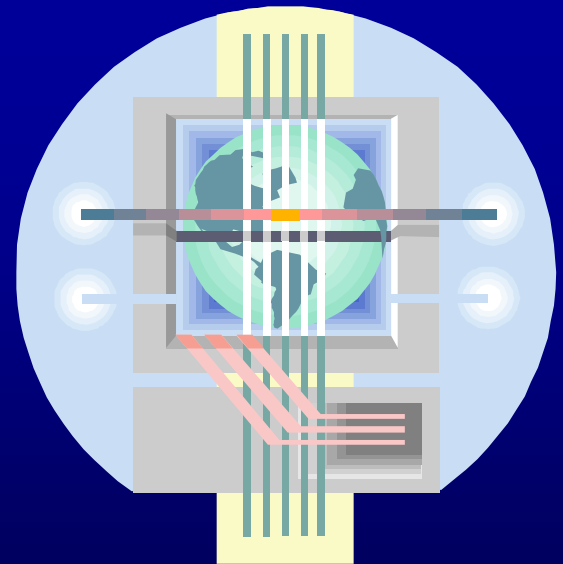# Linear Programming

company makes two types of products, A and B. These products are produced during a 40-hour work week and then shipped out at the end of the week. They require 20 and 5 kg of raw material per kg of product, respectively, and the company has access to 9500 kg of raw material per week. Only one product can be created at a time with production times for each of 0.04 and 0.12 hr, respectively. The plant can only store 550 kg of total product per week. Finally, the company makes profits of $45 and $20 on each unit of A and B, respectively. Each unit of product is equivalent to a kg.

Introduction

❖ Mathematical programming is used to find the best or optimal solution to a problem that requires a decision or set of decisions about how best to use a set of limited resources to achieve a state goal of objectives.

❖ **Steps involved in mathematical programming**

➢ Conversion of stated problem into a mathematical model that abstracts all the essential elements of the problem.

➢ Exploration of different solutions of the problem.

➢ Finding out the most suitable or optimum solution.

❖ **Linear programming** requires that all the mathematical functions in the model be linear functions.

The Linear Programming Model (1)

Let: $X_1, X_2, X_3, \ldots\ldots, X_n$ = decision variables

Z = Objective function or linear function

<u>Requirement:</u> Maximization of the linear function Z.

$$Z = c_1X_1 + c_2X_2 + c_3X_3 + \ldots\ldots + c_nX_n \quad \ldots..Eq\ (1)$$

subject to the following <u>constraints:</u>

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2$$

$$\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \ldots..Eq\ (2)$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_n$$
$$\text{all } x_J \geq 0$$

where a_{ij}, b_i, and c_j are given constants.

The Linear Programming Model (2)

❖ The linear programming model can be written in more efficient notation as:

Maximize

$$Z = \sum_{j=1}^{n} c_j x_j$$

subject to:

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i$$

where

$$i = 1, 2, \ldots, m \qquad \text{.....Eq (3)}$$

and

$$x_j \geq 0$$

where

$$j = 1, 2, \ldots, n$$

The decision variables, x_1, x_2, ..., x_n, represent levels of n competing activities.

Examples of LP Problems (1)

1. A Product Mix Problem

❖ A manufacturer has fixed amounts of different resources such as raw material, labor, and equipment.

❖ These resources can be combined to produce any one of several different products.

❖ The quantity of the ith resource required to produce one unit of the jth product is known.

❖ The decision maker wishes to produce the combination of products that will maximize total income.

Examples of LP Problems (2)

2. A Blending Problem

❖ Blending problems refer to situations in which a number of components (or commodities) are mixed together to yield one or more products.

❖ Typically, different commodities are to be purchased. Each commodity has known characteristics and costs.

❖ The problem is to determine how much of each commodity should be purchased and blended with the rest so that the characteristics of the mixture lie within specified bounds and the total cost is minimized.

Examples of LP Problems (3)

3. A Production Scheduling Problem

❖ A manufacturer knows that he must supply a given number of items of a certain product each month for the next *n* months.

❖ They can be produced either in regular time, subject to a maximum each month, or in overtime. The cost of producing an item during overtime is greater than during regular time. A storage cost is associated with each item not sold at the end of the month.

❖ The problem is to determine the production schedule that minimizes the sum of production and storage costs.

Examples of LP Problems (4)

4. A Transportation Problem

❖ A product is to be shipped in the amounts a_l, a_2, ..., a_m from m shipping origins and received in amounts b_l, b_2, ..., b_n at each of n shipping destinations.

❖ The cost of shipping a unit from the ith origin to the jth destination is known for all combinations of origins and destinations.

❖ The problem is to determine the amount to be shipped from each origin to each destination such that the total cost of transportation is a minimum.

Examples of LP Problems (5)

5. A Flow Capacity Problem

❖ One or more commodities (e.g., traffic, water, information, cash, etc.) are flowing from one point to another through a network whose branches have various constraints and flow capacities.

❖ The direction of flow in each branch and the capacity of each branch are known.

❖ The problem is to determine the maximum flow, or capacity of the network.

Decision and Risk Analysis   *Syed M. Ahmed, Ph.D.*

Developing LP Model (1)

❖ The variety of situations to which linear programming has been applied ranges from agriculture to zinc smelting.

❖ **Steps Involved:**

➢ Determine the objective of the problem and describe it by a criterion function in terms of the decision variables.

➢ Find out the constraints.

➢ Do the analysis which should lead to the selection of values for the decision variables that optimize the criterion function while satisfying all the constraints imposed on the problem.

Developing LP Model (2)

Example: Product Mix Problem

The N. Dustrious Company produces two products: I and II. The raw material requirements, space needed for storage, production rates, and selling prices for these products are given in Table 1.

TABLE .1 Production Data for N. Dustrious Company

	Product	
	I	II
Storage space (ft²/unit)	4	5
Raw material (lb/unit)	5	3
Production rate (units/hr)	60	30
Selling price ($/unit)	13	11

The total amount of raw material available per day for both products is 15751b. The total storage space for all products is 1500 ft², and a maximum of 7 hours per day can be used for production.

Developing LP Model (3)

Example Problem

All products manufactured are shipped out of the storage area at the end of the day. Therefore, the two products must share the total raw material, storage space, and production time. **The company wants to determine how many units of each product to produce per day to maximize its total income.**

Solution

- The company has decided that it wants to maximize its sale income, which depends on the number of units of product I and II that it produces.

- Therefore, the decision variables, x_1 and x_2 can be the number of units of products I and II, respectively, produced per day.

Developing LP Model (4)

- The object is to maximize the equation:

$$Z = 13x_1 + 11x_2$$

 subject to the constraints on storage space, raw materials, and production time.

- Each unit of product I requires 4 ft^2 of storage space and each unit of product II requires 5 ft^2. Thus a total of $4x_1 + 5x_2$ ft^2 of storage space is needed each day. This space must be less than or equal to the available storage space, which is 1500 ft^2. Therefore,

$$4X_1 + 5X_2 \leq 1500$$

- Similarly, each unit of product I and II produced requires 5 and 3 1bs, respectively, of raw material. Hence a total of $5x_I + 3x_2$ lb of raw material is used.

Decision and Risk Analysis   *Syed M. Ahmed, Ph.D.*

Developing LP Model (5)

- This must be less than or equal to the total amount of raw material available, which is 1575 lb. Therefore,

$$5x_1 + 3x_2 \leq 1575$$

- Prouct I can be produced at the rate of 60 units per hour. Therefore, it must take I minute or 1/60 of an hour to produce I unit. Similarly, it requires 1/30 of an hour to produce 1 unit of product II. Hence a total of $x_1/60 + x_2/30$ hours is required for the daily production. This quantity must be less than or equal to the total production time available each day. Therefore,

$$x_1 / 60 + x_2 / 30 \leq 7$$

or $x_1 + 2x_2 \leq 420$

- Finally, the company cannot produce a negative quantity of any product, therefore x_1 and x_2 must each be greater than or equal to zero.

Developing LP Model (6)

- The linear programming model for this example can be summarized as:

Maximize

$$Z = 13x_1 + 11x_2$$

subject to:

$$4x_1 + 5x_2 \leq 1500 \qquad \text{.....Eq (4)}$$
$$5x_1 + 3x_2 \leq 1575$$
$$x_1 + 2x_2 \leq 420$$
$$x_1 \geq 0$$
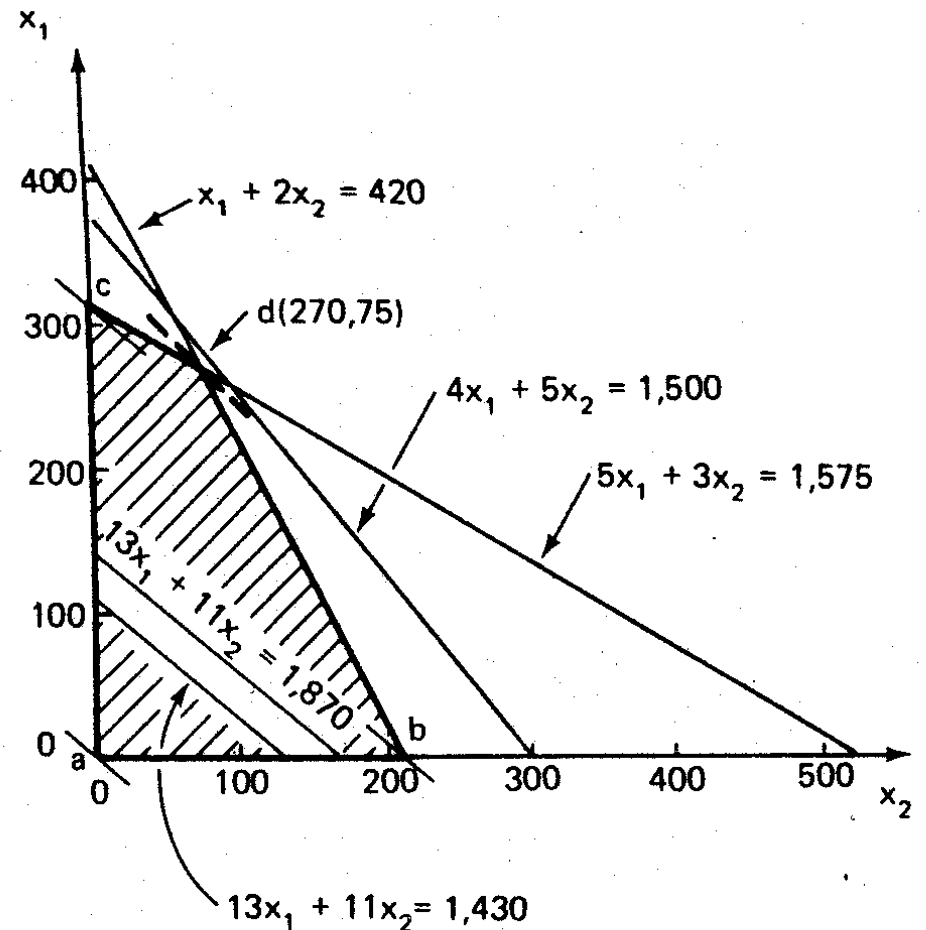$$x_2 \geq 0$$

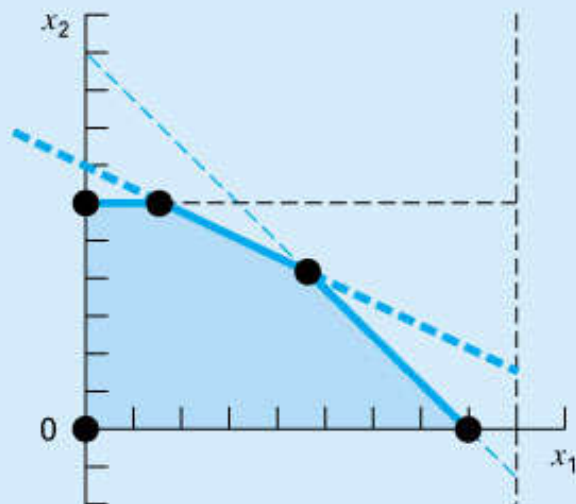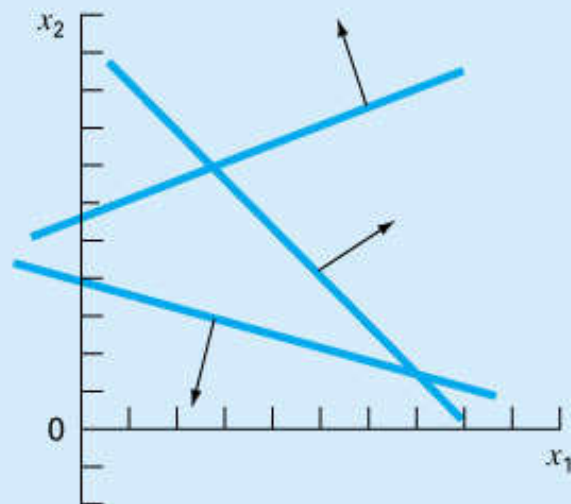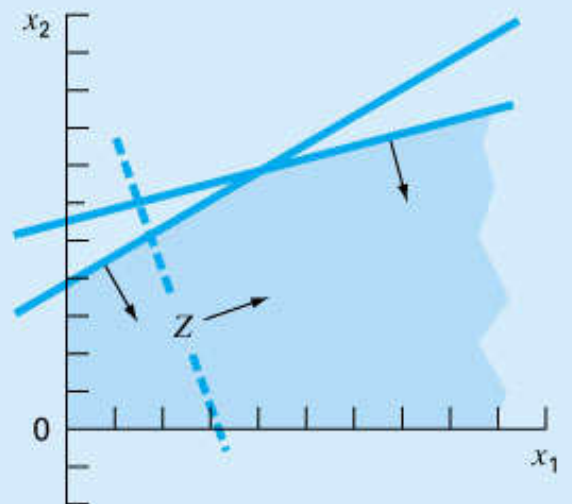# Graphical Solution to LP Problems (1)

Figure   Graphical solution for linear programming problem.

Aside from a single optimal solution; there are three other possible outcomes:

(a) Alternative optima
(b) no feasible solution
(c) an unbounded result

Graphical Solution to LP Problems (2)

- An equation of the form $4x_1 + 5x_2 = 1500$ defines a straight line in the x_1-x_2 plane. An inequality defines an area bounded by a straight line. Therefore, the region below and including the line $4x_1 + 5x_2 = 1500$ in the Figure represents the region defined by $4x_1 + 5x_2 \leq 1500$.

- Same thing applies to other equations as well.

- The <u>shaded area</u> of the figure comprises the area common to all the regions defined by the constraints and contains all pairs of x_1 and x_2 that are feasible solutions to the problem.

- This area is known as the *feasible region* or *feasible solution space.* <u>The optimal solution must lie within this region</u>.

- There are various pairs of x_1 and x_2 that satisfy the constraints such as:

$$\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Graphical Solution to LP Problems (3)

- Trying different solutions, the optimal solution will be:

$$X_1 = 270$$

$$X_2 = 75$$

- This indicates that maximum income of $4335 is obtained by producing 270 units of product I and 75 units of product II.

- In this solution, all the raw material and available time are used, because the optimal point lies on the two constraint lines for these resources.

- However, 1500- [4(270) + 5(75)], or 45 ft^2 of storage space, is not used. Thus the storage space is not a constraint on the optimal solution; that is, more products could be produced before the company ran out of storage space. Thus this constraint is said to be *slack*.

Graphical Solution to LP Problems (4)

- If the objective function happens to be parallel to one of the edges of the feasible region, any point along this edge between the two extreme points may be an optimal solution that maximizes the objective function. When this occurs, there is no unique solution, but there is an infinite number of optimal solutions.

- The graphical method of solution may be extended to a case in which there are three variables. In this case, each constraint is represented by a plane in three dimensions, and the feasible region bounded by these planes is a polyhedron.

The Simplex Method (1)

❖ When decision variables are more than 2, it is always advisable to use Simplex Method to avoid lengthy graphical procedure.

❖ The simplex method is not used to examine all the feasible solutions.

❖ It deals only with a small and unique set of feasible solutions, the set of vertex points (i.e., extreme points) of the convex feasible space that contains the optimal solution.

The Simplex Method (2)

❖ **Steps involved:**

1. Locate an extreme point of the feasible region.

2. Examine each boundary edge intersecting at this point to see whether movement along any edge increases the value of the objective function.

3. If the value of the objective function increases along any edge, move along this edge to the adjacent extreme point. If several edges indicate improvement, the edge providing the greatest rate of increase is selected.

4. Repeat steps 2 and 3 until movement along any edge no longer increases the value of the objective function.

The Simplex Method (3)

Example: Product Mix Problem

The N. Dustrious Company produces two products: I and II. The raw material requirements, space needed for storage, production rates, and selling prices for these products are given below:

	Product	
	I	II
Storage space (ft²/unit)	4	5
Raw material (lb/unit)	5	3
Production rate (units/hr)	60	30
Selling price ($/unit)	13	11

The total amount of raw material available per day for both products is 1575 1b. The total storage space for all products is 1500 ft², and a maximum of 7 hours per day can be used for production. **The company wants to determine how many units of each product to produce per day to maximize its total income.**

The Simplex Method (4)

Solution

❖ **Step 1:** Convert all the inequality constraints into equalities by the use of slack variables. Let:

$$S_1 = \text{unused storage space}$$
$$S_2 = \text{unused raw materials}$$
$$S_3 = \text{unused production time}$$

As already developed, the LP model is:

Maximize

$$Z = 13x_1 + 11x_2$$

subject to:

$$4x_1 + 5x_2 \leq 1500$$

$$x_1 \geq 0 \qquad 5x_1 + 3x_2 \leq 1575 \qquad \text{.....Eq (4)}$$

$$x_2 \geq 0 \qquad x_1 + 2x_2 \leq 420$$

The Simplex Method (5)

❖ Introducing these slack variables into the inequality constraints and rewriting the objective function such that all variables are on the left-hand side of the equation. Equation 4 can be expressed as:

$$Z - 13x_1 - 11x_2 \qquad\qquad = \quad 0 \qquad \text{(A1)}$$
$$4x_1 + 5x_2 + S_1 \qquad\qquad = 1500 \qquad \text{(B1)}$$
$$5x_1 + 3x_2 \qquad + S_2 \qquad = 1575 \qquad \text{(C1)}$$
$$x_1 + 2x_2 \qquad\qquad + S_3 = 420 \qquad \text{(D1)}$$
$$x_i \geq 0, \qquad i = 1, 2$$

.....Eq (5)

From the equations above, it is obvious that one feasible solution that satisfies all the constraints is: $x_1 = 0$, $x_2 = 0$, $S_1 = 1500$, $S_2 = 1575$, $S_3 = 420$, and $Z = 0$.

The Simplex Method (6)

❖ Since the coefficients of x_1 and x_2 in Eq. (A1) are both negative, the value of Z can be increased by giving either x_1 or x_2 some positive value in the solution.

❖ In Eq. (B1), if $x_2 = S_1 = 0$, then $x_1 = 1500/4 = 375$. That is, there is only sufficient storage space to produce 375 units at product I.

❖ From Eq. (C1), there is only sufficient raw materials to produce $1575/5 = 315$ units of product I.

❖ From Eq. (D1), there is only sufficient time to produce $420/1 = 420$ units of product I.

❖ Therefore, considering all three constraints, there is sufficient resource to produce only 315 units of x_1. Thus the maximum value of x_1 is limited by Eq. (C1).

The Simplex Method (7)

❖ **Step 2:** From Equation CI, which limits the maximum value of x_1.

$$x_1 = -\frac{3}{5}x_2 - \frac{1}{5}S_2 + 315 \qquad \text{.....Eq (6)}$$

Substituting this equation into Eq. (5) yields the following new formulation of the model.

$$
\begin{aligned}
Z - \frac{16}{5}x_2 \quad + \frac{13}{5}S_2 \quad &= 4095 \quad \text{(A2)} \\
+ \frac{13}{5}x_2 + S_1 - \frac{4}{5}S_2 \quad &= 240 \quad \text{(B2)} \\
x_1 + \frac{3}{5}x_2 \quad + \frac{1}{5}S_2 \quad &= 315 \quad \text{(C2)} \\
\frac{7}{5}x_2 \quad - \frac{1}{5}S_2 + S_3 &= 105 \quad \text{(D2)}
\end{aligned}
$$

.....Eq (7)

The Simplex Method (8)

❖ It is now obvious from these equations that the new feasible solution is:

$x_1 = 315$, $x_2 = 0$, $S_1 = 240$, $S_2 = 0$, $S_3 = 105$, and $Z = 4095$

❖ It is also obvious from Eq.(A2) that it is also not the <u>optimum solution</u>. The coefficient of x_1 in the objective function represented by A2 is negative (-16/5), which means that the value of Z can be further increased by giving x_2 some positive value.

The Simplex Method (9)

❖ Following the same analysis procedure used in step 1, it is clear that:

❖ In Eq. (B2), if $S_1 = S_1 = 0$, then $x_2 = (5/13)(240) = 92.3$.

❖ From Eq. (C2), x_2 can take on the value $(5/3)(315) = 525$ if $x_1 = S_2 = 0$

❖ From Eq. (D2), x_2 can take on the value $(5/7)(105) = 75$ if $S_2 = S_3 = 0$

❖ Therefore, constraint D_2 limits the maximum value of x_2 to 75. Thus a new feasible solution includes $x_2 = 75$, $S_2 = S_3 = 0$.

The Simplex Method (10)

❖ **Step 3:** From Equation D2:

$$x_2 = \frac{1}{7}S_2 - \frac{5}{7}S_3 + 75 \qquad \text{.....Eq (8)}$$

Substituting this equation into Eq. (7) yield:

$$Z \qquad + \frac{15}{7}S_2 + \frac{16}{7}S_3 = 4335 \quad (A3)$$

$$S_1 - \frac{3}{7}S_2 - \frac{13}{7}S_3 = \quad 45 \quad (B3)$$

$$x_1 \qquad + \frac{2}{7}S_2 - \frac{3}{7}S_3 = \quad 270 \quad (C3) \qquad \text{.....Eq (9)}$$

$$x_2 \qquad - \frac{1}{7}S_2 + \frac{5}{7}S_3 = \quad 75 \quad (D3)$$

From these equations, the new feasible solution is readily found to be: $x_1 = 270$, $x_2 = 75$, $S_1 = 45$, $S_2 = 0$, $S_3 = 0$, $Z = 4335$.

The Simplex Method (11)

❖ Because the coefficients in the objective function represented by Eq. (A3) are all positive, this new solution is also the optimum solution.
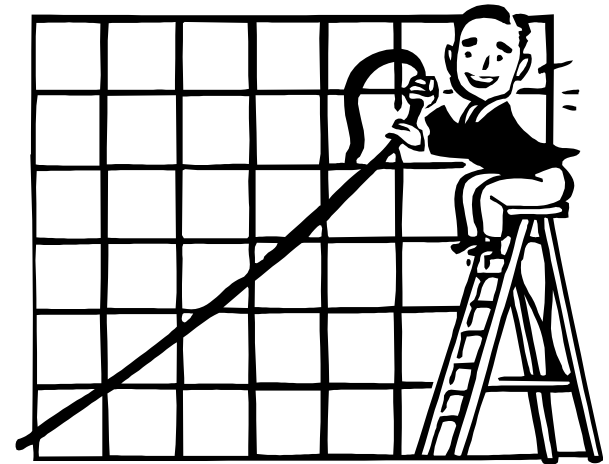
Simplex Tableau for Maximization (1)

❖ **Step I:** Set up the initial tableau using Eq. (5).

$$Z - 13x_1 - 11x_2 \qquad\qquad = \quad 0 \qquad \text{(A1)}$$
$$4x_1 + 5x_2 + S_1 \qquad\qquad = 1500 \qquad \text{(B1)}$$
$$5x_1 + 3x_2 \qquad + S_2 \qquad = 1575 \qquad \text{(C1)}$$
$$x_1 + 2x_2 \qquad\qquad + S_3 = \quad 420 \qquad \text{(D1)}$$
$$x_i \geq 0, \qquad i = 1, 2$$

.....Eq (5)

In any iteration, a variable that has a nonzero value in the solution is called a **basic variable**.

Row Number	Basic Variable	Coefficients of:						Right-Hand Side	Upper Bound on Entering Variable
		Z	x_1	x_2	S_1	S_2	S_3		
Initial tableau									
A1	Z	1	−13	−11	0	0	0	0	
B1	S_1	0	4	5	1	0	0	1500	375
C1	S_2	0	5	3	0	1	0	1575	315
D1	S_3	0	1	2	0	0	1	420	420

Simplex Tableau for Maximization (2)

❖ **Step II:** . Identify the variable that will be assigned a nonzero value in the next iteration so as to increase the value of the objective function. This variable is called the *entering variable*.

➢ It is that nonbasic variable which is associated with the smallest negative coefficient in the objective function.

➢ If two or more nonbasic variables are tied with the smallest coefficients, select one of these arbitrarily and continue.

❖ **Step III:** Identify the variable, called the *leaving variable,* which will be changed from a nonzero to a zero value in the next solution.

Simplex Tableau for Maximization (3)

❖ **Step IV:** . Enter the basic variables for the second tableau. The row sequence of the previous tableau should be maintained, with the leaving variable being replaced by the entering variable.

Row Number	Basic Variable	Coefficients of:						Right-Hand Side	Upper Bound on Entering Variable
		Z	x_1	x_2	S_1	S_2	S_3		
Initial tableau									
A1	Z	1	-13	-11	0	0	0	0	
B1	S_1	0	4	5	1	0	0	1500	375
C1	S_2	0	5	3	0	1	0	1575	315
D1	S_3	0	1	2	0	0	1	420	420
Second tableau at end of first iteration									
A2	Z	1	0	$-\dfrac{16}{5}$	0	$+\dfrac{13}{5}$	0	4095	
B2	S_1	0	0	$\dfrac{13}{5}$	1	$-\dfrac{4}{5}$	0	240	92.3
C2	x_1	0	1	$\dfrac{3}{5}$	0	$\dfrac{1}{5}$	0	315	525
D2	S_3	0	0	$\dfrac{7}{5}$	0	$-\dfrac{1}{5}$	1	105	75

Simplex Tableau for Maximization (4)

❖ **Step V:** Compute the coefficients for the second tableau. A sequence of operations will be performed so that at the end the x_1 column in the second tableau will have the following coefficients:

	x_1
Z	0
S_1	0
x_1	1
S_3	0

The second tableau yields the following feasible solution:

x_1 = 315, x_2 = 0, SI = 240, S2 = 0, S3 = 105, and Z = 4095

Simplex Tableau for Maximization (5)

❖ The row operations proceed as fo1lows:

➤ The coefficients in row C2 are obtained by dividing the corresponding coefficients in row C1 by 5.

➤ The coefficients in row A2 are obtained by multiplying the coefficients of row C2 by 13 and adding the products to the corresponding coefficients in row AI.

➤ The coefficients in row B2 are obtained by multiplying the coefficients of row C2 by -4 and adding the products to the corresponding coefficients in row BI.

➤ The coefficients in row D2 are obtained by multiplying the coefficients of row C2 by -1 and adding the products to the corresponding coefficients in row DI.

Simplex Tableau for Maximization (6)

❖ **Step VI:** Check for optimality. The second feasible solution is also not optimal, because the objective function (row A2) contains a negative coefficient. <u>Another iteration beginning with step 2 is necessary</u>.

❖ In the third tableau (next slide), all the coefficients in the objective function (row A3) are positive. Thus an optimal solution has been reached and it is as follows:

$x_1 = 270, x_2 = 75, SI = 45, S2 = 0, S3 = 0,$ and $Z = 4335$

Row Number	Basic Variable	Z	x_1	x_2	S_1	S_2	S_3	Right-Hand Side	Upper Bound on Entering Variable
Initial tableau									
A1	Z	1	-13	-11	0	0	0	0	
B1	S_1	0	4	5	1	0	0	1500	375
C1	S_2	0	5	3	0	1	0	1575	315
D1	S_3	0	1	2	0	0	1	420	420
Second tableau at end of first iteration									
A2	Z	1	0	$-\frac{16}{5}$	0	$+\frac{13}{5}$	0	4095	
B2	S_1	0	0	$\frac{13}{5}$	1	$-\frac{4}{5}$	0	240	92.3
C2	x_1	0	1	$\frac{3}{5}$	0	$\frac{1}{5}$	0	315	525
D2	S_3	0	0	$\frac{7}{5}$	0	$-\frac{1}{5}$	1	105	75
Third tableau at end of second and final iteration									
A3	Z	1	0	0	0	$+\frac{15}{7}$	$+\frac{16}{7}$	4335	
B3	S_1	0	0	0	1	$-\frac{3}{7}$	$-\frac{13}{7}$	45	
C3	x_1	0	1	0	0	$\frac{2}{7}$	$-\frac{3}{7}$	270	
D3	x_2	0	0	1	0	$-\frac{1}{7}$	$\frac{5}{7}$	75	

Complications in Simplex Method (1)

❖ An objective function to be minimized instead of maximized.

❖ Greater-than-or-equal-to constraints.

❖ Equalities instead of inequalities for constraints.

❖ Decision variables unrestricted in signs.

❖ Zero constants on the right-hand side of one or more constraints.

❖ Some or all decision variables must be integers.

❖ Non-positive constants on the right-hand side of the constraints.

❖ More than one optimal solution, that is, multiple solutions such that there is no unique optimal solution.

Complications in Simplex Method (2)

❖ The constraints are such that <mark>no feasible solution exists</mark>.

❖ The constraints are such that one or more of the variables can increase without limit and never violate a constraint (i.e., the solution is unbounded).

❖ Some or all of the coefficients and right-hand-side terms are given by a probability distribution rather than a single value.

Complications in Simplex Method (3)

Minimization Problem – Solution 1

Minimize

$$Z = c_1x_1 + c_2x_2 + c_3x_3 + \cdots + c_nx_n$$

❖ This objective function can be converted to the standard form of maximization. Let Z' = -Z, so:

$$Z' = -c_1x_1 - c_2x_2 - c_3x_3 - \cdots - c_nx_n$$

❖ Since maximum Z' = minimum (Z), the objective function becomes:

Maximize

$$Z' = -c_1x_1 - c_2x_2 - c_3x_3 - \cdots - c_nx_n$$

❖ After the Z' value is found, replace Z = -Z'

Complications in Simplex Method (4)

Minimization Problem – Solution 2

❖ In the case of a minimization problem, an optimum solution is reached when:

   ➢ All the nonbasic variables have nonpositive coefficients in row 1 of the simplex tableau

   ➢ The entering variable will be one which has the largest positive coefficient in row I.

❖ All the other operations in the simplex method remain unchanged.

Complications in Simplex Method (5)

Greater- Than-Or-Equal- To Constraints

❖ In the standard form of the linear programming model, the constraints are all expressed as less than or equal to ($\leq$) a certain amount, that is,

$$\sum_{j=1}^{n} c_{ij} \leq b_i$$

❖ In many occasions, the constraints must specify the lower bounds rather than the upper bounds such as:

$$\sum_{j=1}^{n} c_{ij} \geq b_i$$

which involves the inequality "greater than or equal to"

Complications in Simplex Method (6)

Example: Greater- Than-Or-Equal- To Constraints

Minimize

$$P = 1500y_1 + 1575y_2 + 420y_3$$

subject to:

$$4y_1 + 5y_2 + y_3 \geq 13 \qquad \text{.......Eq. (1)}$$
$$5y_1 + 3y_2 + 2y_3 \geq 11$$
$$\text{all } y_1 \geq 0$$

❖ To start the solution, slack variables must first be assigned to convert all <u>in-equalities to equalities</u>. Let S_1 and S_2 be slack variables.

❖ Re- arrange the objective function so that all the variables are on the left-hand side of the equation.

Complications in Simplex Method (7)

$$P - 1500y_1 - 1575y_2 - 420y_3 = 0$$
$$4y_1 + 5y_2 + y_3 - S_1 = 13$$
$$5y_1 + 3y_2 + 2y_3 - S_2 = 11$$
$$\text{all } y_i \geq 0, \text{ all } S_i \geq 0$$

…….Eq. (2)

❖ The negative signs for S_1 and S_2 make it no longer feasible to set all the decision variables (i.e., y_1, y_2, y_3) equal to zero as the initial solution.

❖ To assure a starting feasible solution, artificial variables can be added to the greater-than-or-equal-to constraints. Let W_1 and W_2 be two artificial variables. Hence the Eq. (2) becomes:

$$4y_1 + 5y_2 + y_3 - S_1 + W_1 = 13$$
$$5y_1 + 3y_2 + 2y_3 - S_2 + W_2 = 11$$

…….Eq. (3)

Complications in Simplex Method (8)

❖ A starting feasible solution can be easily derived from Eq. (3) as follows:

$$y_1 = y_2 = y_3 = S_1 = S_2 = 0, \quad W_1 = 13, \quad \text{and} \quad W_2 = 11$$

❖ The objective function in Eq. (3) then becomes:

Minimize
$$P = 1500y_1 + 1575y_2 + 420y_3 + 5000W_1 + 5000W_2$$

.……Eq. (4)

❖ From Eq. (3):

$$W_1 = 13 - 4y_1 - 5y_2 - y_3 + S_1$$
$$W_2 = 11 - 5y_1 - 3y_2 - 2y_3 + S_2$$

Complications in Simplex Method (9)

❖ Substituting these expressions in Eq. (4) yields the following new expression for the objective function:

$$P = -43,500y_1 - 38,425y_2 - 14,580y_3 + 5000S_1 + 5000S_2 + 120,000$$

❖ The objective function may now be combined with Eq. (3) to express the problem model as follows:

$$P + 43,500y_1 + 38,425y_2 + 14,580y_3 - 5000S_1 - 5000S_2 = 120,000$$
$$4y_1 + 5y_2 + y_3 - S_1 + W_1 = 13$$
$$5y_1 + 3y_2 + 2y_3 - S_2 + W_2 = 11$$

…….Eq. (5)

❖ The coefficients and constants in Eq. (5) can now be arranged in the Tableau format as shown in next slide.

Complications in Simplex Method (10)

Row Number	Basic Variable	Coefficients of:								Right-Hand Side	Upper Bound on Entering Variable
		P	y_1	y_2	y_3	S_1	S_2	W_1	W_2		
Initial tableau											
A1	P	1	43,500	38,425	14,580	−5,000	−5,000	0	0	120,000	
B1	y_6	0	4	5	1	−1	0	1	0	13	$\frac{13}{4}$
C1	y_7	0	5	3	2	0	−1	0	1	11	$\frac{11}{5}$
Second tableau											
A2	P	1	0	12,325	−2,820	−5,000	3,700	0	−8,700	24,300	
B2	y_6	0	0	$\frac{13}{5}$	$-\frac{3}{5}$	−1	$\frac{4}{5}$	1	$-\frac{4}{5}$	$\frac{21}{5}$	$\frac{21}{13}$
C2	y_1	0	1	$\frac{3}{5}$	$\frac{2}{5}$	0	$-\frac{1}{5}$	0	$\frac{1}{5}$	$\frac{11}{5}$	$\frac{11}{3}$
Third tableau											
A3	P	1	0	0	24.23	−259.62	−92.31	−4,740.38	−4,907.69	4,390.38	
B3	y_2	0	0	1	$-\frac{3}{13}$	$-\frac{5}{13}$	$\frac{4}{13}$	$\frac{5}{13}$	$-\frac{4}{13}$	$\frac{21}{13}$	
C3	y_1	0	1	0	$\frac{7}{13}$	$\frac{3}{13}$	$-\frac{5}{13}$	$-\frac{3}{13}$	$\frac{5}{13}$	$\frac{16}{13}$	$\frac{16}{7}$
Fourth tableau											
A4	P	1	−45	0	0	−270	−75	−4,730	−4,925	4,335	
B4	y_2	0	$\frac{3}{7}$	1	0	$-\frac{2}{7}$	$\frac{1}{7}$	$\frac{2}{7}$	$-\frac{1}{7}$	$\frac{15}{7}$	
C4	y_3	0	$\frac{13}{7}$	0	1	$\frac{3}{7}$	$-\frac{3}{7}$	$-\frac{3}{7}$	$\frac{5}{7}$	$\frac{16}{7}$	

Equality Constraint

❖ An equality constraint has the following general form:

$$c_1 x_1 + c_2 x_2 + c_3 x_3 + \cdots + c_n x_n = b$$

❖ An artificial variable must be assigned to each equality constraint to start the simplex solution. Otherwise, the constraint would be violated when all the decision variables are assumed to be zero.

❖ Sometimes a decision variable may take on either negative or positive values. It x_1 is unrestricted in sign, replace it throughout the model by the difference of two new nonnegative variables:

$$x_j = x_j' - x_j'' \quad \text{where } x_j' \geq 0, \, x_j'' \geq 0$$

Complications in Simplex Method (12)

Equality Constraint

❖ Because x_j' and x_j'' can have any nonnegative values their difference $(x_j' - x_j'')$ can have any value (positive or negative). After substitution, the simplex method can proceed with just nonnegative variables.