Logic and Reasoning in Computer Science SS 2025

Exercise Sheet 2

Instructions for project submissions: You are requested to solve at least one project problem and submit your project solutions by **May 16, 2025**. Project problems are explicitly stated as *Project Problem 2.1* and *Project Problem 2.2*. If you solve both project problems, the deadline for submitting your project solutions is still May 16, 2025.

Please upload only your project problem solutions (any solution to other problems will be ignored). For the project submission, please upload your solutions in a zip archive named using your student ID, e.g. `e123456789.zip`, using the following naming and formatting:

- `proj_1.tptp` : A text file with your TPTP encoding of Project Problem 2.1 if you solved it. Please use comments in your TPTP encoding so that you explain what your encoding expresses.
- `proj_1.out` : The output of Vampire when running it on `proj_1.tptp`.
- `proj_2.tptp` : A text file with your TPTP encoding of Project Problem 2.2 if you solved it. Please use comments in your TPTP encoding so that you explain what your encoding expresses.
- `proj_2.out` : The output of Vampire when running it on `proj_2.tptp`.

Please **follow the file naming convention**: use `.tptp` and `.out` extensions, and file names, as specified. Submit your project solutions as instructed, by May 16, 2025. If the file names are incorrect, your projects may not be credited.

Exercise Problem 2.1. Consider the following formulas

$$\begin{aligned} \varphi_1 &= \forall x \exists y R(x,y) & \varphi_2 &= \forall x \forall y (R(x,y) \rightarrow R(y,x)) \\ \psi_1 &= \exists x \forall y R(x,y) & \psi_2 &= \forall x \forall y \forall z ((R(x,y) \land R(y,z)) \rightarrow R(x,z)). \end{aligned}$$

Answer the questions. To justify each answer you must show a model or argue why it does not exist.

(a) Is the theory $\{\varphi_1, \varphi_2, \psi_1\}$ consistent? (or equivalently, is $\varphi_1 \land \varphi_2 \land \psi_1$ satisfiable?)

(b) What about the theory $\{\neg\varphi_1, \varphi_2, \psi_1\}$, is it consistent?

(c) Finally, consider the theory $\{\varphi_1, \varphi_2, \neg\psi_1, \neg\psi_2\}$. Is it consistent?

Solution.

(a) Let $\mathcal{M} = (D, I)$ be a structure and $\ell : V \rightarrow D$ a variable assignment for $\mathcal{M}$ that

- $D = \{0, 1\}$ and $R^{\mathcal{M}} = I(R) = \{(0,0), (0,1), (1,0)\}$.

So $\{\varphi_1, \varphi_2, \psi_1\}$ is satisfiable in $\mathcal{M}$ wrt ℓ:

- $R^{\mathcal{M}}$ is serial and symmetric, so φ_1 and φ_2 are satisfiable in $\mathcal{M}$ wrt ℓ;
- ψ_1 is satisfiable — there is a $0 \in D$ such that $(0,1)$ and $(0,0)$ are both in $R^{\mathcal{M}}$.

(b) Suppose $\{\neg\varphi_1, \varphi_2, \psi_1\}$ is satisfiable in $\mathcal{M} = (D, I)$ wrt ℓ that $R^{\mathcal{M}} = I(R)$.

- From $\neg\varphi_1$ is satisfiable, there is a $d \in D$ for all $c \in D$ that $(d, c) \notin R^{\mathcal{M}}$ (@$_1$).
- From ψ_1 is satisfiable, there is a $d' \in D$ for all $c \in D$ that $(d', c) \in R^{\mathcal{M}}$ (@$_2$).

- From φ_2 is satisfiable in $\mathcal{M}$ wrt ℓ, $R^{\mathcal{M}}$ is symmetric. In addition with (@$_2$), which gives $(d', d) \in R^{\mathcal{M}}$, so $(d, d') \in R^{\mathcal{M}}$. However, this result is in conflict to (@$_1$).

So $\{\neg\varphi_1, \varphi_2, \psi_1\}$ is **not** satisfiable.

(c) Let $\mathcal{M} = (D, I)$ be a structure and $\ell : V \to D$ a variable assignment for $\mathcal{M}$ that

- $D = \{0, 1\}$ and $R^{\mathcal{M}} = I(R) = \{(0, 1), (1, 0)\}$.

So $\{\varphi_1, \varphi_2, \neg\psi_1, \neg\psi_2\}$ is satisfiable in $\mathcal{M}$ wrt ℓ:

- As $R^{\mathcal{M}}$ is serial and symmetric, φ_1 and φ_2 are satisfiable in $\mathcal{M}$ wrt ℓ;
- From $(0, 0), (1, 1) \notin R^{\mathcal{M}}$, we know $\neg\psi_1$ is satisfiable in $\mathcal{M}$ wrt ℓ;
- From $R^{\mathcal{M}}$ is not transitive — $(0, 1), (1, 0) \in R^{\mathcal{M}}$ but $(0, 0) \notin R^{\mathcal{M}}$, it follows that $\neg\psi_2$ is satisfiable in $\mathcal{M}$ wrt ℓ.

Exercise Problem 2.2. Consider the following formulas.

$$\begin{aligned}
\varphi_1 &= \forall x\ R(x, s(x)) \\
\varphi_2 &= \forall x \forall y\ (R(x, y) \to \neg R(y, x)) \\
\varphi_3 &= \forall x \forall y \forall z (R(x, y) \land R(y, z) \to R(x, z))
\end{aligned}$$

(a) Can you find a model of $\varphi_1 \land \varphi_2 \land \varphi_3$?

(b) Is there a model of $\varphi_1 \land \varphi_2 \land \varphi_3$ with a *finite* domain? Explain your answer!

Solution.

(a) Let $\mathcal{M} = (D, I)$ be a structure and $\ell : V \to D$ a variable assignment for $\mathcal{M}$ that

- $D = \mathbb{N}$ and $I(R) =<$, as well as $\mathsf{s} = I(s) : \mathbb{N} \to \mathbb{N}$ such that $\mathsf{s}(n) = n + 1$ where $n \in \mathbb{N}$.

So $\varphi_1 \land \varphi_2 \land \varphi_3$ is satisfiable in $\mathcal{M}$:

- φ_1 is satisfiable in $\mathcal{M}$ wrt ℓ — $(n, n + 1) \in <$ for all $n \in \mathbb{N}$.
- As $<$ is asymmetric and transitive, therefore the model $\mathcal{M}$ satisfies φ_2 and φ_3 as well.

(b) Suppose there is a finite model $\mathcal{M} = (D, I)$ and $\ell : V \to D$ a variable assignment for $\mathcal{M}$ such that $D = \{d_1, \ldots, d_n\}$, $R^{\mathcal{M}} = \mathsf{R} \subseteq D \times D$, and $s^{\mathcal{M}} = \mathsf{s} : D \to D$. Assume that $\{\varphi_1, \varphi_2, \varphi_3\}$ is satisfiable in $\mathcal{M}$ wrt ℓ. So,

- From $\mathcal{M}, \ell \models \varphi_1$ it follows that for all $d \in D$ that $d\mathsf{R}\mathsf{s}(d)$ (@$_1$);
- From $\mathcal{M}, \ell \models \varphi_2$ it follows that for all $d, c \in D$ that if $d\mathsf{R}c$ then **not** -$c\mathsf{R}d$ (@$_2$).
- From $\mathcal{M}, \ell \models \varphi_3$ it follows that for all $d, c, e \in D$ that if $d\mathsf{R}c$ and $c\mathsf{R}e$ then $d\mathsf{R}e$ (@$_3$).

- We define a new notation $\mathsf{s}^{n+1}(d) = \overbrace{\mathsf{s}(\ldots \mathsf{s}}^{n+1}(d)) = \mathsf{s}(\mathsf{s}^n(d))$ — so, $\mathsf{s}^n(d)$ means that we apply the function s on $d \in D$ in n times.
- By (@$_1$), it follow that $\mathsf{s}^n(d)\mathsf{R}\mathsf{s}^{n+1}(d)$ where $n \in \mathbb{N}$ (@$_4$) — meaning that every $\mathsf{s}^n(d)$ is accessed by its direct successor $\mathsf{s}^{n+1}(d)$.
- By (@$_3$) and (@$_4$), we have $d\mathsf{R}\mathsf{s}^k(d)$ (@$_5$) — meaning that every d is accessed by its k-successor $\mathsf{s}^k(d)$, where $k \in \mathbb{N}$.

- By ($@_2$), given any dRc, it is not possible to have $d = c$. Otherwise, it would lead to dRd which is in conflict to ($@_2$). This means that, every point cannot be its successor nor its ancestor ($@_6$).
- By ($@_5$) and ($@_6$), we have $d\mathsf{R}\mathsf{s}^1(d)$, $d\mathsf{R}\mathsf{s}^2(d), \ldots$, $d\mathsf{R}\mathsf{s}^n(d)$; however, $d, \mathsf{s}^1(d), \mathsf{s}^2(d), \ldots, \mathsf{s}^n(d)$ are $n+1$ different points, which is contrary to the assumption that D only have n elements!

(You can find the same strategy on page 113 of the Lecture 10 slides, or see the example shown below.)

- From ($@_4$), we can enumerate $d\mathsf{R}\mathsf{s}^1(d)$, $\mathsf{s}^1(d)\mathsf{R}\mathsf{s}^2(d), \ldots, \mathsf{s}^{k-1}(d)\mathsf{R}\mathsf{s}^k(d)$, where $k < n$.
- Otherwise, $k \geq n$. Suppose $k = n$ and so $\mathsf{s}^{n-1}(d)\mathsf{R}\mathsf{s}^n(d)$.
- Given that D only have n elements, there must be a $i < n$ such that $\mathsf{s}^n(d) = \mathsf{s}^i(d)$. So $\mathsf{s}^{n-1}(d)\mathsf{R}\mathsf{s}^i(d)$ ($@_7$).
- By ($@_3$) and the enumeration, we have $\mathsf{s}^i(d)\mathsf{R}\mathsf{s}^{n-1}(d)$ ($@_8$).
- However, ($@_7$) and ($@_8$) together are in conflict to ($@_2$).

Exercise Problem 2.3. Consider the formula φ:

$$\exists x P(x) \wedge \forall x(P(x) \rightarrow Q(f(x)))$$

Which of the following sentences are entailed by φ? For each ψ_i provide either a proof of the entailment or a countermodel witnessing the non-entailment.

$$\psi_1 := \exists x P(x) \wedge \exists x Q(x) \qquad \psi_2 := \exists x(P(x) \wedge Q(x))$$
$$\psi_3 := \forall x(\neg P(x) \vee \neg Q(f(x))) \qquad \psi_4 := \forall x(\neg Q(f(x)) \rightarrow \neg P(x))$$

You can use semantic arguments (e.g., arguing about all interpretations) or proof-theoretic arguments (e.g., tableau proofs). You can choose the approach you prefer, provided that you use it correctly!
Solution.

- The case of ψ_1: Suppose $\{\varphi, \neg\psi_1\}$ is satisfiable. However, its semantic tableau is closed. So this set is unsatisfiable, and ψ_1 is entailed from φ.

$$\begin{array}{lccc}
1. & \multicolumn{3}{c}{\mathsf{t} : \exists x P(x) \wedge \forall x(P(x) \rightarrow Q(f(x))) \checkmark} \\
2. & \multicolumn{3}{c}{\mathsf{f} : \exists x P(x) \wedge \exists x Q(x) \checkmark} \\
3. & \multicolumn{3}{c}{\mathsf{t} : \exists x P(x) \checkmark c} \\
4. & \multicolumn{3}{c}{\mathsf{t} : P(c) \checkmark} \\
5. & \multicolumn{3}{c}{\mathsf{t} : \forall x(P(x) \rightarrow Q(f(x))) \checkmark c} \\
6. & \multicolumn{3}{c}{\mathsf{t} : P(c) \rightarrow Q(f(c)) \checkmark} \\
7. & \mathsf{f} : \exists x P(x) \checkmark c & \multicolumn{2}{c}{\mathsf{f} : \exists x Q(x) \checkmark c} \\
8. & \mathsf{f} : P(c) & \multicolumn{2}{c}{\mathsf{f} : Q(f(c)) \checkmark} \\
 & \otimes & & \\
9. & & \mathsf{f} : P(c) & \mathsf{t} : Q(f(c)) \\
 & & \otimes & \otimes
\end{array}$$

- The case of ψ_2. The set $\{\varphi, \neg\psi_2\}$ is satisfiable — Its semantic tableau is **not** closed. So ψ_2 is **not** entailed from φ.

1. $\mathsf{t} : \exists x\ P(x) \land \forall x(P(x) \to Q(f(x)))\ \checkmark$
2. $\mathsf{f} : \exists x(P(x) \land Q(x))\ \checkmark c$
3. $\mathsf{t} : \exists x P(x)\ \checkmark c$
4. $\mathsf{t} : P(c)$
5. $\mathsf{t} : \forall x(P(x) \to Q(f(x)))\ \checkmark c$
6. $\mathsf{t} : P(c) \to Q(f(c))\ \checkmark$
7. $\mathsf{f} : P(c) \land Q(c)\ \checkmark$
8. $\mathsf{f} : P(c)$ $\otimes$ $\qquad$ $\mathsf{f} : Q(c)$
9. $\qquad\qquad$ $\mathsf{f} : P(c)$ $\otimes$ $\qquad$ $\mathsf{t} : Q(f(c))$

There is a counter model of the entailment $\varphi \vDash \psi_2$. Let $\mathcal{M} = (D, I)$ be a structure and $\ell : V \to D$ such that $D = \{a, c\}$, $P^{\mathcal{M}} = \{c\}$, $Q^{\mathcal{M}} = \{a\}$, and $f^{\mathcal{M}}(c) = a$. So $\mathcal{M}$ falsifies this entailment.

- The case of ψ_3: The set $\{\varphi, \neg\psi_3\}$ is satisfiable — Its semantic tableau is **not** closed. So ψ_3 is **not** entailed from φ.

1. $\mathsf{t} : \exists x\ P(x) \land \forall x(P(x) \to Q(f(x)))\ \checkmark$
2. $\mathsf{f} : \forall x(\neg P(x) \lor \neg Q(f(x)))\ \checkmark a$
3. $\mathsf{t} : \exists x P(x)\ \checkmark a$
4. $\mathsf{t} : P(a)$
5. $\mathsf{t} : \forall x(P(x) \to Q(f(x)))\ \checkmark a$
6. $\mathsf{f} : \neg P(a) \lor \neg Q(f(a))\ \checkmark$
7. $\mathsf{f} : \neg P(a)\ \checkmark$
8. $\mathsf{f} : \neg Q(f(a))\ \checkmark$
9. $\mathsf{t} : P(a)$
10. $\mathsf{t} : Q(f(a))$
11. $\mathsf{t} : P(a) \to Q(f(a))\ \checkmark$
12. $\mathsf{f} : P(a)$ $\otimes$ $\qquad$ $\mathsf{t} : Q(f(a))$

There is a counter model of the entailment $\varphi \vDash \psi_3$. Let $\mathcal{M} = (D, I)$ be a structure and $\ell : V \to D$ such that $D = \{a\}$, $P^{\mathcal{M}} = \{a\}$, $Q^{\mathcal{M}} = \{a\}$, and $f^{\mathcal{M}}(a) = a$. So $\mathcal{M}$ falsifies this entailment.

- The case of ψ_4: Suppose $\{\varphi, \neg\psi_4\}$ is satisfiable. However, its semantic tableau is closed. So ψ_4 is entailed from φ.

1.	$\mathsf{t} : \exists x\, P(x) \wedge \forall x(P(x) \rightarrow Q(f(x)))$ ✓
2.	$\mathsf{f} : \forall x(\neg Q(f(x)) \rightarrow \neg P(x))$ ✓ a
3.	$\mathsf{t} : \forall x(P(x) \rightarrow Q(f(x))$ ✓ a
4.	$\mathsf{f} : \neg Q(f(a)) \rightarrow \neg P(a)$ ✓
5.	$\mathsf{t} : \neg Q(f(a))$ ✓
6.	$\mathsf{f} : \neg P(a)$ ✓
7.	$\mathsf{f} : Q(f(a))$
8.	$\mathsf{t} : P(a)$
9.	$\mathsf{t} : P(a) \rightarrow Q(f(a))$
10.	$\mathsf{f} : P(a)$ $\quad$ $\mathsf{t} : Q(f(a))$
	$\otimes$ $\quad$ $\otimes$

Exercise Problem 2.4. Prove the following equivalences in FOL. You can use semantic arguments (e.g., arguing about all interpretations) or proof theory (e.g., tableau proofs). You can choose the approach you prefer, provided that you use it correctly!

$$\neg \forall x \phi \equiv \exists x \neg \phi \qquad \forall x \phi \wedge \forall x \psi \equiv \forall x(\phi \wedge \psi)$$
$$\neg \exists x \phi \equiv \forall x \neg \phi \qquad \exists x \phi \vee \exists x \psi \equiv \exists x(\phi \vee \psi)$$

Solution.

- Let $\mathcal{M} = (D, I)$ be a structure and $\ell : V \rightarrow D$ a variable assignment for $\mathcal{M}$.

$$\begin{array}{ll} & (\mathcal{M}, \ell) \models \neg \forall x \varphi \\ \Leftrightarrow & \text{it is not that: } (\mathcal{M}, \ell) \models \forall x \varphi \\ \Leftrightarrow & \text{it is not that: } (\mathcal{M}, \ell[d/x]) \models \varphi, \text{ for all } d \in D \\ \Leftrightarrow & (\mathcal{M}, \ell[d/x]) \not\models \varphi, \text{ exists } d \in D \\ \Leftrightarrow & (\mathcal{M}, \ell[d/x]) \models \neg \varphi, \text{ exists } d \in D \\ \Leftrightarrow & (\mathcal{M}, \ell) \models \exists x \neg \varphi. \end{array}$$

- Let $\mathcal{M} = (D, I)$ be a structure and $\ell : V \rightarrow D$ a variable assignment for $\mathcal{M}$.

$$\begin{array}{ll} & (\mathcal{M}, \ell) \models \forall x \varphi \wedge \forall x \psi \\ \Leftrightarrow & (\mathcal{M}, \ell) \models \forall x \varphi \text{ and } (\mathcal{M}, \ell) \models \forall x \psi \\ \Leftrightarrow & (\mathcal{M}, \ell[d/x]) \models \varphi, \text{ for all } d \in D; \text{ meanwhile } (\mathcal{M}, \ell[c/x]) \models \psi, \text{ for all } c \in D \\ \Leftrightarrow & (\mathcal{M}, \ell[d/x]) \models \varphi \wedge \psi, \text{ for all } d \in D \\ \Leftrightarrow & (\mathcal{M}, \ell) \models \forall x(\varphi \wedge \psi) \end{array}$$

- Suppose $\{\neg \exists x \varphi, \neg \forall x \neg \varphi\}$ is satisfiable. However, its semantic tableau is closed. So this set is unsatisfiable, and $\forall x \neg \varphi$ is entailed from $\neg \exists x \varphi$.

1.	$\mathsf{t} : \neg \exists x \varphi$ ✓
2.	$\mathsf{f} : \forall x \neg \varphi$ ✓ c
3.	$\mathsf{f} : \exists x \varphi$ ✓ c
4.	$\mathsf{f} : \neg \varphi(c)$ ✓
5.	$\mathsf{t} : \varphi(c)$
6.	$\mathsf{f} : \varphi(c)$
	$\otimes$

Suppose $\{\forall x\neg\varphi, \neg\neg\exists x\varphi\}$ is satisfiable. However, its semantic tableau is closed. So this set is unsatisfiable, and $\neg\exists x\varphi$ is entailed from $\forall x\neg\varphi$.

1. $\mathsf{t} : \forall x\neg\varphi \checkmark c$
2. $\mathsf{f} : \neg\exists x\varphi \checkmark c$
3. $\mathsf{t} : \exists x\varphi \checkmark$
4. $\mathsf{t} : \varphi(c)$
5. $\mathsf{t} : \neg\varphi(c) \checkmark$
6. $\mathsf{f} : \varphi(c)$

$\otimes$

- Suppose $\{\exists x\phi \vee \exists x\psi, \neg\exists x(\phi \vee \psi)\}$ is satisfiable. However, its semantic tableau is closed. So this set is unsatisfiable, and $\exists x(\phi \vee \psi)$ is entailed from $\exists x\phi \vee \exists x\psi$.

1.	$\mathsf{t} : \exists x\phi \vee \exists x\psi \checkmark$	
2.	$\mathsf{f} : \exists x(\phi \vee \psi) \checkmark c, a$	
3.	$\mathsf{t} : \exists x\phi \checkmark c$	$\mathsf{t} : \exists x\psi \checkmark a$
4.	$\mathsf{t} : \phi(c)$	$\mathsf{t} : \psi(a)$
5.	$\mathsf{f} : \phi(c) \vee \psi(c) \checkmark$	$\mathsf{f} : \phi(a) \vee \psi(a) \checkmark$
6.	$\mathsf{f} : \phi(c)$	$\mathsf{f} : \psi(a)$
	$\otimes$	$\otimes$

Suppose $\{\neg(\exists x\phi \vee \exists x\psi), \exists x(\phi \vee \psi)\}$ is satisfiable. However, its semantic tableau is closed. So this set is unsatisfiable, and $\exists x\phi \vee \exists x\psi$ is entailed from $\exists x(\phi \vee \psi)$.

1.	$\mathsf{t} : \exists x(\phi \vee \psi) \checkmark c$	
2.	$\mathsf{f} : \exists x\phi \vee \exists x\psi \checkmark$	
3.	$\mathsf{t} : \phi(c) \vee \psi(c) \checkmark$	
4.	$\mathsf{t} : \phi(c)$	$\mathsf{t} : \psi(c)$
5.	$\mathsf{f} : \exists x\phi \checkmark c$	$\mathsf{f} : \exists x\psi \checkmark c$
6.	$\mathsf{f} : \psi(c)$	$\mathsf{f} : \psi(c)$
	$\otimes$	$\otimes$

Exercise Problem 2.5. Consider the following sentences:

1. All dogs howl at night.
2. Someone who has cats will not have any mice.
3. Light sleepers do not have anything which howls at night.
4. John has either a cat or a dog.

Does the sentence

> If John is a light sleeper, then John does not have any mice

follow from 1-4? Provide either a proof or a counterexample.
Solution.
We use the predicates

- $\mathsf{Dog}(x)$ to represent the property ‘being a dog’;
- $\mathsf{Howl}(x)$ to represent the property ‘howling at night’;
- $\mathsf{Have}(x, y)$ to represent the property ‘x having y’;
- $\mathsf{Cat}(x)$ to represent the property ‘being a cat’;
- $\mathsf{Mice}(x)$ to represent the property ‘being a mice’;
- $\mathsf{LS}(x)$ to represent the property ‘being a light sleeper’;

and the constant

- John to represent the person John.

So the above sentences can be represented as follows:

s_1. $\forall x(\mathsf{Dog}(x) \to \mathsf{Howl}(x))$

s_2. $\forall x \forall y(\mathsf{Have}(x, y) \wedge \mathsf{Cat}(y) \to \neg\exists z(\mathsf{Have}(x, z) \wedge \mathsf{Mice}(z)))$

s_3. $\forall x(\mathsf{LS}(x) \to \neg\exists y(\mathsf{Have}(x, y) \wedge \mathsf{Howl}(y)))$

s_4. $\exists x(\mathsf{Have}(\mathsf{John}, x) \wedge (\mathsf{Cat}(x) \vee \mathsf{Dog}(x))$

s_5. $\mathsf{LS}(\mathsf{John}) \to \neg\exists z(\mathsf{Have}(\mathsf{John}, z) \wedge \mathsf{Mice}(z))$

Suppose $\{s_1, s_2, s_3, s_4, \neg s_5\}$ is satisfiable. However, its semantic tableau is closed. So this set is unsatisfiable, and s_5 is entailed from $\{s_1, s_2, s_3, s_4\}$.

1. $\mathsf{t} : \forall x(\mathsf{Dog}(x) \to \mathsf{Howl}(x)) \checkmark b$
2. $\mathsf{t} : \forall x \forall y(\mathsf{Have}(x, y) \wedge \mathsf{Cat}(y) \to \neg\exists z(\mathsf{Have}(x, z) \wedge \mathsf{Mice}(z))) \checkmark \mathsf{John}, b$
3. $\mathsf{t} : \forall x(\mathsf{LS}(x) \to \neg\exists y(\mathsf{Have}(x, y) \wedge \mathsf{Howl}(y))) \checkmark \mathsf{John}$
4. $\mathsf{t} : \exists x(\mathsf{Have}(\mathsf{John}, x) \wedge (\mathsf{Cat}(x) \vee \mathsf{Dog}(x)) \checkmark b$
5. $\mathsf{f} : \mathsf{LS}(\mathsf{John}) \to \neg\exists z(\mathsf{Have}(\mathsf{John}, z) \wedge \mathsf{Mice}(z)) \checkmark$
6. $\mathsf{t} : \mathsf{LS}(\mathsf{John})$
7. $\mathsf{f} : \neg\exists z(\mathsf{Have}(\mathsf{John}, z) \wedge \mathsf{Mice}(z)) \checkmark$
8. $\mathsf{t} : \exists z(\mathsf{Have}(\mathsf{John}, z) \wedge \mathsf{Mice}(z)) \checkmark a$
9. $\mathsf{t} : \mathsf{Have}(\mathsf{John}, a) \wedge \mathsf{Mice}(a) \checkmark$
10. $\mathsf{t} : \mathsf{Have}(\mathsf{John}, a)$
11. $\mathsf{t} : \mathsf{Mice}(a)$
12. $\mathsf{t} : \mathsf{Have}(\mathsf{John}, b) \wedge (\mathsf{Cat}(b) \vee \mathsf{Dog}(b)) \checkmark$
13. $\mathsf{t} : \mathsf{Have}(\mathsf{John}, b)$
14. $\mathsf{t} : \mathsf{Cat}(b) \vee \mathsf{Dog}(b)$
15. $\mathsf{t} : \mathsf{Cat}(b)$ $\quad$ $\mathsf{t} : \mathsf{Dog}(b)$

The tableau of the branch from $\mathsf{t} : \mathsf{Cat}(b)$ at the 16th step is closed:

1. $\mathsf{t} : \mathsf{Cat}(b)$
2. $\mathsf{t} : \mathsf{Have}(\mathsf{John}, b) \wedge \mathsf{Cat}(b) \rightarrow \neg\exists z(\mathsf{Have}(\mathsf{John}, z) \wedge \mathsf{Mice}(z))$ ✓
3. $\mathsf{f} : \mathsf{Have}(\mathsf{John}, b) \wedge \mathsf{Cat}(b)$ ✓ $\quad$ $\mathsf{t} : \neg\exists z(\mathsf{Have}(\mathsf{John}, z) \wedge \mathsf{Mice}(z))$ ✓
4. $\mathsf{f} : \mathsf{Have}(\mathsf{John}, b)$ $\quad$ $\mathsf{f} : \mathsf{Cat}(b)$
5. $\otimes$ $\quad$ $\otimes$ $\quad$ $\mathsf{f} : \exists z(\mathsf{Have}(\mathsf{John}, z) \wedge \mathsf{Mice}(z))$ ✓ a
6. $\mathsf{f} : \mathsf{Have}(\mathsf{John}, a) \wedge \mathsf{Mice}(a)$ ✓
7. $\mathsf{f} : \mathsf{Have}(\mathsf{John}, a)$ $\quad$ $\mathsf{f} : \mathsf{Mice}(a)$

$\otimes$ $\quad$ $\otimes$

The tableau of the branch from $\mathsf{t} : \mathsf{Dog}(b)$ at the 16th step is closed:

1. $\mathsf{t} : \mathsf{Dog}(b)$
2. $\mathsf{t} : \mathsf{Dog}(b) \rightarrow \mathsf{Howl}(b)$
3. $\mathsf{f} : \mathsf{Dog}(b)$ $\quad$ $\mathsf{t} : \mathsf{Howl}(b)$
4. $\otimes$ $\quad$ $\mathsf{t} : \mathsf{LS}(\mathsf{John}) \rightarrow \neg\exists y(\mathsf{Have}(\mathsf{John}, y) \wedge \mathsf{Howl}(y))$ ✓
5. $\mathsf{f} : \mathsf{LS}(\mathsf{John})$ $\quad$ $\mathsf{t} : \neg\exists y(\mathsf{Have}(\mathsf{John}, y) \wedge \mathsf{Howl}(y))$ ✓
6. $\otimes$ $\quad$ $\mathsf{f} : \exists y(\mathsf{Have}(\mathsf{John}, y) \wedge \mathsf{Howl}(y))$ ✓ b
7. $\mathsf{f} : \mathsf{Have}(\mathsf{John}, b) \wedge \mathsf{Howl}(b)$ ✓
8. $\mathsf{f} : \mathsf{Have}(\mathsf{John}, b)$ $\quad$ $\mathsf{f} : \mathsf{Howl}(b)$

$\otimes$ $\quad$ $\otimes$

Exercise Problem 2.6. Formalize the following sentences in first-order logic:

1. All actors and journalists invited to the party are late
2. There is at least one person on time
3. There is at least one invited person who is neither a journalist nor an actor

Is this reasoning correct? That is, is (3) entailed by (1) and (2)? Provide either a proof or a counterexample.

Solution. We use the predicates

- Actor to represent the property "being an actor";
- $\mathsf{Journalist}$ to represent the property "being a journalist";
- $\mathsf{Invited}$ to represent the property "being invited";
- Late to represent the property "being late" or "not being on time".

The above sentences can be formalised as follows:

s_1. $\forall x((\mathsf{Actor}(x) \vee \mathsf{Journalist}(x)) \wedge \mathsf{Invited}(x) \rightarrow \mathsf{Late}(x))$

s_2. $\exists x(\neg\mathsf{Late}(x))$

s_3. $\exists x(\neg(\mathsf{Actor}(x) \lor \mathsf{Journalist}(x)) \land \mathsf{Invited}(x))$

<u>Method 1</u>: There is a counter model of the entailment $\{s_1, s_2\} \models s_3$ — Someone who is neither an actor nor a journalist is not invited but is on time. Let $\mathcal{M} = (D, I)$ be a structure and $\ell : V \to D$ such that

- $D = \{a\}$
- $\mathsf{Actor}^{\mathcal{M}} = \emptyset$
- $\mathsf{Journalist}^{\mathcal{M}} = \emptyset$
- $\mathsf{Invited}^{\mathcal{M}} = \emptyset$
- $\mathsf{Late}^{\mathcal{M}} = \emptyset$

So $\mathcal{M}$ falsifies this entailment.
<u>Method 2</u>: The set $\{s_1, s_2, \neg s_3\}$ is satisfiable — Its semantic tableau is **not** closed. So s_3 is **not** entailed from $\{s_1, s_2\}$.

1. $\mathsf{t} : \forall x((\mathsf{Actor}(x) \lor \mathsf{Journalist}(x)) \land \mathsf{Invited}(x) \to \mathsf{Late}(x))$
2. $\mathsf{t} : \exists x(\neg\mathsf{Late}(x)) \checkmark a$
3. $\mathsf{f} : \exists x(\neg(\mathsf{Actor}(x) \lor \mathsf{Journalist}(x)) \land \mathsf{Invited}(x)) \checkmark a$
4. $\mathsf{t} : \neg\mathsf{Late}(a) \checkmark$
5. $\mathsf{f} : \mathsf{Late}(a)$
6. $\mathsf{f} : \neg(\mathsf{Actor}(a) \lor \mathsf{Journalist}(a)) \land \mathsf{Invited}(a) \checkmark$
7. $\mathsf{f} : \neg(\mathsf{Actor}(a) \lor \mathsf{Journalist}(a))$ $\quad$ $\mathsf{f} : \mathsf{Invited}(a)$
8. $\mathsf{t} : \mathsf{Actor}(a) \lor \mathsf{Journalist}(a)$
9. $\mathsf{t} : \mathsf{Actor}(a)$ $\quad$ $\mathsf{t} : \mathsf{Journalist}(a)$

The tableau of the branch from $\mathsf{t} : \mathsf{Actor}(a)$ at the 10th step remains open:

1. $\mathsf{t} : \mathsf{Actor}(a)$
2. $\mathsf{t} : (\mathsf{Actor}(a) \lor \mathsf{Journalist}(a)) \land \mathsf{Invited}(a) \to \mathsf{Late}(a) \checkmark$
3. $\mathsf{f} : (\mathsf{Actor}(a) \lor \mathsf{Journalist}(a)) \land \mathsf{Invited}(a) \checkmark$ $\quad$ $\mathsf{t} : \mathsf{Late}(a)$ $\otimes$
4. $\mathsf{f} : \mathsf{Actor}(a) \lor \mathsf{Journalist}(a) \checkmark$ $\quad$ $\mathsf{f} : \mathsf{Invited}(a)$
5. $\mathsf{f} : \mathsf{Actor}(a)$ $\otimes$

The tableau of the branch from $\mathsf{t} : \mathsf{Journalist}(a)$ at the 10th step remains open:

1. $\mathsf{t} : \mathsf{Journalist}(a)$
2. $\mathsf{t} : (\mathsf{Actor}(a) \lor \mathsf{Journalist}(a)) \land \mathsf{Invited}(a) \to \mathsf{Late}(a) \checkmark$
3. $\mathsf{f} : (\mathsf{Actor}(a) \lor \mathsf{Journalist}(a)) \land \mathsf{Invited}(a) \checkmark$ $\quad$ $\mathsf{t} : \mathsf{Late}(a)$ $\otimes$
4. $\mathsf{f} : \mathsf{Actor}(a) \lor \mathsf{Journalist}(a) \checkmark$ $\quad$ $\mathsf{f} : \mathsf{Invited}(a)$
5. $\mathsf{f} : \mathsf{Journalist}(a)$ $\otimes$

The tableau of the branch from $\mathsf{f} : \mathsf{Invited}(a)$ at the 8th step remains open as well:

$$
\begin{array}{lccc}
1. & & \mathsf{f} : \mathsf{Invited}(a) & \\
2. & & \mathsf{t} : (\mathsf{Actor}(a) \vee \mathsf{Journalist}(a)) \wedge \mathsf{Invited}(a) \rightarrow \mathsf{Late}(a)\ \checkmark & \\
3. & \mathsf{f} : (\mathsf{Actor}(a) \vee \mathsf{Journalist}(a)) \wedge \mathsf{Invited}(a)\ \checkmark & & \mathsf{t} : \mathsf{Late}(a) \\
 & & & \otimes \\
4. & \mathsf{f} : \mathsf{Actor}(a) \vee \mathsf{Journalist}(a)\ \checkmark & \mathsf{f} : \mathsf{Invited}(a) & \\
5. & \mathsf{f} : \mathsf{Actor}(a) & & \\
6. & \mathsf{f} : \mathsf{Journalist}(a) & &
\end{array}
$$

Exercise Problem 2.7. Convert the following FOL formulas into prenex normal form:

$$
\begin{aligned}
\psi_1 : &\quad \forall x(P(x) \rightarrow \exists y Q(x,y)) \wedge \exists x P(x) \\
\psi_2 : &\quad \exists x(P(x) \wedge \forall y(R(y) \rightarrow Q(x,y)) \wedge \neg(\exists y Q(y,x))) \\
\psi_3 : &\quad \neg\big(\forall x \exists y(\neg P(y) \vee Q(x, f(y)) \vee Q(y,z))\big)
\end{aligned}
$$

Solution.

- The case of ψ_1:
  1. Eliminate $\rightarrow$: $\forall x(\neg P(x) \vee \exists y Q(x,y)) \wedge \exists x P(x)$;
  2. Move all negations inwards to the atomic level (Nothing need to be done): $\forall x(\neg P(x) \vee \exists y Q(x,y)) \wedge \exists x P(x)$;
  3. Rename all variables apart: $\forall x_1(\neg P(x_1) \vee \exists x_2 Q(x_1,x_2)) \wedge \exists x_3 P(x_3)$;
  4. Move all quantifiers to the front:
     (a) $\forall x_1 \exists x_2(\neg P(x_1) \vee Q(x_1,x_2)) \wedge \exists x_3 P(x_3)$;
     (b) $\forall x_1(\exists x_2(\neg P(x_1) \vee Q(x_1,x_2)) \wedge \exists x_3 P(x_3))$;
     (c) $\forall x_1 \exists x_2((\neg P(x_1) \vee Q(x_1,x_2)) \wedge \exists x_3 P(x_3))$;
     (d) $\forall x_1 \exists x_2 \exists x_3((\neg P(x_1) \vee Q(x_1,x_2)) \wedge P(x_3))$.
- The case of ψ_2:
  1. Eliminate $\rightarrow$: $\exists x(P(x) \wedge \forall y(\neg R(y) \vee Q(x,y)) \wedge \neg(\exists y Q(y,x)))$
  2. Move all negations inwards to the atomic level: $\exists x(P(x) \wedge \forall y(\neg R(y) \vee Q(x,y)) \wedge \forall y \neg Q(y,x))$;
  3. Rename all variables apart: $\exists x_1(P(x_1) \wedge \forall x_2(\neg R(x_2) \vee Q(x_1,x_2)) \wedge \forall x_3 \neg Q(x_3,x_1))$;
  4. Move all quantifiers to the front:
     (a) $\exists x_1 \forall x_2(P(x_1) \wedge (\neg R(x_2) \vee Q(x_1,x_2)) \wedge \forall x_3 \neg Q(x_3,x_1))$;
     (b) $\exists x_1 \forall x_2 \forall x_3(P(x_1) \wedge (\neg R(x_2) \vee Q(x_1,x_2)) \wedge \neg Q(x_3,x_1))$.
- The case of ψ_3:
  1. Eliminate $\rightarrow$ (Nothing need to be done): $\neg\big(\forall x \exists y(\neg P(y) \vee Q(x,f(y)) \vee Q(y,z))\big)$;
  2. Move all negations inwards to the atomic level:
     (a) $\exists x \forall y \neg(\neg P(y) \vee Q(x,f(y)) \vee Q(y,z))$;
     (b) $\exists x \forall y(P(y) \wedge \neg Q(x,f(y)) \wedge \neg Q(y,z))$;
  3. Rename all variables apart: $\exists x_1 \forall x_2(P(x_2) \wedge \neg Q(x_1,f(x_2)) \wedge \neg Q(x_2,x_3))$;
  4. Move all quantifiers to the front (Nothing need to be done): $\exists x_1 \forall x_2(P(x_2) \wedge \neg Q(x_1,f(x_2)) \wedge \neg Q(x_2,x_3))$.

For each project problem below, formalize the problem in TPTP and solve it using the Vampire theorem prover.

You can use Vampire by

- building or downloading Vampire from

  `https://vprover.github.io/download.html`

- running Vampire in the browser at

  `https://tptp.org/cgi-bin/SystemOnTPTP`

Further details on using the Vampire prover are given in Lecture 14.

Project Problem 2.1. Consider the group theory axiomatisation used in the lecture (Lecture 14). Use Vampire to prove that the group's left identity element is also a right identity element and it is unique.

Project Problem 2.2. Let $>$ be a strict order, that is a transitive, anti-symmetric and irreflexive relation. Let $\geq$ be the corresponding non-strict order extension of $>$, that is $x \geq y$ if and only if $x > y$ or $x = y$, for all x, y.

We call a function f monotonically increasing if from $x \geq y$ it follows that $f(x) \geq f(y)$, for all x, y.

Consider the unary functions f, g such that both f and g are monotonically increasing. Further, assume that $f(x) \geq x$ and $g(x) \geq x$ hold, for all x.

Let h be the function resulting from composing f and g, that is $h(x) = f(g(x))$, for all x.

Use Vampire to prove that h is monotonically increasing and $h(x) \geq x$, for all x.

Hint: You need to axiomatize the properties of the orders $>, \geq$.

Instructions for encoding your solutions to project 2.1 and/or project 2.2: Use the TPTP input format for encoding your solutions, using the first-order formula `fof` encoding of TPTP. For each first-order formula, provide a short comment (in natural language) in your TPTP encoding explaining what the formula expresses.