mathbf{w})$ as the sum $\mathbf{F}^-(\mathbf{w}) + \mathbf{F}^+(\mathbf{w})$, such that the two

"conservation laws"

$$\mathbf{w}_t + [\mathbf{F}^-(\mathbf{w})]_x = 0, \tag{38a}$$

$$\mathbf{w}_t + [\mathbf{F}^+(\mathbf{w})]_x = 0, \tag{38b}$$

can be stably computed by the backward- and forward-differencing schemes

$$\mathbf{w}_i^{n+1} - \mathbf{w}_i^n + \frac{\Delta t}{\Delta x}[\mathbf{F}^-(\mathbf{w}_{i+1}^n) - \mathbf{F}^-(\mathbf{w}_i^n)] = 0, \tag{39a}$$

$$\mathbf{w}_i^{n+1} - \mathbf{w}_i^n + \frac{\Delta t}{\Delta x}[\mathbf{F}^+(\mathbf{w}_i^n) - \mathbf{F}^+(\mathbf{w}_{i-1}^n)] = 0, \tag{39b}$$

respectively. This would be equivalent to using an approximate Riemann solution in which

$$\mathbf{F}_{i+1/2}(\mathbf{w}_L, \mathbf{w}_R) = \mathbf{F}^+(\mathbf{w}_L) + \mathbf{F}^-(\mathbf{w}_R) \tag{40}$$

and amounts to requiring that the Jacobian matrices A^-, A^+, defined by

$$A^- = \frac{\partial \mathbf{F}^-}{\partial \mathbf{w}}, \qquad A^+ = \frac{\partial \mathbf{F}^+}{\partial \mathbf{w}},$$

have respectively no positive and no negative eigenvalues. A simple method of splitting $\mathbf{F}$ into two parts that have this property is to define

$$\mathbf{F}_1 = \frac{\rho(u-a)}{2\gamma}\begin{vmatrix} 1 \\ u-a \\ h-ua \end{vmatrix}, \qquad \mathbf{F}_2 = \frac{(\gamma-1)\rho u}{\gamma}\begin{vmatrix} 1 \\ u \\ \frac{1}{2}u^2 \end{vmatrix},$$

$$\mathbf{F}_3 = \frac{\rho(u+a)}{2\gamma}\begin{vmatrix} 1 \\ u+a \\ h+ua \end{vmatrix}. \tag{41}$$

It is easily checked that $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = \mathbf{F}$, and each has an associated wave speed

$$\lambda_1 = (u-a), \qquad \lambda_2 = u, \qquad \lambda_3 = (u+a). \tag{42}$$

Each $\mathbf{F}_i$ is assigned to $\mathbf{F}^-$ if $\lambda_i < 0$, and to $\mathbf{F}^+$ if $\lambda_i > 0$.

This idea seems to have been first used in the context of astrophysical gasdynamics (Sanders & Prendergast 1974), and to have been rediscovered, with a fuller mathematical development, by Steger & Warming (1981). Note that each $\mathbf{F}_i$ vanishes at the moment it is transferred from $\mathbf{F}^-$ to $\mathbf{F}^+$. However, calculations made using this particular splitting showed small

errors around the sonic point, which were attributed to the transition not being made smoothly enough. Steger & Warming had already noted that the requirement of stable differencing made the choice of $\mathbf{F}^-, \mathbf{F}^+$ far from unique. Van Leer (1982) has given an alternative splitting such that each $\mathbf{F}_i$ is "faded out" with zero derivative, and this refinement gives noticeably better results (Andersson et al. 1985).

Flux-vector splitting is not entirely clear in its physical interpretation. Harten et al. (1983) interpret it in terms of a Boltzmann model with three sorts of particles moving with distinctive speeds, so that analyzing a single state $\mathbf{w}$ reveals how those particles are distributed. Clearly this is not a realistic description of a continuum fluid, and yet the model seems to work almost as well as the methods that more plausibly infer the passage of waves from a comparison of two neighboring states. It should be noted that although the incorporation of physical ideas into numerical analysis appears to pay dividends, we have as yet little idea how precise these ideas need to be.

Nonconservative Upwind Schemes

The only convincing demonstrations of captured discontinuities moving at the correct speed all involve conservative discretizations. However, by basing a scheme on the conservation form [Equations (1)], we seemingly must accept some fairly lengthy algebraic expressions, which do not make for efficient code design. (Typically, conservative upwind codes take about three times as long to update a mesh point as do nonupwind codes.) Nonconservative codes, which fit the shock waves by explicitly computed discontinuities, can incorporate upwinding very cheaply. This can be worth doing, because the information about characteristic behavior becomes useful at boundaries and at places where the discontinuities are fitted.

If we choose our dependent variables for convenience, it is natural to choose u and a as two of them, since these quantities must be used in any upwind scheme. Many workers have felt the natural third choice to be entropy $s = p/\rho^\gamma$. In these variables, the Euler equations are

$$a_t + ua_x + kau_x = 0, \tag{43a}$$

$$u_t + uu_x + \frac{a}{k}a_x - \frac{a^2 s_x}{\gamma(\gamma-1)} = 0, \tag{43b}$$

$$s_t + us_x = 0, \tag{43c}$$

where $k = \frac{1}{2}(\gamma-1)$. We can rewrite these equations as follows:

$$a_t + \tfrac{1}{2}(u-a)a_x - \tfrac{1}{2}k(u-a)u_x + \tfrac{1}{2}(u+a)a_x + \tfrac{1}{2}k(u+a)u_x = 0, \tag{44a}$$

$$u_t+\tfrac{1}{2}(u-a)u_x-\frac{1}{2k}(u-a)a_x+\tfrac{1}{2}(u+a)u_x+\frac{1}{2k}(u+a)a_x$$

$$+\frac{a(u-a)}{2\gamma(\gamma-1)}\frac{s_x}{s}-\frac{a(u+a)}{2\gamma(\gamma-1)}\frac{s_x}{s}=0, \tag{44b}$$

$$s_t+us_x=0. \tag{44c}$$

It can be shown that all the terms containing a factor $(u-a)$ correspond to information carried by the $(u-a)$ characteristics, and these terms are evaluated using information from $[i-1, i]$ or $[i, i+1]$, depending upon the sign of $(u-a)$ at $(i\Delta x, u\Delta t)$. The same is true, *mutatis mutandis*, of terms containing $(u+a)$. The term us_x in (44c) represents the information carried along streamlines and is evaluated according to the sign of u. Numerical schemes based on (44) can be found in articles by Moretti (1979) and Gabutti (1983). Some ingenious programming devices to aid the tracking of discontinuities are given by Moretti & DiPiano (1983). An extension to (shock-free) two-dimensional flow is presented by Moretti & Zannetti (1984).

Flow With Area Variation

Until now, our discussion has concerned flow in a parallel pipe with no complicating events. Perhaps the simplest extension we may consider is to flow in a narrow duct of variable area. The Euler equations then read

$$(A\mathbf{w})_t+(A\mathbf{F})_x=\mathbf{S}, \tag{45}$$

where $A(x)$ is the area of the duct, and the source term is $\mathbf{S}=(0, -p\partial A/\partial x, 0)$. Several authors solve this problem by advancing one time step by an upwind method, as though $\mathbf{S}$ were absent, and then adding the effects of $\mathbf{S}$. It seems, however, that this particular form of splitting is inappropriate because it neglects those interactions between the wave propagation and the area changes that are the essence of the problem. Data that would produce a simple wave in a pipe will give rise to strongly reflected waves in a duct. Other authors have applied elaborate schemes to (45). Ben-Artzi & Falcovitz (1984) propose a "generalized Riemann problem," which is the solution of (45) with initial data $\mathbf{w}=\mathbf{w}_L+x\mathbf{C}$ $(x<0)$ and $\mathbf{w}=\mathbf{w}_R+x\mathbf{D}$ $(x>0)$. They find a solution to this problem in the vicinity of the origin, and from it they derive interface fluxes. Glimm et al. (1984) and, independently, Glaz & Liu (1984) consider a similar generalization in which the data are determined by $\mathbf{w}_L, \mathbf{w}_R$ with the condition that, apart from the discontinuity at $x=0$, the data should be a steady solution of (45). This idea was first suggested by van Leer (1981) for single conservation laws.

A far simpler way of accounting for the source term has been developed by the present author. The expressions (34) for the wave strengths in Roe's Riemann solver are modified as follows:

$$\alpha_1 = \frac{1}{2a^2}\left[\Delta p - \rho a \Delta u + \frac{\rho u a^2 \Delta A}{(u-a)A}\right], \tag{46a}$$

$$\alpha_2 = \frac{1}{a^2}[a^2\Delta\rho - \Delta p], \tag{46b}$$

$$\alpha_3 = \frac{1}{2a^2}\left[\Delta p + \rho a \Delta u + \frac{\rho u a^2 \Delta A}{(u+a)A}\right], \tag{46c}$$

where the extra terms in (46a) and (46c) contain a factor ΔA that is the area change over the interval $(i, i+1)$. These expressions have the property that $\alpha_1 = \alpha_2 = \alpha_3 = 0$ when the flow is steady. Note that the predicted wave strengths are infinite for sonic flow unless $\Delta A = 0$, which fits with the well-known property that the only stable position for a sonic point is at a throat. One way to implement this scheme is to evaluate an interface flux from Equation (28) and note that the source terms are already incorporated. This method has been applied by Glaister (1985) to a test problem devised by Noh (1983). An ideal gas with $p = 0$, $\rho = 1$, and $\gamma = 5/3$ pours radially inward toward the center of a sphere with $u = -1$. A shock forms at the center and moves outward. For these idealized conditions, the exact solution can be found and is shown in Figure 5. Glaister's numerical results are also shown and are much better than can be obtained from artificial-viscosity methods.

Higher-Order Accuracy

No variant of Godunov's method yet discussed here has overcome the fundamental limitation apparently imposed by the theorem following Equation (19). The first specific statement of the loophole that allows monotonicity to be achieved simultaneously with better than first-order accuracy was given in this context by van Leer (1974). The marching algorithm must be nonlinear, even when applied to the linear problem (13). We can be slightly more precise; let M be the operator such that $u^{n+1} = Mu^n$ and M approximates a shift through $a\Delta t$. Roe & Baines (1984) stipulate that for any constant k

$$M(ku) = kM(u), \tag{47}$$

but not necessarily that

$$M(u+v) = M(u) + M(v). \tag{48}$$

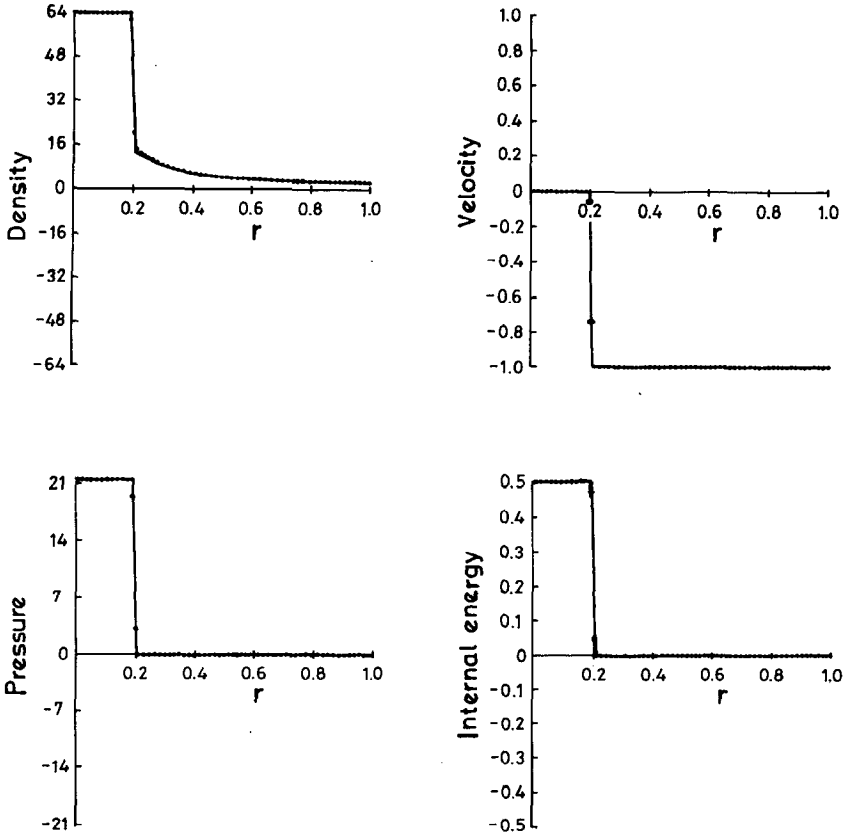

Figure 5 Calculated flow in a spherical shock problem. The initial conditions are $p = 0$, $u = -1$, and $\rho = 1$. The shock is moving outward from the origin after a dimensionless time $t = 0.60$. The number of grid points is 100.

These nonlinear algorithms can be designed to have specified properties for the linear advection problem and then empirically extended to the Euler equations. What is remarkable is that the empirical extensions made by several workers have turned out to be more or less equivalent. We give an account here that does not precisely follow any published method, but that sketches the central idea inherent in them all.

To solve (13) in conservation form (with $f = au$) by the upwind scheme, we could specify

$$u_i^{n+1} - u_i^n = -\frac{\Delta t}{\Delta x}(f_{i+1/2} - f_{i-1/2}), \tag{49}$$

where

$$f_{i+1/2} = \tfrac{1}{2}(f_i + f_{i+1}) - \frac{\operatorname{sgn} v}{2}(f_{i+1} - f_i) \tag{50}$$

and sgn $v = \pm 1$ denotes the sign function. Equation (50) is valid for both positive and negative wave speeds. The simplest second-order scheme is that of Lax & Wendroff (1964), which results from taking

$$f_{i+1/2} = \tfrac{1}{2}(f_i + f_{i+1}) - \frac{v}{2}(f_{i+1} - f_i). \tag{51}$$

One way to guarantee second-order accuracy is to ensure that the expression for $f_{i+1/2}$ differs from (51) by $O(\Delta x^2)$. The difference between (51) and (50) is

$$\tfrac{1}{2}(\operatorname{sgn} v - v)(f_{i+1} - f_i), \tag{52}$$

which may be regarded as a second-order correction term, valid in smooth regions of the flow. In nonsmooth regions, the presence of (52) converts a monotonic scheme into a nonmonotonic scheme, so it is reasonable to limit its effect by multiplying it by some positive factor. Denote the multiplying factor by φ. It has been found that successful algorithms result if φ is made to depend on the ratio of $(u_{i+1} - u_i)$ to $(u_i - u_{i-1})$ if $a > 0$, or the ratio of $(u_{i+1} - u_i)$ to $(u_{i+2} - u_{i+1})$ if $a < 0$. Therefore, define (with $\sigma = \operatorname{sgn} v$)

$$r = \frac{u_{i+1+\sigma} - u_{i+\sigma}}{u_{i+1} - u_i} \tag{53}$$

and set

$$f_{i+1/2} = \tfrac{1}{2}(f_i + f_{i+1}) - \frac{\sigma}{2}(f_{i+1} - f_i) + \frac{\varphi(r)}{2}(\sigma - v)(f_{i+1} - f_i), \tag{54}$$

where the first two terms on the right-hand side produce upwind differencing and the third term enhances the accuracy. If $\varphi = 1$ we recover the Lax-Wendroff scheme, and since $r = 1 + O(\Delta x)$, it follows that if φ is any function satisfying

$$\varphi(1 + \varepsilon) = 1 + O(\varepsilon), \tag{55}$$

the scheme is second-order accurate. The usefulness of Equation (54) resides in the fact that the resulting scheme can be proved to be monotonic under certain additional restrictions on $\varphi(r)$, discussed by van Leer (1974), Roe & Baines (1982, 1984), Roe (1985a), and Sweby (1984). A common "limiter

function" is

$$\varphi(r) = 0, \qquad r < 0$$
$$\varphi(r) = \min(1, r), \qquad r > 0 \tag{56}$$

but a better choice is van Leer's recommendation

$$\varphi(r) = \frac{r + |r|}{1 + r^2}, \tag{57}$$

and even sharper discontinuities are produced by

$$\varphi(r) = 0, \qquad r < 0$$
$$\varphi(r) = \min(2r, 1), \qquad 0 < r < 1 \tag{58}$$
$$\varphi(r) = \min(2, r), \qquad 1 < r.$$

Roe & Baines (1984) report the dramatic result that using (58) allows linear discontinuities to propagate for an indefinitely long time without numerical diffusion.

The generalization of (54) to systems of nonlinear conservation laws is most directly accomplished following the ideas proposed by Roe (1980, 1981, 1982) and somewhat extended by Harten (1983). Corresponding to the first two terms on the right-hand side of (54), we may choose $\mathbf{F}_{i+1/2}$ as computed by any of the conservative upwind schemes, i.e. either Godunov's original suggestion or any of Equations (28), (37), or (40). The remaining term can be approximated by

$$\tfrac{1}{2} \sum_k \varphi(r_k)(\sigma_k - \nu_k)\alpha_k \lambda_k \mathbf{e}_k, \tag{59}$$

where $\nu_k = \lambda_k \Delta t / \Delta x$. Without the factors $\varphi(r_k)$, this formula would yield a second-order Lax-Wendroff scheme. Their inclusion rests on the intuitive notion that the effects of each wave can be treated separately. This is not wholly consistent, since wave-interaction effects are second order and it is the second-order terms that we are discussing, but in practice the treatment is highly successful. The exact definition of r_k does not seem to be critical; it is convenient to take it as

$$r_k = \frac{(\alpha_k)_{i+1/2+\sigma}}{(\alpha_k)_{i+1/2}}. \tag{60}$$

Other nonlinear mechanisms for evading Godunov's theorem are remarkably similar in principle. van Leer's "MUSCL" scheme (1979) works through a preliminary nonlinear smoothing operation on the data. It has since been simplified (Goodman & Leveque 1984) and also elaborated (Colella & Woodward 1984).

The notion of a nonlinear "antidiffusion" step is central to the flux-corrected transport (FCT) algorithms originating from the work of Boris & Book (1973). Steps toward creating a characteristic-based form of FCT are reported by McDonald & Ambrosiano (1984).

Two-Dimensional Extensions

A conservative discretization in two space dimensions is easily accomplished for an arbitrary computational mesh, as shown in Figure 6. The rate at which the conserved variables $\mathbf{w}$ are transferred from cell (i, j) to cell $(i+1, j)$ is given by

$$(\mathbf{F}_{i+1/2,j})(y_P - y_Q) - (\mathbf{G}_{i+1/2,j})(x_P - x_Q), \tag{61}$$

where the vectors $\mathbf{w}$, $\mathbf{F}$, and $\mathbf{G}$ are now given by

$$\mathbf{w} = \begin{vmatrix} \rho \\ \rho u \\ \rho v \\ \rho e \end{vmatrix}, \quad \mathbf{F} = \begin{vmatrix} \rho u \\ p + \rho u^2 \\ \rho uv \\ u(\rho e + p) \end{vmatrix}, \quad \mathbf{G} = \begin{vmatrix} \rho v \\ \rho uv \\ p + \rho v^2 \\ v(\rho e + p) \end{vmatrix} \tag{62}$$

and where u and v are the velocity components in some fixed Cartesian coordinates (x, y). The total change of conserved quantities within $PQRS$ is the sum of four terms such as (61). The characteristic-based schemes developed for one-dimensional flow can be used straightforwardly to estimate appropriate values for the interface-flux functions, provided one crucial assumption is made. This is that conserved quantities are exchanged between cells by means of waves traveling normal to the interfaces. One then solves either exactly or by one of the approximate methods a one-dimensional Riemann problem in which velocities parallel to the interface are neglected. Any difference in the parallel component is assumed to take

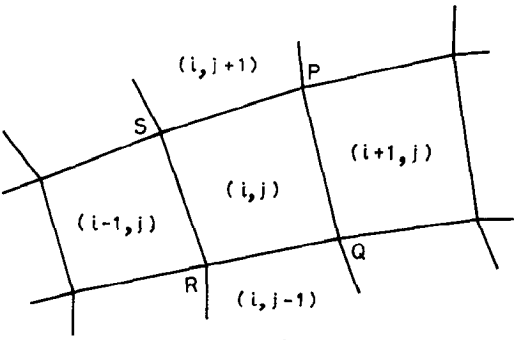

Figure 6 Part of a mesh for two-dimensional calculations.

place across the contact surface. The inclusion of terms analogous to (59) elevates the scheme to second-order accuracy without inducing oscillations, but the details vary from author to author and must be sought in the cited references.

The first practical application of these ideas was made by Sells (1980), who computed the steady, inviscid transonic flow past two-dimensional airfoils as the limit at large time of an unsteady computation. Sells' code, although time-consuming, proved to be very robust in the sense defined in the introduction, and it was extremely useful as a means of calibrating more empirically based methods. Several minor improvements are described in Lytton (1984). Other applications to steady transonic flow can be found in Yee et al. (1985a,b), Chakravarthy & Osher (1985), Eidelmann et al. (1984), and Andersson et al. (1985). A typical example is shown in Figure 7, taken from the last-cited paper. The pressure distribution is shown around a NACA 0012 airfoil at an incidence of 1.25° in a uniform flow with Mach number 0.80. The illustration on the left was obtained by a characteristic-based method using flux-vector splitting. The right-hand side shows results obtained on the same computational mesh by the central-differencing artificial-viscosity scheme of Jameson et al. (1981). The left-hand picture defines the shock waves much more crisply, especially the weak one on the lower surface. However, since Jameson et al.'s scheme is faster, the economics of the comparison is not clear.

At present, the characteristic-based schemes seem to show to better advantage in calculating transient flows. Impressive examples of one- and

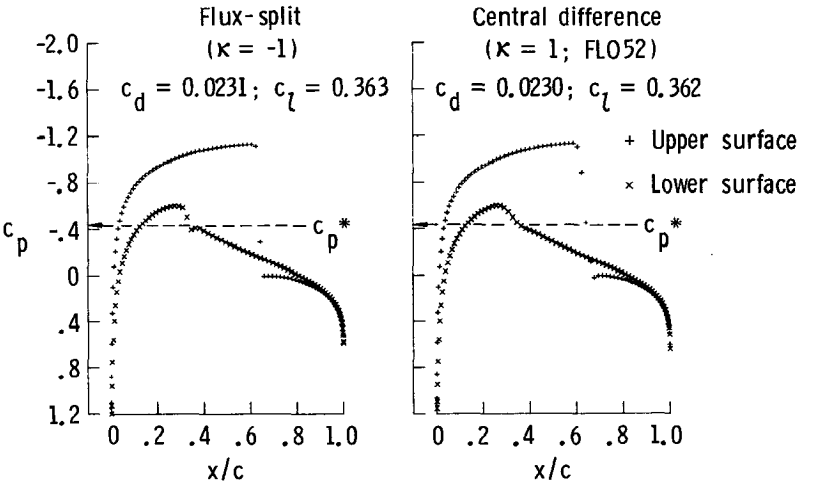

Figure 7 Two different calculations for the pressure distribution over a NACA 0012 airfoil. C_p^* denotes the pressure coefficient at sonic conditions.

two-dimensional unsteady flows can be found in Woodward & Colella (1984). This group is currently engaged in computing diffraction patterns due to a shock passing over a wedge (Glaz et al. 1985). They are able to explore in great detail the variety of patterns that arise. An example is given in Figure 8, where we see contours of density, internal energy, and pressure arising from a shock wave (moving from right to left at Mach number 8 through a perfect gas with $\gamma = 1.3$) that meets a 35° angle wedge. The surface of the wedge is horizontal in the figure, and a short section of the incident shock can be seen at far left. Noteworthy is the clear definition of a vortex generated by shock interactions.

Another illustration of unsteady flow with vorticity appears in Figure 9, which shows two stages in the breakup of a two-dimensional jet (flowing from left to right at Mach number 2). The fluid outside the jet is initially at rest and 10 times denser than the jet. In both fluids the value of γ is 5/3. One can clearly see the development of small-scale structures, roughly periodic, marking the early (inviscid) stages of decline into turbulence.

In Figure 10, we see an example drawn from astrophysics, a subject in

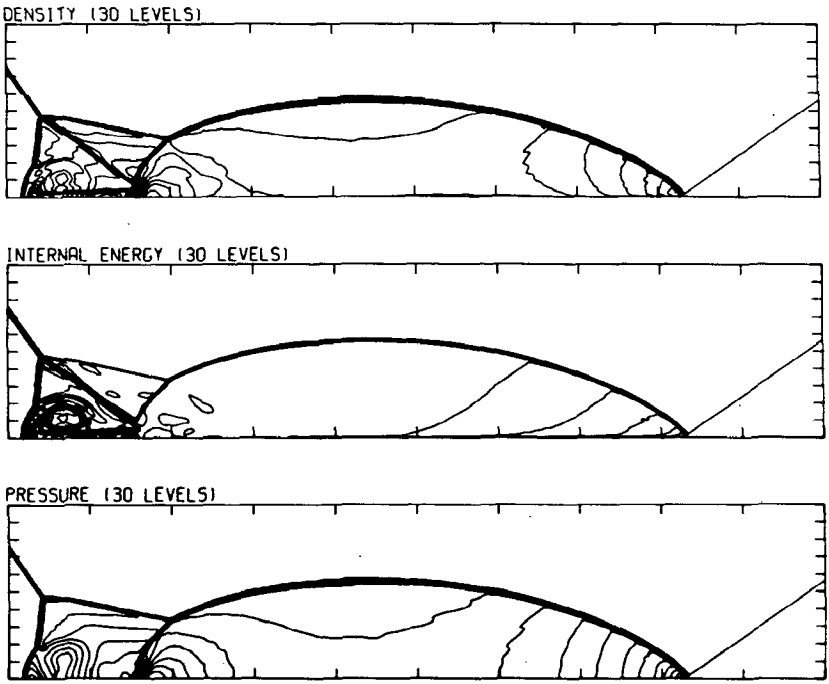

Figure 8 Contours of density, internal energy, and pressure in the flow due to a diffracted shock wave.

which a good deal of computation is done to check the plausibility of explanations offered for the observed phenomena. Here, gravitating matter is supposed to be spread over a thin, rotating elliptical disk. The equations describing the motion of the interstellar gas within this disk are the Euler equations, simplified by assuming isothermal conditions and with source terms representing gravitational forces and star formation. Mulder (1985) solves these equations using a second-order flux-splitting technique. In his solution, the gas swirls around in the nonuniform gravitational field, experiencing accelerations and decelerations, with the latter events causing shock waves and, therefore, large density gradients. In Figure 10, the darker areas represent the denser regions. Mulder claims that a plausible range of parameters can account for all the observed patterns of spiral galaxies.

Our final example returns us to Earth, and to steady flow, but introduces a consideration of viscosity. Figure 11 shows a solution of the Navier-Stokes equations (Yee 1985) for the transonic flow past an RAE 2822 airfoil at a Reynolds number of 6.5×10^6 and angle of incidence of 2.79°. The characteristic-based schemes are used to model the inviscid terms; a similar

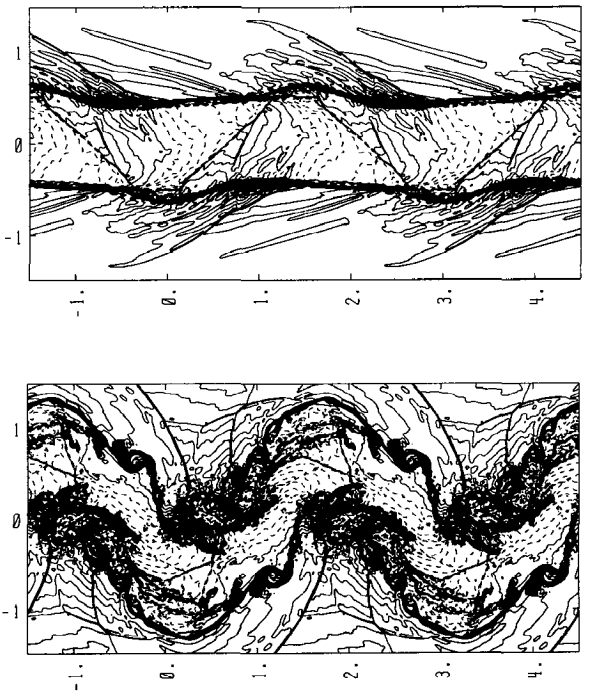

Figure 9 Density contours at two stages in the breakup of a plane jet (P. Woodward, private communication).

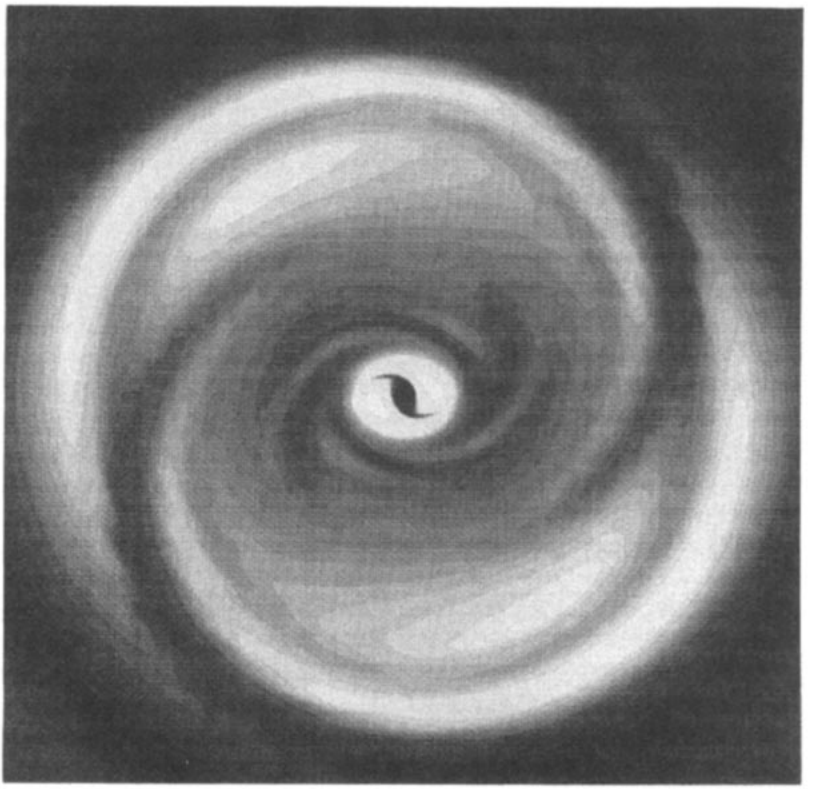

Figure 10 Density contours in a cloud of interstellar gas swirling through the gravitational field of a galaxy.

approach has been taken by Chakravarthy et al. (1985). One can see, in the contours of local Mach number, the developing boundary layer whose interaction with the shock wave brings the shock to the very place observed in experiments by Cook et al. (1979).

Despite these valuable results, however, it is possible that characteristic-based schemes can be applied to problems in two (or more) dimensions more effectively than they have been. All the results in this section derive from the above-mentioned assumption that all waves propagate normal to the cell boundaries, which is by no means as realistic in two dimensions as in one. Possible advantages of a more "grid-free" flow model are discussed by Davis (1984) and Roe (1985b).

Concluding Remarks

In this review I have demonstrated the mixture of mathematical and physical arguments that is serving to clarify the issues involved in computing

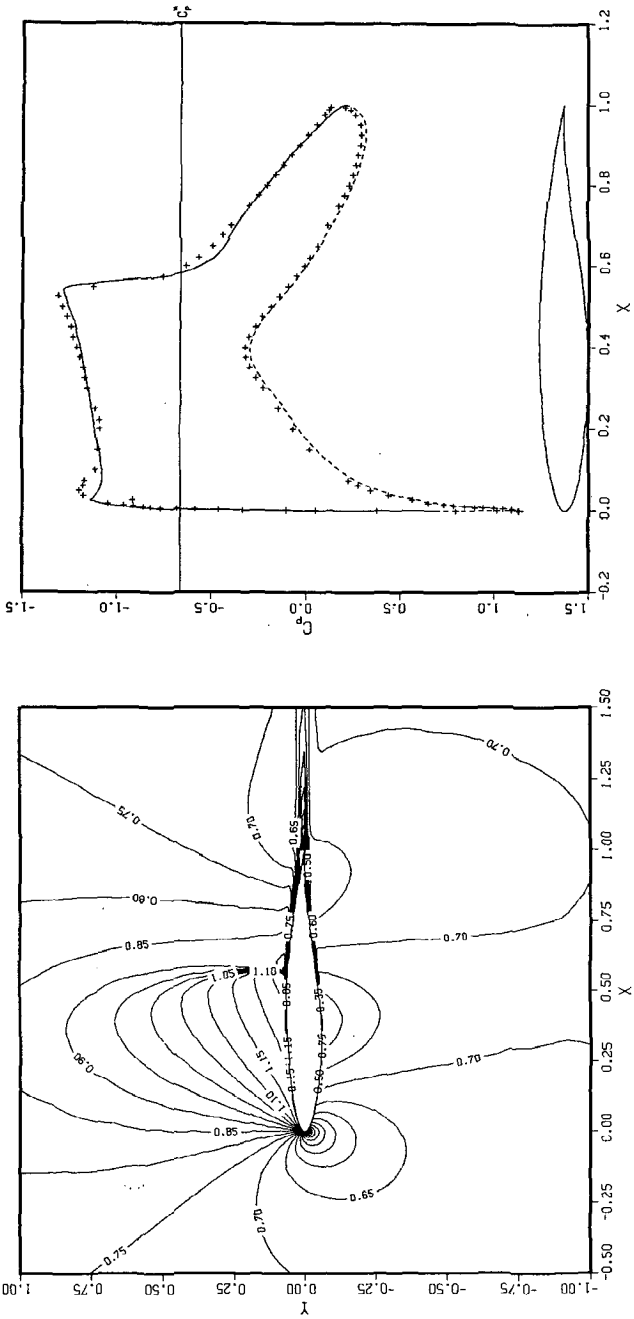

Figure 11 Calculations of viscous flow past an RAE 2822 airfoil. (*Left*) Contours of local Mach number. (*Right*) Calculated pressure distribution (shown as line) compared with experimental values (crosses).

accurate and efficient solutions to problems of inviscid compressible flow. This review comes at a time when publications in this area are still rapidly emerging, and although I hope to have selected from the more permanent material, I am uncomfortably aware of many omissions. Future work will further unify and consolidate the subject, as well as extend its applications to, for example, the hyperbolic problems that arise in dynamic elasticity and magnetohydrodynamics.

Characteristic-based schemes are also showing promise in the context of finite-element methods, as demonstrated by Hughes & Mallet (1985), Morton (1985), and Morton & Sweby (1985). A collection of papers for further reading can be found in the proceedings of the AMS-SIAM Summer Seminar on Large-Scale Computations in Fluid Mechanics held at San Diego in 1983 (published by SIAM in 1985 as number 22 in their series *Lectures in Applied Mathematics*).

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