~ Numerical Differentiation and Integration ~

Integration of Equations

Chapter 22

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Romberg Integration

Successive application of the *trapezoidal rule* to attain efficient numerical integrals of functions.

Richardson's Extrapolation: In numerical analysis, **Richardson extrapolation** is a sequence acceleration method, used to improve the rate of convergence of a sequence. Here we use two estimates of an integral to compute a third and more accurate approximation.

$$I = I(h) + E(h) \qquad h = (b-a)/n \qquad n = (b-a)/h$$

$$I(h_1) + E(h_1) = I(h_2) + E(h_2)$$

I = exact value of integral *E(h)* = the truncation error

I(h): trapezoidal rule (n segments, step size h)

$$E \cong \frac{b-a}{12} h^2 \bar{f}'' = O(h^2)$$

(assume $\bar{f}''$ is constant for different step sizes)

$$\frac{E(h_1)}{E(h_2)} \cong \frac{h_1^2}{h_2^2} \quad \Rightarrow \quad E(h_1) \cong E(h_2)\left(\frac{h_1}{h_2}\right)^2$$

$$I(h_1) + E(h_2)(h_1/h_2)^2 \cong I(h_2) + E(h_2) \quad \Rightarrow \quad E(h_2) \cong \frac{I(h_2) - I(h_1)}{(h_1/h_2)^2 - 1}$$

$$\boldsymbol{I = I(h_2) + E(h_2)}$$

$$\boldsymbol{I \cong I(h_2) + \frac{1}{(h_1/h_2)^2 - 1}\left[I(h_2) - I(h_1)\right]}$$

Improved estimate of the integral. It is shown that the error of this estimate is $O(h^4)$. Trapezoidal rule had an error estimate of $O(h^2)$.

$$I \cong I(h_2) + \frac{1}{(h_1/h_2)^2 - 1}\left[I(h_2) - I(h_1)\right]$$

If $(h_2 = h_1/2) \Rightarrow$

$$I \cong I(h_2) + \frac{1}{2^2 - 1}\left[I(h_2) - I(h_1)\right] = \frac{4}{3} I(h_2) - \frac{1}{3} I(h_1)$$

Example

Evaluate the integral of $f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$ from a=0 to b=0.8.

I (*True Integral value*) = 1.6405

Segments	h	Integral	ε_{tr}%
1	0.8	0.1728	89.5
2	0.4	1.0688	34.9
4	0.2	1.4848	9.5

Segments 1 & 2 combined to give :

$$I \cong \frac{4}{3}(1.0688) - \frac{1}{3}(0.1728) = 1.3675$$

$$E_t = 1.6405 - 1.3675 = 0.273 \quad (\varepsilon_t = 16.6\%)$$

Segments 2 & 4 combined to give :

$$I \cong \frac{4}{3}(1.4848) - \frac{1}{3}(1.0688) = 1.6234$$

$$E_t = 1.6405 - 1.6234 = 0.0171 \quad (\varepsilon_t = 1\%)$$

In each case, two estimates with error $O(h^2)$ are combined to give a third estimate with error $O(h^4)$

In Example 22.1, we computed two improved estimates of **O(h⁴).** These two estimates can, in turn, be combined to yield an even better value with error **$O(h^6)$.**

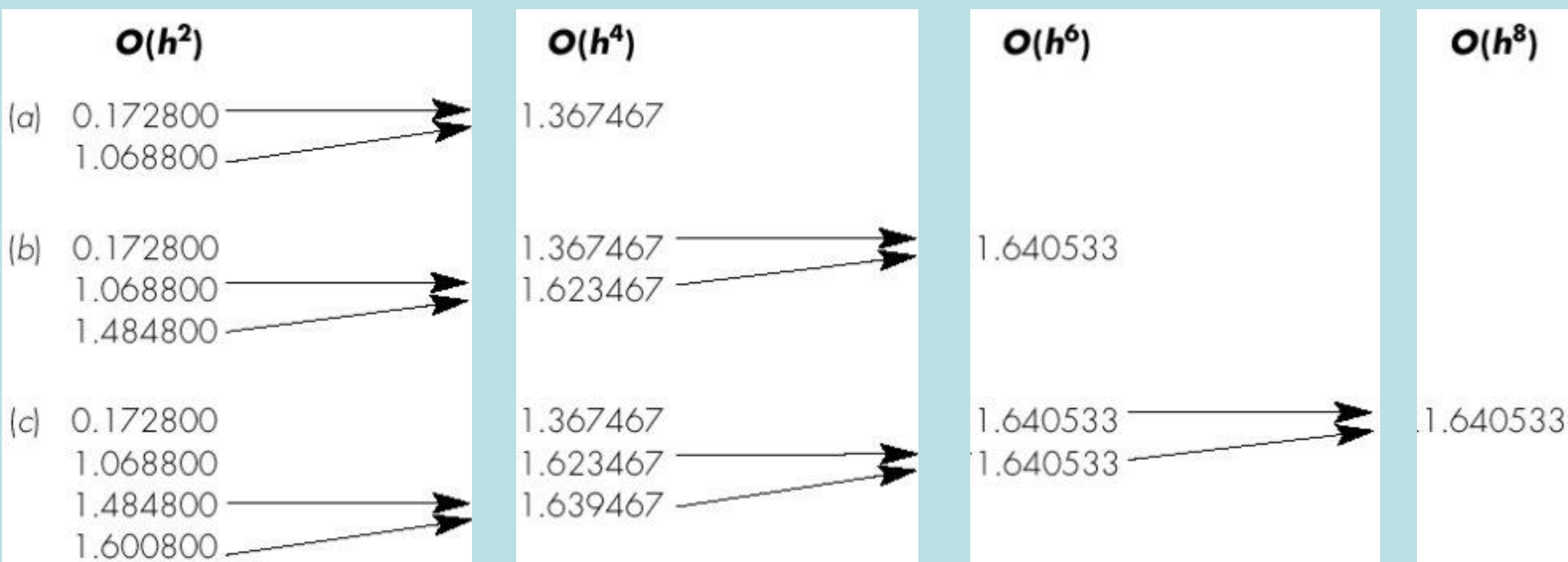
For the special case where the original trapezoidal estimates are based on ***successive halving*** of the step size, the equation used for **$O(h^6)$** accuracy is:

$$\boxed{I \cong \frac{16}{15} I_m - \frac{1}{15} I_l}$$

where I_m and I_l are more and less accurate estimates

Similarly, two $O(h^6)$ estimates can be combined to compute an I that is $O(h^8)$.

$$\boxed{I \cong \frac{64}{63} I_m - \frac{1}{63} I_l}$$



The Romberg Integration Algorithm

$$I_{j,k} \cong \frac{4^{k-1} I_{j+1,k-1} - I_{j,k-1}}{4^{k-1} - 1}$$

k=1 refers to *trapezoidal* rule, hence **O(h²)** accuracy.

k=2 refers to **$O(h^4)$** and k=3 ➔ **$O(h^6)$**

Index j is used to distinguish between the ***more*** (j+1) and the ***less*** (j) accurate estimates.