Linear Transformations

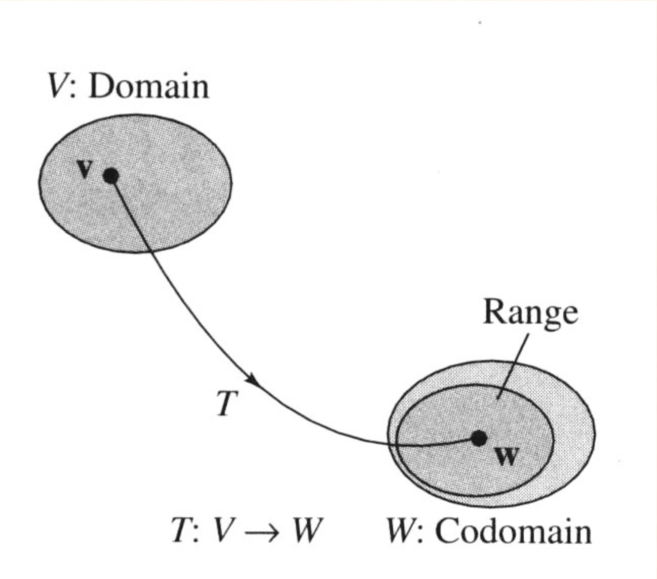
6.1 Introduction to Linear Transformations

- Function T that maps a vector space V into a vector space W:

$$T : V \xrightarrow{\text{mapping}} W, \quad V, W : \text{vector space}$$

V: the domain of T

W: the codomain of T



- Image of **v** under *T*:

  If **v** is in V and **w** is in W such that

  $$T(\mathbf{v}) = \mathbf{w}$$

  Then **w** is called the image of **v** under T .

- the range of *T*:

  The set of all images of vectors in V.

- the preimage of **w**:

  The set of all **v** in V such that *T*(**v**)=**w**.

- Ex 1: (A function from R^2 into R^2)

$$T: R^2 \rightarrow R^2 \qquad \mathbf{v} = (v_1, v_2) \in R^2$$

$$T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2)$$

(a) Find the image of **v**=(-1,2). (b) Find the preimage of **w**=(-1,11)

Sol:

$$(a)\ \ \mathbf{v} = (-1, 2)$$

$$\Rightarrow T(\mathbf{v}) = T(-1, 2) = (-1-2, -1+2(2)) = (-3, 3)$$

$$(b)\ \ T(\mathbf{v}) = \mathbf{w} = (-1, 11)$$

$$T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2) = (-1, 11)$$

$$\Rightarrow v_1 - v_2 = -1$$

$$v_1 + 2v_2 = 11$$

$\Rightarrow v_1 = 3,\ v_2 = 4$ Thus {(3, 4)} is the preimage of **w**=(-1, 11).

- Linear Transformation (L.T.):

V, W: vector space

$T: V \to W$: V to W linear transformation

(1) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}), \quad \forall \mathbf{u}, \mathbf{v} \in V$

(2) $T(c\mathbf{u}) = cT(\mathbf{u}), \quad \forall c \in R$

- Notes:

(1) A linear transformation is said to be operation preserving.

$$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \qquad\qquad T(c\mathbf{u}) = cT(\mathbf{u})$$

Addition in V

Addition in W

Scalar multiplication in V

Scalar multiplication in W

(2) A linear transformation $T: V \to V$ from a vector space into itself is called a **linear operator**.

- Ex 2: (Verifying a linear transformation T from R^2 into R^2)

$$T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2)$$

Pf:

$\mathbf{u} = (u_1, u_2),\ \mathbf{v} = (v_1, v_2)$: vector in R^2, c : any real number

(1) Vector addition :

$$\mathbf{u} + \mathbf{v} = (u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2)$$

$$\begin{aligned} T(\mathbf{u} + \mathbf{v}) &= T(u_1 + v_1, u_2 + v_2) \\ &= ((u_1 + v_1) - (u_2 + v_2), (u_1 + v_1) + 2(u_2 + v_2)) \\ &= ((u_1 - u_2) + (v_1 - v_2), (u_1 + 2u_2) + (v_1 + 2v_2)) \\ &= (u_1 - u_2, u_1 + 2u_2) + (v_1 - v_2, v_1 + 2v_2) \\ &= T(\mathbf{u}) + T(\mathbf{v}) \end{aligned}$$

(2) Scalar multiplication

$$c\mathbf{u} = c(u_1, u_2) = (cu_1, cu_2)$$

$$\begin{aligned}T(c\mathbf{u}) &= T(cu_1, cu_2) = (cu_1 - cu_2, cu_1 + 2cu_2)\\ &= c(u_1 - u_2, u_1 + 2u_2)\\ &= cT(\mathbf{u})\end{aligned}$$

Therefore, T is a linear transformation.

- Ex 3: (Functions that are not linear transformations)

$(a)\ f(x) = \sin x$

$$\sin(x_1 + x_2) \neq \sin(x_1) + \sin(x_2) \Leftarrow f(x) = \sin x \text{ is not}$$

$$\sin(\tfrac{\pi}{2} + \tfrac{\pi}{3}) \neq \sin(\tfrac{\pi}{2}) + \sin(\tfrac{\pi}{3})$$

linear transformation

$(b)\ f(x) = x^2$

$$(x_1 + x_2)^2 \neq x_1^2 + x_2^2$$

$$(1+2)^2 \neq 1^2 + 2^2$$

$\Leftarrow f(x) = x^2$ is not linear transformation

$(c)\ f(x) = x + 1$

$$f(x_1 + x_2) = x_1 + x_2 + 1$$

$$f(x_1) + f(x_2) = (x_1 + 1) + (x_2 + 1) = x_1 + x_2 + 2$$

$$f(x_1 + x_2) \neq f(x_1) + f(x_2) \Leftarrow f(x) = x + 1 \text{ is not}$$

linear transformation

- Notes: Two uses of the term "linear".

(1) $f(x)=x+1$ is called a linear function because its graph is a line.

(2) $f(x)=x+1$ is not a linear transformation from a vector space R into R because it preserves neither vector addition nor scalar multiplication.

- Zero transformation:

$$T: V \to W \qquad T(\mathbf{v}) = 0,\ \forall \mathbf{v} \in V$$

- Identity transformation:

$$T: V \to V \qquad T(\mathbf{v}) = \mathbf{v},\ \forall \mathbf{v} \in V$$

- Thm 6.1: (Properties of linear transformations)

$$T: V \to W,\quad \mathbf{u}, \mathbf{v} \in V$$

$$(1)\ T(\mathbf{0}) = \mathbf{0}$$

$$(2)\ T(-\mathbf{v}) = -T(\mathbf{v})$$

$$(3)\ T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v})$$

$$(4)\ \text{If } \mathbf{v} = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$$

$$\text{Then } T(\mathbf{v}) = T(c_1 v_1 + c_2 v_2 + \cdots + c_n v_n)$$

$$= c_1 T(v_1) + c_2 T(v_2) + \cdots + c_n T(v_n)$$

- Ex 4: (Linear transformations and bases)

Let $T: R^3 \rightarrow R^3$ be a linear transformation such that

$$T(1,0,0)=(2,-1,4)$$

$$T(0,1,0)=(1,5,-2)$$

$$T(0,0,1)=(0,3,1)$$

Find $T(2, 3, -2)$.

Sol:

$$(2,3,-2)=2(1,0,0)+3(0,1,0)-2(0,0,1)$$

$$T(2,3,-2)=2T(1,0,0)+3T(0,1,0)-2T(0,0,1) \qquad (T \text{ is a L.T.})$$

$$=2(2,-1,4)+3(1,5,-2)-2T(0,3,1)$$

$$=(7,7,0)$$

- Ex 5: (A linear transformation defined by a matrix)

The function $T: R^2 \rightarrow R^3$ is defined as $T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

(a) Find $T(\mathbf{v})$, where $\mathbf{v} = (2,-1)$

(b) Show that T is a linear transformation form R^2 into R^3

Sol: $(a)\mathbf{v} = (2,-1)$

$$\begin{array}{ccccc} & & R^2 \text{ vector} & & R^3 \text{ vector} \\ & & \downarrow & & \downarrow \\ T(\mathbf{v}) = A\mathbf{v} = & \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & -2 \end{bmatrix} & \begin{bmatrix} 2 \\ -1 \end{bmatrix} & = & \begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix} \end{array}$$

$\therefore T(2,-1) = (6,3,0)$

$(b)\ T(\mathbf{u}+\mathbf{v}) = A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v} = T(\mathbf{u}) + T(\mathbf{v})$ (vector addition)

$T(c\mathbf{u}) = A(c\mathbf{u}) = c(A\mathbf{u}) = cT(\mathbf{u})$ (scalar multiplication)

- Thm 6.2: (The linear transformation given by a matrix)

  Let A be an $m \times n$ matrix. The function T defined by

  $$T(\mathbf{v}) = A\mathbf{v}$$

  is a linear transformation from R^n into R^m.

- Note:

$$A\mathbf{v} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \overset{R^n \text{ vector}}{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}} = \overset{R^m \text{ vector}}{\begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n \\ a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n \\ \vdots \\ a_{m1}v_1 + a_{m2}v_2 + \cdots + a_{mn}v_n \end{bmatrix}}$$

$$T(\mathbf{v}) = A\mathbf{v}$$

$$T : R^n \longrightarrow R^m$$