~ Roots of Equations ~

Open Methods

Chapter 6

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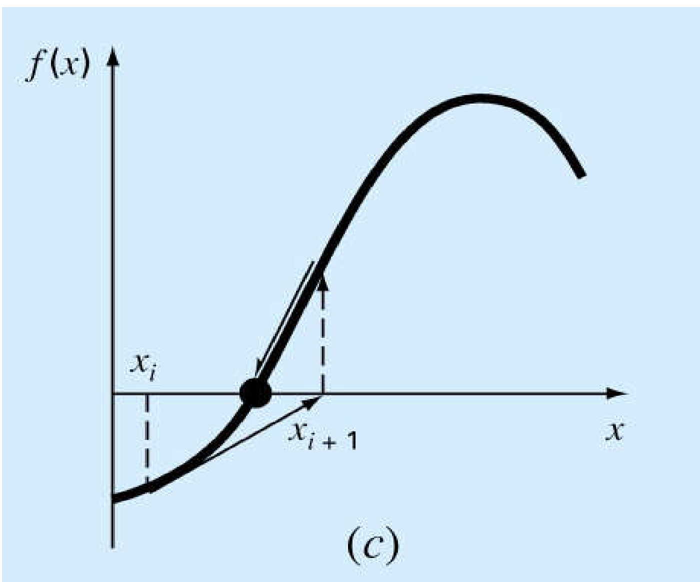
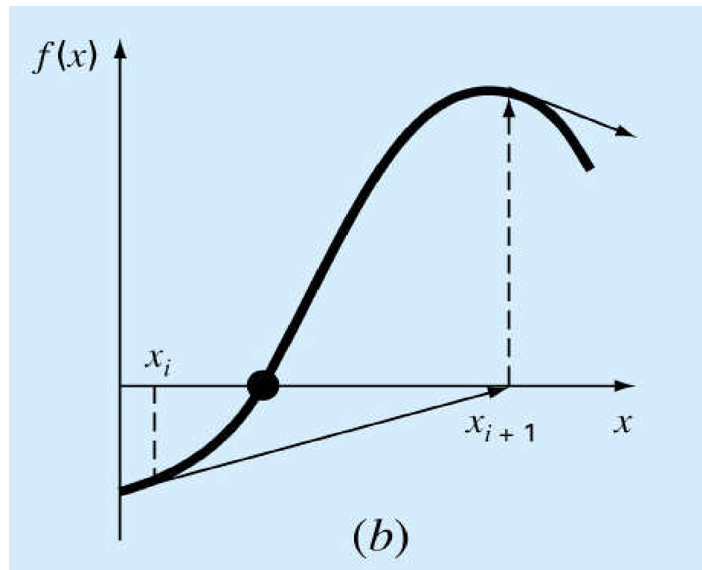
Open Methods

- Generally use a **single starting value** or two starting values that do not need to bracket the root.

Open Method (convergent)



(divergent)



Simple Fixed-point Iteration

- Rearrange the function so that x is on the left-hand side of the equation:

$$f(x) = 0 \quad \Rightarrow \quad g(x) = x$$

$$x_k = g(x_{k-1}) \quad x_o \text{ is given, } \quad k = 1, 2, \ldots$$

- Bracketing methods are "convergent" if you have a bracket to start with
- Fixed-point methods may sometimes "converge", depending on the starting point (initial guess) and how the function behaves.

EXAMPLE:

$$f(x) = x^2 - x - 2 = 0$$

Rewrite it as : $x = g(x)$

$$x = x^2 - 2$$

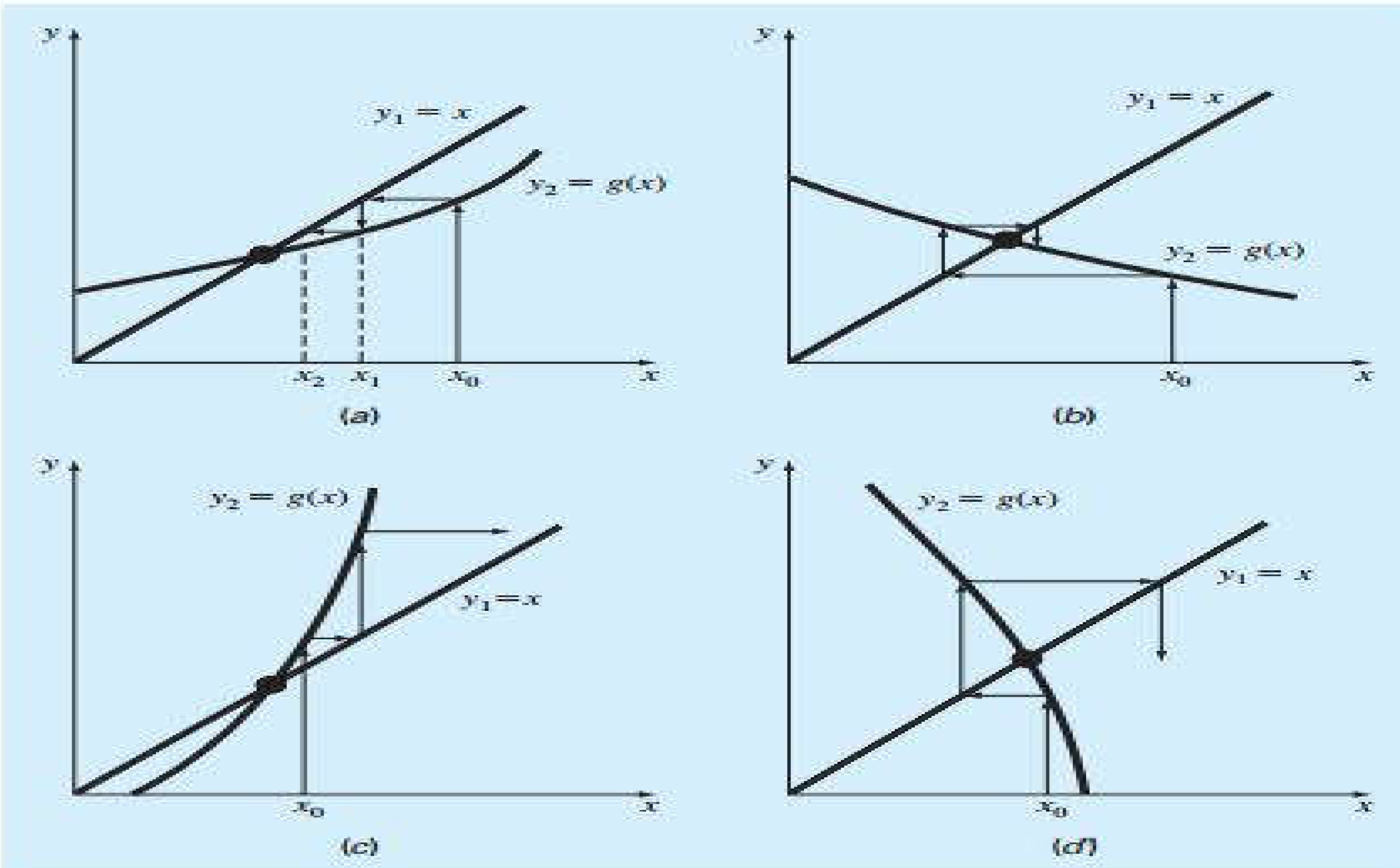
or

$$x = \sqrt{x+2}$$

or

$$x = 1 + \frac{2}{x}$$

$\vdots$



y
$y_1 = x$
$y_2 = g(x)$
x_2
x_1
x_0
x
(a)
y
$y_1 = x$
$y_2 = g(x)$
x_0
x
(b)
y
$y_2 = g(x)$
$y_1 = x$
x_0
x
(c)
y
$y_2 = g(x)$
$y_1 = x$
x_0
x
(d)

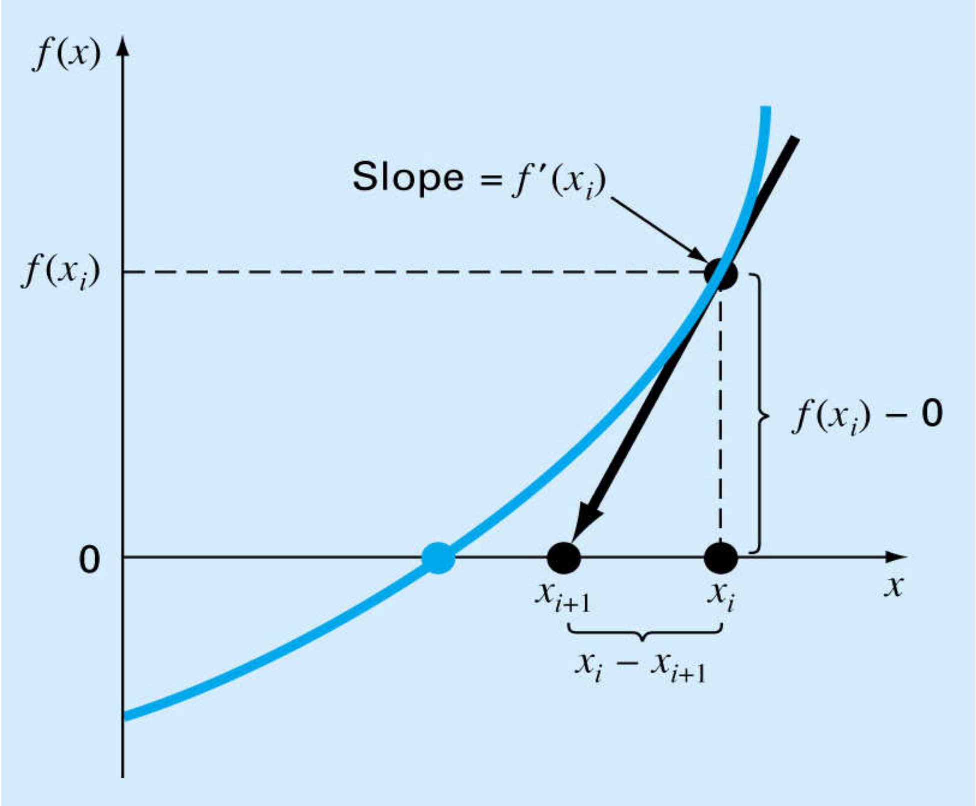
Newton-Raphson Method

- Most widely used formula for locating roots.
- Can be derived using **Taylor series** or the geometric interpretation of the slope in the figure

$$f'(x_i) = \frac{f(x_i) - 0}{(x_i - x_{i+1})}$$

rearrange to obtain :

$$\boxed{x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}}$$



$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

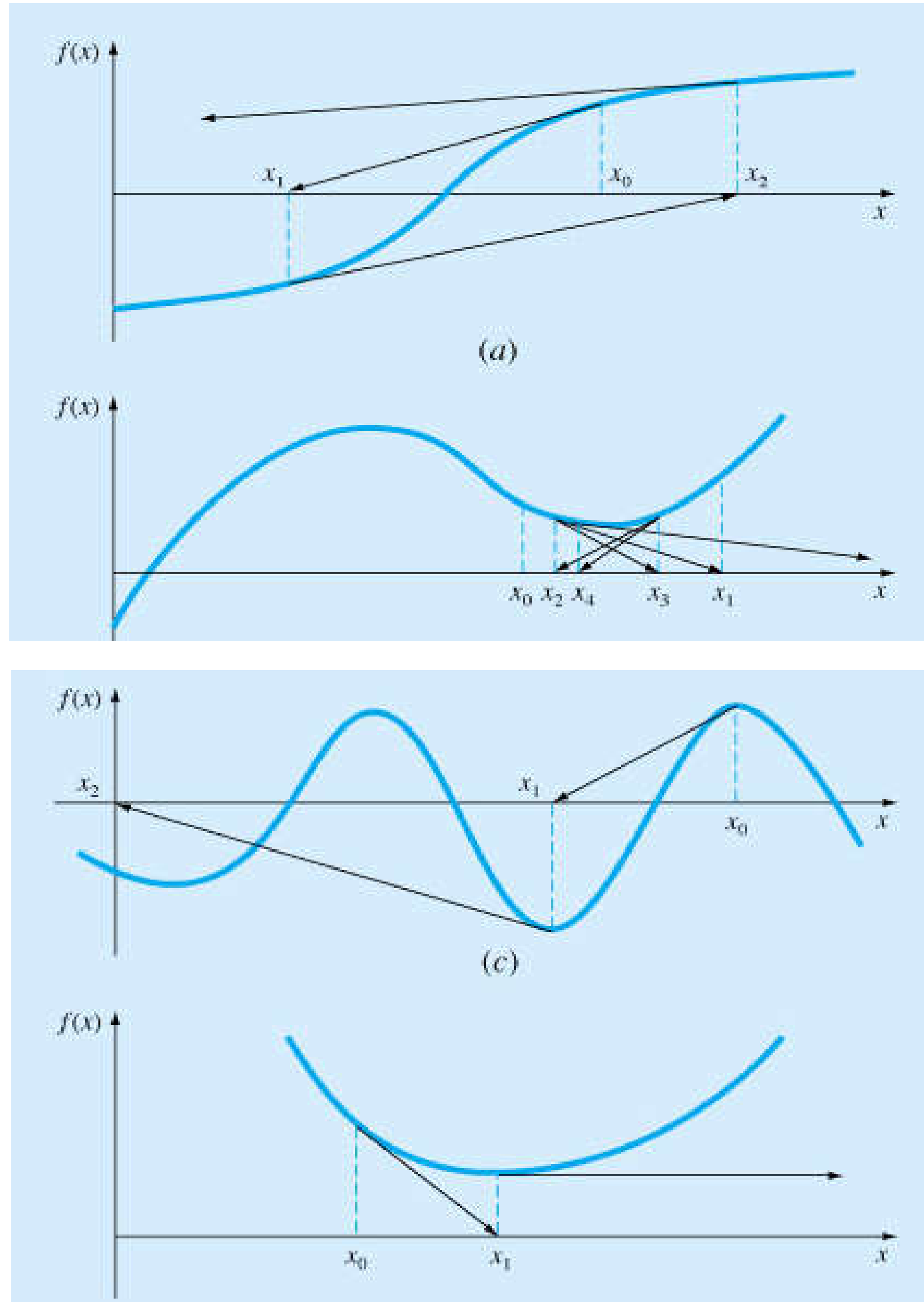
- ***Newton-Raphson*** is a convenient method if *f'(x)* (the derivative) can be evaluated ***analytically***
- Rate of convergence is quadratic, i.e. the error is roughly proportional to the square of the previous error

$$E_{i+1} = O(E_i^2)$$

(proof is given in the Text)

But:

- it does not always converge ➔➔➔
- There is **no convergence criterion**
- Sometimes, it may converge very very slowly (see next slide)



Example : Slow Convergence

Find the positive roots of :

$$f(x) = x^{10} - 1$$

N - R Formula :

$$x_{i+1} = x_i - \frac{x_i^{10} - 1}{10x_i^9} \qquad \text{use} \quad x_o = 0.5$$

Iteration	x
0	0.5
1	51.65
2	46.485
3	41.8365
4	37.65285
5	33.887565
.	
.	
38	1.083
∞	1.0000000

The Secant Method

- If derivative $f'(x)$ can not be computed analytically then we need to compute it numerically (*backward finite divided difference method*)

RESULT: N-R ➔becomes➔ SECANT METHOD

$$\textbf{Newton - Raphson :} \qquad x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$f'(x_i) = \frac{df}{dx} \cong \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$$

$$\textbf{Secant :} \quad x_{i+1} = x_i - f(x_i)\frac{x_i - x_{i-1}}{f(x_i) - f(x_{i-1})} \qquad i = 1,2,3,\ldots$$

The Secant Method

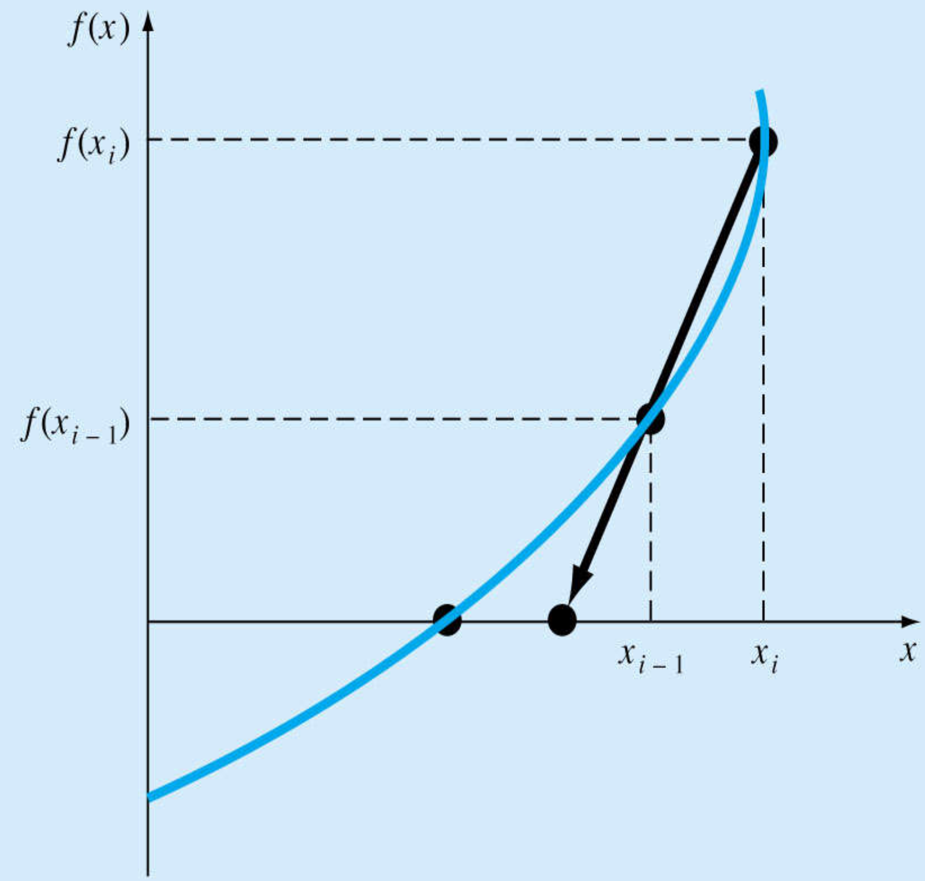
$$x_{i+1} = x_i - f(x_i)\frac{x_i - x_{i-1}}{f(x_i) - f(x_{i-1})}$$

- Requires two initial estimates x_o, x_1.

  However, it is not a "bracketing" method.

- *The Secant Method* has the same properties as *Newton*'s method.

  Convergence is not guaranteed for all x_o, f(x).



Modified Secant Method

Newton - Raphson :

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Original Secant :

$$x_{i+1} = x_i - f(x_i)\frac{x_i - x_{i-1}}{f(x_i) - f(x_{i-1})}$$

Use a small perturbation fraction δ to compute

$$f'(x_i) = \frac{f(x_i + \delta x_i) - f(x_i)}{\delta x_i}$$

in the original N - R formula :

Modified Secant :

$$x_{i+1} = x_i - f(x_i)\frac{\delta x_i}{f(x_i + \delta x_i) - f(x_i)}$$