Numerical Solution of Ordinary Differential Equations

Chapter 25

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Differential Equations

- Differential equations play a fundamental role in engineering. Many physical phenomena are best formulated in terms of their rate of change:

$$\frac{dv}{dt} = g - \frac{c}{m}v$$

v- dependent variable

t- independent variable

- Equations which are composed of an *unknown function* and its *derivatives* are called ***differential equations***.

- **One** independent variable ➔ ***ordinary differential equation*** *(or* ***ODE****)*
- **Two** or **more** independent variables ➔ ***partial diff. equation*** *(or* ***PDE****)*

- *A first order equation* includes a first derivative as its highest derivative
- ***Second order*** *equation* includes second derivative

- Higher order equations can be reduced to a system of first order equations, by redefining the variables.

ODEs and Engineering Practice

Physical law: $F = ma$

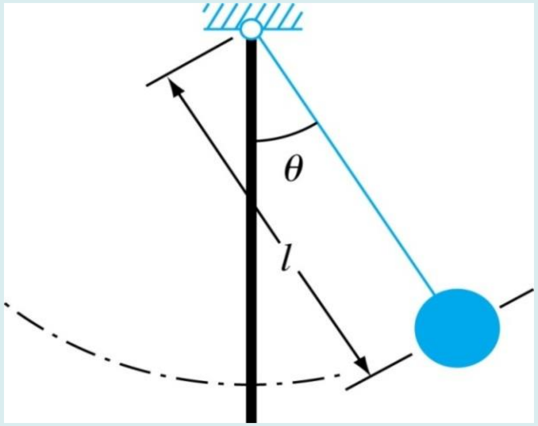
ODE: $\frac{dv}{dt} = g - \frac{c}{m}v$

Solution:

Analytical: $v = \frac{gm}{c}\left(1 - e^{-(c/m)t}\right)$

Numerical: $v_{i+1} = v_i + (g - \frac{c}{m}v_i)\Delta t$

Falling parachutist problem



Swinging pendulum

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0$$

A second-order nonlinear ODE.

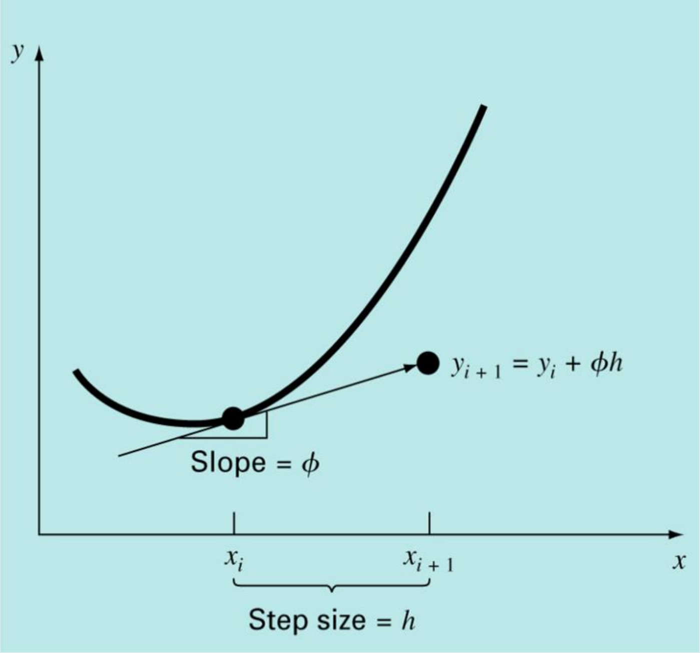
Solving Ordinary Differential Equations (ODEs)

- This chapter is devoted to solving ordinary differential equations (ODEs) of the form

$$\frac{dy}{dx} = f(x, y)$$

$$\text{New value} = \text{old value} + \text{slope} * (\text{step\_size})$$

$$y_{i+1} = y_i + \phi * h$$



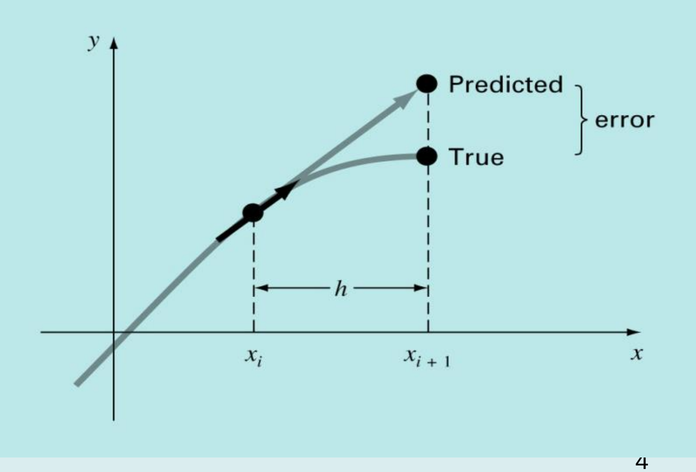
Euler's Method

- First derivative provides a direct estimate of the **slope** at x_i:

$$\phi = f(x_i, y_i) \quad (\text{diff. equ. evaluated at } x_i \text{ and } y_i)$$

then,

$$y_{i+1} = y_i + f(x_i, y_i)h$$



Error Analysis for Euler's Method

- Numerical solutions of ODEs involves two types of error:
  - *Truncation* error
    - ***Local*** *truncation error*
    - *Propagated truncation error*

    The sum of the two is the *total or **global** truncation error*
  - *Round-off* errors (due to limited digits in representing numbers in a computer)
- We can use Taylor series to quantify the ***local truncation error*** in Euler's method.

Given $y' = f(x, y)$

$$y_{i+1} = y_i + y_i' h + \frac{y_i''}{2!} h^2 + \ldots + \frac{y_i^{(n)}}{n!} h^n + R_n$$

$$\underbrace{y_{i+1} = y_i + f(x_i, y_i) h}_{\text{EULER}} + \underbrace{\frac{f'(x_i, y_i)}{2!} h^2 + \ldots + \frac{f^{(n-1)}(x_i, y_i)}{n!} h^n + O(h^{n+1})}_{\text{Local Truncation ERROR}}$$

$$E_a = \frac{f'(x_i, y_i)}{2!} h^2 + R_3$$

$$E_a \cong \frac{f'(x_i, y_i)}{2!} h^2 = O(h^2)$$

- **The error is reduced by 4 times if the step size is halved ➔ *O(h^2)*.**
- In real problems, the derivatives used in the Taylor series are not easy to obtain.
- If the solution to the differential equation is *linear*, the method will provide error free predictions (2nd derivative is **zero** for a straight line).

Example: Euler's Method

Solve numerically : $$\frac{dy}{dx} = -2x^3 + 12x^2 - 20x + 8.5$$

From **x=0** to **x=4** with step size **h=0.5**

initial condition: (x=0 ; y=1)

Exact Solution: $y = -0.5x^4 + 4x^3 - 10x^2 + 8.5x + 1$

Numerical Solution:

$$y_{i+1} = y_i + f(x_i, y_i)h$$

$y(0.5) = y(0)+f(0, 1)0.5 = 1+8.5*0.5 = 5.25$

(true solution at x=0.5 is y(0.5) = 3.22 and ε_t = 63%)

$y(1) = y(0.5)+f(0.5, 5.25)0.5$

$= 5.25+[-2(0.5)^3+12(0.5)^2-20(0.5)+8.5]*0.5$

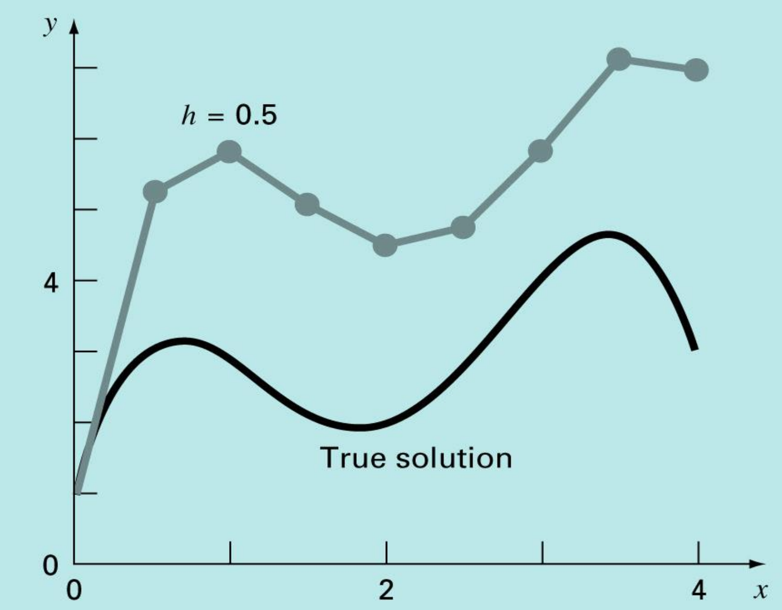
$= 5.25+0.625 = 5.875$

(true solution at x=1 is y(1) = 3 and ε_t = 96%)

$y(1.5) = y(1)+f(1, 5.875)0.5 = 5.125$

....

x	y_{euler}	y_{true}	Error Global	Error Local
0	1	1	%	%
0.5	5.250	3.218	63.1	63.1
1.0	5.875	3.000	95.8	28
1.5	5.125	2.218	131.0	1.41
2.0	4.500	2.000	125.0	20.5



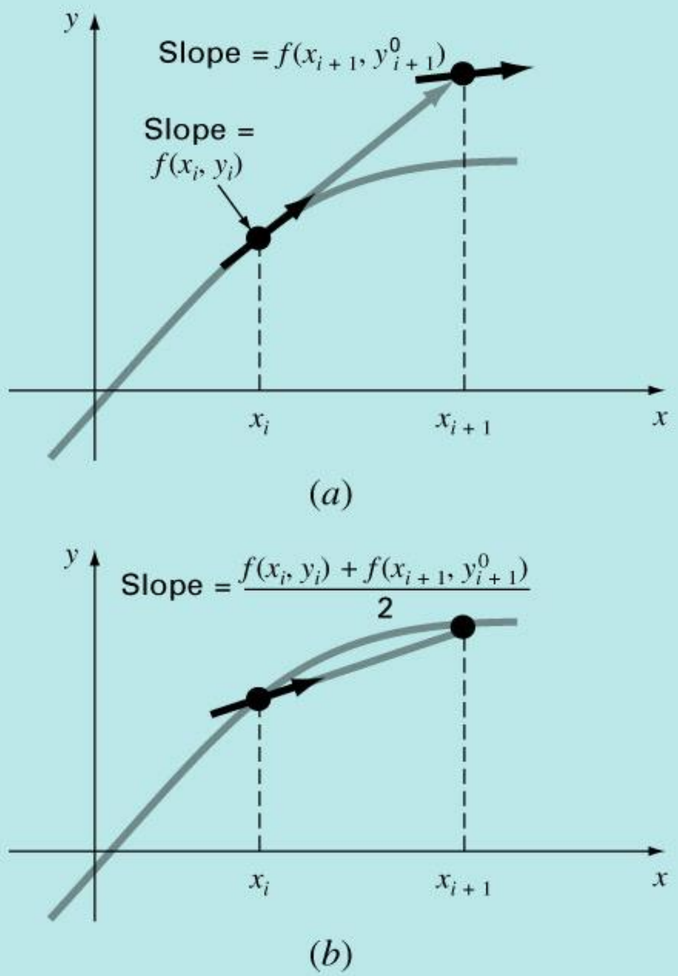
Improvements of Euler’s method

- A fundamental source of error in Euler’s method is that the derivative at the beginning of the interval is assumed to apply across the entire interval.

- Two simple modifications are available to circumvent this shortcoming:
  - **Heun’s Method**
  - **The Midpoint** (or Improved Polygon) Method

Heun's method

- To improve the estimate of the slope, determine two derivatives for the interval:
  - At the initial point
  - At the end point
- The two derivatives are then averaged to obtain an improved estimate of the slope for the entire interval.



$$\textbf{\textit{Predictor}}: \quad y^0_{i+1} = y_i + f(x_i, y_i)h$$

$$\textbf{\textit{Corrector}}: \quad y_{i+1} = y_i + \frac{f(x_i, y_i) + f(x_{i+1}, y^0_{i+1})}{2}h$$

Heun's method (**improved**)

Original Huen's:

$$\textbf{\textit{Predictor}}: \quad y_{i+1}^{0} = y_i + f(x_i, y_i)h$$

$$\textbf{\textit{Corrector}}: \quad y_{i+1} = y_i + \frac{f(x_i, y_i) + f(x_{i+1}, y_{i+1}^{0})}{2} h$$

Note that the corrector can be iterated to improve the accuracy of y_{i+1}.

$$\textbf{\textit{Predictor}}: y_{i+1}^{0} = y_i + f(x_i, y_i)h$$

$$\textbf{\textit{Corrector}}: y_{i+1}^{j} = y_i + \frac{f(x_i, y_i) + f(x_{i+1}, y_{i+1}^{j-1})}{2} h \qquad j = 1, 2, \ldots$$

However, it does not necessarily converge on the true answer but will converge on an estimate with a small error.

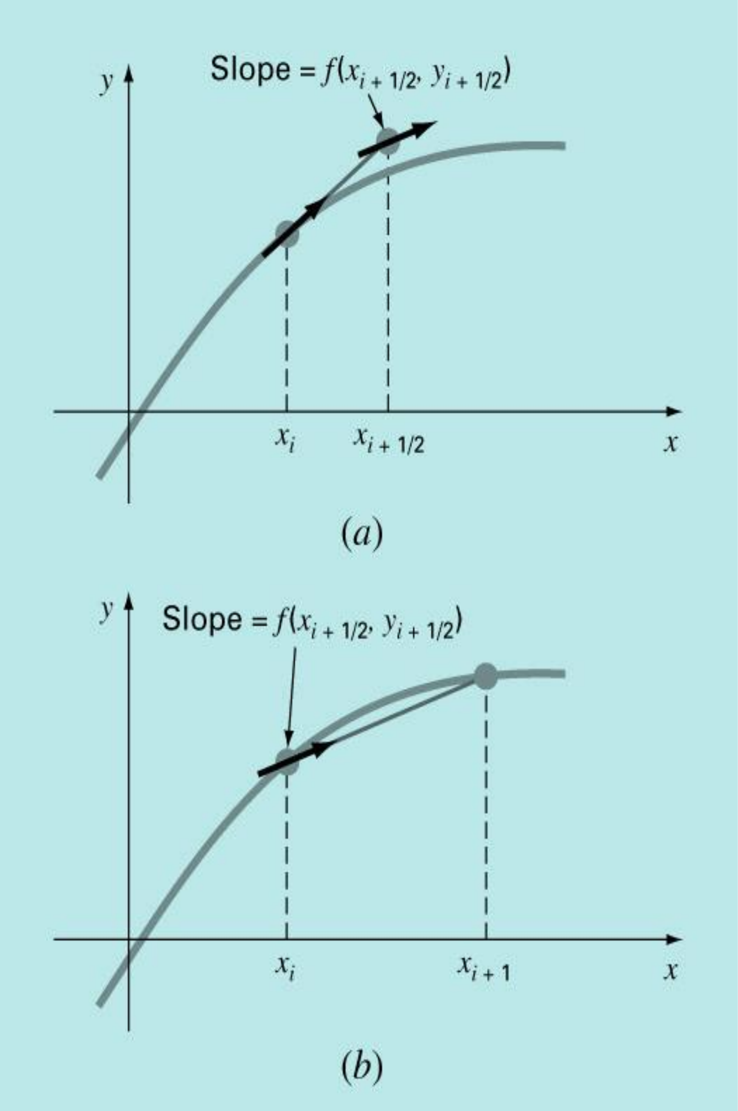
Example 25.5 from Textbook

The Midpoint (or Improved Polygon) Method

- Uses Euler's method to predict a value of *y* using the slope value at the midpoint of the interval:

$$y_{i+1/2} = y_i + f(x_i, y_i)\frac{h}{2}$$

$$y_{i+1} = y_i + f(x_{i+1/2}, y_{i+1/2})h$$



Runge-Kutta Methods (RK)

- Runge-Kutta methods achieve the accuracy of a Taylor series approach without requiring the calculation of higher derivatives.

$$y_{i+1} = y_i + \phi(x_i, y_i, h)h$$

$$\phi = a_1 k_1 + a_2 k_2 + \cdots + a_n k_n$$

Increment Function

***a*'s are constants**

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + p_1 h, y_i + q_{11} k_1 h)$$

***p*'s and *q*'s are constants**

$$k_3 = f(x_i + p_3 h, y_i + q_{21} k_1 h + q_{22} k_2 h)$$

$$\vdots$$

$$k_n = f(x_i + p_{n-1} h, y_i + q_{n-1} k_1 h + q_{n-1,2} k_2 h + \cdots + q_{n-1,n-1} k_{n-1} h)$$

Runge-Kutta Methods (cont.)

- Various types of RK methods can be devised by employing different number of terms in the increment function as specified by *n*.

- **First order** RK method with $n=1$ and $a_1=1$ is in fact **Euler's method**.

$$y_{i+1} = y_i + \phi(x_i, y_i, h)h$$
$$\phi = a_1 k_1 + a_2 k_2 + \cdots + a_n k_n$$
$$k_1 = f(x_i, y_i)$$

choose $\mathrm{n} = 1$ and $\mathrm{a}_1 = 1$, we obtain

$$y_{i+1} = y_i + f(x_i, y_i)h \qquad \textbf{(Euler's Method)}$$

Runge-Kutta Methods (cont.)

Second-order Runga-Kutta Methods:

$$y_{i+1} = y_i + (a_1 k_1 + a_2 k_2)h$$

$$k_1 = f(x_i, y_i) \qquad k_2 = f(x_i + p_1 h, y_i + q_{11} k_1 h)$$

- Values of $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{p_1}$, and $\mathbf{q_{11}}$ are evaluated by setting the above equation equal to a *Taylor series expansion* to the second order term. This way, three equations can be derived to evaluate the four unknown constants (See **Box 25.1** for this derivation).

A value is assumed for one of the unknowns to solve for the other three.

$$\left\{\begin{aligned} a_1 + a_2 &= 1 \\ a_2 p_1 &= \frac{1}{2} \\ a_2 q_{11} &= \frac{1}{2} \end{aligned}\right.$$

$$y_{i+1} = y_i + (a_1 k_1 + a_2 k_2)h$$

$$k_1 = f(x_i, y_i) \qquad k_2 = f(x_i + p_1 h, y_i + q_{11} k_1 h)$$

$$a_1 + a_2 = 1$$

$$a_2 p_1 = \frac{1}{2}$$

$$a_2 q_{11} = \frac{1}{2}$$

- Because we can choose an infinite number of values for a_2, there are an infinite number of second-order RK methods.

Three of the most commonly used methods are:

- **Huen's Method** with a Single Corrector (a_2=1/2)
- **The Midpoint Method** (a_2=1)
- **Ralston's Method** (a_2=2/3)

Huen's Method (a_2= 1/2) ➔ a_1 = 1/2 p_1 = 1 q_{11} = 1

$$y_{i+1} = y_i + (a_1 k_1 + a_2 k_2)h = y_i + (\frac{1}{2} k_1 + \frac{1}{2} k_2)h$$

$$k_1 = f(x_i, y_i) \qquad k_2 = f(x_i + h, y_i + k_1 h)$$

$$y_{i+1} = y_i + (a_1 k_1 + a_2 k_2)h$$

$$k_1 = f(x_i, y_i) \qquad k_2 = f(x_i + p_1 h, y_i + q_{11} k_1 h)$$

$$a_1 + a_2 = 1$$

$$a_2 p_1 = \frac{1}{2}$$

$$a_2 q_{11} = \frac{1}{2}$$

The Midpoint Method ($a_2 = 1$)

➔➔➔ $a_1 = 0$ $p_1 = 1/2$ $q_{11} = 1/2$

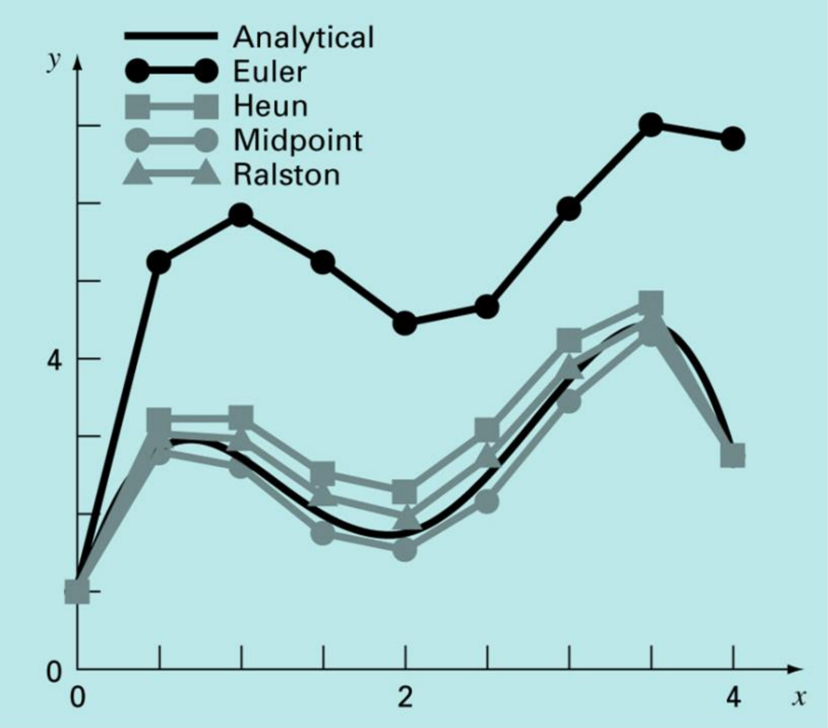
$$y_{i+1} = y_i + (a_1 k_1 + a_2 k_2)h = y_i + (k_2)h$$

$$k_1 = f(x_i, y_i) \qquad k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right)$$

$$y_{i+1} = y_i + (k_2)h = y_i + f\left(x_i + \frac{h}{2}, y_i + \frac{h}{2} f(x_i, y_i)\right)h$$

- Three most commonly used methods:
  - **Huen Method** with a Single Corrector (a_2=1/2)
  - **The Midpoint Method** (a_2=1)
  - **Ralston's Method** (a_2=2/3)

Comparison of Various Second-Order RK Methods



Ralston's Method (a_2=2/3)

$$y_{i+1} = y_i + (a_1 k_1 + a_2 k_2)h = y_i + (\frac{1}{3}k_1 + \frac{2}{3}k_2)h$$

$$k_1 = f(x_i, y_i) \qquad k_2 = f(x_i + \frac{3}{4}h, y_i + \frac{3}{4}k_1 h)$$

Systems of Equations

- Many practical problems in engineering and science require the solution of a system of simultaneous ordinary differential equations (ODEs) rather than a single equation:

$$\frac{dy_1}{dx} = f_1(x, y_1, y_2, \ldots, y_n)$$

$$\frac{dy_2}{dx} = f_2(x, y_1, y_2, \ldots, y_n)$$

$$\vdots$$

$$\frac{dy_n}{dx} = f_n(x, y_1, y_2, \ldots, y_n)$$

- Solution requires that n initial conditions be known at the starting value of x.

  i.e. $(x_0, y_1(x_0), y_2(x_0), \ldots, y_n(x_0))$

- At iteration i, n values $(y_1(x_i), y_2(x_i), \ldots, y_n(x_i))$ are computed.