LU Decomposition and Matrix Inversion

Chapter 10

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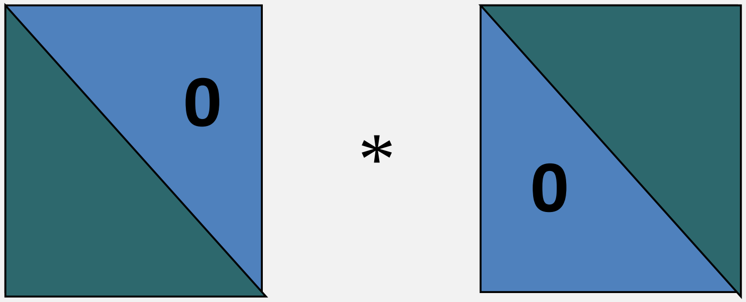
Solve $\boldsymbol{A \cdot x = b}$ *(system of linear equations)*

Decompose $\boldsymbol{A = L \cdot U}$



L : Lower Triangular Matrix U : Upper Triangular Matrix

To solve $[A]\{x\}=\{b\}$

$$[L][U]=[A] \quad \rightarrow \quad [L][U]\{x\}=\{b\}$$

Consider $$[U]\{x\}=\{d\}$$
$$[L]\{d\}=\{b\}$$

1. Solve $[L]\{d\}=\{b\}$ using **forward substitution** to get $\{d\}$

2. Use **back substitution** to solve $[U]\{x\}=\{d\}$ to get $\{x\}$

Both phases, (1) and (2), take $O(n^2)$ steps.

$$[A]\{x\} = \{b\} \Rightarrow [L][U]\{x\} = \{b\}$$

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}}_{[L]} \underbrace{\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}}_{[U]}$$

$$[A]\{x\}=\{b\} \Rightarrow [L][U]\{x\}=\{b\}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Gauss Elimination ➔ ➔

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b''_3 \end{bmatrix}$$

[*U*]

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

Coefficients used during the elimination step

$$\boldsymbol{A}=\begin{bmatrix}\boldsymbol{a}_{11} & \boldsymbol{a}_{12} & \boldsymbol{a}_{13}\\ \boldsymbol{a}_{21} & \boldsymbol{a}_{22} & \boldsymbol{a}_{23}\\ \boldsymbol{a}_{31} & \boldsymbol{a}_{32} & \boldsymbol{a}_{33}\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ \boldsymbol{l}_{21} & 1 & 0\\ \boldsymbol{l}_{31} & \boldsymbol{l}_{32} & 1\end{bmatrix}\begin{bmatrix}\boldsymbol{a}_{11} & \boldsymbol{a}_{12} & \boldsymbol{a}_{13}\\ 0 & \boldsymbol{a}_{22}^{'} & \boldsymbol{a}_{23}^{'}\\ 0 & 0 & \boldsymbol{a}_{33}^{''}\end{bmatrix}$$

$$l_{21}=\frac{a_{21}}{a_{11}}$$

$$l_{31}=\frac{a_{31}}{a_{11}}$$

$$l_{32}=?$$

[$\boldsymbol{L}\cdot\boldsymbol{U}$]

Example: $A = L \cdot U$

$$\begin{bmatrix} -1 & 2.5 & 5 \\ -2 & 9 & 11 \\ 4 & -22 & -20 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -4 & -3 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2.5 & 5 \\ 0 & 4 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

Gauss Elimination

Coefficients

l_{21} = -2/-1= 2

l_{31} = 4/-1= -4

l_{32} = -12/4= -3

[*L*]

$$\begin{bmatrix} -1 & 2.5 & 5 \\ -2 & 9 & 11 \\ 4 & -22 & -20 \end{bmatrix} \Rightarrow \quad \Rightarrow \begin{bmatrix} -1 & 2.5 & 5 \\ 0 & 4 & 1 \\ 0 & -12 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2.5 & 5 \\ 0 & 4 & 1 \\ 0 & -12 & 0 \end{bmatrix} \Rightarrow \boxed{\begin{bmatrix} -1 & 2.5 & 5 \\ 0 & 4 & 1 \\ 0 & 0 & 3 \end{bmatrix}}$$

[*U*]

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -4 & ?? & 1 \end{bmatrix} \Rightarrow\Rightarrow \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -4 & -3 & 1 \end{bmatrix}$$

Example: $A = L \cdot U$

Gauss Elimination with pivoting

$$\left.\begin{bmatrix} -1 & 2.5 & 5 \\ -2 & 9 & 11 \\ 4 & -22 & -20 \end{bmatrix}\right) \Rightarrow pivoting \Rightarrow \begin{bmatrix} 4 & -22 & -20 \\ -2 & 9 & 11 \\ -1 & 2.5 & 5 \end{bmatrix}$$

Coefficients

l_{21} = -2/4= -.5

l_{31} = -1/4= -.25

$$\begin{bmatrix} 4 & -22 & -20 \\ -2 & 9 & 11 \\ -1 & 2.5 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & -22 & -20 \\ 0 & \mathbf{-2} & \mathbf{1} \\ 0 & -3 & 0 \end{bmatrix} \Rightarrow pivoting \Rightarrow \begin{bmatrix} 4 & -22 & -20 \\ 0 & -3 & 0 \\ 0 & \mathbf{-2} & \mathbf{1} \end{bmatrix}$$

Coefficients

l_{32} = -2/-3

$$\begin{bmatrix} 4 & -22 & -20 \\ 0 & -3 & 0 \\ 0 & -2 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & -22 & -20 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

[*U*]

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \mathbf{-0.5} & 1 & 0 \\ -0.25 & ?? & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ -0.25 & 1 & 0 \\ \mathbf{-0.5} & ?? & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ -0.25 & 1 & 0 \\ \mathbf{-0.5} & \mathbf{0.66} & 1 \end{bmatrix}$$

LU decomposition

- *Gauss Elimination* can be used to decompose [A] into [L] and [U]. Therefore, it requires the same total FLOPs as for *Gauss elimination:* In the order of (proportional to) N^3 where N is the # of unknowns.

- l_{ij} values (the factors generated during the elimination step) can be stored in the lower part of the matrix to save storage. This can be done because these are converted to zeros anyway and unnecessary for the future operations.

- Provides efficient means to compute the matrix inverse

MATRIX INVERSE

$$A.\,A^{-1} = I$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solve in n=3 major phases

1

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

2

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

3

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Solve each one
using A=L·U method ➔ e.g.

$$LU \cdot \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Each solution takes **O(n²)** steps.

Therefore,
The Total time = **O(n³)**

Solve Problem 10.6. Solution file is available on the web